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> **To link to this article**: DOI: 10.1109/LSP.2014.2352362 URL: <u>http://dx.doi.org/10.1109/LSP.2014.2352362</u>

To cite this version: Besson, Olivier *On false alarm rate of matched filter under distribution mismatch.* (2015) IEEE Signal Processing Letters, vol. 22 (n° 2). pp. 167-171. ISSN 1070-9908

On False Alarm Rate of Matched Filter Under Distribution Mismatch

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Abstract-The generalized likelihood ratio test (GLRT) is a very widely used technique for detecting signals of interest amongst noise, when some of the parameters describing the signal (and possibly the noise) are unknown. The threshold of such a test is set from a desired probability of false alarm P_{fa} and hence this threshold depends on the statistical assumptions made about noise. In practice however, the noise statistics are seldom known and it becomes crucial to characterize P_{fa} under a mismatched distribution. In this letter, we address this problem in the case of a simple binary composite hypothesis testing problem (matched filter) when the threshold is designed under a Gaussian assumption while the noise actually follows an elliptically contoured distribution. We also consider the inverse situation. Generic expressions for the assumed and actual P_{fa} are derived and illustrated on the particular case of Student distributions for which simple, closed-form expressions are obtained. The latter show that the GLRT based on Gaussian assumption is not robust while that based on Student assumption is.

Index Terms—Distribution mismatch, elliptically contoured distributions, generalized likelihood ratio test, probability of false alarm.

I. INTRODUCTION AND PROBLEM STATEMENT

C ONTROL of the false alarm rate is a crucial issue in most radar systems, as a mismatch between the presumed (designed) probability of false alarm (P_{fa}) and the actual one has very detrimental effects in any post-detection step, e.g., target tracking. Usually, a desired P_{fa} is specified and a threshold, associated to a given detection scheme, is computed based on statistical assumptions regarding the data under test. In practice, however, one cannot ensure that the received data will match its assumed distribution, and therefore it is highly desirable that a detection scheme be robust to a possible mismatch between the assumed and the actual data distribution. In this preliminary study, we investigate the relation between the designed P_{fa} and its actual value under distribution mismatch, for a conventional hypothesis testing problem, namely

$$H_0: \boldsymbol{x} = \boldsymbol{n}$$

$$H_1: \boldsymbol{x} = \alpha \boldsymbol{v} + \boldsymbol{n}.$$
 (1)

In (1), \boldsymbol{v} stands for the signature of the signal of interest and α denotes its unknown complex amplitude. \boldsymbol{n} is a random noise vector, whose assumed probability density function $p_N(\boldsymbol{n})$ is

herein considered to be known. The generalized likelihood ratio test (GLRT) for the problem at hand is given by [1], [2]

$$t = \frac{\max_{\alpha} p_N(\boldsymbol{x} - \alpha \boldsymbol{v})}{p_N(\boldsymbol{x})} \underset{H_0}{\overset{H_1}{\geq}} \eta.$$
(2)

The threshold η is computed from $P_{fa} = \Pr[t \ge \eta | H_0]$ and thus depends on $p_N(.)$. Therefore, P_{fa} can be ensured only if \boldsymbol{n} is distributed according to $p_N(.)$. In this letter, we investigate what becomes of $\Pr[t \ge \eta | H_0]$ when **n** is drawn from $\tilde{p}_N(.)$. More precisely, we first investigate the usual case where $p_N(.)$ corresponds to a Gaussian distribution while $\tilde{p}_N(.)$ corresponds to an elliptically contoured (EC) distribution [3], [4], [5]. EC distributions have been used in a number of engineering applications, including radar where they encompass the widely used compound-Gaussian model [6]. We refer the reader to [5] and references therein for a detailed discussion. In a second step, we consider the inverse situation, viz. the probability of false alarm of the GLRT based on EC distributed data when applied to Gaussian data. We provide closed-form expressions for the actual versus assumed P_{fa} in the case of Student distributions, and show that the GLRT based on a Gaussian assumption is not robust at all, while the GLRT based on a Student assumption is rather robust.

