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# Approximate unconditional maximum likelihood direction of arrival estimation for two closely spaced targets

François Vincent, Olivier Besson and Eric Chaumette

**Abstract**—We consider Direction of Arrival (DoA) estimation in the case of two closely spaced sources. In this case, most high resolution techniques fail to estimate the two DoAs if the waveforms are highly correlated. Maximum Likelihood Estimators (MLE) are known to be more robust, but their excessive computational load limits their use in practice. In this paper, we propose an asymptotic approximation of the Unconditional Maximum Likelihood (UML) procedure in the case of a Uniform Linear Array (ULA) and two closely spaced targets. This approximation is based on an asymptotically (in the number of observations) equivalent formulation of the UML criterion, and on its Taylor series approximation for small DoA separation. This simplified procedure, which requires solving a 1D-optimization problem only, is shown to be accurate for source separation lower than half the mainlobe. Furthermore, it outperforms conventional high resolution algorithms in the case of two correlated sources.

**Index Terms**—Maximum Likelihood, Direction Of Arrival, Uniformly Linear Array

## I. INTRODUCTION

Spatially separating sources is a central problem in many applications such as astronomy, seismology, radar, and sonar, and it has been one of the most addressed signal processing problems in the last forty years. The main goal of these techniques is to improve over the Rayleigh resolution for DoA estimation of plane waves impinging on an array of sensors. Most popular techniques include Capon [1], MUSIC [2] [3] or root-MUSIC [4], [5], ESPRIT [6] and Min-Norm [7] in the case of a ULA. Most of these techniques can achieve near optimal performance whenever sources are uncorrelated [8] but degrade rapidly when source amplitudes tend to be correlated or even coherent. This is the case when one has to deal with multipath which is a common situation in many practical applications (Radar, Sonar, telecommunications and satellite navigation for instance). Some methods such as spatial smoothing [9] are known to improve robustness to correlation, but the price to be paid is a loss of resolution. Moreover this kind of technique is difficult to use with arrays comprising a low number of sensors. In this challenging scenario, one has to turn to optimal techniques such as Maximum Likelihood (ML). Two ML criterion can be derived, depending upon whether the source amplitudes are considered to be unknown deterministic or stochastic. The Conditional ML (CML) estimator is more general as it

doesn't make any assumptions on the source model; however, it is known to be less accurate than the UML which assumes that the source amplitudes are temporally white Gaussian variables [10]. The deterministic and stochastic Cramer Rao Bounds (CRB) have also been derived [10] and it has been shown that the UML is efficient whereas the CML is not, because the number of unknown parameters in the model grows with the number of snapshots. The main drawback of such algorithms remains their computational cost. Indeed one has to solve a non-linear multidimensional optimization problem to estimate the DoA. To reduce this complexity, some recursive simplified procedures have been proposed in the literature such as the Alternating Projection [11], the Expectation Maximization principle [12], the IQML [13] or IMODE [14]. More recently, a data-supported grid search approach has been proposed [15] and the use of genetic algorithms to solve the multidimensional optimization has shown very good results [16] [17]. But the majority of these techniques requires a good initialization and are not guaranteed to converge to the global solution.

In this paper, we consider the case of two closely spaced sources. It is the natural framework when dealing with high resolution problems. Moreover, this scenario usually appears in real life applications such as over-the-sea radar target measurement, where one principal multipath is highly correlated with the direct path, or geostationary satellite communications. This problem has been addressed in several papers, see e.g., [18]. In [19], we proposed a procedure to simplify the CML when dealing with two closely spaced targets. In this paper, we now focus on the UML estimator which is known to have higher performance. For this last criterion, the approximation proposed in [19] cannot be directly applied. We propose to use an equivalent, asymptotic expression of the UML criterion and an approximation of it in case of closely spaced targets approximation, which leads to a simple 1D search algorithm to estimate the two frequencies of interest. These Approximate UML (AUML) estimates are shown to be very close to the exact UML solution and outperform common estimators such as Root-MUSIC or ESPRIT in the case of correlated sources.

