ON THE GIERER-MEINHARDT SYSTEM WITH PRECURSORS

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ABSTRACT. We consider the following Gierer-Meinhardt system with a **precursor** $\mu(x)$ for the activator A in $\mathbb{R}^1$:

$$\begin{cases} A_t = \epsilon^2 A^{''} - \mu(x)A + \frac{A^2}{H} & \text{in } (-1,1), \\ \tau H_t = DH^{''} - H + A^2 & \text{in } (-1,1), \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0. \end{cases}$$

Such an equation exhibits a typical Turing bifurcation of the **second kind**, i.e., homogeneous uniform steady states do not exist in the system.

We establish the existence and stability of $N-$peaked steady-states in terms of the precursor $\mu(x)$ and the diffusion coefficient D. It is shown that $\mu(x)$ plays an essential role for both existence and stability of spiky patterns. In particular, we show that precursors can **give rise to instability**. This is a **new effect** which is not present in the homogeneous case.

Dedicated to Professor M. Mimura on the occasion of his 65th birthday

1. Introduction

Since the work of Turing [43] in 1952, a lot of models have been proposed and studied to explore the so-called *Turing diffusion-driven instability*. One of the most famous models in biological pattern formation is the Gierer-Meinhardt system which after suitable re-scaling can be stated as follows:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{H} & \text{in } \Omega, \\ \tau H_t = D\Delta H - H + A^2 & \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^K, (K \leq 3)$ is a smooth and bounded domain.

Problem (1.1) has been studied by numerous authors. In the one-dimensional case where $\Omega = (-1,1)$, the existence of symmetric $N-$peaked solution was first established by I. Takagi [42]. The existence of *asymmetric* $N-$spikes was first shown by Ward-Wei [45] using matched asymptotic analysis and Doelman-Kaper-van der Ploeg [4] using dynamical system techniques. The stability of symmetric $N-$peaks in the one-dimensional case was established by Iron-Ward-Wei [17] using matched asymptotic expansions. For asymmetric $N-$spikes in $\mathbb{R}^1$, the stability was proved in Ward-Wei [45]. Later we gave a unified rigorous approach to the existence and stability of both symmetric and asymmetric spikes, [55]. In two dimensions, the existence and stability of symmetric and asymmetric N spots were established in a series of papers [56], [57], [58].

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Problem (1.1) can be considered as a typical Turing bifurcation of the **first kind**, i.e., homogeneous uniform steady states exist in the system. If these states have instabilities which are spatially varying but no instabilities which are spatially uniform one says that a Turing instability occurs. This behavior is commonly used to explain the onset of spatial patterns.

Holloway et. al. [13] among others have added precursors to (1.1), i.e. they have added coefficients which are spatially varying. This dramatically changes the behavior of (1.1) so that now a Turing bifurcation of the **second kind** occurs, i.e., homogeneous uniform steady states do not exist in the system and so they cannot be used to explain the onset of pattern formation.

The existence of precursor gradients in the system also changes its behavior fundamentally. In particular, in [13] the authors numerically studied the following Gierer-Meinhardt system with a precursor (inhomogeneity) $\mu(x)$ in the variable A:

$$\begin{cases} A_t = \epsilon^2 A'' - \mu(x)A + \frac{A^2}{H} & \text{in } (-1,1), \\ \tau H_t = DH'' - H + A^2 & \text{in } (-1,1), \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0 \end{cases} \tag{1.2}$$

They consider two classes of precursors: linear precursors ($\mu(x) = Ax + B$ for some constants A, B) and exponential precursors ($\mu(x) = \sum_{i=1}^m A_i e^{-|x-x_i|}$ for some constants $A_i > 0$ and points $x_i \in (-1,1)$). As we shall see in this paper, precursors greatly change the profile and other properties of the peaked solutions.

Precursor gradients have been used in reaction-diffusion models for over thirty years. The original Gierer-Meinhardt [7] model was introduced with precursor gradients. This was effectively used in their first application, localization of the head structure in the coelenterate *Hydra*, and in much subsequent work. Gradients have also been used in the Brusselator to limit pattern formation to some fraction of the system [14]. In that example, the gradient carries the system in and out of the pattern-forming region of the linear parameter space (across the Turing bifurcation), effectively confining the region wherein peak formation can occur. Such localization has been used to model segmentation patterns in the fruit fly, *Drosophila melanogaster* in [22] and [12].

Another important effect of precursors is the appearance of stable asymmetric multi-peak pattern (with irregular spacing and unequal amplitudes), which is frequently observed for real biological applications (such as seashells, spots on fish skins, etc.) and seems to be more common than symmetric peak pattern (with regular spacing and equal amplitudes), which is typical for systems without precursors.

Note that both of these properties are clearly evident in the simulations presented in the last section of this paper, in particular for confinement see Figure 5 and for asymmetric peaks with irregular amplitudes and spacing see Figure 6.

An area of particular interest for precursors is ecology where commonly precursors are included into the model to represent the interaction between the eco-system and its heterogeneous environment. Typical variables considered include temperature, flow of air and water, movement of soil and chemical reactions. Reaction-diffusion systems have been successful in modelling some pattern-forming effects and stability properties in ecosystems. The interplay of different scales often plays a central role. For a survey see [39].

Precursors have also been shown to cause the Brusselator to form striped patterns in two dimensions [20]. We refer to Chapter 4 of the PhD thesis by Holloway [13]. Since we are considering a one-dimensional system we do not investigate this effect here.

Turing systems have mostly been considered with kinetic parameters and diffusion coefficients constant in space. But even Turing himself stated that "most of an organism, most of the time, is developing from one pattern to another, rather than from homogeneity into a pattern" [43]. This fundamental idea can be incorporated into reaction-diffusion models by precursors representing pre-existing spatial structure within a biological system, e.g. a living organism.

The purpose of this paper is to rigorously study the effect of $\mu(x)$ on the existence and stability of N-peaked solutions. In [46], Ward etc. have studied the pinning phenomena for the following problem

$$\begin{cases} A_t = \epsilon^2 \Delta A - \mu_1(x) A + \frac{A^2}{H} & \text{in } \ \Omega, \\ \tau H_t = D\Delta H - \mu_2(x) H + A^2 & \text{in } \ \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \ \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^1$ or $\mathbb{R}^2$. There they only considered one-spike solutions. In this paper we shall consider multiple spikes of (1.2) in the 1-D case. (So we may assume from now on that $N \geq 2$.) Of course, the 2-D case is also very interesting. We shall come to this issue in a future study.

One can certainly generalize the results of this paper to the Gierer-Meinhardt system with precursors in both A and H, for example to the following equation:

$$\begin{cases} A_t = \epsilon^2 (D_1(x) A')' - \mu_1(x) A + \rho_1(x) \frac{A^2}{H} & \text{in } \ (-1,1), \\ \tau H_t = (D_2(x) H')' - \mu_2(x) H + \rho_2(x) A^2 & \text{in } \ (-1,1), \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0. \end{cases} \tag{1.4}$$

But to keep the presentation simple we restrict our attention to (1.2).

The stationary solution to (1.2) satisfies

$$\begin{cases} \epsilon^2 A'' - \mu(x) A + \frac{A^2}{H} = 0 & \text{in } \ (-1,1), \\ DH'' - H + A^2 = 0 & \text{in } \ (-1,1), \\ A'(-1) = A'(1) = H'(-1) = H'(1) = 0 \end{cases} \tag{1.5}$$

We remark that even the existence of N-peaked solutions to (1.5) is not easy as $\mu(x) \not\equiv$ constant. Recall that in the proof of existence of N-peaked solutions, I. Takagi [42] first studied 1-peaked solutions. Then by even extension he obtained N-peaked solutions. If there is a precursor in the system, the symmetry is lost and this method can not be applied. Even in the construction of 1-peaked solutions Takagi used symmetry – he restricted solutions to be in the class of even functions. Here, again, we do not have this symmetry. Instead, we have to work on the whole function space (which greatly increases the difficulty) and then use the method of Liapunov-Schmidt reduction which has been used for the

1-D Schrödinger equation [6], [37], [38] and then been extended to the higher-dimensional Cahn-Hilliard equation [52], [53] and semilinear elliptic equations [10], [11]. This method has also been applied to the 2-D Gierer-Meinhardt system [55]-[58].

