

U N I V E R S I T Y   O F   L O N D O N  
UNIVERSITY COLLEGE - DEPARTMENT OF ECONOMICS

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**EXTENSIONS OF VON NEUMANN'S  
THEORY OF  
ECONOMIC GROWTH**

Ph. D. Thesis  
(Submitted June 91)

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ABSTRACT

In this thesis some simplified assumptions present in the von Neumann expanding economy are examined and replaced by more appropriate ones.

First we explain more clearly the von Neumann original model and the Kemeny Morgenstern and Thompson's extension. We clarify the meaning of the open constrained economy which Morgenstern and Thompson derive from the von Neumann economy as a linear programming problem. Then we examine the exact meaning and relevance, in the ambit of the von Neumann's model, of different definitions of closed economy, in connection with the treatment of labour.

We study critically all the ways of introducing consumption into the model. After we deal with the models which explicitly introduced externalities and processes which produce both goods and bads. Some of the previous generalizations have not been completely free from criticism; however some shortcomings can be solved introducing externalities as different commodities, distinguished on the basis of their users. These public intermediate commodities influence directly the use and the intensity of one or more productive processes.

They are also one of the premises and foundations of the public sector normative and positive theory of De Viti De Marco. In our models we follow his line of reasoning and confine ourselves to the original (or KMT) model. We show

how the rate of growth of the von Neumann's economy, where the prices of public commodities (bads) are determined by the equilibrium solution and paid through taxation, is greater than the one which would be determined by the private market equilibrium where public commodities (bads) are free. It is also possible to have no private market equilibrium, even if it always exists a constrained one which is a mixed economy equilibrium.

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## 1. THE VON NEUMANN MODEL.

### A. Premises

The very concise von Neumann (1937) paper was originally presented at Princeton in 1932. It represents a major advance in economic theory not only for the depth and originality of the mathematical analysis, but also for the profound economic meaning, synthesis and new ideas contained in it. It is in fact the first time that the proof of the existence of at least one equilibrium position, in perfect competition, is given and its maximal properties are studied. Furthermore the free goods, the prices of all private goods, the production scale for each process, the rate of interest and the rate of growth are simultaneously determined in a dynamic setting. Even if some of the assumptions are artificial, the model is regarded as an important innovation in the history of the theory of competitive equilibrium. It is the origin of activity analysis of production as well as a fresh reformulation of Cassel's theory of balanced growth. It allows for joint production and permits to deal explicitly with the problem of economic and physical lifetime of capital goods. This aspect is not fully appreciated, even by Dorfman Samuelson and Solow<sup>(1)</sup> and in spite of von Neumann's and Morishima's emphasis of it,<sup>(2)</sup> it seems to be

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(1) Cfr. Dorfman, Samuelson and Solow (1958) p.383.

(2) Cfr. Morishima (1969).



ignored in most of recent models.

With few exceptions, von Neumann's paper remained unknown for many years to economists. This happened in part because of the unfamiliar character of the mathematics and in part because it was published in German in a mathematical journal. Only when it appeared in an English translation with a comment by Champernowne (1945-46), who explained the economic implications of the paper in plain English, economists began to study the von Neumann type of economy.

Till now however, it has been less known and less studied, than the Leontief input-output model, which was formulated later and which was considerably less sophisticated both from a mathematical and an economic point of view. Nevertheless there has existed a limited circle of academic economists who appreciated von Neumann's contribution, and by their effort the simplified assumptions of the original paper (stated by Champernowne) have been examined and replaced by more general or more appropriate ones. Von Neumann's simplified assumptions can be listed as:

- 1) every good is involved in every process of production,
- 2) workers do not save and capitalists do not consume,
- 3) no consumer choice,
- 4) predetermined wages (at subsistence level),
- 5) perfect competition,

- 6) constant return to scale for every process,
- 7) natural factors of production (including labour) available in unlimited amounts,
- 8) no public sector,
- 9) no export and import sectors,
- 10) no money,
- 11) growth is simply replication (no technical progress).

In what follows we will focus our attention, first, on the contributions that deal with problems related with points 1 to 7 above. In the first section, our aim is just to explain more clearly the Von Neumann original model and the need of replacing the original strong assumption 1 by a more reasonable one due to Kemeny Morgenstern and Thompson. In section 2, we will concern ourselves with clarifying the economic meaning of a simple constrained model due to Morgenstern and Thompson which is obtained from the original von Neumann model by re-interpreting it from the point of view of linear programming. In section 3 we will examine the exact meaning and relevance of different definitions of a closed economy and a closed model.

### 1.1 Von Neumann and Kemeny-Morgenstern-Thompson models.

#### A. Original assumptions

Von Neumann examined an economy with a finite set of  $n$  goods and  $m$  available processes of production, considering



simultaneously the problems of determination of the real and monetary "structure" of the economy. Consequently he wanted to determine the intensity with which each process could be used ( $x_i$ ), the price of each commodity ( $y_j$ ), the rate of growth ( $\alpha-1$ ) and the rate of interest ( $\beta-1$ ). Furthermore these values were determined in an equilibrium position; defined as the state "where the whole economy expands without change of structure" (a kind of long run steady-state).

In order to deal with all these problems and to avoid further complications he built a simplified model.

In his model goods are produced, within a discrete interval of time, from "natural factors of productions", available in unlimited quantities and from other goods, to be produced in a regime of constant returns to scale. Each possible process of production includes "necessities of life consumed by workers and employees" and he assumes that "all income in excess of necessities of life will be reinvested".<sup>(1)</sup> With the joint production assumption von Neumann can deal with the problems of wear and tear

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(1) In this case von Neumann seems not to have stated his assumption completely and unambiguously. One could think that workers are allowed to save and reinvest part of their income (but this is possible only with a sort of socialist economy where all property is owned by the state that appropriates itself with "all income in excess of necessities of life"). This is not the case in a private, perfectly competitive economy as it follows from the original "economic equations" 7 and 8. The same interpretation was made by Champenowne (1945-46) pp. 11-2, KMT (1956) p. 115, Morishima (1960). So only capitalists do save, but they do not consume.

of capital goods.<sup>(2)</sup> In those cases where all the processes which use capital goods  $k$  are not employed at all, the entire stock of  $k$  would be discarded. The economic lifetime of capital goods is determined endogenously within the system, and thus one of the most unreal assumptions of the neoclassical model, that the economic lifetime of a capital good equals its physical lifetime, is eliminated. The  $i$ -th process lasts one unit of time and converts one bundle of goods  $a_{i*} = (a_{i1} \dots a_{in})$  into another bundle of goods  $b_{i*} = (b_{i1} \dots b_{in})$ . The input and output quantities of good  $j$  in the  $i$ -th process,  $a_{ij}$  and  $b_{ij}$ , are non negative constants and their sum is strictly positive  $a_{ij} + b_{ij} > 0$ . Consequently each single process uses as input or produces as output every single good. So defining  $A = [a_{ij}]$  and  $B = [b_{ij}]$  as the  $m$  by  $n$  input and output matrices von Neumann assumes:

$$[A] \quad A \geq 0, \quad B \geq 0 \quad (\text{non-negative matrices hypothesis})$$

$$[B] \quad A + B \geq 0 \quad (\text{positive entries hp.})^{(3)}$$

(2) See von Neumann (1937) and Morishima (1969).

(3) At this stage it is important that this von Neumann point should not be misunderstood. In our opinion it is not possible to sustain the view "that the workers are slaves so that their consumption in the period  $t$  appear as input in  $t+1$ ", as Frish (1969) does. In this way, workers are allowed to consume what is going to be produced. Wages are prepaid in von Neumann's own system, although it is perfectly possible, like in the Walras-von Neumann model of Morishima (1964) that they are payed at the end of the period. With regard to the von Neumann and KMT models, it is not fully correct to argue (like Morgenstern and Thompson (1977) did) that "natural factors must be treated

## B. Equilibrium constraints

If at time  $t$ , the  $m$  processes have been operated at the level of intensities  $\underline{x}(t) = (\underline{x}_1(t), \dots, \underline{x}_m(t))$  then at time  $t+1$  the input of good  $j$ , needed by the economy,  $\underline{x}(t+1) \underline{a}_{*j}$ , must be equal to or less than the output of the same good produced in the previous period,  $\underline{x}(t) \underline{b}_{*j}$ .<sup>(1)</sup> So in the state of balanced growth  $\underline{x}(t+1) = \alpha \underline{x}(t)$  we have  $\underline{x}(t+1) \underline{a}_{*j} \leq \alpha \underline{x}(t) \underline{b}_{*j}$  for all  $j$ . The rate of growth of each good is  $[\underline{x}(t) \underline{b}_{*j} / \underline{x}(t) \underline{a}_{*j}] - 1$ ; but since all goods are produced, or used by the system [B], we find that the economy can expand only at the minimum rate of expansion of any one of the commodities; hence  $\alpha \leq [\underline{x}(t) \underline{b}_{*j} / \underline{x}(t) \underline{a}_{*j}]$ . Consequently, dropping the time index from  $\underline{x}(t)$  we may write this as:

$$[1] \quad \underline{x} ( B - \alpha A ) \geq 0 \quad (\text{expansion constraint})$$

as free, since they are not produced". If so, we are no longer dealing with a von Neumann economy, but with a von Neumann machine, of the sort described by Clark (1978).

Although von Neumann himself gives no detailed explanation of this assumption, we may interpret him as assuming that the prices of natural factors of production are given in the international market and the economy can import as much of them as it wants.

As for the constant vector of consumption, as an economist I am not able to consider the assumption of the consumption specification being predetermined (g.e. by sociological factors as Bauer (1974) puts it) as an honorable way out. See also Champernowne (1945-6) p. 12 and 17 specially.

(1) As the reader may have noticed  $\underline{a}_{*j}$  represents the row vector formed with the  $i$ -th row of  $A$  while  $\underline{a}_{i*}$  is the column vector formed with the  $j$ -th column of  $A$ . The same applies for  $B$  and for any matrix.

Furthermore overproduced goods will be free goods, that is to say, if for the  $i$ -th product strict inequality  $\alpha \underline{a}_{ij} < \underline{b}_{ij}$  applies then its price will be zero  $y_j = 0$ . Thus we obtain condition

$$[2] \quad \underline{x} (B - \alpha A) \underline{y} = 0 \quad (\text{free goods condition})$$

where  $\underline{y} = (y_1 \dots y_n)^T \geq 0$  is the price vector of the  $n$  commodities. On the valuation side outputs are evaluated at  $\underline{y}(t+1)$ , while inputs at  $\underline{y}(t)$ . The value of input must include the interest paid. Let the interest rate be  $\beta(t)$ ; in equilibrium no process will yield a positive profit, so  $\underline{b}_i * \underline{y}(t+1) \leq \beta(t) \underline{a}_i * \underline{y}(t)$ . It is assumed that unproductive processes (i.e. processes which bring about a negative profit) will not be used.

It follows that if strict inequality applies for the  $i$ -th process then its intensity  $x_i$  will be zero. Each process has its own rate of return  $[\underline{b}_i * \underline{y}(t+1) / \underline{a}_i * \underline{y}(t)] - 1$ . In equilibrium, dropping the time index the interest rate will be equal to the maximum of those rates of return because only the most profitable processes are employed.<sup>(5)</sup>

(5) While  $\alpha$  compares the volume of inputs in the current period with the volume of production in the subsequent period,  $\beta$  has the same function with respect of the values of the bundles of goods produced and consumed as inputs, and hence some economists considered it to be a pure technical datum (see Champernowne (1945-46) p.12 and Morgenstern-Thompson (1976) p. 22). However von Neumann himself stated that  $\beta$  is a monetary variable so that should not be incorrect to think of its equilibrium level as a sort of "natural interest rate". Also see Georgescu-Roegen (1951) specially p. 103 and following.

$$\beta = \max(\underline{b}_i * \underline{y} / \underline{a}_i * \underline{y})$$

Thus we have:

$$[3] \quad (B - \beta A) \underline{y} \leq 0 \quad (\text{profit constraint})$$

$$[4] \quad \underline{x} (B - \beta A) \underline{y} = 0 \quad (\text{profitability condition})$$

The non-negative solutions  $(\alpha, \beta, \underline{x}, \underline{y})$  to [1] to [4] will constitute a von Neumann equilibrium. Of course assumptions [A] and [B] are imposed on the model.

### C. A comparison with the KMT model.

On the other hand a Kemeny-Morgenstern-Thompson (KMT) equilibrium is characterized by the further requirement:

$$[5] \quad \underline{x} B \underline{y} > 0 \quad (\text{positive value constraint})$$

Furthermore they replace assumption [B] by:

$$[C] \quad A \underline{f} > 0; \quad \underline{e} B > 0 \quad (\text{economic production system hp.})$$

where  $\underline{f}$  and  $\underline{e}$  are  $n$  by 1 and 1 by  $m$  sum-vectors, with all entries equal to one. Thus the KMT model assumes that each process uses some input,  $\sum_j a_{ij} > 0$  for all  $i$ , and that the economy can produce any single good,  $\sum_i b_{ij} > 0$  for all  $j$ .<sup>(6)</sup> It is necessary to point out at this

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(6) It is important to notice that the set of models (who satisfy the von Neumann assumption [A], [B]) is not a subset of the one for which KMT assumptions [A], [C] are satisfied.



stage, how grave the consequences of [C] may be. In fact [B] guarantees that:

(1) at the equilibrium position  $\alpha$  and  $\beta$  are uniquely determined, preventing the economy to break up into disconnected parts (subeconomies) each of which has its own expansion rate,

(2) a state of the 'end of the world' ( $B = 0$  and  $\alpha = 0$ ) may be consistent with condition [B] while condition [5] is not fulfilled. In fact from [2] we must have:

$$\underline{x}By = \alpha \underline{x}Ay > 0. (7)$$

(3) the expansion and the rate of return of each individual good and process "can never assume the meaningless form  $0/0$ " (as von Neumann says).

On the other hand the KMT assumption [C], which is still sufficient to establish the existence of equilibrium, allows economists to deal with subeconomies and to choose

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For instance the matrices :  $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$   $B = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$  do satisfy von Neumann's assumption but not KMT's, because, although  $A + B > 0$ ,  $\underline{e}B \geq 0$ . Furthermore the KMT assumptions may be too strict when we are dealing with an open economy. So it is not completely correct to say that "the two new conditions admit a much wider class of economic phenomena than the original condition" and that "if a certain good cannot be produced by any processes in the economy then there is no point in including it": Morgenstern - Thompson (1976) p. 28.

(7) It is therefore incorrect to say that condition "[5] follows from conditions [1] - [4] under the assumption" [B] as Bauer (1974) states at p. 14. In the case of note (6), we have in fact,  $\beta = \alpha = 0$   $\underline{x} = (0 \ 1)$  and  $\underline{y} = (0 \ 1)^T$  as von Neumann solutions but [5] is not satisfied.

between different rates of expansion so that it could be regarded as a better assumption compared with von Neumann [B]. Von Neumann's assumption seems, however, quite likely to be satisfied where each process employs workers and workers consume most commodities.

## 1.2 The Morgenstern Thompson linear programming model and their constrained economy.

It is not our aim in this section to demonstrate the existence of a balanced growth equilibrium either in the original von Neumann model, or in its KMT version. Our purpose is simply to show that the Morgenstern and Thompson model of an open expanding economy is nothing else but a constrained economy derived from a linear programming problem connected to the original closed expanding economy. This is not only the simplest approach to their model, it also allows us to examine the original von Neumann problem from a new point of view and enables us to explore some aspects of the original system.

Although the notation we use differs from the original one of Morgenstern-Thompson (1967), our model does not much differ from their latest revised version and our interpretation is similar to the one by Mardon in Los e Los (1974). Our exposition is based on linear programming models in chapters 3 and 4 of Morgenstern-Thompson (1976), and is supported by their first four axioms.<sup>(1)</sup> One should note that axioms [O1] and [O2] hold with strict equality and that axioms [O3] and [O4] give the objective functions of the dual programming problems. For this reason, and also because we are not interested in a reconsideration of

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(1) See Morgenstern and Thompson (1976), pages 59 and 61.



the entire problem of the open economy our comment will be not too extensive.

### A. The formulation of the dual problems

Let us first rewrite conditions [1], [2], [3], [4] of the original closed model in the form:

$$[1'] \quad \underline{x} (B - \alpha A) \geq 0$$

$$[3'] \quad (B - \alpha A) \underline{y} \leq 0$$

This is possible because any solution of the original problem must satisfy  $\alpha = \beta$ , and [1'] and [3'] imply  $\underline{x} (B - \alpha A) \underline{y} = 0$  so that all the four original conditions are fulfilled. In fact, in both models solutions must satisfy  $\underline{x} B \underline{y} = \alpha \underline{x} A \underline{y} = \beta \underline{x} A \underline{y}$ . Consequently, we have  $\alpha = \beta = \underline{x} B \underline{y} / \underline{x} A \underline{y}$  where  $0 \leq \alpha, \beta \leq \infty$ . We can rule out the meaningless form 0/0 under the von Neumann assumption [B] but not the economically meaningless case  $\alpha = \beta = 0$ , while under the KMT assumption [C] we have  $\underline{x} B \underline{y} > 0$  and therefore  $\underline{x} A \underline{y} > 0$ , thus  $\alpha = \beta = \underline{x} B \underline{y} / \underline{x} A \underline{y}$  with  $0 < \alpha, \beta < \infty$ .

It is shown that for an appropriate value of  $\alpha$  there are non-negative, non-zero solutions  $\underline{x}$  and  $\underline{y}$  of [1'] and [3'] respectively. Following Morgenstern and Thompson, this problem is tackled by standardizing the variables such that  $0 \leq \underline{x} \leq \underline{e}^T$ ,  $0 \leq \underline{y} \leq \underline{f}^T$  (where  $\underline{f}$  and  $\underline{e}$  are the usual sum vectors defined in 1.1). The following more general treatment would be useful in the discussion of

international economic equilibrium; throughout the following we impose general constraints:  $0 \leq \underline{x}^m \leq \underline{x} \leq \underline{x}^e$ ,  $0 \leq \underline{y}^e \leq \underline{y} \leq \underline{y}^m$ . Let us now consider a programming problem:

$$\begin{aligned} & \text{Min} \quad \underline{w}^i \underline{y}^i - \underline{w}^e \underline{y}^e && \text{subject to:} \\ [P] \quad & \underline{x}(B - A) + \underline{w}^i - \underline{w}^e = 0 \\ & \underline{x} \geq \underline{x}^m \\ & -\underline{x} \geq -\underline{x}^M \end{aligned}$$

where  $\underline{w}^i \geq 0$ ,  $\underline{w}^e \geq 0$  are 1 by  $n$  slack variable vectors. Obviously it is seen that  $\underline{y}^e = 0$ ,  $\underline{x}^m = 0$  in the a special case of the closed economy. We can show that there are feasible solutions for each  $\alpha$ .

Let  $\underline{x} = \underline{x}^m$  and  $\underline{w}^i_h = -[\underline{x}(B - \alpha A)]_h$  when  $[\underline{x}(B - \alpha A)]_h < 0$  and otherwise  $\underline{w}^i_h = 0$ . Also let  $\underline{w}^e_h = [\underline{x}(B - \alpha A)]_h$  when  $[\underline{x}(B - \alpha A)]_h > 0$  and otherwise put  $\underline{w}^e_h = 0$ . It is evident that these satisfy all the constraints of [P]. In particular, where  $\underline{x}^m = 0$  we have feasible solutions  $\underline{x} = 0$ ,  $\underline{w}^i = 0$ ,  $\underline{w}^e = 0$  which is also an optimal solution, provided that  $\underline{y}^e = 0$ . Hence for all optimal solutions the objective function must attain a zero value.

We obtain similar solutions for the dual problem:

$$\begin{aligned} & \text{Max} \quad \underline{x}^m \underline{p}^m - \underline{x}^M \underline{p}^M && \text{subject to:} \\ [P'] \quad & (B - \alpha A)\underline{y} + \underline{p}^m - \underline{p}^M = 0 \\ & \underline{y} \leq \underline{y}^i \\ & \underline{y} \leq -\underline{y}^e \end{aligned}$$

where  $\underline{p}^m > 0$ ,  $\underline{p}^M > 0$  are  $m$  by 1 slack variable vectors.

Hence, provided  $\underline{x}^m=0$  and  $\underline{y}^e=0$  all optimal solutions must have  $\underline{w}^i = 0$  and  $\underline{p}^M = 0$ , and  $\underline{x}(B-\alpha A) - \underline{w}^e = 0$ ,  $(B-\alpha A)\underline{y} + \underline{p}^m = 0$ , with  $\underline{w}^e \geq 0$  and  $\underline{p}^m \geq 0$ , satisfying [1'] and [3'].

Furthermore both the last terms of the inequalities

$$\underline{x}(B - \alpha A)\underline{y} = \underline{w}^e \underline{y} - \underline{w}^i \underline{y} \geq \underline{w}^e \underline{y}^e - \underline{w}^i \underline{y}^i$$

$$\underline{x}(B - \alpha A)\underline{y} = \underline{x} \underline{p}^M - \underline{x} \underline{p}^m \leq \underline{x}^M \underline{p}^M - \underline{x}^m \underline{p}^m$$

are equal to zero where solutions are optimal with  $\underline{x}^m=0$  and  $\underline{y}^e=0$ . Hence we can derive the conditions

$$y_j = y^e_j \quad \text{for } w^e_j > 0$$

$$x_h = x^m_h \quad \text{for } p^m_h > 0$$

Similarly we obtain  $y_j = y^i_j$  for  $w^i_j > 0$  and  $y_j = y^M_j$  for  $p^M_j > 0$ .

These linear programming results are interpreted, in the context of the closed economy as follows.

$w^e_h > 0$  indicates an excess output of good h that the economy will not use and consequently in our closed economy h is a free good (whose price is  $y_h = y^e_h = 0$ ).

$p^m_k > 0$  indicates that process k is unprofitable; that is, its excess profit is negative and, therefore, its intensity should be  $x_k = x^m_k = 0$ . The fact that the optimal solution has  $\underline{p}^m \geq 0$  means that no process in the economy yields positive profits. On the other hand  $\underline{w}^i = 0$  means that we can only use goods produced in the previous period, so that in a closed economy it is impossible to

obtain a supply greater than the amount of commodities accumulated in the past, within it.

### B. The constrained open economy interpretation

The model in the previous section can be transformed into a model of an open economy, rather than the one of the closed economy, by removing the conditions  $\underline{x}^m = \underline{0}$  and  $\underline{y}^e = \underline{0}$ . In this context Morgenstern and Thompson assume that the rest of the world will import (and consume) the whole vector  $\underline{w}^e$  at a price  $\underline{y} = \underline{y}^e \geq \underline{0}$ , and export the vector  $\underline{w}^i$  at the price  $\underline{y} = \underline{y}^i \geq \underline{y}^e$ .

They assume fixed rates of exchange, between the country concerned and the rest of the world. Accordingly the condition  $\underline{0} \leq \underline{y}^e \leq \underline{y} \leq \underline{y}^i$  is reinterpreted from a new point of view. The first constraint of [P]

$$[10] \quad \underline{x} (B - \alpha A) + \underline{w}^i - \underline{w}^e = 0$$

now means that the output of  $t-1$  plus import must be equal to the input for the next period  $t$  plus the export. The import vector  $\underline{w}^i$  and the export vector  $\underline{w}^e$  of the country are considered to be determined instantaneously at the beginning of period  $t$ . Morgenstern and Thompson choose an equation instead of the inequality  $\underline{x}(B - \alpha A) + \underline{w}^i - \underline{w}^e \geq \underline{0}$  because  $\underline{w}^i$  and  $\underline{w}^e$  are obtained as slack variables, determined so as to make the equality hold.

The only case in which the inequality might be considered to hold would be in the case where we have  $y_j =$

$y^e_j = 0$ . In this case the  $j$ -th good is free in the country concerned and can be exported freely to the rest of the world, so that no excess supply remains in the world economy and thus [10] holds with equality.

The dual problem [P'] may also be interpreted in a similar way. Where we impose a minimum level of activity to 'unprofitable' processes we must subsidize them by an amount  $\underline{x}^m p^m$  at this minimum level. In order to prevent processes to be operated at a level more than a given maximum one, we must reduce their profit to zero by taxing them the amount  $\underline{x}^M p^M$  at this maximum level. Thus the first constraint

$$[20] \quad (B - \beta A) \underline{y} + \underline{p}^m - \underline{p}^M = 0$$

states that the value of output at  $t$  plus the subsidies is equal to the value of input (including interest) plus taxes.

Thus the subsidies  $\underline{p}^m$  and the taxes  $\underline{p}^M$  are instantaneously determined at the beginning of the period  $t$ . We have an equality, instead of the usual inequality, because of the slack variable nature of  $\underline{p}^m$  and  $\underline{p}^M$  in [P']; and because processes are subsidized if necessary even at a zero minimum level of activity.

### C. The balance of payments and balanced budget requirements

As an international economic equilibrium condition, Morgenstern and Thompson add:

$$[30] \quad \underline{w}^e \underline{y}^e - \underline{w}^i \underline{y}^i = 0$$

which means that exports equal imports, if they are evaluated at the external export and import prices.

Where condition [30] prevails in a state of balanced growth equilibrium, the rest of the world can sustain a constant proportion of the capital of the country economy concerned (or of the public sector debt) only in the very special case in which the interest rate  $\beta_w$  of the rest of the world is not different from the rate of growth  $\alpha$  of the economy. Where capital  $D$  expands at the rate  $\alpha-1$  the economy has an in-flow of capital equal to  $(\alpha - 1) D$ , while the interest payments to the rest of the world are  $(\beta_w - 1) D$ . To establish a zero value of the capital account of the balance of payments we must require:

$$(\beta_w - 1) D - (\alpha - 1) D = 0.$$

which holds for  $D > 0$  if and only if  $\beta_w = \alpha$ .

Allowing for no other source of finance in their model Morgenstern and Thompson state that subsidies should be financed by taxes on positive profits, so that where all these processes are operated at their lowest and highest bounds, we have:

$$[40] \quad \underline{x}^M \underline{p}^M - \underline{x}^m \underline{p}^m = 0$$

This condition, imposed by Morgenstern and Thompson, implies that the public sector can "use the tax money to

sustain the unprofitable industry as well as finance its own activities or services".<sup>(2)</sup> This is rather strange. However, even this idea will become completely unsound when we see how in this model the net revenue, available to finance these other public services, can in reality be negative, leading to a public sector debt, being:

$$0 = \underline{x}^M p^M - \underline{x}^m p^m \leq \underline{x} (p^m - p^M)$$

However, taking the dual linear program into account, we can see that these conditions state nothing else but that the optimal value of the dual objective functions must be equal to zero, at some  $\alpha$  appropriately chosen.

We have already seen that prices  $y_j$  and intensities  $x_h$  are equal to one of their respective lowest and highest bounds, where either  $w^i_h > 0$ , or  $w^e_h > 0$  in the case of  $x_h$  and either  $p^m_j > 0$ , or  $p^M_j > 0$  in the case of  $y_j$ . We should however notice that the condition [4o], which is true only when  $\alpha = \beta$ ,<sup>(3)</sup> is no longer required by the model.

#### D. Assumptions and conclusions by Morgenstern and Thompson

To complete the model Morgenstern and Thompson require:

$$[5o] \quad \underline{x} B \underline{y} > 0$$

$$[6o] \quad \underline{x}^m \leq \underline{x} \leq \underline{x}^M$$

(2) Cfr. Morgenstern-Thompson (1976), p. 61.

(3) As Morgenstern and Thompson demonstrate on p.74.



$$[70] \quad \underline{y}^e \leq \underline{y} \leq \underline{y}^i$$

In addition to assumption [C], that is an assumption which can hardly be rational in the case of a model including international trade. Although it implies that, with regard to any commodity, the economy is provided with at least one process which can produce it, Morgenstern and Thompson also assume:

$$[D] \quad \underline{0} \leq \underline{x}^m \leq \underline{x}^M$$

$$[E] \quad \underline{0} \leq \underline{y}^e \leq \underline{y}^i$$

They also demonstrate:

(a) that in the case of  $\underline{x}^m = 0$  and  $\underline{y}^e = 0$  the model is reduced to the original von Neumann closed economy (as we have already seen) and

(b) that where  $\underline{w}^e \underline{y}^e > 0$  we obtain an open economy, while where  $\underline{w}^e \underline{y}^e = 0$ , a closed economy.<sup>(4)</sup>

This is, however, not completely true. In fact we may have  $w^i_j > 0$  for  $y^i_j = 0$ ; in this case the economy would import free goods from the rest of the world which are useful to it. Thus even though  $\underline{w}^e \underline{y}^e = 0$ , the economy may still be open. Finally in order to obtain an open economy model they assume:

$$[F] \quad \underline{x}^m \wedge \underline{y}^e > 0$$

(4) See definition 2, p. 63.



with given  $\underline{x}^m (\geq 0)$  and  $\underline{y}^e (\geq 0)$ . This means according to Morgenstern and Thompson, that, even when the economic system operates at the minimum level of activity  $\underline{x}^m$ , its input requirement is positive even though it is evaluated at the lowest possible price set  $\underline{y}^e$ .

Now taking into account [3o] and [4o] as well as assumptions [D] and [E], we obtain from [1o] and [2o]:

$$\underline{xBy} - \alpha \underline{xAy} \geq 0 \geq \underline{xBy} - \beta \underline{xAy} \quad \text{and hence:}$$

$$\alpha \leq \underline{xBy} / \underline{xAy} \leq \beta$$

On the other hand, from [1o] and [2o], we have  $\underline{w}^1 \underline{y} - \underline{w}^e \underline{y} = 0$  and  $\underline{x}p^m - \underline{x}p^M = 0$ , respectively; hence  $\underline{xBy} - \alpha \underline{xAy} = \underline{xBy} - \beta \underline{xAy} = 0$  so that  $\alpha = \beta = \underline{xBy} / \underline{xAy}$ .

Furthermore it is seen that every set of solutions to the system, [1o] to [7o], with  $\alpha = \beta$  provides a pair of solutions to the two problems [P] and [P'].

Morgenstern and Thompson prove that for any  $\alpha > 0$  there are optimal solutions to the dual linear programming problems, as long as feasible solutions exist. They also show that the optimal values of the objective functions ([3o] and [4o]) will both be zero for some  $\alpha > 0$  if [F] holds. Consequently, given their assumptions, they succeeded in demonstrating that there exists at least one  $\alpha > 0$  that satisfies [1o] through [7o] with  $\alpha = \beta$ . This gives an open economy equilibrium which is obtained by solving the dual linear programming problems.

### E. The budget and balance of payment constraints

Morgenstern and Thompson concentrate their attention to the solutions with  $\alpha = \beta$ , because they are interested in obtaining an international economic equilibrium by solving the linear programming problems.

When we deal with a small economy, however, we need a balance of payment condition, which is different from the one in [30] which corresponds to the objective function of the linear program [P], because in such a case we should "treat the export and import prices as exogenous variables" as they themselves recognize.<sup>(5)</sup>

This is because the economy cannot import (or export) positive quantities from (or to) the rest of the world at a price lower (or higher) than the one at which the rest of the world is ready to export (or import) them. In this model Morgenstern and Thompson almost ignore the side of the rest of the world, and implicitly assume that it allows the economy to export and import any volume of goods at fixed export and import prices, and say that these amounts "do not influence world prices". Therefore, they should require that  $\underline{y}_j = \underline{y}^e_j$  for  $\underline{w}^e_j > 0$  and  $\underline{y}_j = \underline{y}^i_j$  for  $\underline{w}^i_j > 0$ , as it is satisfied by the solutions of the linear programming.

Morgenstern and Thompson however do not require this condition in their model, in fact from their axioms we

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(5) Morgenstern and Thompson (1976) p.59.

have:

$$0 = \underline{w}^e \underline{y}^e - \underline{w}^i \underline{y}^i \leq \underline{w}^e \underline{y} - \underline{w}^i \underline{y} = \underline{x} (B - \alpha A) \underline{y}$$

Therefore it is possible that  $y_j < y^i_j$  for  $w^i_j > 0$  and  $y_j > y^e_j$  for  $w^e_j > 0$ . It will be shown below that the growth factor may be lower than the minimum level of interest rate, where "the internal balance of payment index" defined as  $r = (\underline{w}^e - \underline{w}^i) \underline{y}$  is positive.<sup>(6)</sup>

In their paper, however, the economic reason why the "external" balance of payment can be in equilibrium whereas the internal one is simultaneously positive is unclear.

