A NEW APPLICATION OF CRAPPER'S EXACT SOLUTION TO WAVES IN CONSTANT VORTICITY FLOWS

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Abstract. In 1957, Crapper found an exact solution for capillary waves in an irrotational flow of infinite depth. Here we provide asymptotic and numerical evidence that a Crapper wave makes the profile of a periodic traveling wave in a constant vorticity flow when the effects of gravity and surface tension are negligible. This is achieved by explicitly constructing successive approximate solutions for small amplitude waves and by numerically computing arbitrary amplitude waves.

1. Introduction

We consider periodic traveling waves propagating in a two dimensional, infinitely deep and constant vorticity flow.

For gravity waves in an irrotational flow (zero vorticity), Stokes [11,12] observed that crests become sharper and troughs flatter as the amplitude increases and that the so-called wave of greatest height or *extreme* wave possesses a 120° angle at the crest. For strongly positive vorticity, by contrast, the numerical computations [10] indicate that the amplitude increases, decreases and increases in the (wave speed) $\times$ (amplitude) plane – namely, a "fold" – and that the profile becomes vertical and overturns as the amplitude increases along the fold. Moreover, a limiting configuration is either an *extreme* wave, like in an irrotational flow, or a *touching* wave, which encloses a bubble of air at the trough. Here we distinguish positive vorticity for waves propagating upstream, whereas negative vorticity for downstream. The latter author [14] numerically found, among others, a new solution branch limited by a touching wave as gravitational acceleration vanishes, whose crest-to-trough height for the unit period – namely, the steepness – is approximately 0.7.

For capillary waves (nonzero surface tension and zero gravity) in an irrotational flow, on the other hand, Crapper [2] derived an exact solution formula for arbitrary amplitude and thereby deduced that crests become flatter and troughs more curved as the amplitude increases toward a touching wave whose steepness is approximately 0.7.

This fortuitous coincidence is no accident. Recently, Dyachenko and the former author [4,5] (see also [3]) numerically found, among others, that touching waves in positive vorticity flows approach the limiting Crapper wave as the vorticity strength increases indefinitely or, equivalently, gravitational acceleration vanishes.

The present purpose is to provide asymptotic and numerical evidence that *any* Crapper wave, not necessarily the limiting one, makes an exact solution for periodic traveling waves in positive vorticity flows when the effects of gravity and surface tension are negligible. For small amplitude, following a Stokes expansion procedure

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[11], we verify that the profile of a periodic traveling wave in a positive vorticity flow agrees with that of a capillary wave in an irrotational flow up to the third order, although the fluid flows beneath the wave are different. For arbitrary amplitude, following the procedure of [14], we numerically compute the solutions and compare with Crapper's exact solution [2].

The zero gravity assumption might be physically unrealistic for surface water waves. On the other hand, there are few exact solutions for free surface flows and they would describe highly idealized situations. For instance, Crapper waves exist for zero gravity. Nevertheless, such solutions are significant for rigorous analysis and numerical computation. Here we intend to emphasize an unexpected and remarkable link between rotational and capillary effects. In addition, numerical evidence (see [4, 5, 14], for instance) suggests that for steep and rounded waves, vorticity is more important than gravitational acceleration. Moreover, our numerical solutions for zero vorticity well approximate those for nonzero gravity for strongly positive vorticity [14] (see also [4, 5]).

2. Formulation

We consider a two dimensional, infinitely deep and constant vorticity flow of an incompressible inviscid fluid, or else an irrotational flow under the influence of surface tension, and waves propagating at the fluid surface. We neglect gravitational acceleration. Although an incompressible fluid may have variable density, we assume for simplicity the unit density.