Before we state our main results in Section 2, we introduce some notation. Throughout this paper, we always assume that $\Omega = (-1,1)$. With $L^2(\Omega)$ and $H^2(\Omega)$ we denote the usual Sobolev spaces. With the variable w we denote the unique homoclinic solution of the following problem:

$$\begin{cases} w'' - w + w^2 = 0 & \text{in } \mathbb{R}^1, \\ w > 0, \quad w(0) = \max_{y\in\mathbb{R}} w(y), & w(y) \to 0 \ \text{ as } |y| \to \infty \end{cases} \tag{1.6}$$

Note that w is an even function and $w^{'}(y) < 0$ if $y > 0$. An explicit representation is

$$w(y) = \frac{3}{2}\left(\cosh\frac{y}{2}\right)^{-2}.$$

Elementary calculations give

$$\int_{\mathbb{R}} w^2(z)\,dz = 6, \quad \int_{\mathbb{R}} w^3(z)\,dz = 7.2, \quad \int_{\mathbb{R}} \left(w^{'}\right)^2(z)\,dz = 1.2. \tag{1.7}$$

We assume that the precursor $\mu(x)$ satisfies

$$\mu(x) \in C^3(\Omega), \quad \mu(x) > 0 \text{ in } \Omega. \tag{1.8}$$

Let $G_D(x,z)$ be Green's function given by

$$\begin{cases} DG_D(x,z)'' - G_D(x,z) + \delta_z = 0 \quad \text{in} \ \ (-1,1), \\ G_D'(-1,z) = G_D'(1,z) = 0. \end{cases} \tag{1.9}$$

We can calculate

$$G_D(x,z) = \begin{cases} A(z)\cosh[\theta(1+x)]/\cosh[\theta(1+z)], & -1 < x < z, \\ A(z)\cosh[\theta(1-x)]/\cosh[\theta(1-z)], & z < x < 1. \end{cases} \tag{1.10}$$

Here

$$A(z) = \frac{1}{\sqrt{D}}(\tanh[\theta(1-z)] + \tanh[\theta(1+z)])^{-1}, \theta = D^{-\frac{1}{2}}. \tag{1.11}$$

We set

$$K_D(|x-z|) = \frac{1}{2\sqrt{D}} e^{-\frac{1}{\sqrt{D}}|x-z|}$$

to be the singular part of $G_D(x,z)$ and by $G_D = K_D - H_D$ we define the regular part H_D of G_D. Note that H_D is C^∞ in both x and z.

We use the notation e.s.t. to denote an exponentially small term of order $O(e^{-d/\epsilon})$ for some $d > 0$ in the corresponding norm. By C we denote a generic constant which may change from line to line.

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2. Main results: Existence of $N-$peaked Solutions

Let $-1 < t_1^0 < \cdots < t_j^0 < \cdots < t_N^0 < 1$ be N points in $(-1,1)$ and $\mu_i^0 = \mu(t_i^0)$, $i = 1, \ldots, N$. We assume that $D < +\infty$ is a fixed number.

By a simple scaling argument, the function

$$w_a(y) = aw(a^{1/2}y), \tag{2.1}$$

where w satisfies (1.6), is the unique solution of the following problem:

$$\begin{cases} w_a^{''} - aw_a + w_a^2 = 0 \quad \text{in } \mathbb{R}, \\ w_a > 0, \quad w_a(0) = \max_{y\in\mathbb{R}} w_a(y), \quad w_a(y) \to 0 \ \text{ as } |y| \to \infty. \end{cases} \tag{2.2}$$

We compute

$$\int_{\mathbb{R}} w_a^2(y)\,dy = a^{3/2}\int_{\mathbb{R}} w^2(z)\,dz, \ \int_{\mathbb{R}} w_a^3(y)\,dy = a^{5/2}\int_{\mathbb{R}} w^3(z)dz, \ \int_{\mathbb{R}} (w_a^{'})^2(y)\,dy = a^{5/2}\int_{\mathbb{R}} (w^{'})^2(z)\,dz \tag{2.3}$$

Put

$$\xi_\epsilon := \left(\epsilon\int_{\mathbb{R}} w^2(z)\,dz\right)^{-1}. \tag{2.4}$$

We introduce several matrices for later use: For $\mathbf{t} = (t_1, ..., t_N) \in (-1,1)^N$ let

$$\mathcal{G}_D(\mathbf{t}) = (G_D(t_i, t_j)). \tag{2.5}$$

Recall that

$$G_D(t_i, t_j) = K_D(|t_i - t_j|) - H_D(t_i, t_j).$$

Let us denote $\frac{\partial}{\partial t_i}$ as ∇_{t_i}. When $i \neq j$, we can define $\nabla_{t_i} G(t_i, t_j)$ in the classical way. When $i = j$, $K_D(|t_i - t_j|) = K_D(0) = \frac{1}{2\sqrt{D}}$ is a constant and we define

$$\nabla_{t_i} G_D(t_i, t_i) := -\frac{\partial}{\partial x}|_{x=t_i} H(x, t_i).$$

Similarly, we define

$$\nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j) = \begin{cases} -\frac{\partial}{\partial x}|_{x=t_i}\frac{\partial}{\partial y}|_{y=t_i} H_D(x,y) \ \text{ if } \ i = j, \\ \nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j) \ \text{ if } \ i \neq j. \end{cases} \tag{2.6}$$

Now the derivatives of $\mathcal{G}$ are defined as follows:

$$\nabla \mathcal{G}_D(\mathbf{t}) = (\nabla_{t_i} G_D(t_i, t_j)), \quad \nabla^2 \mathcal{G}_D(\mathbf{t}) = (\nabla_{t_i}\nabla_{t_j} G_D(t_i, t_j)). \tag{2.7}$$

We now have our first assumption:

(H1) There exists a solution $(\hat{\xi}_1^0, \ldots, \hat{\xi}_N^0)$ of the equation

$$\sum_{j=1}^N G_D(t_i^0, t_j^0)(\hat{\xi}_j^0)^2(\mu_j^0)^{3/2} = \hat{\xi}_i^0, \quad i = 1, ..., N. \tag{2.8}$$

Next we introduce the following matrix

$$b_{ij} = G_D(t_i^0, t_j^0)(\hat{\xi}_j^0)(\mu_j^0)^{3/2}, \quad \mathcal{B} = (b_{ij}). \tag{2.9}$$

Our second assumption is the following:

(H2) It holds that

$$\frac{1}{2} \notin \sigma(\mathcal{B}), \tag{2.10}$$

where $\sigma(\mathcal{B})$ is the set of eigenvalues of $\mathcal{B}$.

Remark 2.1. Since the matrix $\mathcal{B}$ is of the form $\mathcal{G}\mathcal{D}$, where $\mathcal{G}$ is symmetric and $\mathcal{D}$ is a diagonal matrix, it follows that the eigenvalues of $\mathcal{B}$ are real.

By the assumption **(H2)**, for $\mathbf{t} = (t_1, ..., t_N)$ near $\mathbf{t}^0 = (t_1^0, ..., t_N^0)$ and $\mu_j = \mu(t_j)$, $i = 1, ..., N$, by the implicit function theorem there exists a (locally) unique solution $\hat{\xi}(\mathbf{t}) = (\hat{\xi}_1(\mathbf{t}), ..., \hat{\xi}_N(\mathbf{t}))$ of the following equation

$$\sum_{j=1}^{N} G_D(t_i, t_j)\hat{\xi}_j^2 \mu_j^{3/2} = \hat{\xi}_i, \quad i = 1, ..., N. \tag{2.11}$$

Moreover, $\hat{\xi}(\mathbf{t})$ is C^1 in $\mathbf{t}$.