It is noted furthermore, that where there are profitable and unprofitable processes at the same time, it is clear that the first will maximize their intensities while the intensities of the second will tend to be zero. For this reason unprofitable processes must be subsidized in order to be run at a positive intensity, while profitable ones must be taxed in order to be limited at their maximum intensities. These taxes provide the necessary funds for the subsidies.

It is thus required that the optimal solutions to the linear programming problem [P'] satisfy  $\underline{x}(p^M - p^m) = 0$ .

Furthermore, under the axioms made by Morgenstern and Thompson we have:

$$0 = \underline{x}^M p^M - \underline{x}^m p^m \geq \underline{x} p^M - \underline{x} p^m = \underline{x} (B - \alpha A) \underline{y};$$

(6) As Morgenstern Thompson show at p.82.

consequently, the "internal balance of profit index"  
 $s = \underline{x}(p^M - p^m)$  may be negative, see p.82.

On these points Morgenstern and Thompson only state that where  $\alpha < \beta$ , "the solution is not efficient" and consequently "it seem economically uninteresting" p.82.

### 1.3 On the closed economy: a digression.

#### A. A definitional problem

Usually a closed economy is defined as an economy that does not trade with other economies; it has neither exports, nor imports. This concept is useful because it enables economists to ignore all other economic systems.

However we may alternatively define a closed economy in the sense of the closed Leontief model. That is to say an economy is closed if:

- a) there is no autonomous demand for any commodity (that is independent from the level of activity),
- b) each commodity can be produced by the system.

Also one may define a closed model as a system consisting of homogeneous equations or inequalities, without specifying economic meaning to it.

In this last sense, however, even the Morgenstern-Thompson open economy may be regarded as a closed model.

Thus the last definition is too general and unsuitable for our purposes.<sup>{1}</sup>

#### B. The von Neumann treatment of Labour

Due to his own peculiar treatment of labour and consumption, it might be contended that von Neumann's

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{1} In constructing his model von Neumann was careful about its economic implications as he did not just built a purely abstract axiomatic model.

economy could be formulated as a closed economy. But in what sense is it "closed"? It is particularly important to examine this point carefully.

Von Neumann may be interpreted, as by many economists, as assuming that natural factors of production are free and disposable in unlimited quantities so that their prices are zero in his model. However the same is not true for labour. He states that "consumption of goods takes place only through the process of production which includes necessities of life consumed by workers and employees". This way of viewing labour and consumption is different from the usual treatment of them. It should be clearly noted that labour is not an ordinary good which is subject to the free good rule, and also on the other side, that the process of consumption is not subject to the rule of profitability.

Corresponding to this, labour, unlike all other goods, appears in the von Neumann model only implicitly, whilst workers' consumption processes are not even stated explicitly and independently. It has been already discussed (note 3 of 1.1.A) how labour cannot be free, in spite of its availability in unlimited amounts. Workers receive wages sufficient for all the necessities of life without any reference to the excess supply. (We can postulate that where supply exceeds -or falls short of- demand, then the gap between them is eliminated by emigration -or immigration. Thus the fact that labour



supply can be expanded indefinitely does not make labour a free good.

### C. Is the von Neumann economy open?

Let us consider the first definition. Assuming that the reproduction of labour is made outside von Neumann's economy, we can eventually imagine a sort of trade between the economy and families, i.e. a trade between labour and consumption goods. A unit of labour is exchanged for a given bundle of consumption goods, the terms of trade being fixed. Thus consumption goods produced within the von Neumann economy are traded for labour produced outside of it, so that it cannot be considered as closed according to the first definition.

The possibility that labour may be a free good can be ruled out by making the assumption that the consumption bundle contains every commodity (this assumption would be consistent with his hypothesis that  $A+B > 0$ ). As one of the commodities is scarce, the value of the consumption bundle is always positive, so that wages which are enough to buy the bundle are also positive. Therefore, labour is not a free good. As von Neumann assumes that labour is imported, where there is an excess demand for it. Thus he assumes the existence of other economies, so that his model cannot be closed. Only if we restrict it to the trade of produced goods, can we speak of a model of a closed economy. Now let us consider the second definition. Although the demand for each commodity depends

on intensities of the production processes and no autonomous demand is included in the model, it is also clear that some of the commodities (especially labour and some kinds of materials) are not produced within the economy. A special consideration is made on these commodities.

It is true that the special treatment reserved by von Neumann to labour has not been accepted by most economists, who are accustomed to treating it as an ordinary good, the price of which is determined by the market forces of demand and supply. These economists regard labour as an output which is produced from the consumption goods consumed by the workers. Concerning the production function they assume that constant returns to scale prevail.

For instance Georgescu-Roegen (1951) defines the closed economy as "the totality of all processes of production and consumption - including consumption of consumers' good as a labour producing process as well as any other achievable process".<sup>(1)</sup>

Afterwards in section 5 and 6 he considers von Neumann's model as "a closed economy where all processes are of the von Neumann type".<sup>(2)</sup>

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(1) Cfr. Georgescu-Roegen (1951), p.101, my italics.

(2) Cfr. Georgescu-Roegen (1951), pp.108-9.



The attitude, as well as the conclusion, of Koopmans is similar. He says: "In ... von Neumann [1937,1945], labour has been treated as an output of an activity, of which the consumption of various commodities constitutes the set of inputs. The model thus becomes a closed one."<sup>(3)</sup>

In the same way Dorfman, Samuelson and Solow (1958) argue that: "In first place we are now dealing with a closed system, a pure production model. It is 'closed' in the sense that there is no final demand and no fixed factor. We can think of labour as being 'produced' by households with consumption good as input... The important thing is that there is no autonomous demand for commodities and no resources which cannot be produced like other goods."<sup>(4)</sup>

It is not clear whether KMT (1956) follow the same reasoning (which has been made explicit later by Dorfman, Samuelson and Solow) when they allow for consumption processes in their model. This is because, if labour is produced, its price should be determined by the economic system, but KMT assume that labour is free. Morgenstern and Thompson in their book, however, repeat the same statement as it is made in the KMT article, that workers

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(3) Cfr. Koopmans (1951), p. 39

(4) Cfr. Dorfman, Samuelson and Solow (1958) at p.382.

are produced as an output.<sup>(5)</sup> Thus Morgenstern and Thompson (1976) make clear what might have been intended by KMT (1956). But it has to be added that their statement that "the processes may also represent consumption" is in open contradiction with KMT's insistence that labour "must be treated as free".<sup>(6)</sup>

This shows how easily these authors followed, and even attributed to von Neumann himself, Champernowne's suggestion "to allow several alternative production processes for obtaining 'labour', each process requiring a different bundle of goods as 'real wages', between which the labourer may be supposed indifferent".<sup>(7)</sup>

Although Champernowne's suggestion may seem natural, one may more ask the more basic question why von Neumann wanted to differentiate labour from others goods.

Answering this question Morishima (1969) writes: "We would in fact depart from reality if we regarded our homes, however humble, as 'pigsties' where 'hogs and pigs' are fed and bred according to the Rule of Profitability.

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(5) See Morgenstern and Thompson (1976) with their example at p. 27.

(6) Cfr. KMT (1956), p. 116 and ch.2 of Morgenstern Thompson (1976).

(7) The suggestion has had a long lasting influence. For instance, though with no reference to Champernowne (1944-5) Weil (1967a) uses the same idea to find a new method of introducing 'extra consumption' into the von Neumann model maintaining its general properties (for example  $\alpha = \beta$ ).

It seems that there is no rationale for requiring that 'rate of profit' be equalized throughout all industries and families."<sup>(8)</sup>

This passage clearly states that the view of consumption as a labour producing process is unrealistic and, hence, unsatisfactory.

Rejecting Georgescu-Roegen's and Koopmans' views, Morishima pursues an entirely different approach which leads to a more appropriate interpretation of von Neumann's economy.

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In practice he introduces a new column for the good (labour) that he calls 'extra consumption' and  $k > 1$  a new row representing "alternative means for producing one unit of extra consumption" (p.44). Then he explains how the "labour force can decide on alternative bundles of goods that will impart equal amount of satisfaction and the price mechanism will select the means of producing extra-consumption [labour] that is cheapest from society's point of view" (p.45). Naturally the influence of Champernowne's (1944-5) article is indirect it being cited neither in note, nor found in the References.

(8) Cfr. Morishima (1969).

## 2. CONSUMPTION IN THE VON NEUMANN MODEL.

### A. Premises

In the original von Neumann equilibrium we have an economy expanding at a uniform rate of growth  $\alpha$  (equal to the interest rate  $\beta$ ) with workers' but no capitalists' consumption. It is easy to see how this result,  $\alpha = \beta$ , is related to the restrictive hypothesis on consumption. We will show that the rates of interest and growth are equal to each other when the value of total consumption is equal to the total income of the natural factors of production. This implicit restriction has already been criticized by Champernowne and the consequences of its removal have been investigated by other economists.

A number of different ways of introducing consumption into the von Neumann model have been proposed, and reviewed in some articles.<sup>(1)</sup> Nevertheless, an important part of the literature [like Morishima (1969), Haga-Otsuki (1965)], has not been seriously examined. Furthermore, the generalized models obtained have not been studied critically.

Let us therefore review once again all the principal models. To simplify the problem we classify first the models from the viewpoint of the relationship of  $\alpha$  to  $\beta$ . We postpone to the end of the chapter the analysis of models which explicitly introduce proper demand functions for goods and supply functions of primary factors (say labour).

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(1) Tomasini (1968), Frish (1969), Bauer (1974), Zamagni (1977).

With the introduction of consumption, we are confronted with the problem of consumption lags. We must formulate an explicit hypothesis concerning the behaviour of economic agents (i.e. their choice of consumers' goods). Also where some agents have different sources of revenue, there is a possibility that the so called Pasinetti paradox may occur.

We shall begin our inquiry with the analysis of the following models: KMT (1956), Malinvaud (1959) and Morgenstern-Thompson (1967). One way to treat these models systematically is perhaps to start by investigating the way of treating consumption and saving which has been proposed in Morgenstern Thompson (1976).<sup>(2)</sup> We will focus our attention on the new part which did not appear in Morgenstern Thompson (1967) but in their (1976).<sup>(3)</sup> This would be justified because we are more interested in the economics of the model. In the new version we find some possible ways of choosing the surplus or income matrix  $H > 0$ . According to the authors,  $H$  may be chosen "such that it will be certain to lead to economic reasonable models".<sup>(4)</sup> We hope that by using the specific Morgenstern Thompson  $H$

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(2) See especially pp.22-3, and p.30 ff. See also ch.6. Note that the Morgenstern Thompson model is, as the authors themselves claim, a more "general model that includes as special cases both the Malinvaud model and the KMT outside demand model" Morgenstern Thompson(1967).

(3) Morgenstern Thompson (1967) pp. 118 - 121

(4) See Morgenstern-Thompson (1976) p.118. Morgenstern-Thompson (1967) clearly state at p.388: "We leave further interpretations of our new results to another occasion,



matrix as an example, the results obtained by them and by KMT in the case of the 'outside demand' model could be better explained and more clearly compared.

### B. A brief summary of the criticisms

Let us explain the nature of the criticisms that will be made on these models and sum up the results of the analysis.

To start with, it is important to notice that  $xHy$  does not represent total income, but additional income, over the subsistence one. It is furthermore noticed that this income matrix may contain negative elements if the intensity vector (or the price vector, depending on what definition will be adopted) is not at its equilibrium value, or the rate of expansion of the von Neumann model is not positive. In order to avoid this problem  $H$  is regarded not as a function of  $x$  or  $y$ , but as a fixed matrix, like in Morgenstern-Thompson (1967). Consequently, it will be interpreted as additional income matrix only where balanced growth prevails.

In our opinion the introduction of 'outside demand' does not lead to a generalization of the von Neumann model. Instead it can only show how the variation of the level of workers' consumption will affect the level of growth and

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believing that setting forth basic, elementary, modifications of the assumptions is indicating before extending upon more detailed interpretation. The main task is still to explore what is logically possible".



interest rate, preserving their equality. Where the public sector's demand for private goods and taxes does not depend on prices and the budget is balanced, the public sector can be accommodated in the von Neumann model by the method of introducing 'outside demand'.

The first model to allow  $\alpha \neq \beta$  is Malinvaud (1959). It distinguishes between demand for goods and that for primary factors of production. It makes clear that 'outside demand' which cannot be capitalists' demand in perfect competition, is only compatible with a sort of mark-up pricing (and extra profits). In any case, Malinvaud, like KMT, but unlike Morishima and Haga-Otsuki, does not allow for demand and supply which are functions of prices and income.

As for the Morgenstern-Thompson model it may be made consistent with perfect competition, allowing for some time lags, only when the saving ratio (out of surplus income) is equal to the ratio of wages income (out of surplus income).

Wages and consumption ratios are given constants as the demands for goods, they are not variables or functions as claimed by Frish and Bauer. Following Frish's interpretation we shall later see that  $\beta^*$  does no longer represent the interest rate, where Morgenstern's and Thompson's unit cost equation holds, and workers' additional consumption and income is zero. However, if the assumption of perfect competition is removed, one other interpretation is possible. In this more general

interpretation the saving ratio must equal the sum of wages and extra-profit income ratios. [Eventually, the model can be made also time consistent even if the distributive aspect is in part undetermined and not all additional income is disposable for consumption.]

### 2.1 Morgenstern Thompson Additional income matrix.

In Morgenstern and Thompson (1976) the income, consumption and saving matrices are derived within the framework of the original von Neumann model.

Consumption  $kH$  and capital accumulation (or savings)  $zH$  (of workers and stockholders) are a fixed fraction of  $H$  which is surplus in real terms obtained as the product of the surplus vector:  $\underline{h} = \underline{x} (B - A)$  and the unit sum vector  $\underline{e}^T$ ; that is to say,  $H$  is defined as  $H = \underline{e}^T \underline{h}$ .<sup>(1)</sup> So that  $\underline{x}H = \underline{x}(B-A)$ , because  $\underline{x} \underline{e}^T = 1$  by normalization.

Then how should the  $A$  matrix be interpreted? Is it still the same as the  $A$  of the original von Neumann model? The answer is positive, in fact in Morgenstern Thompson (1976) it is stated that the inputs include the necessities of life of workers.<sup>(2)</sup>

(1) This is the 'excess production' definition of  $H$  but quite analogously  $H$  may be defined according to the 'excess value' definition as  $H = \underline{h} \underline{f}$  and  $\underline{h} = (B-A)\underline{y}$ , where  $\underline{f}$  is a 1 by  $m$  sum row vector, so that  $\underline{y}\underline{f} = 1$ . Therefore  $\underline{x}H = \underline{h} = \underline{x}(B-A)$  but  $H \neq B-A$ .

(2) Morgenstern Thompson (1976) p.102. Furthermore the entries  $h$  are "interpreted as the maximum amount of optional consumption or surplus, in real terms, that are permitted in the model."

Thus  $\underline{x}Hy$  must be considered as **additional** (not total) income and optional consumption  $kH$  excludes the necessary consumption included in the  $A$  matrix.

However, this way of defining  $H$  as the maximum surplus matrix does not assure the initial assumption  $H \geq 0$ , for all possible  $\underline{x} \geq 0$ ,  $A$  and  $B$ , with  $A+B > 0$ . Generally speaking, an arbitrary level of activity  $\underline{x}$  will not satisfy the von Neumann assumptions. Therefore the Morgenstern-Thompson model differs from von Neumann's.

Hence  $\underline{x} (B - \alpha A) > 0$  is not necessarily satisfied (and even if it was, it does not mean  $\alpha - 1 > 0$ ). For some  $\underline{x} \geq 0$  we may have a negative consumption of good  $j$ ,  $\underline{h} = \underline{x} (\underline{b}_{*j} - \alpha \underline{a}_{*j}) < 0$ , though this is unintelligible from the economic point of view. Even for those  $\underline{x}$  which satisfy the von Neumann inequalities,  $H \geq 0$  is not generally obtained with  $\alpha < 1$ . Of course, negative  $H$  should be avoided; positiveness was notably the only restriction "on the choice of the optional consumption matrix  $H$ " imposed when  $H$  was introduced.<sup>(3)</sup> Consequently, the 'excess production' way of introducing  $H$  will "lead to economically reasonable models", only under very restrictive conditions. Where  $A$ ,  $B$  and  $\underline{x}$  cannot be further restricted, it should be assumed that in equilibrium  $\alpha > 1$ .

This is a remarkable assumption, especially because in the other parts of the Morgenstern-Thompson book we are

(3) See Morgenstern-Thompson (1976), Ch.6 (p.101).

told that we are not able to determine the condition for  $\omega > \alpha = \beta > 1$  in the von Neumann model.<sup>(4)</sup> Hence Morgenstern-Thompson do not say anything about how to choose  $H$  so as to produce a reasonable model encompassing consumption.

Let us point out again that as  $A$  does already include the subsistence consumption of workers,  $\underline{x}_H$  clearly is the vector of goods "in excess over the amount needed to maintain a stationary economy (including the reproduction of workers)".<sup>(5)</sup> This as the fact of  $\underline{x}_H \underline{y}$  being additional income will however be valid only in a state of balanced growth.

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(4) See for instance Morgenstern Thompson (1976) p. 33.

(5) See Morgenstern-Thompson (1976), p.118.

## 2.2 The KMT and Malinvaud consumption model.

### A. The KMT consumption model

Before fully considering Morgenstern Thompson (1976) let us analyze, within this framework, the so called 'outside demand case' (first treated in KMT (1956) and re-examined in Morgenstern-Thompson).<sup>(1)</sup> We will represent the  $m$  by  $n$  consumption matrix  $e^T d$  (where  $d$  is the  $1$  by  $n$  row vector of quantities of goods produced by the economy and supplied to consumers and  $e^T$  the  $m$  by  $1$  sum column vector) by the consumption matrix  $kH$ .<sup>(2)</sup> The new model can be represented by the following equations:

$$[1^{\wedge}] \quad \underline{x}[B - \alpha^{\wedge}(A+kH)] \geq 0 \quad (\text{expansion constraint})$$

$$[2^{\wedge}] \quad \underline{x}[B - \alpha^{\wedge}(A+kH)]\underline{y} = 0 \quad (\text{free good rule})$$

$$[3^{\wedge}] \quad [B - \beta^{\wedge}(A+kH)]\underline{y} \leq 0 \quad (\text{profit constraint})$$

$$[4^{\wedge}] \quad \underline{x}[B - \beta^{\wedge}(A+kH)]\underline{y} = 0 \quad (\text{profitability rule})$$

We will still have  $\alpha^{\wedge} = \beta^{\wedge}$ . Moreover, as  $A^{\wedge} = A+kH$ , we obtain  $\underline{x}B\underline{y}/(\underline{x}A\underline{y}) > \alpha^{\wedge} = \underline{x}B\underline{y}/(\underline{x}A^{\wedge}\underline{y})$  because  $\underline{x}A^{\wedge}\underline{y} > \underline{x}A\underline{y}$  for all values of  $\underline{x}$  and  $\underline{y}$ .<sup>(3)</sup> In particular where von Neumann solutions  $\underline{x}$ ,  $\underline{y}$  equal Morgenstern's and Thompson's  $\underline{x}^{\wedge}$ ,  $\underline{y}^{\wedge}$ , as discussed in footnote (5) below we have  $\underline{x}B\underline{y}/(\underline{x}A\underline{y}) = \alpha$ .

(1) See Morgenstern-Thompson (1976), pp. 37-38.

(2) This is equivalent to  $k = z$  in the Ch.6 model.

(3) In the light of this and of [5] it is quite odd that Morgenstern and Thompson have not realized that in Fig.6.6 this model is on the  $45^{\circ}$  line, rather than the point  $k=z=0$  (p. 110).

Notice also that (14) at p. 37 is wrong.



In any case we still have a system which is the same as the von Neumann model as already seen by Champernowne.<sup>(4)</sup>

In view of the fact that  $\hat{A}$  is an augmented input matrix containing the consumption of workers, it is not surprising at all to see that we do really go back to the von Neumann model. In this sense the original model does not require, as KMT put it, "the restriction of the consumption by workers to the level of subsistence". This formulation of demand implies that total wages equal to total consumption  $\underline{x}Lv = k\underline{x}Hy = \underline{x}LCy$  where  $v$  are the real wages paid per unit of labour and  $L$  is the labour input coefficient vector. The excess of wages over subsistence consumption,  $\underline{x}LCy > 0$ , where  $C$  denotes the consumption bundle of workers, is spent for additional consumption  $k \underline{x} H$ , so that workers do not save. It is also possible to think of a tax imposed on the processes of production by the government. We may assume a balanced budget, so that the tax is entirely spent on citizens' consumption items or transferred abroad as foreign aid.

Where the tax revenue is not consumed but reinvested in the production sectors, the values of  $\hat{\alpha}$  and  $\hat{\beta}$  are affected though they keep their equality. This however is not true if the tax burden is not imposed exclusively on the processes of production. In making a simple model where 'balanced budget' means a zero current budget surplus, we

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(4) See Champernowne (1945-46) p. 16. See also note (1) of ch.1.



can show that  $\underline{y}$  and  $\underline{x}$  are not affected in the 'excess value' case.<sup>(5)</sup>

This may be regarded as a 'non distortionary tax system' for the von Neumann model (when  $\alpha > 1$ )<sup>(6)</sup> and this type of interpretation seems more satisfactory than the Morgenstern-Thompson one in terms of workers' wages and consumption. If the model should be extended to include workers' additional consumption over the subsistence level consumer choice should have been formulated such that it is consistent with the utility maximization by workers, like in Morishima (1964).

But, where the income matrix is determined by the excess production of goods,  $\underline{x}$  and  $\underline{y}$  will remain constant, as we have seen. Furthermore, even if we may accept the fact that workers consume all their income, it would be difficult to accept that their demands do not depend also on prices, as well as that the marginal propensity to consume of capitalists is always equal to zero. This causes a decreased rate of expansion in step with an equally decreased rate of interest. This "result seems

(5)  $\underline{x} = \underline{x}^{\wedge}$ , follows from Theorem 6-3 p.120 of Morgenstern Thompson (1976), and we will have (taking  $k=z$ )  $\alpha^{\wedge} = \beta^{\wedge} = \alpha / [1 + k(\alpha - 1)]$  from (31) and (33) p.119. On the other hand  $\underline{x}$  and  $\underline{y}$  with the "excess value" case will not be affected because of Th. 6-4 p.120. Furthermore, from  $[B - \beta^{\wedge}(A + kH)] \underline{y} \leq 0$  we obtain  $\{B - [\beta^{\wedge}(1-k)/(1 - \beta^{\wedge}k)] A\} \underline{y} < 0$  (for  $k < 1/\beta^{\wedge}$ ) which is satisfied by von Neumann's by  $\underline{y}$  and  $\beta = \beta^{\wedge}(1-k)/(1 - \beta^{\wedge}k) = \alpha$ .

(6) It is however worth noting that, where the processes are taxed differently it is possible that the real and monetary structures of the economy ( $\alpha$ ,  $\underline{x}$ ,  $\beta$ ,  $\underline{y}$ ) are affected.

unsatisfactory, because the way in which final demand is introduced into the von Neumann model is rather peculiar".<sup>(7)</sup>

### B. The Malinvaud consumption model

A slight variation on the theme is found in the Malinvaud (1959) article, whose particular parts, connected with the problem of consumption, will fit well in the previous framework. Malinvaud's demand for the  $n$  goods at time  $t$   $\underline{d}_t$  can be obtained as a difference of the output and the input vectors  $\underline{b}_t$  and  $\underline{a}_t$

$$[1m] \quad \underline{d}_t = \underline{b}_t - \underline{a}_t, \quad \underline{a}_t \geq \underline{0}, \quad \underline{b}_t \geq \underline{0} \quad (\text{fundamental equation})$$

So the demand for good  $j$  could be negative according to the corresponding element of  $\underline{a}_t$  being greater than the one of  $\underline{b}_t$ . In fact he is dealing with net demand, from the productive sector, for the factor of production (i.e. labour).

The processes of production are defined in terms of  $A$  and  $B$  as usual, even though it is not clear whether Malinvaud has included workers' subsistence level of consumption in  $A$ . While assumption  $[A]$  is maintained,  $[B]$  is substituted by:

$$[D] \quad \underline{A}\underline{f} > \underline{0}, \quad \underline{B}\underline{f} > \underline{0} \quad (\text{economic processes hp.})$$

The new hypothesis means that each process needs some input and produces some output. The economy is no longer

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(7) Cfr. Morishima (1960) p. 352.

required to produce every single good. In fact, the good  $j$ , needed as input, but not produced as output, will be one of those having a net negative demand  $d_j < 0$ .

The feasibility of the global process of production is guaranteed if there exists an intensity vector  $\underline{x} > 0$  such that:

$$[2m] \quad \begin{cases} \underline{a}_t \geq \underline{x}_t A \\ \underline{b}_{t+1} \leq \underline{x}_t B \end{cases} \quad (\text{feasibility constraint})$$

This means that the quantities produced  $\underline{x}_t B$  are equal to or more than the output  $\underline{b}_{t+1}$  and the quantities used  $\underline{x}_t A$  are equal to or less than the input  $\underline{a}_t$ . Dropping the time indexes and assuming:

$$[3m] \quad \underline{a}_t = (\alpha')^t \underline{a}; \quad \underline{b}_t = (\alpha')^t \underline{b}; \quad \underline{d}_t = (\alpha')^t \underline{d}; \quad \underline{x}_t = (\alpha')^t \underline{x}$$

we will obtain from [1m] and [2m]

$$[4m] \quad \begin{cases} \underline{b} = \underline{a} + \underline{d} & \underline{a} > \underline{0}, \quad \underline{b} > \underline{0} \\ \underline{a} > \underline{x} A \\ \alpha' \underline{b} < \underline{x} B \end{cases}$$

With one period of production lag that implies

$$[5m] \quad \underline{x}(B - \alpha' A) \geq \alpha' \underline{d} \quad \underline{x} \geq \underline{0}, \quad \alpha' \geq 0$$

with equality holding for [6m]  $\underline{a} = \underline{x}A$ ,  $\alpha' \underline{b} = \underline{x}B$ .

Malinvaud now assumes that the net demand is given by:

$$[7m] \quad \underline{d} = \underline{x} \underline{q} \underline{h}$$

where  $\underline{q}$  is the  $m$  by  $1$  column vector of weights given to the processes of production, to compute the total volume of production  $\underline{xq}$ , while  $\underline{h}$  is the  $1$  by  $n$  row vector of demand of goods which distributes the total volume of production over  $n$  goods. Therefore, the sum of the elements of  $\underline{h}$  is equal to one.

Assuming now  $\underline{kH} = \underline{q} \underline{h}$  we can see how  $\underline{q}$  plays the same role as  $\underline{ke}^T$  in the KMT model. Malinvaud's condition (9) associated with the maximum feasible value of  $\alpha'$  is equivalent to the Morgenstern-Thompson conditions,  $[\hat{1}]$ ,  $[\hat{2}]$ ,  $[\hat{3}]$ ,  $[\hat{4}]$  which imply  $\hat{\alpha} = \hat{\beta}$ . However this system can no longer be considered to be a model of the original von Neumann type because in this case  $\hat{A} > 0$  no longer holds. In fact  $\underline{kh}_{ij}$  could be negative and greater in absolute value than  $a_{ij}$ , even though  $\sum_i (\underline{kh}_{ij} - a_{ij}) \geq 0$  from [4m].

Moreover (being the cost of production fully accounted by  $\underline{Ay}$ ), the vector of excess profit  $\underline{p}$  ( $m$  by  $1$ ) will be no longer equal to zero, because

$$[8m] \quad \underline{p} = (\underline{B} - \hat{\alpha} \underline{A}) \underline{y} = \hat{\alpha} \underline{kH} \underline{y} \underset{\leq}{\geq} \underline{0} \quad \text{for} \quad \underline{hy} \underset{\leq}{\geq} \underline{0}$$

The total amount of excess profits will be zero only when the total value of net demand is zero. This happens when the value of consumers' demand equals the value of the demand of factors of production. In this case we of course have  $\hat{\alpha} = \underline{xBy} / (\underline{xAy} + \underline{xkHy}) = \alpha' = \beta'$ . It is obvious that the  $\hat{\beta}^{-1}$  determined being equal to  $\hat{\alpha}^{-1}$  differs from the

equilibrium rate of interest in perfect competition  $\beta^{-1}$ , which satisfies the original von Neumann's equations [3] and [4].

Consequently the conditions of an efficient economic system must consist of  $[1^{\wedge}]$   $[2^{\wedge}]$ , [3], [4]; where  $[1^{\wedge}]$  holds with equality in the particular case of Malinvaud (1959), p.22: thus

$$[9m] \quad \underline{x}[B - \alpha^{\wedge}(A+kH)] \underline{y} \geq 0$$

$$[10m] \quad \underline{x}[B - \alpha^{\wedge}(A+kH)] \underline{y} = 0$$

$$[11m] \quad [B - \beta A] \underline{y} \leq 0$$

$$[12m] \quad \underline{x} [B - \beta A] \underline{y} = 0$$

Frish and Bauer referred to the Malinvaud - von Neumann model as the pure consumption case. This because the introduction of the demand vector does not change the unit cost and, consequently, the original profit constraint and the rule of profitability.

From [12m] and [10m] we obtain  $\beta \underline{xAY} = \alpha^{\wedge} (\underline{xAY} + \underline{dy})$  from which  $\beta - \alpha^{\wedge} = \alpha^{\wedge} \underline{dy} / \underline{xAY}$  and finally:

$$[13m] \quad \beta \underset{<}{\geq} \alpha^{\wedge} \quad \text{for} \quad \underline{hy} \underset{<}{\geq} 0$$

Interpreting good  $j$ , with  $h_j < 0$ , to be a natural factor of production, we can say that the interest rate equals the rate of expansion when and only when the whole income of natural factors of production is consumed so that  $\underline{hy} = 0$ . Where capitalists consume and, therefore a value of consumption is greater than the income of natural factors,



the rate of interest would be greater than the rate of expansion. Thus this model allows for capitalists' consumption as well as savings from the income of natural factors of production. It also allows for savings by workers in the case of labour being one of the  $n$  goods.

If  $\underline{h} > 0$  then  $\underline{d}$  represents only capitalists' consumption (as Frish (1969) implicitly assumes), whilst, as  $\underline{d}$  is constant, demand functions for final goods are not given in an appropriate form as Malinvaud has no difficulty to admit (see p.218).

This is also true for  $kH$ . Since the weights of processes of production  $\underline{q}$  can be taken arbitrarily in the definition  $kH = \underline{q} \underline{h}$ , we may take  $\underline{q}$  and  $\underline{h}$  as:

$$[14m] \quad \underline{q} = A\underline{y}; \quad \underline{h} = \underline{d} / \underline{x}A\underline{y}$$

In this case,  $\underline{q}$  is equal to the vector of effective costs and the system will achieve the maximum feasible rate of expansion. Naturally this is only an arbitrary way of obtaining the  $kH$  matrix such that  $\underline{y}$  satisfies  $[B - \alpha^{\wedge}(A + kH)]\underline{y} \leq 0$ ; and we have  $(\beta - \alpha)A\underline{y} = (\alpha^{\wedge}\underline{h}\underline{y})\underline{q} = \alpha^{\wedge}kH\underline{y}$  in this case. We find that the vector of excess profits is proportional to costs  $\underline{p} = \underline{d}\underline{y} (A\underline{y}/\underline{x}A\underline{y})$  and we obtain a sort of mark-up  $(1 + \underline{d} \underline{y})$ .



### 2.3 The Morgenstern and Thompson closed model

Morgenstern's and Thompson's closed model of Ch.6 in their book enables us to relax the following assumptions which exist behind the KMT outside demand: a) workers consume all their income, b) capitalists' income is all invested.

