Edge-disjoint rainbow trees in properly coloured complete graphs

Alexey Pokrovskiy

Department of Mathematics, ETH, 8092 Zurich, Switzerland

Benny Sudakov

Department of Mathematics, ETH, 8092 Zurich, Switzerland

Abstract

A subgraph of an edge-coloured complete graph is called rainbow if all its edges have different colours. The study of rainbow decompositions has a long history, going back to the work of Euler on Latin squares. We discuss three problems about decomposing complete graphs into rainbow trees: the Brualdi-Hollingsworth Conjecture, Constantine's Conjecture, and the Kaneko-Kano-Suzuki Conjecture. The main result which we discuss is that in every proper edge-colouring of K_n there are $10^{-6}n$ edge-disjoint isomorphic spanning rainbow trees. This simultaneously improves the best known bounds on all these conjectures. Using our method it is also possible to show that every properly $(n-1)$-edge-coloured K_n has $n/9$ edge-disjoint spanning rainbow trees, giving a further improvement on the Brualdi-Hollingsworth Conjecture.

1 Introduction

We consider the following question: Can the edges of every properly edge-coloured complete graph be decomposed into edge-disjoint rainbow spanning trees. Here a properly edge-coloured complete graph K_n means an assignment of colours to the edges of K_n so that no two edges at a vertex receive the same colour. A rainbow spanning tree in K_n is a tree containing every vertes of K_n, all of whose edges have different colours.

The study of rainbow decompositions dates back to the 18th century when Euler studied the question "for which n does there exist a properly n-edge-coloured $K_{n,n}$ which can be decomposed into n edge-disjoint rainbow perfect matchings [1]." Euler constructed such proper n-edge-colourings of $K_{n,n}$ whenever $n \not\equiv 2 \pmod 4$, and conjectured that these are the only values of n for which they can exist. The $n = 6$ case of this conjecture is Euler's famous "36 officers problem", which was eventually proved by Tarry in 1901. For larger n, Euler's Conjecture was disproved in 1959 by Parker, Bose, and Shrikhande. Together these results give a complete description of the values of n for which there exists a properly n-edge-coloured $K_{n,n}$ which can be decomposed into n edge-disjoint rainbow perfect matchings.

Decompositions of properly $(2n-1)$-edge-coloured K_{2n} into edge-disjoint rainbow perfect matchings have also been studied. They were introduced by Room in 1955 [2], who raised the question of which n they exist for. Wallis showed that such decompositions of K_{2n} exist if, and only if, $n \neq 2$ or 4. Rainbow perfect matching decompositions of both $K_{n,n}$ and K_{2n} have found applications in scheduling tournaments and constructing experimental designs (see eg [9].)

Euler and Room wanted to determine the values of n for which there exist colourings of $K_{n,n}$ or K_n with rainbow matching decompositions. However given an *arbitrary* proper edge-colouring of $K_{n,n}$ or K_n it is not the case that it must have a decomposition into rainbow perfect matchings. A natural way of getting around this is to consider decompositions into rainbow graphs other than perfect matchings. In the past decompositions into rainbow subgraphs such as cycles and triangle factors have been considered [7].

Here we consider decompositions into rainbow trees. In contrast to the perfect matching case, it is believed that every properly edge coloured K_n can be decomposed into edge-disjoint rainbow trees. This was conjectured by three different sets of authors.

[1] Euler studied the values of n for which a pair of $n \times n$ orthogonal Latin squares exists. Using a standard argument, it is easy to show that $n \times n$ orthogonal Latin squares are equivalent objects to rainbow perfect matching decompositions of $K_{n,n}$.

[2] Room actually introduced objects which are now called "Room squares". It is easy to show that Room squares are equivalent objects to decompositions of $(2n-1)$-edge-coloured K_{2n} into edge-disjoint rainbow perfect matchings.

Conjecture 1.1 (Brualdi and Hollingsworth, [5]) *Every properly $(2n-1)$-edge-coloured K_{2n} can be decomposed into edge-disjoint spanning rainbow trees.*

Conjecture 1.2 (Kaneko, Kano, and Suzuki, [13]) *Every properly edge-coloured K_n contains $\lfloor n/2 \rfloor$ edge-disjoint isomorphic spanning rainbow trees.*

Conjecture 1.3 (Constantine, [8]) *Every properly $(2n-1)$-edge-coloured K_{2n} can be decomposed into edge-disjoint isomorphic spanning rainbow trees.*

There are many partial results on the above conjectures. It is easy to see that every properly coloured K_n contains a single rainbow tree—specifically the star at any vertex will always be rainbow. Strengthening this, various authors have shown that more disjoint trees exist under assumptions of Conjectures 1.1–1.3.