Suppose for definiteness that in Cartesian coordinates, the x axis points in the direction of wave propagation and the y axis vertically upwards. Suppose that the fluid at time t occupies a region in the (x, y) plane, bounded above by a free surface $y = \eta(x, t)$. We assume for now that η is single-valued (but see Section 4). Let $\mathbf{u} = \mathbf{u}(x, y, t)$ denote the velocity of the fluid at the point (x, y) and time t, and $P = P(x, y, t)$ the pressure. They satisfy the Euler equations for an incompressible fluid:

$$\left.\begin{aligned} &\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P \\ &\nabla\cdot\mathbf{u} = 0 \end{aligned}\right\} \quad \text{in } -\infty < y < \eta(x,t). \tag{1a}$$

Let

$$\Omega = \nabla \times \mathbf{u} \tag{1b}$$

denote constant vorticity. Note that if the vorticity is constant in the fluid region at the initial time then Kelvin's circulation theorem implies that it remains so at all later times. The kinematic and dynamic conditions:

$$\left.\begin{aligned} &\eta_t + \mathbf{u}\cdot\nabla(\eta - y) = 0 \\ &P = -T\frac{\eta_{xx}}{\sqrt{1+\eta_x^2}^{\,3}} \end{aligned}\right\} \quad \text{at } y = \eta(x,t) \tag{1c}$$

express that each fluid particle at the surface remains so at all times, and that there is a jump in the pressure across the fluid surface, proportional to the curvature, where T is the coefficient of surface tension.

When $T = 0$ (zero surface tension), for any Ω, note that

$$\eta(x,t) = 0, \quad \mathbf{u}(x,y,t) = (-\Omega y, 0) \quad \text{and} \quad P(x,y,t) = 0 \tag{2}$$

solve (1) at all times. They make a linear shear flow, for which the fluid surface is flat and the fluid velocity varies linearly with depth. We assume that some external effects such as wind produce a flow of the kind and restrict the attention to waves propagating in (2).

Suppose that

$$\mathbf{u} = (-\Omega y, 0) + \nabla\Phi, \tag{3}$$

whence the latter equation of (1a) implies that $\nabla^2\Phi = 0$ in $-\infty < y < \eta(x,t)$. We pause to remark that for non-constant vorticity, such a velocity potential is no longer viable to use. Substituting (3) into the former equation of (1a) and using the latter equation of (1c), we make an explicit calculation to show that

$$\Phi_t + \frac{1}{2}(\Phi_x - \Omega y)^2 - \frac{1}{2}\Omega^2 y^2 + \frac{1}{2}\Phi_y^2 + \Omega\Psi - T\frac{\eta_{xx}}{\sqrt{1+\eta_x^2}^3} = B(t) \quad \text{at } y = \eta(x,t)$$

for some function $B(t)$, where Ψ be a harmonic conjugate of Φ.

We turn the attention to traveling waves of (1). That is, $\mathbf{u}$ and P are functions of $(x - ct, y)$ while η is a function of $x - ct$ for some $c > 0$, the wave speed. Under the assumption, we will go to a moving coordinate frame, changing $x - ct$ to x, whereby t completely disappears. The result becomes:

$$\nabla^2\Phi = 0 \qquad \text{in } y < \eta(x), \tag{4a}$$

$$(\Phi_x - \Omega y - c)\eta' = \Phi_y \qquad \text{at } y = \eta(x), \tag{4b}$$

$$\frac{1}{2}(\Phi_x - \Omega y - c)^2 + \frac{1}{2}\Phi_y^2 - T\frac{\eta''}{\sqrt{1+(\eta')^2}^3} = B \qquad \text{at } y = \eta(x) \tag{4c}$$

for some constant B, where the prime means differentiation in the x variable. The boundary condition in the infinite depth is that

$$\Phi \to 0 \qquad \text{as } y \to -\infty. \tag{4d}$$

This agrees with the boundary condition of [4, 10, 14] and others. When $\Omega = 0$ (zero vorticity), (4) becomes the capillary wave problem [2].

In what follows, we assume that Φ and η are periodic, Φ is odd and η even in the x variable.