However it does not describe the way in which income  $\underline{xHy}$  is distributed. Instead it only distinguishes between aggregate consumption  $k \underline{xHy}$ , and capital accumulation  $z \underline{xHy}$  (aggregate savings). In this way equations  $[1^{\wedge}]$ ,  $[2^{\wedge}]$  are maintained, but  $[3^{\wedge}]$  and  $[4^{\wedge}]$  are changed into:

$$[3*] \quad [B - \beta^*(A+zH)]\underline{y} \leq 0$$

$$[4*] \quad \underline{x}[B - \beta^*(A+zH)]\underline{y} = 0$$

Now  $[3*]$  and  $[4*]$  together imply that in equilibrium "the value of the outputs must be enough to cover the capitalized value of the inputs plus the excess profits".<sup>(1)</sup> Before we analyze the cost of production equations, we spend a little more time to explain consumers' demand for goods.<sup>(2)</sup> It seems that it is one of the special features of the Morgenstern-Thompson model that their  $k H$  can contain stockholders' consumption.<sup>(3)</sup>

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(1) Morgenstern Thompson (1976) p.102.

(2) We will later examine the method of deriving  $[1^{\wedge}]$  and  $[2^{\wedge}]$  from an assumed set of demand functions which has been adopted by Frisch (1969) pp.479 - 481, and by Bauer (1974) p.22, who heavily depends on Frisch's approach.

(3) In this respect it is strange how for Morgenstern-

### A. The demand function in Morgenstern-Thompson model.

In our view it is wrong to say that, in a more general interpretation, the vector  $\underline{d} = k \underline{x} H$  can be generated from a set of demand functions, although Morgenstern and Thompson never explicitly made this claim, in their (1967) article, or in their (1976) book.

On the other hand, Frish (1969) and Bauer (1974) pretend to have derived Morgenstern's and Thompson's demand vector  $\underline{d}$  from a general system of demand for goods:  $d_j = f_j(y) \underline{x} H \underline{y}$  (for  $j = 1, \dots, n$ ), satisfying  $\underline{d}(\underline{y}) \underline{y} = k \underline{x} H \underline{y}$ .

However, after this derivation, Frish himself writes down (in terms of our notation)  $d_j = k \underline{x} \underline{h}_{*j}$ .<sup>(4)</sup> This formula is clearly different from  $d_j = f_j(\underline{y}) \underline{x} H \underline{y}$ .

Moreover, as these two formulas the former does not show any direct dependence of demand on prices  $\underline{y}$  as the latter does. Note that  $\underline{h}$  represents a given amount of good,  $j$ , that is obtained as income by the economic agents (workers and entrepreneurs) from a unit operation of activity  $i$ .<sup>(5)</sup> In order to obtain equation [1<sup>^</sup>], the formula,  $d_j = k \underline{x} \underline{h}_{*j}$ , is unavoidable, whereas it is not strictly necessary for equation [2<sup>^</sup>] for which restriction

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Thompson equation [1<sup>^</sup>] "states that the output of a given time period must be enough to provide the input of next time period (including the reproduction of workers) plus the surplus consumption of the workers" only. p.102

(4) Frish (1969) at p.48 (first row).

(5) Frish (1969) p.479.

$\underline{d} \underline{y} = k$  is enough. It is obvious that, where  $\underline{d} = c \underline{x} H$ , the consumer has no voice in determining the demand functions,<sup>(6)</sup> apart from the choice of a fixed level of  $k$  which is independent of any level of interest rate and prices. This is a very restrictive and unrealistic hypothesis.

### B. Consumption and saving in Morgenstern and Thompson

Let us now assume  $z = 1 - k$  seeming unreasonable the possibility  $1 < k + z \leq 2$  to dispose of more of the current income  $\underline{x} H \underline{y}$ . We make this assumption because for us it is difficult to understand the economics behind Morgenstern and Thompson's statement that condition  $k+z > 1$  could be interpreted as capital consumption, while we still have  $k < 1$  or  $k+z < 1$  implies capital formation, and not simply  $z > 0$ . Consequently, in our opinion,  $k + z = 1$  seems the only economically meaningful region where  $k$  and  $z$  are located in their graphical analysis.<sup>(7)</sup>

The reason why  $k = 1 - z$  has not always been imposed can be attributed to their intention to obtain the most general model, or to the fact that  $z$  can represent also something different from the saving ratio.<sup>(8)</sup>

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(6) The proportion of the goods is already given, from  $\underline{x} H$ .

(7) Morgenstern Thompson (1976) fig. 6.6 p.111.

(8) In any case for Morgenstern-Thompson in reality: "The purpose of studying models of the type under investigation is not at first to make them 'realistic', but rather to see whether such models are at all logically possible and what their properties would be..."p.387.

Nevertheless in their book (1976) Morgenstern and Thompson "frequently make the stronger but economically reasonable assumption that a fraction  $k$  of the surplus [from the input-output process] is consumed ... and the remainder  $z = 1 - k$  is added to capital stock."<sup>(9)</sup>

### C. Wages and the cost function: a possible interpretation.

In their model Morgenstern and Thompson "do not describe the exact process by which these goods are declared surplus and available for consumption or the way they are distributed...".<sup>(10)</sup> Now, in order to allow for stockholders' consumption and in order to examine the cost of production in depth let us assume that capitalists' consumption equals to  $u kH$  and total wages cost minus the reproduction cost of workers equal to  $wH = (1-u) kH + (1-u')zH$  where  $0 < u, u' < 1$ , are respectively the fractions of consumption and saving of capitalists. If we subtract the input and total wages cost from the revenue, then condition [3\*] and [4\*] could be written as:

$$[3'] \quad [B - \beta' (A+wH)]\underline{y} \leq 0$$

$$[4'] \quad \underline{x}[B - \beta' (A+wH)]\underline{y} = 0$$

[3'] means that the value of output is equal to or less than the capitalized value of input necessary to produce

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(9) Morgenstern Thompson (1976) p.101, our italic. We will come back again later on this point.

(10) Morgenstern Thompson (1976) p.101.

it. Capital grows from  $(A+wH)\underline{y}$  to  $B\underline{y}$ .<sup>(11)</sup> This formulation naturally follows from the KMT outside demand model.

In any case, where  $\underline{h} = \underline{x} (B - A)$ , wages are no longer given but an additional payment is made in each process in proportion to the value added of the whole system.<sup>(12)</sup> It is a payment in real terms and is indexed to the bundle of goods  $\underline{h}$ . However with this interpretation it must be determined how  $H$  is to be distributed among workers and stockholders so that there are at least three choice parameter or two if we assume  $u = u'$  (one class society).<sup>(13)</sup>

#### D. Frish's interpretation of Morgenstern and Thompson

In the light of the previous formulation it is important to see if Morgenstern's and Thompson's closed economy can be interpreted in a different way which assumes  $z \neq$  saving rate  $\neq w$ ? Also we may ask: can their unit production cost function [which is apparently different from the definitions by Marx and Walras, applied by Morishima (1960) and (1964)] have any economic meaning? Morgenstern and Thompson do not explicitly justify their choice nor derive

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(11) Note that this is equal to [3\*] and [4\*] if  $w = z$ .

(12) In the excess value case the payment is equal to the value added for each process. But no account is taken, in both cases, of the possible differences in the hours of work needed in each process.

(13) This assumption seems at least different from what Morgenstern and Thompson stated. See Morgenstern-Thompson (1976) p.101 (line 6 and ff.).



it from other economist' works. Consequently it is important to try and examine an interpretation of the Morgenstern-Thompson cost function given by Frish (1969), discussing its consequences and analysing whether it is economically reasonable.

Frish's interpretation is completely different from the one that has been already proposed by us, which, due to M.C. Lovell, has been accepted more or less by Morgenstern and Thompson (1967) as an 'alternative interpretation'.

Frish seems to consider that the true interpretation of Morgenstern-Thompson (1976) would take  $h$  as "the amount of good  $j$  obtained by the workers in the  $i$ -th industry if that industry is run at unit intensity."<sup>(14)</sup>  $z$  and  $k$  are their saving and consumption ratios. However he defines  $H$  as the income matrix (this interpretation seems to be confirmed by Morgenstern's and Thompson's derivation of  $H$ ) so that  $H \underline{y}$  is the vector of income, created by each process, and attributed to the economic agents involved.  $zH \underline{y}$  represent the value re-invested in these processes at an unit level of activity and  $k \underline{x} H$  the total demand for the  $n$  goods. Consequently in the case of the  $i$ -th process the amount  $zh_{i*} \underline{y}$  will be reinvested, earning interest of the amount of  $(\beta^* - 1)zh_{i,j} \underline{y}$  and both must be added to the previous production cost  $\beta^* a_{i*} \underline{y}$  to obtain the total cost including interest. Thus we have equation [3\*].

(14) Frish (1969) p.479.



Clearly  $sHy$  differs from  $wHy$  which is the reproduction cost of workers.

Accepting this new definition of unit cost of production, it is not clear at all why, if we can produce output  $\underline{x} B$  from input  $\underline{x} A$ , we incur an additional cost  $zHy$  to produce the same output value  $\underline{x} By$ .<sup>(15)</sup> To avoid this,  $zHy$  should be interpreted as an additional payment to some factor of production (wages in [3']), rather than reinvested savings. In fact the unit input cost of each process will remain the same that is  $Ay$ , and total cost comprises already the interest income  $(\beta-1)\underline{x}Ay$ . If under perfect competition  $zH$  is subtracted from the production process this means that to produce  $B$  we must incur an additional cost. But in this case some production factor is paid  $zHy$  if it is paid at the beginning of the period, or  $\beta zHy$  if it is paid at the end, in addition to the interest  $(\beta-1)\underline{x}Ay$  on  $\underline{x}Ay$ .<sup>(16)</sup>

#### E. The meaning of $\beta^*$ in the Frish's interpretation

To clarify what has been said, and to see that as long as we follow this interpretation  $\beta^*$  is no longer an interest rate, in the Morgenstern-Thompson model, we will

(15) "Axiom II stipulates 'full competition' (no profit)", Frish (1969) p.481. Hence  $zHy$  cannot be an excess profit.

(16) It is worth noting that there is an implicit assumption in the Morgenstern-Thompson article to the effect that saving = investment = capital accumulation, as is evidenced by the fact  $zH$  is subtracted from  $B$  in [1^] and [2^].

compare the original von Neumann model and Frish's version of the Morgenstern-Thompson model, by assuming a balanced growth equilibrium (with  $\alpha > 1$ ) in both. To make things easier let us follow Morgenstern and Thompson (1976) in assuming that income is equal to  $\underline{xHy} = \underline{xBy} - \underline{xAy}$ . In the von Neumann model the income

$$[a] \quad \underline{xHy} = (\beta - 1) \underline{xAy}$$

is fully reinvested, while in Morgenstern and Thompson the reinvestment is

$$[b] \quad z \underline{xHy} = (\alpha^* - 1) \underline{xAy}$$

Equating, at  $t+1$ , the value of total disposable capital  $\underline{xAy} + z \underline{xHy}$  to the value of the inputs available for production  $\alpha^* \underline{xAy}$ , we obtain:

$$[c] \quad \underline{xHy} = (\alpha^* - 1) \underline{xAy} / z = \underline{x}(B - A)y \quad \text{Consequently:}$$

$$[d] \quad \underline{xBy} = (\alpha^* + z - 1) \underline{xAy} / z = \beta \underline{xAy}$$

which is the original von Neumann equation.

In this way from [a] and [c] we obtain the relation between the rate of growth and the rate of interest:

$$[6] \quad z(\beta - 1) = \alpha^* - 1$$

which is the result of Morishima (1964), where only capitalists are allowed to save. If instead we follow Morgenstern-Thompson, cost equation [4\*] being  $\underline{xAy} = [\underline{xAy} + z \underline{xHy}] / \alpha^*$  from [d] we have:

$$\underline{xBy} = (\alpha^* + z - 1) [\underline{xAy} + z\underline{xHy}] / z \quad \alpha^* = \beta^* [\underline{xAy} + z\underline{xHy}]$$

that is Morgenstern's and Thompson's [4\*].

So, following Frisch's interpretation, from [d], the value,  $\beta^* = \beta \neq \alpha^*$ , of the Morgenstern Thompson model is not the rate of interest in 'full competition', as Frisch thinks, but it is the ratio of the rate of interest and of the rate of growth. Furthermore only capitalists are entitled to additional consumption  $k\underline{xHy}$ .<sup>(17)</sup> The general process of production and investment during two periods is illustrated in fig. 2.1, in the case of the von Neumann

FIG. 2.1

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	PERIOD t		PERIOD t+1
	goods input	output	goods input
$\underline{xBy}/\alpha =$	$\underline{xAy}$	$\underline{xBy}$	$\alpha \underline{xAy}$
	" "	" "	" "
	$\underline{xAy}/\alpha + z\underline{xHy}/\alpha$	$\beta(\underline{xAy}/\alpha + z\underline{xHy}/\alpha)$	$\underline{xAy} + z\underline{xHy}$
	old + new capital	" "	old + new capital
		$\rightarrow$	$\rightarrow$
			$\beta(\underline{xAy} + z\underline{xHy})$

model, and in fig.2.2, in the case of the Morgenstern-Thompson model as is interpreted by Frisch.

(17) This contradicts the fact that for Morgenstern and Thompson this model is equivalent to Morishima (1964) for  $z = 0$ ; see pp.398-9.

FIG. 2.2

PERIOD t goods capitalist                      output input consumption $\underline{xBy}/\alpha = \underline{xAy} + k\underline{xHy}$ $\underbrace{\hspace{10em}}_{\substack{\text{old} + \text{new capital}}} \rightarrow \beta^* (\underline{xAy}/\alpha^* + z\underline{xHy}/\alpha^*)$	PERIOD t+1 goods capitalist                      output input consumption $= \alpha^* (\underline{xAy} + k\underline{xHy})$ $\alpha^* \underline{xBy}$ $\underbrace{\hspace{10em}}_{\substack{\text{old} + \text{new capital}}} \rightarrow \beta^* (\underline{xAy} + z\underline{xHy})$
--	---

One other interesting and direct evidence of the absurdity, to which this interpretation could lead, is the fact that, to make it coherent, we must suppose that capitalists' savings in the Morishima (1964) model are invested outside the economy. An interpretation of this kind was also given by Tomasini (1968) to Morishima (1964).<sup>(18)</sup>

#### F. On the available capital

On the other hand if it is wrongly considered that  $\underline{xAy} + z\underline{xHy}$  is the capital available at time t we would obtain from [c] and [4\*]  $\beta^* (\alpha^* - 1) \underline{xAy} = \underline{xBy} - \beta^* \underline{xAy}$  and consequently  $\underline{xBy} = \alpha^* \beta^* \underline{xAy}$ . That is to say, the capitalized value of the new capital  $\beta^* (\alpha^* \underline{xAy})$  (available for productive purpose) at time t+1 equals the value of the output at time t  $\underline{xBy}$  implying a stationary process (or equivalently using [2\*]  $\underline{xBy} - \alpha^* (\underline{xAy} + k \underline{xHy}) = 0$  we obtain  $(\beta^* - 1) \underline{xAy} = k \underline{xHy}$  that all interest income is consumed).

(18) Tomasini (1968) at p. 330.

But it was never mentioned by Morgenstern and Thompson (1967) as the fact that excess profits are equal to zero. In fact  $k$  and  $z$  are defined also as "consumption and profit coefficients", p.393, while condition [3\*] as we have already said "states that the value of the outputs must be enough to cover the capitalized value of the inputs plus the excess profits", that is  $\underline{xAy} + z\underline{xHy}$ .<sup>(19)</sup> Nevertheless it is not clear at all why the excess profit has to be equal to the investment. In any case the presence of extra profits will explain why it does not matter who (workers or capitalists) receives this amount and why the Morishima (1964) model should correspond to a zero value of  $s$ .

#### G. The extra profit hypothesis

Let us explain the non-zero profit interpretation in more detail and compare it to von Neumann himself. Let  $\pi$   $\underline{xHy}$  be the discounted value of extra profits at the beginning of the current period and let  $\underline{xHy} = \underline{x}(B - A)\underline{y}$  and  $k\underline{xHy}$  be income and consumption. Assume for simplicity sake that additional wages are equal to zero  $w = 0$ . Then at the end of the period:

$$[7] \quad \underline{xHy} \begin{cases} = k\underline{xHy} + (1-k)\underline{xHy} \\ = (\beta^* - 1)(\underline{xAy} + \pi \underline{xHy}) + \pi \underline{xHy} \end{cases}$$

The current capital invested at the beginning of the

<sup>(19)</sup> Morgenstern Thompson (1967)p.393, (1976)p. 102.

period will be  $\underline{x}_{Ay}$  and its capitalized value at the end of the period  $\underline{x}_{By} = \beta \underline{x}_{Ay} = \beta^* (\underline{x}_{Ay} + \pi \underline{x}_{Hy})$ . Being  $\underline{x}_{Hy} = (\beta - 1) \underline{x}_{Ay}$  we have:<sup>(20)</sup>

$$[8] \quad \beta^* = \beta / [1 + \pi (\beta - 1)] = \beta / [1 + z(\beta - 1)], \quad \text{for } \pi = z$$

Note that  $\pi = 1 - k$  is not assumed.

However in order to obtain the same result from the following equation: savings equal to total income  $\beta^* (\underline{x}_{Hy} + \underline{x}_{Ay}) - \underline{x}_{Ay}$  less consumption  $k \underline{x}_{Hy}$  <sup>(21)</sup>  $z \underline{x}_{Hy} = (\beta^* \pi - k) \underline{x}_{Hy} + (\beta^* - 1) \underline{x}_{Ay} = [\beta^* \pi - k + (\beta^* - 1)/(\beta - 1)] \underline{x}_{Hy}$  (being  $\underline{x}_{Hy} = (\beta - 1) \underline{x}_{Ay}$ ) we must assume also  $k + z = 1$ . In fact:

$$[9] \quad \beta^* = [(k+z)(\beta-1) + 1] / [1 + \pi(\beta-1)] = \beta / [1+z(\beta-1)]$$

only for  $k+z = 1$  and  $\pi=z$ . This is illustrated in fig. 2.3.

FIG. 2.3

PERIOD t		PERIOD t+1	
goods total	output	goods total	output
input consumption		input consumption	
$\underline{x}_{By}/\alpha = \underline{x}_{Ay} + k\underline{x}_{Hy}$	$\underline{x}_{By}$	$\alpha^* (\underline{x}_{Ay} + k\underline{x}_{Hy})$	$\alpha^* \underline{x}_{By}$
$\underline{x}_{Ay}/\alpha^* + \pi \underline{x}_{Hy}/\alpha^* \rightarrow \beta^* (\underline{x}_{Ay}/\alpha^* + \pi \underline{x}_{Hy}/\alpha^*)$		$\underline{x}_{Ay} + \pi \underline{x}_{Hy} \rightarrow \beta^* (\underline{x}_{Ay} + \pi \underline{x}_{Hy})$	
total + present value		total + present value	
capital of profits		capital of profits	

(20) Formula (33) at p.119 of Morgenstern Thompson (1976).

(21) Note that this and the income equation are pure accounting equations, if we assume that interest income of capitalists and profit income of entrepreneurs are disposable for consumption and saving only at the end of each period.



In a more general model which allows for additional wages and positive excess profits we have  $z = w + \pi$  which reduces to  $z = w$  for  $\pi = 0$ . That is the interpretation of section 2.2.C.

#### H. Further considerations

What has just been said will become self evident in view of the consequences of [3\*] and [4\*] under the assumption of no additional wages or consumption. Let us assume that we have pure subsistence wages so that  $w = 0 = k$  and no capitalists' consumption, then [3'] and [4'] are reduced to the von Neumann model, and thus all income is reinvested. In the Morgenstern-Thompson model however, where  $z \neq 0$ , the investment decision will distort the interest factor and possibly the price structure. Furthermore as long as the demand-supply equation of the original model without consumption continue to hold, we have, where  $z=1$ , the maximum possible distortion for the interest factor  $\beta^* = 1$ , for both excess production and excess value cases. In fact from [8]  $\beta^* = \beta/[1+z(\beta-1)] = \alpha/[1 + z(\alpha-1)]$  we obtain  $\lim \beta^* = 1$  (for  $z \rightarrow 1$ ). This is because the investment  $zHy$  is included in the cost of the production processes.<sup>(22)</sup> This can be justified only

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(22) It is quite astonishing that even confronted with the consequences of this example Morgenstern-Thompson try to justify it by saying: "the decision to reinvest rather than to consume now resides in the consumers hands. The very fact that they have made this consumption-investment decision in this way has caused the extreme distortion in the interest factor and price structure as compared to the former interest and price structure." p.105.

if additional wages or profits are rewarded or if  $\beta^* = \beta/\alpha^*$ . Another consequence of [3\*] is that  $\beta^*$  is decreasing with respect to  $z$  and, therefore, it is no longer a pure technological interest rate as it is in the original von Neumann and the KMT models.

### I. Morgenstern Thompson model and perfect competition

The hypothesis  $z = w$  does not constrain the expansion rate to be less than or equal to the interest rate. We have an expansion rate greater (or less) than the interest rate according as the proportion of income saved by the workers is less (or greater) than the proportion of income consumed by the capitalists.

$$\begin{array}{lll} \alpha' > \beta' & \text{for } (1 - u')s > u k & \text{or } w > k \\ \alpha' = \beta' & \text{for } (1 - u')s = u k & \text{or } w = k \\ \alpha' < \beta' & \text{for } (1 - u')s < u k & \text{or } w < k \end{array}$$

Finally let us examine the relationship between the rate of growth ( $\alpha' - 1$ ) and the rate of interest ( $\beta' - 1$ ). The equation that we derive is not far from the one found by Morishima (1964)<sup>(23)</sup> and similar to the one reached by Kaldor (1957) in his model of economic growth.

First of all from [4'] we easily obtain  $\underline{xHy} = \underline{x}(B - A)\underline{y} = w\underline{xHy} + (\beta' - 1)\underline{x}(A + wH)\underline{y}$  which just distinguishes income from work from income from interest on capital. Therefore  $\underline{xHy} = (\beta' - 1)\underline{x}(A + wH)\underline{y} / (1 - w)$ .

(23) We limit ourselves to the case  $\alpha = \beta > 1$ .

On the other hand from [2<sup>^</sup>] and [4'] we have:

$$\begin{aligned} \beta' \underline{xAY} + \beta' w \underline{xHY} &= \alpha' \underline{xAY} + \alpha' k \underline{xHY} \\ &= \alpha' \underline{xAY} + \alpha' w \underline{xHY} + \alpha' (k-w) (\beta'-1) \underline{x(A+wH)Y} / (1-w). \end{aligned}$$

Consequently, in viewing  $k = 1-z$ , we have  $\beta' = \alpha' [1 - z/(1-w)] (\beta'-1)$  which gives the relationship between the interest and growth rates.

$$(\alpha' - 1) = [ z/(1-w) ] (\beta' - 1)$$

Here  $s/(1-w)$  is the ratio between aggregate saving and the interest income that is equal to the capitalists' average propensity to save  $s_c$  if they are the only agent to save  $u' = 0$ , as in the Morishima Model)<sup>(24)</sup> and the rate of profit on invested capital is

$$(\beta'-1) = (\underline{xBY} - \underline{xAY} - w \underline{xHY}) / (\underline{xAY} + w \underline{xHY})$$

So "Barring negative expansion, the economy grows at a rate equal to the product of the capitalists' average propensity to save [or the ratio of aggregate saving to interest income] and the rate of profit on capital"<sup>(25)</sup> invested at the beginning of the period.

(24) Cfr. last formula of p.150 Morishima (1964)

$$\alpha^{**} - 1 = z (\beta^{**} - 1).$$

However ours is a purely accounting equation as the Morgenstern-Thompson definition of income ignores production lags as was indicated previously.

(25) Morishima (1964) p.151; our text between the brackets.

This interpretation, however, does not affect the lack of economic analysis of the aggregate demand  $k \underline{x} H$  and the peculiarity of the definition of aggregate income matrix  $H$  which we have previously pointed out. It only reveals that Morgenstern and Thompson have just constructed a formal model, without clarifying its economic implications appropriately. Its microeconomic foundations, that is, its relationships to individual agents' behaviour are left entirely obscure.

## 2.4 The Morishima and Haga-Otsuki models

In what follows we shall examine the treatment of consumption by Morishima (1964) (1969),<sup>(1)</sup> and by Haga-Otsuki (1965).

### A. The Marx - Von Neumann version

In Morishima (1964) we find a general consumption model, in which a part of interest income is consumed but there are two demand vector functions (for capitalists  $\underline{d}^c(t)$  and for workers  $\underline{d}^w(t)$ ) depending on income and prices  $p(t)$  and the supply of labour  $N(t)$  grows at a finite rate:  $N(t) = r^t N$ .

Morishima explicitly distinguishes in the original von Neumann model the material inputs,  $a_{ij}$ , technologically required, from those for workers' consumption  $D^w = \underline{l} \underline{c}$  where  $\underline{l}$  is the labour input column vector of  $m$  by 1 and  $\underline{c}$  the bundle vector of goods "needed to persuade a man to work", which is of 1 by  $n$ . Consequently his  $A$  is a purely technological input matrix. He maintains assumption [A] and adds:

$$[E] \quad \underline{f}B > 0, \quad \underline{l} > 0$$

This implies that all goods are produced by some

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(1) Morishima discusses the Walras-von Neumann version as well but we deal only with the Marx - von Neumann version. He discusses the Marx-von Neumann model in his (1969) as well. In order to cover both contribution, we modify his notation in this section.

process in the economy and that the homogeneous labour is necessary for all processes.<sup>(2)</sup>

Differently from Morishima we, from the beginning, discuss this model in a balanced growth context. Let us assume that the economy is in a state of equilibrium. We denote real wages by  $w$ .<sup>(3)</sup> Where  $w$  prevails, a normalized price vector  $\underline{y}$ , an interest rate  $\beta - 1$ , rate of growth  $\alpha - 1$  and a normalized intensity vector  $\underline{x}$  establish the balanced growth equilibrium.

Morishima assumes that all capitalists have identical tastes. Also he makes a similar assumption concerning workers. He writes their  $n$  by  $1$  demand vectors for a unit income as:

$$[G] \quad \underline{g}(\underline{y}) > \underline{0}, \quad \underline{h}(\underline{y}) > \underline{0}$$

respectively, which are single valued and continuous in the set  $\{\underline{y}: \underline{y} > \underline{0}, \underline{e}^T \underline{y} = 1\}$ . The capitalists real income is:  $E^c = E = \underline{x}[B\underline{y} - (A\underline{y} + \underline{1} w)] = (\beta - 1) \underline{x}(A\underline{y} - w \underline{1})$  which is equal to the whole interest income, because workers do not save.

(2) The exclusion of  $A\underline{e} > \underline{0}$  from [E] is justifiable, not only by the fact that it is not needed, but also by the fact that this condition is satisfied by the original von Neumann matrix  $(A+D)$ . In fact in view of  $\underline{c} \underline{e} > 0$  and  $A\underline{e} \geq 0$  we get  $A\underline{e} + \underline{1} \underline{c} \underline{e} > 0$  so that [E] is equivalent to KMT [C].

(3)  $\underline{e}^T p(t)$  (where  $p$  is the absolute price vector and  $e$  a  $n$  by  $1$  sum vector) can be seen as the price index of a bundle of the  $n$  goods which can be sufficiently approximated to the price index of the consumption goods transforming, in a convenient way, the units of measurement of the various goods. See Morishima (1969), p. 100.



The demand functions of capitalists are specified as:

$$[H] \quad \underline{d}^c = \max(0, E^c) \underline{q}(y) \quad \text{with} \quad \underline{q}(y)y = c_c$$

Thus capitalists' consumption is zero where interest income is negative, while it is proportional to income, where  $E^c$  is positive  $\underline{d}^c y = E^c c_c$ .

The demand for labour is  $\underline{x} \underline{l}$  and the workers' consumption function is

$$[I] \quad \underline{d}^w = w \underline{x} \underline{l} \underline{h}(y) \quad \text{with} \quad \underline{h}(y) y = 1$$

Thus  $\underline{d}^w y = \underline{x} \underline{l}$ , as workers don't save.

The equations of Morishima (1964) are:

$$[1M] \quad \underline{x} \{ B - \alpha [A + w \underline{l} \underline{h}(y)] - \max(0, \beta - 1) [A y + w \underline{l}] \underline{q}(y) \} \geq 0$$

$$[2M] \quad \underline{x} \{ B y - [\alpha + c_c \max(0, \beta - 1)] [A y + w \underline{l}] \} = 0$$

$$[3M] \quad B y - \beta [A y + w \underline{l}] \leq 0$$

$$[4M] \quad \underline{x} \{ B y - \beta [A y + w \underline{l}] \} = 0$$

$$[5M] \quad \underline{x} B y > 0$$

$$[6M] \quad \alpha^t \xi \underline{x} \underline{l} = r^t N \quad \text{for all } t$$

Equation [6M] represents the balance in the labour market. It is equivalent to the conditions  $\alpha = r$  and  $\xi = N/\underline{x} \underline{l}$  (where  $\xi$  and  $N$  are the initial values of the volume of production and of the supply of labour force).

It is easy to see that, if a solution exists, from [4M] and [2M] we obtain  $\beta = \alpha + \max(0, \beta - 1) c_c$ , so that viewing  $c_c = 1 - s_c$  we have the relationship between the rates of growth and interest

$$[7M] \quad \begin{cases} \beta = \alpha = r & \text{if } \beta - 1 \leq 0 \\ s_c(\beta - 1) = \alpha - 1 = r - 1 & \text{if } \beta - 1 > 0 \end{cases}$$

which is equivalent to [7'] as already explained.<sup>(4)</sup>

Consequently if the state of balanced growth exists, the rates of interest and of growth are uniquely determined by the rate of growth of the labour supply (and by the capitalists propensity to save if  $r > 1$ ) and are independent from the prices, the wage level and the intensities of processes.

#### B. Morishima (1969) model

In Morishima (1969) workers can save a part of their income and so they too may earn a part of the interest income  $E^w$ . Since workers' saving is given as  $s_w(\alpha w \underline{x}_1 + E^w)$ , we have a new specification of the demand functions

$$[I'] \quad \underline{d}^w = (E^w + \alpha w \underline{x}_1) \underline{h}(\underline{y}) \quad \text{with } 0 < \underline{h}(\underline{y}) \underline{y} = c_w \leq 1$$

Let us now assume for the sake of simplicity  $\beta > 1$ , so that capitalists consume a constant portion of their income  $E^c$ :

$$[H'] \quad \underline{d}^c = E^c \underline{q}(\underline{y}) \quad \text{with } 0 \leq \underline{q}(\underline{y}) \underline{y} = c_c < c_w$$

(4) Equation (3d) at p.19 of Bauer (1974) is wrong because the interest income of period  $t$  (differently from the wage-income) is realized and disposable for consumption only in period  $t+1$ . Consequently the capitalists' total consumption is  $\underline{d}^c$  and not  $\alpha \underline{d}^c$ . However the final relation between  $\alpha$  and  $\beta$  is right because  $\alpha$  vanishes without any apparent algebraic reason.

Naturally equations [1M] and [2M] are affected by these new assumptions and are replaced by

$$[8M] \quad \underline{x}B - \alpha [\underline{x}A + w \underline{x}lh(\underline{y})] - (E^w h(\underline{y}) + E q(\underline{y})) \geq 0$$

$$[9M] \quad \underline{x}B\underline{y} - \alpha [\underline{x}A\underline{y} + c_w w \underline{x}l] - [c_w E^w + c_c E^c] = 0$$

### C. Consequences of workers savings in Morishima (1969)

In this new steady state equilibrium  $E^w$  and  $E^c$  cannot be arbitrary, in view of the condition, introduced by Pasinetti, that the distribution of capital ownership between capitalists and workers must remain unchanged. Thus the capital assets owned by capitalists and workers which are  $E^c/(\beta-1)$  and  $E^w/(\beta-1)$ , respectively, must grow at the same rate.