This paper is organized as follows. In the following section, we introduce the framework at hand and we recall the expressions of the UML and the corresponding CRB. Then, as described in [19], we perform a Taylor expansion of the noise

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subspace projection with respect to the spatial frequency difference in section III. This approximation, applied to an equivalent UML expression [20] allows us to calculate an estimate of the frequency difference. Then, the two spatial frequencies can be easily estimated using a 1D-minimization. Numerical simulations illustrate the performances of the AUML in section IV and our conclusions are drawn in section V.

## II. DATA MODEL

We assume that two closely spaced sources impinge on a narrow-band uniform linear array of  $M$  sensors with inter-element spacing  $d$ . Their respective DoA  $\theta_1$  and  $\theta_2$  correspond to two spatial frequencies  $f_i = \frac{d}{\lambda} \sin \theta_i$ ,  $i = 1, 2$ . For convenience, we prefer to use  $f_1$  and  $\Delta_f = f_2 - f_1$ , where  $\Delta_f$  is much smaller than the Rayleigh resolution ( $\Delta_f \ll 1/M$ ), so that the model at hand can be written as:

$$\mathbf{x}_t = \mathbf{A}(f_1, \Delta_f) \mathbf{s}_t + \mathbf{n}_t \quad t = 0, \dots, (N-1) \quad (1)$$

where

- $N$  is the number of snapshots.
- $\mathbf{A}(f_1, \Delta_f) = [\mathbf{a}(f_1) \quad \mathbf{a}(f_1 + \Delta_f)] \in \mathbb{C}^{M \times 2}$  with  $\mathbf{a}(f) = \frac{1}{\sqrt{M}} [1 \quad e^{2i\pi f} \dots e^{2i\pi f(M-1)}]^T$  denoting the normalized steering vector.
- $\mathbf{s}_t \in \mathbb{C}^2$  stands for the vector of the Gaussian amplitudes of the sources.  $\mathbf{s}_t$  is supposed zero mean circularly and temporally white:  $\mathcal{E}\{\mathbf{s}_t \mathbf{s}_{t'}^H\} = \mathbf{R}_s \delta_{t-t'}$  and  $\mathcal{E}\{\mathbf{s}_t \mathbf{s}_{t'}^T\} = \mathbf{0}$  where  $\mathbf{R}_s$  is non-singular ( $|\mathbf{R}_s| \neq 0$ ).
- $\mathbf{n}_t \in \mathbb{C}^M$  denotes the noise vector and is assumed to be zero-mean circularly Gaussian with covariance matrix  $\sigma^2 \mathbf{I}$  where  $\sigma^2$  is an unknown scalar and independent from  $\mathbf{s}_t$ . Moreover,  $\mathbf{n}_t$  is supposed to be temporally white, so that  $\mathcal{E}\{\mathbf{n}_t \mathbf{n}_s^H\} = \sigma^2 \mathbf{I} \delta_{t-s}$  and  $\mathcal{E}\{\mathbf{n}_t \mathbf{n}_s^T\} = \mathbf{0}$ .

The problem at hand consists in estimating  $f_1$  and the frequency difference  $\Delta_f$ . The UML solution is obtained by maximizing the log-likelihood function with respect to the unknown parameters. Concentrating the likelihood function with respect to  $\sigma^2$  and  $\mathbf{R}_s$ , it is well known that the UML estimator of  $f_1$  and  $\Delta_f$  is given by [21]

$$\hat{f}_1, \hat{\Delta}_f = \arg \min_{f_1, \Delta_f} \left| \mathbf{P} \hat{\mathbf{R}} \mathbf{P} + \frac{\text{Tr}\{\mathbf{P}^\perp \hat{\mathbf{R}}\} \mathbf{P}^\perp}{M-2} \right| \quad (2)$$