Brualdi and Hollingsworth [5] showed that every properly $(2n-1)$-coloured K_{2n} has 2 edge-disjoint spanning rainbow trees. Krussel, Marshall, and Verrall [14] showed that there are 3 spanning rainbow trees under the same assumption. Kaneko, Kano, and Suzuki [13] showed that 3 edge-disjoint spanning rainbow trees exist in any proper colouring of K_n (with any number of colours.) Akbari and Alipour [1] showed that 2 edge-disjoint spanning rainbow trees exist in any colouring of K_n with $\leq n/2$ edges of each colour. Carraher, Hartke, and Horn [6] showed that under the same assumption, $\lfloor n/1000 \log n \rfloor$ edge-disjoint spanning rainbow trees exist. In particular this implies that every properly coloured K_n has this many edge-disjoint spanning rainbow trees. Horn [12] showed that there is an $\epsilon > 0$ such that every $(2n-1)$-coloured K_{2n} has ϵn edge-disjoint spanning rainbow trees. Subsequently, Fu, Lo, Perry, and Rodger [11] showed that every $(2n-1)$-coloured K_{2n} has $\lfloor \sqrt{6m+9}/3 \rfloor$ edge-disjoint spanning rainbow trees. For Conjecture 1.3, Fu and Lo [10] showed that every $(2n-1)$-coloured K_{2n} has 3 isomorphic edge-disjoint spannind trees. In addition to these results, there has been a fair ammount of work showing that edge-coloured complete graphs with certain specific colourings can be decomposed into spanning rainbow trees (see eg [2,**?**]).

To summarize the best known results for these problems for large n: Horn proved for the Brualdi-Hollingsworth Conjecture that ϵn edge-disjoint spanning rainbow trees exist. For the Kaneko-Kano-Suzuki Conjecture, Carraher, Hartke, and Horn proved that $\lfloor n/1000 \log n \rfloor$ edge-disjoint spanning rainbow trees exist. For Constantine's Conjecture, Fu and Lo proved that 3 edge-disjoint isomorphic spanning rainbow trees exist.

We are able to substantially improve the best known bounds for all three conjectures. Define a t-spider to be a radius 2 tree with t degree 2 vertices (or equivalently a tree obtained from a star by subdividing t of its edges once.) In [16] we prove the following.

Theorem 1.4 *Every properly coloured K_n contains $10^{-6}n$ edge-disjoint spanning rainbow t-spiders for any $0.0007n \leq t \leq 0.2n$.*

Beyond improving the bounds on Conjectures 1.1–1.3, the above theorem is qualitatively stronger than all of them. Firstly, the isomorphism class of the spanning trees in Theorem 1.4 is independent of the colouring on K_n (whereas Constantine's Conjecture allows for such a dependency.) Additionally Theorem 1.4 produces isomorphic spanning trees under a weaker assumption than Constantine's Conjecture (namely we do not specify that K_n is $(n-1)$-coloured.)

Balogh, Liu and Montgomery [4] independently proved the existence of $\Omega(n)$ edge-disjoint spanning rainbow trees in every properly edge-colored K_n.

The method used in [16] to prove Theorem 1.4 is quite flexible. For any one of the three conjectures, it is easy to modify the method to give a further improvement on the $10^{-6}n$ bound from Theorem 1.4. For example in [16] we show that in the case of the Brualdi-Hollingsworth Conjecture one cover over 20% of the edges by spanning rainbow trees.

Theorem 1.5 *Every properly $(n-1)$-edge-coloured K_n has $n/9$ edge-disjoint spanning rainbow trees.*

2 Proof ideas

In this section we give a sketch of the proof of Theorem 1.4. Throughout the section, we fix a properly coloured complete graph K_n and let $m = 10^{-6}n$ be the number of edge-disjoint spiders we are trying to find.

Recall that a graph D is a t-spider if $V(S) = \{r, j_1, \dots, j_t, x_1, \dots, x_t, y_1, \dots, y_{|S|-2t-1}\}$ with $E(S) = \{rj_1, \dots, rj_t\} \cup \{ry_1, \dots, ry_{|S|-2t-1}\} \cup \{j_1x_1, \dots, j_tx_t\}$. The vertex r is called the *root* of the spider D. The vertices $y_1, \dots, y_{|S|-2t-1}$ are called *ordinary leaves* of the spider.