3. Small amplitude: Stokes expansion

We proceed to a Stokes expansion procedure for small amplitude solutions of (4) when either $T = 0$ (nonzero vorticity and zero surface tension) or $\Omega = 0$ (zero vorticity and nonzero surface tension). We may assume without loss of generality that η is of mean zero, and let

$$\eta(x) = \epsilon\cos(kx) + \epsilon^2\eta_2\cos(2kx) + \epsilon^3\eta_3\cos(3kx) + \cdots, \tag{5}$$

where $\epsilon \ll 1$ is an amplitude parameter and k the wave number, $\eta_2, \eta_3, \dots$ are to be determined. Let

$$\Phi(x,y) = \epsilon\Phi_1 e^{ky}\sin(kx) + \epsilon^2\Phi_2 e^{2ky}\sin(2kx) + \epsilon^3\Phi_3 e^{3ky}\sin(3kx) + \cdots \tag{6}$$

for $\epsilon \ll 1$, where $\Phi_1, \Phi_2, \Phi_3, \dots$ are to be determined. Note that (6) solves (4a) and (4d). Moreover, let

$$c = c_0 + \epsilon^2 c_2 + \cdots \tag{7}$$

for $\epsilon \ll 1$, where $c_0, c_2, \dots$ are to be determined.

We substitute (5)-(7) to (4b)-(4c) and explicitly determine successive approximate solutions when either $T = 0$ or $\Omega = 0$. In the latter, one may resort to Crapper's exact solution [2] (see (13)), but it is simpler and more instructive to use directly the equations.

The convergence of successive approximate solutions can be proved, for instance, by extending the argument of [6, 7] to the infinite depth (see [9], for instance, for gravity waves). Here we are interested in explicit solutions and, hence, we omit the details.

At the order of ϵ, we find that

$$\Phi_1 = c_0 = \begin{cases} \Omega/k & \text{for } T = 0, \\ \sqrt{Tk} & \text{for } \Omega = 0. \end{cases} \tag{8}$$

The result agrees with the dispersion relation (see* [1, (13)], for instance) for gravity waves in a constant vorticity flow when gravitational acceleration vanishes. It is consistent with [8, (3.7)] (see also references therein) for gravity-capillary waves in a constant vorticity flow, which may be written

$$kc_0^2 - \Omega c_0 - (g + Tk^2) = 0,$$

where g is the constant due to gravitational acceleration. Note that when $T = 0$, $c_0 = (\Omega \pm \sqrt{\Omega^2 + 4gk})/2k \to \Omega/k$ as $g \to 0$. When $\Omega = 0$, $c_0 = \sqrt{g/k + Tk} \to \sqrt{Tk}$ as $g \to 0$.

At the order of ϵ^2, (4b)-(4c) become

$$\begin{aligned} c_0\eta_2 - \Phi_2 =& \frac{1}{4}c_0k - \frac{1}{4}(c_0k - \Omega), \\ c_0(2k\Phi_2 - \Omega\eta_2) - 4k^2T\eta_2 =& -\frac{3}{4}c_0^2k^2 + \frac{1}{4}(c_0k - \Omega)^2. \end{aligned}$$

Here we use (6) and (5), and we make a Taylor series expansion to find that

$$\begin{aligned} \Phi_x(x, \eta(x)) =& \epsilon k\Phi_1 \cos(kx) + \epsilon^2k^2\Phi_1 \cos^2(kx) + 2\epsilon^2k\Phi_2 \cos(2kx) \\ &+ \epsilon^3k^2\Phi_1\eta_2 \cos(kx)\cos(2kx) + \tfrac{1}{2}\epsilon^3k^3\Phi_1 \cos^3(kx) \\ &+ 4\epsilon^3k^2\Phi_2 \cos(kx)\cos(2kx) + 3\epsilon^3k\Phi_3 \cos(3kx) + \cdots, \end{aligned}$$

and similarly for $\Phi_y(x, \eta(x))$. We substitute (8) and make an explicit calculation to find that