where  $\text{Tr}\{\cdot\}$  and  $|\cdot|$  are respectively the trace and the determinant of matrix between braces.  $\mathbf{P} = \mathbf{P}(f_1, \Delta_f)$  is the projection onto the subspace spanned by the columns of  $\mathbf{A}(f_1, \Delta_f)$  (signal subspace) and  $\mathbf{P}^\perp = \mathbf{I} - \mathbf{P}$  is the projection onto the noise subspace.  $\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=0}^{N-1} \mathbf{x}_t \mathbf{x}_t^H$  is the signal covariance matrix estimate of  $\mathbf{R} = \mathcal{E}\{\mathbf{x}_t \mathbf{x}_t^H\} = \mathbf{A} \mathbf{R}_s \mathbf{A}^H + \sigma^2 \mathbf{I}$ . Based on this stochastic model, one can also derive the associated CRB [10]:

$$\mathbf{B}_u = \frac{\sigma^2}{2N} \left( \text{Re} \left[ \mathbf{R}_s \mathbf{A}^H \mathbf{R}^{-1} \mathbf{A} \mathbf{R}_s \odot \mathbf{H}^T \right] \right)^{-1} \quad (3)$$

where  $\mathbf{H} = \Delta^H \mathbf{P}^\perp \Delta$  with  $\Delta = \left[ \frac{\partial \mathbf{a}(f)}{\partial f} \Big|_{f_1} \quad \frac{\partial \mathbf{a}(f)}{\partial f} \Big|_{f_1 + \Delta_f} \right]$ .

## III. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION FOR TWO CLOSELY SPACED SOURCES

As stated before, we focus on the case of two close spatial frequencies. We can then conduct a Taylor series expansion of the UML criterion with respect to the frequency difference  $\Delta_f$ . However, it appears to be difficult to directly apply this Taylor expansion to the UML criterion from eq. (2). Therefore, we turn to the following asymptotic (for large  $N$ ) equivalent expression,  $V(f_1, \Delta_f)$  of the UML criterion in (2), namely [20]:

$$\frac{V(f_1, \Delta_f)}{(M-2)} = \frac{\text{Tr}\{\mathbf{P}^\perp \hat{\mathbf{R}}\}}{(M-2)} - \frac{(M-2)}{\text{Tr}\{\mathbf{P}^\perp \hat{\mathbf{R}}^{-1}\}} \quad (4)$$

As stated in [20], it is noteworthy that this equivalent criterion is the difference of two estimators of the noise power  $\sigma^2$ . The first term,  $\frac{\text{Tr}\{\mathbf{P}^\perp \hat{\mathbf{R}}\}}{(M-2)}$  corresponds to the CML criterion, to be minimized when the sources are supposed deterministic [8]. The second one,  $\frac{(M-2)}{\text{Tr}\{\mathbf{P}^\perp \hat{\mathbf{R}}^{-1}\}}$  is another noise power estimator as it can be noticed that  $(\mathbf{P}^\perp \hat{\mathbf{R}}^{-1})$  tends towards  $(\frac{\mathbf{P}^\perp}{\sigma^2})$  when the number of snapshots increases.

This alternative criterion shows a simpler dependence with respect to the noise subspace projection matrix which contains the frequencies we are looking for. Considering the following relationship between the two steering vectors,

$$\mathbf{a}(f_1 + \Delta_f) = \mathbf{D}(\Delta_f) \mathbf{a}(f_1) \quad (5)$$

where  $\mathbf{D}(\Delta_f) = \text{diag}([1 \quad e^{2i\pi\Delta_f} \dots e^{2i\pi\Delta_f(M-1)}])$  in the case of ULA, we can easily develop the expression of  $\mathbf{P}$  and exhibit its dependence on  $\Delta_f$ :

$$\begin{aligned} \mathbf{P}(f_1, \Delta_f) &= \frac{1}{1 - |c(\Delta_f)|^2} [\mathbf{a}(f_1) \mathbf{a}^H(f_1) + \\ &\mathbf{D}(\Delta_f) \mathbf{a}(f_1) \mathbf{a}^H(f_1) \mathbf{D}^H(\Delta_f) - c(\Delta_f) \mathbf{a}(f_1) \mathbf{a}^H(f_1) \mathbf{D}^H(\Delta_f) \\ &\quad - c(\Delta_f)^* \mathbf{D}(\Delta_f) \mathbf{a}(f_1) \mathbf{a}^H(f_1)] \quad (6) \end{aligned}$$

where  $c(\Delta_f) = \frac{\text{Tr}\{\mathbf{D}(\Delta_f)\}}{M}$ .

