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A Modified Differential Transform Method for Solving Nonlinear Systems

Saber T.R. Syouri[1], I. M. Sulaiman[2, *], Ibrahim M.M. Alghrouz[3], M. Mamat[4]

[1,2,4]Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu, Malaysia.
[3]Mathematics Department, Faculty of Science and Technology Al-Quds University Palestine.

Abstract

Many natural phenomena are now being modelled by fractional calculus, however, there is still need for improvements of the present numerical approaches due to the non-local property of the fractional derivative and difficulty in solving problems related to physical units. In this paper, we present an improved numerical method based on Newton-Raphson fractional method (NFM) for solution of some nonlinear systems in complex space. Unlike in the Newton-Raphson fractional method, the commands of fractional derivatives are replaced with functions which is valid for one and several variables. The conformable fractional derivative of fractional order α *was employed to replace the first order derivative in the Newton method. Numerical results have been presented which show that the proposed numerical approach is efficient and promising.*

Keywords: *Newton's Method, Fractional Calculus, Conformable Fractional Derivative, Fractional Newton Method*

1. Introduction

Fractional calculus is a branch of mathematical analysis that has been in existence since 17th century. However, only recently has engineers and scientist realize that the fractional calculus provides an efficient and better alternative to natural phenomena whose solutions has proven to be complex [1]. Some of these complex phenomena includes identification of systems, non-Brownian motion, material viscoelasticity, processing of signal, and many more. The non-local properties of the fractional derivatives enable it to describe complex systems involving long-memory in an improved way. These led to the study of numerical method for analyzing the computational data described in a fractional way [2].

However, the two the major difficulties faced when solving problems in fractional calculus include: solving problems related physical units such as determining the velocity or acceleration of a particle whose physical units of meters and second appear to non-integer exponents rather than the differential operators of integer order [3-5]. This makes the fractional derivatives seems to have no practical meaning. The second problem is knowing the order "α" that should be used as an optimal solution when solving problems related to fractional operators. Recently, the second problem has been the topic of discussion in the study of fractional calculus by numerous researchers [4]. To overcome these difficulties, for the first case of solving problems related to physical unit, any equation involving fractional operators would be dimensionless. This led to the study of applications of a dimensionless nature [5]. In the second case, different "α" orders would be used in fractional operators to solve some of the problems before applying consistent standard in choosing the order "α" that provides the optimal solution.

Motivated by these difficulties, many researches have studied several numerical methods such as the Newton method specifically for problems related to the search for roots in the complex polynomial space. To obtain the complex root of a polynomial using the Newton's method, a complex initial condition x_0 that would lead to a complex solution is required. However, this initial condition can also lead to a real solution. Suppose the

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subsequent root is real, then, the initial condition needs to be replaced in order to obtain the complex solution. This process is continued until the desired solution is obtained. This recurrence process is known as iteration and its similar to what happens in fractional operators which also involves computing using various values of α until a desired solution is obtained. For further reading on numerical methods for fractional calculus, please refer to [6-10, 13].

2. Modified Differential Transform Method

Considering the Newton's method (NM) from fractional calculus perspective, it is presumed that a fixed α order exists, in such case, $\alpha = 1$, with varying condition of x_0 until the desired solution is obtained. Contrarily, changing the order of α and leaving the initial condition x_0 fixed would produce the fractional Newton's method (FNM). This result is obtained using any definition of fractional derivative on Newton's method based on the function one is working with. Applying the real initial conditions would leads to obtaining the roots of the problems in complex space since the fractional operators do not generally carry polynomials to polynomials.

Let $P^n(\mathcal{R})$ denote a polynomial space of degree less than or equal to *"n"* with real coefficients. The zeros z of a function $f \epsilon P^n(\mathcal{R})$ are usually referred to as roots. The Newton's method is one of the famous methods used for obtaining the root of the function $f \epsilon P^n(\mathcal{R})$. However, this method cannot find the roots of $z \epsilon\, C$, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, i = 0,1,2, \dots \quad (1)$$

has an initial condition $x_0 \epsilon\, C$ [11]. To overcome this drawback of the Newton's method, we develop an efficient method that is capable of finding the real and complex roots of a polynomial provided the initial condition x_0 is real. Given the fractional derivative $D_\alpha f(x)$ being applied to solve $f(z) = 0$ in both real and complex domains, this proposed scheme replace the first order derivative $f'(z) = 0$ in Newton's method by the conformable fractional derivative of fractional order α [8]:

$$f^{\alpha}(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \forall x > 0, 0 < \alpha < 1 \quad (2)$$

