Robust Moving Horizon State Estimation for Uncertain Linear Systems using Linear Matrix Inequalities

Anastasis Georgiou, Furqan Tahir*, Simos A. Evangelou and Imad M. Jaimoukha

Abstract— **This paper investigates the problem of state estimation for linear-time-invariant (LTI) discrete-time systems subject to structured feedback uncertainty and bounded disturbances. The proposed Robust Moving Horizon Estimation (RMHE) scheme computes at each sample time tight bounds on the uncertain states by solving a linear matrix inequality (LMI) optimization problem based on the available noisy input and output data. In comparison with conventional approaches that use offline calculation for the estimation, the suggested scheme achieves an acceptable level of performance with reduced conservativeness, while the online computational time is maintained relatively low. The effectiveness of the proposed estimation method is assessed via a numerical example.**

Index Terms— **Robust Moving Horizon Estimation, Uncertain Systems, Semidefinite programming, LMI optimization**

I. INTRODUCTION

In most industrial applications the states that characterise the dynamics of a system are not physically measurable and only noisy output measurements through sensors are available. Thus, state estimation plays an important role in different engineering areas such as feedback control, fault detection, system monitoring, as well as system optimization. One of the most popular approaches for state estimation, in a general context, is the Kalman filtering, which is based on the minimization of the variance of the estimation error [1]. However, the main assumptions in the standard Kalman filter approach are that the state-space model of the linear system does not include any uncertainty and thus it accurately represents the real system, and also there are no constraints on the states. As these premises are not satisfied in many industrial applications, the standard Kalman filter may not have robust properties against an uncertain model with disturbances [2].

Recent studies in the literature, which investigate output-feedback control schemes, mostly employ a fixed stable linear observer, such as a Luenberger observer, to compute an estimate of the linear system-state, which is subsequently used within the control scheme (see for example [3], [4], [5], [6]). One of the major advantages of schemes shown in [3] and [5] is that their online computational complexity is similar to that of (full-state) nominal model predictive control (MPC) schemes. The main assumption in [3] and [5] is that the observer has to run for a sufficiently long time before implementing the control scheme, in order to allow the estimation error to enter an invariant set. It is clear that the choice of observer gain has an impact on the estimation error bounds and, therefore, on the overall control algorithm. However, in most of the aforementioned schemes the observer is designed offline (to ensure stability). Consequently, all of the aforementioned offline calculations can potentially add to the conservatism of the corresponding control algorithm.

A very promising online approach to the estimation problem is the so called Moving Horizon Estimation (MHE). Originally proposed by [7] in the early 90s, the estimation scheme suggests estimating the state of a dynamic system by using only the input/output information of the system over the most recent time interval. MHE is a filtering scheme that can be solved online and it can successfully overcome the previously mentioned problems introduced by offline calculations. In the last decade, MHE has become a very popular topic of investigation and its application to linear and non-linear systems has achieved significant success [8]–[13].

Despite the plethora of MHE algorithms proposed in the literature, the contributions when the system is uncertain are scarce. One such contribution is [14], in which the minimization of an upper bound on a worst-case quadratic cost defined over a moving horizon window allows one to construct a filter for uncertain linear systems. This design method is based on the solution of min-max regularized least-squares problems [15]. However, robust least-squares problems are known to have computational difficulties reaching a solution, since they are in general NP-hard [16]. Reduction of the excessive online computational burden can be achieved by reformulating the optimization problem as an equivalent semidefinite programming (SDP) problem. SDP is concerned with optimization problems that have solutions over the cone of all positive semidefinite matrices. SDP is a well-established methodology that allows the solution of a class of problems within a given accuracy in polynomial time using interior-point methods [17].

In the present work, instead of employing an offline linear observer, the past input/output data window is used, in a manner similar to Receding Horizon Estimation (RHE) described in [7], to compute online (tight) bounds on the current state. The main contribution of the paper is the generalization of MHE from systems subject to disturbances only (see [8]) to systems subject to structured feedback uncertainty (see Section II), as well as external disturbances, which is more realistic for applications. In addition, the proposed estimation method reduces conservativeness as compared to observer

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based methods by solving online an optimization problem through LMIs, while keeping the computational burden low. Finally, in the proposed method at every sample time hard bounds on the estimated state are given rather than only the estimated state values, which most of the estimation schemes in the literature compute. Very importantly, hard bounds on the estimated states can potentially be used in a control scheme and improve significantly the robust properties of the controller.

The remainder of this paper is organized as follows. In Section II the problem description is presented. The proposed state estimation is explained in Section III. In Section IV the overall proposed algorithm is given together with simulation results for an exemplary case study from the literature involving a paper-making process. Finally, conclusions are drawn in Section V.

