Recovering the initial state of a Well-Posed Linear System with skew-adjoint generator

Ghislain Haine

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For instance:

$$A = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \text{ (+ Dirichlet boundary conditions) on } \Omega \subset \mathbb{R}^n \text{ and}$$

$$X = H_0^1(\Omega) \times L^2(\Omega).$$

$$\Downarrow$$

the classical wave equation.

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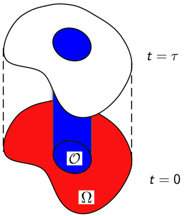
The classical wave equation, with $C = \begin{bmatrix} 0 & \chi_{\mathcal{O}} \end{bmatrix}$:

$$\begin{aligned} y(t) &= \begin{bmatrix} 0 & \chi_{\mathcal{O}} \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, \quad && \forall t \in [0, \tau], \\ &= \chi_{\mathcal{O}} \dot{w}(t), && \forall t \in [0, \tau]. \end{aligned}$$



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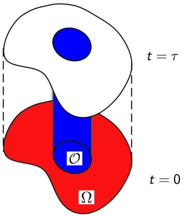
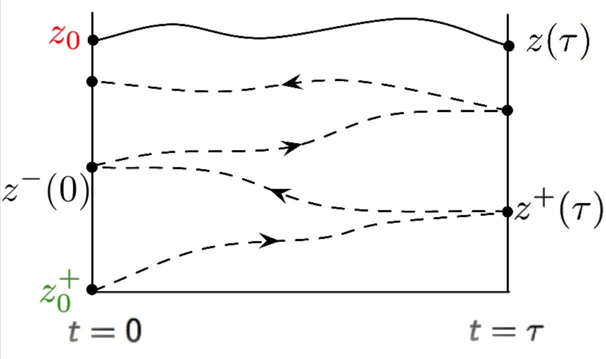


Our problem

Reconstruct the unknown z_0 from the measurement $y(t)$.

K. Ramdani, M. Tucsnak, and G. Weiss
Recovering the initial state of an infinite-dimensional system using observers (Automatica, 2010)

Intuitive representation



2 iterations, observation on $[0, \tau]$.

Some bibliography

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- **2010:** Ramdani, Tucsnak and Weiss (*Automatica*) generalized the TRF, based on the generalization of Luenberger's observers

We construct the **forward observer**

$$\begin{cases} \dot{z}^+(t) = Az^+(t) - C^*Cz^+(t) + C^*y(t), & \forall\, t \in [0, \tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

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which is known to be exponentially stable if and only if (A, C) is exactly observable, *i.e.*

$$\exists \tau > 0, \exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

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After a time reversal $Z^-(t) = Я_\tau z^-(t) := z^-(\tau - t)$, we get

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And from similar computations for $A^- := -A - C^*C$ as those for $A^+ := A - C^*C$:

$$t\|z^-(0) - z_0\| \leq Me^{-\beta\tau}\|z^+(\tau) - z(\tau)\| \leq M^2e^{-2\beta\tau}\|z_0^+ - z_0\|.$$

If the system is exactly observable in time $\tau > 0$, that is if:

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Iterating n-times the forward–backward observers with $z_n^+(0) = z_{n-1}^-(0)$ leads to

$$\|z_n^-(0) - z_0\| \leq \alpha^n \|z_0^+ - z_0\|.$$

This is the iterative algorithm of Ramdani, Tucsnak and Weiss to reconstruct z_0 from $y(t)$.

Outline

1. Introduction
2. The reconstruction algorithm
3. **Main result**
   - With bounded observation operator
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4. Application
5. Conclusion

In this work, the exact observability assumption in time τ

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Questions

- Given arbitrary C and $\tau > 0$, does the algorithm converge ?
- If it does, what is the limit of $z_n^-(0)$ and how is it related to z_0 ?

Decomposition of X:

- Let us denote Ψ_τ the following continuous linear operator

$$\begin{array}{rccc} \Psi_\tau & : & X & \longrightarrow & L^2\left([0,\tau],Y\right), \\ & & z_0 & \mapsto & y(t). \end{array}$$

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- We decompose $X = \mathrm{Ker}\ \Psi_\tau \oplus \left(\mathrm{Ker}\ \Psi_\tau\right)^\perp$ and define

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  Note that the exact observability assumption is equivalent to Ψ_τ is bounded from below and then $\Rightarrow X = \mathrm{Ran}\ \Psi_\tau^*$.

