The Scenario Approach Meets Uncertain Game Theory and Variational Inequalities

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***Abstract*— Variational inequalities are modeling tools used to capture a variety of decision-making problems arising in mathematical optimization, operations research, game theory. The scenario approach is a set of techniques developed to tackle stochastic optimization problems, take decisions based on historical data, and quantify their risk. The overarching goal of this manuscript is to bridge these two areas of research, and thus broaden the class of problems amenable to be studied under the lens of the scenario approach. First and foremost, we provide out-of-samples feasibility guarantees for the solution of variational and quasi variational inequality problems. Second, we apply these results to two classes of uncertain games. In the first class, the uncertainty enters in the constraint sets, while in the second class the uncertainty enters in the cost functions. Finally, we exemplify the quality and relevance of our bounds through numerical simulations on a demand-response model.**

I. Introduction

Variational inequalities are a very rich class of decision-making problems. They can be used, for example, to characterize the solution of a convex optimization program, or to capture the notion of saddle point in a min-max problem. Variational inequalities can also be employed to describe complementarity conditions, nonlinear systems of equations, or equilibrium notions such as that of Nash or Wardrop equilibrium [1]. With respect to the applications, variational inequalities have been employed in countless fields, including transportation networks, demand-response markets, option pricing, structural analysis, evolutionary biology [2]–[5].

Many of these settings feature a non-negligible source of uncertainty, so that any planned action inevitably comes with a degree of risk. While deterministic models have been widely used as a first order approximation, the increasing availability of raw data motivates the development of data-driven techniques for decision-making problems, amongst which variational inequalities are an important class. As a concrete example, consider that of drivers moving on a road traffic network with the objective of reaching their destination as swiftly as possible. Based on historical data, a given user would like to i) plan her route, and ii) estimate how likely she is to reach the destination within a given time.

Towards this goal, it is natural to consider variational inequalities whose solutions are robust against a set of observed realizations of the uncertainty, as formalized in what follows. Given a collection of sets $\{\mathcal{X}_{\delta_i}\}_{i=1}^N$ in $\mathbb{R}^n$, where $\{\delta_i\}_{i=1}^N$ are independent observations from the probability space $(\Delta, \mathcal{F}, \mathbb{P})$, and given $F : \mathbb{R}^n \to \mathbb{R}^n$, we consider the following variational inequality problem:

$$\text{find } \; x^\star \in \mathcal{X} \doteq \bigcap_{i=1}^N \mathcal{X}_{\delta_i} \;\; \text{s.t. } \; F(x^\star)^\top (x - x^\star) \geq 0 \;\; \forall x \in \mathcal{X}. \tag{1}$$

We assume that no information is available on the distribution $\mathbb{P}$ with which the δ_i's are drawn, and ask the following fundamental question: *how robust is a solution of* (1) *against unseen realizations?*

In this respect, our main objective is to provide probabilistic bounds on the feasibility of a solution to (1), while ensuring that such solution can be computed using a tractable algorithm. Later, we will show how to apply the obtained results to game theoretic models to, e.g., quantify the probability of incurring a higher cost compared to what originally predicted.

Related work. Two formulations are typically employed to incorporate uncertainty into variational inequality models [6]. A first approach, termed *expected-value formulation*, captures uncertainty arising in the corresponding operator F in an average sense. Given $\mathcal{X} \subseteq \mathbb{R}^n$, $F : \mathcal{X} \times \Delta \to \mathbb{R}^n$, and a probability space $(\Delta, \mathcal{F}, \mathbb{P})$, a solution to the expected-value variational inequality is an element $x^\star \in \mathcal{X}$ such that

$$\mathbb{E}[F(x^\star, \delta)]^\top (x - x^\star) \geq 0 \qquad \forall x \in \mathcal{X}. \tag{2}$$

Naturally, if the expectation can be easily evaluated, solving (2) is no harder than solving a deterministic variational inequality, for which much is known (e.g., existence and uniqueness results, as well as algorithms [1]). If this is not the case, one could employ sampling-based algorithms to compute an approximate solution of (2), see [7].

A second approach, which we refer to as the *robust formulation*, is used to accommodate uncertainty both in the operator, and in the constraint sets. Consider the collection $\{\mathcal{X}_\delta\}_{\delta \in \Delta}$, where $\mathcal{X}_\delta \subseteq \mathbb{R}^n$, and let $\mathcal{X} \doteq \cap_{\delta \in \Delta} \mathcal{X}_\delta$. A solution to the robust variational inequality is an element $x^\star \in \mathcal{X}$ s.t.

$$F(x^\star, \delta)^\top (x - x^\star) \geq 0 \qquad \forall x \in \mathcal{X}, \quad \forall \delta \in \Delta. \tag{3}$$

It is worth noting that, even when the uncertainty enters only in F, a solution to (3) is unlikely to exist.[1] The above requirement is hence weakened employing a formulation termed *expected residual minimization* (ERM), see [8].

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[1] To understand this, consider the case when the variational inequality is used to describe the first order condition of a convex optimization program. Within this setting, (3) requires $x^\star$ to solve a *family* of different optimization problems, one per each $\delta \in \Delta$. Thus, (3) only exceptionally has a solution.

