Reliable H_∞ Control for Discrete-Time Piecewise Linear Systems with Infinite Distributed Delays ⋆

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Abstract

In this paper, the reliable H_∞ control problem is investigated for discrete-time piecewise linear systems with time delays and actuator failures. The time delays are assumed to be infinitely distributed in the discrete-time domain, and the possible failure of each actuator is described by a variable varying in a given interval. The aim of the addressed reliable H_∞ control problem is to design a controller such that, for the admissible infinite distributed delays and possible actuator failures, the closed-loop system is exponentially stable with a given disturbance attenuation level γ. The controller gain is characterized in terms of the solution to a linear matrix inequality that can be easily solved by using standard software packages. A simulation example is exploited in order to illustrate the effectiveness of the proposed design procedures.

Key words: Piecewise linear systems; H_∞ control; reliable control; infinite distributed delays; actuator failure.

1 Introduction

The control systems with piecewise components, such as dead-zone, saturation and relays, are called piecewise linear (PWL) systems that have become an important area of research. It has also been shown that the PWL systems could approximate nonlinear ones in an efficient way. Therefore, over the past decade, The stability analysis, control and filtering synthesis problems for PWL systems have been extensively investigated, see e.g. [1,7]. In particular, for the *discrete-time* PWL systems, the analysis and synthesis problems have been dealt with by some researchers [1, 7]. On the other hand, reliable controller problems for linear or nonlinear systems have gained considerable attention and a number of results have been reported in the literature, see, example, [5,8].

Recently, there has been significant progress on both the analysis and synthesis issues for linear/nonlinear systems with various types of delays, see e.g. [2, 4]. It is worth pointing out that the *distributed delay* occurs very often in reality which has drawn increasing research attention, see e.g. [3, 4]. However, *almost all existing works* on distributed delays have exclusively focused on continuous-time systems that are described in the form of either finite or infinite integral. With the increasing application of digital control systems, the distributed delays may appear digitally (i.e. in a discrete-time manner) and therefore it becomes desirable to study the discrete-time systems with distributed delays. Very recently, in [4], some pioneering work has been carried out on the formulation of discrete-time distributed delays. Up to date, to the best of the authors' knowledge, the research on *discrete-time* piecewise linear systems with *discrete-time* distributed delays is still an open problem that deserves further investigation.

Motivated by the above discussion, in this paper, we consider the reliable H_∞ control problem for a class of discrete-time PWL systems with infinite distributed delays and actuator failures. The objective is to design a reliable H_∞ controller such that, in the presence of time delays as well as actuator failure, the closed-loop PWL control system is exponentially stable and also satisfies a prescribed H_∞ disturbance attenuation index.

2 Problem Formulation

Consider the following discrete-time piecewise linear systems with infinite distributed delays:

$$\begin{aligned} x(k+1) &= A_l x(k) + D_l \sum_{d=1}^{\infty} \mu_d x(k-d) \\ &\quad + B_l u(k) + E_l v(k), \qquad (1) \\ z(k) &= C_l(k) x(k), \qquad \text{for } x \in S_l,\ l = 1, \dots, L, \qquad (2) \\ x(k) &= \phi(k), \quad \forall\, k \in \mathbb{Z}^-, \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is the state, $\{S_l\}_{l \in L}$ denotes a partition of the state space into a number of closed polyhedral subspaces, L is the index set of subspaces, $z(k) \in \mathbb{R}^q$ is the controlled output vector, $u(k) \in \mathbb{R}^m$ is the control input, $v(k) \in l_2[0, \infty)$ is the disturbance input, $\phi(k)$ ($\forall\, k \in \mathbb{Z}^-$) is the initial state, and A_l, B_l, C_l, D_l and E_l are all constant matrices with appropriate dimensions corresponding to the l-th local model of the systems. When the state of the system transits from one region to another at the time k, the dynamics is governed by the local model of the former one. $\mu_d \geq 0$ is the convergence constants that satisfy the following condition:

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$$\bar{\mu} := \sum_{d=1}^{\infty} \mu_d \leq \sum_{d=1}^{\infty} d\mu_d < +\infty. \qquad (3)$$

As in [1], for presentation convenience, we define $\Omega := \{l, j | x(k) \in S_l, x(k+1) \in S_j, j \neq l\}$ to represent all possible transitions from one region to itself or another.