This implies  $s_w (\alpha w \underline{x}l + E^w)/E^w = s_c$ ; hence

$$[10M] \quad \begin{cases} E^w = s_w \alpha w \underline{x}l / (s_c - s_w) \\ E^c = E - E^w \end{cases}$$

Pasinetti assumes  $\alpha w \underline{x}l s_w / (s_c - s_w) < E$ , so that both  $E^c$  and  $E^w$  are positive. Consequently in view of [I'] and [H'], we have  $\underline{d}^c + \underline{d}^w = E q(\underline{y}) + [s_c h(\underline{y}) - s_w q(\underline{y})] \alpha w \underline{x}l / (s_c - s_w)$  in the "Pasinetti case", so:

$$[1'M] \quad \underline{x}\{B - \alpha[A + w\underline{l}][s_c h(\underline{y}) - s_w q(\underline{y})] / (s_c - s_w)\} - (\beta - 1)[A\underline{y} + w\underline{l}]q(\underline{y}) \geq 0$$

$$[2'M] \quad \underline{x}\{B\underline{y} - [\alpha + (\beta - 1)c_c](A\underline{y} + w\underline{l})\} = 0$$

Note that [2'M] is equal to [2M] (because of the

assumption  $\beta - 1 > 0$  made to assure [H']) and the presence of workers' saving does not change the free good rule. Consequently from [4M] and [2'M] and [6M] we obtain again:

$$[7'M] \quad \alpha - 1 = r - 1 = (\beta - 1) s_c$$

and therefore  $\beta - 1 = (\alpha - 1) / s_c$ .

However to satisfy the condition of full employment we must require a further assumption on the rate of growth of labour supply. Using the previous result and multiplying equation [2'M] by  $\underline{y}$ , we find  $\underline{x}B\underline{y} \geq \{(r - c_c) / s_c\} \underline{x}A\underline{y}$ , if  $w = 0$ . So, as Morishima says, if  $\underline{x}B < \{(r - c_c) / s_c\} \underline{x}A$  (for all  $x \geq 0$ ), "then it would be impossible for the economy to grow in balance at the natural rate  $\alpha$  even if the wage rate were reduced to 0."

Consequently we must assume  $r$  to be so low that

$$[J] \quad \underline{x}B > [(r - c_c) / s_c] \underline{x}A \quad \text{for at least one } x > 0$$

However if  $E < \alpha w \underline{x} \underline{1} s_w / (s_c - s_w)$  then  $E^c = 0$ , because it is meaningless to assume that capitalists hold a negative amount of capital. In this "anti-Pasinetti case" we get:

$$[1''M] \quad \underline{x}\{B - \alpha[A + w \underline{1} h(\underline{y})] - (\beta - 1)[A + w \underline{1}]h(\underline{y})\} \geq 0$$

$$[2''M] \quad \underline{x}\{B\underline{y} - \alpha[A\underline{y} + c_w w \underline{1}] - (\beta - 1)[A\underline{y} + w \underline{1}]c_w\} = 0$$

Obviously from [2''M] and [4M] and [6M], condition [7'M] is no longer obtained.

#### D. The Haga-Otsuki model

Haga and Otsuki (1965) start from the initial Morishima model (1960) and in some respects generalize it. They allow for:

- a) a variable propensity to save by capitalists which is a function of monetary variables such as prices and the interest rate
- b) a finite number 'o' of primary factors of production being used in the production processes,
- c) a vector of supply functions of these factors which are dependent on prices, the total value of these supplies being identical with the total value of the consumption function; both growing in each period at a common constant rate  $r$ .
- d) a non-negative rate of interest being non-negative and savings subject to the rule of free goods.

In this model the amount of the  $k$ -th primary factor required for the  $i$ -th process is denoted by  $l_{ik}$  which is a non-negative element of the matrix  $L$  ( $m$  by  $o$ ). Similarly with Morishima, they assume:

$$[K] \quad \underline{f} B > \underline{0}, \quad L \underline{e} > \underline{0}$$

where  $\underline{e}$  is the  $o$  by  $1$  column vector. Furthermore they assume that the economy is capable to produce all the  $n$  goods in quantities greater than their inputs:

$$[M] \quad \underline{x} (B - A) > \underline{0} \quad \text{for some } \underline{x} \geq \underline{0}.$$

The vector of prices of goods and primary factors of production  $[\underline{y}, \underline{w}]$  is normalized so as to be contained in  $S = \{ [\underline{y}, \underline{w}] : \underline{y} \geq \underline{0}, \underline{w} \geq \underline{0}, \underline{f} \underline{y} + \underline{e}^T \underline{w} = 1 \}$ . By assumption the value of consumption of the primary factors  $\underline{h}^{\wedge} \underline{y} = \underline{h}^{\wedge}(\underline{y}, \underline{w}) \underline{y}$  is identically equal to their income,  $\underline{k}^{\wedge} \underline{w} = \underline{k}^{\wedge}(\underline{y}, \underline{w}) \underline{w}$ .  $\underline{h}^{\wedge}(>\underline{0})$  and  $\underline{k}^{\wedge}(>\underline{0})$  are single valued functions shifting "autonomously at a common rate which is greater than 1", so that:

$$[N] \quad \underline{h}^{\wedge} \underline{y} - \underline{k}^{\wedge} \underline{w} = 0, \quad \text{for } [\underline{y}, \underline{w}] \in S, \quad \text{with } \underline{h}^{\wedge} \geq \underline{0} \text{ for } \underline{w} \geq \underline{0}$$

In equilibrium capitalists receive at the beginning of  $t$  an income  $(\beta-1)\underline{x}(A\underline{y} + L\underline{w})$ , which is in part consumed and in part saved as long as it is non negative. Thus

$$[O] \quad \underline{q}^{\wedge}(\underline{y}, \underline{w}, \beta) \underline{y} + \underline{s}^{\wedge}(\underline{y}, \underline{w}, \beta) \underline{y} = c_c + s_c = 1$$

This means that consumption and saving are not only single valued and continuous but also homogeneous of degree one in income. Furthermore with respect to saving Haga and Otsuki assume:

$$[P] \quad \underline{s}^{\wedge}(\underline{y}^0, \underline{w}^0, \beta'') \geq \underline{s}^{\wedge}(\underline{y}^0, \underline{w}^0, \beta^0) > 0, \quad \text{for } \beta'' > \beta^0 > 1$$

As  $\underline{s}^{\wedge}(\underline{y}^0, \underline{w}^0, \beta^0) \underline{y}^0 = s_c$ , this means that the propensity to save is an increasing function of the rate of interest.

The conditions of balanced equilibrium growth for  $[\underline{y}, \underline{w}] \in S$ ,  $\underline{x} \geq \underline{0}$ ,  $(\alpha-1) > 0$ ,  $(\beta-1) > 0$ , are now stated:



- [1]  $\underline{x}B - [\alpha \underline{x}A + r \underline{h} + (\beta-1)q^{\wedge}\underline{x}(A\underline{y} + L\underline{w})] \geq 0$
- [2]  $\{\underline{x}B - [\alpha \underline{x}A + r \underline{h} + (\beta-1)c_c \underline{x}(A\underline{y} + L\underline{w})]\}\underline{y} = 0$
- [3]  $B\underline{y} - \beta (A\underline{y} + L\underline{w}) \leq 0$
- [4]  $\underline{x}[B\underline{y} - \beta (A\underline{y} + L\underline{w})] = 0$
- [5]  $\underline{x}B\underline{y} > 0$
- [6]  $\alpha^t \underline{x}L \leq r^t \underline{k}^{\wedge}$  (process feasibility)
- [7]  $[\alpha^t \underline{x}L - r^t \underline{k}^{\wedge}]\underline{w} = 0$  (factor market equilibrium)
- [8]  $(\alpha-1)\underline{x}(A\underline{y}+L\underline{w}) \leq s(\beta-1)\underline{x}(A\underline{y}+L\underline{w})$  (saving requirement)
- [9]  $(\beta-1)[(\alpha-1)\underline{x}(A\underline{y}+L\underline{w}) - s(\beta-1)\underline{x}(A\underline{y}+L\underline{w})] = 0$   
(savings-investment equilibrium)

This model, apart from the non-saving assumption from primary factors, seems quite a general one, although this assumption rules out the Pasinetti problem which is discussed in Morishima (1969). We must also point out that it is not clear why the real interest rate should be constrained to non negative values  $(\beta-1) \geq 0$ , although this restriction is ineffective in the present model because Haga and Otsuki make another strong assumption,  $r > 1$ , that is, supplies of primary factors of production always grow at a constant rate  $r$ , irrespective of  $\underline{y}$  and  $\underline{w}$ . Where  $r$  is flexible, there is a possibility of balanced decay of the economy as Morishima (1969) has discussed in the chapter on flexible population (pp.255-67, see especially p.259, footnote), so that  $\beta$  may be less than 1 ( $\beta \geq \alpha = r > 1$ ).

Also the assumption of the labour market satisfying the free goods rule should be subject to criticism.

In reality it seems more reasonable to assume that the lower bound for  $\beta$  is set by the maximum rate of decay of non free goods,  $\delta$ , which is obtained when they are kept in an appropriate storage process.

In the Haga-Otsuki model, provided an equilibrium exists, we have  $\alpha = r > 1$ , from [7]. Note that  $\underline{x} \geq \underline{0}$  from [5], and  $\underline{x}L > \underline{0}$  by [k]. Hence for  $\alpha > 0$  we have  $\underline{k}^* > \underline{0}$  and therefore  $\underline{k}^* \underline{w} > 0$ , by [n].

In the Haga-Otsuki system the usual relation between the rates of growth and of interest,  $(\alpha-1) = s(\beta-1)$ , is established, because  $\underline{x}(A\underline{y} + L\underline{w}) > 0$  from [5] and [4] and hence  $0 < \alpha-1 \leq s(\beta-1)$  from [8], and this must hold with equality, because the rule of free goods holds for saving and investment.

Finally let  $\mu = (\beta-1)(A\underline{y} + L\underline{w})\underline{q}$  be the consumption demand of capitalists. In view of  $\alpha-1 = r-1 = s(\beta-1)$  and  $(\beta-1)\underline{y} = (1-s)\underline{y}(\beta-1)$  obtained from [0], we discover an interesting feature of the model. First, it is clear that we may rewrite [1] and [3] as

$$[1'] \quad \underline{x}(B - rA - \mu) \geq r \underline{h}^*$$

$$[3'] \quad (B - rA - \mu)\underline{y} \leq r L\underline{w}$$

and taking [6] into account, we may formulate [1]-[9] as a dual problem of linear programming: for  $\underline{h}^*$ ,  $L\underline{w}$ ,  $\mu$  fixed at any equilibrium level.

$$\max \quad \underline{h}^{\wedge} \underline{y} - \underline{k}^{\wedge} \underline{y}$$

subject to:

$$(\underline{B} - r\underline{A} - \mu)\underline{y} - \underline{L}\underline{w} \leq 0$$

$$\underline{f} \underline{y} + \underline{e}^T \underline{w} \leq 1$$

$$-\underline{f} \underline{y} - \underline{e}^T \underline{w} \leq -1$$

$$\min \quad \underline{u} - \underline{v}$$

subject to:

$$\underline{x}(\underline{B} - \underline{A} - \mu) + \underline{f}\underline{u} - \underline{f}\underline{v} \geq r \underline{h}^{\wedge}$$

$$-\underline{x} \underline{L} + \underline{e}^T \underline{u} - \underline{e}^T \underline{v} \geq -r \underline{k}^{\wedge}$$

Solving these we obtain equilibrium values of  $\underline{h}^{\wedge}$ ,  $\underline{L}\underline{w}$ ,  $\mu$ . This suggests that a planned-economy interpretation of the Haga-Otsuki model is possible.

Finally, it must be emphasized that Morishima and Haga and Otsuki derive consumption functions of workers and capitalists from the conventional theory of consumer behaviour, that is, from indifference curve analysis. Therefore, their models have sound microeconomic foundations. In this respect, they are definitely superior to Morgenstern, Thompson and Malinvaud, whose models are based on merely arbitrary foundations. Moreover, it is seen that Morishima, Haga and Otsuki, unlike von Neumann, have coefficient matrices which are not constant but depend on  $\underline{x}$  and  $\underline{y}$ . Their basic inequalities [1M] and [3M] or [1'] and [3'] are not symmetric. In spite of these differences from von Neumann, their models are, as they claim, a generalization of the von Neumann model. They are economically and mathematically more general than the latter.

## 2.5 PUBLIC SECTOR MODEL IN THE VON NEUMANN ECONOMY

### A. The Morgenstern Thompson Public sector model

In their 1967 article Morgenstern and Thompson tried to introduce, for the first time the public sector into the von Neumann model. Their analysis is in parallel to their treatment of private consumption and saving. Consequently the criticisms that we have made on the income matrix  $H$  and on the general structure of the model apply 'mutatis mutandis' to the  $G$  matrix "which indicates the real goods payment of the economy to the government" (Morgenstern-Thompson (1976)p. 122) and the general structure of their public sector model. The situation is not very different as  $G$  is predetermined (as  $H$  in the original article) without any reference to the excess production or excess value, and as the condition imposed on  $k'$  and  $z'$  ("the consumption and saving coefficients of the public sector") is only their non-negativeness:  $k', z' \geq 0$ . Consequently the amount of savings of the public sector is always non-negative even though the budget is in deficit (Note that  $k'$  and  $z'$  are not bounded from above by one, in the authors' model).

As in the private sector of their model, we meet also here a strange concept of savings. In particular non-negative savings of the public sector seem really strange. Concerning the private sector we have already stated that negative savings are possible where  $k > 1$ . The same should be true for the public sector; we may have  $z' < 0$ .

We may regard the fact that this rule does not apply as a sign that  $z'$  may have, even in this context, another economic interpretation. However it would be strange to have a public sector consumption  $k'G$  greater than the amount of output "in real goods"  $G$  subtracted from the private sector. The fundamental equations of the Morgenstern Thompson public sector model are:

$$[1g] \quad \underline{x} [ B - \alpha(A + kH + k'G) ] \quad \geq \underline{0}$$

$$[2g] \quad \underline{x} [ B - \alpha(A + kH + k'G) ] \underline{y} = 0$$

$$[3g] \quad [ B - \beta(A + zH + z'G) ] \underline{y} \leq \underline{0}$$

$$[4g] \quad \underline{x} [ B - \beta(A + zH + z'G) ] \underline{y} = 0$$

This means that the existence of a public sector does not add any process or good to the economic system. Further we note that the consumption and saving decisions of the public sector seems do not affect the private consumption-investment decisions. The surplus  $H$  of which the private sector can dispose of is also independent of them. There is no mention at all of these influences. No hint of how the revenue is collected, no explicit reference to taxes is given. The interpretation of  $z'$  as purely the public sector propensity to save would also mean that only the expenditure side, consumption and investment  $(k' + z')G$  do matter.

Thus what Musgrave (1959) p.207 called, the "resource transfer" problem is examined but the way in which expenditure is financed is left entirely unexamined, so

that no "incidence" problem is discussed.

It is clear that the model needs an entirely different interpretation. The model, as it stands, seems to make no particular economic sense, even though it is taken as a model for a non-monetary economy. In what follows we shall try, keeping the previous equations, to re-interpret the economic meaning of  $k'$  and  $z'$ . This will be done in a model that takes into account the effects of different sources of finance (whose effects may vary depending on the point of impact at which the revenue flow is inserted) and considering the final distribution of the burden not to differ from the distribution of initial liabilities (for simplicity's sake). The limits of our work of re-interpretation will be examined at the final stage of the analysis.

Let us now consider the budget equation distinguishing between the different sources of taxation. We can write:

$$T_f + T_c + T_s + T_w + \delta D = \underline{xGy} = c_g \underline{xGy} + s_g \underline{xGy}$$

where  $T_f$ ,  $T_c$ ,  $T_s$ ,  $T_w$ , are the net taxes (i.e. taxes minus subsidies) on the production processes, on consumption, on interest income, and on wages, with  $T_u = t_u \underline{xGy}$  for  $u = f, c, s, w$ ;  $c_g$ ,  $s_g$  the ratio of public sector consumption and saving to the total revenue  $\underline{xGy}$ , and  $\delta D$  is the interest paid on the public debt. Thus  $c_g + s_g = 1 = t_f + t_c + t_s + t_w + \delta d$  where  $d = D/\underline{xGy}$ .

In this model we can assume that  $k \underline{xGy}$  is consumed,



outside the private sector, directly by citizens, or is used as input in producing public goods, with no consequences on the demand for private goods. To make things as simple as possible we set  $k = z = w$ . So we assume that capitalists' consumption is zero and workers' savings are zero. The amount of taxes is consistent with the capacity of those who carry their burden. It is assumed that  $k$  and  $w$  includes taxes on consumption and on wages, respectively.

Then on the basis of these assumptions it is evident that  $(t_c + t_w)\underline{xGy}$  is subtracted from private consumption, while  $t_f \underline{xGy}$  and  $t_s \underline{xGy}$  are additional production costs as provided;  $\beta$  is the interest factor net of taxes. Consequently conditions [2g] and [4g] may be rewritten as:

$$[2'g] \quad \underline{x}\{B - \alpha [A + cH + (c_g - t_c - t_w)G]\}\underline{y} = 0$$

$$[4'g] \quad \underline{x}\{B - \beta [A + wH + (t_f + t_s)G]\}\underline{y} = 0$$

We now denote consumption net of taxes by  $\hat{k}\underline{xHy} = \underline{z}\underline{xHy} - (t_w + t_c)\underline{xGy}$ . Then by writing  $A' = A + \hat{k}H + c_g G$ , being  $k = w$ , [2g] and [4g] can be put in the form:

$$\underline{x} [B - \alpha A'] \underline{y} = 0$$

$$\underline{x}[B - \beta (A' + (s_g - \delta d)G)]\underline{y} = 0$$

where  $(s_g - \delta d)\underline{xGy}$  is the budget surplus. With no budget surplus or deficit we go back to our outside demand interpretation.

A budget surplus would mean a rate of growth greater

than the interest rate. Let us now consider the possibility of a budget deficit. Assuming  $A'' = A + k^{\wedge}H + G - \delta D$  conditions [2'g] and [4'g] can be written as:

$$\begin{aligned} \underline{x}[B - \alpha (A'' + (\delta d - s_g)G)]\underline{y} &= 0 \\ \underline{x} [ B - \beta A'' ] \underline{y} &= 0 \end{aligned}$$

So that a budget deficit  $(\delta d - s_g)\underline{xGy} > 0$  means an interest rate greater than the rate of growth of the economy.

Let us now assume, for simplicity's sake, perfect substitution between public debt and private capital. This means  $\delta = \beta - 1$ . Given the private capital of the economy  $E$ , the public debt (or capital) cannot exceed certain limits which we must take explicitly into account. Let us determine the entity of the public sector's debt by taking into account its growth equation in the state of balanced growth. As  $\alpha D = \delta D - s_g \underline{xGy}$  we have:

$$D = (-s_g \underline{xGy}) / (\alpha - \beta)$$

In order to assure that the private sector of the economy can sustain the public debt we must then impose  $E > |D|$ . Hence:

$$\begin{aligned} 0 < s_g < E(\alpha - \beta) / \underline{xGy} & \quad \text{or} \\ 0 > s_g > E(\alpha - \beta) / \underline{xGy} \end{aligned}$$

depending on whether we have a budget deficit or surplus.

With the alternative assumption that private

consumption and wages are exclusive of taxes, we can interpret  $k'x_{Gy}$  as the public sector's consumption of private goods,  $z'x_{Gy}$  as the taxes levied on the production process,  $1-k'$  as the budget surplus and  $1-z'$  as the interest paid out of the public sector debt (if all tax burden rests on production processes).

However even if equations [1g],[2g],[4g] can be interpreted in this way the G matrix should be constructed in such a way that  $s'Gy$  is the vector of the tax burden on the different processes which are operated at the unit intensity level. This means that the burden on the processes should always be proportional to the intensity level at which they are run. Nevertheless, apart from the previous remarks, the analysis is quite unsatisfactory because more flexible consumption and saving functions are needed, depending, for example, on the goods offered by the public sector and the way in which the expenditure are financed. Furthermore new goods which are produced by the public sector, should be introduced, and the processes should be rewritten in such a way that their profitability and availability is influenced by the activity of the public sector. All these changes are needed in order to recognize that the operations of the public sector and of the private sector are interdependent; they co-exist within the same general equilibrium system.

#### B. A very simple public sector model.

The purpose of this section is to introduce the public

sector into a generalized von Neumann model due to Morishima(1969) and Haga-Otsuki(1965). In this model we assume that the public sector, as well as workers and capitalists of the private sector, is allowed to save and has a proper demand vector function. The maximum advantage is taken of the restrictions on capitalists and workers demands in simplifying the tax incidence problem in the model.

Let propensity to consume  $c_w$  and  $c_c$  be fixed for every price set and for every level of income and interest rate. To simplify the tax revenue functions we assume, as is implicit in Morishima (1969), both  $\alpha, \beta > 1$ . Our model only deviates from Morishima in that it accommodates Haga's and Otsuki's input matrix of 'natural factors'.

The processes of production and income redistribution within the public sector will not be explicitly taken into account; also ignored are the possible reactions of workers and capitalists to the public sector services. Therefore the expenditure of the public sector does not affect the private demand functions as well as the demand for natural factors of productions. We also assume that the surplus of the public sector balance is non negative. Of course current consumption is required to be less than or equal to the disposable revenue at the start of each period.

In our economy there exist value added tax  $\tau^v$ , income taxes  $\tau^y = \tau^\pi + \tau^w$  on profit  $\tau^\pi$  and on wages  $\tau^w$  and

consumption taxes  $\tau^c$ . The tax rates are collectively denoted by diagonal matrices,  $T^v$ ,  $T^\pi$ ,  $T^w$ ,  $T^c$ , respectively.

The value added tax revenue  $\tau^v = \underline{x} T^v (B - A)\underline{y}$  would be subtracted from the gross profits to obtain the taxable profit vector  $B\underline{y} - A\underline{y} - L\underline{w} - T^v (B - A)\underline{y}$ . So the profit tax revenue will be  $\tau^\pi = \underline{x} T^\pi [(I - T^v)(B - A)\underline{y} - L\underline{w}]$ , while the wages tax revenue is  $\tau^w = \underline{x} T^w L\underline{w}$ .

Thus disposable wages and profit income are

$$\Omega = \underline{x}(I_v - T^w)L\underline{w} \text{ and}$$

$$E = \underline{x}(I_m - T^\pi)[(I_m - T^v)(B - A)\underline{y} - L\underline{w}]$$

and the total tax revenue of the public sector is given as:  $\tau = \tau^v + \tau^\pi + \tau^w + \tau^c$ , with  $\tau^c = \underline{x}(d^c + d^w)T^c\underline{y}$ .

The public sector is assumed to save a part of the revenue  $\sigma_g(\tau + E^g)$  (given as the sum of tax and interest revenue  $E^g$ ) which it can dispose of at the beginning of period  $t$  and its current consumption function vector is given by:

$$[a] \quad \underline{d}^g = (\tau + E^g) \underline{g}(\underline{y}) \quad 0 < \underline{g}(\underline{y})\underline{y} = c_g \leq 1$$

The suppliers of natural factors can save a part of their income  $\sigma_w(\Omega + E^w)$ , and have the following demand functions:

$$[b] \quad \underline{d}^w = (\Omega + E^w) \underline{h}(\underline{y}^0) \quad 0 < \underline{h}(\underline{y}^0)\underline{y}^0 = c_w \leq 1$$

where  $\underline{y}^0 = (I_n + T^c)\underline{y}$ .

Capitalists save a greater part of their income than the public sector and the workers so that their

consumption functions:

$$[d] \quad \underline{d}^c = E^c \underline{q}(y') \quad c_c < c_w, c_g$$

satisfy the condition  $0 < \underline{q}(\underline{y}^0) \underline{y}^0 = c_c < c_w, c_g$ .

We may then write the expansion and profit constraint and the free goods and profitability rules in the following form:

$$[1p] \quad \underline{x}B - \alpha^g [\underline{x}A - \Omega^w \underline{h} - E^w \underline{h} - E^c \underline{q} - (\tau + E^g) \underline{g}] \geq 0$$

$$[2p] \quad \underline{x}B\underline{y} - \alpha^g [\underline{x}A\underline{y} - c_w \Omega - c_w E^w - c_c E^c + \tau^c - c_g (\tau + E^g)] = 0$$

$$[3p] \quad B\underline{y} - \beta^g [(A\underline{y} + L\underline{w}) - T^v (B - A)\underline{y}] \leq 0$$

$$[4p] \quad \underline{x}\{B\underline{y} - \beta^g [(A\underline{y} + L\underline{w}) - T^v (B - A)\underline{y}]\} = 0$$

Where workers and the public sector can save, the Pasinetti problem may arise. In a state of balanced growth capital ownership is distributed among the public sector, capitalists, and workers in such a way that it remains unchanged over time. This implies that their profit incomes,  $E^g$ ,  $E^w$ ,  $E^c$ , cannot be arbitrary but they must satisfy  $\sigma_g (\tau + E^g)/E^g = \sigma_w (\Omega + E^w)/E^w = \sigma_c$ . Where  $\sigma_g = 1 - c_g$ ,  $\sigma_w = 1 - c_w$ ,  $\sigma_c = 1 - c_c$ . Hence under  $\sigma_g$ ,  $\sigma_w < \sigma_c$  we obtain the following equations:

$$E^w = \sigma_w \Omega / (\sigma_c - \sigma_w)$$

$$[e] \quad E^g = \sigma_g \tau / (\sigma_c - \sigma_g)$$

$$E^c = E - E^w - E^g$$

which similarly to [10M] give economically meaningful



results only for  $E^w + E^g < E$ . Where these conditions are verified, the total consumption vector is:  $\underline{d}^c + \underline{d}^w + \underline{d}^g = E\underline{q} + \Omega(\sigma_c \underline{h} - \sigma_w \underline{q}) / (\sigma_c - \sigma_w) + (\sigma_c \underline{g} - \sigma_g \underline{q}) / (\sigma_c - \sigma_g)$ , and its value  $(\Omega + E^w)c_w + E^c c_c + (\tau + E^g)c_g - \tau^c = \Omega + E c_c + \tau - \tau^c$ .

Furthermore substituting the value of B from [4p] in the definition of E we find

$$E = (\beta^g - 1) \underline{x}(\underline{I}_m - T^\pi)(\underline{A}\underline{y} + \underline{L}\underline{w}).$$

We now can rewrite equations [1p] and [2p] as:

$$[1p'] \quad \underline{x}B - \alpha^g \underline{x}A - \Omega(\sigma_c \underline{h} - \sigma_w \underline{q}) / (\sigma_c - \sigma_w) - (\sigma_c \underline{g} - \sigma_g \underline{q}) / (\sigma_c - \sigma_g) - (\beta^g - 1) \underline{x}(\underline{I}_m - T^\pi)(\underline{x}\underline{A}\underline{y} + \underline{L}\underline{w}) \underline{q} \geq 0$$

$$[2p'] \quad \underline{x}B\underline{y} - [\alpha^g + c_c(\beta^g - 1)](\underline{x}\underline{A}\underline{y} + \underline{x}\underline{L}\underline{w}) - (1 - c_c)\tau^\pi - \tau^v = 0$$

Together with [4p] these enables us to derive a new relation between the rate of expansion and the rate of interest. This is because we have from [4p] and [2p'],  $\tau^v + \beta^g(\underline{x}\underline{A}\underline{y} + \underline{x}\underline{L}\underline{w}) = [\alpha^g + c_c(\beta^g - 1)](\underline{x}\underline{A}\underline{y} + \underline{x}\underline{L}\underline{w}) + \tau^v + (1 - c_c)\tau^\pi$ . This gives

$$[7p] \quad (\beta^g - 1)\sigma^{\wedge}c = (\alpha^g - 1)$$

where  $\sigma^{\wedge}c = \sigma_c(1 - t^\pi)$  where  $t^\pi = \underline{x}T^\pi(\underline{A}\underline{y} + \underline{L}\underline{w}) / (\underline{x}\underline{A}\underline{y} + \underline{x}\underline{L}\underline{w})$  is the average tax rate on capital which enables the public sector to receive a revenue of the same amount as the profit income tax.

Even in this particular case, of this very simplified model, it can be shown that the introduction of the public sector does not automatically lead to a reduction in the

rate of growth provided that capitalists savings are unaffected. From [2p'] we have  $\alpha^e = [\underline{x}B\underline{y} - c(\beta - 1)\underline{x}(A\underline{y}+L\underline{w}) - (1-c_c)(\tau^\pi + \tau^v)] / \underline{x}(A\underline{y} + L\underline{w})$  and consequently the rate of growth is given as:

$$[8p] \quad \alpha^e = \alpha - (1-c_c)(\tau^\pi + \tau^v) / \underline{x}(A\underline{y} - L\underline{w})$$

The equilibrium remains unchanged if there are no profit and value added taxes, so that  $\alpha^e = \alpha$  and  $\underline{y}$  and  $\underline{w}$  remain constant. In this case the public sector merely substitutes private agents in their consumption demand.

Apart from some differences the above formulation is consistent with Morishima's elementary model in his THE ECONOMICS OF INDUSTRIAL SOCIETY (1984), and with his econometric specification in Morishima-Nosse (1972). Although he does not discuss the problem in the context of the von Neumann model, so that he does not allow for alternative processes, or for balanced growth, his model can be extended along those lines. The model we have just examined above may be regarded as one possible extension. Conversely the present model draws from Morishima's, demonstrating that this sort of model is not just an academic exercise but can be effectively applied to the actual world. In this sense, the two models are complementary to each other.

### 3. BADS AND EXTERNALITIES IN THE VON NEUMANN'S MODEL

#### A. Premises

In this chapter we examine externalities and deal with the processes which produce both goods and bads. It is assumed that production processes in the system may affect other production processes or the welfare (the utility functions) of the agents of the system directly. For instance the cost for removing garbage and pollutants has, throughout the previous sections, not been taken into account in the profitability condition of each process and in the computation of the prices of outputs. However, in reality, this cost could be so high as to make some processes no longer feasible or profitable and could decrease economic agents' welfare substantially.

This type of problem is discussed under the title of 'market failure' and the solution to it is not found within a pure private economy. In what follows we show that by revising the interpretation of the von Neumann model and by introducing a public sector in the system, we may deal with the problem of externality. It is pointed out that in introducing non-pure private goods, we must be careful to deal with the following question: What are the new types of goods (or bads) to be produced in the new model? What type of equations should the model satisfy?

### 3.1 The model of Creamans (1969)

The article of J.E. Creamans (1969) follows Leontief's input-output method and uses it to tackle the environmental problems by adapting it to the context of a von Neumann economy. He deals with the pollution problem and the problem of bads in the KMT framework.

#### A. The necessity of introducing a public sector

To tackle these problems, we must, first of all, have a public sector which produces services to control the private sector. Alternatively, the activity of eliminating pollution produces a good to be consumed by workers; but there is a new problem of free riding. The problems which arise, are general, because it is impossible to exclude some particular individual agents from consuming these goods. In general, unless this exclusion could be made, no private agent would operate the pollution-eliminating processes. He would have no power to force the community to pay him for the activity and any agent or process who did not want to pay could not be excluded from consuming services of eliminating pollution. This means that only the public sector, raising the revenue by taxation, can finance the provision of such services. Creamans' model must be examined bearing this point in mind.

To reduce pollution it is necessary for the public sector to compel the agents running the polluting processes to reduce the amount of pollutants

produced, by operating, at appropriate intensities, certain of the pollution-eliminating processes simultaneously. As can be easily seen the amount of pollutants to be eliminated may be introduced into the model as an outside demand coming from the public sector.

In order to maintain all the condition of the KMT model the  $n$  goods now include  $q$  pollutants and  $r$  'eliminated pollutants'. (Usually  $r$  may be equal to  $q$ , but we do not make such an assumption.) The  $m$  processes include 'dirty' methods of production (producing pollutants as joint output) and pollution-eliminating processes. To find the actual net output of pollutants the quantities of eliminated pollutants produced must be subtracted from the output of pollutants produced by the 'dirty' processes.

### B. Pollution control

In Creamans' model the public sector controls the dirty processes, so that they are obliged "to buy the appropriate quantities of eliminated pollutant", in order to reduce their net output of pollution.<sup>(1)</sup>

It should be noticed that contrarily to what Creamans states, it may not be possible to reduce net pollution of all processes at a fixed rate in the way he proposes.<sup>(2)</sup> This occurs because the inputs for

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(1) Creamans (1969) p.530.