Within this setting, given a probability space $(\Delta, \mathcal{F}, \mathbb{P})$, a solution is defined as $x^\star \in \arg\min_{x\in\mathcal{X}} \mathbb{E}[\Phi(x,\delta)]$, where $\Phi : \mathcal{X} \times \Delta \to \mathbb{R}$ is a residual function.[2] In other words, we look for a point that satisfies (3) as best we can (measured through Φ), on average over Δ. Sample-based algorithms for its approximate solution are derived in, e.g., [8].

While the subject of our studies, defined in (1), differs in form and spirit from that of (2), it can be regarded as connected to (3). Indeed, our model can be thought of as a sampled version of (3), where the uncertainty enters only in the constraints. In spite of that, our objectives significantly depart from that of the ERM formulation, as detailed next.

Contributions. The goal of this manuscript is that of *quantifying the risk* associated with a solution of (1) against unseen samples $\delta \in \Delta$, while ensuring that such solution can be computed tractably. Our main contributions are as follows.

i) We provide a-priori and a-posteriori bounds on the probability that the solution of (1) remains feasible for unseen values of $\delta \in \Delta$ (out-of-sample guarantees).
ii) We show that the bounds derived in i) hold for the broader class of *quasi variational inequality* problems.
iii) We leverage the bounds obtained in i) to study Nash equilibrium problems with uncertain constraint sets.
iv) We employ the bounds derived in ii) to give concrete probabilistic guarantees on the performance of Nash equilibria, relative to games with uncertain payoffs, as originally defined by Aghassi and Bertsimas in [9].
v) We consider a simple demand-response scheme and exemplify the applicability and quality of our probabilistic bounds through numerical simulations.

Our results follow the same spirit of those derived within the so-called *scenario approach*, where the sampled counterpart of a robust optimization program is considered, and the risk associated to the solution is bounded in a probabilistic sense [10]–[16]. To the best of the authors' knowledge, our contribution is the first to enlarge the applicability of the scenario approach to the broader class of variational inequalities, and hence to the class of Nash equilibrium problems. While variational inequalities are used to model a wide spectrum of problems, we discuss the impact of our results limitedly to the class of uncertain games, due to space considerations.

Organization. In Section II we introduce the main subject of our analysis, as well as some preliminary notions. Section III contains the main result and its extension to quasi variational inequalities. In Section IV we show the relevance of the bounds previously derived in connection to uncertain games. In Section V we test our results on a demand-response scheme through exhaustive numerical simulations. All the proofs are included in [17], for reasons of space.

II. The scenario approach to variational inequalities

Motivated by the previous discussion, in the remainder of this paper we consider the variational inequality (VI) introduced in (1) and reported in the following:

$$\text{find } \; x^\star \in \mathcal{X} \doteq \bigcap_{i=1}^{N} \mathcal{X}_{\delta_i} \;\; \text{s.t. } \; F(x^\star)^\top (x - x^\star) \geq 0 \;\; \forall x \in \mathcal{X},$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{X}_{\delta_i} \subseteq \mathbb{R}^n$ for $i \in \{1, \ldots, N\}$ are elements of a family of sets $\{\mathcal{X}_\delta\}_{\delta\in\Delta}$. Throughout the presentation we assume that $\{\delta_i\}_{i=1}^N$ are independent samples from the probability space $(\Delta, \mathcal{F}, \mathbb{P})$, though no knowledge is assumed on $\mathbb{P}$. In order to provide out-of-sample guarantees on the feasibility of a solution to (1), we begin by introducing two concepts that play a key role: the notion of risk and that of support constraint.

Definition 1 (Risk)**.** *The* risk *of a given $x \in \mathcal{X}$ is given by $V(x) \doteq \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin \mathcal{X}_\delta\}$.*

The quantity $V(x)$ measures the violation of the constraints defined by $x \in \mathcal{X}_\delta$ for all $\delta \in \Delta$. As such, $V : \mathcal{X} \to [0, 1]$ and, for fixed x, it constitutes a deterministic quantity. Nevertheless, since $x^\star$ is a random variable (through its dependance on $(\delta_1, \ldots, \delta_N)$), the risk $V(x^\star)$ associated with the solution $x^\star$ is also a *random variable*.[3] Our objective will be that of acquiring deeper insight into its distribution.

Standing Assumption (Existence and uniqueness)**.** *For any N and for any tuple $(\delta_1, \ldots, \delta_N)$, the variational inequality* (1) *admits a unique solution identified with $x^\star$.*

Throughout the manuscript, we assume that the Standing Assumption is satisfied, so that $x^\star$ is well defined and unique. It is worth noting that the existence of a solution to (1) is guaranteed under very mild conditions on the operator F and on the constraints set $\mathcal{X}$. Uniqueness of $x^\star$ is instead obtained under structural assumptions on F (e.g., strong monotonicity). While these cases do not encompass all possible variational inequalities arising from (1), the set-up is truly rich and includes important applications such as traffic dispatch [2], cognitive radio systems [18], demand-response markets [3], and many more. Sufficient conditions guaranteeing the satisfaction of the Standing Assumption are presented in Proposition 1, included at the end of this section.