Remark 1 *Distributed time-delays have been widely recognized and intensively studied for continuous-time systems, see e.g. [3, 4]. However, the corresponding results for discrete-time systems have been very few due mainly to the difficulty in formulating the distributed delays in a discrete-time domain. In (1)-(2), we introduce the distributed delay term* $\sum_{d=1}^{\infty} \mu_d x(k-d)$ *that can be regarded as the discretization of the infinite integral form* $\int_{-\infty}^{t} k(t-s)x(s)ds$ *for the continuous-time system, and a special inequality is employed to facilitate the subsequent mathematical analysis.*

When the actuators experience failures, we use $u^F(k)$ to describe the control signal sent from actuators. Consider the actuator failure model [8] with failure parameter F:

$$u^F(k) = Fu(k), \qquad (4)$$

where $0 \leq \underline{F} = \text{diag}\{\underline{f}_1, \cdots, \underline{f}_m\} \leq F = \text{diag}\{f_1, \cdots, f_m\} \leq \bar{F} = \text{diag}\{\bar{f}_1, \cdots, \bar{f}_m\} \leq I$, in which the variables f_i $(i = 1, \cdots, m)$ quantify the failures of the actuators. Let

$$\begin{aligned} F_0 &= \text{diag}\{f_{01}, \cdots, f_{0m}\} := \{\underline{F} + \bar{F}\}/2 \\ &= \text{diag}\left\{\{\underline{f}_1 + \bar{f}_1\}/2, \cdots, \{\underline{f}_m + \bar{f}_m\}/2\right\}, \qquad (5) \\ \tilde{F} &= \text{diag}\{\tilde{f}_1, \cdots, \tilde{f}_m\} := \{\bar{F} - \underline{F}\}/2 \\ &= \text{diag}\left\{\{\bar{f}_1 - \underline{f}_1\}/2, \cdots, \{\bar{f}_m - \underline{f}_m\}/2\right\}. \qquad (6) \end{aligned}$$

Apparently, $\tilde{f}_i = \{\bar{f}_i - \underline{f}_i\}/2$, $(i = 1, \ldots, m)$ and we can rewrite F as follows:

$$F = F_0 + \Delta = F_0 + \text{diag}\{\delta_1, \cdots, \delta_m\}, \qquad (7)$$

where $|\delta_i| \leq \tilde{f}_i, \ \ i = 1, \ldots, m.$

We consider the following controller $u(k) = K_l x(k)$ $(l = 1, \ldots, L)$ where K_l is the controller gain matrix to be designed. Then, the closed-loop delayed piecewise linear system with actuator failures can be given by

$$\begin{aligned} x(k+1) &= \mathcal{A}_l x(k) + D_l \sum_{d=1}^{\infty} \mu_d x(k-d) + E_l v(k), \qquad (8) \\ z(k) &= C_l(k)x(k), \qquad \text{for } x \in S_l,\ l = 1, \ldots, L, \qquad (9) \end{aligned}$$

where $\mathcal{A}_l := A_l + B_l F K_l$.

Definition 1 Given a scalar $\gamma > 0$, the piecewise linear system (8)-(9) is said to be *exponentially stable with disturbance attenuation level* γ if it is exponentially stable and, under zero initial conditions, $\|z(k)\|_2 < \gamma^2 \|v(k)\|_2$ holds for all nonzero $v(k) \in l_2[0, \infty)$.

The objective of this paper is to design a reliable H_∞ controller for the discrete-time piecewise linear system (8)-(9) such that the closed-loop system (8)-(9) is exponentially stable with disturbance attenuation level γ.

3 Main Results

Lemma 1 *[4] Let* $M \in \mathbb{R}^{n \times n}$ *be a positive semi-definite matrix,* $x_i \in \mathbb{R}^n$ *and constant* $a_i > 0 (i = 1, 2, \ldots)$*. If the series concerned is convergent, then we have*

$$\left(\sum_{i=1}^{\infty} a_i x_i\right)^T M \left(\sum_{i=1}^{\infty} a_i x_i\right) \leq \left(\sum_{i=1}^{\infty} a_i\right) \sum_{i=1}^{\infty} a_i x_i^T M x_i. \qquad (10)$$

The inequality (10), which is actually an extension of the Hölder-Minsowski inequality, plays a key role in establishing the LMI framework. Based on inverse Hölder-Minsowski inequality, we might know how conservative the relaxation is. It would be interesting to further improve (10) so that the resulting conservatism can be reduced.