As we are interested in the case where  $\Delta_f \ll 1$ , we can then conduct a similar Taylor series expansion as we have done in [19] both for  $\mathbf{D}$  and  $c$  (where, for the sake of notational simplicity, we have temporarily dropped the dependence with respect to  $f_1$  or  $\Delta_f$ ):

$$\mathbf{D} = \sum_k \mathbf{D}_k \Delta_f^k; \quad (7)$$

$$c = \sum_k c_k \Delta_f^k; \quad c_k = \frac{\text{Tr}\{\mathbf{D}_k\}}{M} \quad (8)$$

where  $\mathbf{D}_k = \frac{(2i\pi)^k}{k!} \text{diag}([0^k \quad 1^k \dots (M-1)^k])$ .

Substituting these expressions into equation (6), it is straightforward to show that

$$\mathbf{P} = - \frac{\sum_{n=1} \Delta_f^n \mathbf{M}_n}{\sum_{n=1} \Delta_f^n d_n} \quad (9)$$

with

$$M_n = \sum_{k=0}^n D_k \mathbf{a} \mathbf{a}^H D_{n-k}^H - c_k \mathbf{a} \mathbf{a}^H D_{n-k}^H - c_k^* D_{n-k} \mathbf{a} \mathbf{a}^H$$

$$d_n = \sum_{k=0}^n c_k c_{n-k}^*.$$

Since  $c_1^* = -c_1$  (due to pure complex phase terms in the steering vector), it follows that  $M_1 = \mathbf{0}$  and  $d_1 = 0$ . Therefore,

$$P^\perp = \mathbf{I} + \frac{M_2 + M_3 \Delta_f + M_4 \Delta_f^2 + O(\Delta_f^3)}{d_2 + d_3 \Delta_f + d_4 \Delta_f^2 + O(\Delta_f^3)}$$

$$\simeq \mathbf{I} + \frac{1}{d_2} (M_2 + M_3 \Delta_f + (M_4 - \frac{d_4}{d_2} M_2) \Delta_f^2) + O(\Delta_f^3)$$
(10)

where we used the fact that  $d_3 = 0$ . Substituting (10) in (4) (retaining only the terms up to  $\Delta_f^2$ ) and differentiating with respect to  $\Delta_f$ , the following **closed-form expression** of the frequency difference is obtained:

$$\Delta_f^{AUML}(f_1) = -\frac{\beta_1 + \alpha_1 \beta_0^2}{2(\beta_2 + \alpha_1 \beta_0 \beta_1 + \alpha_2 \beta_0^2)} \quad (11)$$

where

$$\alpha_1 = \frac{\text{Tr} \{ \mathbf{M}_3 \hat{\mathbf{R}} \}}{d_2 (M-2)} \quad (12)$$

$$\alpha_2 = \frac{\text{Tr} \{ (\mathbf{M}_4 - \frac{d_4}{d_2} \mathbf{M}_2) \hat{\mathbf{R}} \}}{d_2 (M-2)} \quad (13)$$

$$\beta_0 = \frac{\text{Tr} \{ (\mathbf{I} + \frac{\mathbf{M}_2}{d_2}) \hat{\mathbf{R}}^{-1} \}}{(M-2)} \quad (14)$$

$$\beta_1 = \frac{\text{Tr} \{ \mathbf{M}_3 \hat{\mathbf{R}}^{-1} \}}{d_2 (M-2)} \quad (15)$$

$$\beta_2 = \frac{\text{Tr} \{ (\mathbf{M}_4 - \frac{d_4}{d_2} \mathbf{M}_2) \hat{\mathbf{R}}^{-1} \}}{d_2 (M-2)} \quad (16)$$

To estimate the lower frequency,  $f_1$ , we just have to substitute this expression into (4) and to minimize the remaining 1D-criterion, i.e.,

$$\hat{f}_1^{UML} = \arg \min_{f_1} V(f_1, \Delta_f^{UML}(f_1)) \quad (17)$$

Hence, we have replaced the computationally intensive 2D UML procedure by a simpler 1D search algorithm providing an equivalent estimation precision, as shown in section IV.