(2) Creamans (1969) p.531 table VII.

producing eliminated pollutants are fixed and not inversely proportional to the rate of growth. In fact the  $A$  matrix and the inputs for producing eliminated pollutants are multiplied by the coefficient of growth  $\alpha$ .

In the first case examined by Creamans, dirty processes have to buy eliminated pollutants in different quantities, proportional to the amount of their pollutant outputs. Then the net outputs of pollutants produced by the economy are reduced because the dirtiest processes become less profitable and hence their levels of activity are decreased.

This implication of the first case is not fully appreciated by the author who points out the lower rate of growth as a relevant factor in order to reduce the level of pollution.

His reasoning seems, however, in part incorrect, in the context of the von Neumann economy. Apparently the surest way to reduce pollution would be a reduction of the equilibrium rate of growth. More drastically we may reach a state of balanced decay of the economy. But as Champernowne notices in the state of "equilibrium there is no progress or change of production per head of population: growth merely consists of replication, and the economic system expands like a crystal suspended in a solution of its own salt".<sup>(3)</sup> Thus in this model it seems more relevant to compare only the reduction of the output



of pollutants in a given period, as we have tried to do.

### C. Other methods of pollution control

On the other hand, when the demand for eliminated pollutants is introduced in the form of outside demand into the system the same amount of these goods is required by each production process (even by the pollution-eliminating processes). In respect of the previous case the burden of reducing pollutants will be lighter for the dirtiest processes and for the same rate of growth we will not reach the previous degree of reduction of pollution. It should be noted that this difference, would probably be greater where private consumption functions depend on the prices of goods.

A last method proposed by Creamans is the introduction of a linear constraint on the maximum amount of pollutant output in the economy.

Creamans presents examples to show how "the goals of economic growth and of pollution control compete for the same scarce resources". That is because a positive amount of the goods, 'eliminated pollutants' must be bought, or the intensity of some processes must be reduced. In the outside demand case, without technological change, saving and investment must be proportionally reduced and therefore, both the rate of growth and of interest fall.

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(3) Champernowne (1944-5) p.11.

In the last case the intensity levels of dirty processes are not allowed to expand beyond a certain point so that the rate of growth of certain goods may be scaled down to a lower level.

Creamans fails to appreciate the pollution problem as a problem of a public good. In particular there is no perception of the fact that: (1) the private sector may fail to eliminate pollution because of free riding behaviour, (2) pollution may affect the use of some processes of production and no agent and no process can evade this (exclusion is not possible) so that pollution is a good different from ordinary private goods.

Another interesting remark may be made regarding the public sector's control of pollution. Creamans offers an example of the trade-off between pollution control and the rate of growth of the economy. However he never discusses which of the ways of controlling pollution he proposes will minimize the reduction in the rate of growth. Consequently he has no way of choosing between the various possible ways of combatting pollution.

### 3.2 Fisher's analysis of externalities (1977)

#### A. Principal features and problems

In his (1977) article, D. Fisher attempted to introduce externalities in the von Neumann model. He defines externality as a phenomenon satisfying the following conditions:

- 1) that an agent may use, at no charge, some input without producing it; and
- 2) that outputs accrue to an agent, other than the producer of them, independently from his will.<sup>(1)</sup>

Though he never explicitly states these conditions, it is noted that inputs and outputs are always positive quantities. This is stressed because we are not told of their negative value, whereas we are told of the possibility of negative prices. Thus he maintains the non-negativity assumption of the KMT model.<sup>(2)</sup>

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(1) In this way Fisher seems to distinguish between two different externalities: the first making more profitable a given process of production, because the producer can get some needed inputs freely, and the second reducing the profit because the producer cannot control all the outputs produced by the process.

These are, however, two faces of the same medal. In fact in the von Neumann model an agent cannot use a commodity as input unless it is produced in the previous period, and it is available at the beginning of the period when the new production process starts.

Finally we add that Fisher deals only with externalities in the production process, as he says in footnote (\*) at p. 274.

(2) However we disagree with Fisher in his claim that he introduces bads as "harmful" output. See Fisher (1977), p.267. In any case he does not say anything about how in

It is implicitly assumed that these goods (externalities) are free to the first agent who can use them as inputs or receive them as outputs, whereas he can exchange them at the current price, where trade is possible. (This is apparent from equations [1f] and [2f] below.)

This assumption is a most controversial one among those made by Fisher. It would be no wonder that if commodities received as free externalities could be sold by the receiving agent, they would be sold at a positive price by the producer himself. Hence they would not be underproduced. Furthermore why should they be overused if instead of using them one can sell them at a positive market price? Finally we may ask why externalities are not traded and priced, in his example, in spite of the possibility of trade there?

#### B. Description of the models

Fisher assumes the existence of  $k$  independent agents  $r = 1, \dots, k$ , each running his set of  $m(r)$  processes at intensities  $\underline{x}^r = (x^r_1 \dots x^r_{m(r)})$  and producing some of the

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practice bads are distinguished from goods in his model and in which way they affect the processes of production. The only explicit reference to bads (garbage) in a footnote at p.280 simply tells us that rather than directly considering bads, "of which no more than a certain maximum may be produced, (namely the amount that can be disposed during the next period)" he considers the 'eliminated bad' as a good and assumes that "a certain minimum amount of it must be provided during each period, which can be considered as a necessary input for the production of other goods". Otherwise he argues we will find it "necessary to introduce a negative price".

$n$  goods at prices  $\underline{y}^r = (y^{r_1} \dots y^{r_n})$ . Then he defines  $A^{rs} = (a^{rs_{ij}})$  as the "matrix of inputs taken from agent  $s$  by agent  $r$ " and  $B^{rs} = (b^{rs_{ij}})$  as "the matrix of outputs given by agent  $r$  to agent  $s$ ".<sup>(3)</sup>

He then makes the following assumptions: "if  $r$  runs process  $i$  with unit intensity, he will use  $a^{rs_{ij}}$  units of good  $j$  from agent  $s$ ", p.274. From this it follows that  $r$  uses  $\sum_i x^{r_i} a^{rs_{ij}}$  units of good  $j$  taken from agent  $s$ , while  $s$  is given good  $j$  of the amount  $\sum_i x^{r_i} b^{rs_{ij}}$  by  $r$ .

Let us now briefly examine the consequences of the assumption made on a process of a given agent. The  $i$ -th process of the  $r$ -th agent, run at an unit level, will transform the  $k$  bundles of goods  $\underline{a}^{rs_{i*}} = (a^{rs_{i1}}, \dots, a^{rs_{in}})$  for  $s = 1, \dots, k$  into the  $k$  bundles of good  $\underline{b}^{rs_{i*}} = (b^{rs_{i1}}, \dots, b^{rs_{in}})$ .<sup>(4)</sup>

D. Fisher, then, distinguishes the case of excludable goods from the one of non-excludable goods. Allowing for free externalities, in both cases, he considers the following two polar cases only.

(3)  $A^{rs}, B^{rs}$  are the matrices of externalities inputs and outputs when  $r \neq s$  and have nothing to do with trade.

(4) Or equivalently the bundle of goods  $\sum_i \underline{a}^{rs_{i*}} = (a_{i1}, \dots, a_{in})$  is transformed into  $\sum_i \underline{b}^{rs_{i*}} = (b_{i1}, \dots, b_{in})$ , if we are not interested in differentiating between the different sources of input or destinations of the output. However the  $r$ -th agent needs to produce by himself (or eventually buy in the market at the current price if trade is admitted) the input vector  $\underline{a}^{r_{i*}}$  and will be able to keep only the output vector  $\underline{b}^{r_{i*}}$  to use in the next period as input (or eventually to sell it in the market).

Case 1 where all goods are excludable and traded, and case 2 where none of the goods can be traded. In both cases only the conditions of the Nash equilibrium are examined whereby it is meant that "each agent short-sighted tries to maximize his own profit, assuming that his actions will have no influence on the behaviour of the others".<sup>(5)</sup>

### C. The trade case

In the first case of all goods being traded all price vectors are the same  $\underline{p}^r = \underline{p}$ . Only the processes which produce profits at the maximum rate  $\beta-1$  are adopted. The  $r$ -th agent evaluates costs (at the unit level of activity) at the market price to buy the input vector (he must provide by himself to run the process), and revenues also at the market selling prices. This enables Fisher to write:

$$[1f] \quad \beta \geq \beta^r = \left( \sum_{j=1}^n b^{r1j} y_j \right) / \left( \sum_{j=1}^n a^{r1j} y_j \right)$$

which means that for each agent  $r$  only non-externalities matter in evaluating the rate of profit, as the prices paid for the goods received as externalities are all zero. One can immediately notice that the existence of externalities would decrease the rate of profit, since it reduces the output of paid goods  $B^{r1}$ ; or increase it indirectly, since it reduces the necessary input of paid

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(5) Cfr. Fisher (1977), p. 278.



goods  $A^{rr}$ .<sup>(6)</sup>

On the other hand, in determining the rate of expansion all goods play a role, including those received as externalities. Thus, the growth factor of the  $j$ -th good is given by the ratio of the total output of good  $j$  produced to its total input required by the prevailing activities  $(\underline{x}^1, \dots, \underline{x}^k)$ . And the rate of expansion of the economy is the minimum one among the rates of expansion of all goods. Following Fisher we may therefore write

$$[2f] \quad \alpha \leq \alpha_j = \left( \sum_{s=1}^k \sum_{r=1}^k \sum_{i=1}^{m(r)} x_i^r b^{rs_{ij}} \right) / \left( \sum_{s=1}^k \sum_{r=1}^k \sum_{i=1}^{m(r)} x_i^r a^{rs_{ij}} \right)$$

The rule of profitability that Fisher states verbally is:

$$x_i^r \underline{b}^{rs_{ij}} y_j = \beta x_i^r \underline{a}^{rs_{ij}} y_j \quad \text{for all } i \text{ and } r$$

and likewise we had the free goods rule:

$$\left( \sum_{s=1}^k \sum_{r=1}^k \underline{x}^r \underline{b}^{rs_{*j}} \right) y_j = \alpha \left( \sum_{s=1}^k \sum_{r=1}^k \underline{x}^r \underline{a}^{rs_{*j}} \right) y_j \quad \text{for all } j$$

Fisher's conditions do not rule out the use of output  $b^{rs_{vj}} > 0$  when  $a^{sh_{ij}} > 0$  is required and  $h/r$ . This means that for  $s$  it is indifferent who is the producer of the externality.

On the other hand Fisher makes a self destructive assumption to the effect that "all goods can be traded".

(6) This is however a really superficial and misleading observation. Where externality have no market price and are not tradeable,  $\beta^r$  are not affected by the existence of externalities as long as  $a^{rr}$  and  $b^{rr}$  remain unchanged as they should.

Thus goods received as externalities can be traded (as we can see from [2f]) so his externalities are not real ones but nothing else than gifts, or fruits of theft. Then the natural question arises: if goods received as externalities can be sold at positive prices why are they not evaluated at the market prices when used in the productive processes?(7)

#### D. The no trade case

When Fisher considers the conditions for the Nash-equilibrium without trade he assumes that prices and profits factors may differ between agents. In this case prices are nothing else than the accounting prices of each agent and enable him to calculate the profitability of each process and to choose the most profitable ones.

(7) In order to exclude  $a^{rs}_{ij}$  from the valuation inequality [1f] we must assume that it is not tradeable because this is the only way to exclude a positive market price. Furthermore in order to distinguish  $\underline{x}^r b^{rs}_{xj}$  from  $x^s b^{ss}_{xj}$ , when  $a^{ss}_{ij} > 0$  we must regard externalities as being different from private goods produced by each individual agent. Thus even if they are equal in quality, they must be treated, in the model, as different goods. However we do not think it is worthwhile to distinguish between  $a^{sr}_{ij}$  and  $a^{sh}_{ij}$  when  $r, h \neq s$ . Consequently it is needless to have all the different  $A^{sr}$  but just one  $A^{s*}$ , the externality input matrix.

These considerations, which are necessary for the working of the model, are never stated by Fisher.

We can show easily that the conditions [1f] and [2f] do not allow for bads. Where we have  $b^{rs}_{ij} > 0$  this in general represents a non-positive externality to  $r$  as well as a non-negative externality to  $s$ . In fact the latter can use it in the  $h$ -th process  $a^{sr}_{hj} > 0$  if the process is profitable, or sell it in the market if its price is positive  $y_j > 0$ . If it is not beneficial (i.e.  $y_j \leq 0$ ), and all processes with  $a^{sr}_{hj} > 0$  are unprofitable then it will be simply ignored.

Consequently he writes:

$$[3f] \quad \beta^r \geq \beta^{r_i} = \left( \sum_{j=1}^n b^{r_i j} y^{r_j} \right) / \left( \sum_{j=1}^n a^{r_i j} y^{r_j} \right)$$

On the other hand the  $r$ -th agent can use as input the amount  $\sum_i x_i b^{s_i j}$  of good  $j$  which is made available by the production activity of agent  $s$  in the previous period. Thus the rate of expansion of the activities of the  $r$ -th agent is limited by the available amount of good  $j$ . There are  $n$  conditions of this sort, each defining a rate of expansion, among which the least one gives the rate of expansion of the economy. To take this into account Fisher writes:

$$[4f] \quad \alpha^r \leq \alpha^{r_j} = \left( \sum_{s=1}^k \sum_{i=1}^{m(s)} x_i^s b^{s_i j} \right) / \left( \sum_{s=1}^k \sum_{i=1}^{m(s)} x_i^s a^{r s_i j} \right)$$

because " $r$  is only concerned with the growth factor in the portion of good  $j$  that is accessible to himself" while "changes in the amount of good  $j$  available to other agents do not concern him since trade is not possible", p. 279.

Consequently  $r$  "takes all external effects ... as given, and adjusts the intensities of his own processes to maintain balanced growth in his stocks of goods" p.279.

It is first noticed that in the case where agent  $r$  takes all goods available to him into account he would use the good  $j$  received from  $s$  as input in any process  $i$  as long as it requires this good, regardless of the source of the good. It is needless to say that this is possible in the present model as it was in the previous one.

We may notice how probably the Fisher formula includes a misprint. In fact from the previous definition of  $a^{rs}_{ij}$  the term  $\sum_s \sum_i x^s_i a^{rs}_{ij}$  of the denominator, should be read  $\sum_s \sum_i x^r_i a^{rs}_{ij}$  which is the units of good  $j$  taken by  $r$  from all agent  $s$ . The availability constraints imply that  $(\alpha^r)^t \sum_s \sum_i x^r_i a^{rs}_{ij}$  be less than or equal to the amount of output  $\sum_s \sum_i (\alpha^s)^{t-1} x^s_i b^{sr}_{ij}$  produced by all the agents in the economy. Where all  $\alpha^r$  are equal to each other we have:

$$[4'f] \quad \alpha^r \leq \alpha^r_j = \left( \sum_{s=1}^k \sum_{i=1}^{m(s)} x^s_i b^{sr}_{ij} \right) / \left( \sum_{s=1}^k \sum_{i=1}^{m(s)} x^r_i a^{rs}_{ij} \right)$$

Otherwise, agents may expand the intensities of their processes at different rates and condition [4'f] won't be verified for all future periods in the balanced growth equilibrium. Like [2f], however, equation [4f] assumes that agent  $r$  uses  $\underline{x}^s \underline{b}^{sr}_{*j}$  when  $\underline{a}^{rs}_{ij}$  is required. Consequently the profit constraint and the rule of profitability have no rationale unless we exclude goods for which  $a^{rr}_{ij} > 0$  implies  $x^h \underline{b}^{hr}_{*j} > 0$  for some  $h \neq r$  and some  $p = 1, 2, \dots, m$ ; or unless we impose some further constraint. Otherwise we would have a situation similar to the one observed in the previous case.

#### E. Bads and negative externalities in the model

With regard to bads our previous consideration may be applied to the no-trade case.

Let us consider, for the completeness of argument, the possibility of negative  $\underline{b}^{sr}_{*j}$ . We consider this

possibility, in spite of the existence of no hint of it in the original article, for the purpose of examining how far the Fisher model can deal with bads.

The results of our analysis are not completely satisfactory. When only a few  $\underline{b}^{sr*}_j$  are non-positive we have a sort of commodity destruction phenomena. Instead when all  $\underline{b}^{sr*}_j < 0$  for  $s = r$ , an economic equilibrium for a single agent or for the whole economy exists only if the condition [2f] of a maximum level of production of this bad is satisfied. One may argue that the presence of processes with  $a^{sr}_{ij} < 0$  (bads-eliminating processes) would solve the problem. This interpretation is however valid only if some special assumptions are satisfied.<sup>(8)</sup> Even in this case, being free of externalities, those processes would not, in general be profitable and hence would not be used.

$b^{sr}_{ij} < 0$  simply has a negative effect on the value of  $\sum_s \sum_i x^s_i b^{sr}_{ij}$ . Thus some of the outputs of good  $j$  available to  $r$  are destroyed while no agent is apparently receiving any beneficial effect from it.

If  $\underline{b}^{sr*}_j < 0$  for all  $s \neq r$  and  $\underline{a}^{rr*}_j$  and  $\underline{b}^{rr*}_j$  are zero whilst  $\underline{a}^{sr*}_j$  is negative too, we will have  $\sum_{s \neq r} (\alpha^r)^{t-1} \underline{x}^s$   
 $\underline{b}^{sr*}_j \geq \sum_{s \neq r} (\alpha^r)^t \underline{x}^r \underline{a}^{rs*}_j$  in the case of the goods externality) being not traded, or equivalently:

(8) We will state this rule explicitly for externalities, see ch.4.3 paragraph d.



$$\sum_{s \neq r} (\alpha^s)^{t-1} \underline{x}^s (-\underline{b}^{sr * j}) \leq (\alpha^r)^t \sum_{s \neq r} \underline{x}^r (-\underline{a}^{rs * j})$$

Thus if the available quantity of the  $j$ -th bad is greater than a maximal permissible level, then the  $r$ -th agent cannot undertake any productive activity. This maximum level depends on the maximum level of each process  $(-\underline{a}^{rs * j})$ , as well as the rate of growth  $\alpha^r$  and the real production structure  $\underline{x}^r$  of agent  $r$ .

A similar condition (derived from [2f]) will hold for the whole economy, when trade is possible:

$$\sum_{r=1} \sum_{s=1} \underline{x}^s (-\underline{b}^{sr * j}) \leq \sum_{r=1} \sum_{s=1} \underline{x}^r (-\underline{a}^{rs * j})$$

If it is not satisfied the whole economy will collapse.

In general no self adjusting process will be enacted, by any agent, to limit the level of the processes, which produce this bad jointly; neither is it possible, under our hypothesis, to produce a positive amount of that commodity.

#### F. Final remarks

Fisher never deals with the question of the existence of equilibrium in his models; he does not even demonstrate what the consequences of the presence of externalities are. To this effect he only builds up a very simple, but not very clear, example of an economy where the public sector runs the garbage disposal industry, by financing its activity by taxes levied on the interest income. He shows that the rate of growth and of interest (after tax) are less in this economy than those in the original von



Neumann model. Then he concludes that "we see, at least in this example, some of the non-optimal behaviour which may be responsible for the excessive generation of waste in decentralized economy" p.284. In the whole article only this part is concerned with showing how inputs from external sources are under-valued and over-used (as in the case of garbage disposal provided free by the public sector). Similarly outputs provided externally to other agents would be under-valued and under-produced. Nevertheless the original von Neumann solution would be obtained if the public sector taxes only the (third) unprofitable processes. We may, on the other hand, recognize that the general approach adopted in Fisher's model has some advantage over Creamans' approach because it explicitly allows for the public goods nature of pollution and it can deal with the non rivalry aspect of the problem. For instance we can impose, for the  $j$ -th good, condition  $b^{s^*}_{i,j} = b^{s^r}_{i,j} = b^{s^u}_{i,j}$  for all  $r$  and  $u$ , so that each agent can consume the full amount of public good produced by  $s \sum_i x^{s_i} b^{s^*}_{i,j}$ . However the polar cases, examined by Fisher, are not interesting, because they are too artificial.

In fact the hypothesis of all goods being traded is not compatible with the existence of externalities, because in this case the only possible interpretation of  $x^{r_i} b^{s^r}_{i,j}$  would be a stolen quantity of good  $j$  by  $s$  from the  $i$ -th process of agent  $r$ . Furthermore in both models (with

and without trade), as has already been stated above, we have some reservations concerning conditions [1f] and [2f]. Nor can we accept Fisher's claim that he has shown "how externalities can be introduced into a von Neumann growth model and how the resulting price distortion can be calculated", in spite of his numerical example.

Fisher's and Creamans' models have in common, two negative features:

1. the treatment of bads, transformed in goods, required as inputs of the dirty processes (as seen in the last example of garbage disposal),

2. the lack of any specific analysis of the public sector: it is simply regarded as being similar to all other agents and subject to the same conditions.

We shall now propose a general treatment of external diseconomies in the following sections. Subsequently, in section 4.3 of the chapter of public intermediate goods we shall attempt to solve the problems left unsolved by Fisher, under the assumption that trade is possible only in the case of private goods. At that stage it will also become clear that the apparatus and notation of Fisher are too complicated to achieve his aim of showing that: "if there are externalities, the myopic profit maximization by competing individuals will generally lead to a less than optimal growth rate" p.267.

### 3.3 Morishima's and Thompson's treatment of external economies

We have already remarked how externalities in production can be represented, in the context of a von Neumann economy, in terms of the intensities of processes.

External economies (or diseconomies) of scale may be discussed as one case, under the broad item of externalities in production. In the case of a single firm, for example, external economies (or diseconomies) are obtained if an expansion of its production level results in a fall (or an increase) in the required unit inputs (outputs) of its production process. To apply this concept to the von Neumann economy, it is necessary to generalize it to  $m$  processes and to take production lag into account.

In the ambit of general phenomena, in which the levels of production of processes depend on the levels of production in the previous period, Morishima and Thompson (1960) analyzed a model involving those external economies (diseconomies) which Meade (1952) called of the "unpaid-factor" type. They show that balanced growth solutions exist so that all processes may grow at the same rate. As they claim, their work is a generalization of the original von Neumann economy in which there is no externality.

Their steady state model is explicitly derived from a temporary equilibrium model, based on static expectations, which is Morishima's usual approach to the problem;

unfortunately not followed by other economists in building their models. In our exposition we will depart from the authors' convention of postdating the outputs of each production process and continue to use the usual matrix notation.

Let  $D(t)$  and  $E(t)$  be the  $m$  by  $n$  matrices of outputs and inputs and  $\underline{q}(t)$  the  $m$  by 1 vector of workers employed in  $m$  processes in period  $t$ . In the "unpaid-factor" case hypothesis the production function of the  $i$ -th process  $\phi_i(\dots) = 0$  depends, not only on the conventional variables such as  $d_{ij}(t)$ ,  $e_{ij}(t)$ ,  $q_i(t)$ , but also on outputs  $D(t-1)$ , inputs  $E(t-1)$ , and the level of production  $q(t-1)$  of all commodities in the previous period. Some of them may have no effect, but others do have effect because of the existence of externalities. Thus

$$\phi_i[\underline{d}_{i*}(t), \underline{e}_{i*}(t), \underline{q}_i(t); D(t-1), E(t-1), \underline{q}(t-1)] = 0$$

where  $\phi$  is homogeneous of degree zero in all arguments. Let us denote the relative intensity vector in terms of the use of labour by  $x_i(t) = q_i(t) / \sum_i q_i(t)$  and define  $b_{ij}(t) = d_{ij}(t)/q_i(t)$ ,  $c_{ij}(t) = e_{ij}(t)/q_i(t)$ ,  $\alpha(t) = q_i(t)/q_i(t-1)$ . Then dividing the arguments of  $\phi_i$  by  $q_i(t-1)$  we have

$$\phi_i[\alpha(t)\underline{b}_{i*}(t), \alpha(t)\underline{c}_{i*}(t), \alpha(t); XB(t), XC(t), X] = 0$$

where  $X$  is a diagonal matrix whose diagonal elements are  $x_r(t-1)/x_r(t-1)$  with  $r = 1, \dots, m$ . Thus we can write:

$$\hat{\phi}_i [\alpha(t) \underline{b}_i x(t), \alpha(t) \underline{c}_i x(t), \alpha(t); B(t-1), C(t-1), \underline{x}(t-1)] = 0$$

We then assume that each process maximizes the capitalized value of profit

$$[1/\beta(t)][\underline{b}_i x(t) y(t)] - \underline{c}_i x(t) y(t) - w(t) x_i(t)$$

with  $\alpha(t), \beta(t), \underline{y}(t), \underline{x}(t)$ , as well as  $B(t-1), C(t-1), \underline{x}(t-1)$  being all given subject to its production function  $\hat{\phi}_i$ . This gives

$$b_{i,j}(t) = f_{i,j}[\underline{y}(t), \beta(t), \alpha(t), \underline{x}(t), B(t-1), C(t-1), \underline{x}(t-1)] / \alpha(t)$$

$$c_{i,j}(t) = g_{i,j}[\underline{y}(t), \beta(t), \alpha(t), \underline{x}(t), B(t-1), C(t-1), \underline{x}(t-1)] / \alpha(t)$$

At this point Morishima and Thompson examine the balanced growth state where :  $\underline{x}_i(t) = \alpha^t \underline{x}$ ;  $\underline{y}(t) = \underline{y}$ ;  $\beta(t) = \beta$  and the technological coefficients are constant over time  $B(t) = B$ ,  $C(t) = C$ . Under these conditions we have from [1]:

$$B = B(\underline{x}, \underline{y}, \alpha, \beta)$$

$$C = C(\underline{x}, \underline{y}, \alpha, \beta)$$

In addition to this, they assume that capitalists only save and workers only consume and define  $k$  as the unit vector of consumption goods which a worker consumes. Where each worker buys  $\mu$  units of  $k$ , we have  $w = \mu k \underline{y}$ , where the proportionality factor  $\mu$  is referred to as the level of consumption. They then define  $a_{i,j}$  as:

$$a_{i,j}(\underline{x}, \underline{y}, \alpha, \beta, \mu) = c_{i,j}(\underline{x}, \underline{y}, \alpha, \beta) + \mu k$$



They thus see that, where all processes expand at constant rate  $\alpha$ , the original condition of the von Neumann model are satisfied

$$\begin{aligned} \underline{x} [ B(\underline{x}, \underline{y}, \alpha, \beta) - A(\underline{x}, \underline{y}, \alpha, \beta, \mu) ] &\geq 0 \\ \underline{x} [ B(\underline{x}, \underline{y}, \alpha, \beta) - A(\underline{x}, \underline{y}, \alpha, \beta, \mu) ] \underline{y} &= 0 \\ [ B(\underline{x}, \underline{y}, \alpha, \beta) - A(\underline{x}, \underline{y}, \alpha, \beta, \mu) ] \underline{y} &\leq 0 \\ \underline{x} [ B(\underline{x}, \underline{y}, \alpha, \beta) - A(\underline{x}, \underline{y}, \alpha, \beta, \mu) ] \underline{y} &= 0 \end{aligned}$$

To prove the existence of equilibria they assume that:

(i) functions  $a_{ij}(\underline{x}, \underline{y}, \alpha, \beta, \mu)$  and  $b_{ij}(\underline{x}, \underline{y}, \alpha, \beta)$  are continuous with respect to each of  $(\underline{x}, \underline{y}, \alpha, \beta)$  in the region  $\underline{x} \geq 0$ ,  $\underline{y} \geq 0$ , and  $0 < \alpha, \beta < \sigma$  ( $\sigma =$  constant).

(ii) for all  $(\underline{x}, \underline{y}, \alpha, \beta)$  in the set the following inequalities hold:  $A(\underline{x}, \underline{y}, \alpha, \beta, \mu) > 0$ ,

$$B(\underline{x}, \underline{y}, \alpha, \beta) > 0,$$

$$A(\underline{x}, \underline{y}, \alpha, \beta, \mu) + B(\underline{x}, \underline{y}, \alpha, \beta) > 0,$$

$$v[B(\underline{x}, \underline{y}, \alpha, \beta)] > 0 \text{ and}$$

$$v[ B(\underline{x}, \underline{y}, \alpha, \beta) - A(\underline{x}, \underline{y}, \alpha, \beta, \mu) ] < 0,$$

where  $v\{ \cdot \}$  is the value of a matrix game.

Finally it is shown how  $\underline{x}$  and  $\underline{y}$  do not enter as argument of  $A(\cdot)$  and  $B(\cdot)$  in absence of any external economy and how "The presence of external economies gives rise to an increase in the rate of balanced growth", or equivalently "the highest allowable level of consumption in the presence of external economies is higher than that in



their absence".

The Morishima-Thompson model may be seen as a sort of generalization of the approach used by Samuelson (1951) for the Leontief input-output model. However Samuelson was not concerned with the von Neumann model; also in his analysis external economies or diseconomies were absent.

The Morishima-Thompson model is a very general one, apart from the restrictions necessary for the existence of a balanced growth equilibrium, and it is quite more comprehensive than the special cases of Fisher (1977). This will become obvious when it is shown later in chapter 4.3 that the Fisher equilibrium is reduced to a constrained von Neumann equilibrium.

The Morishima-Thompson model can also deal in a much more satisfactory and meaningful way, with bads and more generally, external diseconomies (harmful externalities), because here, differently from the Creamans (1969) or Fisher (1977) models, these are allowed to reduce directly the efficiency of single production processes.

However in Morishima and Thompson externalities remain hidden behind the scene and we are not able to identify them with specific commodities. Consequently we are not able to attribute accounting prices to them or to determine demand for those externalities or to set for them a maximum level of production (as is required, for instance, by the public sector).

Following von Neumann and KMT, we obtain a balanced

growth equilibrium that is established by private profit maximization, activities being carried out by private agents alone even though the situation is improved, at least theoretically, by the public sector intervention.

A more efficient equilibrium and a greater rate of growth may be reached by a planned economy in which externality producing processes are subsidized and bads producing processes are taxed.

Although Morishima Thompson limit themselves to the comparison of the economy with external economies and a similar one where the external effects are absent, this kind of comparison is less interesting from a normative point of view than the one between the equilibrium of the Morishima Thompson model with external economies and an optimal one reached by a planned economy. This comparison however has not been attempted yet.

In order to deal with the problem of external diseconomies and bads in a more specific way a different model may be required. For this purpose we must built a new model and analyse it in a way similar to the one used by Arrow (1951) to generalize Leontief's model. Samuelson (1951) used instead a different approach in order to solve the very same problem. Notwithstanding the difference of topic between Samuelson's and Arrow's articles on one side and Morishima Thompson and my analysis on the other we may use the same approach which Arrow used in the analysis of

externalities.<sup>(1)</sup>

In this way we can make an explicit reference to produced externalities, which are joint outputs of some processes and whose level of production in the previous period determines the intensities of production in the present one. Thus the external diseconomies problem, which remained somewhat hidden in the Morishima Thompson treatment, may now be explicitly analyzed.

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(1) This comparison was suggested to me by prof. Morishima, and can be helpful in resuming the differences of our approach.

### 3.4 Introduction to an alternative approach.

We have argued that the most general and correct treatment of externalities in the von Neumann regime still seems to be the article by Morishima and Thompson of 1960. Nevertheless their model has many limitations. For example they are concerned only with the case classified by Meade (1952) as external economies due to the "unpaid factors" and do not discuss fully the problem of the maximal rate of growth. In the recent literature these undervalued problems are considered to play a relevant role even in many environmental application.<sup>(1)</sup> Furthermore Morishima and Thompson do not discuss the problem of efficiency. In what follows we re-examine the entire problem of external economies in the context of the von Neumann model, referring to the two cases distinguished by Meade (1952). Being as faithful as possible to the spirit of the original von Neumann model, we reformulate the problem from a different analytical point of view.<sup>(2)</sup>

The substantial difference between the Morishima-Thompson approach and ours lies in that we treat the

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{1} Cfr. A. Sandmo (1978) p. 176.

