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Solving Thermoacoustic Tomography with an observer-based algorithm

Institut Supérieur de l'Aéronautique et de l'Espace

Ghislain Haine

Department of Mathematics, Computer Science, Control

Introduction

In medical imaging, we mostly need to recover the initial (or final) state of a physical system from partial observation over some finite time interval. In general, measurements are performed outside the body. This constraint leads to incomplete data. For instance in breast or kidney imaging, we cannot expect a measurement all around the object of interest. In this paper we investigate the problem of data recovery with this lack of information. In other words, we investigate systems which are not *exactly observable* (more than one initial state lead to the same observation).

In the last decade, new algorithms based on time reversal (see Fink [5, 6]) have been proposed for data recovery. We can mention, for instance, the Back and Forth Nudging proposed by Auroux and Blum [2], the Time Reversal Focusing by Phung and Zhang [18], the algorithm proposed by Ito, Ramdani and Tucsnak [12] and finally, the one we will consider, the algorithm studied in [20].

In thermoacoustic tomography, the problem is to recover from surface measurements the initial state of a wave equation (see Gebauer and Scherzer [7]), see for instance Kuchment and Kunyansky [13].

Various methods have been used to tackle the problem of thermoacoustic tomography, such as inverse source concepts in Fourier domain [1], Fourier series [14, 15, 16] and time reversal method [11]. A new method has been proposed in [21], based on time reversal and leading to a Neumann series. It has been studied in recent works [19, 17]. Finally, observer-based algorithm for data assimilation [2] has been successfully applied to thermoacoustic tomography [3].

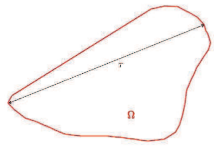
We propose the use of the iterative observer-based algorithm of [20], which also leads to a Neumann series. However, it involves only the resolution of direct wave equations in practice. Our main result shows that the algorithm converges at least polynomially to the initial state. Moreover, in the case of incomplete data, we prove that it converges to the *observable part* of the initial state.

The wave system

Let us state our mathematical inverse problem. We consider the wave equation in the whole domain $\mathbb{R}^3$, with initial position compactly supported in a bounded open set $\Omega \subset \mathbb{R}^3$. More precisely, let $w_0 \in H_0^1(\Omega)$, and consider the following system

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} w(x,t) = \Delta w(x,t), & \forall\, x \in \mathbb{R}^3, t \geq 0, \\ w(x,0) = w_0(x), & \forall\, x \in \Omega, \\ w(x,0) = 0, & \forall\, x \in \mathbb{R}^3 \setminus \Omega, \\ \dfrac{\partial}{\partial t} w(x,0) = 0, & \forall\, x \in \mathbb{R}^3. \end{cases} \tag{1}$$

The observation is performed on a surface surrounded the initial state. We then suppose that we observe the state w on $\partial\Omega$, during a time interval $[0,\tau]$, with $\tau \geq \operatorname{diam}(\Omega)$, where $\operatorname{diam}(\Omega)$ is the supremum of the path rays from boundary to boundary Ω.



Cut in the plane containing $\operatorname{diam}(\Omega)$ of an example of configuration.

This leads to

$$y(x,t) = w(x,t), \quad \forall\, x \in \partial\Omega,\ t \in [0,\tau]. \tag{2}$$

This last assumption will make the inverse problem well-posed. However, our method allows to consider ill-posed cases. For instance, we could observe only on a part of the boundary, as it is done in the numerical tests.

The inverse problem we consider is to recover w_0 from y.

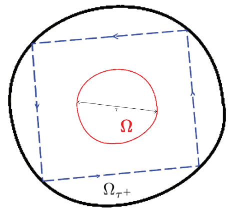
The algorithm

We consider a bounded domain Ω_{τ^+} *sufficiently large*, *i.e.* such that any information in Ω does not hit $\partial\Omega_{\tau^+}$ in time τ (using Huygens' principle). We then construct the n-th couple of systems

