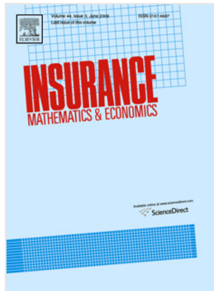
# Accepted Manuscript

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PII: S0167-6687(10)00043-0
DOI: 10.1016/j.insmatheco.2010.04.004
Reference: INSUMA 1494

To appear in: *Insurance: Mathematics and Economics*

Received date: August 2009
Revised date: March 2010
Accepted date: 14 April 2010

Please cite this article as: Date, P., Mamon, R., Jalen, L., Wang, I.C., A linear algebraic method for pricing temporary life annuities and insurance policies. *Insurance: Mathematics and Economics* (2010), doi:10.1016/j.insmatheco.2010.04.004

# A linear algebraic method for pricing temporary life annuities and insurance policies

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**Abstract**

We recast the valuation of annuities and life insurance contracts under mortality and interest rates, both of which are stochastic, as a problem of solving a system of linear equations with random perturbations. A sequence of uniform approximations is developed which allows for fast and accurate computation of expected values. Our reformulation of the valuation problem provides a general framework which can be employed to find insurance premiums and annuity values covering a wide class of stochastic models for mortality and interest rate processes. The proposed approach provides a computationally efficient alternative to Monte Carlo based valuation in pricing mortality-linked contingent claims.

*Key words:* stochastic interest rate models, stochastic mortality models, annuity, insurance premium

## 1 Introduction

In recent years, there have been extensive studies examining the issue of pricing annuity and insurance products under a stochastic mortality setting. The first milestone in this field was brought about by the contribution of Lee & Carter (1992) who developed a model for central mortality rates as a random process. This was later improved by various authors (e.g., Renshaw & Haberman (2003) and Brouhns et al. (2002)). The evolution of mortality as a stochastic variable in discrete time was proposed by Lee (1992) and Cairns et al. (2006*b*). Since then, continuous-time models for mortality emerged (see for example, Carriere (1994), Milevsky & Promislow (2001), Dahl (2004) and Luciano & Vigna (2005)).

On the other hand, the theory of stochastic modelling of interest rates is a well-developed area nowadays. We note, however, that the classic literature on pricing annuities started with deterministic discount factors, see Kellison (1991). Bowers et al. (1997) introduced the valuation of annuities when interest rates are random variables but no specific dynamics are given. Random interest rate formulation was also previously explored in Zaks (2001) whose characterisation focused on the mean and variance of the accumulation factor assuming rates are independent and identically distributed random variables. Certain results in Zaks (2001) were later modified by Burnecki et al. (2003).

More recently, the pricing of certain derivatives subject to both mortality and financial risks has generated considerable attention. For instance, Schrager (2006) valued guaranteed annuity options using affine term structure models, which lead to closed-form solutions in certain cases. In Ballotta & Haberman (2006), a different approach is taken based on Heath-Jarrow-Morton methodology, as proposed in Heath et al. (1992), of modelling the evolution of arbitrage-free forward interest rate curve. This leads to Monte Carlo-based evaluation of prices of guaranteed annuity options. Brigo & Mercurio (2006) put forward the use of a more general framework when processes follow affine dynamics. Lin & Cox (2005) and Gaillardetz (2008) valued life insurance products under stochastic interest rates in a discrete time set-up. Jalen & Mamon (2009) employed the change of reference probability technique together with the Bayes' rule for conditional expectations to price life insurance contracts under stochastic mortality and interest rates assumed not independent of each other. The problem of hedging insurance derivatives is discussed in Milevsky & Promislow (2001) who argued the possibility of hedging the risks due to interest rates as well as mortality by using a replicating portfolio involving insurance contracts, annuities and default-free bonds. Apart from the papers mentioned here, authors of several other papers exploit the similarities between the force of mortality and instantaneous interest rate to develop mortality derivatives pricing methodologies; see Dahl (2004), Cairns et al. (2006*a*), Oliveri & Pitacco (2008) and the references therein, among others.

This paper introduces a new method to evaluate the fair price of annuity and to determine the life insurance premiums under stochastic interest rates and stochastic mortality. We assume the existence of risk-neutral specification as explained in Cheyette (1998). Hamilton (1988) also utilised this framework to test the unbiased expectation hypothesis of the term structure of interest rates. We offer an alternative approach to those employed in the above-mentioned papers by reducing the valuation problem under stochastic interest rate and force of mortality into the problem of solving a system of simultaneous linear equations with random coefficients. A method for solution to problems of this type was developed in Date et al. (2007), which is used here to derive formulae for accurate approximation of annuity and insurance premiums in terms of the conditional moments of one-period future spot rates and one-period force of mortality. We show how to obtain the conditional moments of future interest rates and force of mortality in terms of the parameters of standard affine term structure models.

The issue of having interest rates and mortality rates that are both positive almost surely is also addressed by our approach as we can begin with positive rates and control the perturbation at each time step. Note that in some classical models, interest rates can become negative with a positive probability (e.g., Vasicek (1977)) and certain modelling assumptions (e.g., mortality governed by affine processes assumed in Luciano & Vigna (2005)) can also lead to negative mortality rates. We give conditions in our formulation that ensure both the interest and the mortality rates remain positive. This augments with greater generality the perspectives embedded in the studies conducted by Koch & De Schepper (2004) and De Schepper et al. (1997) attempting to restrict interest rate evolution in order to meet special types of financial or actuarial constraints.