II. ANALYSIS OF GAUSSIAN GLRT (MATCHED FILTER)

When *n* follows a complex Gaussian distribution, i.e., $n \sim \mathbb{CN}(0, R)$, and the covariance matrix R > 0 is known, the GLRT, also referred to as matched filter is given by [1], [2], [7]

$$t_{\rm G} = \frac{|\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{v}|^2}{\boldsymbol{v}^H \boldsymbol{R}^{-1} \boldsymbol{v}} \underset{H_0}{\overset{H_1}{\geq}} \eta_{\rm G}.$$
 (3)

The probability of false alarm is related to $\eta_{\rm G}$ through $P_{fa}^{\rm G}(t_{\rm G},\eta_{\rm G}) = e^{-\eta_{\rm G}}$ where the superscript ^G over P_{fa} indicates that the probability of false alarm is obtained for a Gaussian distributed \boldsymbol{x} .

Our aim is to study the robustness of the test (3) when applied to elliptically distributed \boldsymbol{x} . Since we address probability of false alarm in this paper, we consider only hypothesis H_0 for which $\boldsymbol{x} = \boldsymbol{n}$, and we assume that \boldsymbol{x} , under H_0 , admits the following stochastic representation [3], [4], [5]

$$\boldsymbol{x} \stackrel{d}{=} \sqrt{\mathcal{Q}} \boldsymbol{R}^{1/2} \boldsymbol{u} \tag{4}$$

where $\stackrel{d}{=}$ means "as the same distribution as". In (4), the vector \boldsymbol{u} is uniformly distributed on the complex M-sphere, where M is the size of vector \boldsymbol{x} . This means that \boldsymbol{u} can be written as $\boldsymbol{u} = \boldsymbol{n}_w / \|\boldsymbol{n}_w\|$ with $\boldsymbol{n}_w \sim \mathbb{CN}(\boldsymbol{0}, \boldsymbol{I})$. Note that \boldsymbol{n}_w and $\|\boldsymbol{n}_w\|$ are independent [3], [8]. \mathcal{Q} is a random positive scalar whose probability density function (p.d.f.) is $p(\mathcal{Q}) = \delta_{M,g}^{-1} \mathcal{Q}^{M-1} g(\mathcal{Q})$ where $g(\cdot)$ is the so-called density generator of the elliptic distribution and $\delta_{M,g}$ is a constant which ensures that $p(\mathcal{Q})$ integrates

to one. Assuming that \mathbf{R} is positive definite, the probability density function of \mathbf{x} , under H_0 , is given by

$$p(\boldsymbol{x}|H_0) \propto |\boldsymbol{R}|^{-1} g(\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{x})$$
(5)

where \propto means "proportional to". We first examine how robust is the test in (3) when applied to \boldsymbol{x} in (4). Let $\boldsymbol{v}_w = \boldsymbol{R}^{-1/2}\boldsymbol{v}$, $\boldsymbol{P}_{\boldsymbol{v}_w} = (\boldsymbol{v}_w^H \boldsymbol{v}_w)^{-1} \boldsymbol{v}_w \boldsymbol{v}_w^H$ denote the projection onto \boldsymbol{v}_w , and $\boldsymbol{P}_{\boldsymbol{v}_w}^{\perp} = \boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{v}_w}$ the projection onto its orthogonal complement. Then, one can write that