Stability of the decomposition under the algorithm:
Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $A^+ := A - C^*C$ (resp. $A^- := -A - C^*C$) on X.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}^-_\tau \mathbb{T}^+_\tau$, *i.e.*

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$$\mathbb{S}_\tau z_0 = \mathbb{T}^+_\tau z_0 + \int_0^\tau \mathbb{T}^+_{\tau - t} C^* \underbrace{C \mathbb{S}_t z_0}_{\Psi_\tau z_0} dt, \qquad \forall\ z_0 \in X.$$

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The algorithm preserves the decomposition of X !

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Furthermore, it is easy to prove that:

$$z_0^+ \in \mathrm{V}_{\mathrm{Obs}} \Longrightarrow z_n^-(0) \in \mathrm{V}_{\mathrm{Obs}},\ \forall n \geq 1.$$

Theorem

Denote by Π the orthogonal projection from X onto $\mathrm{V}_{\mathrm{Obs}}$. Then the following statements hold true for all $z_0 \in X$ and $z_0^+ \in \mathrm{V}_{\mathrm{Obs}}$:

1. For all $n \geq 1$,
$$\left\|(I-\Pi)\left(z_n^-(0) - z_0\right)\right\| = \left\|(I-\Pi) z_0\right\|.$$
2. The sequence $\left(\left\|\Pi\left(z_n^-(0) - z_0\right)\right\|\right)_{n\geq 1}$ is strictly decreasing and
$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| = \left\|z_n^-(0) - \Pi z_0\right\| \underset{n\to\infty}{\longrightarrow} 0.$$
3. There exists a constant $\alpha \in (0,1)$, independent of z_0 and z_0^+, such that for all $n \geq 1$,
$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| \leq \alpha^n \left\|z_0^+ - \Pi z_0\right\|,$$
if and only if $\mathrm{Ran}\ \Psi_\tau^*$ is closed in X.

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- Well-posed linear system

$$\begin{bmatrix} z(t) \\ y|_{[0,t]} \end{bmatrix} = \Sigma_t \begin{bmatrix} z_0 \\ u|_{[0,t]} \end{bmatrix}, \qquad \forall\, t \geq 0,$$

where $u \in \mathcal{U} := L^2([0,\infty), U)$ and $y \in \mathcal{Y} := L^2([0,\infty), Y)$ are the control and the observation (with U and Y two Hilbert spaces).

What happens if C is unbounded ?

- Main issue $\Longrightarrow$ $A - C^*C$ has no more meaning (as a generator). How to close the system ?
- Main tool $\Longrightarrow$ Stabilization by colocated feedback law for well-posed linear system (Curtain and Weiss 2006) allowing admissible C.
- Well-posed linear system

$$\begin{bmatrix} z(t) \\ y|_{[0,t]} \end{bmatrix} = \Sigma_t \begin{bmatrix} z_0 \\ u|_{[0,t]} \end{bmatrix}, \qquad \forall\, t \geq 0,$$

where $u \in \mathcal{U} := L^2([0,\infty), U)$ and $y \in \mathcal{Y} := L^2([0,\infty), Y)$ are the control and the observation (with U and Y two Hilbert spaces).

- **Well-posedness** means that for all $t \geq 0$:

$$\Sigma_t = \begin{bmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix} \in \mathcal{L}\left(X \times \mathcal{U}, X \times \mathcal{Y}\right).$$

M. Tucsnak and G. Weiss
Well-posed systems – The LTI case and beyond (Automatica, 2014)

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Let $A \in \mathcal{L}(\mathcal{D}(A), X)$ be the infinitesimal generator of $\mathbb{T}$.
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Associated triple (A, B, C)**:** There exist a control operator $B \in \mathcal{L}(U, X_{-1})$ and a observation operator $C \in \mathcal{L}(X_1, Y)$ such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds, \qquad \forall\, u \in \mathcal{U},$$

and

$$\Psi_t z_0(s) = \begin{cases} C\mathbb{T}_s z_0, & \forall\, s \in [0,t] \\ 0, & \forall\, s > t \end{cases} \qquad \forall\, z_0 \in X_1.$$

Let Σ be associated with (A, C^*, C), with A skew-adjoint.

Theorem (Curtain and Weiss 2006)

There exists $\kappa \in (0, \infty]$ such that for all $\gamma \in (0, \kappa)$, the feedback law $u = -\gamma y + v$ (v is the new control) leads to a closed-loop system Σ^γ which is well-posed. Furthermore:

$$\Sigma^\gamma - \Sigma = \Sigma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma^\gamma = \Sigma^\gamma \begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \Sigma.$$

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Let z^+ be the trajectory of Σ^+ with control $v = \gamma y$ (for simplicity we suppose $u \equiv 0$), then we have

$$z^+(t) - z(t) = \mathbb{T}_t^+ \left(z_0^+ - z_0 \right), \qquad \forall\, t \geq 0, z_0^+ \in X,$$

where $\mathbb{T}^+$ is the semigroup of Σ^+.