Set

$$\mathcal{H}(\mathbf{t}) = (\hat{\xi}_i(\mathbf{t})\delta_{ij}), \quad \mu(\mathbf{t}) = (\mu(t_i)\delta_{ij}), \quad \mu^{'}(\mathbf{t}) = (\mu^{'}(t_i)\delta_{ij}). \tag{2.12}$$

We introduce the following vector field:

$$F(\mathbf{t}) = (F_1(\mathbf{t}), ..., F_N(\mathbf{t})),$$

where

$$F_i(\mathbf{t}) = \frac{5}{4}\hat{\xi}_i \mu_i^{-1} \mu^{'}(t_i) + \sum_{l=1}^{N} \nabla_{t_i} G_D(t_i, t_l)\hat{\xi}_l^2 \mu_l^{3/2}, \quad i = 1, \ldots, N. \tag{2.13}$$

Set

$$\mathcal{M}(\mathbf{t}) = \left(\frac{\partial F_i(\mathbf{t})}{\partial t_j} \right). \tag{2.14}$$

Our final assumption concerns the vector field $F(\mathbf{t})$.

(H3) We assume that at $\mathbf{t}_0 = (t_1^0, ..., t_N^0)$

$$F(\mathbf{t}_0) = 0, \quad \det\, (\mathcal{M}(\mathbf{t}_0)) \neq 0. \tag{2.15}$$

Remark 2.2. By the same reasoning as for the matrix $\mathcal{B}$, the eigenvalues of $\mathcal{M}$ are all real.

Our first result can be stated as follows:

Theorem 2.1. *Assume that assumptions* **(H1)**, **(H2)** *and* **(H2)** *hold. Then for* $\epsilon << 1$, *problem (1.2) has an* N*-peaked solution centered at* $t_1^\epsilon, \cdots, t_N^\epsilon$. *More precisely, it satisfies*

$$A_\epsilon(x) \sim \sum_{i=1}^{N} \xi_\epsilon \hat{\xi}_i^0 w_i \left(\frac{x - t_i^\epsilon}{\epsilon} \right), \tag{2.16}$$

where w_i *is given by (2.2) for* $a = \mu(t_i^0)$, ξ_ϵ *has been defined in (2.4),* $\hat{\xi}_i^0$ *has been introduced in* **(H1)**,

$$H_\epsilon(t_i^\epsilon) \sim \xi_\epsilon \hat{\xi}_i^0, \quad i = 1, \cdots, N, \tag{2.17}$$

$$t_i^\epsilon \to t_i^0, \quad i = 1, \cdots, N. \tag{2.18}$$

The next theorem reduces the stability to the conditions on the two matrices $\mathcal{B}$ and $\mathcal{M}$.

Theorem 2.2. *Let (A_ϵ, H_ϵ) be the solutions constructed in Theorem 2.1. Assume that $\epsilon << 1$.*

(1) (Stability) If

$$\min_{\sigma\in\sigma(\mathcal{B})} \sigma > \frac{1}{2} \tag{2.19}$$

and

$$\sigma(\mathcal{M}) \subseteq \{\sigma | Re(\sigma) \geq c > 0\}, \tag{2.20}$$

there exists $\tau_0 > 0$ such that (A_ϵ, H_ϵ) is linearly stable for $0 \leq \tau < \tau_0$.

(2) (Instability) If

$$\min_{\sigma\in\sigma(\mathcal{B})} \sigma < \frac{1}{2}, \tag{2.21}$$

then (A_ϵ, H_ϵ) is linearly unstable for all $\tau > 0$.

(3) (Instability) If there exists

$$\sigma \in \sigma(\mathcal{M}),\ Re(\sigma) < 0, \tag{2.22}$$

then (A_ϵ, H_ϵ) is linearly unstable for all $\tau > 0$.

We end this section with a few remarks.

Remark 2.3. Generally speaking, if $\mu(x) \not\equiv$ constant, $\hat{\xi}_i^0 \neq \hat{\xi}_j^0$ for $i \neq j$. Thus the height of different peaks may be different. This is strikingly different from the symmetric solutions constructed by I. Takagi in the homogeneous case [42].

Remark 2.4. For the linear gradient case, we have

$$\mu'(t_i^0) = c_0, \quad \mu''(t_i^0) = 0.$$

Condition **(H3)** corresponds to a shift of $(t_1^0, \cdots, t_N^0)$ from the centered position since the first term of $F_i(\mathbf{t})$ in (2.13) is constant.

Let us now calculate $\mathcal{M}(\mathbf{t}^0)$. As a preparation we first compute the derivatives of $\hat{\xi}(\mathbf{t})$. It is easy to see that $\hat{\xi}(\mathbf{t})$ is C^1 in $\mathbf{t}$. Now from (2.11) we calculate:

$$\nabla_{t_j}\hat{\xi}_i = 2\sum_{l=1}^N G_D(t_i,t_l)\hat{\xi}_l\mu_l^{3/2}\nabla_{t_j}\hat{\xi}_l + \frac{\partial}{\partial t_j}(G_D(t_i,t_j))\hat{\xi}_j^2\mu_j^{3/2} + \frac{3}{2}G_D(t_i,t_j)\hat{\xi}_j^2\mu_j^{1/2}\mu_j' \quad \text{for } i \neq j,$$

$$\begin{aligned}
\nabla_{t_i}\hat{\xi}_i &= 2\sum_{l=1}^N G_D(t_i,t_l)\hat{\xi}_l\mu_j^{3/2}\nabla_{t_i}\hat{\xi}_l + \sum_{l=1}^N \frac{\partial}{\partial t_i}(G_D(t_i,t_l))\hat{\xi}_l^2\mu_l^{3/2} + \frac{3}{2}G_D(t_i,t_j)\hat{\xi}_j^2\mu_j^{1/2}\mu_j' \\
&= 2\sum_{l=1}^N G_D(t_i,t_l)\hat{\xi}_l\mu_j^{3/2}\nabla_{t_i}\hat{\xi}_l + \nabla_{t_i}G_D(t_i,t_i)\hat{\xi}_i^2\mu_i^{3/2} - \frac{5}{4}\hat{\xi}_i\mu_i^{-1}\mu'(t_i) \\
&\quad + \frac{3}{2}G_D(t_i,t_j)\hat{\xi}_j^2\mu_j^{1/2}\mu_j' + O\left(\sum_{j=1}^N |F_j(\mathbf{t})|\right) \quad \text{for } i = j,
\end{aligned}$$

where we have used the definition of (2.13).

Note that

$$(\nabla_{t_j} G_D(t_i,t_j)) = (\nabla \mathcal{G}_D)^T.$$

Therefore introducing matrix notation

$$\nabla \xi = (\nabla_{t_j} \hat{\xi}_i), \quad \mathcal{P} = (I - 2\mathcal{G}_D \mathcal{H} \mu^{3/2})^{-1}, \tag{2.23}$$

we have

$$\nabla \xi(\mathbf{t}) = \mathcal{P}\left[(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mu^{3/2} - \frac{5}{4}\mathcal{H}\mu^{-1}\mu' + \frac{3}{2}\mathcal{G}_D \mathcal{H}^2 \mu^{1/2}\mu' \right] + O\left(\sum_{j=1}^N |F_j(\mathbf{t})| \right). \tag{2.24}$$

Let

$$\mathcal{Q} = (q_{ij}) = \left(\left(\frac{1}{D}\hat{\xi}_i^{-1}\mu_i^{-3/2} - \frac{1}{2D^{3/2}} \right) \delta_{ij} \right). \tag{2.25}$$