$$t_{\rm G} = \frac{|\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{v}|^2}{\boldsymbol{v}^H \boldsymbol{R}^{-1} \boldsymbol{v}} \stackrel{d}{=} \mathcal{Q} \frac{|\boldsymbol{n}_w^H \boldsymbol{v}_w|^2}{(\boldsymbol{v}_w^H \boldsymbol{v}_w) (\boldsymbol{n}_w^H \boldsymbol{n}_w)}$$
$$= \mathcal{Q} \frac{\boldsymbol{n}_w^H \boldsymbol{P} \boldsymbol{v}_w \boldsymbol{n}_w}{\boldsymbol{n}_w^H \boldsymbol{P} \boldsymbol{v}_w \boldsymbol{n}_w + \boldsymbol{n}_w^H \boldsymbol{P}_{\boldsymbol{v}_w}^{\perp} \boldsymbol{n}_w}$$
$$\stackrel{d}{=} \mathcal{Q} \frac{\mathbb{C} \chi_1^2(0)}{\mathbb{C} \chi_1^2(0) + \mathbb{C} \chi_{M-1}^2(0)} \stackrel{d}{=} \mathcal{Q} \times \mathbb{C} \mathcal{B}_{M-1,1}(0) \quad (6)$$

where $\mathbb{C}\chi_q^2(0)$ denotes the complex central chi-square distribution with q degrees of freedom (d.o.f.) whose p.d.f. is

$$p_{\mathbb{C}\chi^2_q(0)}(x) = \Gamma(q)^{-1} x^{q-1} e^{-x}$$

and $\mathbb{C}\mathcal{B}_{p,n}(0) = (1 + \frac{\mathbb{C}\chi_p^2(0)}{\mathbb{C}\chi_n^2(0)})^{-1}$ is the complex beta distribution, whose p.d.f. is

$$p_{\mathbb{C}\mathcal{B}_{p,n}(0)}(x) = \frac{\Gamma(p+n)}{\Gamma(p)\Gamma(n)} x^{n-1} (1-x)^{p-1}; \quad x \in [0,1].$$

Consequently, if the threshold $\eta_{\rm G}$ is used as in (3), the actual probability of false alarm is given by (the superscript ^g means that the data follows an EC distribution with $g(\cdot)$ as the density generator)

$$P_{fa}^{g}(t_{\rm G},\eta_{\rm G}) = \Pr\left[\mathcal{Q} \times \mathbb{C}\mathcal{B}_{M-1,1}(0) \ge \eta_{\rm G}\right]$$
$$= \int_{0}^{\infty} \Pr\left[\mathbb{C}\mathcal{B}_{M-1,1}(0) \ge \frac{\eta_{\rm G}}{\mathcal{Q}}\right] p(\mathcal{Q}) d\mathcal{Q}$$
$$= \int_{\eta_{\rm G}}^{\infty} \left[1 - \frac{\eta_{\rm G}}{\mathcal{Q}}\right]^{M-1} p(\mathcal{Q}) d\mathcal{Q}$$
$$= \delta_{M,g}^{-1} \int_{\eta_{\rm G}}^{\infty} \left[1 - \frac{\eta_{\rm G}}{\mathcal{Q}}\right]^{M-1} \mathcal{Q}^{M-1} g(\mathcal{Q}) d\mathcal{Q}$$
$$= \delta_{M,g}^{-1} \int_{0}^{\infty} x^{M-1} g(x + \eta_{\rm G}) dx.$$
(7)

The previous formula allows one to recover the usual Gaussian case for which $g(t) = e^{-t}$, $\delta_{M,g} = \Gamma(M)$, which yields

$$P_{fa}^{\rm G}(t_{\rm G},\eta_{\rm G}) = \delta_{M,g}^{-1} \int_0^\infty x^{M-1} e^{-(x+\eta_{\rm G})} dx = e^{-\eta_{\rm G}}.$$
 (8)



Fig. 1. Actual probability of false alarm of the Gaussian GLRT when x follows a Student distribution, as a function of d. M = 32 and $P_{fa}^G(t_G, \eta_G) = 10^{-4}$.