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Under some additional assumptions (namely optimizability and estimatability), the closed-loop system is exponentially stable. In other words, the associated semigroup is: z^+ is a **forward observer** of z.

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Dual system

Define Σ^d by

$$\Sigma_t^d = \begin{bmatrix} \mathbb{T}_t^d & \Phi_t^d \\ \Psi_t^d & \mathbb{F}_t^d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \text{Я}_t \end{bmatrix} \begin{bmatrix} \mathbb{T}_t^* & \Psi_t^* \\ \Phi_t^* & \mathbb{F}_t^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \text{Я}_t \end{bmatrix}.$$

Then Σ^d is a well-posed linear system with input space Y, state space X and output space U, associated with (A^*, C^*, B^*).

Where $\text{Я}_t u(s) := u(t-s)$ is the time reversal operator.

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We then obtain the same theorem as for bounded C, using z^+ and z^-, the respective trajectories of Σ^+ and Σ^-, as forward and backward observers.

Example

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Consider the following wave system

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, \quad \forall x \in \Omega, t > 0, \\ w(x,t) = 0, \qquad\qquad \forall x \in \Gamma_0, t > 0, \\ w(x,t) = u(x,t), \quad \forall x \in \Gamma_1, t > 0, \\ w(x,0) = w_0(x), \ \dot{w}(x,0) = w_1(x), \ \forall x \in \Omega, \end{cases}$$

with u the control, and (w_0, w_1) the initial state.

Observation

Let ν be the unit normal vector of Γ_1 pointing towards the exterior of Ω, we observe the system *via*

$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu}, \quad \forall x \in \Gamma_1, t > 0.$$

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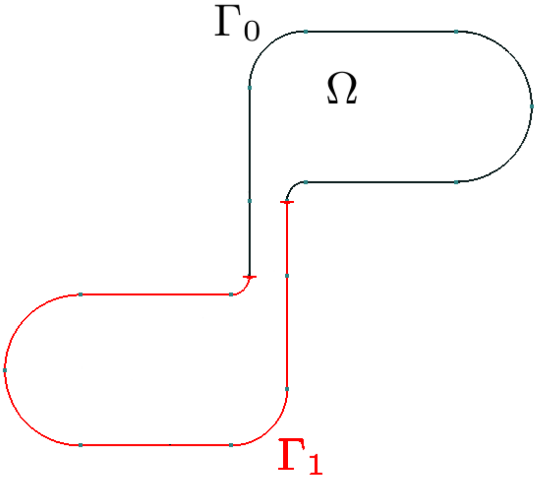
- Curtain and Weiss (SIAM J. Control Optim., 2006) $\Rightarrow$ construction of forward and backward observers (formally $A^{\pm} = \pm A - C^*C$).
- So we can use the algorithm.

For instance, let us consider the following configuration



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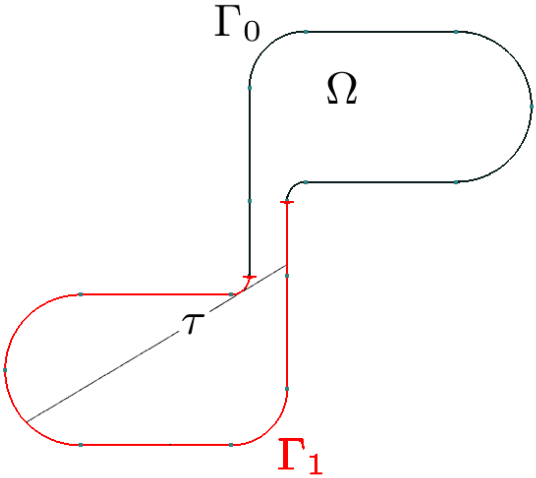