A. Notation

The notation used is fairly standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the space of n-dimensional (column) vectors whose entries are in $\mathbb{R}$, $\mathbb{R}^{n\times m}$ denotes the space of all $n\times m$ matrices whose entries are in $\mathbb{R}$, and $\mathbb{D}^n$ denotes the space of diagonal matrices in $\mathbb{R}^{n\times n}$. For $A\in\mathbb{R}^{n\times m}$, A^T denotes the transpose of A. If $A\in\mathbb{R}^{n\times n}$ is symmetric, $\underline{\lambda}(A)$ denotes the smallest eigenvalue of A and we write $A\succeq 0$ if $\underline{\lambda}(A)\geq 0$ and $A\succ 0$ if $\underline{\lambda}(A)>0$. Analogous definitions apply to the largest eigenvalue $\overline{\lambda}(A)$, with respect to $A\preceq 0$ and $A\prec 0$. We define the norm of $A\in\mathbb{R}^{n\times m}$ as $\|A\|=\sqrt{\overline{\lambda}(AA^T)}$. For $x,y\in\mathbb{R}^n$, the inequality $x<y$ (and similarly $\leq$, $>$ and $\geq$) is interpreted element-wise. The notation I_q denotes the $q\times q$ identity matrix with the subscript omitted when it can be inferred from the context. Let $z\in\mathbb{R}^n$ and denote the i-th element of z by z_i. Then, $\text{diag}(z)$ is the diagonal matrix whose (i,i) entry is z_i. For square matrices $A_1,\ldots,A_m$, $\text{diag}(A_1,\ldots,A_m)$ denotes a block diagonal matrix whose i-th diagonal block is A_i. The symbol e_i denotes the i-th column of the identity matrix of appropriate dimension. The symbol $\otimes$ denotes the Kronecker product. If $\boldsymbol{U}\subseteq\mathbb{R}^{p\times q}$ is a subspace, then operator $\mathscr{B}$ is such that $\mathscr{B}\boldsymbol{U}$ denotes the unit ball of $\boldsymbol{U}$, e.g. $U\in\mathscr{B}\boldsymbol{U}$ denotes that $U\in\boldsymbol{U}$, $\|U\|\leq 1$.

$$\Delta_k\in\mathscr{B}\boldsymbol{\Delta},\quad \mathscr{B}\boldsymbol{\Delta}=\{\Delta\in\boldsymbol{\Delta}:\ \|\Delta\|\leq 1\}.$$

In the proposed formulation (Sections III), we make use of the Schur complement argument. This refers to the result that if $C\succ 0$ then $\begin{bmatrix} A & B\\ \star & C\end{bmatrix}\succeq 0$ if and only if $A-BC^{-1}B^T\succeq 0$, where $\star$ denotes a term easily inferred from symmetry.

To deal with norm-bounded structured uncertainties (usually having repeated and/or full blocks on the diagonal entries), we use the following lemma based on the results in [18].

***Lemma** 1.1:* Let $R=R^T$, F, E, and H be real matrices of appropriate dimensions. Let $\widehat{\boldsymbol{\Delta}}$ be a linear subspace and define the associated linear subspace:

$$\widehat{\Psi}=\{(S,G): S\succ 0,\ S\Delta=\Delta S,\ \Delta G+G^T\Delta^T=0,\ \forall\Delta\in\widehat{\boldsymbol{\Delta}}\}.$$

Then, $R+F\Delta(I-H\Delta)^{-1}E+E^T(I-\Delta^TH^T)^{-1}\Delta^TF^T\succ 0$ and $det(I-H\Delta)\neq 0$ for every $\Delta\in\mathscr{B}\widehat{\boldsymbol{\Delta}}$ if there exists $(S,G)\in\widehat{\Psi}$ such that:

$$\begin{bmatrix} R & E^T+FG^T & FS\\ \star & S+HG^T+GH^T & HS\\ \star & \star & S\end{bmatrix}\succ 0. \tag{1}$$

Finally, we refer to the S-procedure in Section III. This is used to derive simple sufficient (in some cases necessary and sufficient) LMI conditions for the non-negativity or non-positivity of a quadratic function on a set described by quadratic inequality constraints [19].

II. Problem Statement

In this section, the system description including system dynamics, initial condition, disturbances and uncertain signals, is first provided. Then the problem of moving horizon estimation for discrete-time systems subject to bounded disturbances and structured uncertainties is presented as an optimization problem.