Definition 2 (Support constraint)**.** *A constraint $x \in \mathcal{X}_{\delta_i}$ is of support for* (1)*, if its removal modifies the solution $x^\star$. We denote with $S^\star$ the set of support constraints associated to $x^\star$.*

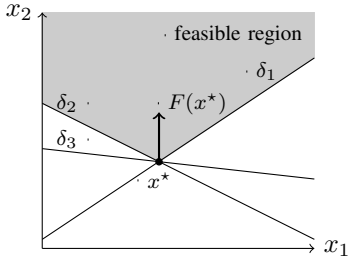

Fig. 1. Example of variational inequality (1) in dimension two. The grey area describes the set $\mathcal{X}$ defined by $\{\delta_i\}_{i=1}^3$. Note that $x^\star$ is a solution of (1) as $x^\star \in \mathcal{X}$, and the inner product between $F(x^\star)$ with any feasible direction at $x^\star$ is non-negative. While δ_1 is a support constraint, δ_2 or δ_3 are not, for the removal of one of them does not change the solution.

[2] A function $\Phi : \mathcal{X} \times \Delta \to \mathbb{R}$ is a residual function if, $\Phi(x, \delta) = 0$ whenever x is a solution of (3) for given δ, and $\Phi(x, \delta) > 0$ elsewhere.

[3] We assume measurability of all the quantities introduced in this paper.

Within the example in Figure 1, note that the removal of the constraints $\mathcal{X}_{\delta_2}$ or $\mathcal{X}_{\delta_3}$ - *one at a time* - does not modify the solution: hence, neither $\mathcal{X}_{\delta_2}$, nor $\mathcal{X}_{\delta_3}$ are support constraints. Nevertheless, the *simultaneous* removal of $\mathcal{X}_{\delta_2}$ and $\mathcal{X}_{\delta_3}$ does change the solution. To rule out degenerate conditions such as this, we introduce the following assumption.

Assumption 1 (Non-degeneracy)**.** *The solution $x^\star$ coincides $\mathbb{P}^N$ almost-surely with the solution obtained by eliminating all the constraints that are not of support.*

Finally, we provide sufficient conditions that guarantee the existence and uniqueness of the solution to (1).

Proposition 1 (Existence and uniqueness, [1])**.**
- *If $\mathcal{X}$ is nonempty compact convex, and F is continuous, then the solution set of* (1) *is nonempty and compact.*
- *If $\mathcal{X}$ is nonempty closed convex, and F is strongly monotone on $\mathcal{X}$, then* (1) *admits a unique solution.*[4]

III. Main result: probabilistic feasibility for variational inequalities

The aim of this section is to provide bounds on the risk associated to the solution of (1) that hold with high confidence. Towards this goal, we introduce the map $t : \mathbb{N} \to [0, 1]$.

Definition 3. *Given $\beta \in (0, 1)$, for any $k \in \{0, \dots, N-1\}$ consider the polynomial equation in the unknown t*

$$\frac{\beta}{N+1} \sum_{l=k}^{N} \binom{l}{k} t^{l-k} - \binom{N}{k} t^{N-k} = 0. \tag{4}$$

Let $t(k)$ be its unique solution in the interval $(0, 1)$.[5] *Further, let $t(k) = 0$ for any $k \geq N$.*

The distribution of the risk $V(x^\star)$ is intimately connected with the number of support constraints at $x^\star$, which we identify with $s^\star = |S^\star|$. Given a confidence parameter $\beta \in (0, 1)$, we wish to construct a function $\varepsilon(s^\star)$ so that

$$\mathbb{P}^N[V(x^\star) \leq \varepsilon(s^\star)] \geq 1 - \beta$$

holds for any variational inequality (1) satisfying the required assumptions. Theorem 1 employs $t(s^\star)$ to construct $\varepsilon(s^\star)$.

Theorem 1 (Probabilistic feasibility for VI)**.** *Given $\beta \in (0, 1)$, consider $t : \mathbb{N} \to \mathbb{R}$ as per Definition 3.*

(i) *Under the Standing Assumption, and Assumption 1, for any Δ and $\mathbb{P}$ it holds that*

$$\mathbb{P}^N[V(x^\star) \leq \varepsilon(s^\star)] \geq 1-\beta \quad \textit{with} \quad \varepsilon(s^\star) \doteq 1-t(s^\star), \tag{5}$$

(ii) *If, in addition, the constraint sets $\{\mathcal{X}_{\delta_i}\}_{i=1}^N$ are convex, then $s^\star \leq n$ (dimension of the decision variable x), and the following a-priori bound holds for all Δ and $\mathbb{P}$*

$$\mathbb{P}^N[V(x^\star) \leq \varepsilon(n)] \geq 1 - \beta \quad \textit{with} \quad \varepsilon(n) \doteq 1 - t(n).$$

[4] An operator $F : \mathcal{X} \to \mathbb{R}^n$ is strongly monotone on $\mathcal{X}$ if there exists $\alpha > 0$ such that $(F(x)-F(y))^\top (x-y) \geq \alpha ||x-y||^2$ for all $x, y \in \mathcal{X}$. If F is continuously differentiable, a sufficient (and easily checkable) condition amounts to requiring the Jacobian of F to be uniformly positive definite, that is $y^\top JF(x)y \geq \alpha ||y||^2$ for all $y \in \mathbb{R}^n$ for all $x \in \mathcal{X}^{\circ}$, where $\mathcal{X}^{\circ}$ is an open superset of $\mathcal{X}$, see [1, Prop. 2.3.2].