{2} The present analysis is related with the type of approach used by Arrow (1951) [while the one followed by Morishima-Thompson may recall Samuelson (1951)] in his alternative proof of the substitution theorem for Leontief models. In fact like Arrow and Koopmans, we will not require that "the alternative processes available to each industry can be subsumed in a production function possessing derivatives" Koopmans (1951) p.147.



external economies which a given industry produces through its production processes explicitly, as outputs of distinct commodities, which are different from the usual output of private goods.<sup>(3)</sup> We consider such an approach as a meaningful extension of the von Neumann model and believe that it would be useful, in order to scrutinize the substance and the implications of the externality.

#### A. Meade's classification of externalities

According to Meade<sup>(4)</sup> external economies are due to:

- (1) the "creation of atmosphere" by an industry,
- (2) the existence of an "unpaid factor".

The essential difference between the two seems to be that the effects of external economies on a given industry do or

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{3} If externalities arise from specific inputs, but the effects of those inputs depend on other inputs with which they are combined and on the processes used, then our approach dealing explicitly with the different processes may be closer to reality and more useful. On the other hand even if in reality we have differentiable production functions similar to the ones described by Morishima and Thompson a linear approximation may still be useful since it enables us to take advantage of using the method of linear programming.

{4} This distinction may be of some use to optimal taxation analysis, Cfr. A. Sandmo (1978) p.176. Meade's article is recalled also in Mishan (1971).

In the first (atmosphere) case "there are constant returns in each industry to those factors which it controls and pays for, but ... there are no constant returns for the two industries taken together, the scale of operations being important in the one industry because of the atmosphere which it creates for the other." In the second (unpaid-factor) case "there are constant returns in society but not ... in each industry to the factors which each industry employs and pays for" Meade (1952) p.57.

do not depend upon its scale of operation.<sup>(5)</sup>

In the "creation of atmosphere" case the conditions which affect the output of a given industry are independent from its production scale. In the "unpaid factor" case the effects on the production function are smaller when the scale of operation of the given (affected) industry increases.

Let us first consider Meade's two industry model in which externalities, produced by the first industry, affect the output of the second industry. Write  $x_1$  and  $x_2$  for the products of industry 1 and industry 2 respectively. The two factors  $l$  and  $c$ , or labour and capital, employed in both industries are assumed to be fully employed:  $l_1 + l_2 = l$  and  $c_1 + c_2 = c$ .<sup>(6)</sup>

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{5} Cfr. Meade (1952) p.61. In the meantime this terminology has been applied to public inputs and subjected to criticism. For Negishi (1979) the unpaid factor case corresponds in reality to the case of the free supply of factors, in which free factors are allocated between users being unavailable simultaneously to several industries; while for him the relevant feature should be that the unpaid factors enter several production processes at the same time. Kohli (1985) more correctly points out that this label is not fully appropriate because it emphasizes the fact that these factors are supplied free of charge, rather than the fact that their effects are "public". He also states: in the unpaid factor case "the input is public between industries, but not within industries, while in the second case [the atmosphere case] the input is public for use by any private factor". We will come back later to this.

{6} Cfr Meade (1952) p.54.



$$\begin{aligned} \text{Let} \quad x_1 &= H_1 (l_1, c_1) \\ x_2 &= H_2 (l_2, c_2; l_1, c_1, x_1) \end{aligned} \quad [1]$$

where  $H_1$  is a homogeneous function of the first degree<sup>(7)</sup>, expressing the fact that of constant return to scale.

In analytical terms we may say that we have the "creation of atmosphere" when  $H_2$  is homogeneous of the first degree in the variables  $l_2$  and  $c_2$ ; while we have the unpaid factor case when  $H_2$  is an homogeneous function of the first degree in all variables,  $l_2, c_2; l_1, c_1, x_1$  on which  $x_2$  depends. It is obvious that the scale of operation of industry 2 is relevant only in the second case.

#### B. A possible subdivision of the atmosphere case

Where we examine the nature of external economies due to the creation of atmosphere it may be useful to distinguish two further categories:

- a) local (or relative) atmosphere related to a given relative production structure of the economy, whatever the absolute level of output and the input of the entire economy may be,

-----  
 {7} Let  $f$  be a scalar field (real valued function of a vector variable) defined on  $R^n$ .  $f$  is said to be an homogeneous function of degree  $p$  over  $S$  if  $t^p f(\underline{x}) = f(t\underline{x})$  for any  $t > 0$  and every  $\underline{x}$  in  $S$  for which  $t\underline{x} \in S$ . Let  $\underline{x} = (\underline{y}, \underline{z})$ ;  $f$  is said to be homogeneous of degree  $p$  in the variables  $\underline{y}$  over  $S$  if  $f(t\underline{y}, \underline{z}) = t^p f(\underline{x})$  for every  $t > 0$  and every  $(\underline{y}, \underline{z})$  in  $S$  for which  $(t\underline{y}, \underline{z}) \in S$ . The previous statement may eventually be extended if necessary to vector fields (vector-valued functions of a vector variable).

- b) global (or absolute) atmosphere depending on the absolute level of output and/or input of one or more industries.

The first case is obtained in an economy which consists of a number of distinct (even if structurally identical) disconnected economic units (islands), which grows by aggregation of new units, while the second case corresponds to an entirely connected economy. Clearly, like Meade (1952), we cannot claim that this subdivision of the atmosphere into local and global is logically complete. In the study of externalities, however, it should be useful to abstract, at first, from economies of scale that are related with the growth of the economy. This abstraction would be satisfactory for a broad range of the levels of operation of the economy within which the change in the global atmosphere may have no significant effect on the production function.

We may study the case of local atmosphere and assume that  $H_2$  is a homogeneous function of the first degree in order to remain in the context of the von Neumann model. In this way we introduce in the model the external economies due to the creation of local atmosphere also.

### C. Some examples

Let us illustrate some of the types of external economies and diseconomies in terms of a simple model.

Confining ourselves to the von Neumann dynamic model

and assuming that the input and output coefficients are constant, we may simply refer to the intensities of the production processes.

Like Morishima and Thompson, we may assume that the current output of the  $i$ -th process depends on the activities of all the processes in the previous period.

Let us examine first the simple case of two private goods, labelled 1 and 2, produced by two industries  $V$  and  $Z$ .

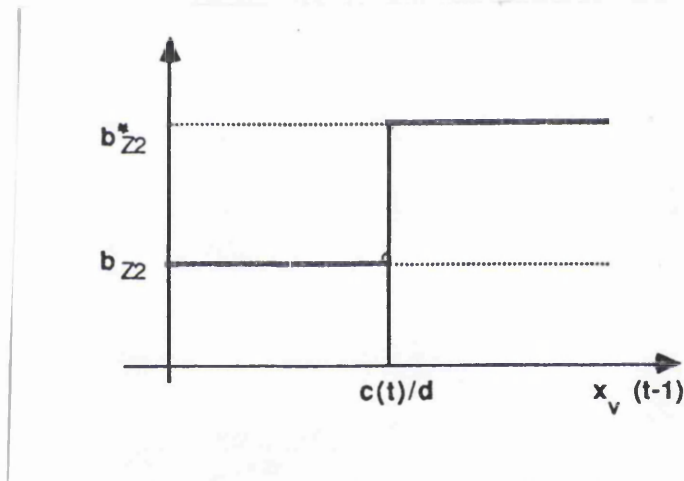
$V$  transforms the bundle of goods  $\underline{a}_v = (a_{v1}, a_{v2})$  into  $(b_{v1}, 0)$ , while  $Z$  has two processes  $z$  and  $z^*$  which transform  $\underline{a}_z = (a_{z1}, a_{z2})$  into  $(0, b_{z2})$  or  $(0, b^*_{z2})$ , respectively. Clearly, where  $b^*_{z2} > b_{z2}$ , process  $z^*$  always dominates  $z$ , as it is more profitable than  $z$  at all positive prices. Let  $x_v(t)$ ,  $x_z(t)$ ,  $x^*_z(t)$  be the intensities of the three production processes  $v$ ,  $z$ ,  $z^*$ , respectively, and let us examine the case when the  $V$ -th industry is the one generating externalities.

#### (1) The "creation of atmosphere" case.

Let  $d > 0$  be the output of the third good "externality" generated by process  $v$  at the unit level of activity.

In the case of an external economy being present, we may assume that the most profitable process  $z^*$  can be operated when and only when the output of externality (commodity 3) of the previous period,  $x_v(t-1)d$ , is not

less than the minimal operative level,  $c(t) > 0$ , i.e.  $x_v(t-1) d \geq c(t)$ . Figure 1 represents the level of output of industry Z, corresponding to the input  $a_z$  as a function of the level of intensity of process v in the previous period. Where  $x_v(t-1) < c(t)/d$ , we can only operate process z, so that the level of output is  $b_{z2}$ , while it is  $b^*_{z2}$  for  $x_v(t-1) \geq c(t)/d$ .



Thus whether the Z industry can use process  $z^*$  or not it depends on the other V industry's activity in the past. The critical point  $c(t)/d$  would move leftwards or rightwards where  $c(t)$  changes in the passage of time. We may assume that the rate of growth of  $c(t)$  is equal to the equilibrium rate of growth of production:  $c(t) = c \alpha^t$ . Then, as we have  $x_v(t) = x_v \alpha^t$  in the state of balanced growth, we may write the condition for process  $z^*$  to be operational, in the form  $x_v d \geq c$ . In this case the critical point,  $c/d$  is fixed.

On the other hand in the case of an external diseconomy the most profitable process  $z^*$  can be operated only where the output of the externality (commodity 3) of the previous period,  $x_v(t-1)d$ , does not exceed  $c(t) > 0$ , the maximal operative level (or the lethal level) of output 3 for process  $z^*$ . Thus  $x_v(t-1)d \leq c(t)$ .

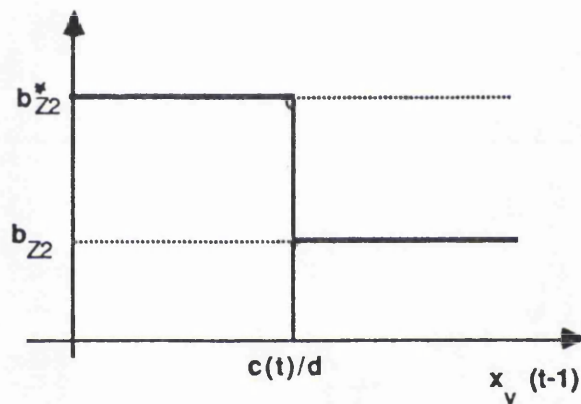


Figure 2 represents this case. When input  $a_z$  is available, good 2 is produced up to a quantity of  $b_{z2}^*$  as long as  $x_v(t-1)d < c(t)/d$  but it decreases to the level  $b_{z2}$  afterwards.

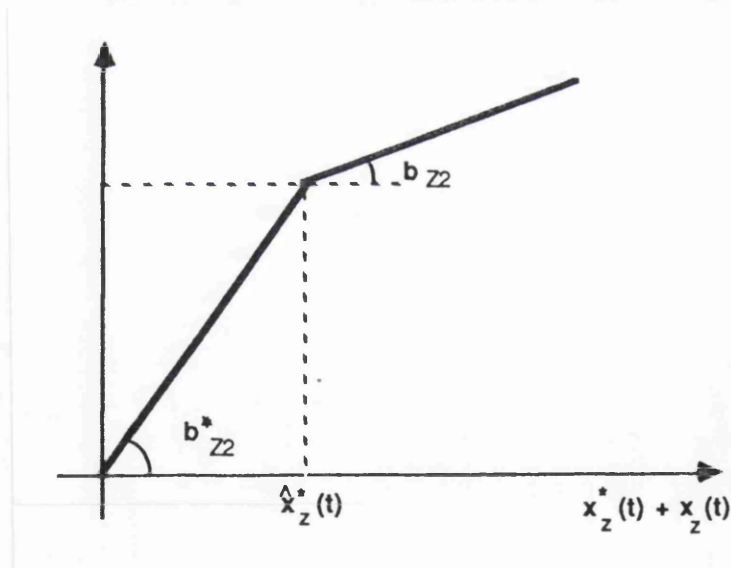
## (2) The "unpaid factor" case.

Let  $d (> 0)$  be the output of the third good "externality" produced by process  $v$  when it is operated at the unit level of activity. We assume that process  $z^*$  (whose intensity is denoted by  $x_{z^*}$ ) could be operated only if  $x_v(t-1)d \geq x_{z^*}(t)c$ , in other words, the output of



commodity 3 obtained from the V industry's activity in the previous period,  $x_v(t-1)d$ , is not less than the amount of it needed by the Z industry,  $x_z^*(t)c$ , [where  $c (>0)$  represents now the input of commodity 3 required per unit operation of process  $z^*$ ].

Process  $z^*$ , consequently can be operated at any intensity level not exceeding the critical level  $\hat{x}_z^*(t) = d x_v(t-1)/c$ , where  $d x_v(t-1)$  is the maximum amount of the available externality. It is clear that when this level is reached, the Z industry can only use inferior process  $z$ , so that output of good 2 increases more slowly than it does while process  $z^*$  is utilized.

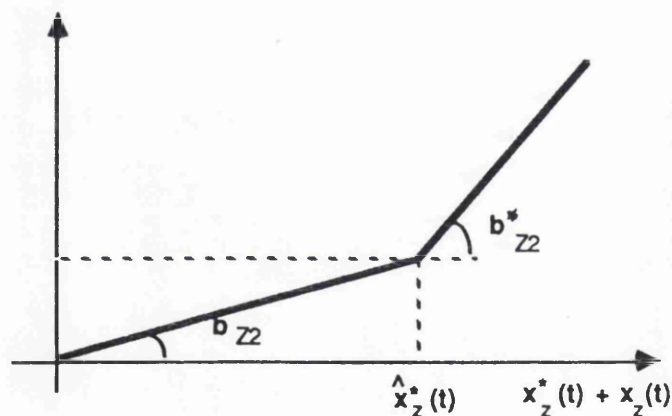


In figure 3 the total output of good 2 is represented as a function of the total activity level,  $x_z(t) + x_z^*(t)$ , of the Z industry. (It is noted that input  $\underline{a}_z$  is fixed in drawing figures 1 and 2, while input is proportional to output,  $\underline{a}_z[x_z(t) + x_z^*(t)]$ , in figure 3 and 4 below.) It



is also noted that in the von Neumann equilibrium where the rate of growth is equal to  $\alpha - 1$  the condition for process  $z^*$  to be operational is given as:  $x_v d \geq \alpha x_z^* c$ .

In the case of an external diseconomy process  $z^*$  becomes operational, after bads produced by  $V$ ,  $x_v(t-1) d$ , have entirely been cleared. While bads remain,  $Z$  industry has to employ process  $z$ , which clears bads, or absorb commodity 3, by the amount of  $c (> 0)$  per unit operation of process  $z$ . Then the employment of  $z^*$  starts after  $x_z(t)$  reaches to a level such that  $x_z(t)c = d x_v(t-1)$ .



Let  $\hat{x}_z(t) = d x_v(t-1)/c$ . Then, as is shown in figure 4 the production curve of industry  $Z$  traces out a curve which has as 'kink' at the critical point,  $\hat{x}_z(t)$ .

### 3.5 Bads and externalities: a further consideration.

In what follows we are concerned with the problem of bads which affect production activities. We suppose that they are generated, as joint output, by different processes of production. In the private market economy they are ignored and the full cost of production of goods does not take damages caused by bads into account. Almost all processes produce bads and suffer damages from them, but they are neglected.

In perfect competition, the output price is calculated by the producers, strictly from the viewpoint of private cost accounting. It does not include the social cost of the damage, inflicted on the rest of the economy, due to the previous running of the given process of production. Thus the problem of bads arises when they affect not only their productive processes but also most other processes. For this reason we do not consider bads as private goods and view them as local external diseconomies.

#### A. The nature of the problem

It is very difficult, first of all, to evaluate the damage and even more difficult to attribute it to the respective, responsible processes. In general this of course depends on the interactions and interdependencies of different individual economic activities, as well as on the type of economic models, which are used to analyze this problem.

First we must discover where bads are produced as joint outputs; we must also know in what quantities these bads are produced per unit activity of each process. Then we must determine the 'lethal' level of each process of production, that is, the maximum level of bads at which the process is operational. When the public sector confines itself to the activity of legal enforcement, this identification and demonstration process is carried out by the damaged private producer in order to claim a refund of the damage. The victim must identify the harmful externality and prove that his damage is a consequence of some bads, produced by a given process. This task is quite difficult and expensive, because any private has no authority to get any information from the supposed polluter. Here there may be a possible role for the public sector.

For instance there was no general knowledge of the existence and harmfulness of dioxine before the case of Seveso. Furthermore the fact that the strength of the bads, jointly produced by waste disposal processes, depends on the combustion temperature was also unknown some years ago (we have many different type of dioxine). Different levels of production of these bads may affect cattle production processes, milk production processes and even the health of human beings.

Furthermore, apart from the producers affected, a general constraint on the production level of such bads

would be required for health reasons, according to public demand. Because of this and other considerations (e.g. the free rider problem), it is likely that no private agreement will be reached between different producers.

Even a public intervention would not be completely satisfactory because the gain to be expected for the economy as a whole may be lower than high administrative costs involved.

### B. The nature of the model

In tackling the problem of bads in the framework of von Neumann-like models we must set some limits to the range of phenomena and on the ways of describing them in terms of assumptions imposed on the model. In what follows we examine a case of local atmosphere on the assumption that the economy is a mixed one.

We show that some intervention by the public sector may be necessary to ensure the existence of an equilibrium of the market economy, in the case where public intermediate commodities are creating a negative atmosphere (bads). We assume that the public sector has complete and detailed information on the technology, so that it knows the input and output coefficients of every single process. In the model its role is limited to controlling the intensity vector of processes. This can be done by imposing taxes on some sectors and granting subsidies to others, in order to make processes with a relatively high output of bads less profitable and processes producing a relatively

small amount of bads more profitable. This of course assumes that the government has the ability to impose and to collect taxes.

The solutions of this constrained von Neumann equilibrium model indicate which processes should be subsidized and which ones should be banned. For technical reasons the government may establish a public enterprise which is responsible for running some processes. In this case subsidies should be interpreted as the financial support of the enterprise from the public sector budget, and taxes as additional profits transferred from the enterprise to the public sector budget.

In a sub-system of the economy which is obtained by ignoring some processes (such as those which are forbidden to run) we may find a constrained equilibrium which is a private market equilibrium, that is, a state of affairs where only private sector units are active. In this case all the other processes of the economy that are ignored should not even be run, if available, because they are either unfeasible or unprofitable given the equilibrium prices and the equilibrium levels of bads. From the viewpoint of a mixed economy this is seen to be equivalent to an equilibrium with zero taxes and zero subsidies. It is possible to have no private marked equilibrium, but it always exists a constrained equilibrium which is a mixed economy equilibrium.

In the mixed economy, if all constrained equilibria



can be attained then all private market equilibria can. It is, therefore, possible that even though a private equilibrium exists, a non-private equilibrium is the one which maximizes the rate of growth; in this case the public sector clearly contributes to the increase in the rate of growth of the economy.

### C. A simple model with a single lethal vector

Let us examine a simple model with bads of a temporary nature. In each period the amount of bads which was at the beginning disappears at the end. To simplify the matter we assume that the maximum permissible level of bads is the same for all production processes.

Let  $n$  be the number of goods,  $p$  the number of bads which create a negative atmosphere and  $m$  the number of production processes. The  $i$ -th process converts the bundle of goods  $\underline{a}_i^*$  into the bundle of goods  $\underline{b}_i^*$  and the bundle of bads  $\underline{b}^*_{i^*} = (b_{i^*1}^*, \dots, b_{i^*p}^*)$ . Each process can operate at a positive level  $x_i > 0$  in period  $t$ , only if the amount of bads produced in the previous period and present in the economy is less than or equal to  $\underline{a}^* = (a_1^*, \dots, a_p^*)$  the maximum level of bads permissible in the economy provided that  $\underline{x} \underline{f} = x_1 + \dots + x_m = 1$  in the previous period. The intensity level is normalized, like in Morishima and Thompson, such that it corresponds to the required input of workers.

Let  $B^* = (\underline{b}_{ij}^*)$  be the bad-output matrix of the economy and  $A, B$  the usual goods input and output matrices;



as usual  $a_{ij}$ ,  $b_{ij}$ ,  $b_{ij}^*$  are non-negative constants. We assume that  $\underline{a}^*$  would increase in each period at a rate which is equal to the actual rate of growth of the economy as was explicitly stated previously. This would be intuitively acceptable.<sup>(1)</sup>

Consequently in this simplified version the amount of bads present, at the beginning of period  $t$ ,  $\underline{x}B^*$ , which is determined by the output of bads produced in the previous period, is required to be less than or equal to the lethal vector  $\underline{a}^*$ . Hence on top of the original von Neumann constraints  $\underline{x}(B-\alpha A) \geq \underline{0}$  and  $(B-\alpha A)\underline{y} \leq \underline{0}$  this model must satisfy:

$$\underline{x} B^* \leq \underline{a}^* \quad (\text{lethal level constraint}) \quad [a]$$

too, in order to establish an equilibrium intensity vector  $\underline{x} \geq 0$ .

This case seems not very interesting because the lethal vector is equal for all processes and each industry is just able to run one basic production process. Nevertheless it is important to notice that the new constrained von Neumann equilibrium corresponds to the private market equilibrium only when the constraint [a] is already satisfied by the original (non-constrained) von

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(1) Where the maximum level of bads  $w$ , contained in grain output of period  $t-1$  is  $a_w^*$ , then in the following period  $t$  when grain output has been increased by  $\alpha-1$  the lethal amount should be  $\alpha a_w^*$ , in order to maintain the same amount of bads  $w$  per unit of grain.

Neumann equilibrium solutions.

When this condition is not satisfied by the original equilibrium intensity vector  $\underline{x}^0$  it is necessary -in order to reduce the output of bads to the level compatible with the running of the economy- to reduce the relative intensities of some processes and increase those of some others. The constrained von Neumann economy can be formulated in the form of linear programming problems. Let  $\underline{f}$  and  $\underline{e}$  be, as usual, the  $m$  by  $1$  and  $1$  by  $n$  unit vectors; then the two dual linear problems may be written as follows:

$$\begin{array}{ll}
 \min v & \max -u - \underline{a}^* \underline{y}^* \\
 \underline{x}(B - \alpha A) + v \underline{e} \geq \underline{0} & (B - \alpha A) \underline{y} - u \underline{f} - B^* \underline{y}^* \leq \underline{0} \\
 -\underline{x} \underline{f} \geq -1 & \underline{e} \underline{y} \leq 1 \\
 -\underline{x} B^* \leq -\underline{a}^* &
 \end{array}$$

As  $\underline{x} \underline{f} \leq 1$ , then we may reformulate the last constraint of the minimization problem as  $\underline{x}(\underline{f} \underline{a}^* - B^*) \geq \underline{0}$ , taking into account the fact that the lethal vector grows at the same rate as intensity vector.

The previous problem has one optimal solution identically zero (even if economically meaningless) being  $v = -u - \underline{a}^* \underline{y}^* = 0$ . Consequently  $v = u = 0$ ,  $\underline{y}^* = \underline{0}$  are necessary conditions for all optimal solutions. Clearly where the unconstrained solution  $\underline{x}^0$  satisfies the additional constraint [a], so that  $\underline{y}^{*0} = \underline{0}$ , then unconstrained solutions  $(\underline{x}^0, \underline{y}^0)$  with  $\underline{y}^{*0} = \underline{0}$  establish a constrained equilibrium which is in turn identical with the

von Neumann equilibrium and the condition [4] is not a binding constraint.

#### D. A comparison with the Morgenstern-Thompson constrained economy

In order to connect the present model with the one of Morgenstern and Thompson of par. 1.3, let us reformulate the previous linear programming problems as:

$$\begin{array}{ll}
 \min \quad \underline{w}^i \underline{y}^i - \underline{w}^e \underline{y}^e & \max \quad \underline{x}^m \underline{p}^m - \underline{x}^M \underline{p}^M \\
 \underline{x}(B-\alpha A) - \underline{w}^e + \underline{w}^i = \underline{0} & (B-\alpha A)\underline{y} + (\underline{f}\underline{a}^* - B^*)\underline{y}^* + \underline{p}^m - \underline{p}^M = \underline{0} \\
 \underline{x}(\underline{f}\underline{a}^* - B^*) \geq \underline{0} & - \underline{y} \leq -\underline{y}^e \\
 \underline{x} \geq \underline{x}^m & \underline{y} \leq \underline{y}^i \\
 - \underline{x} \geq \underline{x}^M &
 \end{array}$$

Where  $\underline{x}^m = \underline{0}$ ,  $\underline{x}^M = \tau \underline{f}$ ,  $\underline{y}^e = \underline{0}$ ,  $\underline{y}^i = \tau \underline{e}$ .<sup>(2)</sup>

The process  $i$  produces bads of the value  $\underline{b}_i \star \star \underline{y}^*$ , when it is operated at unit intensity. The difference  $p_i = (\underline{a}^* - \underline{b}_i \star \star) \underline{y}^*$  represents the opportunity cost for the economy of running the  $i$ -th process at a unit intensity. The amount  $x_i p_i$  is paid by the  $i$ -th process when  $p_i < 0$ , otherwise it is paid to the  $i$ -th process.

The solutions  $\underline{x}'$ ,  $\underline{y}'$ ,  $\alpha'$  to the above constrained von Neumann system, can be implemented by the public sector through the original Morgenstern-Thompson linear programming model.

(2) These problems, as well as the previous ones, have one set of optimal, but economically meaningless, solutions.

This is done by putting  $x_i^M = x'_i$  for those  $i$  for which  $-p_i^M = p'_i < 0$  and  $x_i^M = x'_i$  for  $i$  with  $p_i^M = p'_i > 0$ . The government then levies taxes and grants subsidies to each process depending on the negative or positive value of  $p'_i$ .

The solutions to our minimization problem  $(\underline{x}', \underline{w}^e', \underline{w}^i')$  are also solutions to the original Morgenstern-Thompson model, and as  $\underline{w}^i = \underline{0}$  and  $\underline{y}^e = \underline{0}$ , the solutions are optimal; being the value of the objective function zero. Consequently the corresponding solutions to the Morgenstern-Thompson dual problem,  $(\underline{y}', \underline{p}^m', \underline{p}^M')$ , attain the maximum  $\underline{x}^m' \underline{p}^m' - \underline{x}^M' \underline{p}^M' = 0$ . This guarantees a balanced budget, the public sector budget being the same in the two models. We thus have found an economic reason for the fact that the public sector imposes some minimum and maximum intensity levels to some of the processes of the economy and obtains their optimum value by solving the programming problem.

### E. Introducing alternative processes

A first generalization of the previous model may be obtained by introducing alternative processes which are not affected by the level of bads. These alternative processes would have the lethal level of bads which is infinity, and, therefore the corresponding lethal vector  $\underline{\tilde{a}}^* = \infty$ .

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(3) Cfr. Ch.1 par.2. above

Let 'A, 'B be the input and output matrices only of these alternative processes. In what follows we call the economy which operates only these processes the alternative von Neumann economy, while the model with all the processes A, B, B\*, a\* is called the general von Neumann economy. The bads output matrix 'B\* of the alternative economy, always satisfies the lethal level constraint because  $\hat{a}^* = \infty$ . But if

$$'x \ 'B^* \geq \underline{a}^* \quad [a']$$

the other processes are infeasible. Clearly, when the condition [a'] above is not satisfied, the alternative economy does not provide a private market equilibrium for the general economy if more profitable processes are feasible.

It is very reasonable to assume that these alternative processes are no longer profitable when the other processes with  $\underline{a}^* < \infty$  are feasible (as shown in fig.2 of section 3.4). However if we assume that the equilibrium of the original von Neumann economy with matrices A, B containing all the processes does not satisfy constraint [a], then in equilibrium a pure private economy would run only alternative processes and the alternative von Neumann equilibrium would prevail when condition [a'] is satisfied.

#### F. Some possible comparisons

In the framework of this simplified model the



comparison (between the general and the alternative economy), originally made by Morishima and Thompson, implies that they set  $B^* = 0$  (for the general economy) so that each industry can run the more profitable processes. It is obvious that the rate of growth they obtain in this way will be greater or equal to the one of the alternative economy, the latter being contained in the general economy (provided negative externalities (bads) are produced). When externalities are positive, the opposite result is of course obtained.

Clearly the private economy equilibrium is obtained, even though a public sector is present, if it does not place on the private sector any restriction which the economy does not satisfy automatically.

It is clear that the economy with public sector intervention can always attain the rate of growth of the alternative von Neumann model so that we may call  $\alpha^{-1}$  the minimum rate of growth. This is easily demonstrated by referring back to the Morgenstern and Thompson linear programming model, examined in 1.3 above. Where the alternative equilibrium is the only pure private equilibrium, we obtain it in the general economy by setting the maximum intensity level  $x_i^M$  at zero for all processes with  $a_i^* < \infty$ , though with public intervention it may be possible to increase some of these  $x_i^M$  to a strict positive level. With increasing some of these maximum intensity levels the rate of growth for the economy



would not decrease -it would either remain unchanged or be increased- because some of the more profitable processes now become available. In this way we have demonstrated that by imposing taxes and granting subsidies the public sector can eventually lead the economy to an equilibrium state where rates of growth and of interest are greater than in the pure competitive economy.

#### G. The general case

It is easy to generalize further the model, following the previous line of reasoning. We may allow each industry to run a finite number of basic production processes. Then the  $i$ -th process may be operated at a positive level  $x_i > 0$  during period  $t$ , if and only if the amount of bads produced by the economy in period  $t-1$  is less than or equal to  ${}^i \underline{a} = (a_1^i, \dots, a_p^i)$  which represents the maximum level of bads-process  $i$ .

We now have a finite number of different lethal bads vectors, which is less than or equal to the number of processes; process  $i$  is operable if and only if condition

$$\underline{x} B^* \leq {}^i \underline{a} \quad [b]$$

is fulfilled. Then the problem of obtaining an equilibrium becomes more complicated than in the previous case but it is possible, like in the model with only two maximum levels of bads, to refer back to the original model with only one constraint.

Let us examine the case where process  $q$  is operable, with  $\bar{a}_q < \infty$ . We can easily find all the processes that are not operable when the amount of bads present in the economy is not less than  $\bar{a}_q$ . These are excluded from the input and output matrices, and new matrices are formed. Let  $\{k : k_{\bar{a}} \geq \bar{a}_q\}$  be the set of indices of production processes that are included in the new matrices  $\bar{A}$ ,  $\bar{B}$ ,  $\bar{B}^*$ . Clearly all these processes, included in these  $q$ -subsystem matrices are operable whenever:

$$\bar{x} \bar{B}^* \leq \bar{a} \quad [c]$$

In this way we go back to the initial model with  $\bar{a}^* = \bar{a}_q$ .

This procedure may be used for all production processes with a different lethal bads vector. There are  $w$  ( $\leq m$ ) different economic subsystems which have distinct feasible areas in the  $p$  dimensional space  $R_+^p$ . The space itself is also a feasible area because the alternative processes are always feasible ( $\bar{a}^* = \infty$ ).

It is important to specify the conditions under which the equilibrium of the present model is reduced to the one of a pure private economy. Suppose that the  $q$ -th economic subsystem has an unconstrained von Neumann equilibrium that satisfies the constraint [b]. This is reduced to an equilibrium of a private economy if the processes that are not present in the output and input matrices of the  $q$ -th economic subsystem are either not feasible, because  $\bar{x} \bar{B}^* \leq \bar{a}$  is not satisfied for any  $w \in \{k : k_{\bar{a}} \leq \bar{a}_q\}$  is

not satisfied) or if they are feasible they are not profitable at the equilibrium prices.