We can compute $\mathcal{M}(\mathbf{t}^0)$ by using (2.24): we have for $i \neq j$

$$\sum_{l=1}^N \nabla_{t_j} \left(\nabla_{t_i} G_D(t_i,t_l) \right) \hat{\xi}_l^2 \mu_l^{3/2} = \left(\nabla_{t_j} \nabla_{t_i} G_D(t_i,t_j) \right) \hat{\xi}_j^2 \mu_j^{3/2} \tag{2.26}$$

and for $i = j$

$$\begin{aligned}
\sum_{l=1}^N \nabla_{t_i} \left(\nabla_{t_i} G_D(t_i,t_l) \right) \hat{\xi}_l^2 \mu_l^{3/2} &= \sum_{l=1,\dots,N, l\neq i} \nabla_{t_i} \nabla_{t_i} G_D(t_i,t_l) \hat{\xi}_l^2 \mu_l^{3/2} \\
&\quad - \left(\frac{\partial^2}{\partial x^2}\Big|_{x=t_i} H_D(x,t_i) \right) \hat{\xi}_i^2 \mu_i^{3/2} - \left(\frac{\partial^2}{\partial x \partial y}\Big|_{x=t_i,\, y=t_i} H_D(x,y) \right) \hat{\xi}_i^2 \mu_i^{3/2} \\
&= \frac{1}{D} \sum_{l=1,\dots,N, l\neq i} G_D(t_i,t_l) \hat{\xi}_l^2 \mu_l^{3/2} - \frac{1}{D} H_D(t_i,t_i) \hat{\xi}_i^2 \mu_i^{3/2} \\
&\quad - \left(\frac{\partial^2}{\partial x \partial y}\Big|_{x=t_i,\, y=t_i} H_D(x,t_i) \right) \hat{\xi}_i^2 \mu_i^{3/2} \\
&= \frac{1}{D} \sum_{l=1}^N G_D(t_i,t_l) \hat{\xi}_l^2 \mu_l^{3/2} - \frac{1}{D} K_D(0) \hat{\xi}_i^2 \mu_i^{3/2} + (\nabla_{t_i} \nabla_{t_i} G_D(t_i,t_i)) \hat{\xi}_i^2 \mu_i^{3/2} \\
&= \frac{1}{D} \hat{\xi}_i - \frac{1}{D} K_D(0) \hat{\xi}_i^2 \mu_i^{3/2} + (\nabla_{t_i} \nabla_{t_i} G_D(t_i,t_i)) \hat{\xi}_i^2 \mu_i^{3/2} \qquad (2.27)
\end{aligned}$$

by (2.8), and hence

$$\begin{aligned}
\mathcal{M}(\mathbf{t}^0) &= (\nabla^2 \mathcal{G}_D + \mathcal{Q}) \mathcal{H}^2 \mu^{3/2} + 2\nabla \mathcal{G}_D \mathcal{H} \nabla \xi \mu^{3/2} \qquad (2.28) \\
&\quad + \frac{5}{4}\left[\nabla \hat{\xi} \mu^{-1}\mu' - \mathcal{H}\mu^{-2}(\mu')^2 \right] + \frac{5}{4}\mathcal{H}\mu^{-1}\mu'' + \frac{3}{2}\nabla \mathcal{G}_D \mathcal{H}^2 \mu^{1/2}\mu'.
\end{aligned}$$

Using

$$\nabla \xi(\mathbf{t}^0) = \mathcal{P}\left[(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mu^{3/2} - \frac{5}{4}\mathcal{H}\mu^{-1}\mu' + \frac{3}{2}\mathcal{G}_D \mathcal{H}^2 \mu^{1/2}\mu' \right], \tag{2.29}$$

which follows from **(H3)** and (2.24), we obtain

$$\mathcal{M}(\mathbf{t}^0) = (\nabla^2 \mathcal{G}_D + \mathcal{Q}) \mathcal{H}^2 \mu^3 + 2\nabla \mathcal{G}_D \mathcal{H} \mathcal{P} (\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mu^3$$

$$+\frac{5}{4}\mathcal{P}\left[(\nabla\mathcal{G}_D)^T\mathcal{H}^2\mu^{1/2}\mu^{'} - \frac{5}{4}\mathcal{H}\mu^{-2}(\mu^{'})^2 + \frac{3}{2}\mathcal{G}_D\mathcal{H}^2\mu^{-1/2}(\mu^{'})^2\right]$$

$$+\frac{5}{4}\mathcal{H}[\mu^{-1}\mu^{''} - \mu^{-2}(\mu^{'})^2] + 3\nabla\mathcal{G}_D\mathcal{H}\mathcal{P}\mathcal{G}_D\mathcal{H}^2\mu^2\mu^{'} - \frac{5}{2}\nabla\mathcal{G}_D\mathcal{H}\mathcal{P}\mathcal{H}\mu^{1/2}\mu^{'}.$$

Remark 2.5. Let us consider the following special quadratic gradient case

$$\mu(x) = A + \sum_{j=1}^{N} B_j(x - t_j^0)^2, \quad t_j^0 = -1 + \frac{(2j-1)}{N}, \quad j = 1, ..., N.$$

We take symmetric $N-$spikes: $\xi_1 = \xi_2 = ... = \xi_N$. For this choice of t_j^0 the assumptions **(H1)** and **(H2)** are satisfied. In fact, we have $\mu_i^0 = A$. The matrix $\mathcal{M}$ becomes

$$\mathcal{M} = (m_{ij}^1 + m_{ij}^2) = \mathcal{M}^1 + \mathcal{M}^2,$$

where

$$\begin{aligned}\mathcal{M}^1 &= \frac{5}{4}\mathcal{H}\mu^{-1}\mu^{''},\\ \mathcal{M}^2 &= (\nabla^2\mathcal{G}_D + \mathcal{Q})\mathcal{H}^2\mu^{3/2} + 2\nabla\mathcal{G}_D\mathcal{H}\mathcal{P}(\nabla\mathcal{G}_D)^T\mathcal{H}^2\mu^3.\end{aligned}$$

Note that $\mathcal{H} = \xi_0 I, \mu = AI$. So

$$m_{ij}^1 = c_0 B_i \delta_{ij}.$$

The second matrix $\mathcal{M}^2 = (m_{ij}^2)$ does not depend on B_i and its eigenvalues have been computed in [17] and [55]. Thus if A is fixed and $B_i = -B < 0$, then for B sufficiently large we have instability. We conclude that precursors may **give rise to instability**. This is a new effect which is not present in the homogeneous case.

Remark 2.6. Numerical studies of the precursor case can be found among others in [13], [40] and [41]. In the last section, we shall perform some numerical experiments to verify our theory.

The proof of both Theorem 2.1 and Theorem 2.2 will follow the same line as in [55], where we considered the existence, stability and classification of $N-$symmetric spikes.

3. Some preliminaries

In this section, we study a system of nonlocal linear operators. We first recall

Theorem 3.1. [51]*: Consider the following nonlocal eigenvalue problem*

$$\nabla^2\phi - \phi + 2w\phi - \gamma\frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2}w^2 = \alpha\phi, \quad \phi \in H^2(\mathbb{R}). \tag{3.1}$$

(1) If $\gamma < 1$, then there is a positive eigenvalue to (3.1).

(2) if $\gamma > 1$, then for any nonzero eigenvalue α of (3.1), we have

$$Re(\alpha) \le -c_0 < 0 \quad \textit{for some } c_0 > 0.$$

(3) If $\gamma \neq 1$ and $\alpha = 0$, then

$$\phi = c_0\frac{\partial w}{\partial y}$$

for some constant c_0.

Next, we consider the following system of linear operators

$$L\Phi := \nabla^2\Phi - \Phi + 2w\Phi - 2\left(\int_{\mathbb{R}} w\mathcal{B}\Phi\right)\left(\int_{\mathbb{R}} w^2\right)^{-1} w^2, \tag{3.2}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N.$$

Set

$$L_0 u := \nabla^2 u - u + 2wu,$$

where $u \in H^2(\mathbb{R})$.