We say that a family of spiders $\mathcal{D} = \{D_1, \dots, D_m\}$ is *root-covering* if the root of D_i is in $V(D_j)$ for any $i, j \in \{1, \dots, m\}$. The basic idea of the proof of Theorem 1.4 is to first find a root-covering family of non-spanning, non-isomorphic, spiders $\mathcal{D} = \{D_1, \dots, D_m\}$. Then, for each i, the spider D_i is modified into an spanning, isomorphic rainbow spider. The reason for considering root-covering families is that the roots are the highest degree vertices in spiders. Because of this, they are intuitively the most difficult vertices to cover in the spiders we are looking for. Thus in the proof we first find a family of spiders which is root-covering, and then worry about making them spanning and isomorphic.

The proof of Theorem 1.4 naturally splits into three parts:

(i) Find a root-covering family of large edge-disjoint rainbow spiders $D_1, \ldots, D_m$ in K_n.

(ii) Modify the spiders from (1) into a root-covering family of *spanning*, edge-disjoint, rainbow spiders $D'_1, \ldots, D'_m$.

(iii) Modify the spiders from (2) into a root-covering family of spanning, edge-disjoint, rainbow, *isomorphic* spiders $D''_1, \ldots, D''_m$.

Part (1) is the easiest part of the proof. To prove it, we first finding a family of disjoint rainbow *stars* $S_1, \ldots, S_m$ rooted at $r_1, \ldots, r_m$ in K_n. Then by exchanging some edges between these stars, we obtain spiders $D_1, \ldots, D_m$ rooted at $r_1, \ldots, r_m$ which is root-covering.

Part (2) is the hardest part of the proof. It involves going through the spiders $D_1, \ldots, D_m$ from part (1) one by one and modifying them. For each i, we modify D_i into a spanning spider D'_i with D'_i edge disjoint from the spiders $D'_1, \ldots, D'_{i-1}$, $D_{i+1}, \ldots, D_m$ and D'_i having the same root as D_i. In order to describe which edges we can use in D'_i, we make the following definition.

Definition 2.1 Let $\mathcal{D} = \{D_1, \ldots, D_m\}$ be a family of edge-disjoint spiders in a coloured K_n. Let $D_i = S_i \cup \hat{D}_i$ where S_i is the star consisting of the ordinary leaves of D_i. We let $G(D_i, \mathcal{D})$ denote the subgraph of K_n formed by deleting the following:

- All the roots of the spiders $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_m$.
- All the edges of the spiders $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_m$.
- All edges sharing a colour with $\hat{D}_i$.
- All vertices of $\hat{D}_i$ except the root.

The intuition behind this definition is that we can freely modify D_i using edges from $G(D_i, \mathcal{D})$ without affecting the other spiders $D_1, \ldots, D_{i-1}$, $D_{i+1}, \ldots, D_m$. The following observation makes this precise.

Observation 1 *Let $\mathcal{D} = \{D_1, \ldots, D_m\}$ be a family of rainbow spiders in a coloured K_n. Let $D_i = S_i \cup \hat{D}_i$ where S_i is the star consisting of the ordinary leaves of D_i. Then for any rainbow spider $\hat{S}_i$ in $G(D_i, \mathcal{D})$ with S_i and $\hat{S}_i$ having the same root, we have that $\hat{S}_i \cup \hat{D}_i$ is a rainbow spider in K_n.*

In addition if $\mathcal{D}$ was edge-disjoint and root-covering, then $\mathcal{D} \setminus \{D_i\} \cup \{\hat{S}_i \cup$

$\hat{D}_i\}$ *is edge-disjoint and root-covering.*

A crucial feature of $G(D_i, \mathcal{D})$ is that it has high minimum degree.

Observation 2 *For a family of spiders* $\mathcal{D} = \{D_1, \dots, D_m\}$ *in a properly coloured* K_n *with* D_i *a* t*-spider we have* $\delta(G(D_i, \mathcal{D})) \geq n - 3m - 4t - 1$.