A. System Description

The following linear discrete-time system, subject to norm-bounded structured uncertainty and external disturbances, is considered (see for example [20]):

$$\begin{bmatrix} x_{k+1}\\ q_k\\ y_k\end{bmatrix}=\begin{bmatrix} A & B_u & B_w & B_p\\ C_q & D_{qu} & 0 & 0\\ C_y & D_{yu} & D_{yw} & D_{yp}\end{bmatrix}\begin{bmatrix} x_k\\ u_k\\ w_k\\ p_k\end{bmatrix},\quad p_k=\Delta_k q_k, \tag{2}$$

where $k=0,1,2,\ldots$ is the time instant, $\Delta_k\in\mathscr{B}\boldsymbol{\Delta}$, where $\boldsymbol{\Delta}\subseteq\mathbb{R}^{n_p\times n_q}$ is a subspace that captures the uncertainty structure. Furthermore, $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^{n_u}$, $y_k\in\mathbb{R}^{n_y}$, $w_k\in\mathbb{R}^{n_w}$ are the state, input, output and disturbance signal, respectively, at time instant k. Here $p_k\in\mathbb{R}^{n_p}$ and $q_k\in\mathbb{R}^{n_q}$ represent the uncertainty vectors and all other symbols in capital letters denote the appropriate distribution matrices. Only the input u_k and the noisy output y_k are measured and it is assumed that (A,C_y) is detectable and (A,B_u) is stabilizable.

Furthermore, lower and upper bounds $\underline{x}_0$ and $\bar{x}_0$ on the initial state are given a-priori such that (see also Section III):

$$x_0\in\mathscr{X}_0:=\left\{x\in\mathbb{R}^n: \underline{x}_0\leq x\leq\bar{x}_0\right\}. \tag{3}$$

Finally, the unmeasured additive disturbances w_k are bounded by a given nonnegative vector r so that

$$w_k\in\mathscr{W}:=\left\{w\in\mathbb{R}^{n_w}: -r\leq w\leq r\right\}. \tag{4}$$

B. Estimation Problem

The objective of the proposed RMHE algorithm is to compute tight upper/lower bounds on the states using a moving and fixed-size window of past input and output data. The information vectors for the inputs and output are defined as follows:

$$\begin{aligned}\tilde{u}&=[u_{k-N_e}^T,\cdots,\ u_{k-1}^T]^T,\\ \tilde{y}&=[y_{k-N_e}^T,\cdots,\ y_k^T]^T,\end{aligned} \tag{5}$$

where $N_e > 0$ denotes a given estimation horizon. The information vectors are updated every sample time by removing the oldest input/output data while the new output measurement and the latest control input are added. Then the estimation problem can be transform into an optimazation problem as follows:

Problem 2.1: At the time instant k, for given information vectors $(\tilde{u}, \tilde{y})$ and pre-computed state bounds values $(\underline{x}_{k-N_e}, \bar{x}_{k-N_e})$, it is required to find lower and upper bound $(\underline{x}_k,\ \bar{x}_k)$ on the current state that solve the min/max and max/min problems

$$\max_{\underline{x}_k \le x_k} \quad \min_{w_k \in \mathscr{W}_k,\ \Delta_k \in \mathscr{B}\mathbf{\Delta}} \quad e_i^T \underline{x}_k, \tag{6}$$

$$\min_{x_k \le \bar{x}_k} \quad \max_{w_k \in \mathscr{W}_k,\ \Delta_k \in \mathscr{B}\mathbf{\Delta}} \quad e_i^T \bar{x}_k. \tag{7}$$

such that the dynamics in (2) are satisfied.

Such a strategy is developed in this paper as described in more detail in Section III.

Remark 1: Note that uncertainty is allowed in all the problem data including the state and the output signal. It is easy to verify that the state dynamics in (2) can be re-written in the form: $x_{k+1} = (A + B_p \Delta_k C_q) x_k + (Bu + B_p \Delta_k D_{qu}) u_k + B_w w_k$.

Remark 2: For the sake of clarity of exposition, both the state-disturbance (η_k) and output-disturbance (ν_k) are combined into a single vector in (2), namely $w_k := [\eta_k^T \quad \nu_k^T]^T$.

III. State bounds Estimation

This section formulates an optimization problem which uses the past N_e inputs and outputs (as well as the current output y_k) to compute upper and lower state bounds ($\underline{x}_k$ and $\bar{x}_k$), as briefly presented in Section II-B.