Then the conjugate operator of L under the scalar product in $L^2(\mathbb{R})$ is

$$L^*\Psi = \nabla^2\Psi - \Psi + 2w\Psi - 2\left(\int_{\mathbb{R}} w^2\right)^{-1}\left(\int_{\mathbb{R}} w^2\mathcal{B}^T\Psi\right) w, \tag{3.3}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N.$$

We then have the following

Lemma 3.2. *(Lemma 3.2 of* [55]*.) If $\frac{1}{2} \notin \sigma(\mathcal{B})$, then*

$$Ker(L) = X_0 \oplus X_0 \oplus \cdots \oplus X_0, \tag{3.4}$$

where

$$X_0 = span\left\{\frac{\partial w}{\partial y}\right\}$$

and

$$Ker(L^*) = X_0 \oplus X_0 \oplus \cdots \oplus X_0. \tag{3.5}$$

As a consequence of Lemma 3.2, we have

Lemma 3.3. *The operator*

$$L : (H^2(\mathbb{R}))^N \to (L^2(\mathbb{R}))^N$$

is an invertible operator if it is restricted as follows

$$L : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(\mathbb{R}))^N \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(\mathbb{R}))^N.$$

Moreover, L^{-1} is bounded.

Proof: This follows from the Fredholm Alternative and Lemma 3.2.

□

4. Study of approximate solutions

Let $-1 < t_1^0 < \cdots < t_j^0 < \cdots t_N^0 < 1$ be the N points satisfying the assumptions **(H1)** – **(H2)**. Let $\hat{\xi}^0 = (\hat{\xi}_1^0, ..., \hat{\xi}_N^0)$ be the (locally) unique solution of (2.8). Let $\mu_i^0 = \mu(t_i^0)$ and

$$\mathbf{t}^0 = (t_1^0, \cdots, t_N^0).$$

We now construct an approximate solution to (1.5) which concentrates near these prescribed N points.

Let $-1 < t_1 < t_2 < \cdots < t_j < \cdots < t_N < 1$ be such that $\mathbf{t} = (t_1, \cdots, t_N) \in B_{\epsilon^{3/4}}(\mathbf{t}^0)$.

Set

$$w_j(x) = \mu_j w\left(\sqrt{\mu_j}\left(\frac{x - t_j}{\epsilon}\right)\right). \tag{4.1}$$

Let r_0 be such that

$$r_0 = \frac{1}{10}\left(\min\left(t_1^0 + 1, 1 - t_N^0, \min_{i \neq j} |t_i^0 - t_j^0|\right)\right). \tag{4.2}$$

Introduce a smooth cut-off function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\chi(x) = 1 \quad \text{for} \quad |x| < 1 \quad \text{and} \quad \chi(x) = 0 \quad \text{for} \quad |x| > 2. \tag{4.3}$$

We now define our approximate solution

$$\tilde{w}_j(x) = w_j(x)\chi\left(\frac{x - t_j}{r_0}\right). \tag{4.4}$$

Then $\tilde{w}_j(x)$ satisfies

$$\epsilon^2 \nabla^2 \tilde{w}_j - \mu_j \tilde{w}_j + \tilde{w}_j^2 + \text{e.s.t.} = 0. \tag{4.5}$$

Recall that, by assumption **(H2)**, for $\mathbf{t} \in B_{\epsilon^{3/4}}(\mathbf{t}^0)$ there exists a unique solution $\hat{\xi}_{\mathbf{t}} = (\hat{\xi}_1, ..., \hat{\xi}_N)$ such that

$$\sum_{j=1}^{N} G_D(t_i, t_j)\hat{\xi}_j^2 \mu_j^{3/2} = \hat{\xi}_i, \quad i = 1, ..., N. \tag{4.6}$$

Moreover, such a solution is also C^1 in $\mathbf{t}$.

Put

$$w_{\epsilon,\mathbf{t}}(x) = \sum_{j=1}^{N} \xi_j \tilde{w}_j(x), \tag{4.7}$$

where

$$\xi_j = \xi_\epsilon \hat{\xi}_j \tag{4.8}$$

and ξ_ϵ has been introduced in (2.4).

For a function $A \in H^2(-1, 1)$ we define $T[A]$ to be the solution of

$$\begin{cases} D\nabla^2 T[A] - T[A] + A^2 = 0, & -1 < x < 1, \\ T[A]'(-1) = T[A]'(1) = 0. \end{cases} \tag{4.9}$$

The solution $T[A]$ is positive and unique.

For $A = w_{\epsilon,\mathbf{t}}$, where $\mathbf{t} \in B_{\epsilon^{3/4}}(\mathbf{t}^0)$, let us now compute

$$\tau_i := T[A](t_i). \tag{4.10}$$

From (4.9), we have

$$\begin{aligned}
\tau_i &= T[A](t_i) = \int_{-1}^{1} G_D(t_i, z) A^2(z)\, dz \\
&= \sum_{j=1}^{N} \xi_j^2 \int_{-1}^{1} G_D(t_i, z) \tilde{w}_j^2(z)\, dz \\
&= \sum_{j=1}^{N} \xi_j^2 \epsilon \left[G_D(t_i, t_j) \int_{-\infty}^{+\infty} w_j^2(y)\, dy + O(\epsilon) \right] \\
&= \sum_{j=1}^{N} \xi_j^2 \epsilon \left[G_D(t_i, t_j) \mu_j^{3/2} \int_{-\infty}^{+\infty} w^2(y)\, dy + O(\epsilon) \right] \qquad \text{(by (2.3))} \\
&= \xi_\epsilon \left[\sum_{j=1}^{N} G_D(t_i, t_j) \hat{\xi}_j^2 \mu_j^{3/2} + O(\epsilon) \right] \qquad \text{(by (2.4), (4.8))} \\
&= \xi_\epsilon [\hat{\xi}_i + O(\epsilon)] \qquad \text{(by (4.6))}.
\end{aligned} \tag{4.11}$$

Let $x = t_i + \epsilon y$, where $x \in B_{\epsilon^{3/4}(t_i)}$. We calculate for $A = w_{\epsilon,\mathbf{t}}$:

$$\begin{aligned}
T[A](x) - T[A](t_i) &= \int_{-1}^{1} [G_D(x,z) - G_D(t_i,z)] A^2(z)\,dz \\
&= \xi_i^2 \int_{-1}^{1} [G_D(x,z) - G_D(t_i,z)] \tilde{w}_i^2(z)\,dz + \sum_{j\neq i} \xi_j^2 \int_{-1}^{1} [G_D(x,z) - G_D(t_i,z)] \tilde{w}_j^2(z)\,dz \\
&= \xi_i^2 \int_{-1}^{1} [K_D(|x-z|) - K_D(|t_i - z|)] \tilde{w}_i^2(z)\,dz - \xi_i^2 \int_{-1}^{1} [H_D(x,z) - H_D(t_i,z)] \tilde{w}_i^2(z)\,dz \\
&\quad + \sum_{j\neq i} \xi_j^2 \int_{-1}^{1} [G_D(x,z) - G_D(t_i,z)] \tilde{w}_j^2(z)\,dz \quad \text{(letting } z = t_j + \epsilon\tilde{z}) \\
&= \epsilon^2 \xi_i^2 \left[\int_{-\infty}^{+\infty} \left[\frac{1}{2D}|\tilde{z}| - \frac{1}{2D}|y - \tilde{z}| \right] w_i^2(\tilde{z})\,d\tilde{z} + O(\epsilon y^2 + \epsilon^2) \right] \\
&\quad + \epsilon^2 \xi_i^2 \left[-\nabla_x H_D(x,t_i)|_{x=t_i}\, y \int_{-\infty}^{+\infty} w_i^2(\tilde{z})\,d\tilde{z} + O(\epsilon y^2 + \epsilon^2) \right] \\
&\quad + \sum_{j\neq i} \epsilon^2 \xi_j^2 [\nabla_x G_D(x,t_j)|_{x=t_i} y \int_{-\infty}^{+\infty} w_j^2(\tilde{z})\,d\tilde{z} + O(\epsilon y^2 + \epsilon^2)] \\
&= \epsilon^2 \xi_i^2 P_i(y) + \sum_{j\neq i} \epsilon^2 \xi_j^2 \int_{-\infty}^{+\infty} w_i^2(\tilde{z})\,d\tilde{z}\, [\nabla_x G_D(x,t_j)|_{x=t_i}] y \\
&\quad + \epsilon^2 \xi_i^2 \int_{-\infty}^{+\infty} w_i^2(|\tilde{z}|)\,d\tilde{z}\, [-\nabla_x H_D(x,t_i)|_{x=t_i}]\, y + O(\epsilon y^2 + \epsilon^2) \\
&= \epsilon \xi_\epsilon \Bigg\{ \hat{\xi}_i^2 \frac{P_i(y)}{\int_{-\infty}^{\infty} w^2(\tilde{z})\,d\tilde{z}} + \sum_{j\neq i} \hat{\xi}_j^2 \mu_j^{3/2} [\nabla_x G_D(x,t_j)|_{x=t_i}]\, y \\
&\quad + \hat{\xi}_i^2 \mu_i^{3/2} [-\nabla_x H_D(x,t_i)|_{x=t_i}]\, y + O(\epsilon y^2 + \epsilon^2) \Bigg\},
\end{aligned} \tag{4.12}$$