To solve (2) we consider the graph $G(D_i, \mathcal{D})$ for $\mathcal{D} = \{D'_1, \dots, D'_{i-1}, D_{i+1}, \dots, D_m\}$. Using Observation 1 to solve (2) it is enough to find a spanning rainbow spider D'_i in $G(D_i, \mathcal{D})$ having the same root as D_i. From Observation 2 we know that $G(D_i, \mathcal{D})$ has high minimum degree. Thus, to solve (2) it would be sufficient to show that "every properly coloured graph with high minimum degree and a vertex r has a spanning rainbow spider rooted at r." Unfortunately this isn't true since it is possible to have have a properly coloured graph G with high minimum degree which has $< |G| - 1$ colours (and hence has no spanning rainbow tree.)

However, in a sense, "having too few colours" is the only barrier to finding a spanning rainbow spider in a high minimum degree graph. In [16], we show that as long as there are enough edges of colours not touching r, then it is possible to find a spanning rainbow spider rooted at r in a high minimum degree graph. This turns out to be sufficient to complete the proof of (2) since it is possible to ensure that the graphs $G(D_i, \mathcal{D})$ have a lot of edges of colours outside D_i. The details of this are somewhat complicated and explained in [16].

Part (3) is similar in spirit to part (2). It consists of going through the spiders $D'_1, \dots, D'_m$ one by one, and modifying D'_i into a spanning spider D''_i with D''_i edge disjoint from the spiders $D''_1, \dots, D''_{i-1}, D'_{i+1}, \dots, D'_m$ and D''_i having the same root as D'_i. We once again consider the graph $G(D'_i, \mathcal{D})$ for $\mathcal{D} = \{D''_1, \dots, D''_{i-1}, D'_{i+1}, \dots, D'_m\}$ and notice that it has high degree. Because of this, to prove (3) it is sufficient to show that "in every properly coloured graph G with high minimum degree and a spanning rainbow star S, there is a spanning rainbow t-spider for suitable t." This turns out to be true for $t \geq 3$, and is proved by replacing edges of D'_i for suitable edges outside D'_i.

References

[1] Akbari, S., A. Alipour, Multicolored trees in complete graphs. *J. Graph Theory*, **54**, (2007), 221–232.

[2] Akbari, S. , A. Alipour, H. L. Fu, and Y. H. Lo, Multicolored parallelisms of isomorphic spanning trees. *SIAM J. Discrete Math.*, **20**, (2006), 564–567.

[3] Alon, N., A. Pokrovskiy, and B. Sudakov, Random subgraphs of properly edge-coloured complete graphs and long rainbow cycles. *Israel J. Math.*, to appear (2017).

[4] Balogh, J., H. Liu and R. Montgomery, Rainbow spanning trees in properly coloured complete graphs. *arXiv:1704.07200*, (2017)

[5] Brualdi R. and S. Hollingsworth, Multicolored trees in complete graphs. *J. Combin. Theory Ser. B*, **68**, (1996), 310–313.

[6] Carraher, J., S. Hartke, and P. Horn, Edge-disjoint rainbow spanning trees in complete graphs. *European J. Combin.*, **57**, (2016), 71–84.

[7] Colbourn, C. and E. Mendelsohn, Kotzig factorizations: existence and computational results. *North-Holland Mathematics Studies*, **60** (1982) 65–78.

[8] Constantine, G., Multicolored parallelisms of isomorphic spanning trees. *Discrete Math. Theor. Comput. Sci.*, **5**, (2002), 121–126.

[9] Dinitz, J., D. Stinson, Contemporary Design Theory: A Collection of Surveys. *North-Holland Mathematics Studies*, (1992).

[10] Fu, H. L. and Y. H. Lo, Multicolored isomorphic spanning trees in complete graphs. *arXiv:1410.0445*, (2014).

[11] Fu, H. L., Y. H. Lo, K. E. Perry, and C. A. Rodger, On the Number of Rainbow Spanning Trees in Edge-Colored Complete Graphs. *arXiv:1605.04501*, (2016).

[12] Horn, P., Rainbow spanning trees in complete graphs colored by matchings. *preprint*, (2013).

[13] Kaneko, A., M. Kano, K. Suzuki, Three edge-disjoint Multicolored Spanning Trees in Complete Graphs. *preprint*, (2002).

[14] Krussel, J., S. Marshall, H. Verrall, Spanning trees orthogonal to one-factorizations of K_{2n}. *Ars Combin.*, **57** (2000) 77–82.

[15] Maamoun, M. and H. Meyniel. On a problem of G. Hahn about coloured Hamiltonian paths in K_{2^t}. *Discrete Math.* **51** (1984), 213–214.

[16] Sudakov, B. and A. Pokrovskiy, Linearly many rainbow trees in properly edge-coloured complete graphs, *arXiv:1703.07301,* (2017).