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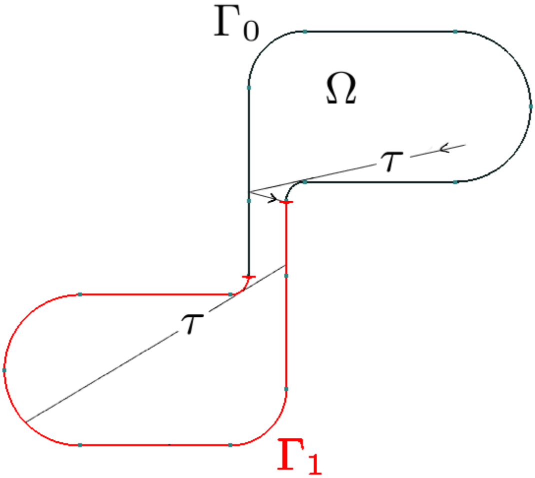

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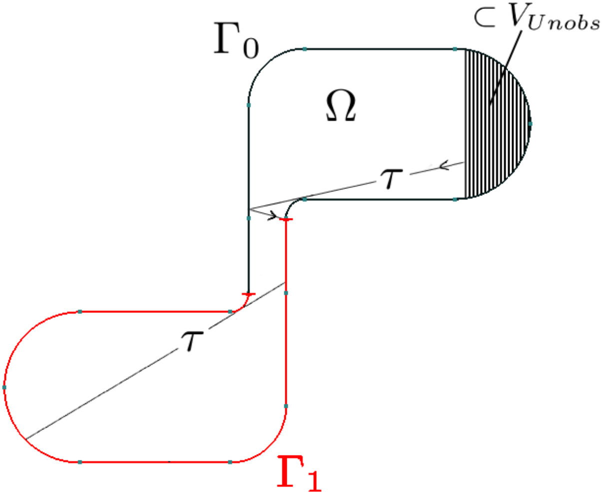

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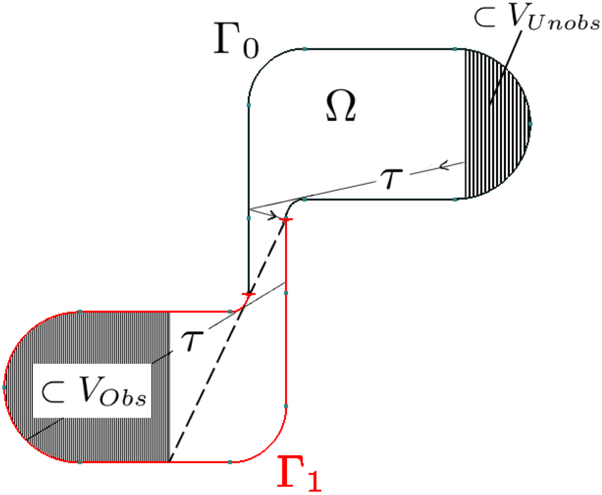

Choosing a suitable initial data

- Supp(w_0) has three components W_1, W_2 and W_3, such that
  - $W_1 \subset \mathrm{V}_{\mathrm{Obs}}$
  - $W_2 \subset \mathrm{V}_{\mathrm{Unobs}}$
  - $W_3 \cap \mathrm{V}_{\mathrm{Obs}} \neq \emptyset$ and $W_3 \cap \mathrm{V}_{\mathrm{Unobs}} \neq \emptyset$
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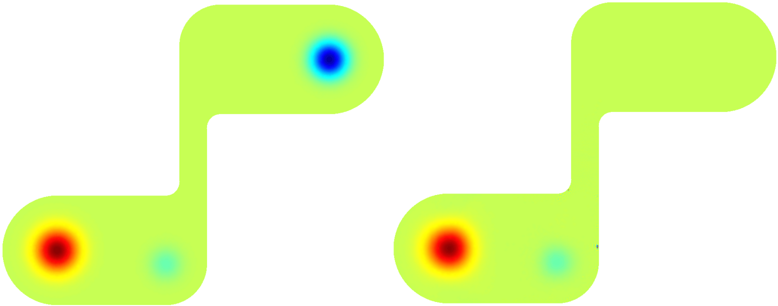
To perform the test, we use

- Gmsh: a 3D finite element grid generator
- GetDP: a general finite element solver

G. Haine and K. Ramdani
Reconstructing initial data using observers: error analysis of the semi-discrete and fully discrete approximations
(Numerische Mathematik (Numer. Math.), *2012*)



The initial position (Left) and its reconstruction (Right) after 3 iterations

$\Rightarrow$ 6% of relative error in $L^2(\Omega)$ on the "observable part".

Conclusion

More ?

G. Haine
Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator
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Still to be done:

- Stability of $\mathrm{V}_{\mathrm{Obs}}$ and $\mathrm{V}_{\mathrm{Unobs}}$ with noisy observation y
- Generalization ($A^* \neq -A$)
- Optimization of γ