by (2.3), (2.4), where

$$P_i(y) = \int_{-\infty}^{+\infty} \left[\frac{1}{2D}|\tilde{z}| - \frac{1}{2D}|y - \tilde{z}| \right] w_i^2(\tilde{z})\,d\tilde{z}. \tag{4.13}$$

Let us define

$$S_\epsilon[A] = \epsilon^2 \nabla^2 A - \mu(x) A + \frac{A^2}{T[A]}, \tag{4.14}$$

where $T[A]$ is given by (4.9). We now compute $S_\epsilon[w_{\epsilon,\mathbf{t}}]$. In fact,

$$\begin{aligned} S_\epsilon[w_{\epsilon,\mathbf{t}}] &= \epsilon^2\nabla^2 w_{\epsilon,\mathbf{t}} - \mu(x) w_{\epsilon,\mathbf{t}} + \frac{w_{\epsilon,\mathbf{t}}^2}{T[w_{\epsilon,\mathbf{t}}]} \\ &= \sum_{j=1}^N \xi_j(\epsilon^2\nabla^2\tilde{w}_j - \mu(x)\tilde{w}_j) + \frac{w_{\epsilon,\mathbf{t}}^2}{T[w_{\epsilon,\mathbf{t}}]} + \text{e.s.t.} \\ &= -\sum_{j=1}^N \xi_j(\mu(x)-\mu_j)\tilde{w}_j + \left[\frac{(\sum_{j=1}^K \xi_j\tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}]} - \sum_{j=1}^K \xi_j\tilde{w}_j^2\right] + \text{e.s.t.} \\ &= E_1 + E_2 + \text{e.s.t.}, \end{aligned} \tag{4.15}$$

where

$$E_1 = -\sum_{j=1}^N \xi_j(\mu(x)-\mu(t_j))\tilde{w}_j \tag{4.16}$$

and

$$E_2 = \left[\frac{(\sum_{j=1}^N \xi_j\tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}]} - \sum_{j=1}^N \xi_j\tilde{w}_j^2\right]. \tag{4.17}$$

As we shall see, E_1 and E_2 contribute separately and they are competing with each other.

We first estimate E_1:

$$\xi_\epsilon^{-1}E_1 = -\sum_{j=1}^N \left(\mu'(t_j)(x-t_j) + \frac{1}{2}\mu''(t_j)(x-t_j)^2 + O(|x-t_j|^3)\right)$$

$$\times\hat{\xi}_j w_j\left(\frac{x-t_j}{\epsilon}\right)\chi\left(\frac{x-t_j}{r_0}\right). \tag{4.18}$$

Therefore

$$\xi_\epsilon^{-1}\|E_1\|_{L^2(\mathbb{R})} = O(\epsilon). \tag{4.19}$$

For E_2, we calculate

$$
\begin{aligned}
(4.20)\quad \xi_\epsilon^{-1}E_2 &= \frac{(\sum_{j=1}^N \xi_j\tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}]}\xi_\epsilon^{-1} - \sum_{j=1}^N \hat{\xi}_j\tilde{w}_j^2 \\
&= \sum_{j=1}^N \frac{(\xi_j\tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}]}\xi_\epsilon^{-1} - \sum_{j=1}^N \hat{\xi}_j\tilde{w}_j^2 \\
&= \sum_{j=1}^N \frac{(\xi_j\tilde{w}_j)^2}{T[w_{\epsilon,\mathbf{t}}](t_j)}\xi_\epsilon^{-1} - \sum_{j=1}^N \hat{\xi}_j\tilde{w}_j^2 - \sum_{j=1}^N \frac{(\xi_j\tilde{w}_j)^2}{(T[w_{\epsilon,\mathbf{t}}](t_j))^2}(T[w_{\epsilon,\mathbf{t}}] - T[w_{\epsilon,\mathbf{t}}](t_j))\xi_\epsilon^{-1} \\
&\quad +O\left(\sum_{j=1}^N |T[w_{\epsilon,\mathbf{t}}] - T[w_{\epsilon,\mathbf{t}}](t_j)|^2\tilde{w}_j^2\right) \\
&= \sum_{j=1}^N \tilde{w}_j^2\left(\frac{\hat{\xi}_j^2}{\hat{\xi}_j} - \hat{\xi}_j\right) - \sum_{j=1}^N \hat{\xi}_j\tilde{w}_j^2\frac{T[w_{\epsilon,\mathbf{t}}] - T[w_{\epsilon,\mathbf{t}}](t_j)}{T[w_{\epsilon,\mathbf{t}}](t_j)} + O\left(\epsilon^2\sum_{j=1}^N \tilde{w}_j^2\right) \\
&= -\epsilon\sum_{j=1}^N \tilde{w}_j^2\Bigg\{\hat{\xi}_j^2\frac{P_j(y_j)}{\int_{-\infty}^\infty w^2(\tilde{z})\,d\tilde{z}} + \sum_{k\neq j}\hat{\xi}_k^2\mu_k^{3/2}[\nabla_x G_D(x,t_k)|_{x=t_j}]\,y_j \\
(4.21)\quad &\quad +\hat{\xi}_j^2\mu_j^{3/2}[-\nabla_x H_D(x,t_j)|_{x=t_j}]\,y_j\Bigg\} + O\left(\epsilon^2\sum_{j=1}^N \tilde{w}_j^2\right) \qquad \text{(by (4.12))},
\end{aligned}
$$

where for $x \in B_{\epsilon^{3/4}}(t_j)$ we have denoted $y_j = \frac{x-t_j}{\epsilon}$. This implies that

$$(4.22)\qquad \xi_\epsilon^{-1}\|E_2\|_{L^2(\mathbb{R})} = O(\epsilon).$$

Combining (4.19) and (4.22), we conclude that

$$(4.23)\qquad \xi_\epsilon^{-1}\|S_\epsilon\|_{L^2(\mathbb{R})} = O(\epsilon).$$

The estimates derived in this section will enable us to carry out the existence proof in the next two sections.