Theorem 1 *Consider the closed-loop piecewise linear system (8)-(9) with known actuator failure parameter matrix* F *and a prescribed* H_∞ *performance index* $\gamma > 0$*. If there exist matrices* $X_l > 0$, $\bar{Q} > 0$ *and* $\bar{K}_l$ *such that the following linear matrix inequalities*

$$\mathbb{M}_l := \begin{bmatrix} \Theta_{l11} & * & * & * & * \\ 0 & -\frac{1}{\bar{\mu}}\bar{Q} & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ \Theta_{l41} & D_l X_l & E_l & -X_j & * \\ C_l X_l & 0 & 0 & 0 & -I \end{bmatrix} < 0, \qquad (11)$$

for $(l, j) \in \Omega$ *hold, where* $\Theta_{l11} = -X_l + \bar{\mu}\bar{Q}$, $\Theta_{l41} = A_l X_l + B_l F \bar{K}_l$*, then the closed-loop system (8)-(9) is exponentially stable with disturbance attenuation level* γ*. In this case, the desired controller is given as follows:*

$$K_l = \bar{K}_l X_l^{-1}. \qquad (12)$$

Proof: Let $P_l = X_l^{-1}$ and define the following Lyapunov-Krasovskii functional candidate:

$$V(k) = x^T(k) P_l x(k) + \sum_{d=1}^{\infty} \mu_d \sum_{\tau=k-d}^{k-1} x^T(\tau) Q x(\tau). \qquad (13)$$

Given the set Ω representing possible transitions from one region to itself or another, the difference of $V(k)$ can be calculated as follows:

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) \\ &= \left[\mathcal{A}_l x(k) + D_l \sum_{d=1}^{\infty} \mu_d x(k-d) + E_l v(k)\right]^T P_j \\ &\quad \times \left[\mathcal{A}_l x(k) + D_l \sum_{d=1}^{\infty} \mu_d x(k-d) + E_l v(k)\right] \\ &\quad - x^T(k) P_l x(k) + x^T(k) \bar{\mu} Q) x(k) \\ &\quad - \sum_{d=1}^{\infty} \mu_d x^T(k-d) Q x(k-d). \qquad (14) \end{aligned}$$

From Lemma 1, it can be easily seen that

$$
\begin{aligned}
&-\sum_{d=1}^{\infty}\mu_d x^T(k-d)Qx(k-d) \\
&\quad \le -\frac{1}{\bar\mu}\left(\sum_{d=1}^{\infty}\mu_d x(k-d)\right)^T Q\left(\sum_{d=1}^{\infty}\mu_d x(k-d)\right), \qquad (15)
\end{aligned}
$$

where $\bar\mu$ is defined in (3). It follows from (14) and (15) that

$$
\begin{aligned}
\Delta V(k) \le& \left[\mathcal{A}_l x(k) + D_l\sum_{d=1}^{\infty}\mu_d x(k-d) + E_l v(k)\right]^T P_j \\
&\times\left[\mathcal{A}_l x(k) + D_l\sum_{d=1}^{\infty}\mu_d x(k-d) + E_l v(k)\right] \\
&-\frac{1}{\bar\mu}\left(\sum_{d=1}^{\infty}\mu_d x(k-d)\right)^T Q\left(\sum_{d=1}^{\infty}\mu_d x(k-d)\right) \\
&-x^T(k)P_l x(k) + \bar\mu x^T(k)Qx(k). \qquad (16)
\end{aligned}
$$

Denoting the following matrix variables:

$$
\begin{aligned}
\xi(k) &= \left[x^T(k)\ \ \sum_{d=1}^{\infty}\mu_d x^T(k-d)\ \ v^T(k)\right]^T, \\
\zeta(k) &= \left[x^T(k)\ \ \sum_{d=1}^{\infty}\mu_d x^T(k-d)\right]^T, \qquad (17)
\end{aligned}
$$

it follows from (16) that $\Delta V(k) \le \zeta^T(k)\Pi_{l1}\zeta(k)$, where

$$
\Pi_{l1} := \begin{bmatrix} -P_l + \bar\mu Q + \mathcal{A}_l^T P_j \mathcal{A}_l & * \\ D_l^T P_j \mathcal{A}_l & -\frac{1}{\bar\mu}Q + D_l^T P_j D_l \end{bmatrix}.
$$

With the given controller parameters in (12), it follows that (11) is true. Therefore, we know from Schur complement lemma that $\Delta V(k) < 0$. It can now be concluded from Lemma 1 of [6] that the discrete-time piecewise linear systems (8)-(9) with $v(k) = 0$ is exponentially stable.