The two equations (7) and (8) enables one to relate the actual and assumed P_{fa} . Let us illustrate the following general formula on the specific case of a Student distribution with d degrees of freedom, for which $g(t) = (1 + d^{-1}t)^{-(M+d)}$ and $\delta_{M,g}^{-1} = \frac{\Gamma(M+d)}{\Gamma(M)\Gamma(d)}d^{-M}$. Then, one has (9), shown at the bottom of the page, where, to obtain the last line, we made use of [9, 3.194.3]. Therefore, the actual probability of false alarm is related to the one assuming \boldsymbol{x} is Gaussian distributed through the relation

$$P_{fa}^{\rm S}(t_{\rm G},\eta_{\rm G}) = \left(\frac{d}{d - \log P_{fa}^{\rm G}(t_{\rm G},\eta_{\rm G})}\right)^d.$$
 (10)

The dependence of $P_{fa}^{\rm S}(t_{\rm G},\eta_{\rm G})$ versus *d* is illustrated in Fig. 1, for $P_{fa}^{\rm G}(t_{\rm G},\eta_{\rm G}) = 10^{-4}$ and M = 32. Accordingly, Fig. 2 presents $P_{fa}^{\rm S}(t_{\rm G},\eta_{\rm G})$ versus $P_{fa}^{\rm G}(t_{\rm G},\eta_{\rm G})$ for various values of *d*. Clearly, the Gaussian matched filter is not robust to deviation from the assumed data distribution as the actual P_{fa} can significantly depart from the presumed one, making this detector not desirable in practical situations.

III. ANALYSIS OF EC GLRT

Suppose now that, under H_0 , \boldsymbol{x} is distributed according to (4). In this case, assuming that g(.) is decreasing, the matched filter takes the form [10]

$$t_g = \frac{g\left(\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{x} - \frac{|\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{v}|^2}{\boldsymbol{v}^H \boldsymbol{R}^{-1} \boldsymbol{v}}\right)}{g\left(\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{x}\right)} \overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\overset{H_1}{\underset{H_1}{\overset{H_1}{\overset{H_1}{\underset{H_1}{\overset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_0}{\overset{H_1}{\overset{H_1}{\underset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\overset{H_1}{\underset{H_1}}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}}{\underset{H_1}{\underset{H_1}{\underset{H_1}{\underset{H_1}}{\underset{H_1$$

$$P_{fa}^{S}(t_{G},\eta_{G}) = \delta_{M,g}^{-1} \int_{0}^{\infty} x^{M-1} \left(1 + \frac{x + \eta_{G}}{d}\right)^{-(M+d)} dx$$

$$= \delta_{M,g}^{-1} \left(\frac{d}{d + \eta_{G}}\right)^{M+d} \int_{0}^{\infty} x^{M-1} \left(1 + \frac{x}{d + \eta_{G}}\right)^{-(M+d)} dx$$

$$= \left(\frac{d}{d + \eta_{G}}\right)^{d}$$
(9)



Fig. 2. Actual probability of false alarm of the Gaussian GLRT when x follows a Student distribution, as a function of $P_{fa}^G(t_G, \eta_G)$. Varying d. M = 32.

Then, using the fact that

$$\boldsymbol{x}^{H}\boldsymbol{R}^{-1}\boldsymbol{x} - \frac{|\boldsymbol{x}^{H}\boldsymbol{R}^{-1}\boldsymbol{v}|^{2}}{\boldsymbol{v}^{H}\boldsymbol{R}^{-1}\boldsymbol{v}} \stackrel{d}{=} \mathcal{Q}\left[1 - \frac{|\boldsymbol{n}^{H}\boldsymbol{v}_{w}|^{2}}{(\boldsymbol{v}_{w}^{H}\boldsymbol{v}_{w})(\boldsymbol{n}^{H}\boldsymbol{n})}\right]$$
$$\stackrel{d}{=} \mathcal{Q}\left[1 - \frac{\mathbb{C}\chi_{1}^{2}(0)}{\mathbb{C}\chi_{1}^{2}(0) + \mathbb{C}\chi_{M-1}^{2}(0)}\right]$$
$$\stackrel{d}{=} \mathcal{Q}\left[1 - \frac{\mathbb{C}\chi_{1}^{2}(0)}{\mathbb{C}\chi_{1}^{2}(0) + \mathbb{C}\chi_{M-1}^{2}(0)}\right]$$
$$\stackrel{d}{=} \mathcal{Q} \times \mathbb{C}\mathcal{B}_{1,M-1}(0)$$
(12)