To demonstrate the applicability and advantage of the proposed method in this paper, we compare our approximate valuation method of annuity and insurance products with valuation using Monte Carlo simulation method, given the risk-neutral dynamics of interest rate and force of mortality.

The scheme of this paper is as follows. In section 2, we set up the equivalent problem of solving a system of linear equations with random coefficients as mentioned above and certain notation will be defined. Section 3 outlines the results on approximate solution of such systems of equations from Date et al. (2007), which are relevant in the context of this paper. Section 4 brings together the results of the two previous sections to provide a constructive procedure for approximate pricing of annuities and temporary life policies under mortality and interest rate risk. In section 5, numerical examples are presented to illustrate the implementation of our pricing approach. The final section summarises our contributions and outlines some further research directions.

## 2 A linear algebraic formulation of annuity and insurance valuation problem

### 2.1 Notation

Throughout this paper, boldface characters indicate real vectors whilst matrices will be represented by capitalised letters. Let

$r_i =$ one-period interest rate (or short rate) during the time interval $[t_{i-1}, t_i]$,
$\lambda_i =$ one-period force of mortality during the time interval $[t_{i-1}, t_i]$,
$p_i =$ "running" present value of future cash flows at time $t_i$.

We assume that all of our processes are well-defined under a complete probability space $(\Omega, \mathcal{F}, P)$ where $P$ is risk-neutral and all expectations in the succeeding discussion are understood to be taken under this probability measure. We suppose $r_i$ and $\lambda_i$ are non-negative random variables. Write $\mathcal{F}_i^r$ for the information set generated by the interest rate process $r := \{r_i : i \geq 1\}$ and $\mathcal{F}_i^\lambda$ for the information set generated by the mortality rate process $\lambda := \{\lambda_i : i \geq 1\}$. Furthermore, define $\mathcal{F}_i := \mathcal{F}_i^r \vee \mathcal{F}_i^\lambda = \sigma(\mathcal{F}_i^r \cup \mathcal{F}_i^\lambda)$.

In the absence of mortality risk, the present value at time $t_i$ of a cash flow of 1 unit payable at time $t_N > t_i$ is given by

$$D_r(N, i) := \frac{1}{\prod_{j=i}^{N-1}(1 + r_{j+1})},$$

which is a random variable adapted to $\mathcal{F}_i^r$. Clearly, the conditional expected value of $D_r(N, i)$ refers to the price at time $t_i$ of a zero-coupon bond having a face value of 1 unit at time $t_N$.

On the other hand, if the interest rates are identically zero and the mortality risk is the only risk, the present value at time $t_i$ of 1 unit cash flow payable at time $t_N > t_i$ is given by

$$D_\lambda(N, i) := \frac{1}{\prod_{j=i}^{N-1}(1 + \lambda_{j+1})},$$

which is also a random variable but it is adapted to $\mathcal{F}_i^\lambda$. The conditional expected value of $D_\lambda(N, i)$ is referred to as the the survival probability, which is the probability that an individual who is alive at time $t_i$ survives until time $t_N$.

The conditional expectation of the product of $D_r(N, i)$ and $D_\lambda(N, i)$ with respect to the joint filtration $\mathcal{F}_i$ given by

$$\mathbb{E}(D_\lambda(N,i)D_r(N,i)|\mathcal{F}_i) = \mathbb{E}\left(\left.\frac{1}{\prod_{j=i}^{N-1}(1+\lambda_{j+1})(1+r_{j+1})}\right|\mathcal{F}_i\right) \quad (1)$$

is the valuation formula in obtaining the price at time $t_i$ of a pure endowment contract; i.e., a contract that entitles the contract holder 1 unit if he survives the time period $t_N - t_i$. The discrete-time framework in modelling the evolution of interest and mortality rates in this paper is similar to the one used in Milevsky & Promislow (2001).

We assume that the market for mortality products is arbitrage-free. Furthermore, we suppose that the dynamics of $r_i$ and $\lambda_i$ are specified under a risk-neutral probability measure. Note that the markets for mortality-related products are seldom complete as not all derivative prices can be spanned by tradable securities. However, we shall focus on obtaining an accurate approximation of price under a pre-specified risk-neutral measure.

In what follows, the short rate $r_i$ and the force of mortality are assumed to be of the form

$$r_{i+1} = g_1(r_i) + f_1(r_i)\, v_{i+1}, \quad (2)$$
$$\lambda_{i+1} = g_2(\lambda_i) + f_2(\lambda_i)\, w_{i+1}, \quad (3)$$

where the functions $g_i(\cdot) : [0,1) \mapsto [0,1)$, $f_i(\cdot) : [0,1) \mapsto [0,1)$, $i = 1,2$ are known and deterministic. The sequences of random variables $\{v_i\}$ and $\{w_i\}$ are independent, identically distributed and satisfy $\mathbb{E}(v_i) = \mathbb{E}(w_i) = 0$ where $\mathbb{E}$ denotes the risk-neutral expectation operator, as before. The initial short rate $r_0 \in [0,1)$ and the initial force of mortality $\lambda_0 \in [0,1)$ are assumed to be known. The processes governing the dynamics of interest rate and force of mortality are typically assumed to be Markovian and whose specifications are sufficiently general; most standard, single-factor models employed in modelling the short rate will reduce to this structure after discretisation, apart from the restriction on the domains of $f_i$ and $g_i$, which we comment upon later. Continuous-time analogues of models of this type have been employed for the dynamics of force of mortality in Dahl (2004) and Schrager (2006). The function $g_2(\cdot)$ is parametrised by age in practice, since the probability that a person alive at time $t_{i-1}$ will survive until time $t_i$ depends on the age of that person at time $t_i$; see Schrager (2006) for an example of such a process. The results in this paper assume a flat term structure with respect to age and can easily be generalised for an age-dependent process $g_2(\cdot)$. It is also assumed that $v_i$ and $w_i$ are defined on a time-varying finite support such that