[5] Existence and uniqueness of the solution to (4) is shown in [16, Thm. 2].

The first statement in Theorem 1 provides an a-posteriori bound, and requires no additional assumptions other than the Standing Assumption and Assumption 1 (e.g., no convexity of the constraint sets is required). In practice, one computes a solution to (1), determines $s^\star$, and is then given a probabilistic feasibility statement for any choice of $\beta \in (0, 1)$.[6] In this respect, we are typically interested in selecting β very small (e.g., 10^{-6}) so that the statement $V(x^\star) \leq \varepsilon(s^\star)$ holds with very high confidence (e.g., $1-10^{-6} = 0.999999$). Upon assuming convexity of the constraints sets, the second statement provides an a-priori bound of the form (5) where $s^\star$ is replaced by n (the dimension of the decision variable). Overall, Theorem 1 shows that the upper bound on the risk derived in [16, Thm. 4] is not limited to optimization programs, but holds for variational inequality problems too.

Computational aspects. While Theorem 1 provides certificates of probabilistic feasibility, its result is of practical interest especially if it is possible to determine a solution of (1) *efficiently*. With respect to the computational aspects, much is known for monotone variational inequalities, i.e., those variational inequalities where the operator F is monotone or strongly monotone (see footnote 4). Examples of efficient algorithms applicable to this class include projection methods, proximal methods, splitting and interior point methods, see [1, Chap. 12]. On the contrary, if the operator F is *not monotone*, the problem is intractable to solve in the worst-case. Indeed, non-monotone variational inequalities hold non-monotone linear complementarity problems as a special case. The latter class is $\mathcal{NP}$-complete [19].

A. Extension to quasi variational inequalities

In this section we show how the results of Theorem 1 carry over to the case when (1) is replaced by a more general class of problems known as quasi variational inequality (QVI). Quasi variational inequalities extend the notion of variational inequality by allowing the decision set $\mathcal{X}$ to be parametrized by x, see [20]. QVIs are important tools used to model complex equilibrium problems arising in various fields including game theory, transportation network, solid mechanics, biology [5], [21], [22]. As we shall see in Section IV, this generalization will be used to provide concrete performance guarantees for robust Nash equilibrium problems, when the uncertainty enters in the agents' cost functions. Let $\mathcal{X}_{\delta_i} : \mathbb{R}^n \rightrightarrows 2^{\mathbb{R}^n}$ be elements of a collection of set-valued maps $\{\mathcal{X}_\delta\}_{\delta \in \Delta}$, for $i \in \{1, \dots, N\}$. Given $F : \mathbb{R}^n \to \mathbb{R}^n$, we consider the following quasi variational inequality problem: find $x^\star \in \mathcal{X}(x^\star) \doteq \cap_{i=1}^N \mathcal{X}_{\delta_i}(x^\star)$ such that

$$F(x^\star)^\top (x - x^\star) \geq 0 \qquad \forall x \in \mathcal{X}(x^\star). \tag{6}$$

Once more, we assume that $\{\delta_i\}_{i=1}^N$ are independent samples from the probability space $(\Delta, \mathcal{F}, \mathbb{P})$. Additionally, we assume that (6) admits a unique solution. The notion of support constraint carries over unchanged from Definition 2, while the notion of risk requires a minor adaptation.

[6] Computing the number of support constraints can be easily achieved by solving the original problem where constraints are removed one at a time.

Definition 4 (Risk for QVI)**.** *The* risk *associated to x is*

$$V(x) \doteq \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin \mathcal{X}_\delta(x)\}$$

The next theorem shows that the main result presented in Theorem 1 extends to quasi variational inequalities.