Let us now prove the H_∞ performance. Assuming zero initial condition, we have from (16) that:

$$
\begin{aligned}
J_N &= \sum_{k=0}^{N}\left[z^T(k)z(k) - \gamma^2 v^T(k)v(k)\right] \\
&\le \sum_{k=0}^{N}\left[z^T(k)z(k) - \gamma^2 v^T(k)v(k) + \Delta V(k)\right] \\
&\le \xi^T(k)\Pi_{l2}\xi(k), \qquad (18)
\end{aligned}
$$

where

$$
\Pi_{l2} := \begin{bmatrix} \Pi_{l211} & * & * \\ D_l^T P_j \mathcal{A}_l & -\frac{1}{\bar\mu}Q + D_l^T P_j D_l & * \\ E_l^T P_j \mathcal{A}_l & E_l^T P_j D_l & -\gamma^2 I + E_l^T P_j E_l \end{bmatrix}.
$$

with $\Pi_{l211} := -P_l + \bar\mu Q + \mathcal{A}_l^T P_j \mathcal{A}_l + C_l^T C_l$. Pre- and post-multiplying (11) by diag$\{P_l,\ P_l,\ I,\ I,\ I\}$, we can conclude from Schur complement that $J_N < 0$. Based on Definition 1, the proof is complete.

It is worth mentioning that, when $F \equiv I$ in Theorem 1, the corresponding results for the "no-fault" case can be obtained easily. Furthermore, when F is *unknown* but satisfies the constraints (4)-(7), we have the following theorem.

Theorem 2 *Consider the closed-loop discrete-time piecewise linear system (8)-(9). For a prescribed constant $\gamma > 0$, if there exist positive definite matrices $X_l > 0$, $\bar Q > 0$, a diagonal matrix $R > 0$ and a matrix $\bar K_l$ such that the following linear matrix inequalities*

$$
\begin{bmatrix}
\Theta_{l11} & * & * & * & * & * & * \\
0 & -\frac{1}{\bar\mu}\bar Q & * & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * & * \\
\bar\Theta_{l41} & D_l X_l & E_l & -X_j & * & * & * \\
0 & 0 & 0 & RB_l^T & -R & * & * \\
C_l X_l & 0 & 0 & 0 & 0 & -I & * \\
\bar K_l & 0 & 0 & 0 & 0 & 0 & -R\tilde F^{-2}
\end{bmatrix} < 0, \qquad (19)
$$

for $(l, j) \in \Omega$ hold, where $\bar\Theta_{l41} := A_l X_l + B_l F_0 \bar K_l$ and Θ_{l11} has been defined in Theorem 1, then the resulting discrete-time piecewise linear system (8)-(9) is exponentially stable with disturbance attenuation γ. In this case, the parameters of the desired controller are given in (12).

Proof: Noticing (7), we know that $\mathbb{M}_l$ in (11) can be rewritten as

$$
\begin{aligned}
\mathbb{M}_l :=& \mathbb{M}_{l0} + [0, 0,\ 0,\ B_l^T,\ 0]^T\Delta[\bar K_l, 0,\ 0,\ 0,\ 0] \\
&+[\bar K_l, 0,\ 0,\ 0,\ 0]^T\Delta[0, 0,\ 0,\ B_l^T,\ 0], \qquad (20)
\end{aligned}
$$

where

$$
\mathbb{M}_{l0} := \begin{bmatrix}
-X_l + \bar\mu\bar Q & * & * & * & * \\
0 & -\frac{1}{\bar\mu}\bar Q & * & * & * \\
0 & 0 & -\gamma^2 I & * & * \\
A_l X_l + B_l F_0 \bar K_l & D_l X_l & E_l & -X_j & * \\
C_l X_l & 0 & 0 & 0 & -I
\end{bmatrix}.
$$

From (7) and the elementary inequality $x^T y + y^T x \le \varepsilon x^T x + \varepsilon^{-1} y^T y$, we have

$$
\begin{aligned}
\mathbb{M}_l \le& \mathbb{M}_{l0} + [0, 0,\ 0,\ B_l^T,\ 0]^T R[0, 0,\ 0,\ B_l^T,\ 0] \\
&+[\bar K_l, 0,\ 0,\ 0,\ 0]^T R^{-1}\tilde F^2[\bar K_l, 0,\ 0,\ 0,\ 0] \\
=& \begin{bmatrix}
\tilde\Theta_{l11} & * & * & * & * \\
0 & -\frac{1}{\bar\mu}\bar Q & * & * & * \\
0 & 0 & -\gamma^2 I & * & * \\
\bar\Theta_{l41} & D_l X_l & E_l & -X_j + B_l R B_l^T & * \\
C_l X_l & 0 & 0 & 0 & -I
\end{bmatrix},
\end{aligned}
$$

where $\tilde{\Theta}_{l11} := -X_l + \bar{\mu}\bar{Q} + \bar{K}_l^T R^{-1} \tilde{F}^2 \bar{K}_l$.