$$\mathbb{P}\left(f_1(r_{i-1})v_i \in (-g_1(r_{i-1}), 1-g_1(r_{i-1}))\right) = 1, \text{ and}$$
$$\mathbb{P}\left(f_2(\lambda_{i-1})w_i \in (-g_2(\lambda_{i-1}), 1-g_2(\lambda_{i-1}))\right) = 1$$

holds at each time $t_i$. This condition ensures that the one-period interest rate and the force of mortality stay within the interval $[0, 1)$. From a practical point of view, this is a reasonable requirement.

With this notation, we shall now consider the annuity pricing problem and the insurance premium valuation problem separately. In the next two subsections, we show that both problems may be solved by determining the solution of a system of linear equations with random coefficients. The method of finding approximate solution of such systems of linear equations is discussed later in section 3.

### *2.2 Temporary life annuity valuation problem*

Let $x_i$ denote the payment which the annuitant (e.g., a pensioner) receives at the end of the period $(t_{i-1}, t_i)$, $i = 1, 2, \cdots, N$. Then the discounted present values $p_i$'s of the future annuity payments may be defined by a recursive relation

$$\begin{aligned} p_{N-1} &= \frac{x_N}{(1+\phi_N)}, \\ p_i &= \frac{p_{i+1} + x_{i+1}}{(1+\phi_{i+1})},\ i \in [0, N-2], \end{aligned} \tag{4}$$

where $\phi_i = r_i + \lambda_i + r_i\lambda_i$, so that $1 + \phi_i = (1 + r_i)(1 + \lambda_i)$. This relationship may be written as a system of linear equations as follows.

**Lemma 1** *The future annuity payments and its present value at each time $t_i$ may be shown to be related by*

$$\mathbf{x} = (Q + \Phi)\mathbf{p} \tag{5}$$

*where*

$$\begin{aligned} [Q]_{ij} &= 1 \text{ if } i = j, \\ &= -1 \text{ if } i = j+1, \\ &= 0 \text{ otherwise.} \end{aligned} \tag{6}$$

$$\begin{aligned} [\Phi]_{ij} &= \phi_{N-i+1} \quad \text{if } i = j, \\ &= 0 \text{ otherwise.} \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{x} &:= \begin{bmatrix} x_N \ x_{N-1} \ \cdots \ x_1 \end{bmatrix}^\top, \\ \mathbf{p} &:= \begin{bmatrix} p_{N-1} \ p_{N-2} \ \cdots \ p_0 \end{bmatrix}^\top, \end{aligned} \tag{8}$$

*where $\top$ denotes transpose of a matrix or a vector.*

**Proof :** This may easily be proved using (4) . ∎

Given the information concerning the distribution of $r_i$ and $\lambda_i$, the solution to the system of random linear equations in (5) provides us with the statistics of the running present value $p_i$ at time $t_i$.

### *2.3 Insurance valuation problem*

Let $y_i$, $i = 0, 1, \cdots, N-1$ represent the insurance premium payable at the beginning of period $(t_i, t_{i+1})$ and let $m_i$ be the death benefit payable to the beneficiary at the end of period $(t_{i-1}, t_i)$ for $i = 1, 2, \ldots, N$. Note that $y_i$ and $m_i$ are defined on different, but adjacent time intervals, so that both the actual payoffs occur at time $t_i$. The death benefit $m_i$ need not be a constant for all $t_i$. As an example of non-constant death benefits, *mortgage life insurance* products in the UK have a death benefit which decreases over time. Using (1), the present value of the premium payments at time $t_0$ in our set-up is given by

$$y_0 + \sum_{i=1}^{N-1} y_i D_\lambda(i,0) D_r(i,0).$$

On the other hand, the present value of death benefit at time $t_0$ is given by the summation of discounted payoffs as

$$\sum_{i=1}^{N} m_i D_\lambda(i-1,0) D_r(i,0)\left(1 - \frac{1}{1+\lambda_i}\right),$$

where we assume that $D_\lambda(0,0) = 1$. Note that, for $i \geq 1$, $D_\lambda(i-1,0)$, $D_r(i,0)$ and $\left(1 - \frac{1}{1+\lambda_i}\right)$ are independent random variables. Further, the expected value of the last term, *viz.* $\mathbb{E}\left(1 - \frac{1}{1+\lambda_i}\right)$ is the probability that a person who is alive at time $t_{i-1}$ dies before time $t_i$ and therefore triggers payoff $m_i$ at time $t_i$. Using the basic actuarial principle:

$$\begin{aligned} \text{Expected Present Value} \;&=\; \text{Expected Present Value} \qquad (9)\\ \text{of Premiums} \;&\phantom{=}\; \text{of Death Benefit,} \qquad (10) \end{aligned}$$

we can determine the premium payments $y_i$ from the equation

$$\mathbb{E}\left(y_0 + \sum_{i=1}^{N-1} \frac{y_i}{\prod_{j=1}^{i}(1+r_j)(1+\lambda_j)}\right) = \mathbb{E}\left(\sum_{i=1}^{N} \frac{1}{\prod_{j=1}^{i}(1+r_j)} \frac{1}{\prod_{j=1}^{i-1}(1+\lambda_j)} \frac{m_i \lambda_i}{1+\lambda_i}\right). \tag{11}$$