We start by iterating the process dynamics in (2) to obtain:

$$\begin{bmatrix} x_k \\ \tilde{q} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_u & \tilde{B}_w & \tilde{B}_p \\ \tilde{C}_q & \tilde{D}_{qu} & \tilde{D}_{qw} & \tilde{D}_{qp} \\ \tilde{C}_y & \tilde{D}_{yu} & \tilde{D}_{yw} & \tilde{D}_{yp} \end{bmatrix} \begin{bmatrix} x_{k-N_e} \\ \tilde{u} \\ \tilde{w} \\ \tilde{p} \end{bmatrix}, \quad \tilde{p} = \tilde{\Delta}\tilde{q}, \tag{8}$$

where the input/output data vectors $\tilde{u}$ and $\tilde{y}$ (defined in (5)) are known, and $\tilde{w} = [w_{k-N_e}^T \cdots\ w_k^T]^T$, $\tilde{q} = [q_{k-N_e}^T \cdots\ q_{k-1}^T]^T$, $\tilde{p} = [p_{k-N_e}^T \cdots\ p_{k-1}^T]^T$ and $\tilde{\Delta} = \mathrm{diag}(\Delta_{k-N_e}, \cdots, \Delta_{k-1})$.All the matrices in (8) are the stacked coefficient matrices, which can be computed easily through iteration using (2).

By using the definition of $\tilde{q}$ in (8), the vector $\tilde{p}$ $(:= \tilde{\Delta}\tilde{q})$ can be rearranged as:

$$\tilde{p} = \tilde{\Delta}(I - \tilde{D}_{qp}\tilde{\Delta})^{-1}(\tilde{C}_q x_{k-N_e} + \tilde{D}_{qu}\tilde{u} + \tilde{D}_{qw}\tilde{w}). \tag{9}$$

Then, using (9) to eliminate $\tilde{p}$ from (8) gives:

$$\begin{bmatrix} x_k \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_d + \tilde{B}_p \overline{\Delta} \tilde{C}_d & \tilde{B}_u + \tilde{B}_p \overline{\Delta} \tilde{D}_{qu} \\ \tilde{C}_{yd} + \tilde{D}_{yp} \overline{\Delta} \tilde{C}_d & \tilde{D}_{yu} + \tilde{D}_{yp} \overline{\Delta} \tilde{D}_{qu} \end{bmatrix} \begin{bmatrix} d \\ \tilde{u} \end{bmatrix}, \tag{10}$$

where $\overline{\Delta} := \tilde{\Delta}(I - \tilde{D}_{qp}\tilde{\Delta})^{-1}$, $A_d := [\tilde{A} \quad \tilde{B}_w]$, $\tilde{C}_{yd} := [\tilde{C}_y \quad \tilde{D}_{yw}]$, $\tilde{C}_d := [\tilde{C}_q \quad \tilde{D}_{qw}]$ and $d := [x_{k-N_e}^T \quad \tilde{w}^T]^T$ such that

$$\begin{bmatrix} \underline{x}_{k-N_e} \\ -\tilde{r} \end{bmatrix} =: \underline{d} \le d \le \overline{d} := \begin{bmatrix} \bar{x}_{k-N_e} \\ \tilde{r} \end{bmatrix}, \tag{11}$$

where $\tilde{r} = \mathbf{1} \otimes r$ and where $\mathbf{1}$ represents the N_e–dimensional vector of ones.

By using (10) and (11), upper- and lower-bounds on x_k are derived in the following theorem.

Theorem 3.1: Let all variables be as defined above. Then, an upper-bound on the ith element of x_k, i.e. $e_i^T x_k$, can be computed if there exist $(\bar{S}_i, \bar{G}_i) \in \widehat{\Psi}$, $\mu_i \in \mathbb{R}^{N_e n_y}$, $0 \prec \overline{D}_x^i \in \mathbb{D}$, $\forall i \in \mathscr{N}_n := \{1, \cdots, n\}$, by minimizing $e_i^T \bar{x}_k$ subject to the LMI

$$\begin{bmatrix} \overline{D}_x^i & \Pi_{12} & \tilde{C}_d^T \overline{G}_i^T & \tilde{C}_d^T \overline{S}_i \\ \star & \Pi_{22} & \mu_i^T \tilde{D}_{yp} - \frac{1}{2} e_i^T \tilde{B}_p + (\tilde{D}_{qu}\tilde{u})^T \overline{G}_i & (\tilde{D}_{qu}\tilde{u})^T \overline{S}_i \\ \star & \star & \overline{S}_i + \tilde{D}_{qp}^T \overline{G}_i^T + \overline{G}_i \tilde{D}_{qp} & \tilde{D}_{qp}^T \overline{S}_i \\ \star & \star & \star & \overline{S}_i \end{bmatrix} \succeq 0, \tag{12}$$

where $\Pi_{12} = -\frac{1}{2}\overline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}A_d^T e_i + \tilde{C}_{yd}^T \mu_i$ and $\Pi_{22} = e_i^T \bar{x}_k + \overline{d}^T \overline{D}_x^i \underline{d} - e_i^T \tilde{B}_u \tilde{u} - 2\mu_i^T y^{\tilde{u}}$.