$$\eta_2 = -\frac{1}{4}k \quad \text{when either } T = 0 \text{ or } \Omega = 0, \tag{9}$$

and

$$\Phi_2 = \begin{cases} -\frac{1}{2}c_0k & \text{for } T = 0 \\ -\frac{3}{4}c_0k & \text{for } \Omega = 0, \end{cases} \tag{10}$$

where c_0 is in (8). Therefore, the profile of a periodic traveling wave in a constant vorticity flow coincides with that of a capillary wave in an irrotational flow up to the second order of a small amplitude parameter, although the fluid flows underneath the wave are different. The result agrees with [8, (3.17)-(3.18)] (see also references therein) for gravity-capillary waves in a constant vorticity flow of finite depth, as the fluid depth increases indefinitely, after returning to dimensional variables.

*$c_0^2(k - \Omega/c_0) = g$ in our notation, where g is the constant due to gravitational acceleration.

To continue, at the order of ϵ^3, (4b)-(4c) become

$$\begin{aligned}3c_0k\eta_3-3k\varPhi_3=&\frac{1}{4}c_0k^2-\frac{1}{4}k(c_0k-\Omega)+2k\varPhi_2+\frac{1}{2}(2k\varPhi_2-\Omega\eta_2),\\ c_0(3k\varPhi_3-\Omega\eta_3)-9k^2T\eta_3=&-\frac{1}{4}c_0^2k^3+\frac{1}{4}c_0k^2(c_0k-\Omega)\\ &-3c_0k^2\varPhi_2+\frac{1}{2}(c_0k-\Omega)(2k\varPhi_2-\Omega\eta_2)+\frac{9}{8}Tk^4.\end{aligned}$$

We use (8), (9), (10) and make an explicit calculation to find that

$$\eta_3=\frac{1}{16}k^2\quad\text{when either } T=0 \text{ or } \Omega=0, \tag{11}$$

while

$$\varPhi_3=\begin{cases}\frac{7}{16}c_0k^2 & \text{for } T=0\\ \frac{13}{16}c_0k^2 & \text{for } \Omega=0,\end{cases}$$

where c_0 is in (8). Therefore, the profiles of a periodic traveling wave in a constant vorticity flow and a capillary wave in an irrotational flow coincide up to the third order of a small amplitude parameter. The result agrees† with [8, (3,28) and (3.30)] in the infinite depth limit, after adaptation to the present notation.

4. Arbitrary amplitude: numerical computation

We take matters further to arbitrary amplitude waves. In what follows, let

$$x\mapsto\frac{k}{2\pi}x\quad\text{and}\quad y\mapsto\frac{k}{2\pi}y,$$

where k is the wave number, and we use dimensionless variables

$$\phi=\frac{k}{2\pi}\frac{\varPhi}{c}\quad\text{and}\quad\omega=\frac{2\pi}{k}\frac{\Omega}{c}. \tag{12}$$

We allow that the fluid surface is multi-valued, given in parametric form. We may assume without loss of generality that the crest is at the origin in the (x, y) plane.

4.1. **Crapper waves.** When $\omega = 0$ (zero vorticity and nonzero surface tension), recall Crapper's exact solution [2], written $x = X(\phi)$ and $y = Y(\phi)$, $0 \leqslant \phi \leqslant 1$, where

$$\begin{cases}X(\phi)=\phi-\dfrac{2}{\pi}\dfrac{A\sin 2\pi\phi}{1+A^2-2A\cos 2\pi\phi},\\ Y(\phi)=-\dfrac{2}{\pi}\dfrac{1}{1+A}+\dfrac{2}{\pi}\dfrac{1+A\cos 2\pi\phi}{1+A^2-2A\cos 2\pi\phi},\end{cases} \tag{13}$$

and

$$A=\frac{2}{\pi s}\left(\left(1+\frac{1}{4}\pi^2s^2\right)^{1/2}-1\right).$$

Here and elsewhere, s denotes the steepness, the crest-to-trough height divided by the period.