it follows that

$$t_g = \frac{g\left(\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{x} - \frac{|\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{v}|^2}{\boldsymbol{v}^H \boldsymbol{R}^{-1} \boldsymbol{v}}\right)}{g\left(\boldsymbol{x}^H \boldsymbol{R}^{-1} \boldsymbol{x}\right)} \stackrel{d}{=} \frac{g(\mathcal{Q}\beta)}{g(\mathcal{Q})}.$$
 (13)

Consequently

$$P_{fa}^{g}(t_{g},\eta_{g}) = \Pr\left[\frac{g(\mathcal{Q}\beta)}{g(\mathcal{Q})} \ge \eta_{g}\right]$$
$$= \int_{0}^{\infty} \Pr\left[g(\mathcal{Q}\beta) \ge \eta_{g}g(\mathcal{Q})|\mathcal{Q}\right]p(\mathcal{Q})d\mathcal{Q}$$
$$= \int_{0}^{\infty} \Pr\left[\beta \le \mathcal{Q}^{-1}g^{-1}\left[\eta_{g}g(\mathcal{Q})\right]|\mathcal{Q}\right]p(\mathcal{Q})d\mathcal{Q}.$$
(14)

Since $0 \le \beta \le 1$, and $g(\cdot)$ is decreasing, we necessarily have $\eta_g \ge 1$ and thus

$$\eta_g g(\mathcal{Q}) \ge g(\mathcal{Q}) \Rightarrow g^{-1} \left[\eta_g g(\mathcal{Q}) \right] \le \mathcal{Q}$$
$$\Rightarrow \mathcal{Q}^{-1} g^{-1} \left[\eta_g g(\mathcal{Q}) \right] \le 1$$

Therefore, the integral in the previous equation is over the domain $\mathcal{D} = \{\mathcal{Q}; g^{-1}[\eta_g g(\mathcal{Q})] \ge 0\}$. Finally, we get

$$P_{fa}^{g}(t_{g},\eta_{g}) = \int_{\mathcal{D}} \left(\mathcal{Q}^{-1}g^{-1} \left[\eta_{g}g(\mathcal{Q}) \right] \right)^{M-1} p(\mathcal{Q}) d\mathcal{Q}$$
$$= \delta_{M,g}^{-1} \int_{\mathcal{D}} \left(g^{-1} \left[\eta_{g}g(\mathcal{Q}) \right] \right)^{M-1} g(\mathcal{Q}) d\mathcal{Q}.$$
(15)

Suppose now that \boldsymbol{x} is EC distributed with a density generator $\tilde{g}(.)$. Since the distribution of β is independent of g(.), the representation in (13) holds with now $p(\mathcal{Q}) = \delta_{M,\tilde{g}}^{-1} \mathcal{Q}^{M-1} \tilde{g}(\mathcal{Q})$ and hence

$$P_{fa}^{\tilde{g}}(t_g,\eta_g) = \delta_{M,\tilde{g}}^{-1} \int_{\mathcal{D}} \left(g^{-1} \left[\eta_g g(\mathcal{Q}) \right] \right)^{M-1} \tilde{g}(\mathcal{Q}) d\mathcal{Q}.$$
(16)