Pre- and Post-Multiplying (19) by diag$\{I, I, I, I, R^{-1}, I, I\}$ and by Schur complement, we have $\mathbb{M}_l < 0$ for $l, j = 1, \ldots, L$ and therefore we can know from Theorem 1, (19) and (12) that the system (8)-(9) is exponentially stable with the prescribed disturbance attenuation level γ and given controller parameters in (12).

Remark 2 *As in [1] and the references therein, the region partition information $\{S_l\}_{l\in L}$ is assumed to be known. In the case that the set Ω can be determined in the analysis, the number of LMIs involved in Theorem 1 can be reduced. However, for PWL systems, it is generally difficult to determine the set before the control law is designed. A possible solution is to design the control law and switching set simultaneously, and this gives rise to a challenging topic for future research.*

Remark 3 *The kernel information μ_d for the distributed delays is included in the conditions in Theorem 1. Since the distributed delays considered are infinite, it is theoretically difficult to make our main results dependent on the individual time-delays. Nevertheless, if we consider bounded distributed delay described by $\sum_{d=1}^{\tilde{d}} x(k-d)$ and construct appropriate Lyapunov functional such as $V(k) = x^T(k)P_l x(k) + \sum_{d=1}^{\tilde{d}} \sum_{\tau=k-d}^{k-1} x^T(\tau)Qx(\tau)$, delay-dependent conditions could be established, thereby reducing the conservatism with respect to the time-delays.*

4 Numerical Example

Consider a piecewise linear systems (1)-(2) with $L \triangleq \{1, 2\}$, where $l = 1$ if $x_1(k) \geq 0$ and $l = 2$ if $x_1(k) < 0$. The model parameters are given as follows:

$$A_1 = \begin{bmatrix} 0.81 & 0.3 & 0.5 \\ -0.2 & 0 & -0.3 \\ 0.1 & 0 & 0.81 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5 & 0.2 & 0.2 \\ -0.2 & 0 & -0.4 \\ 0.2 & 0 & 0.5 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.03 \\ 0 & 0 & 0.17 \end{bmatrix}, D_2 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.3 \\ 0.1 & 0.4 & 0.17 \end{bmatrix},$$

$$B_i = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}^T, E_i = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix}^T, \quad (i = 1, 2)$$
$$C_1 = [0.36 \;\; 0.15 \;\; 0.25], \; C_2 = [0.36 \;\; 0.5 \;\; 0.25],$$
$$\gamma^2 = 1, \; \underline{F} = 0.1, \; \bar{F} = 0.9.$$

According to Theorem 2, by using the LMI toolbox, the controller parameters can be calculated as follows:

$$K_1 = [-0.9058 \;\; -0.4009 \;\; -1.0941],$$
$$K_2 = [-0.8734 \;\; -0.2880 \;\; -0.7357].$$

Fig. 1 depicts the state responses for the uncontrolled systems, which are apparently unstable. Fig. 2 gives the simulation results of the responses of the closed-loop piecewise linear systems, which confirm that the closed-loop system is indeed stable.

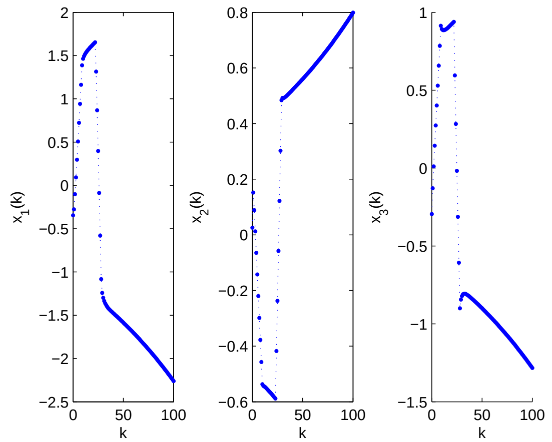

Fig. 1. The state evolution $x(k)$ of uncontrolled systems

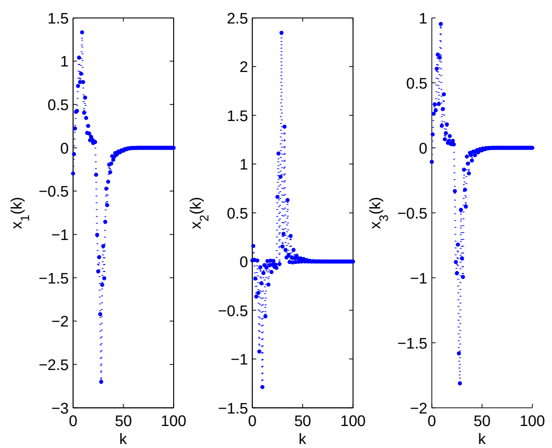

Fig. 2. The state evolution $x(k)$ of controlled systems

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