5. The Liapunov-Schmidt Reduction Method

In this section, we study the linear operator defined by

$$\tilde{L}_{\epsilon,\mathbf{t}} := S_\epsilon'[A]\phi = \epsilon^2\nabla^2\phi - \mu(x)\phi + \frac{2A\phi}{T[A]} - \frac{A^2}{(T[A])^2}(T'[A]\phi),$$

$$\tilde{L}_{\epsilon,\mathbf{t}} : H^2(\Omega) \to L^2(\Omega),$$

where $A = w_{\epsilon,\mathbf{t}}$ and for $\phi \in L^2(\Omega)$ the function $T'[A]\phi$ is defined as the unique solution of

$$(5.1)\qquad \begin{cases} D\nabla^2(T'[A]\phi) - (T'[A]\phi) + 2A\phi = 0, & -1 < x < 1, \\ (T'[A]\phi)(-1) = (T'[A]\phi)(1) = 0. \end{cases}$$

We denote $\Omega_\epsilon = (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$. We define the approximate kernel and co-kernel of the operator $\tilde{L}_{\epsilon,\mathbf{t}}$, respectively, as follows:

$$\mathcal{K}_{\epsilon,\mathbf{t}} := \text{span}\ \left\{\frac{d\tilde{w}_i}{dx}\middle| i = 1,\dots,N\right\} \subset H^2(\Omega),$$

$$\mathcal{C}_{\epsilon,\mathbf{x}} := \text{span}\ \left\{ \frac{d\tilde{w}_i}{dx} \middle| i = 1, \ldots, N \right\} \subset L^2(\Omega).$$

Recall the definition of the following system of linear operators from (3.2):

$$L\Phi := \nabla^2\Phi - \Phi + 2w\Phi - 2\left(\int_{\mathbb{R}} wB\Phi\right)\left(\int_{\mathbb{R}} w^2\right)^{-1} w^2, \tag{5.2}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N.$$

By Lemma 3.3, we know that

$$L : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(\mathbb{R}))^N \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(\mathbb{R}))^N$$

is invertible with a bounded inverse, where $X_0 = \text{span}\left\{\frac{dw}{dy}\right\}$

We also introduce the orthogonal projection $\pi^\perp_{\epsilon,\mathbf{t}} : L^2(\Omega) \to \mathcal{C}^\perp_{\epsilon,\mathbf{t}}$ and study the operator $L_{\epsilon,\mathbf{t}} := \pi^\perp_{\epsilon,\mathbf{t}} \circ \tilde{L}_{\epsilon,\mathbf{t}}$. By letting $\epsilon \to 0$, we will show that $L_{\epsilon,\mathbf{t}} : \mathcal{K}^\perp_{\epsilon,\mathbf{t}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{t}}$ is invertible with a bounded inverse provided ϵ is sufficiently small. In proving this, we will use the fact that this system is a limit of the operator $L_{\epsilon,\mathbf{t}}$ as $\epsilon \to 0$. This statement is contained in the following proposition, whose proof is given in Proposition 5.1 of [55].

Proposition 5.1. *There exist positive constants $\bar{\epsilon}$, $\bar{\delta}, \lambda$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and all $\mathbf{t} \in \Omega^N$ with $|1 + t_1| + |1 - t_N| + \min_{i\neq j} |t_i - t_j| > \bar{\delta}$ we have*

$$\|L_{\epsilon,\mathbf{t}}\phi_\epsilon\|_{L^2(\Omega_\epsilon)} \geq \lambda\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)}. \tag{5.3}$$

Furthermore, the map

$$L_{\epsilon,\mathbf{t}} = \pi_{\epsilon,\mathbf{t}} \circ \tilde{L}_{\epsilon,\mathbf{t}} : \mathcal{K}^\perp_{\epsilon,\mathbf{t}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{t}}$$

is surjective.

Now we are in a position to solve the equation

$$\pi^\perp_{\epsilon,\mathbf{t}} \circ S_\epsilon(w_{\epsilon,\mathbf{t}} + \phi) = 0. \tag{5.4}$$

Since $L_{\epsilon,\mathbf{t}} : \mathcal{K}^\perp_{\epsilon,\mathbf{t}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{t}}$ is invertible (call the inverse $L^{-1}_{\epsilon,\mathbf{t}}$) we can rewrite

$$\phi = -(L^{-1}_{\epsilon,\mathbf{t}} \circ \pi^\perp_{\epsilon,\mathbf{t}})(S_\epsilon[w_{\epsilon,\mathbf{t}}]) - (L^{-1}_{\epsilon,\mathbf{t}} \circ \pi^\perp_{\epsilon,\mathbf{t}})(N_{\epsilon,\mathbf{t}}[\phi]) \equiv M_{\epsilon,\mathbf{t}}[\phi], \tag{5.5}$$

where

$$N_{\epsilon,\mathbf{t}}[\phi] = S_\epsilon[w_{\epsilon,\mathbf{t}} + \phi] - S_\epsilon[w_{\epsilon,\mathbf{t}}] - S'_\epsilon[w_{\epsilon,\mathbf{t}}]\phi$$

and the operator $M_{\epsilon,\mathbf{t}}$ is defined by (5.5) for $\phi \in H^2_N(\Omega_\epsilon)$, where

$$\Omega_\epsilon = \frac{\Omega}{\epsilon} = \left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right). \tag{5.6}$$

We are going to show that the operator $M_{\epsilon,\mathbf{t}}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\phi \in H^2(\Omega_\epsilon) | \|\phi\|_{H^2(\Omega_\epsilon)} < \delta\}$$

if δ and ϵ are sufficiently small. We have by (4.19), (4.22), and Proposition 5.1

$$\begin{aligned}\xi_\epsilon^{-1}\|M_{\epsilon,\mathbf{t}}[\phi]\|_{H^2(\Omega_\epsilon)} &\le \lambda^{-1}\xi_\epsilon^{-1}(\|\pi^\perp_{\epsilon,\mathbf{t}}(N_{\epsilon,\mathbf{t}}[\phi])\|_{L^2(\Omega_\epsilon)}+\left\|\pi^\perp_{\epsilon,\mathbf{t}}(S_\epsilon[w_{\epsilon,\mathbf{t}}])\right\|_{L^2(\Omega_\epsilon)})\\ &\le \lambda^{-1}C\left(\xi_\epsilon^{-1}c(\delta)\delta+\epsilon\left(\sum_{j=1}^N|\mu'(t_j)|+\sum_{j=1}^N\sum_{k=1}^N\left|\nabla_{t_j}G_D(t_j,t_k)\right|\right)\right),\end{aligned}$$

where $\lambda>0$ is independent of $\delta>0$, $\epsilon>0$ and $c(\delta)\to0$ as $\delta\to0$. Similarly, we show

$$\xi_\epsilon^{-1}\left\|M_{\epsilon,\mathbf{t}}[\phi]-M_{\epsilon,\mathbf{t}}[\phi']\right\|_{H^2(\Omega_\epsilon)}\le\lambda^{-1}C\left(\xi_\epsilon^{-1}c(\delta)\delta+\epsilon\left(\sum_{j=1}^N|\mu'(t_j)|+\sum_{j=1}^N\sum_{k=1}^N\left|\nabla_{t_j}G_D(t_j,t_k)\right|\right)\right)\|\phi-\phi'\|_{H^2(\Omega_\epsilon)},$$

where $\lambda>0$ is independent of $\delta>0$, $\epsilon>0$ and $c(\delta)\to0$ as $\delta\to0$. By the previous two estimates, if we choose δ and ϵ sufficiently small, then $M_{\epsilon,\mathbf{t}}$ is a contraction on $B_{\epsilon,\delta}$. The existence of a fixed point $\phi_{\epsilon,\mathbf{t}}$ now follows from the contraction mapping principle and $\phi_{\epsilon,\mathbf{t}}$ is a solution of (5.5).

We have thus proved

Lemma 5.2. *There exist $\overline{\epsilon}>0$ $\overline{\delta}>0$ such that for every pair of $\epsilon,\mathbf{t}$ with $0<\epsilon<\overline{\epsilon}$ and $\mathbf{t}\in B_{\epsilon^{3/4}}(\mathbf{t}^0)$, $|t_i-t_j|>\overline{\delta}$ there is a unique $\phi_{\epsilon,\mathbf{t}}\in K^\perp_{\epsilon,\mathbf{t}}$ satisfying $S_\epsilon(w_{\epsilon,\mathbf{t}}+\phi_{\epsilon,\mathbf{t}})\in\mathcal{C}_{\epsilon,\mathbf{t}}$. Furthermore, we have the estimate*

$$\xi_\epsilon^{-1}\|\phi_{\epsilon,\mathbf{t}}\|_{H^2(\Omega_\epsilon)}\le C\epsilon. \tag{5.7}$$

6. The reduced problem

In this section we solve the reduced problem and prove our main existence result, Theorem 2.1.