Equivalently, from the above we have

$$\mathbb{E}\left(\sum_{i=0}^{N-1} \frac{y_i}{\prod_{j=0}^{i}(1+r_j)(1+\lambda_j)}\right) = \mathbb{E}\left(\sum_{i=1}^{N} \frac{m_i \lambda_i}{\prod_{j=1}^{i}(1+r_j)(1+\lambda_j)}\right), \tag{12}$$

where we assume $r_0 = \lambda_0 = 0$ in order to make the left hand side of equation (12) well-defined. To simulate the process $r_i$ and $\lambda_i$, we could start with respective initial values $r_1$ and $\lambda_1$, both of which apply to the interval $[t_0, t_1]$. Note that equation (1) is just a special case of the left hand side of (12) where the valuation time $t_i$ in (1) is $t_0$ in (12) and the last payment date of $t_N$ in (1) is replaced by $t_{N-1}$ in (12). In the pure endowment case, all the $y_i$'s are zero except at time $t_N$, which is a unit amount.

We can express the above equation as a systems of linear equations using an argument similar to the one used in section 2.2. The discounted present values of future insurance premiums, $\tilde{p}_i$ at time $t_i$, may be defined using a recursive relation

$$\begin{aligned} \tilde{p}_{N-1} &= y_{N-1}, \\ \tilde{p}_i &= y_i + \frac{\tilde{p}_{i+1}}{(1+\phi_{i+1})}, \ i \in [0, N-2]. \end{aligned} \tag{13}$$

Similarly, the discounted present values of death benefit, $\tilde{d}_i$ at time $t_i$, may be defined by

$$\begin{aligned} \tilde{d}_{N-1} &= \frac{\lambda_N m_N}{(1+\phi_N)}, \\ \tilde{d}_i &= \frac{\tilde{d}_{i+1} + \lambda_{i+1} m_{i+1}}{(1+\phi_{i+1})}, \ i \in [0, N-2]. \end{aligned} \tag{14}$$

Now define vectors

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} y_{N-1} \ y_{N-2} \ \cdots \ y_0 \end{bmatrix}^\top, \\ \tilde{\mathbf{p}} &= \begin{bmatrix} \tilde{p}_{N-1} \ \tilde{p}_{N-2} \ \cdots \ \tilde{p}_0 \end{bmatrix}^\top, \\ \tilde{\mathbf{d}} &= \begin{bmatrix} \tilde{d}_{N-1} \ \tilde{d}_{N-2} \ \cdots \ \tilde{d}_0 \end{bmatrix}^\top, \\ \mathbf{m} &= \begin{bmatrix} m_N \ m_{N-1} \ \cdots \ m_1 \end{bmatrix}^\top, \quad \text{and} \\ \boldsymbol{\lambda} &= \begin{bmatrix} \lambda_N \ \lambda_{N-1} \ \cdots \ \lambda_1 \end{bmatrix}^\top. \end{aligned}$$

Finally, given the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ with $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_N \ \alpha_{N-1} \cdots \alpha_1 \end{bmatrix}^\top$ and $\boldsymbol{\beta}$ defined similarly, let $vec(\alpha\beta)$ denote a vector with the element-wise product $\alpha_i \beta_i$ as its $i^{th}$ element. With this notation, the insurance premium and the death benefit can be linked through a system of linear equations as follows.

**Lemma 2** *The following relationships hold:*

$$(I + \Phi)\mathbf{y} = (Q + \Phi)\tilde{\mathbf{p}} \quad \textit{and} \tag{15a}$$
$$vec(\boldsymbol{\lambda}\mathbf{m}) = (Q + \Phi)\tilde{\mathbf{d}}, \tag{15b}$$

*where the matrices $Q$ and $\Phi$ are as in (6) and $I$ is the identity matrix. Further, (12) may be written as*

$$\mathbf{e}_1^\top \tilde{\mathbf{p}} = \mathbf{e}_1^\top \tilde{\mathbf{d}}, \tag{16}$$

*where* $\mathbf{e}_1 = \begin{bmatrix} 0 \; 0 \cdots \; 0 \; 1 \end{bmatrix}^\top$.

**Proof :** This may be proved using (13)-(14) and re-arranging (12). ∎

Note that $\tilde{\mathbf{p}}$ is affine in premium $\mathbf{y}$ and $\tilde{\mathbf{d}}$ is affine in death benefit $\mathbf{m}$. This crucial fact in the above lemma allows us to compute a fair, constant insurance premium $y_c$ for a given mortality and interest rate dynamics and for a given death benefit vector $\mathbf{m}$ as follows.