Theorem 2 (Probabilistic feasibility for QVI)**.** *Let $x^\star$ be the (unique) solution of* (6) *and $s^\star$ be the number of support constraints. Let Assumption 1 hold. Given $\beta \in (0,1)$, let $t : \mathbb{N} \to \mathbb{R}$ be as per Definition 3. Then, for any Δ, $\mathbb{P}$, it is*

$$\mathbb{P}^N[V(x^\star) \leq \varepsilon(s^\star)] \geq 1-\beta \quad \textit{where} \quad \varepsilon(s^\star) \doteq 1 - t(s^\star). \tag{7}$$

If, in addition, the sets $\{\mathcal{X}_{\delta_i}(x^\star)\}_{i=1}^N$ are convex, then $s^\star \leq n$ and the bound (7) *holds a-priori with n in place of $s^\star$.*

IV. Application to robust game theory

A. Uncertainty entering in the constraint sets

We begin by considering a general game-theoretic model, where agents aim to minimize private cost functions, while satisfying uncertain local constraints *robustly*. Formally, each agent $j \in \mathcal{M} = \{1, \dots, M\}$ is allowed to select $x^j \in \mathcal{X}^j \doteq \cap_{i=1}^N \mathcal{X}^j_{\delta_i} \subseteq \mathbb{R}^m$, where $\{\mathcal{X}^j_{\delta_i}\}_{i=1}^N$ is a collection of sets from the family $\{\mathcal{X}^j_\delta\}_{\delta\in\Delta}$, and $\{\delta_i\}_{i=1}^N$ are independent samples from the probability space $(\Delta, \mathcal{F}, \mathbb{P})$. Agent $j \in \mathcal{M}$ aims at minimizing the cost function $J^j : \mathcal{X}^j \to \mathbb{R}$. To ease the notation, we define $x^{-j} = (x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^M)$, for any $j \in \mathcal{M}$. We consider the notion of Nash equilibrium.

Definition 5 (Nash equilibrium)**.** *A tuple $x_{\rm NE} = (x^1_{\rm NE}, \dots, x^M_{\rm NE})$ is a Nash equilibrium if $x_{\rm NE} \in \mathcal{X}^1 \times \cdots \times \mathcal{X}^M$ and $J^j(x^j_{\rm NE}, x^{-j}_{\rm NE}) \leq J^j(x^j, x^{-j}_{\rm NE})$ for all deviations $x^j \in \mathcal{X}^j$ and for all agents $j \in \mathcal{M}$.*

Assumption 2. *For all $j \in \mathcal{M}$, the cost function J^j is continuously differentiable, and convex in x^j for any fixed x^{-j}. The sets $\{\mathcal{X}^j\}_{j=1}^M$ are non-empty, closed, convex for every tuple $(\delta_1, \dots, \delta_N)$, for every N.*

The next proposition, adapted from [1] draws the key connection between Nash equilibria and variational inequalities.

Proposition 2 ([1, Prop. 1.4.2])**.** *Let Assumption 2 hold. A point $x_{\rm NE}$ is a Nash equilibrium iff it solves* (1)*, with*

$$F(x) \doteq \begin{bmatrix} \nabla_{x^1} J^1(x) \\ \vdots \\ \nabla_{x^M} J^M(x) \end{bmatrix}, \quad \mathcal{X}_{\delta_i} \doteq \mathcal{X}^1_{\delta_i} \times \cdots \times \mathcal{X}^M_{\delta_i}. \tag{8}$$

Within the previous model, the uncertainty described by $\delta \in \Delta$ is meant as *shared* among the agents. This is indeed the most common and challenging situation. In spite of that, our model also includes the case of *non-shared* uncertainty, i.e., the case where $\mathcal{X}^j_\delta$ is of the form $\mathcal{X}^j_{\delta^j}$ as δ can represent a vector of uncertainty. Limitedly to the latter case, it is possible to derive probabilistic guarantees on each agent's feasibility by direct application of the scenario approach [11] to each agent optimization $x^j_{\rm NE} \in \arg\min_{x^j \in \mathcal{X}^j} J^j(x^j, x^{-j}_{\rm NE})$, after having fixed $x^{-j} = x^{-j}_{\rm NE}$. Nevertheless, for the case of shared uncertainty, a direct application of [16] provides no answer. Instead, the following corollary offers feasibility guarantees for $x_{\rm NE}$. In this context, a constraint is of support for the Nash equilibrium problem, if its removal alters the solution.

Corollary 1 (Probabilistic feasibility for $x_{\rm NE}$)**.**

- *Let Assumption 2 hold. Then, a Nash equilibrium exists.*
- *Further assume that the operator F defined in* (8) *is strongly monotone. Then, $x_{\rm NE}$ is unique.*
- *Fix $\beta \in (0,1)$, and let $t : \mathbb{N} \to [0,1]$ be as in Definition 3. In addition to the previous assumptions, assume that $x_{\rm NE}$ coincides $\mathbb{P}^N$ almost-surely with the Nash equilibrium of a game obtained eliminating all the constraints that are not of support. Then, the following a-posteriori and a-priori bounds hold for any Δ and $\mathbb{P}$*

$$\mathbb{P}^N[V(x_{\rm NE}) \leq \varepsilon(s^\star)] \geq 1-\beta \text{ with } \varepsilon(s^\star) \doteq 1 - t(s^\star),$$
$$\mathbb{P}^N[V(x_{\rm NE}) \leq \varepsilon(n)] \geq 1-\beta \text{ with } \varepsilon(n) \doteq 1 - t(n),$$

where n is the dimension of the decision variable x, and $s^\star$ is the number support constraints of $x_{\rm NE}$.