Figure 1 shows the wave profiles for several values of s. When s is small, we find that the profile is single-valued and well approximated by (5), (9) and (11), after returning to dimensionless variables. As s increases, we confirm that crests become more rounded and troughs more curved, and as $s \to s^* \approx 0.7298$, the profile

†The steepness is used as a small amplitude parameter in [8] whereas here we use the first Fourier coefficient, notwithstanding different notation.

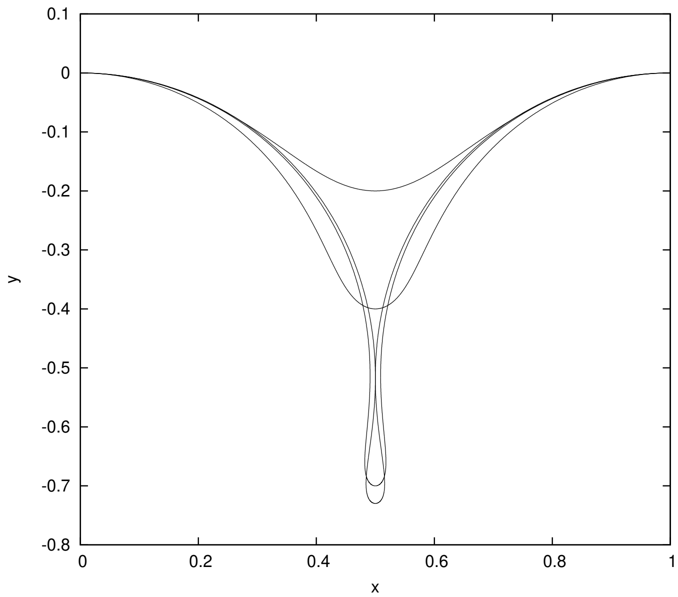

FIGURE 1. Profiles of Crapper waves (see (13)), and also numerical solutions for periodic traveling waves in constant vorticity flows (see (14)), for $s = 0.2$, 0.4, 0.7 and 0.7298. The corresponding values of ω are 5.47, 3.91, 2.14 and 2.02, respectively.

approaches a limiting configuration which encloses a bubble of air at the trough. For $s > s^*$, the fluid surface intersects itself and the flow becomes multi-valued, whence the solution is not physically realistic.

4.2. **Waves in constant vorticity flows.** When $T = 0$ (nonzero vorticity and zero surface tension), on the other hand, there is no exact solution formula to the best of the authors' knowledge, and we follow [14] (see also [10, 13]) to numerically compute the solutions. Here we discuss the main ideas for completeness and for some differences in the choice of the parameters. Readers are referred to [14] and references therein for details.

Suppose that $x = X(\tau)$ and $y = Y(\tau)$ parametrize the fluid surface, where τ is the arclength, and that $\tau = 0$ at the crest. That is,

$$X'(\tau)^2 + Y'(\tau)^2 = 1 \quad \text{for } -L/2 \leqslant \tau \leqslant L/2, \tag{14a}$$

where L is the length of the fluid surface over one period, and

$$X(0), Y(0) = 0. \tag{14b}$$

Let

$$u = \phi_x \quad \text{and} \quad v = \phi_y.$$

Note that $u - iv$ is an analytic function of $x + iy$ and $u - iv \to 0$ as $y \to -\infty$, and let

$$(u - iv)(\tau) = (u - iv)(X(\tau) + iY(\tau)).$$

An application of the Cauchy integral formula leads to

$$\begin{aligned}\pi u(\tau) = &- \operatorname{Im} \int_0^{L/2} \frac{(u(\sigma) - iv(\sigma))(-2\pi i X'(\sigma) + 2\pi Y'(\sigma))}{1 - \exp(-2\pi i(X(\tau) - X(\sigma)) + 2\pi(Y(\tau) - Y(\sigma)))}\, d\sigma \\ &- \operatorname{Im} \int_0^{L/2} \frac{(u(\sigma) + iv(\sigma))(-2\pi i X'(\sigma) - 2\pi Y'(\sigma))}{1 - \exp(-2\pi i(X(\tau) + X(\sigma)) + 2\pi(Y(\tau) - Y(\sigma)))}\, d\sigma.\end{aligned} \tag{14c}$$