(1) Find an approximation to $\mathbb{E}\left(\mathbf{e}_1^\top \tilde{\mathbf{d}}\right)$.
(2) Find an approximation to $\mathbb{E}\left(\mathbf{e}_1^\top \tilde{\mathbf{p}}\right)$, corresponding to a unit insurance premium (*i.e.* $y_i = 1, i = N-1, N-2, \ldots, 0$).
(3) The constant insurance premium is then given by

$$y_c = \frac{\mathbb{E}\left(\mathbf{e}_1^\top \tilde{\mathbf{d}}\right)}{\mathbb{E}\left(\mathbf{e}_1^\top \tilde{\mathbf{p}}\right)}.$$

Conversely, we can use the equality (16) to compute the death benefit for a given insurance premium. The method to construct an approximation to $\mathbb{E}\left(\mathbf{e}_1^\top \tilde{\mathbf{d}}\right)$ or $\mathbb{E}\left(\mathbf{e}_1^\top \tilde{\mathbf{p}}\right)$ for given interest rate and mortality dynamics will be discussed in the next two sections and the approximation procedure will be illustrated through a numerical examples in section 5.

## 3 Approximate solution of system of linear equations with random coefficients

The essence of Lemmas 1 and 2 is the respective re-formulation of the annuity valuation and fair insurance premium problems as problems of solving systems of linear equations with random coefficients. Specifically, both results require solving systems of the form

$$\mathbf{f} = (Q + \Psi)\mathbf{z}, \tag{17}$$

where $\mathbf{z} = \begin{bmatrix} z_N & z_{N-1} & \cdots & z_1 \end{bmatrix}^\top$ is the random vector with unknown statistics, $Q$ is defined in (6), $\mathbf{f} = \begin{bmatrix} f_N & f_{N-1} & \cdots & f_1 \end{bmatrix}^\top$ is a known vector and $\Psi$ is a diagonal matrix with random elements $\psi_i$ along the diagonal. For the two applications of this methodology considered in this paper, $\psi_{N-i+1} = \phi_i = r_i + \lambda_i + r_i\lambda_i$. We outline a method (first proposed in Date et al. (2007)) of constructing uniformly convergent approximations to the solution of such systems. In what follows, we use the standard definitions of a vector induced matrix norm denoted by $\|\cdot\|$ and the matrix 2−norm denoted by $\|\cdot\|_2$ (cf. chapter 5 of Horn & Johnson (1999)):

$$\|A\| = \sup_{\|\mathbf{z}\|=1} \|A\mathbf{z}\|,$$
$$\|A\|_2 = \sup_{\|\mathbf{z}\|_2=1} \|A\mathbf{z}\|_2 = \sqrt{\mathrm{eig}\,(A^\top A)},$$

where $\|\mathbf{z}\|$ is the corresponding vector norm for a vector $\mathbf{z}$. Any function that maps the space of matrices to the non-negative real line and satisfies the axioms of a matrix norm is denoted by $|||\cdot|||$.

The next theorem, which summarises the relevant results from Date et al. (2007), provides a uniform approximation of the statistics of vector $\mathbf{z}$.

**Theorem 3** *Suppose* $\max_i |\psi_i| < 1$, $f_i$ *satisfies the condition* $\max_i |f_i| < \gamma$ *for some* $\gamma < \infty$ *and the inverse of* $(Q + \Psi)$ *exists with probability* 1. *Also assume that* $\mathbb{P}\left(\|Q^{-1}\Psi\|_2 < 1\right) = 1$. *Write*

$$\mathbf{z} := (Q+\Psi)^{-1}\mathbf{f} \quad \textit{and} \quad \mathbf{z}^{(L)} := \sum_{i=0}^{L} (-Q^{-1}\Psi)^i Q^{-1}\mathbf{f} \tag{18}$$

*with* $(-Q^{-1}\Psi)^0 = I$. *Then*

*(i)* $\quad \mathbf{z}^{(L)} \to \mathbf{z}$ *with probability* 1.

*(ii)* $\quad \lim_{L\to\infty} \mathbb{E}\left(\|\mathbf{z}^{(L)} - \mathbf{z}\|_2^2\right) = 0.$

*Furthermore, the following statements hold:*

*(iii) With probability* 1,

$$\|\mathbf{z}^{(L)} - \mathbf{z}\| \le \frac{\|Q^{-1}\Psi\|^{L+1}}{1 - \|Q^{-1}\Psi\|}\|Q^{-1}\mathbf{f}\| \tag{19}$$

*for any vector-induced norm* $\|\cdot\|$, *provided* $\|Q^{-1}\Psi\| < 1$.

*(iv)* *If* $\psi_{min}, \psi_{max}$ *are positive constants such that,* $\psi_i \in (\psi_{min}, \psi_{max})\, \forall i$*, then*

$$\|Q^{-1}\Psi\|_2 \in \left(\psi_{min}\sqrt{\frac{N+1}{2}},\ \psi_{max}\sqrt{\frac{N(N+1)}{2}}\right), \tag{20}$$

*almost surely.*

*(v)* *If* $\psi_{min}, \psi_{max}$ *are defined as above, then for any* $\epsilon > 0$*, there exists a matrix norm* $|||\cdot|||$ *s.t.* $|||Q^{-1}\Psi||| \in (\psi_{min}, \psi_{max} + \epsilon)$.