A consequence of Corollary 1 is the possibility to bound the infeasibility risk associated to any agent $j \in \mathcal{M}$. Indeed, let $V^j(x) \doteq \mathbb{P}\{\delta \in \Delta \text{ s.t. } x^j \notin \mathcal{X}^j\}$. Since $V^j(x) \leq V(x)$, Corollary 1 ensures that $\mathbb{P}^N[V^j(x_{\rm NE}) \leq \varepsilon(s^\star)] \geq 1-\beta$.

B. Uncertainty entering in the cost functions

We consider a game-theoretic model where the cost function associated to each agent depends on an uncertain parameter. Within this setting, we first revisit the notion of *robust equilibrium* introduced in [9]. Our goal is to exploit the results of Section III and bound the probability that an agent will incur a higher cost, compared to what predicted.

Let $\mathcal{M} = \{1, \dots, M\}$ be a set of agents, where $j \in \mathcal{M}$ is constrained to select $x^j \in \mathcal{X}^j$. Denote $\mathcal{X} \doteq \mathcal{X}^1 \times \cdots \times \mathcal{X}^M$. The cost incurred by agent $j \in \mathcal{M}$ is described by the function $J^j(x^j, x^{-j}; \delta) : \mathcal{X} \times \Delta \to \mathbb{R}$. Since J^j depends both on the decision of the agents, and on the realization of $\delta \in \Delta$, the notion of Nash equilibrium is devoid of meaning. Instead, [9], [23] propose the notion of *robust equilibrium* as a robustification of the former.[7] While a description of the uncertainty set Δ is seldom available, agents have often access to past realizations $\{\delta_i\}_{i=1}^N$, which we assume to be independent samples from $(\Delta, \mathcal{F}, \mathbb{P})$. It is therefore natural to consider the "sampled" counterpart of a robust equilibrium.

Definition 6 (Sampled robust equilibrium)**.** *Given samples $\{\delta_i\}_{i=1}^N$, a tuple $x_{\rm SR}$ is a sampled robust equilibrium if $x_{\rm SR} \in \mathcal{X}$ and $\max_{i\in\{1,\dots,N\}} J^j(x^j_{\rm SR}, x^{-j}_{\rm SR}; \delta_i) \leq \max_{i\in\{1,\dots,N\}} J^j(x^j, x^{-j}_{\rm SR}; \delta_i)$, $\forall x^j \in \mathcal{X}^j$, $\forall j \in \mathcal{M}$.*

Observe that $x_{\rm SR}$ can be thought of as a Nash equilibrium with respect to the worst-case cost functions

$$J^j_{\max}(x) \doteq \max_{i\in\{1,\dots,N\}} J^j(x; \delta_i). \tag{9}$$

In parallel to what discussed in Section IV-A, the uncertainty should be regarded as *shared* amongst the agents. In this

[7] A feasible tuple $x_{\rm R}$ is a robust equilibrium if $\forall j \in \mathcal{M}$, $\forall x^j \in \mathcal{X}^j$, it is $\max_{\delta\in\Delta} J^j(x^j_{\rm R}, x^{-j}_{\rm R}; \delta) \leq \max_{\delta\in\Delta} J^j(x^j, x^{-j}_{\rm R}; \delta)$, see [9], [23].

context, we are interested in bounding the probability that a given agent $j \in \mathcal{M}$ will incur a higher cost, compared to what predicted by the empirical worst case $J^j_{\max}(x_{\rm SR})$.

Definition 7 (Agent's risk)**.** *The risk incurred by agent $j \in \mathcal{M}$ at the given $x \in \mathcal{X}$ is*

$$V^j(x) = \mathbb{P}\left\{\delta \in \Delta \text{ s.t. } J^j(x;\delta) \geq J^j_{\max}(x)\right\}$$

In addition to existence and uniqueness results, the following corollary provides a bound on such risk measure.

Corollary 2 (Probabilistic feasibility for $x_{\rm SR}$)**.** *Assume that, for all $j \in \mathcal{M}$, the cost function J^j is continuously differentiable, as well as convex in x^j for fixed x^{-j} and δ. Assume that the sets $\{\mathcal{X}^j\}_{j=1}^M$ are non-empty, closed, convex.*

- *Then, a sampled robust equilibrium exists.*
- *Further assume that, for all tuples $(\delta_1, \dots, \delta_N)$, and N,*

$$F(x) \doteq \begin{bmatrix} \partial_{x^1} J^1_{\max}(x) \\ \vdots \\ \partial_{x^M} J^M_{\max}(x) \end{bmatrix} \tag{10}$$

  is strongly monotone.[8] *Then $x_{\rm SR}$ is unique.*
- *Fix $\beta \in (0,1)$. Let $\varepsilon(k) = 1 - t(k)$, $k \in \mathbb{N}$, with $t : \mathbb{N} \to [0,1]$ as in Definition 3. In addition to the previous assumptions, assume that $x_{\rm SR}$ coincides $\mathbb{P}^N$ almost-surely with the robust sampled equilibrium of a game obtained by eliminating all the constraints that are not of support. Then, for any agent $j \in \mathcal{M}$, any Δ, $\mathbb{P}$*

$$\begin{aligned} &\mathbb{P}^N[V^j(x_{\rm SR}) \leq \varepsilon(s^\star)] \geq 1 - \beta, \\ &\mathbb{P}^N[V^j(x_{\rm SR}) \leq \varepsilon(n+M)] \geq 1 - \beta, \end{aligned} \tag{11}$$

  where $s^\star$ is the number support constraints of $x_{\rm SR}$.