Similarly, a lower-bound on $e_i^T x_k$, if there exist $(\underline{S}_i, \underline{G}_i) \in \widehat{\Psi}$, $\mu_i \in \mathbb{R}^{N_e n_y}$, $0 \succ \underline{D}_x^i \in \mathbb{D}$, $\forall i \in \mathscr{N}_n := \{1, \cdots, n\}$, can be obtained by maximizing $e_i^T \underline{x}_k$ subject to the LMI

$$\begin{bmatrix} \underline{D}_x^i & \Lambda_{12} & \tilde{C}_d^T \underline{G}_i^T & \tilde{C}_d^T \underline{S}_i \\ \star & \Lambda_{22} & \mu_i^T \tilde{D}_{yp} - \frac{1}{2} e_i^T \tilde{B}_p + (\tilde{D}_{qu}\tilde{u})^T \underline{G}_i & (\tilde{D}_{qu}\tilde{u})^T \underline{S}_i \\ \star & \star & \underline{S}_i + \tilde{D}_{qp}^T \underline{G}_i^T + \underline{G}_i \tilde{D}_{qp} & \tilde{D}_{qp}^T \underline{S}_i \\ \star & \star & \star & \underline{S}_i \end{bmatrix} \preceq 0, \tag{13}$$

where $\Lambda_{12} = -\frac{1}{2}\underline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}A_d^T e_i + \tilde{C}_{yd}^T \mu_i$ and $\Lambda_{22} = e_i^T \underline{x}_k + \overline{d}^T \underline{D}_x^i \underline{d} - e_i^T \tilde{B}_u \tilde{u} - 2\mu_i^T y^{\tilde{u}}$.

Proof: In order to take account of the available past input/output data $(\tilde{u}, \tilde{y})$ in the proposed formulation, the following equality constraint is considered, based on the expression for $\tilde{y}$ in (10):

$$y^{\tilde{\Delta}} - C_d^{\tilde{\Delta}} d = 0, \tag{14}$$

where $y^{\tilde{\Delta}} := \tilde{y} - (\tilde{D}_{yu} + \tilde{D}_{yp}\overline{\Delta}\tilde{D}_{qu})\tilde{u}$ and $C_d^{\tilde{\Delta}} := (\tilde{C}_{yd} + \tilde{D}_{yp}\overline{\Delta}\tilde{C}_d)$. For convenience, we also define $y^{\tilde{u}} = \tilde{y} - \tilde{D}_{yu}\tilde{u}$. Now by considering $\bar{x}_k$ as an upper-bound on x_k in (10), it is required $\forall i \in \mathscr{N}_n$:

$$e_i^T x_k - e_i^T \bar{x}_k = e_i^T (A_d^{\tilde{\Delta}} d + B_{\tilde{u}}^{\tilde{\Delta}} \tilde{u}) - e_i^T \bar{x}_k \le 0, \tag{15}$$

where $A_d^{\tilde{\Delta}} := A_d + \tilde{B}_p \overline{\Delta} \tilde{C}_d$ and $B_{\tilde{u}}^{\tilde{\Delta}} = \tilde{B}_u + \tilde{B}_p \overline{\Delta} \tilde{D}_{qu}$.

By incorporating (14), it can then be verified that for any diagonal $\overline{D}_x^i \succ 0$ and $\mu_i \in \mathbb{R}^{N_e n_y}$

$$\begin{aligned} e_i^T x_k - e_i^T \bar{x}_k = &-(\overline{d} - d)^T \overline{D}_x^i (d - \underline{d}) \\ &- \left(\mu_i^T (y^{\tilde{\Delta}} - C_d^{\tilde{\Delta}} d) + (y^{\tilde{\Delta}} - C_d^{\tilde{\Delta}} d)^T \mu_i\right) \\ &- \hat{d}^T \mathscr{L}_i(\overline{D}_x^i, \tilde{\Delta}, \mu_i)\hat{d},\ \forall i \in \mathscr{N}_n, \end{aligned} \tag{16}$$

where $\hat{d} := [d^T \ 1]^T$ and $\overline{\mathscr{L}}_i(\overline{D}_x^i, \tilde{\Delta}, \mu_i)$ is defined as:

$$\begin{bmatrix} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d}+\underline{d}) - \frac{1}{2}(A_d^{\tilde{\Delta}})^T e_i + (C_d^{\tilde{\Delta}})^T \mu_i \\ \star & e_i^T \overline{x}_k - e_i^T B_{\tilde{u}}^{\tilde{\Delta}} \tilde{u} + \overline{d}^T \overline{D}_x^i \underline{d} - 2\mu_i^T y^{\tilde{\Delta}} \end{bmatrix} \tag{17}$$