#### IV. NUMERICAL ILLUSTRATIONS

In this section, we compare the performance of this approximate UML procedure with that of the exact UML estimator and its asymptotic formulation,  $V(f_1, \Delta_f)$  from (4), based on a 2D grid-search over  $f_1$  and  $f_2$ . For comparison purposes, we also display the results from the CML estimator and its approximate version for two closely spaced targets from [19] (ACML), as well as the performance of two popular algorithms, namely ESPRIT and root-MUSIC. We use the

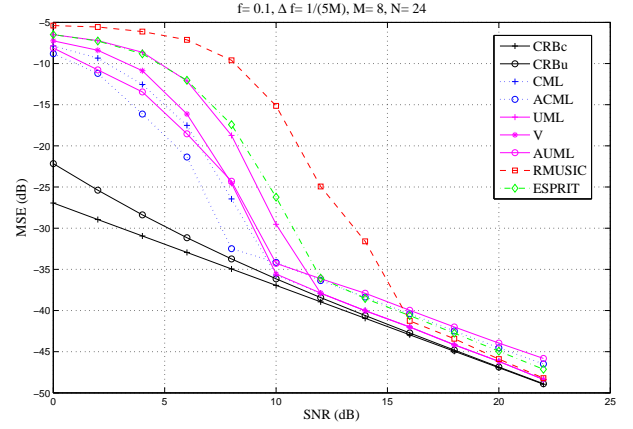


Fig. 1. Uncorrelated sources, SNR variation

frequency vector Mean Square Error (MSE) (calculated from  $[f_1 \ f_2]^T$ ) as a figure of merit to compare the different algorithms and the deterministic and stochastic CRB.

For all the following simulations we consider a uniformly spaced linear array of  $M = 8$  isotropic sensors. The spatial frequencies of the sources are  $f_1 = 0.1$  and  $f_1 + \Delta_f$  with  $\Delta_f = \frac{1}{5M}$ . The sample covariance matrix is estimated from  $N = 3 \times M$  snapshots. The MSE are computed from 1000 Monte-Carlo runs where the Gaussian vectors  $\mathbf{n}_t$  and  $\mathbf{s}_t$  vary in each trial. The signal to noise ratio is defined as

$$SNR = \frac{\text{Tr} \{ \mathbf{A}^H \mathbf{A} \mathbf{R}_s \}}{\sigma^2 M}. \quad (18)$$

We first compare all the above mentioned algorithms for Gaussian and uncorrelated sources, as a function of the SNR, in order to identify the so-called threshold region where the MSE departs from the CRB. According to Fig. 1, the following comments are provided. First of all, in the asymptotic region, as it is well known in the literature, all algorithms are close to the CRB and the two CRB tend one towards the other. We can also notice that the deterministic CRB ( $CRB_c$ ) is always lower than the stochastic one ( $CRB_u$ ) as mentioned in [10]. The first algorithm to depart from the CRB is root-MUSIC, then ESPRIT and the non-approximated UML algorithms some 4dB less, the approximated UML procedure, one dB less and finally the deterministic ML algorithms. It is first noteworthy that the UML is less accurate than the CML in the threshold region, but performs similarly in the asymptotic region. Hence, the superiority of the UML over the CML is questionable as the former requires stronger hypotheses about the sources, and even in this case, its performance is close to that of the CML in the asymptotic region and degrades more rapidly when SNR decreases. Secondly, in our case of interest of two closely spaced targets, we can notice that the asymptotic expression of the UML criterion in (4) gives more precise results than the exact formulation showing a kind of robustness in the threshold region. To finish, the AUML algorithm, proposed in this paper, exhibits similar and very