For instance, when x is Gaussian distributed, $Q \sim C\chi^2_M(0)$ and it ensues that

$$P_{fa}^{G}(t_{g}, \eta_{g})$$

$$= \frac{1}{\Gamma(M)} \int_{0}^{\infty} \Pr\left[g(\mathcal{Q}\beta) \geq \eta_{g}g(\mathcal{Q})|\mathcal{Q}\right] \mathcal{Q}^{M-1} e^{-\mathcal{Q}} d\mathcal{Q}$$

$$= \frac{1}{\Gamma(M)} \int_{0}^{\infty} \Pr\left[\beta \leq \mathcal{Q}^{-1} g^{-1} \left[\eta_{g}g(\mathcal{Q})\right]|\mathcal{Q}\right] \mathcal{Q}^{M-1} e^{-\mathcal{Q}} d\mathcal{Q}$$

$$= \frac{1}{\Gamma(M)} \int_{\mathcal{D}} \left(g^{-1} \left[\eta_{g}g(\mathcal{Q})\right]\right)^{M-1} e^{-\mathcal{Q}} d\mathcal{Q}.$$
(17)

Let us now apply these general formulas to the case of a Student distribution with d degrees of freedom, for which $g(t) = (1 + d^{-1}t)^{-(M+d)}$ and $\delta_{M,g}^{-1} = \frac{\Gamma(M+d)}{\Gamma(M)\Gamma(d)}d^{-M}$. It is straightforward to show that, in this case, $\mathcal{D} = \{\mathcal{Q}; \mathcal{Q} \ge \alpha^{-1}(1-\alpha)d\}$ where $\alpha_{\rm S} = \eta_{\rm S}^{-1/(M+d)}$. Consequently, $P_{fa}^{\rm S}(t_{\rm S}, \eta_{\rm S})$ is obtained as (18), shown at the bottom of the page. Hence, the threshold $\eta_{\rm S}$ can be computed very easily from $P_{fa}^{\rm S}(t_{\rm S}, \eta_{\rm S})$. Let us examine now what happens if $t_{\rm S}$ is used with the same threshold $\eta_{\rm S}$ but \boldsymbol{x} is now Gaussian distributed. Then, we simply have

$$P_{fa}^{\rm G}(t_{\rm S},\eta_{\rm S}) = \frac{1}{\Gamma(M)} \int_{\mathcal{D}} \left(\mathcal{Q}^{-1} g^{-1} \left[\eta_{\rm S} g(\mathcal{Q}) \right] \right)^{M-1} \mathcal{Q}^{M-1} e^{-\mathcal{Q}} d\mathcal{Q}$$

$$P_{fa}^{S}(t_{S},\eta_{S}) = \int_{\mathcal{D}} \left(\mathcal{Q}^{-1}g^{-1} \left[\eta_{S}g(\mathcal{Q}) \right] \right)^{M-1} p(\mathcal{Q}) d\mathcal{Q}$$

$$= \delta_{M,g}^{-1} \int_{d\frac{(1-\alpha_{S})}{\alpha_{S}}}^{\infty} \left[\alpha_{S}\mathcal{Q} + d(\alpha_{S}-1) \right]^{M-1} \left(1 + d^{-1}\mathcal{Q} \right)^{-(M+d)} d\mathcal{Q}$$

$$= \delta_{M,g}^{-1} \alpha_{S}^{M+d-1} \int_{0}^{\infty} x^{M-1} \left(1 + d^{-1}x \right)^{-(M+d)} dx$$

$$= \alpha_{S}^{M+d-1}.$$
(18)



Fig. 3. Actual probability of false alarm of the Student GLRT when x follows a Gaussian distribution. M = 32 and $P_{fa}^{S}(t_{S}, \eta_{S}) = 10^{-4}$.

$$= \frac{1}{\Gamma(M)} \int_{d\frac{(1-\alpha_{\rm S})}{\alpha_{\rm S}}}^{\infty} \left[\alpha_{\rm S} \mathcal{Q} + d(\alpha_{\rm S} - 1) \right]^{M-1} e^{-\mathcal{Q}} d\mathcal{Q}$$
$$= \frac{1}{\Gamma(M)} \frac{e^{d\frac{(1-\alpha_{\rm S})}{\alpha_{\rm S}}}}{\alpha_{\rm S}} \int_{0}^{\infty} x^{M-1} e^{-x/\alpha_{\rm S}} dx$$
$$= \alpha_{\rm S}^{M-1} e^{d\frac{(1-\alpha_{\rm S})}{\alpha_{\rm S}}}.$$
(19)