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} w_n^+(x,t) = \Delta w_n^+(x,t), & \forall\, x \in \Omega_{\tau^+}, t \in [0,\tau], \\ w_n^+(x,t) = y(x,t), & \forall\, x \in \partial\Omega, t \in [0,\tau], \\ w_n^+(x,t) = 0, & \forall\, x \in \partial\Omega_\tau, t \in [0,\tau], \\ w_1^+(x,0) = 0, & \forall\, x \in \Omega_{\tau^+}, \\ \dfrac{\partial}{\partial t} w_1^+(x,0) = 0, & \forall\, x \in \Omega_{\tau^+}, \\ w_n^+(x,0) = w_{n-1}^-(x,0), & \forall\, x \in \Omega_{\tau^+}, n \geq 2, \\ \dfrac{\partial}{\partial t} w_n^+(x,0) = \dfrac{\partial}{\partial t} w_{n-1}^-(x,0), & \forall\, x \in \Omega_{\tau^+}, n \geq 2, \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} w_n^-(x,t) = \Delta w_n^-(x,t), & \forall\, x \in \Omega_{\tau^+}, t \in [0,\tau], \\ w_n^-(x,t) = y(x,t), & \forall\, x \in \partial\Omega, t \in [0,\tau], \\ w_n^-(x,t) = 0, & \forall\, x \in \partial\Omega_\tau, t \in [0,\tau], \\ w_n^-(x,\tau) = w_n^+(x,\tau), & \forall\, x \in \Omega_{\tau^+}, n \geq 1, \\ \dfrac{\partial}{\partial t} w_n^-(x,\tau) = \dfrac{\partial}{\partial t} w_n^+(x,\tau), & \forall\, x \in \Omega_{\tau^+}, n \geq 1. \end{cases} \tag{4}$$

Remark that the system is not exactly observable in this case. So the results of [20] is not sufficient.



The dashed ray is trapped.

Main results

We prove in [10, 9] that systems (3)–(4) reconstruct w_0. More precisely, we have

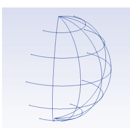
Theorem . *Assume that $w_0 \in H_0^1(\Omega)$ and let y given by (2) be the observation of the solution of (1). Let w_n^+ and w_n^- be the solutions of (3) and (4) respectively, for each $n \geq 1$. Then*

$$\int_\Omega \|w_n^-(x,0) - w_0(x)\|^2 dx + \int_\Omega \|\nabla w_n^-(x,0) - \nabla w_0(x)\|^2 dx + \int_\Omega \left\| \frac{\partial}{\partial t} w_n^-(x,0) \right\|^2 dx = o\left(\frac{1}{n}\right).$$

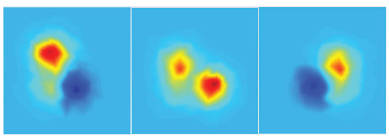
When the observation is not complete, that is when we do not observe on a surface surrounding the object to image, the inverse problem is ill-posed. Our theoretical result shows that the previous theorem remains valid, with a projector Π on the subspace of all *observable data* (*i.e.* leading to non-zero measurement). We then rewrite the above reconstruction error with Πw_0 instead of w_0. It is important to note that we do not need to know Π to reconstruct the observable part Πw_0 of the initial data, if the first initial guess lies in the subspace of the *observable data* (for instance, we choose here zero).

3D numerical simulations

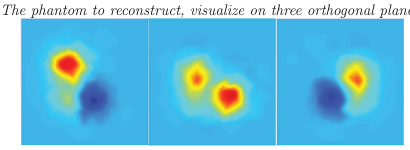
We implement the algorithm on GMSH [8] and GetDP [4]. In these first tests performed on a coarse mesh, we observe on a sphere of radius 0.5 surrounding the support of the initial data. We add Gaussian noise to the observation, with 0.25 of deviation. We test three cases: full reconstruction from complete observation on a sphere surrounding the initial data, partial reconstruction from observation on a half-sphere (not shown on this poster) and partial reconstruction from observation on the following grid of lines detector.



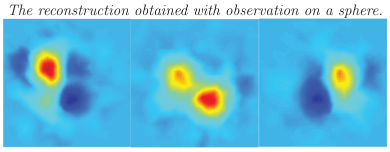
Comparison of reconstructed data



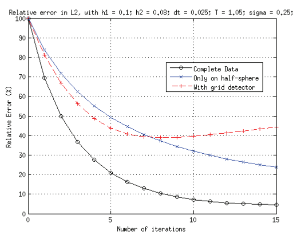
The phantom to reconstruct, visualize on three orthogonal planes.



The reconstruction obtained with observation on a sphere.



The reconstruction obtained with observation on the grid.



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