See [14] for details. Moreover, (4b) and (4c) become

$$(u - 1 - \omega Y)Y'(\tau) = vX'(\tau), \tag{14d}$$

$$(u - 1 - \omega Y)^2(\tau) + v^2(\tau) = B \tag{14e}$$

for $-L/2 \leqslant \tau \leqslant L/2$.

For ω given, we seek a numerical solution, u and v, X' and Y', of (14), where B and L are determined as part of the solution.

Let

$$\tau_j = \frac{L}{2}\frac{j-1}{N-1}, \quad j = 1, 2, \dots, N,$$

define N uniform mesh points over $[0, L/2]$, and let

$$u_j = u(\tau_j) \quad \text{and} \quad v_j = v(\tau_j), \quad j = 1, 2, \dots, N, \tag{15a}$$

$$X'_j = X'(\tau_j) \quad \text{and} \quad Y'_j = Y'(\tau_j), \quad j = 1, 2, \dots, N, \tag{15b}$$

make $4N$ unknowns. Let

$$\tau_{j-1/2} = \frac{\tau_{j+1} + \tau_j}{2}, \quad j = 1, 2, \dots, N-1.$$

We approximate $X_j = X(\tau_j)$ and $Y_j = Y(\tau_j)$, $j = 1, 2, \dots, N$, using (15b) and the trapezoidal rule. That is, $X_1 = 0$, $Y_1 = 0$ (see (14b)), and

$$X_j = X_{j-1} + X'(\tau_{j-3/2})\frac{L}{2}\frac{1}{N-1},$$

$$Y_j = Y_{j-1} + Y'(\tau_{j-3/2})\frac{L}{2}\frac{1}{N-1},$$

$j = 2, 3, \dots, N$, where we evaluate $X'(\tau_{j-3/2})$ and $Y'(\tau_{j-3/2})$ using (15b) and a four-point interpolation formula.

We satisfy (14a), (14d) and (14e) at $\tau = \tau_j$, $j = 1, 2, \dots, N$. They make $3N$ equations. We then satisfy (14c) at $\tau = \tau_j$, $j = 2, 3, \dots, N-1$, using the trapezoidal rule and summing over $\sigma = \tau_{j-1/2}$, $j = 1, 2, \dots, N-1$. This makes $N-2$ equations. We pause to remark that the symmetry of the discretization and of the trapezoidal rule with respect to the singularity of the integrand at $\sigma = \tau$ enables us to evaluate the Cauchy principal value integral with an accuracy no less than a non-singular integral. Lastly, we require two symmetry conditions