**Proof :** See Date et al. (2007). ∎

The expression for $\mathbf{z}^{(L)}$ is a multivariate polynomial in $\psi_i$ and $f_i$ (involving only the inverse of a deterministic matrix $Q$) and it is therefore significantly simpler than the expression for $\mathbf{z}$ which involves a direct inversion of a random matrix $(Q + \Psi)$. In the annuity valuation problem, this suggests a simple way of approximating the expected value of the vector $\mathbf{p}$ in terms of $\mathbb{E}\left(\sum_{i=0}^{L}(Q^{-1}\Phi)^i Q^{-1}\mathbf{x}\right)$, which, in turn can be expressed in terms of the moments of $r_i$ and $\lambda_i$. In a similar fashion, one may use (15a) to approximate the expected value of $\tilde{\mathbf{p}}$ in terms of the moments of $r_i$ and $\lambda_i$ as

$$\tilde{\mathbf{p}} = \mathbb{E}\left(\sum_{i=0}^{L}(Q^{-1}\Phi)^i Q^{-1}\mathbf{y} + \sum_{i=1}^{L+1}(Q^{-1}\Phi)^i\mathbf{y}\right).$$

This formulation is independent of the specific choice of stochastic processes assumed for $r_i$ and $\lambda_i$ and depends only on the availability of expressions for joint conditional moments.

The computation involved in finding low order moments of $\mathbf{z}^{(L)}$ is simpler than it appears. The matrices $Q^{-1}\Psi$, $(Q^{-1}\Psi)^2$ and the vector $Q^{-1}\mathbf{f}$ have particularly simple forms, as shown in Date et al. (2007). As an example, for $N = 3$ we have

$$Q^{-1}\Psi = \begin{bmatrix} \psi_3 & 0 & 0 \\ \psi_3 & \psi_2 & 0 \\ \psi_3 & \psi_2 & \psi_1 \end{bmatrix},$$

$$Q^{-1}\mathbf{f} = \begin{bmatrix} f_3 \\ f_3 + f_2 \\ f_3 + f_2 + f_1 \end{bmatrix}.$$

Using the above expressions for $Q^{-1}\Psi$ and $Q^{-1}\mathbf{f}$ along with the definition of $\mathbf{z}^{(L)}$ in (18), it is possible to establish the expressions for $\mathbb{E}(\mathbf{e}_1\mathbf{z}^{(L)})$ for the cases when $L \leq 3$, i.e., the first, second and third order approximations, to the current price of a general cash flow with $\psi_j$ as the discounting factor for the period $(t_{j-1}, t_j)$:

$$\mathbb{E}\left(\mathbf{e}_1^\top \mathbf{z}\right) = \mathbb{E}\left(\sum_{i=1}^{N} \frac{f_i}{\Pi_{j=1}^{i}(1+\psi_j)}\right). \tag{21}$$

The method presented here thus provides a simple yet rigorous and accurate approximation to the solution of pricing a general cash flow (which could be a stream of annuity payments or life insurance premiums) under stochastic interest rates and stochastic force of mortality.

Higher order approximations ($\mathbf{z}^{(L)}$ for $L > 3$) may also be derived in a straightforward manner although the resulting expressions may have an unwieldy form. However, such expressions involving $(-Q^{-1}\Psi)^{(L)}$ can be calculated easily using any standard symbolic algebra package (e.g., MATLAB's Symbolic Math Toolbox).

Finally, from part (iv) of Theorem 3, we note that $\psi_{max}$ is a bound on $\psi_i$ *per period* which is *not* annualised and this is likely to be a small number. Since $\frac{N(N+1)}{2} \leq N^2$, a sufficient condition for the bound $\|Q^{-1}\Psi\|_2 < 1$ to be satisfied is $N\psi_{max} < 1$. However, the error bound in (20) is still conservative. As may be seen from its proof in Date et al. (2007), this conservative error bound stems from the use of trace of a positive semi-definite matrix bounding from above its maximum eigenvalue. Whilst this bound is an equality in the worst case, it is very conservative for well-conditioned matrices. It is hard to impose a constraint on the condition number of $Q^{-1}\Psi$ in terms of a relevant constraint on $\psi_i$ or in terms of constraints on $\lambda_i$ and $r_i$. However, the errors in practice seem to be far less than those suggested by (19), as demonstrated in the latter section of this paper containing our numerical experiments.

## 4 Approximation of annuity prices and insurance premiums

So far, we made two contributions in the previous two sections. First, it was shown in section 2 that annuity pricing and insurance premium valuation problems can be modelled as problems of solving systems of linear equations with random coefficients. Second, an approximation to the solution of such systems of equations was provided in section 3 using the results in Date et al. (2007). In particular, the expected value of the solution were shown to be expressible in terms of the moments of the random coefficients, *viz.* $\phi_i = r_i + \lambda_i + r_i\lambda_i$. In this section, we will provide the expressions for moments of $\phi_i$ for commonly used models of interest rate and mortality dynamics. This finally allows us to evaluate the approximate solutions to annuity pricing and insurance premium valuation problems in closed-form. We consider a class of generic affine term structure models of the following form:

$$r_{i+1} = (a_1 + b_1 r_i) + \sqrt{c_1 + d_1 r_i}\, v_i, \tag{22}$$
$$\lambda_{i+1} = (a_2 + b_2 \lambda_i) + \sqrt{c_2 + d_2 \lambda_i}\, w_i, \tag{23}$$

where $a_j, b_j, c_j$ and $d_j, j = 1, 2$, are non-negative deterministic functions of time and $v_i$, $w_i$ are bounded, uncorrelated and zero mean random variables. We assume that $|r_i| < 1$ and $|\lambda_i| < 1$ hold almost surely. Apart from these restrictions, the models above are similar to standard affine term structure models in the literature. In particular, the model specified in (22) with appropriate time-varying coefficient $a_1$ is similar to the CIR++ model as discussed in Brigo & Mercurio (2006) whereas $a_2$ may be chosen to be a function of age of the annuitant or the insured person; see Schrager (2006) and the references therein.