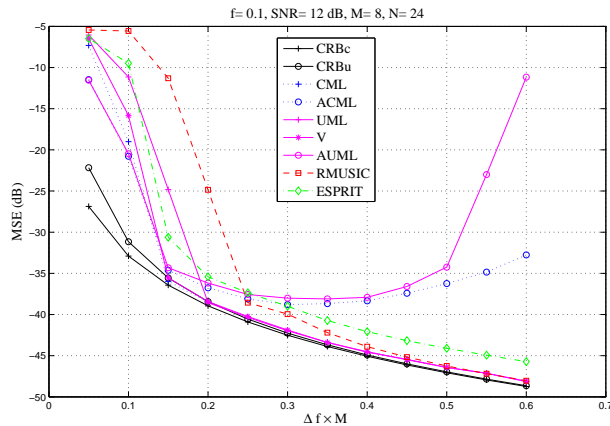


Fig. 2. Uncorrelated sources, frequency difference variation

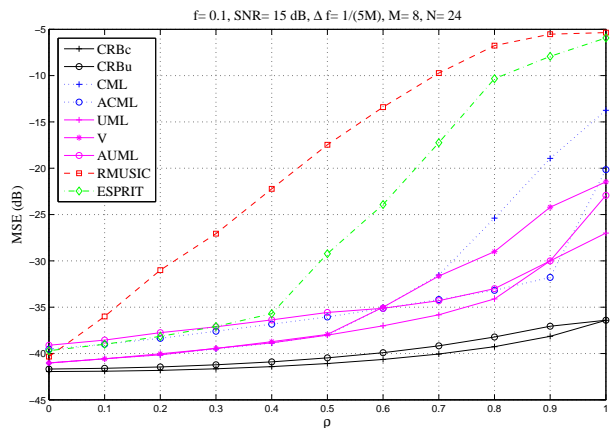


Fig. 3. Correlated sources, correlation coefficient variation

good results proving the validity of the procedure and its efficiency.

Now, we examine the robustness of the proposed algorithm with respect to the spatial frequency difference between the two targets. Fig. 2 represents the MSE as a function of  $\Delta_f$  for  $SNR = 12dB$  and allows to analyze to what extent the Taylor expansion derived in this paper is valid. We observe that the MSE of the AUML departs from that of the exact UML when the frequency difference reaches approximately a third of main lobe width and does not degrade too much up to half the main lobe. It can be noticed that over this point high resolution methods may not be necessary. On the left side of the  $\Delta_f$  axis, we observe, again, a kind of robustness of the proposed algorithm compared to the exact UML as it can resolve 30% more closely spaced targets for a given SNR. This point has to be emphasized and should be of great importance when one has to resolve two very close frequencies. Moreover, the behavior of the AUML is similar to that of the ACML.

Finally, we examine the robustness of the algorithms to source correlation in Fig. 3. This last simulation has been

conducted with a frequency difference  $\Delta_f = \frac{1}{5M}$  and for  $SNR = 15dB$ . First of all, we can see that the optimal performance degradation when the correlation coefficient increases is very progressive as the two CRB only increase of 7dB when  $\rho = 1$ . We can notice that the UML is more robust to source correlation than the CML. Although the UML assumes that the sources are not coherent ( $R_s$  must be full rank), Stoica and al. [22] has shown that it gives as precise results as the UML tailored for coherent sources. Once again, the AUML is as robust as the original UML and provides outstanding performance in this complicated scenario where the best of the classical high resolution techniques cannot resolve the two targets if  $\rho > 0.4$ .

## V. CONCLUSIONS

This paper proposed to exploit a little known asymptotic formulation of the UML criterion together with an asymptotic approximation for small separation between two frequencies. As a result, a 1D-minimization procedure only is needed to estimate the two frequencies, which yields a computationally simple algorithm. The latter exhibits near efficient performance up to half the main lobe width frequency difference and an even better resolution than UML in case of very close sources. Moreover, we also get equivalent UML performance in case of highly correlated or coherent sources where the proposed method outperforms the CML algorithm as well as common high resolution techniques.

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