where $f^{(\alpha)}(x)$ denotes any fractional derivative that meets the following condition of continuity in the order of the derivative

$$\lim_{\alpha \to 1} f^{\alpha}(x) = f'(x) \quad (3)$$

From (1), (2) and (3), we get

$$x_{i+1} = \emptyset(x_i, \alpha) = x_i - \frac{f(x_i)}{f^{\alpha}(x_i)}, \forall x > 0, 0 < \alpha < 1, i = 0,1,2, \dots \quad (4)$$

where:

$$f^{\alpha}(x_i) = D_\alpha f(x)|_{x-x_i}. \quad (5)$$

such that:

$$\lim_{x \to z}\left(f^{\alpha}(x)\right) = f^{\alpha}(z), \ \|f^{\alpha}(z)\| \neq 0. \quad (6)$$

To ensure there exist a simple root with a convergent arrangement [3], observe that the right hand-side of the equation of fractional Newton method can be treated as an iteration function $\phi(x,\alpha)$

$$\emptyset(x,\alpha) = x - \frac{f(x_i)}{f^{\alpha}(x)}, \forall x > 0, 0 < \alpha < 1 \tag{7}$$

where α is the parameter. When ϕ is regular enough, the first and the second derivatives can be calculated and Taylor's series expansion of the function ϕ around a zero 'z' of P can be performed with a parameter of α. But, when ϕ is sufficiently regular, then, the first and second derivative can be determined and Taylor's series expansion of the function ϕ around zero z of $f \epsilon P^n(\mathcal{R})$ can be achieved. On the other hand, if ϕ is not sufficiently normal, then some regularization with the use of integral operators is necessary and the expansion of Taylor's series is obtained for regularized ϕ. The fractional Newton method converges at least linearly when $\alpha \neq 1$ and at least quadratically when $\alpha = 1$ for polynomials with zeros of multiplicity one [12].

The general procedure for generating the function graph using the proposed conformable fractional derivative in Fractional Newton's method is presented in Algorithm 1. In the algorithm, the convergence test is defined as (x_{i+1}, x_i, δ),

$$|P(x_{k+1})| < \delta, \ |x_{k+1} - x_k| < \delta, \tag{8}$$

where δ is any arbitrary numbers. This convergence test will return $TRUE$ if the method has converged to the root, and *FALSE* otherwise. The algorithm searches for the nearest root for the approximation x_{i+1} by using the metric module and colors the starting point with the root color. If the number of iterations performed is equal to M, the starting point will receive the color black. The root gets a color other than black in the attraction coloring basins. Next, if the number of iterations performed is less than M, the algorithm will be used for the approximation x_{i+1}. However, if the number of iterations performed is equal to M, the starting point will obtain the color black. The starting point is colored in iteration color by calculating the number of iterations performed on the color in the graph.

***Algorithm* 1: Function graph generation**

Input: $f \in \mathbb{C}$ such that f is a fractional differentiable function, α- derivative order, $A \subset \mathbb{C}$ area
M - the maximum number of iterations, δ − accuracy, color graph $[0..C-1]$ with C colors.
Output: the suitable function's root and function graph for the area A.

1. for $x_0 \in A$ do
2. $i = 0$
3. while $i < M$ do
4. $x_{i+1} = x_i - \frac{f(x_i)}{f^{\alpha}(x_i)}$
5. if $convergence\ Test(x_{i+1}, x_i, \delta) = TR$
6. $\left| \begin{array}{l} break \\ i = i + 1 \end{array} \right.$
7. $color\ z$ "$with\ the\ used\ of\ FNM$" $as\ a\ root$

The following conditions are necessary when solving for the zeros z of a function f using the modified fractional Newton's method based on Algorithm 1.

(a). A partition of the interval [-2,2] is formed as follows without considering the -1 and 0 entries,

$$-2 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_s < \alpha_{s+1} = 2,$$

and using the partition, the following sequence $\{\alpha_n\}_{n=1}^{s}$ is established.

(b). A non-negative tolerance *TOL* < 1 and an iteration limit $L_{IT} > 1$ are set for all α_n.

(c). A value $\delta > TOL > 0$ is set to use $\alpha_{f(xi)}$ for $\delta > 0$, given by:

$$\alpha_{f\,(xi)} := \begin{cases} \alpha, & if\ \|f(x_i)\| \geq \delta\ and\ \ \|x_i\| \neq 0, \\ 1, & if\ \|f(x_i)\| \geq \delta\ or\ \ \|x_i\| = 0\ . \end{cases}$$

In addition, a fractional derivative is taken that satisfies the condition of continuity as the following:

$$\lim_{\alpha \to 1} f^{(\alpha)}(x) = f'(x),$$

(d). An initial condition $x_0 \neq 0$ and the value of *M*, such that $M > L_{IT}$ are set for all α_n.