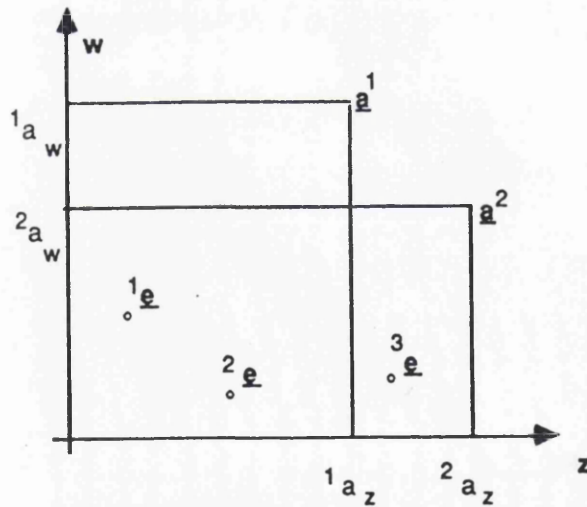
Unfortunately there is no reason for the existence of a general private equilibrium. However, a private equilibrium can always be attained either in a planned economy or by the public sector's intervention, as it was in the previous model.

Furthermore all constrained equilibria can be achieved by the public sector's intervention so that an equilibrium always exists for the planned or the mixed economy. In the case of existence of many equilibria the public sector may of course choose among them.

What has been said may be represented graphically in the case of two bads  $w$  and  $z$ . It is possible to draw the feasibility area in a cartesian plane. Let us assume that there are three maximum levels of bads  $\underline{a}^1, \underline{a}^2 < \underline{a}^3 = \infty$ .

We may distinguish between three economic sub-systems. It is clear that the processes of the  $i$ -th sub-economy are feasible when condition [c] is satisfied. Let  ${}^i \underline{e}$  and  ${}^i \underline{y}$  be the bads-output and price vectors, respectively, of the unconstrained von Neumann equilibrium which coincides, in our case, with the constrained equilibria of the  $i$ -th economy. Let also  $\underline{e}$  and  $\underline{y}$  be the output and price vectors of the unconstrained von Neumann equilibrium of the general economy.

In this case, as is shown in the figure, all these bads-outputs do not exclude, as infeasible, all the processes that are not present in their respective sub-system. Consequently if any of the feasible processes, not contained in the  $i$ -th sub-system, yields a positive profit at prices  $^1y$ , then the  $i$ -th equilibrium is not a private equilibrium of the general economy. In this case, however, if we change  $^3e$  for  $^2e$  then the new  $^2e$  will represent a pure private equilibrium, in which only those processes (which are present in the second sub-system) are feasible.



#### 4. DE VITI DE MARCO AND THE THEORY OF PUBLIC COMMODITY

##### 4.1 De Viti De Marco's theory

In modern economies a substantial share of the national product is absorbed by the public sector budget and the activities of the public sector affect equilibrium of the private sector in many ways.<sup>(1)</sup>

The history of the analysis of the peculiar nature of public goods and that of the conditions for equilibrium of the public sector's revenue-expenditure process may be traced back, at least, to the end of the nineteenth century.<sup>(2)</sup> The theory of public goods, was first formulated as part of a theory of the public sector (operational rather than normative), in the 1880's by such economists as Pantaleoni, De Viti De Marco, Mazzola and others.<sup>(3)</sup>

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(1) For a survey of expenditure and taxes in some European economies see Vagliasindi (1983).

(2) E.g. the indivisibility as well as the non-excludability properties of public goods with the consequence of a different pricing rule were discussed in 1880 by Mazzola. The unified character of the public sector revenue-expenditure problem, and the necessity to add the public sector to the Walrasian system, were discussed by Pantaleoni in 1883. A similar view, that the theory of Public Finance is an integral part of the economic equilibrium theory, was also systematically investigated in 1888 by De Viti De Marco. In the following we refer to the 1936 translation of his work.

(3) However Samuelson's recent mathematical reformulation (1954) was purely normative and only dealt with the conditions of Pareto optimality concerning 'collective consumption' goods. These goods were defined by



Many criticisms are made against these 19<sup>th</sup> century economists by Samuelson and Musgrave; but it seems to me that most of them are unfair, at least as far as De Marco is concerned.<sup>(4)</sup> I believe that his view of the nature of public goods and the economic activities of the public

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Samuelson as the goods whose quantity "enters two or more persons 'utility'" and "production-possibility frontier and industry production functions in the standard way", as he explicitly stated in his 1969 article. Thus Samuelson's social goods include the pure public consumption goods, as well as other externalities, which cannot be optimally handled by the market, even if it could deal with public consumption goods.

(4) For instance in Musgrave (1985) we find: "By framing the efficiency rule in terms of benefit taxation, attention was diverted from specifying just how indivisibility affects efficiency condition, condition which may be met with or without benefit finance... Moreover, by focusing on the benefit rule as an analogue to market pricing attention was diverted from the political, non market process needed to reach an efficient solution." From his account it seems also that the Italian scholars have ignored the distribution and preference aspect of the problem as well as the examination of the process which would, in reality, lead to an approximate optimal solution, attributed to Wicksell (1896), while the relevance of externalities and mixed goods as well as the analysis of coercion and its cost is attributed to Pigou (1928), (1932).

We do not aim to defend earlier formulations of fiscal doctrine, rediscovered by Musgrave (1938), nor to become involved in discussing the details of De Viti De Marco's analysis. However we hope that it will become clear that his emphases placed on the "exchange relationship in the production and distribution" of public goods and on the attribution to fees and taxes of the same functions as prices (p.111) had been a reaction against the total lack of public expenditure analysis in traditional economic. Furthermore his theory would be vindicated by our use of the von Neumann model. Even in the 'pure theory' of a cooperative regime he is concerned with the state, its political constitution and decision process; and, in general, his analysis is profoundly rooted in history. Finally the free-rider problem is examined as one of De Viti De Marco's two fundamental axioms, and his treatment of the distribution problem, of public goods, mixed goods and externalities is explicit.



sector may be useful for the construction of a model.<sup>(5)</sup>

In the following we shall refer to the English translation of De Viti De Marco (1932), *FIRST PRINCIPLES OF PUBLIC FINANCE*, which contains the final, most refined exposition of the 1888 essay.

#### A. The provision of public goods and private production

De Viti De Marco emphasized how the public sector "takes part in exchange by buying and selling private goods". It transforms these "and personal services into public goods", which in turn "come to influence production, exchange and consumption of private goods". In fact "the equilibrium of production and of exchange as between private individuals is different according as one has a good or bad set of roads, an efficient or an inefficient system for the defence of property, a protectionist or liberal economic policy, and so on".<sup>(6)</sup> Consequently the

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(5) The lack of a general model in the theory of the public sector's economic behaviour is recognized by contemporary students of finance. For instance, at p.205 of the *HANDBOOK OF PUBLIC ECONOMICS*, BOS (1985) says: "- We still lack economic and political theories of (de)nationalization and (de)regulation. Comparing the efficiency of private and public sector (including regulated enterprises) may indeed give some hints about the reason for such decisions. Ideological arguments may also be presented. Yet all this is far from being a theory which integrates property rights problems, lack of competition, different objectives of public authorities and further relevant economic and political determinants in some model of sufficient generality" "- The usual models are typically restricted to static analysis".

(6) Cfr. De Viti De Marco p.51. Furthermore he tries to show when this final effect of the public sector on the private process of production may be ignored for

theory of the public sector must deal with the "theory of production and consumption of public goods". It must investigate De Viti De Marco says "the conditions to which the productive activity of the state must be subjected in order that the choice of the public services which are to be produced, the determination of their respective amounts, the distribution of the cost among consumers, etc., may take place ... with the least possible waste of private wealth, in order to attain the greatest satisfaction of collective needs."<sup>(7)</sup>

Collective wants arise from "the very fact of social life" and from "a certain 'conflict' of interest between the groups that make up the national and international social structure" p.38. This depends on the fact that individuals differ in their estimates of the utilities of these goods.<sup>(8)</sup>

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methodological reasons. He states: "one may abstract from the service of providing public security if one assumes ... that the amount of security provided is equal for all private productive enterprises that exchange their products" and that they "utilize public services in" the same degree. See op. cit. p.52.

(7) See op. cit. p.36

(8) He classifies, on p.40, collective wants into three groups:

(1) original collective wants "the satisfaction of which represents a function that has long been necessary in all states" (e.g. defence of the territory),

(2) transformed private wants "which were originally individual wants, but which had been modified by the fact that people live together" (e.g. public health)

(3) private wants with a collective element (e.g. "economic defence against private monopoly").

In De Viti De Marco's view public finance studies economic activities of the state and in particular "the productive activities which are directed towards the satisfaction of collective wants" p.34. They are useful to "individuals who make up the social organism" and "represent the factual presupposition underlying the problems of Public Finance", and must "be analyzed for what they are".<sup>(9)</sup>

From the hedonistic and 'egoistic' premise he derives:

"(a) that members of society agree in desiring that public goods shall be produced according to the law of least cost",

"(b) that every citizen tends to maximize his consumption of public goods, at the same time attempting to pay the least possible amount" (the free rider problem).<sup>(10)</sup>

The fact that the "calculation of financial advantage and disadvantage is a resultant of the individual evaluations of the members or part of the members that make

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(9) Cfr, p.35

(10) Cfr. p.36. He also clearly mentions that "in reality there does not exist ... a democratic constitution in which the class that governs does not have a position of relative monopoly" p.43, and that "there is always an element of compulsion in every legal association of individuals ... and in the State, where it reaches maximum dimensions with respect to the power with which, and the period for which, the compulsion is exercised" p.50. This also emerges in the theory of the fee where he recognizes how the public sector strengthens his market "position by the establishment of a legal monopoly, forbidding and penalizing private competition", p.82, or in the decision process of the distribution of the burden of fees, p.84, and taxes, p.126.

up the political group" leads De Viti De Marco "to take account of the political constitution that is in force in a given country at a given time"<sup>(11)</sup> He examines two "extreme types", "abstract hypotheses" of the state:

(1) absolute, in which "the dominant caste has exclusive power and uses it under conditions of monopoly"<sup>(12)</sup>

(2) democratic, "where there is personal identity between producers and consumers".

In a democratic setting the production cost of 'collective wants' should be kept, if possible, "at the level at which a private enterprise may make". "For other categories the cost to the collectivity decides whether it is better that production should be carried on by private enterprise or assumed by the state".<sup>(13)</sup> These rules are enforced in a democratic state by the pressure of public

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(11) Cfr. p.41

(12) Cfr. p.42

(13) Cfr. p.47. In De Viti De Marco's opinion the other cases may be studied as a departure from the previous case.

For instance, as a first approximation, for the public sector it would be profitable to sell its patrimonial goods at a price greater than or equal to their net income capitalized at the rate of interest paid on public debt p.65. Also in a second approximation, "the repercussion which the sale may have on the general economic system", p.66, should be taken into account.

Naturally the political interests of the dominant class or of pressure-groups (army, bureaucracy, and so on) may dislike this rule and try to modify it. Nevertheless, "the fact that the state tends to specialize in the production of a given category of goods" and services, would not, according to De Viti De Marco, be altered.



opinion.<sup>(14)</sup>

In his general theory of expenditure and revenue fees and taxes play a role of "covering and distributing among the citizen-consumers the cost of production of public goods".<sup>(15)</sup>

In the case of "general public services" individual consumption is unknown. Two presumptions become the "premise necessary for the construction of the pure theory of the tax"

(i) "all the members of the community ... are consumers"

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(14) De Viti De Marco does not, however, rule out overlapping between the private and public sectors. In fact he distinguishes between "Domain goods ... the result of productive activity of the State ... intended for the direct satisfaction of collective needs" and "Patrimonial goods" (p.58) that may satisfy individual needs and are to be transformed into public goods. He also allows for mixed goods. It is necessary for him to examine the conditions that determine the division of production between public and private sector.

(15) Cfr. p.111. Fee finance is feasible when the service is "technically divisible into saleable units" and is "constantly demanded by individual" p.79 ("special public services").

The tax covers the cost of a service "which is not divisible and which is not felt desirable to divide" p.112 ("general public services").

It is the aim of a democratic state to maximize consumption of the public service and to minimize its cost of production. The cost is just covered in the aggregate by the fee. However fees can be used to distribute the cost to consumers, not evenly but rather discriminating by (as "monopoly price") in order to favour some group (distributive aim) or for efficiency reason. Surpluses are allowed in some service in order to finance others. In this case a part of fee is essentially an indirect tax. (A similar idea was later expounded by Myrdal (1929).)

De Viti De Marco has also been concerned with the problem of income discrimination. He discusses a "differential treatment" of net incomes by differentiating deductions among the various groups of individuals.



(ii) "the consumption of general public services is proportional to the income of each citizen".<sup>(16)</sup>

He shows that the distribution of the burden "will start a conflict among the very same taxpayers who have decided upon the expenditure a struggle in which each person will attempt to pay the smallest possible tax" p.126 (see premise (b) ).

#### B. The dynamic aspect of De Marco's theory.

De Marco's analysis is not purely static: the public sector "obtains its capital just as any other enterprise ... first of all, [it] has an original patrimony, from which it will draw an annual income ... it may have recourse to borrowing ... or to an annual levy"<sup>(17)</sup> Furthermore "in offering for the consumption of citizens

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(16) Cfr. p. 113 and p.116. De Viti De Marco, in viewing taxes as a 'subscription' price for the total of public services, considers that "almost all general public services are instrumental in the production and necessary for the consumption of the goods produced by the private persons" and consequently indirectly useful to all family groups p.115.

However in the first case the public sector "does not run the risk of producing anti-economically" (p.87), since it is somehow regulated by citizen demand, while in the second it may "continue for a long time in anti-economic production" (p.117). This depends on the efficiency of the democratic process of decision, and, even if he is optimistic for the long run he shows that even in the optimal case a minority may not agree with majority choice. He measures the "damage suffered by the minority ... by the difference between the tax which it pays and the lesser tax which it would have been willing to pay" (p.124).

(17) Cfr. p.54.

the public goods produced annually, the state exacts a corresponding payment, with which it reconstitutes its working capital and begins a new productive cycle". De Viti De Marco's analysis of net income has an explicit dynamical setting. He starts with a stationary economy with five production processes, each employing the same number of workers and producing different intermediate and final commodities: wheat, plow, road and public safety, flour, and a single final good bread. Then, the output of bread (the annual final product of the economy) is equally divided among workers. If the process producing both road and security is run by the public sector, then it is entitled to a fifth of the annual product. Thus, the public sector may "cash its fiscal claim against all the specialized enterprises by levying the 20 per cent of the" value added, or net income, of each enterprise, including the public sector itself.<sup>(18)</sup> From this it "follows that each part of income, no matter how small, comes into existence bearing the corresponding tax-debt" and "that the sum of individual incomes or net incomes must be equal to the national income".<sup>(19)</sup>

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(18) See p.220

(19) Cfr. p.223. An interesting problem arises in the general balanced growth equilibrium. That is to say: must savings (equal to the non consumed profits) be taxed? De Viti De Marco's answer is yes, because "they are income the consumption of which may be transformed into capital if and when it is used to acquire machines, raw materials and

Von Neumann's model of an expanding economy may be regarded as a natural framework for examining this aspect of De Viti De Marco's dynamic theory if we introduce public intermediate commodities.

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human labour" and because otherwise "a person who saves at compound interest ceases to be a taxpayer", p.231.

On this ground he criticizes J. Mill's well known theory of double taxation of savings; arguing that "there is a fusing into a single productive cycle of what are really two productive cycles: that in which the saved income was produced, and the following one in which the interest, which is a new income, is produced" and in "each period, public services are utilized and [therefore, have to be] paid for" p.232.

## 4.2 The concept of public intermediate commodities

### A. What is a public intermediate commodity?

Public goods have been presented by Mishan (1971), pp. 11-3, "as the limiting case of an external economy in which the spillover on others is identical to the good enjoyed by the generator himself." This can be naturally applied also to public factors of production and public intermediate commodities (pic in shorthand form).

To De Viti De Marco (1888) and also Pigou (1932) many public goods were producer goods and hence used in production processes. Sandmo (1972) believes that it is just as easy to find examples of public intermediate goods as to find public final goods. Kohli (1985) examined the problem of public inputs but he deals only with primary factors of production of the 'unpaid factor' type, supplied exogenously in a neoclassical model.

In our simplified model in section 3.4.C, we have identified the creation of each externality with the production of a distinct commodity. Consequently we have already dealt with the limiting case of a public good, where the output of the externality (commodity 3 created by us fictitiously) is produced in the previous period and is regarded as available in the present period. We can name produced externalities public intermediate commodities, because we are interested only in the production side and assume that the effects of such commodities last for one

period only.

Public factors of the non-rival type which are universally available to various production functions of different industries have already been analyzed in the context of a static competitive equilibrium. For instance, Milleron (1972) used a model where there is no distinction between final and intermediate public goods and where such public goods are accumulative, that is the initial endowments of public goods are added to their output even if these endowments are used in the production process.<sup>(1)</sup>

In the case of external economies of the "unpaid factor" type  $c$  may be regarded as the unit input (a technical coefficient) of public intermediate service<sup>(2)</sup> needed, together with the input of private goods  $\underline{a}_2$ , to produce the output  $b^*_{22}$ . Let us suppose that the economy has three goods and three processes which transform

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{1} The case treated by Milleron is thus the case of capital public goods and seems the opposite of our case of public intermediate services. We shall see that our previous treatment of externality is a very general approach to the problem of public intermediate goods.

{2} In the discussion of public goods, it is important to note that:

- a) it is meaningful to sum the outputs of such a good produced by different processes because they have the same effects,
- b) it is meaningless to sum together the minimal operative levels required by different processes to obtain the level required to run all of them,
- c) the output required to operate all processes is given by the maximum among their minimal operative levels.



the bundles of goods  $(a_{v1}, a_{v2}, 0)$ ,  $(a_{z1}, a_{z2}, 0)$ ,  $(a_{z1}, a_{z2}, c)$  into  $(b_{v1}, 0, d)$ ,  $(0, b_{z2}, 0)$ ,  $(0, b^*_{z2}, 0)$ , respectively. However, even in this simple case of no problem of aggregating supply of or demand for the public good, it is imposed, on top of the usual conditions, that the price of the third good be zero. In general, therefore, unless the third good is a free good, the private competitive equilibrium is not a von Neumann equilibrium, and, thus not an optimal state.

#### B. "Factor augmenting" and "firm augmenting" pic

A number of writers derived efficiency conditions for the supply of pic which are similar to Samuelson's summation rule (for collective consumption). These rules are valid in the "factor augmenting" case. We shall examine, for a von Neumann-like economy with pic, the rule concerning the maximal rate of growth under the assumption that the public sector knows the production processes and has the power to enforce payment of taxes on the net income of each process.<sup>{3}</sup>

One may think of an artificial lake that can be used for multi-purposes such as generating electricity, flood control and irrigation as well as providing recreational

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{3} This assumption seems to be similar to the one made by De Viti De Marco, though he additionally assumes that all processes use the same amount of the public input. Cfr. Petretto (1987). A similar assumption is made by McMillan (1979) and Kohli (1985). Given this assumption it does not matter whether pic are excludable.

services. It is assumed that the use by each different industry (process) does not detract from use by other industries (other processes), while the use by different firms of an industry (using the same process) detracts from use by other firms in the same industry. Thus each industry operates under diminishing return to scale. This case seems to correspond to the case of "factor augmenting public good" by McMillan (1979) and is opposite to the case of "firm-augmenting public good" that may correspond instead to the case where congestion is ruled out within each industry; to do so, for example, some kinds of legal (or weather) services are provided by trade associations.<sup>{4}</sup>

The origin of the discussion of this category seems to be traced back to the article by Henderson (1974) who examined the case under which the "Samuelson summation rule" is no longer meaningful. He finds out that it is necessary to assume that: firms, operating in a competitive market, cannot be excluded from utilizing the public inputs, and their production functions are not homogeneous in the private inputs only. Thus Meade's original distinction (examined in 3.4.A) is now applied to the analysis at the level of firms, rather than at the level of industry.

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{4} It seems to exist, however, a terminological confusion between pic of the "firm-augmenting" type and that of the "creation of atmosphere" type, Cfr. Kohli (1985) p.380.

We thus examine the case of many firms belonging to the same industry, which was not possible in Meade's previous analysis. However a simple application of Meade's original distinction to firms may create confusion if one analyzes the problem at both industry and firm levels simultaneously. In the rest of the thesis I shall call the cases of "creation of atmosphere" and "unpaid factor" at the industry level, "firm-augmenting" and "factor-augmenting" respectively.

### C. Durable collective goods and collective assets

Mishan says: "The term collective (or public) good is used in two senses in the literature: sometimes to designate the physical asset itself, say a bridge, and sometimes to designate the services provided by that asset."<sup>{5}</sup> In the following, however, we need clearly distinguishing between the public intermediate services and the assets that provide these services through their existence or through a more or less complicated production process.

At first, abstracting from congestion costs and externalities, we examine in the context of a von Neumann model the case of an everlasting public capital good, proposed by Milleron. Let us introduce productive process  $p$  transforming the unit input of an everlasting public asset (commodity 4, i.e.  $a_{p4} = 1$ ) into an unit of

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{5} Cfr Mishan (1971) p.14

the public intermediate service  $b_{p3} = 1$ , and the same asset  $b_{p4} = 1$  as a joint output. We also assume that the processes producing asset 4 produce, as a joint output, an equal amount of the pic (commodity 3).

The public asset is treated in the same way as private assets are, the only difference being in whether the associated service is public or private. Since the public asset is required only in the production process of the pic, no private agent obtains any profit in producing such asset. However if this asset is produced as a joint output, then process  $p$  must satisfy the profitability conditions being at a zero price of the pic.

This special case may be complicated in many ways. Like for private capital goods each asset can have its own fixed physical lifetime, while following the lines of von Neumann and of Morishima (1969) the economic lifetime may be decided at the balanced-growth equilibrium.

Other inputs, apart from the public asset, may be required in order to produce a public service, and the same asset may produce different services, without rivalry in the use of the asset.

If these different services require only the existence of the asset we may treat all these as joint outputs. In the case in which these services require other and different inputs, we may introduce several joint pic produced by the asset, each one required in some production processes in the place of the public asset itself.

Clearly where the use of a public asset is shared between rival processes the asset may be treated just like a private one.

Furthermore, there are cases where a public asset may produce private goods or where it may be transformed into a private asset, and consequently, be priced by the market. Thus the present model enables us to determine, by finding an equilibrium point, not only the choice between private and public services and the processes to produce them, but also the degree of public use of any given asset.



### 4.3 De Viti De Marco - Von Neumann

#### A. "Unpaid" PIC in the Von Neumann Model

Before returning to De Viti De Marco, let us identify the economic units influenced by pic in the von Neumann model, with externality of the "unpaid factor" type [3.4.c]. Among the things that may be regarded as the elementary units (atoms) of the economy there are:

- (1) the intensities at which the processes are operated,
- (2) the economic agents which run certain processes.

In the first model it is assumed that the use of pic by distinct processes is non-rival, and that the quantities of pic used depend on the intensity of each process. The availability of pic will influence the level of each process, thus we do not allow each single process to be present twice in the model. (We assume all processes to be distinct because the use of pic by two identical processes is rival.<sup>(1)</sup>)

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[1] If we allow all the processes to be present twice in the input and output matrices the new economy will be able to obtain the same total output of private goods with half of the previous input of public goods. In fact, subdividing the intensity of each single previous process into the intensities of the corresponding two identical new processes, each new process will require half of the previous amount of pic and the total demand will be half of the previous amount, because the use of pic is non-rival among the two processes. Then keeping on duplicating each process the demand for pic will decrease tending to zero. This seems to agree with Henderson's proposition (1974) that industries consisting of a large number of firms of infinitesimal size would benefit from saving the use of pic. This is possible only in the special case in which pic are public to firms, so that the reduction of the firm size would increase the amount of pic which is usable with

One can think of a reservoir that can be used for generating electricity, flood control, irrigation and recreational services. We can assume that the use by each different industry (process) does not detract from the use by other industries (processes) while the use by different firms of the same industry (using the same process) detracts from use by the other ones. Hence each industry is operating under diminishing return to scale.

Thus case (1) corresponds to McMillan's "factor augmenting public good" (1979). It is opposite to the case of firm-augmenting public goods" that may correspond to case (2), provided there are a given number of firms.<sup>(2)</sup>

In this latter case, if all agents are able to run all processes, the  $k$ -th agent may allocate in some way the whole amount of public goods to the processes he runs, so that from his point of view they resemble private goods [apart from the fact that the aggregate output of pic is

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private factors. In the case we are examining each different process represents an industry and there is no exclusion of use among different industries (i.e. non-rivalry on the industry level) though it exists among firms of the same industry, cfr. Milleron (1972). That is, there is no congestion between industries but may be possible to have congestion within industries; for interesting examples see Tawada (1980). Consequently, the assumption that we do not allow each single process to be present twice in the model may be a way of representing this type of pic in the von Neumann framework.

{2} Pic are classified into "firm-augmenting" and "factor-augmenting", depending on the fact that the 'publicness' of the pic is at the firm or at the industry level (see also the previous footnote). Consequently we are dealing with the "firm-augmenting" type of pic in case (2), although case (2) may also include other different situations.

given by his production multiplied by the number of the economic agents, because all agents behave in the same way.<sup>{3}</sup>

A number of writers have derived efficiency conditions for the supply of pic similar to Samuelson's summation rule for collective consumption which is valid in the "factor augmenting" case. We shall examine the rule concerning the maximal rate of growth, for a von Neumann economy with pic, on the assumption that the public sector knows the production processes and has the power to enforce payment of taxes on the net income of each process.<sup>{4}</sup> Here, with interdependence between the various economic activities stronger because of the presence of pic, the circular nature of production, one of the fundamental features captured by the original von Neumann model, is increased. We concentrate upon the supply side of the economy so as to determine the pseudo-price of the pic and the pseudo-rates of interest and growth.<sup>{5}</sup>

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{3} In case (2) we focus on the general model that Fisher (1977) left unexamined. We start from the trade case in which a set of assumptions are necessary, in order to guarantee that Fisher's conditions are satisfied.

As we shall see later, this leads us directly to what Fisher thought to be the general case, where private goods are traded and Fisher's free externalities (our pic) are not traded in the market as their use is non excludable.

{4} This is the hypothesis nearest to the ones made by De Viti De Marco, although he assumes that all processes have the same use of the public input, Cfr. Petretto (1987). Similar assumption are made by McMillan (1979) and Kohli (1985). Once given these assumptions, it doesn't matter whether pics are excludable or not.

{5} Clearly, the rates of interest and growth will depend

### B. Case (1) "Factor augmenting".<sup>{6}</sup>

We suppose that industry  $k$  operates processes  $i = n_{k-1} + 1, \dots, n_k$  and that each process  $i$  is feasible at the beginning of the new period when the level of  $\text{pic}$  is greater than the  $i$ 's minimum level of use of  $\text{pic}$  multiplied by the growth factor  $\alpha$ .

$\text{Pic}$  may represent external economies or public services like defence. We suppose that the quantity of  $\text{pic } j$  required by the  $i$ -th process increases at the same rate as the economic system grows. Assuming that the consumption of  $\text{pic } j$  by the  $i$ -th process is non-rival with respect to the use of the same good by different processes, the required quantity of input of the  $j$ -th  $\text{pic}$  by the economy will be determined by the processes requiring the highest amount of input  $j$ .

In this model the use of public goods for each given

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on the processes which are operated as well as on the production of public goods in the past. Even though the expansion constraint for  $\text{pic}$  must be satisfied by a private competitive economy, other conditions which involve pseudo-prices may not hold, or should be modified, because private producers, in general, take into account only the cost of private goods.

{6} The nature of the models that will be examined hereafter, and their similarity to Samuelson's model, need to be clarified. As Samuelson argued, in a purely private competitive economy, all pseudo-prices usually take on zero values.

The rules we formulate, however, are valid where all assets are owned by the public sector which wants to make a better use of the available resources or where a planning authority has a power to impose them on the private sector. In what follows, unless otherwise mentioned we assume that each process pays the pseudo-prices for the use of  $\text{pic}$ , so that the growth and interest rates are 'pseudo-rates' too.



process is proportional to its level of production.<sup>{7}</sup>

Let us partition the input and output matrices A and B into A' and B' which contain the inputs and outputs of the pic, and A'' and B'' which contain only the inputs and outputs of the private goods, and partition similarly the price vector.

As pic are non-rival in consumption, each process may use the full amount of the available pic. Consequently the expansion constraints for public and private commodities are written as:

$$[1.a'] \quad \tau_e \underline{x} B' \geq \alpha X A'$$

$$[1.a''] \quad \underline{x} B'' \geq \alpha \underline{x} A''$$

{7} That is to say, the needed quantity of public goods depends not only on which processes are employed but also on their intensities. We will see in the following paragraph C that the present model is an economy satisfying all conditions of the original von Neumann equilibrium.

The i-th process (operated at a unit intensity) may require a minimal level  $\alpha a'_{ij}$  of the j-th pic (say environmental protection) to be used. Consequently we have  $\underline{x} b'_{*j} \geq \alpha \underline{x} a'_{*j}$  where  $\underline{x} b'_{*j}$  is the level of production of environmental protection in the past period by the whole economy.

In the case of transport we may define  $a'_{kh}$  as the required road network and  $a_{kh}$  as its capacity, requiring  $\underline{x} b'_{*h} \geq \alpha x_h a'_{kh}$  and  $\underline{x} b_{*h} \geq \alpha \underline{x} a_{*h}$  where  $\underline{x} b'_{*h}$  and  $\underline{x} b_{*h}$  are the available level of network and of capacity before congestion (joint outputs of some production process). Positive externalities which the i-th process gives to the r-th must satisfy:  $\underline{x} b'_{*h} = x_i b'_{ih} \geq \alpha x_r a'_{rh}$ ; no further conditions are required. This is because the requirement of the h-th good increases automatically with the level of production of the r-th process. Finally mixed goods may also be allowed for, by introducing a pic as a joint output of the amount (say  $b'_{kj}$ ) which is proportional to the amount of the private good (say  $b_{kj}$ ).



where  $X$  is the diagonal matrix whose diagonal elements are the intensities of the production processes ( $X_{ij} = x_i$  for  $i=j$  and zero otherwise) so that  $\underline{e}X = \underline{x}$  and  $\underline{e}$  is a sum row vector.

The pseudo price of a public good is zero when its demand is less than its supply, i.e.,  $\alpha \max_i (x_i A'_{ij}) < \sum_i x_i B'_{ij}$ . Only the processes which use the maximal amount of this public good (for which  $\alpha x_i A'_{ij} = \sum_i x_i B'_{ij}$ ) will contribute, through  $0 \leq \alpha x_i A_{ij} P_{ji} = (\sum_i x_i B'_{ij}) P_{ji}$ , to the total production cost  $\sum_i x_i B'_{ij} y_j$ .

Therefore the free good conditions may be written:

$$[2.a'] \quad \alpha \text{dia}(PXA') = \text{dia}(P\underline{e}'\underline{x}B') = XB'\underline{y}'$$

$$[2.a''] \quad \alpha \underline{x} A'' \underline{y}'' = \underline{x} B'' \underline{y}''$$

where  $\text{dia}(C)$  is defined as the column vector  $\underline{c}$  whose elements are the diagonal elements of the matrix  $C$ ,  $\underline{c}_i = C_{ii}$ , and  $P$  is the pseudo-price matrix for the use of pic, with  $P_{ji}$  the pseudo-price paid for a unit of good  $j$  by the process  $i$ . Thus  $[\text{dia}(PXA')]_j = \sum_i P_{ji} A'_{ij} x_i$  is the total amount paid by all processes for the pic  $j$ .

Similarly the profit constraint becomes:

$$[3.'] \quad B \underline{y} \leq \beta [ \text{dia}(A'P) + A''\underline{y}'' ]$$

where  $[\text{dia}(A'P)]_i = \sum_j A'_{ij} P_{ji}$  is the total price paid by process  $i$  for its consumption of all pics.

Finally, the profitability rule may be written as:

$$[4.'] \quad \underline{x}B''\underline{y} = \beta \underline{x} [ \text{dia}(A'P) + A''\underline{y}'' ]$$

In view of  $\underline{e} \text{ dia}(PXA') = \sum_j \sum_i P_{ji} A'_{ij} x_i = \sum_i x_i (\sum_j A'_{ij} P_{ji}) = \underline{x} \text{ dia}(A'P)$ , it can be shown that in this model too, we have  $\alpha = \beta$ . We will show in the following section C that this model can be reformulated as the original von Neumann model. It is therefore seen that the price rule that we have just formulated for pic is actually an efficient rule and equivalent with the original von Neumann one.