By using the constraints (11) and (14) in (16), together with the S-procedure (Farkas' Theorem) [19], it follows that $\overline{\mathscr{L}}_i(\overline{D}_x^i, \tilde{\Delta}, \mu_i) \succ 0$, $\forall i \in \mathscr{N}_n$, is a sufficient condition for (15). By applying a Schur complement argument followed by a re-arrangement, shows that, for all $i \in \mathscr{N}_n$, this sufficient condition can be written as:

$$R_i + F_i\tilde{\Delta}(I - H\tilde{\Delta})^{-1}E + E^T(I - \tilde{\Delta}^T H^T)^{-1}\tilde{\Delta}^T F_i^T \succ 0, \tag{18}$$

where

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{cc|c} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d}+\underline{d}) - \frac{1}{2}A_d^T e_i + \tilde{C}_{yd}^T \mu_i & 0 \\ \star & e_i^T \overline{x}_k + \overline{d}^T \overline{D}_x^i \underline{d} - e_i^T \tilde{B}_u \tilde{u} - 2\mu_i^T y^{\tilde{u}} & \mu_i \tilde{D}_{yp}^T - \frac{1}{2} e_i^T \tilde{B}_p \\ \hline \tilde{C}_d & \tilde{D}_{qu}\tilde{u} & \tilde{D}_{qp}^T \end{array}\right].$$

Using Lemma 1.1 yields the LMI (12) as a sufficient condition for (18) for all $\tilde{\Delta}$. A similar procedure can be used to derive LMI (13) for the lower-bound i.e. $-e_i^T x_k \leq -e_i^T \underline{x}_k$, $\forall i \in \mathscr{N}_n$. ■

***Remark** 3:* The estimated value of the state $\hat{x}_k$ is selected to be the mid-point of the upper and lower bounds of the state computed by the LMIs (12) and (13), i.e $\hat{x}_k = \frac{1}{2}(\overline{x}_k + \underline{x}_k)$. Note that, at the time $k = 0$ the initial estimated value $\hat{x}_0$ is arbitrarily selected to be the mid-point of the known a-priori initial bounds $(\overline{x}_0, \underline{x}_0)$.

***Remark** 4:* Note that the LMIs (12) and (13) always have feasible solutions since they are are used to evaluate upper bounds on $e_i^T \underline{x}_k$ and lower bounds on $e_i^T \overline{x}_k$. The main issue is the tightness of these bounds. The quality of the bounds is illustrated in the example below.

IV. Algorithm Outline and Simulations

In this section the overall proposed strategy is presented and its effectiveness is demonstrated by a benchmark example.

A. Implementation Strategy

The proposed estimation scheme computes online hard upper and lower bounds on the state x_k based on past input/output data. However, at sample time $k=0$ there is no past data to compute the state bounds and the state estimation value. Thus, at the time point $k=0$ the a-priori bounds on x_0 are used and $\hat{x}_0$ is computed (see Section II-A). Subsequently, while more data is collected from the input/output at each iteration, the estimation horizon $\tilde{N}_e$ is incremented until it reaches the pre-specified estimation horizon N_e. During this period the current state bounds $\underline{x}_k$, $\overline{x}_k$ and the estimated state $\hat{x}_k$ are computed by considering all available past data. By the time that $\tilde{N}_e$ is equal to N_e the bounds and the estimated state are calculated by the moving horizon framework presented in Section III. The overall approach can therefore be outlined as follows.

***Algorithm** 1:* Robust Moving Horizon Estimation scheme

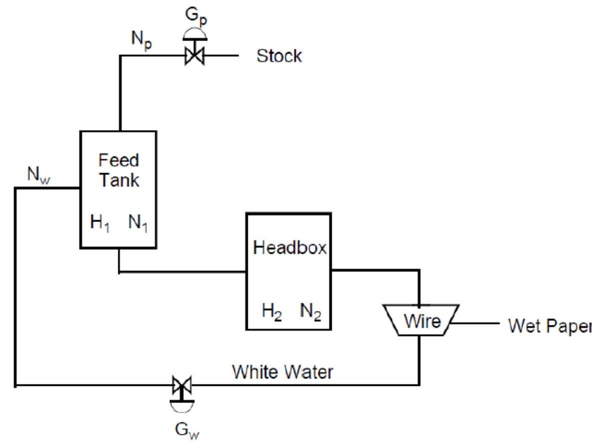

Fig. 1. Schematic of Paper Machine Headbox

(1) Initially at $k = 0$, given a-priori bounds on x_0 compute the estimated state $\hat{x}_0$. Then apply the first control action u_0 onto the system.
(2) Update the vectors $\tilde{u}$, $\tilde{y}$ with the newly available input/output data from the current and previous step (u_{k-1}, y_k).
(3) If $\tilde{N}_e < N_e$, increment $\tilde{N}_e$, else fix $\tilde{N}_e = N_e$. Then, using vectors $\tilde{u}$, $\tilde{y}$ and state bounds $\underline{x}_{k-N_e}$, $\overline{x}_{k-N_e}$ solve the LMI's problem in Theorem 3.1 to compute bounds and estimated state $\hat{x}_k$ of the current state x_k.
(4) Go to step (2).