$$v_1, v_N = 0.$$

In addition, since the crest is at the origin,

$$Y_N = -s,$$

where s is the steepness. Since $u - iv$ vanishes in the infinite depth,

$$\int_0^{L/2} (uX' + vY')(\tau)\, d\tau = 0.$$

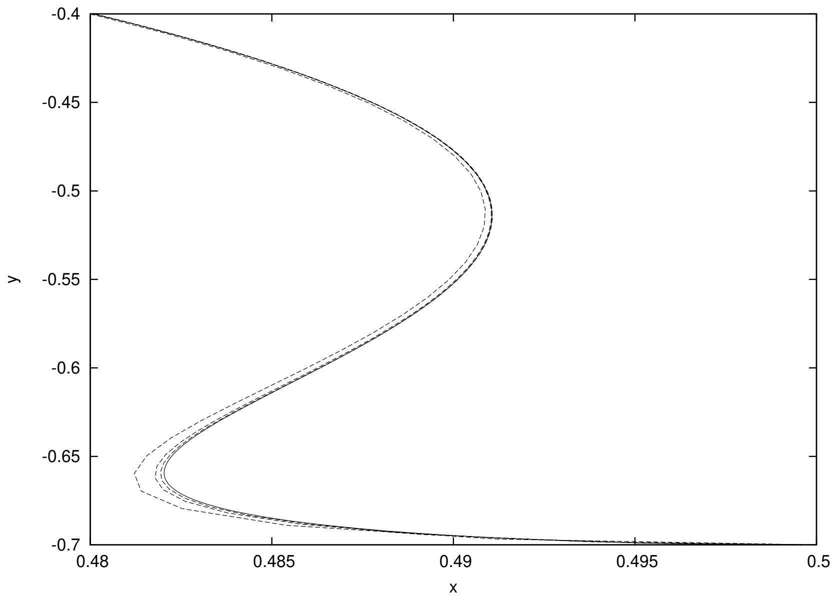

FIGURE 2. $s = 0.7$ ($\omega = 2.14$); the Crapper wave (solid), and the numerical solutions for a periodic traveling wave in the constant vorticity flow for $N = 100, 150, 200$ (dashed) near $x = 0.5$.

Moreover, we may require

$$X_N = 1/2$$

– namely, the unit period. Together, they make $4N + 3$ equations for $4N + 3$ unknowns (15), plus B, L and ω. We will solve them by Newton's method, with s a parameter. See [14] for details.

4.3. **Numerical results.** We follow the method of Section 4.2 and numerically compute the solutions of (14) for various values of s. For sufficiently large N ($N > 200$), we find that the wave profiles are indistinguishable within graphical accuracy from the Crapper's exact solution (see (13)). See, for instance, Figure 1 for $s = 0.2$, 0.4, 0.7 and 0.7298. The corresponding values of ω are 5.47, 3.91, 2.14 and 2.02, respectively.

For $s = 0.7$ (and $\omega = 2.14$), for instance, Figure 2 compares Crapper's exact solution and the numerical solutions of (14) for $N = 100, 150$ and 200 near $x = 0.5$, the trough, where it turns out that the numerical errors are the largest.

To make comparisons more quantitative, we compute the length of a Crapper wave over one period, by numerically evaluating

$$L = 2\int_0^{1/2} \sqrt{X'(\phi)^2 + Y'(\phi)^2}\, d\phi, \tag{16}$$

where X and Y are in (13), using the trapezoidal rule, which is spectrally accurate for periodic functions. Table 1 provides a comparison of the lengths for the numerical solutions of (14) and the numerical evaluation of (16).

	$s = 0.2$	$s = 0.4$	$s = 0.7$
$N = 100$	1.09640843	1.36221253	1.97561843
$N = 200$	1.09638286	1.36206379	1.97324907
$N = 300$	1.09637804	1.36203852	1.97282934
$N = 400$	1.09637633	1.36203002	1.97269855
$N = 500$	1.09637553	1.36202619	1.97264290
Crapper ($N = \infty$)	1.09637405	1.36201963	1.97255882

Table 1. Lengths for the numerical solutions in constant vorticity flows vs. Crapper's exact solution.

5. Conclusion

Asymptotic and numerical evidence suggests that Crapper's celebrated solution for capillary waves in an irrotational flow makes an exact solution for periodic traveling waves in constant vorticity flows when gravitational acceleration is negligible. In Section 3, explicit approximate solutions have been constructed for small amplitude waves, while in Section 4, numerical solutions in constant vorticity flows have been compared with Crapper's exact solution for arbitrary amplitude. While the zero gravity assumption might not be physically realistic for surface water waves, an exact solution in such an idealized situation would be an invaluable tool for rigorous analysis and numerical computation of gravity waves in rotational flows.

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