(e). The iteration function *"FNM"* is used with all the values of the partition $\{\alpha_n\}_{n=1}^{s}$, and for each value α_n a sequence $\{n_{x_i}\}_{i=1}^{T_n}$, is generated, where

$$T_n = \begin{cases} I_1 \leq L_{IT}, & \exists i > 0\ such\ that\ \|f(n_{x_i})\| \geq M\ \forall i \geq k \\ I_2 \leq L_{IT}, & \exists\ i > 0\ such\ that\ \|f(n_{x_i})\| < TOL\ \forall i \geq k \\ L_{IT}, & if\ \|f(n_{x_k})\| \geq TOL\ \forall k \geq 0. \end{cases}$$

Then a sequence $\{x_{T_n}\}_{n=1}^{r}$ is generated, with $r \leq s$, such that

$$\|f(x_{T_n})\| \leq TOL, \text{ for all } n \geq 1.$$

(f). A value $\varepsilon > 0$ is set and the values x_{T_1} and x_{T_2} are taken. $X_1 = x_{T_1}$ is defined, so if this is the case,

$$\|X_1 - x_{T_2}\| \leq \varepsilon \quad \text{and} \quad T_2 \leq T_1\,, \qquad X_2 = x_{T_2} \quad \text{is defined.}$$

On the other hand, if,

$$\|X_1 - x_{T_2}\| > \varepsilon\,, X_2 \leq x_{T_2} \qquad \text{is defined.}$$

Without loss of generality, the second condition can be believed to be met, then if we take $X_3 = x_{T_3}$ and test the above conditions for X_1 *and* X_2 values. The above process is repeated for all values x_{T_n}, with $n \geq 5$, and that generates a sequence $\{X_n\}_{n=1}^{t}$, with $t \leq r$, such that:

$$\|X_i - X_j\| > \varepsilon, \text{ for all } i \neq j$$

Following the above steps to apply the Newton fractional method, a subset of the solution set of roots can be obtained from function *f*, both real and complex, [11].

3. Numerical Results and Discussion

The results of graphic and numerical experiments are discussed in this section. The experiments are conducted on several functions using the development of the definitions of the fractional derivative of Caputo and Riemann-Liouville (2) to the conformable fractional derivative definition with order *α*. Various values of parameter $\alpha \epsilon (0,2)$ are considered. The graphical examples are obtained through the methods of polynomial or any function's graph.

Example 1
Let the following function:

$$f(x) = x^3 - 8 \,,$$

then the following values are set to use iteration function $\phi(x_i, \alpha)$ in fractional Newton methods such that:

$$TOL = 5.0000e - 08, \quad L_{IT} = 12, \quad \delta = 0.005, \quad x_0 = 1, \quad M = e + 3, \quad \alpha = \frac{1}{3}$$

Solution:
The roots using the iteration function:

$$x_{i+1} = \phi(x_i, \alpha) = x_i - \frac{f(x_i)}{f^{\alpha}(x_i)}, \forall x > 0, 0 < \alpha < 1, i = 1,2, \ldots$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{\left(\frac{1}{3}\right)}(x_i)}, \forall x > 0, 0 < \alpha < 1, i = 1,2, \ldots, 8.$$

Then, through the conformable fractional derivative, we'll find;

$$f^{\frac{1}{3}}(x) = 3x^{\frac{8}{3}}$$

$$x_{i+1} = x_i - \frac{(x_i)^3 - 8}{3(x_i)^{\frac{8}{3}}}, \forall x > 0, i = 1,2, \ldots, 8.$$

$fractional\ \alpha - derivative\ (alpha)\ \alpha := 1/3$
$Enter\ the\ number\ of\ decimal\ places := 7$
$epsilon\ =\ 5.0000e - 08$
$Enter\ the\ initial\ approximation\ x_0 := 1$
$The\ Root\ is:\ 2.0000$

$No. of\ Iterations:\ 22$.
The obtained result is presented in Table 2 below

Table 1: Results obtained using Modified Fractional Newton's Method

i	x_{i+1}	$f(x_i)$	$f^{\frac{1}{3}}(x_i)$	$\phi(x_i, \alpha)$	$\lvert x_{i+1} - x_i \rvert$
0		-7	3	3.333333	2.333333
1	3.333333	29.03704	74.38144	2.942953	0.39038
2	2.942953	17.48884	53.35931	2.615197	0.327756
3	2.615197	9.886002	38.94632	2.36136	0.253837
4	2.36136	5.167001	29.66338	2.187173	0.174188
5	2.187173	2.46283	24.1811	2.085323	0.101849
6	2.085323	1.06818	21.29366	2.035159	0.050164
7	2.035159	0.429368	19.95493	2.013642	0.021517
8	2.013642	0.164824	19.39727	2.005145	0.008497
9	2.005145	0.061897	19.17976	2.001918	0.003227
10	2.001918	0.023033	19.09756	2.000712	0.001206