For the models described in equations (22)-(23), we can compute the first three moments of $\phi_i$ as follows.

**Lemma 4** *Let $\phi_i = r_i + \lambda_i + r_i\lambda_i$. Suppose that $r_0, \lambda_0$ are given and the dynamics of $r_i, \lambda_i$ are governed by (22)-(23). Then the following holds:*

$$\mathbb{E}(\phi_i) = \mathbb{E}(r_i) + \mathbb{E}(\lambda_i) + \mathbb{E}(r_i)\mathbb{E}(\lambda_i), \tag{24}$$
$$\begin{aligned}\mathbb{E}(\phi_i\phi_j) = {} & \mathbb{E}(r_i r_j)\left(1 + \mathbb{E}(\lambda_i) + \mathbb{E}(\lambda_j) + \mathbb{E}(\lambda_i\lambda_j)\right) + \\ & \mathbb{E}(\lambda_i\lambda_j)\left(1 + \mathbb{E}(r_i) + \mathbb{E}(r_j)\right) + \mathbb{E}(r_i)\mathbb{E}(\lambda_j) \\ & + \mathbb{E}(r_j)\mathbb{E}(\lambda_i),\end{aligned} \tag{25}$$
$$\mathbb{E}(\phi_i\phi_j\phi_k) = \mathbb{E}(\phi_i)\mathbb{E}(\phi_j)\mathbb{E}(\phi_k), \tag{26}$$

*where the expectations are taken conditional on available information up to time $t_0$, and*

$$\mathbb{E}(r_i) = a_1 \sum_{k=0}^{i-1} b_1^k + b_1^i r_0, \tag{27}$$
$$\mathbb{E}(r_i r_j) = \mathbb{E}(r_i)\mathbb{E}(r_j) + \sum_{k=1}^{j} b_1^{i+j-2k}(c_1 + d_1\mathbb{E}(r_{k-1})), \tag{28}$$
$$\mathbb{E}(\lambda_i) = a_2 \sum_{k=0}^{i-1} b_2^k + b_2^i \lambda_0, \tag{29}$$
$$\mathbb{E}(\lambda_i\lambda_j) = \mathbb{E}(\lambda_i)\mathbb{E}(\lambda_j) + \sum_{k=1}^{j} b_2^{i+j-2k}(c_2 + d_2\mathbb{E}(\lambda_{k-1})). \tag{30}$$

*Without loss of generality, we assume $i \geq j$ in (28) and (30).*

**Proof:** This follows by straightforward algebraic manipulation of (22)-(23). In particular, the simple formula for third order moment follows from the assumptions about symmetry and independence of the random variables $v_k$ and $w_k$. ∎

Lemmas 1, 2 and 4 together with Theorem 3 enable us to build closed-form approximate solutions to annuity valuation and insurance premium computation problems. Provided the model structures allow us to express the conditional moments of short rate and the force of mortality in terms of model parameters, these approximate solutions can be computed in closed-form. It is worth stressing here that (24)-(26) are valid for virtually *any* choices of functions $g_i$ and $f_i$ in (2)-(3). The subsequent parametric expressions for conditional moments are valid for the specific and practically relevant class of models given in (22)-(23). Further generalisation of our approach to multi-factor models is conceptually straightforward so long as the model in question allows us to compute joint conditional moments. We have focussed here on single factor models mainly for notational simplicity; for instance, see Wang (2008) for linear algebraic approximations to bond prices using multi-factor models. In contrast to the proposed approach, exact closed-form solutions are only possible for very specific forms of $f_i$ and $g_i$ (*e.g.*, linear Gaussian models or CIR type models). Our approach allows us to use more flexible and potentially more accurate models for interest rate and force of mortality whilst retaining numerical tractability.

The next section demonstrates this fact with numerical examples.

## 5 Numerical examples

We compare the results of our method for annuity valuation and insurance premium calculation with those generated from Monte Carlo experiments. All the numerical experiments in this section were performed on a desktop computer with 1.83 GHz dual core processor and 2GB RAM. The software used was MATLAB R2009b running under Windows 7.

It is assumed that $r_i$ follows the dynamics in (22) with the following constant parameters, taken from the example in Date et al. (2007): $a_1 = 0.0027, b_1 = 0.2634, c_1 = 0, d_1 = 0.000024$, with $r_1 = 0.0041$. For the mortality risk dynamics, we take $a_2 = 0, b_2 = 0.10859, c_2 = 0.0000002304, d_2 = 0$, and $\lambda_1 = 0.000734$ in equation (23). This gives the same linear Gaussian spot mortality process as the one used in Luciano & Vigna (2005), on a monthly, rather than annual scale.