Corollary 2 ensures that, for any given agent $j \in \mathcal{M}$, the probability of incurring a higher cost than $J^j_{\max}(x_{\rm SR})$ is bounded by $\varepsilon(s^\star)$, with high confidence.

V. An application to demand-response markets

In this section, we consider a demand response scheme where electricity scheduling happens 24-hours ahead of time, agents are risk-averse and self-interested. Formally, given a population of agents $\mathcal{M} = \{1, \dots, M\}$, agent $j \in \mathcal{M}$ is interested in the purchase of x^j_t electricity-units at the discrete time $t \in \{1, \dots, T\}$, through a demand-response scheme. Agent $j \in \mathcal{M}$ is constrained in his choice to $x^j \in \mathcal{X}^j \subseteq \mathbb{R}^T_{\geq 0}$ convex, as dictated by its energy requirements. Let $\sigma(x) = \sum_{j=1}^n x^j$ be the total consumption profile. Given an inflexible demand profile $\mathbf{d} = [d_1, \dots, d_T] \in \mathbb{R}^T_{\geq 0}$ corresponding to the non-shiftable loads, the cost incurred by each agent j is given by its total electricity bill

$$J^j(x^j, \sigma(x); \mathbf{d}) = \sum_{t=1}^T (\alpha_t \sigma_t(x) + \beta_t d_t) x^j_t, \tag{12}$$

where we have assumed that, at time t, the unit-price of electricity $c_t\sigma_t(x) + \beta_t d_t$ is a sole function of the shiftable load $\sigma_t(x)$ and of the inflexible demand d_t (with $\alpha_t, \beta_t > 0$), in the same spirit of [24], [25]. In a realistic set-up, each agent has access to a history of previous profiles $\{\mathbf{d}_i\}_{i=1}^N$ (playing the role of $\{\delta_i\}_{i=1}^N$), which we assume to be independent samples from the probability space $(\Delta, \mathcal{F}, \mathbb{P})$, though $\mathbb{P}$ is not known. We model the agents as self-interested and *risk-averse*, so that the notion of sampled robust equilibrium in Definition 6 is well suited. Assumption 2 is satisfied, while the operator F defined in (10) is strongly monotone for every N and tuple $(\mathbf{d}_1, \dots, \mathbf{d}_N)$.[9] By Corollary 2, $x_{\rm SR}$ exists and is unique. Additionally, under the non-degeneracy assumption, we inherit the probabilistic bounds (11), whose quality and relevance we aim to test in the following numerics.

We use California's winter daily consumption profiles (available at [26]), as samples of the inflexible demand $\{\mathbf{d}_i\}_{i=1}^N$, on top of which we imagine to deploy the demand-response scheme. In order to verify the quality of our bounds - and only for that reason - we fit a multidimensional Gaussian distribution $\mathcal{N}(\mu, \Sigma)$ to the data. Figure 2 displays 100 samples from the dataset [26] (left), and 100 synthetic samples from the multidimensional Gaussian model (right).

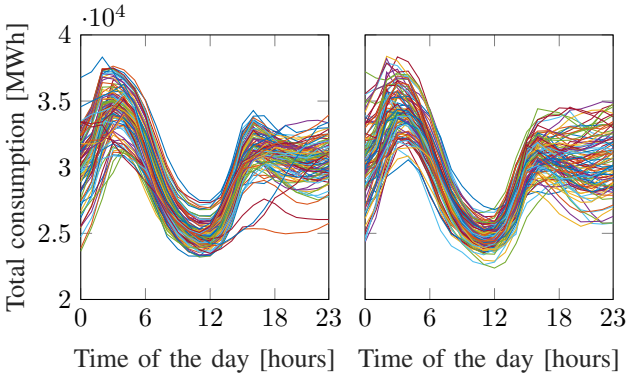

Fig. 2. Left: data samples from [26]. Right: synthetic samples $\sim \mathcal{N}(\mu, \Sigma)$.

We assume that the agents' constraint sets are given by $\mathcal{X}^j = \{x^j \in \mathbb{R}^{24}_{\geq 0}, \text{ s.t. } \sum_{t=1}^{24} x^j_t \geq \gamma^j\}$, where γ^j is randomly generated according to a truncated gaussian distribution with mean 480, standard deviation 120 and $400 \leq \gamma^j \leq 560$, all in MWh. We set $\alpha_t = \beta_t = 500\$/\text{MWh}^2$, and consider $M = 100$ agents representing, e.g., electricity aggregators. We let $N = 500$ samples (i.e., a history of 500 days) to make the example realistic. Since $n+M = 2500$, the a-priori bound in (11) is not useful. On the other hand, the values of $s^\star$ observed after extracting $\{\mathbf{d}_i\}_{i=1}^N$ from $\mathcal{N}(\mu, \Sigma)$ and computing the solution $x_{\rm SR}$, are in the range $3 \leq s^\star \leq 7$. Considering the worst-case with $s^\star = 7$, and setting $\beta =$

[8] $\partial_{x^j} J^j_{\max}(x)$ denotes the subgradient of $J^j_{\max}$ with respect to x^j, computed at x. While the operator $F(x)$ is now set valued, the definition of strong monotonicity given in footnote 4 can be easily generalized.