B. Numerical Example

The benchmark problem of the control of a paper-making process (see for example [21]–[23]) is considered in this subsection to investigate the performance of the proposed estimation scheme. The system, shown in Fig. 1, consists of process states $x = [H_1 \ H_2 \ N_1 \ N_2]^T$, where H_1 and N_1 denote liquid level and composition of the feed tank, respectively, and H_2 and N_2 denote liquid level and composition of the headbox, respectively. The control input vector is given by $u = [G_p \ G_w]^T$, where G_p is the flow rate of stock entering the feed tank and G_w is the recycled white water flow rate. All variables are normalized (i.e. they are zero at steady state) and only noisy measurements of H_2 and N_2 are available. The consistency and composition of white water is a source of uncertainty within the dynamics, particularly in the state N_1 and input G_w. Moreover, disturbance ζ_k affects all four states and v_k denotes the output measurement noise (see Remark 2 to describe the system as shown in (2)).

The discrete-time dynamics (including uncertainty description), sampled at 2 minutes (see [21]), are given by (2) with:

$$A = \begin{bmatrix} 0.0211 & 0 & 0 & 0 \\ 0.1062 & 0.4266 & 0 & 0 \\ 0 & 0 & 0.2837 & 0 \\ 0.1012 & -0.6688 & 0.2893 & 0.4266 \end{bmatrix}$$

$$B_u = \begin{bmatrix} 0.6462 & 0.6462 \\ 0.2800 & 0.2800 \\ 1.5237 & -0.7391 \\ 0.9929 & 0.1507 \end{bmatrix}, \; B_w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \; B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0 & 0 & 0.2 & 0 \end{bmatrix}, \; D_{qu} = \begin{bmatrix} 0 & 0.2 \end{bmatrix}$$

$$C_y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; D_{yw} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

The process disturbance and output measurement noise are respectively characterized by the sets:

$$\zeta_k \in Z := \left\{ \zeta \in \mathbb{R} : -0.1 \leq \zeta \leq 0.1 \right\}$$

$$v_k \in V := \left\{ v \in \mathbb{R} : -0.05 \leq v \leq 0.05 \right\}$$

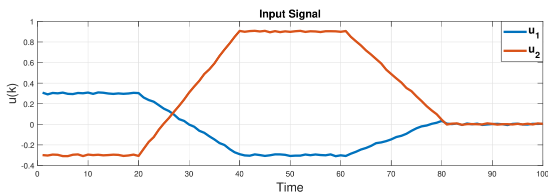

Fig. 2. Control Input

Finally, the estimation horizon for the above set-up is set at $N_e = 15$. For an arbitrary control input sequence u, the objective of the estimation scheme is to compute tight bounds on the states of the system, while the estimation error ($error_i = x_i - \hat{x}_i$) between the actual states and the estimated states is computed.

Figures 2 to 5 show the simulation results. For the sake of comparison with previous works, the classic Luenberger observer and Receding Horizon Estimation (RHE) method proposed by Alessandri in [8], are considered. The control input signal applied to the paper making machine for all estimation algorithms under consideration in this case study is presented in Fig. 2. The value of Δ_k is set equal to 0.5 for all k. Figure 3 shows the state bounds for the measured states (x_2 and x_4), while Fig. 4 illustrates the state bounds for the unmeasured states (x_1 and x_3). For comparison purposes, in these figures the estimated state by utilizing Luenberger observer (dashed light blue) and RHE (dashed black), as well as the actual states (solid blue lines) of the process (not measurable in real time) are also included in these plots. It is noted that the computed bounds almost touch the actual states at some points, which demonstrates their tightness and the effectiveness of the new estimation scheme. It is also important to observe that for both algorithms considered from the literature, sometimes the estimated states are outside the hard bounds provided by the proposed MHE algorithm, which again demonstrates the superiority of the proposed scheme.

Figures 5 and 6 show the state estimation error for the measured and unmeasured states, respectively, where it can be seen that the estimation error using the proposed RMHE converges faster and into a smaller set around zero as compared to the other methods under consideration from the literature.