For the annuity valuation problem, the future annuities are assumed to be increasing at a constant rate ($x_0 = 1$, $x_i = 1.004074 \times x_{i-1}, i = 1, 2, \ldots, N$), which corresponds to a 5% annual increase. Results for second-order and third-order approximation to the present value of this annuity using Lemma 1 and Theorem 3 are compared with Monte Carlo simulation results using 50000 sample paths for different values of time horizon $t_N$. The time-step for simulation, which is also the time-step for computation of one period conditional moments in our approximation, is one month. This comparison is exhibited in Table 1. The second column of table 1 gives Monte Carlo (MC) value of the annuity whilst the third and the fourth columns give the second and the third order approximations respectively. Table 2 provides computation times for Monte Carlo valuation as well as for closed-form approximations proposed in this paper. It can be seen that the percentage difference between the third order approximation of present value and the Monte Carlo evaluation of the same is less than 0.2% for all the values of $N$ considered, while the computation time for the proposed approximations is less than 2% of the time required for Monte Carlo evaluation for each value of $N$ considered.

These experiments were repeated for a deterministic mortality risk with the same values of $a_2, b_2, d_2$ and $\lambda_1$ as above but with $c_2 = 0$ and the results obtained were quite similar; with the difference between Monte Carlo valuation and the third order approximation being less than 0.2% in all cases and the time required for third order approximation being less than 15% of that required for Monte Carlo valuation. Detailed results for deterministic mortality case are omitted for brevity.

**Table 1: Comparison of present values of annuities**

| $N$ | MC val | 2nd ord. val | 3rd ord. val |
|---|---|---|---|
| 12 | 11.9172 | 11.9174 | 11.9172 |
| 24 | 23.7749 | 23.7775 | 23.7758 |
| 36 | 35.5642 | 35.5723 | 35.5636 |
| 48 | 47.2766 | 47.2961 | 47.2675 |
| 60 | 58.8868 | 58.9470 | 58.8735 |
| 72 | 70.3590 | 70.5289 | 70.3665 |
| 84 | 81.7330 | 82.0532 | 81.7303 |
| 96 | 92.9519 | 93.5417 | 92.9468 |
| 108 | 104.0980 | 105.0300 | 103.9960 |
| 120 | 115.0250 | 116.5730 | 114.8550 |

**Table 2: Comparison of time in seconds for computation of present values of annuities**

| N | MC time | 2nd ord. time | 3rd ord. time |
|---|---|---|---|
| 12 | 3.666 | $< 0.001$ | $< 0.001$ |
| 24 | 7.441 | 0.016 | 0.016 |
| 36 | 11.607 | 0.015 | 0.015 |
| 48 | 15.116 | 0.032 | 0.032 |
| 60 | 18.892 | 0.046 | 0.062 |
| 72 | 22.496 | 0.078 | 0.109 |
| 84 | 26.286 | 0.141 | 0.172 |
| 96 | 30.046 | 0.202 | 0.249 |
| 108 | 33.884 | 0.281 | 0.343 |
| 120 | 37.736 | 0.39 | 0.468 |

For the computation of insurance premium, we consider a fixed death benefit $m_i = M = 100,000$ and find a constant $y_i = y$ using Lemma 2 and Theorem 3. The parameters for the interest rate and the mortality dynamics are the same as those assumed above in the annuity valuation case. Table 3 compares the monthly premium obtained by Monte Carlo simulation with 50000 sample paths with the premium obtained using second order closed-form approximation proposed in this paper. It can be seen that the percentage difference between the second order approximation of the insurance premium and a Monte Carlo evaluation of the same is less than 0.7% for all values of $N$ considered. Table 4 shows that the computation time for the proposed approximation method is less than 5% of the time required for Monte Carlo evaluation of insurance premium, for each value of $N$ considered.

**Table 3: Comparison of insurance premiums**

| $N$ | MC val | 2nd ord. val |
|---|---|---|
| 12 | 76.7524 | 77.1250 |
| 24 | 80.7469 | 81.2383 |
| 36 | 85.0584 | 85.4454 |
| 48 | 89.1674 | 89.7265 |
| 60 | 93.7498 | 94.0580 |
| 72 | 98.2253 | 98.4128 |
| 84 | 102.7650 | 102.7610 |
| 96 | 106.9440 | 107.0710 |
| 108 | 110.8890 | 111.3110 |
| 120 | 115.0910 | 115.4550 |

**Table 4: Comparison of time in seconds for computation of insurance premiums**

| N | MC time | 2nd ord. time |
|---|---|---|
| 12 | 3.978 | $< 0.001$ |
| 24 | 7.862 | 0.016 |
| 36 | 11.731 | 0.047 |
| 48 | 15.538 | 0.124 |
| 60 | 19.407 | 0.234 |
| 72 | 23.103 | 0.375 |
| 84 | 26.957 | 0.608 |
| 96 | 30.966 | 0.904 |
| 108 | 34.788 | 1.295 |
| 120 | 38.423 | 1.779 |

## 6 Conclusion

In this paper, we extended our recent research on a linear algebraic approach to pricing deterministic cash flows under stochastic interest rates to pricing temporary life policies and annuities under stochastic interest rates and mortality risk. Numerical examples illustrate the accuracy and substantial advantage in terms of speed for this pricing method when compared to Monte Carlo simulation.

Implementing this method to price more complex insurance products such as guaranteed annuity options is a topic of current research. An equally important and challenging research investigation is the pricing of perpetuities under this framework. The conditions on existence of stable distributions as $N$ goes to infinity were studied in the past by Cairns (1995) and Dufresne (1990). It would be practically relevant to examine whether the approximations similar to the ones suggested in this paper for temporary policies may be derived for the limiting distributions when perpetuities are considered.

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