11	2.000712	0.008541	19.06689	2.000264	0.000448
12	2.000264	0.003163	19.05551	2.000098	0.000166
13	2.000098	0.001171	19.05129	2.000036	0.000061
14	2.000036	0.000433	19.04973	2.000013	0.000023
15	2.000013	0.00016	19.04915	2.000005	0.000008
16	2.000005	0.000059	19.04894	2.000002	0.000003
17	2.000002	0.000022	19.04886	2.000001	0.000001
18	2.000001	0.000008	19.04883	2	0
19	2	0.000003	19.04882	2	0
20	2	0.000001	19.04882	2	0
21	2	0	19.04881	2	0
22	2	0	19.04881	2	0

Example 2

Let the following function:

$$f(x) = \sin(x) - \frac{3}{2x},$$

then the following values are set to use iteration function $\phi(x_i, \alpha)$ in fractional Newton methods such that:

$$TOL = 5e - 10, \quad L_{IT} = 12, \quad x_0 = 0.29, \quad M = e + 3, \ \alpha = \frac{1}{2}$$

***Solution*:**

The roots using the iteration function: And using the conformable fractional derivative the results of the following Tables are obtained

$$x_{i+1} = \emptyset(x_i, \alpha) = x_i - \frac{f(x_i)}{f^{\alpha}(x_i)}, \forall x > 0, 0 < \alpha < 1, i = 1,2, \dots$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{\left(\frac{1}{2}\right)}(x_i)}, \forall x > 0, \alpha = \frac{1}{2}, i = 1,2, \dots, 8.$$

Then, through the conformable fractional derivative, we'll find;

$$f^{\frac{1}{2}}(x) = (x)^{\frac{1}{2}} \cos(x) - \frac{3}{2}(x)^{\frac{-3}{2}}$$

$$x_{i+1} = x_i - \frac{sin(x_i) - 2(x_i)^2}{(x_i)^{\frac{1}{2}} \cos(x) + \frac{3}{2}(x)^{\frac{-3}{2}}}, \forall x_i > 0, i = 1,2, \dots, 8.$$

$fractional\ \alpha - derivative\ (alpha)\ \alpha := 0.5$
$Enter\ the\ number\ of\ decimal\ places := 9$
$epsilon\ =\ 5.0000e - 10$
$Enter\ the\ initial\ approximation\ x_0 := 0.29$
$The\ Root\ is:\ 1.503412$

$No. of\ Iterations:\ 16$

The obtained result is presented in Table 2 below.

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Table 2: Results obtained using Modified Fractional Newton's Method for Problem 2

i	x_{i+1}	$f(x_i)$	$f^{\frac{1}{2}}(x_i)$	$\phi(x_i,\alpha)$	$\lvert x_{i+1}-x_i\rvert$
0		-4.88646	10.12096	0.772806	0.482806
1	0.772806	-1.24283	2.837322	1.210836	0.43803
2	1.210836	-0.3029	1.5134	1.410983	0.200147
3	1.410983	-0.07583	1.083997	1.480939	0.069955
4	1.480939	-0.01691	0.941515	1.498894	0.017956
5	1.498894	-0.00332	0.905353	1.502563	0.003669
6	1.502563	-0.00062	0.897983	1.503255	0.000692
7	1.503255	-0.00012	0.896594	1.503383	0.000128
8	1.503383	-2.1E-05	0.896337	1.503407	0.000024
9	1.503407	-4E-06	0.896289	1.503411	0.000004
10	1.503411	-1E-06	0.896281	1.503412	0.000001
11	1.503412	0	0.896279	1.503412	0
12	1.503412	0	0.896279	1.503412	0
13	1.503412	0	0.896279	1.503412	0
14	1.503412	0	0.896279	1.503412	0
15	1.503412	0	0.896279	1.503412	0
16	1.503412	0		1.503412	0

4. Conclusions

This paper presents a modified FNM method for solving nonlinear systems. The proposed algorithm replaced the first order derivative in the Newton method with conformable fractional derivative of fractional order α and the results obtained are encouraging compare results obtained by [11]. This implies that the proposed method is effective and can be used as alternatives for obtaining polynomial roots using only real initial conditions.

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