[9] This can be seen upon noticing that $J^j_{\max}(x) = \sum_{t=1}^T (\alpha_t \sigma_t(x)) x^j_t + \max_{i \in \{1,\dots,N\}} (B\mathbf{d}_i)^\top x^j$, where $B = \text{diag}(\beta_1, \dots, \beta_T)$. Correspondingly, the operator F is obtained as the sum of two contributions $F = F_1 + F_2$. The operator F_1 is relative to a game with costs $\{\sum_{t=1}^T (\alpha_t \sigma_t(x)) x^j_t\}_{j=1}^M$, and F_2 is relative to a game with costs $\{\max_{i \in \{1,\dots,N\}} (B\mathbf{d}_i)^\top x^j\}_{j=1}^M$. While F_1 has been shown to be strongly monotone in [3, Lem 3.], F_2 is monotone as it is obtained stacking one after the other the subdifferentials of the convex functions $\{\max_{i \in \{1,\dots,N\}} (B\mathbf{d}_i)^\top x^j\}_{j=1}^M$. Thus, F is strongly monotone.

10^{-6}, the a-posteriori bound in (11) gives $V^j(x_{\rm SR}) \leq \varepsilon(7) = 6.49\%$ for all agents, with a confidence of 0.999999. Since the cost $J^j(x^j, \sigma(x); \mathbf{d})$ is linear in $\mathbf{d}$, and $\mathbf{d} \sim \mathcal{N}(\mu, \Sigma)$, it is possible to compute the risk at the solution $V^j(x_{\rm SR})$ in closed form. This calculation reveals that the highest risk over all the agents is $0.16\% \leq 6.49\% = \varepsilon(7)$, in accordance to Corollary 2 (the lowest value is 0.11%).

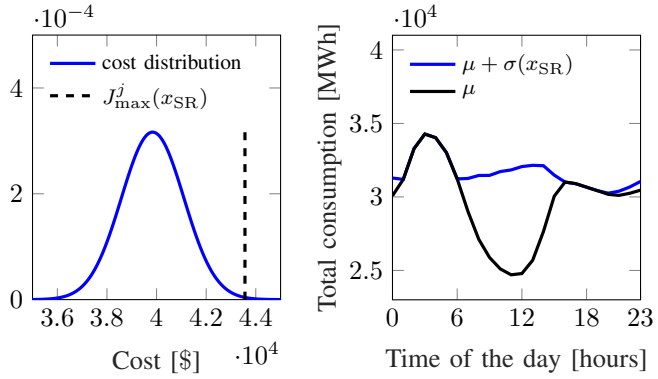

Fig. 3. Left: cost distribution for the agent with the highest risk. Right: sum of average inflexible demand $\mu = \mathbb{E}[\mathbf{d}]$, and flexible demand $\sigma(x_{\rm SR})$.

Figure 3 (left) shows the distributions of the cost for the agent with the highest risk. Figure 3 (right) shows the sum of the average inflexible demand μ, and the flexible demand $\sigma(x_{\rm SR})$. The difference between $\varepsilon(7) = 6.49\%$ and $0.11\% \leq V^j(x_{\rm SR}) \leq 0.16\%$, $j \in \mathcal{M}$ is partly motivated, by the request that the bound $V^j(x_{\rm SR}) \leq \varepsilon(7) = 6.49\%$ holds true for very high confidence 0.999999. While an additional source of conservatism might be ascribed to having used $V(x_{\rm SR}) \leq \varepsilon(s^\star)$ to derive $V^j(x_{\rm SR}) \leq \varepsilon(s^\star)$ (see the proof of Corollary 2 in Appendix), this is not the case relative to the setup under consideration. Indeed, Monte Carlo simulations show that $V(x_{\rm SR}) \approx 0.17\%$, comparably with $V^j(x_{\rm SR})$. In other words, a realization that renders $x_{\rm SR}$ unfeasible for agent j is also likely to make $x_{\rm SR}$ unfeasible for agent $l \neq j$.

VI. Conclusion

In this manuscript, we aimed at unleashing the power of the scenario approach to the rich class of problems described by variational and quasi variational inequalities. As fundamental contributions, we provided a-priori and a-posteriori bounds on the probability that the solution of (1) or (6) remains feasible against unseen realizations. We then showed how to leverage these results in the context of uncertain game theory. While this work paves the way for the application of the scenario approach to a broader class of real-world applications, it also generates novel research questions. An example that warrants further attention is that of *tightly* bounding the risk incurred by individual players, when taking data-driven decisions in multi-agent systems.

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