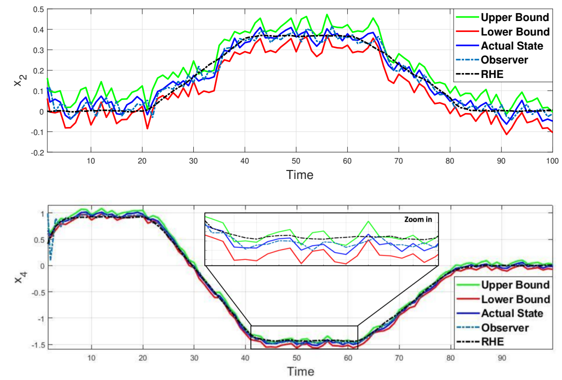

Fig. 3. The observed states x_2 and x_4 for two different estimation schemes (Luenberger observer, RHE), as well as the actual states evolution with their respective computed upper and lower bounds by the proposed RMHE.

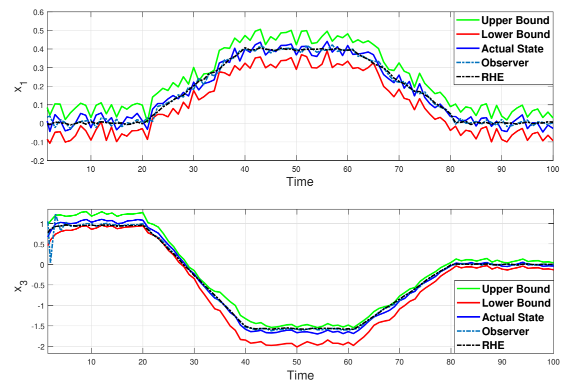

Fig. 4. The unobserved states x_1 and x_3 for two different estimation schemes (Luenberger observer, RHE), as well as the actual states evolution with their respective computed upper and lower bounds by the proposed RMHE.

All the simulations are performed using MATLAB R2017b on a computer with 2.40GHz Intel Xeon(R) CPU and 64.0GB memory. The average online computation time at each sampling time (2 minutes) for the MHE estimation problem at the presented example is 0.8 seconds. Note that the estimation horizon is directly related with the estimation error and computational burden. Although selecting a short estimation horizon results in less online computation time, the estimation error is larger due to the lack of information considered to the estimation problem. On the other hand, continuing to increase the estimation horizon does not improve further the estimation error due to data overfeeding. Therefore, choosing a suitable value of estimation horizon depends on the sampling time of the system (maximum available computation time) and the estimation error improvement that you get by increasing the estimation horizon. In the presented example the maximum estimation horizon is $N_e = 45$, however it is chosen to be $N_e = 15$ since there is no improvement in the estimation error above this value.

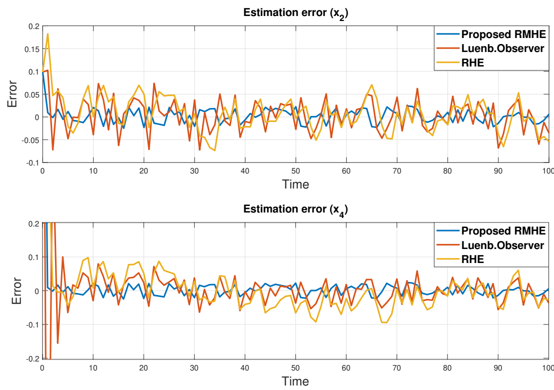

Fig. 5. Estimation error for the observed states x_2 and x_4 using the proposed RMHE method, Luenberger observer and Receding Horizon Estimation [8].

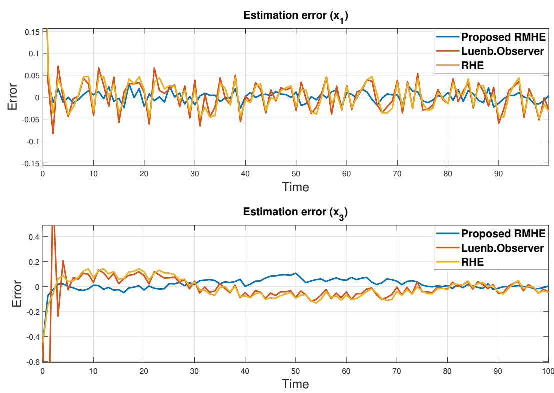

Fig. 6. Estimation error for the unobserved states x_1 and x_3 using the proposed RMHE method, Luenberger observer and Receding Horizon Estimation [8].

V. Conclusion

In this paper an investigation of the estimation problem based on past input/output data of linear discrete-time systems subject to model-uncertainties and bounded disturbances is presented. An online algorithm that computes estimates of the state alongside with tight bounds is suggested, while conservativeness is reduced and computation complexity is maintained low. Importantly, the proposed robust moving horizon estimation (RMHE) algorithm is formulated in a convex form and optimality is guaranteed at every sample time by solving an LMIs optimization problem. Finally, the effectiveness and superior performance of the proposed MHE algorithm as compared to state-of-the-art algorithms in the literature is demonstrated by an industrial process example.

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