This allows to obtain the following relation between the designed probability of false alarm $P_{fa}^{\rm S}(t_{\rm S}, \eta_{\rm S})$ and the actual one $P_{fa}^{\rm G}(t_{\rm S}, \eta_{\rm S})$:

$$\log P_{fa}^{\rm G}(t_{\rm S},\eta_{\rm S}) = \frac{M-1}{M+d-1} \log P_{fa}^{\rm S}(t_{\rm S},\eta_{\rm S}) - d \frac{1 - \left(P_{fa}^{\rm S}(t_{\rm S},\eta_{\rm S})\right)^{1/(M+d-1)}}{\left(P_{fa}^{\rm S}(t_{\rm S},\eta_{\rm S})\right)^{1/(M+d-1)}}.$$
 (20)

For illustration purposes, in Fig. 3, we display $P_{fa}^{\rm G}(t_{\rm S}, \eta_{\rm S})$ versus d for $P_{fa}^{\rm S}(t_{\rm S}, \eta_{\rm S}) = 10^{-4}$. Accordingly, Fig. 4 presents $P_{fa}^{\rm G}(t_{\rm S}, \eta_{\rm S})$ versus $P_{fa}^{\rm S}(t_{\rm S}, \eta_{\rm S})$ for various values of d. Clearly, the Student matched filter, when applied to Gaussian distributed data, exhibits a very good robustness, since the actual P_{fa} is very close to the designed one. In other words, even if the data is truly Gaussian distributed, there is no significant loss in assuming that it is Student distributed as the resulting P_{fa} will be quite close to the desired one.

Finally, as a last example, consider a mismatch in the degrees of freedom of the Student distribution. In other words, the GLRT is constructed assuming a Student distribution with d_1 degrees of freedom (and subsequently the threshold is computed based on this hypothesis) and it is applied to data following a Student distribution with d_2 degrees of freedom. Using (7) and (16), and after some straightforward calculations it can be shown that

$$P_{fa}^{S_2}(t_{S_1},\eta_{S_1}) = P_{fa}^{S_1}(t_{S_1},\eta_{S_1})\alpha_{S_1}^{-d_1} \left(1 - \frac{d_1}{d_2}\frac{\alpha_{S_1}-1}{\alpha_{S_1}}\right)^{-d_2}.$$
(21)



Fig. 4. Actual probability of false alarm of the Student GLRT when x follows a Gaussian distribution, as a function of $P_{fa}^{s}(t_{\mathbb{S}}, \eta_{\mathbb{S}})$. Varying d. M = 32.



Fig. 5. Actual probability of false alarm of the Student GLRT (assuming d_1 d.o.f.) when x follows a Student distribution with d_2 d.o.f., as a function of d_2 . $P_{fa}^{S1}(t_{S_1}, \eta_{S_1}) = 10^{-4}, d_1 = 10$ and M = 32.

The relation between these two P_{fa} is illustrated in Fig. 5. It can be observed that the GLRT is not very sensitive to a wrong guess of the number of d.o.f. in the Student distribution, which is a desirable feature.

IV. CONCLUDING REMARKS

In this letter, we considered the (non-adaptive) detection of a signal of interest embedded in noise. We studied the sensitivity of the probability of false alarm of the GLRT, when the latter is designed based on some noise distribution, while the data obeys another distribution. More precisely, we considered the case where data is assumed to be Gaussian [respectively elliptically] distributed while it is actually elliptically [respectively Gaussian] distributed. Conclusions are that the Gaussian matched filter is not robust while the GLRT based on elliptical assumption guarantees a P_{fa} rather close to the designed one. This preliminary work should be pursued and extended to the more relevant case of adaptive detectors.

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