### C. Reduction of case (1) to a von Neumann economy

Let us now consider a particular von Neumann model, which is related to the previous model (1). Let  $A^\wedge = [A' | A'']$  and  $B^\wedge = [B' | B'']$  be the input and output matrix of this economy. Corresponding to each pic in the previous model, we introduce  $m$  different new commodities into this economy. In this way we distinguish between the  $j$ -th pic consumed by the  $r$ -th process and the same consumed by the  $g$ -th process; since they are regarded as different goods.

Then the  $h$ -th process which produces the output  $b'_{hj}$  of the  $j$ -th pic produces a  $1$  by  $m$  vector  $\underline{b}^*_{hj}$  of the  $j$ -th pic consumed by the  $m$  different processes. We denote the  $s$ -th component of this vector  $\underline{b}^*_{hj}$  by  $b^*_{hjs}$  so that  $\underline{b}^*_{hj} = (b^*_{hjs})$  with  $s = 1, 2, \dots, m$ . Then, the amount produced of the new commodity being equal to the amount of the original pic, we require:

$$b^*_{hjs} = b'_{hj} \quad \text{for } s = 1, \dots, m \text{ and for all } j.$$

This describes how each process produces  $m$  different private goods in correspondence of each pic of the previous model.

In order to make the new outputs correspond to the required inputs in each of the  $m$  production process let  $a^*_{hjs} = a'_{hj}$  for  $s = h$  and zero otherwise. Then we have the pic matrices  $B^* = [b^*_{hjs}]$  and  $A^* = [a^*_{hjs}]$  corresponding to the previous  $B'$  and  $A'$ . (Notice that  $B^*$  and  $A^*$  have  $m$  columns in correspondence to each column of  $B'$  and  $A'$ .) Then condition [1.a'] may be rewritten as

$$[1.A] \quad \underline{x} B^* \geq \alpha \underline{x} A^*$$

Let us now define  $y^*_{js} = P_{js}$  as the pseudo-price of this newly introduced good  $j$  used by process  $s$ . Then, if we write:

$$\underline{y}^* = \text{vec} (\text{TP}) = [P_{11}, P_{12}, \dots, P_{1m}; P_{21}, \dots, P_{2m}; \dots] = \\ = [y^*_{11}, y^*_{12}, \dots, y^*_{1m}; y^*_{21}, \dots, y^*_{2m}; \dots] = [y^*_{js}];$$

the condition [2.a'] can be rewritten as :

$$[2.A] \quad \underline{x} B^* \underline{y}^* = \alpha \underline{x} A^* \underline{y}^*$$

$$\text{being} \quad \sum_i x_i b'_{ij} y'_j = \sum_i (\sum_s x_i b'_{ijs}) P_{js} = \sum_s (\sum_i x_i b^*_{ijs}) P_{js} = \sum_s \sum_i x_i b^*_{ijs} y^*_{js} \quad \text{and} \\ \sum_i P_{ji} a'_{ij} x_i = \sum_i y^*_{ji} a^*_{ij} x_i = \sum_{js} \sum_i y^*_{js} a^*_{ijs} x_i.$$

Similarly conditions [3.'] and [4.'] may be put in the form:

$$[3.A] \quad B^* \underline{y}^* + B'' \underline{y}'' \leq \beta [A^* \underline{y}^* + A'' \underline{y}'']$$

$$[4.A] \quad \underline{x} (B^* \underline{y}^* + B'' \underline{y}'') = \beta \underline{x} [A^* \underline{y}^* + A'' \underline{y}'']$$

where  $\sum_j A'_{ij} P_{ji} = \sum_j A^*_{ij} y^*_{ji}$ , that is  $\text{dia}(A' P) = A^* \underline{y}^*$ .

It is obvious that together with [1a''] and [2a''], [1.A], [2.A], [3.A] and [4.A] form a von Neumann economy. All these conditions may be expressed in von Neumann's original way, in terms of  $B^+ = [B^* \mid B'']$ ,  $A^+ = [A^* \mid A'']$ ,  $\underline{x}^+ = [\underline{x}]$  and  $\underline{y}^+ = [\underline{y}^* \mid \underline{y}'']$ .

Thus our economy with pic has the same properties as the KMT economy has. However, even though this model is equivalent to a von Neumann economy, if the allocation of costs is not enforced by the public sector in a compulsory way, there is no reason why a balanced growth equilibrium should be realized by the private system.

In this case we are able not only to compare the pure private economy with external economies to the one without them, as Morishima and Thompson did, but also, we can compare the equilibrium with pseudo-prices to the pure private economy.

In the first comparison we will obtain, like in the previous model, a result similar to the one obtained by Morishima and Thompson. Let us consider a balanced growth equilibrium where we set to zero the maximum level of intensity of all processes which use a positive amount of pic, and ignore their output, being  $\underline{y}^* = 0$ . If we now reintroduce the processes that uses pic then the economy

become able to use some of these processes, as long as they satisfy condition [1.A]  $\underline{x}_B^* \geq \alpha \underline{x}_A^*$ , so that while the previous rate of growth is feasible we may also be able to obtain perhaps a greater one by employing more profitable processes.

In the second type of comparison it is clear from the properties of von Neumann equilibrium that this economic system realizes a growth factor which is greater than or equal to the one of the pure private economy. This is because in the latter case  $\underline{y}^* = 0$  because of the behaviour of the private producers; consequently we have a vector price different from the one of the von Neumann equilibrium.

#### D. DE VITI DE MARCO's theory reformulated

We have already seen that  $\text{pic}$  cannot be produced in the required amount in the private market equilibrium, if any of these commodities has a positive price. To avoid this and other sorts of problems, we follow De Viti De Marco's idea in his example; that is he introduces into his stationary economy the public sector as the producer of  $\text{pic}$ .<sup>{8}</sup> In what follows we extend this idea introducing the public sector into the more general steady state economy of the von Neumann type. We provide a simplified, but substantially faithful, mathematical formulation of the

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{8} To this purpose we may use our model with a very simple public sector model. Cfr. section. 2.5.B.



ideas of De Viti De Marco, assuming that the supply of  $p_i c$  is the only aim of the public sector. This is a fully justified assumption, because, as it has been seen for De Viti De Marco the production of private goods by the public sector does not require special treatment and the influence of the public sector on the private is one of his major concerns. We assume that the public sector has a given set of  $m^g$  processes of production and runs them with intensities  $\underline{x}^g$ , producing  $d$   $p_i c$ s with some of the  $n'$  private goods as joint outputs. Naturally, some of the  $p_i c$  can be produced only by the public sector. The public sector sets an optimal level for  $p_i c$ s available to the entire economy, at the end of the period, which is given by the vector  $\alpha \underline{z}^0$ . Thus, vector  $\alpha \underline{z}^0$  has to be produced and, for this purpose, the net input of private goods  $\underline{g}^0$  is required.

In the present model, the  $m^g$  production processes are defined in terms of input matrix  $L^g$  for the primary factors, and goods input and output matrices  $G$  and  $Z$ .  $Z^1_{hk}$  ( $G^1_{hk}$ ) is the output (or input) of the  $k$ -th  $p_i c$  produced (required) by the  $h$ -th process of production of the public sector, and  $Z^2_{hk}$  ( $G^2_{hk}$ ) is the output (input) of the  $k$ -th private good produced (required) by the  $h$ -th public process of production. We will define  $\underline{x}^g$  as the intensity vector of these processes and  $X^g$  will be the diagonal matrix with  $X^g_{i,j} = \underline{x}^g_i$  on the diagonal ( $i=j$  and zero otherwise), while  $y^* = [0 ; y^2]$  will be the  $1$  by  $n$

(where  $n=d+n'$ ) price vector composed by  $d$  zeros and  $n'$  prices of the private goods  $y^2$ , (taking into account the free rider problem). The minimal amount of pic that the public sector should produce is  $\underline{z}^g = \underline{z}^o - \underline{x}^p D^1$  (where  $\underline{x}^p D^1$  is the amount of pic produced by the private sector of the economy). At the end of each period we must have  $\underline{z}^o \leq \underline{x} Z$  which implies that the quantities of pics produced by the public sector and available at the beginning of the next period are greater than or equal to the minimal quantities it must produce. At the beginning of the period the public sector has a net demand vector for private goods which is equal to the difference between the input of private goods, which the public sector requires, and the joint output of the private goods produced by the public sector in the previous period; that is:

$$[a'] \quad \underline{g}^o = \underline{x}^g (\alpha G^2 - Z^2)$$

The cost of pic produced by the public sector is  $\underline{x}^g [\delta (G^2 \underline{y}^2 + L^g w) - Z^2 \underline{y}^2]$ , where  $\delta$  is the interest factor of the public sector, equal to  $\beta$  in the present model (assuming private and public debt to be perfect substitutes) and  $w$  is the real wage,  $w = \underline{c} \underline{y}^2$ .

Let  $\underline{z}^A = \underline{x}^g Z^1 + \underline{x}^p D^1$ , be the vector of the available quantities of pic at the beginning of the period and  $Z^A$  is an  $(m \text{ by } o)$  matrix whose rows are given by the vector  $\underline{z}^A$  (which corresponds to  $x B'$  in model 1). The intensity vector  $\underline{x}^g$  satisfies:

$$[p] \quad \underline{x}^g Z^1 \geq \underline{z}^o \quad (\text{minimal production requirement})$$

$$[q] \quad Z^A \geq \alpha X^g G^1 \quad (\text{expansion constraint for pic})$$

We then define the budget deficit as

$$-S = g^o y^2 + \underline{x}^g L^g w - T - E^g$$

where  $g^o y^2$  and  $\underline{x}^g L^g w$  are the current expenditure on private goods and wages,  $T$  the tax revenue net of subsidies and  $E^g$  the net interest income of the public sector.  $E^g = s E$  is a share of the total interest income  $E$  of the capital employed in the private sector (with  $s \leq 1$ ). A negative value of  $s$  implies that a part of the capital, required for public production, is financed by issuing bonds. Capital ownership remains unaltered in time because, in the present model, the propensities to consume out of the net income of the public sector and of capitalists are equal to the unity, while workers do not save. In fact the whole net income of the public sector  $S$  and of capitalists, that is  $E^c = (1 - sE) = (1-s)\underline{x}^p (D^2 + C^2) y^2 = (1-s)(1-\beta) \underline{x}^p C^2 y^2$  is saved and invested. In the above equation,  $C^2$  and  $D^2$  represents the private-goods input and output matrices and  $\underline{x}^p$  the intensity vector, of the private sector. In order to obtain the new expansion constraint and the free goods rule we partition the private-sector input and output matrices into  $C = (C^1 \mid C^2)$ ,  $D = (D^1 \mid D^2)$ ; the new price vector is also similarly partitioned. In this way the private sector produces pic of the amount  $\underline{x}^p D^1$ . The expansion constraint and the free goods rule which

apply to the private sector are then given by:

$$[1p'] \quad Z^A - \alpha X^P C^1 \geq \underline{Q}$$

$$[1p''] \quad \underline{x}^P D^2 - \alpha \underline{x}^P C^2 - \underline{g}^0 \geq \underline{Q}$$

$$[2p] \quad \underline{x}^P D^2 \underline{y}^2 - \alpha \underline{x}^P C^2 \underline{y}^2 - \underline{g}^0 \underline{y}^2 = 0$$

Consequently, the public sector would minimize the use of certain processes that are efficient but require public inputs.

The expansion constraint of the whole economy is now stated as:

$$[1e'] \quad \underline{e}^T \underline{x} B' \geq \alpha X A'$$

$$[1e''] \quad \underline{x} (B'' - \alpha A'') \geq \underline{Q}$$

$$\text{where } \underline{x} = \begin{pmatrix} \underline{x}^g \\ \underline{x}^p \end{pmatrix}; \quad B' = \begin{pmatrix} Z^1 \\ D^1 \end{pmatrix}; \quad A' = \begin{pmatrix} G^1 \\ C^1 \end{pmatrix}; \quad B'' = \begin{pmatrix} Z^2 \\ D^2 \end{pmatrix}; \quad A'' = \begin{pmatrix} G^2 + L^g \underline{c} \\ C^2 \end{pmatrix}$$

The partitioned matrices of the whole economy are, as usual, written as  $A = (A' | A'')$ ,  $B = (B' | B'')$ . The free goods rule for private goods remains unchanged, and being  $\underline{y}^1 = \underline{Q}$ , we may write:

$$[2e] \quad \underline{x} B \underline{y} = \alpha \underline{x} A \underline{y}$$

while the other constraint for the private sector are:

$$[3e] \quad D^2 \underline{y}^2 - \beta C^2 \underline{y}^2 - T^v (D^2 - C^2) \underline{y}^2 \leq \underline{Q}$$

$$[4e] \quad \underline{x}^P (D^2 \underline{y}^2 - \beta C^2 \underline{y}^2) - T = 0$$

In these expressions we assume that the net tax revenue of

the public sector,  $T = \underline{x}^p (D^2 - \beta C^2) \underline{y}$ , is obtained through a system of value added taxes net of subsidy. The elements of the diagonal net tax rate matrix  $T^v$  are  $t^v_{ij} = 0$  for  $i \neq j$  and  $t^v_{ij} = t^v_i$  for  $i=j$ .

The question then is which processes should the public sector run, at what intensities should they be run, and how should the net VAT tax rates be determined, in order to establish an equilibrium position of model 1 (in sections B and C above)? The intensities at which the public sector should run processes are given by the equilibrium intensities of the corresponding processes of model 1. Following De Viti De Marco, the net losses produced by the public processes are covered by the revenue from value added taxes, net of subsidies.

The net rates of VAT applied to the private processes are determined as follows. In the equilibrium of model 1 the amount paid to the public sector by each single process is equal to the amount paid for the use of pic minus the revenue derived from pic sales. In this way the profitability rule of the previous model with the equilibrium price vector  $\underline{y}''$  of the private goods remains unchanged and thus the previous unprofitable processes remain unprofitable.

The  $u$ -th private process, which corresponds to the  $i$ -th process of model 1, pays, or is granted, a tax, or a subsidy, on its value added amounting to  $t^v_u \underline{x}_i (D^2_{i*} - C^2_{i*}) \underline{y}^2 = \alpha x_i \underline{A}'_{i*} \underline{P}_{*i} - x \underline{B}'_{i*} \underline{y}'$ , according as  $t^v_u$  is



positive or negative. The net cost of the use of  $pic$  in model 1 and the value of  $t^v_u$  in the present model are both determined in this way. This is true, as has been explained, for each private process of production. Hence, the profitability condition [3e] is equivalent to [3.'], and all the constraints of this model are satisfied.

Our model is fully consistent with the theory and the example given by De Viti De Marco. In fact, the public sector takes part in the production and exchange of private goods and, therefore, through the production of  $pic$ , influences them. Moreover, for those  $pic$  which can be produced only by the public sector, the "private production cost" is considered in order "to keep expenditure at a level at which a private enterprise may make" them, at the prevailing prices. For all other  $pic$ , however, the actual "private production cost" decides whether they should be produced by the private or the public sector. In the present model, like in De Viti De Marco's example, the public sector claims payment against the enterprises which use the full amount of its product, by taxing a share of the value added (or net income) of each productive process.

This solution may be regarded as a generalization of De Viti De Marco's example. In fact in his example the benefit from  $pic$  is equal for all processes and consequently the net income of all processes should be taxed at an equal rate. His approximate solution is based

on his belief that it would work well (as a second best) in the case where the public sector has no complete information about the technology of every firm, and should consequently be satisfied with equal treatment of all processes.<sup>(9)</sup> Furthermore, (using De Viti De Marco's methodology) we may construct a hypothetical private economy which is in a state of perfect competitive equilibrium (see section C above). In this way we may choose the processes, to be run by the public sector, and calculate the amount of pic to be produced, together with the tax rates to be levied on each single process.

In this generalization (relying upon von Neumann's model and using his equilibrium approach) of De Viti De Marco's example the very heart of his doctrine of public finance, in a democratic setting, is perfectly represented.

In fact as a by-product, we have also shown that the public sector's activities decrease the production cost of private goods and, therefore, they increase the rate of interest on private capital, as well as the rate of growth. Taxes are levied on the production processes as the prices to be paid for the use of pic.

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{9} For a similar approach to the same problem one can refer to Petretto (1987). Although his economy is in a state of static equilibrium and has differentiable production functions, his analysis seems to agree, at least in substance, with my interpretation of De Viti De Marco.

E. Case (2): Fisher's model with trade.

In what follows we show that case (2) or a reinterpretation of Fisher's model with trade is mathematically similar to case (1) above.

From the examination of Fisher's contribution (1977) we see that some questions remain partially unanswered. For instance: how can we specify additional assumptions in an economically meaningful way, in order to construct an economic system satisfying all the axioms, written and unwritten, of the Fisher model with trade? In order to answer this question we adapt to our needs a trade-model devised by M. Morishima.<sup>{10}</sup> Condition [2f] may be written in matrix notation as

$$(x^1 \dots x^k) \begin{bmatrix} B^{11} & \dots & B^{1k} \\ \vdots & & \vdots \\ B^{k1} & \dots & B^{kk} \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \geq \alpha(x^1 \dots x^k) \begin{bmatrix} A^{11} & \dots & A^{1k} \\ \vdots & & \vdots \\ A^{k1} & \dots & A^{kk} \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}$$

where as one may recall  $\underline{x}^r$  is the intensity vector of agent  $r$ ,  $B^{rs}$  the matrix of those outputs which agent  $r$  provides to agent  $s$  as free externalities,  $A^{rs}$  is the matrix of those inputs which agent  $r$  obtains from agent  $s$  and  $I$  the identity matrix of a suitable dimension.

If we define appropriately the input and output matrices  $A^{rs}$  and  $B^{rs}$ , we can rewrite the previous

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{10} This model has been exposed to me in the course of a conversation at LSE in 1985. In expressing to Prof. M. Morishima my gratitude I do not want to involve him in possible mistakes due to my use of his model.

expression as  $\underline{x} B \geq \alpha \underline{x} A$ . We thus obtain conditions similar to the original von Neumann ones.

We should now rule out the possibility that agents use exexternalities rather than private goods that they produce or buy in the market. Thus we require:

$$B^{rs} = \begin{cases} [*B^{rs} | 0] & \text{for } r \neq s \\ [0 | 'B^{rs}] & \text{for } r = s \end{cases} \quad \text{and} \quad A^{rs} = \begin{cases} [*A^{rs} | 0] & \text{for } r \neq s \\ [0 | 'A^{rs}] & \text{for } r = s \end{cases}$$

where  $*B^{rs}$  and  $*A^{rs}$  are the output and input sub-matrices for externalities and  $'B^{rs}$ ,  $'A^{rs}$  the output and input sub-matrices for private goods.

The previous condition distinguishes between externality and private goods (which cannot be received from other agents as externalities). It simply states that the goods which an agent receives from other agents as externalities cannot be produced and used by the same agent. Otherwise it would be possible that the agent may give a positive value to goods received as externalities. This value would be equal to the value of the same goods he himself produces.

This assumption may be stated, more precisely, as  $a^{sr_{ij}}, b^{rs_{ij}} > 0$  if  $j = 1, \dots, n_k$  and  $r \neq s$ ; or for  $j = n_k + 1, \dots, n$  if  $r = s$ ; and zero otherwise. Thus commodities 1 to  $n_k$  are (untradeable) externalities while commodities  $n_k + 1$  to  $n$  are tradeable private goods. Once we specify these first requirements, unformulated in the Fisher model, but needed in order to define a meaningful

economic system let us try to impose the untradeability constraint on externalities and then to get rid of Fisher's notation.

In general, goods received as externalities are not tradeable between different agents. Consequently we regard the good  $j$  received as a free externality by  $r$  as being different from the same good  $j$  received as a free externality by a different agent  $s$ . By defining these goods as different, we consider that the good  $j$  received by  $r$  as a free externality is required as input only by the processes that agent  $r$  runs.

In order to guarantee this let us re-define and order externalities into  $k$  groups distinguished on the basis of their users. From now on externalities with  $j = n_{s-1} + 1, \dots, n_s$  will be the only externalities used by agent  $s$ ; clearly  $n_0 = 0$ . Then we have for the output of the newly defined externalities,  $*b^{rs}_{i,v} = b^{rs}_{i,j}$  for  $j = 1 \dots n_k$  with  $v = n_s(s-1) + j$ , while for the inputs we have  $*a^{sr}_{i,h} = a^{sr}_{i,j} \geq 0$  for  $j = 1, \dots, n_k$  with  $h = n_k(r-1) + j$ .

The previous condition distinguishes between externalities used or received from other different externalities, redefining them as different commodities. It states that the goods which an agent receives as externalities cannot be used by a different agent. Otherwise it would be possible that the receiving agent may sell them at a positive price.



In order to simplify the notation, eliminating subscripts, let us now define two sub-matrices  $*B$  and  $*A$ . Let  $*B = (*B_{z v})$  with  $*B_{z v} = *b^{r s_i v}$  for  $z = \sum_{h < r} m_h + i$ , and similarly  $*A = (*A_{z v})$ , with  $*A_{z v} = *a^{r s_i v}$  for  $z = \sum_{h < r} m_h + i$ . In this way we are no longer explicitly distinguishing between the agents who run each process, and are ordering them in input and output matrices of the whole economy. These matrices may be partitioned as:

$$*A = \begin{bmatrix} N & 0 & 0 & \dots & 0 \\ 0 & N & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & N \end{bmatrix} \quad *B = \begin{bmatrix} 0 & N & N & \dots & N \\ N & 0 & N & \dots & N \\ \cdot & \cdot & \cdot & & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ N & N & N & \dots & 0 \end{bmatrix}$$

where  $N$  are non-negative submatrices of suitable dimensions (not necessarily equal to each other) and  $N > 0$  if the von Neumann assumption ( $A + B > 0$ ) is satisfied.

Finally we may proceed to the same simplification with private goods. Thus let us define  $'B = ('B_{z v})$  for  $v = n_k + 1, \dots, n$  and  $z = 1, \dots, \sum_h m_h$ ; where  $'B_{z v} = b^{r r_i v}$  for  $z = \sum_{h < r} m_h + i$ , and analogously  $'A = ('A_{z v})$  with  $'A_{z v} = a^{r r_z v}$  for  $z = \sum_{h < r} m_h + i$ .

We may now write  $B = [*B \mid 'B]$  and  $A = [*A \mid 'A]$  and define the price vector as  $y = (*y^T \mid 'y^T)^T$ .

In the traditional von Neumann economy the profitability condition is stated as  $B y \leq \beta A y$ . This corresponds to  $'B 'y \leq \beta 'A 'y$  with  $*y = 0$  which Fisher assumed for externalities. In fact as  $'B 'y =$

$\sum_j b^r s_{ij} \cdot y_j$  and  $\cdot A \cdot y = \sum_j a^r r_{ij} \cdot y_j$ ; the above inequality represents Fisher's [1f]. Then the remaining conditions take the usual form:

$$\underline{x} ( B - \alpha A ) \underline{y} = \underline{x} ( B - \beta A ) \underline{y} = 0$$

Apart from the apparent similarity between the conditions, we can observe that the departure from the original von Neumann conditions consists in that  $\cdot y$  is no longer determined by the equilibrium conditions, but it is set equal to zero from the start.

The von Neumann indecomposability assumption,  $A + B > 0$  is satisfied because we assume that  $\cdot A + \cdot B > 0$ , and that  $\cdot A + \cdot B > 0$ . Thus, in our model, the cases  $A = 0$  or  $B = 0$  are excluded by  $A + B > 0$ , but in Fisher's equilibrium, due to assumption  $\cdot y = 0$ , we could have  $\alpha = 0$ , where all positive outputs are externalities ( $\cdot B = 0$ ), in this case  $\underline{x} B \underline{y} = \underline{x} \cdot B \cdot y = 0$ , and also we have  $\underline{x} A \underline{y} = \underline{x} \cdot A \cdot y > 0$ , because  $\cdot A > 0$ , and hence  $\alpha = 0$ .

In this way, by comparing the original equilibrium with the constrained one, we have shown that in this re-interpretation of Fisher's model, inputs and outputs of externalities are undervalued, since some of them are not free in the von Neumann equilibrium. Furthermore it is noted that whilst a von Neumann equilibrium is associated with the maximal rate of growth for the economy, the Fisher equilibrium where  $\cdot p$  are regarded as free goods (corresponding to the market equilibrium) is characterized

by a smaller, at least equal, rate of growth.<sup>{14}</sup>

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{14} These results cannot be obtained in this form in the case of the KMT decomposable economy, which was referred to by Fisher (1977), due to the existence of multiple equilibria.

### 5. FINAL REMARKS

The main aim of this thesis has been to expand the original von Neumann theory of economic growth, especially taking into account the presence of externalities and of a public sector. After examining how von Neumann's work has been interpreted and extended by Champernowne, Kemeny, Morgenstern, Thompson, Haga-Otsuki, Morishima, Fisher and many other economists with varying degrees of success and in a variety of directions, one feels that these and further developments are still possible. Consequently, it may be useful to summarize what has been said in the previous chapters (reviewing the work already done) and to point to possible future perspectives for research.

In particular we wanted to examine and replace (by a more general hypothesis) the following simplification of von Neumann's assumptions:

- a) workers do not save and capitalists do not consume,
- b) consumer choice is not allowed for and wages are predetermined,
- c) perfect competition is assumed in the absence of any type of externalities and public goods,
- d) there is no public sector or any kind of public intervention.

The preceding exposition (of the thesis) has followed closely the actual evolution of the author's ideas, while he was confronting previous interpretations and models. These contributions represent extensions of von Neumann's model, almost exclusively oriented towards incorporating additional consumption by non-production sectors (such as families). In relation to assumptions (a) and (b) the work started with Champernowne, Kemeny, Morgenstern and Thompson and was virtually completed with Haga-Otsuki's and Morishima's contributions. In fact Morishima incorporates proper consumption demand and primary factor supply functions into the von Neumann economy.

On the other hand, no major progress has been achieved towards an economically meaningful introduction of the public sector in the consumption and production side of the original model. Rather, externalities and public commodities have not been satisfactorily dealt with as market failure and in connection with public intervention. After a critical examination of the existing contributions it was natural to generalize and integrate them into comprehensive models and to deal with different cases and comparison of possible balanced growth equilibria. In this context lies the main original contribution of the previous work.

On these problems we have a specialist literature in the area of public finance. Usually it assumes smooth



production functions and examines partial equilibrium situations. Consequently, it is quite interesting to deal with these problems in a general equilibrium growth setting, using von Neumann production functions. That is true from at least two possible perspectives. First, this approach may allow more flexibility in empirical application and the use of linear programming methods.

Second, von Neumann production functions with joint products permit the economic lifetime of capital goods to be determined endogenously and achieve a clear distinction between public services (of a one period duration) and public assets (which last for more than one period). These are useful contributions that develop further the usual public finance treatment of the subject. Indeed they enable us to take into account problems which are quite relevant from a theoretical and practical point of view.

However, before entering into these types of extensions it was important to re-examine thoroughly the possible interpretations of the original von Neumann model and to compare them with the original paper. This work has mainly been done in the first two chapters and it has in itself some original merit. Not only are some interesting aspects pinpointed but also new views and interpretations are offered. Our general thesis is that the original von Neumann growth model is a real breakthrough in economics for more than the mathematical formalization alone. It has

a lot of merit in the treatment of capital, in the rigorous economic solution of the problems of free goods and choice of techniques and in the interpretation of classical and neoclassical economic thought. Hence von Neumann is not a pure formalist but a first class mathematical economist not interested in axiomatization for its own sake.

In the first chapter it has been shown how:

- In von Neumann  $\beta$  is a monetary variable so that it would be correct to think of its equilibrium level as a sort of "natural interest rate".
- It is not completely true that the KMT assumptions are more general than the von Neumann original ones. In fact the set of models which satisfy von Neumann's original assumptions is not contained in the set for which KMT assumptions are satisfied. Furthermore KMT assumptions may be too strict for an open economy.
- The Morgenstern-Thompson model of an open expanding economy is just a linear programming model of what we may call a constrained economy. It was originally achieved as a linear programming problem related to the von Neumann economy. Unfortunately the new axioms, derived in this way, do not have an immediately meaningful economic interpretation. Nevertheless, it may be useful as a first step towards an open economy

and, above all, the treatment of public sector constraints on the production system.

This contribution shows how we may deal with an open economy in a closed model. This analytical definition is too general and it should be clarified whether the von Neumann economy is closed in a stricter economic sense, following Georgescu-Roegen, Koopmans and Samuelson. The problem is connected with the rather special treatment reserved by von Neumann for labour which was not accepted by those great economists, accustomed to treat labour as an ordinary good whose price is determined in the market. Von Neumann never assumes consumption as a labour producing process (which is an unsatisfactory hypothesis). Consequently he deals with an open economy because labour is not produced by the system.

In the second chapter we examine the different ways of introducing consumption in von Neumann's model. This work is needed in order to examine some contributions unduly ignored in previous review articles and to study all the extensions critically from an economic point of view.

One should make a distinction between the models that introduce proper demand and supply functions and the ones which do not. Differently from previous interpretations (due to Frisch and Bauer) we demonstrate how the Morgenstern-Thompson model belongs to the last category, even if we take into account their latest contribution of

1976, that is their additional income matrix which should "lead to economically reasonable models".

In this context we support and generalize the interpretation hinted at by M. C. Lovell against Frisch and Bauer. That, however, does not affect the lack of economic analysis in Morgenstern-Thompson's aggregate demand and the peculiarity of the income matrix. In comparison the positive interest rate hypothesis and the introduction of the free good rule in the labour market by Haga-Otsuki are just minor slips.

In the very same paper Morgenstern and Thompson also tried to introduce a public sector as a new consumer and saver in the von Neumann model. We propose our own economic interpretation of that model as well as a new original model. This is consistent with the previous Morishima-Nosse econometric specification and, consequently, can be effectively applied to the actual world.

For a more general treatment of the public sector in the von Neumann model it is necessary to introduce externalities and public commodities. This has been done by Morishima-Thompson and attempted by Fisher in an activity analysis way. We review Morishima and Thompson and we critically examine Fisher, trying to interpret his models in an economically reasonable way.

Subsequently in the very same third chapter we introduce an alternative approach, which does not use smooth production functions, and examine a different externality (of the creation of atmosphere type as opposed to the unpaid factor type assumed by Morishima and Thompson).

In order to find the equilibria of this model we can apply a modified version of the Morgenstern-Thompson linear programming model of a constrained economy. In our case, however, the presence of a public sector increases the rate of growth and of interest. Furthermore, differently from previous models, there may not exist a private market equilibrium. All the equilibria can be achieved only by a mixed economy (through public intervention).

In the fourth chapter we deal with public intermediate commodities and we build the De Viti De Marco-von Neumann model, in honour of this great public finance scholar. We also discuss another possible model with positive unpaid factor externalities. We introduce each externality as a different von Neumann commodity, distinguished on the basis of its user. These public intermediate commodities (mainly related to the production side of the economy) influence directly the use and the intensity of one or more productive processes.

It is quite surprising that those economies can be reduced to a von Neumann economy. Thus the very heart of



the Italian doctrine of public finance can be reformulated. Taxes are levied on the production processes as the prices to be paid for the use of public commodities. In the private market equilibrium those pseudo-prices are no longer determined by the equilibrium conditions but are simply set equal to zero. We have shown how the rate of growth of the von Neumann economy, where the prices of public commodities are determined by the equilibrium solution and paid through taxation, is greater than the one which would be determined by the private market equilibrium where public commodities are free. The very fact that public commodities are handled within the von Neumann model may also help to clarify the problems encountered by the "Samuelson summation rule" in the case of "firm-augmenting public goods".

In this context one may feel that there is still plenty of space for new useful and economically meaningful extensions. The only real limits are given by the steady state equilibrium setting, not by the linear processes, as has been shown in the previous analysis. One interesting possibility is to integrate in the family and the public sector better, fully allowing for consumers' public goods. The "very simple public sector model" of chapter two may represent a first step in this direction. In any case even if we decide to abandon the steady state equilibrium, the von Neumann production functions remain a very useful instrument of analysis.

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