By Lemma 5.2, for every $\mathbf{t}\in B_{\epsilon^{3/4}}(\mathbf{t}_0)$, there exists a unique solution $\phi_{\epsilon,\mathbf{t}}\in\mathcal{K}^\perp_{\epsilon,\mathbf{t}}$ such that

$$S_\epsilon[w_{\epsilon,\mathbf{t}}+\phi_{\epsilon,\mathbf{t}}]=v_{\epsilon,\mathbf{t}}\in\mathcal{C}_{\epsilon,\mathbf{t}}. \tag{6.1}$$

Our idea is to find $\mathbf{t}^\epsilon=(t_1^\epsilon,\dots,t_N^\epsilon)\in B_{\epsilon^{3/4}}(\mathbf{t}^0)$ such that also

$$S_\epsilon[w_{\epsilon,\mathbf{t}^\epsilon}+\phi_{\epsilon,\mathbf{t}^\epsilon}]\perp\mathcal{C}_{\epsilon,\mathbf{t}^\epsilon}. \tag{6.2}$$

Then from (6.1) and (6.2) we get that $S_\epsilon[w_{\epsilon,\mathbf{t}^\epsilon}+\phi_{\epsilon,\mathbf{t}^\epsilon}]=0$. To this end, we let

$$\begin{aligned}W_{\epsilon,i}(\mathbf{t}) &:= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega S[w_{\epsilon,\mathbf{t}}+\phi_{\epsilon,\mathbf{t}}]\frac{d\tilde{w}_i}{dx}\,dx,\\ W_\epsilon(\mathbf{t}) &:= (W_{\epsilon,1}(\mathbf{t}),...,W_{\epsilon,N}(\mathbf{t})):B_{\epsilon^{3/4}}(\mathbf{t}_0)\to\mathbb{R}^N.\end{aligned}$$

Then $W_\epsilon(\mathbf{t})$ is a map which is continuous in $\mathbf{t}$ and (6.2) is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{t})$.

Let us now calculate $W_\epsilon(\mathbf{t})$. By (4.18) and (4.21), we have

$$\begin{aligned}\xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega S[w_{\epsilon,\mathbf{t}}+\phi_{\epsilon,\mathbf{t}}]\frac{d\tilde{w}_i}{dx}\,dx &= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega S[w_{\epsilon,\mathbf{t}}]\frac{d\tilde{w}_i}{dx}\,dx\\ &\quad+\xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega S'_\epsilon[w_{\epsilon,\mathbf{t}}]\phi_{\epsilon,\mathbf{t}}\frac{d\tilde{w}_i}{dx}\,dx+\xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega N_\epsilon(\phi_{\epsilon,\mathbf{t}})\frac{d\tilde{w}_i}{dx}\,dx+O(\epsilon^2)\\ &= I_1+I_2+I_3+O(\epsilon^2),\end{aligned}$$

where I_1, I_2 and I_3 are defined by the last equality.

The computation of I_3 is the easiest: note that the first term in the expansion of N_ϵ is quadratic in $\phi_{\epsilon,\mathbf{t}}$ and so

$$I_3 = O(\epsilon^2). \tag{6.3}$$

We will now compute I_1 and I_2. The result will be that I_1 is the leading term and $I_2 = O(\epsilon)$.

For I_1, we have

$$I_1 = \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega (E_1+E_2)\frac{d\tilde{w}_i}{dx}\,dx + O(\epsilon) = I_{11} + I_{12},$$

where E_1 and E_2 have been defined in (4.16) and (4.17), respectively.

We calculate, using (4.18),

$$\begin{aligned}
I_{11} &= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega E_1\frac{d\tilde{w}_i}{dx}\,dx\\
&= -\mu'(t_i)\int_{\mathbb{R}} y\hat{\xi}_i w_i(y)w_i'(y)\,dy + O(\epsilon)\\
&= \mu'(t_i)\hat{\xi}_i\int_{\mathbb{R}} \frac{1}{2}\left(w_i(y)\right)^2\,dy + O(\epsilon)\\
&= \mu'(t_i)\hat{\xi}_i\mu_i^{3/2}\int_{\mathbb{R}}\frac{1}{2}w^2(y)\,dy + O(\epsilon) \qquad \text{(by (2.3))}.
\end{aligned}$$

Next, we calculate by (4.21)

$$\begin{aligned}
I_{12} &= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega E_2\frac{d\tilde{w}_i}{dx}\,dx\\
&= -\int_{\mathbb{R}}\tilde{w}_i^2\left[\sum_{k\neq i}\hat{\xi}_k^2\mu_k^{3/2}[\nabla_x G_D(x,t_k)|_{x=t_i}]y + \hat{\xi}_i^2\mu_i^{3/2}[-\nabla_x H_D(x,t_i)|_{x=t_i}]y\right]\tilde{w}_i'\,dy + O(\epsilon)\\
&= -\int_{\mathbb{R}}\left(y\tilde{w}_i^2\tilde{w}_i'\right)dy\left\{\sum_{k=1}^N\hat{\xi}_k^2\mu_k^{3/2}\left\{[\nabla_x G_D(x,t_k)|_{x=t_i}](1-\delta_{ik}) - [\nabla_x H_D(x,t_i)|_{x=t_i}]\delta_{ik}\right\}\right\} + O(\epsilon)\\
&= \mu_i^{5/2}\frac{1}{3}\int_{\mathbb{R}} w^3dy\left\{\sum_{k=1}^N\hat{\xi}_k^2\mu_k^{3/2}\left\{[\nabla_x G_D(x,t_k)|_{x=t_i}](1-\delta_{ik}) - [\nabla_x H_D(x,t_i)|_{x=t_i}]\delta_{ik}\right\}\right\} + O(\epsilon)
\end{aligned}$$

since $P_i(y)$ is an even function. Using (1.7), we have

$$I_1 = \mu_i^{3/2}\left[c_i\mu'(t_i) - d_{ii}\nabla_{t_i}H_D(t_i,t_i) + \sum_{j\neq i} d_{ij}\nabla_{t_i}G_D(t_i,t_j)\right] + O(\epsilon), \tag{6.4}$$

where

$$c_i = 3\hat{\xi}_i, \quad d_{ij} = 2.4\mu_i\hat{\xi}_j^2\mu_j^{3/2}$$

For I_2, we calculate

$$\begin{aligned} I_2 &= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega S_\epsilon'[w_{\epsilon,\mathbf{t}}](\phi_{\epsilon,\mathbf{t}})\frac{d\tilde{w}_i}{dx}\,dx \\ &= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega \left[\epsilon^2\nabla^2\phi_{\epsilon,\mathbf{t}} - \mu(x)\phi_{\epsilon,\mathbf{t}} + \frac{2w_{\epsilon,\mathbf{t}}\phi_{\epsilon,\mathbf{t}}}{T[w_{\epsilon,\mathbf{t}}]} - \frac{w_{\epsilon,\mathbf{t}}^2}{(T[w_{\epsilon,\mathbf{t}}])^2}(T'[w_{\epsilon,\mathbf{t}}]\phi_{\epsilon,\mathbf{t}})\right]\frac{d\tilde{w}_i}{dx}\,dx \\ &= \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega \left[\epsilon^2\nabla^2\frac{d\tilde{w}_i}{dx} - \mu(t_i)\frac{d\tilde{w}_i}{dx} + \frac{2w_{\epsilon,\mathbf{t}}}{T[w_{\epsilon,\mathbf{t}}](t_i)}\right]\phi_{\epsilon,\mathbf{t}}\,dx \\ &\quad +\xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega 2\frac{w_{\epsilon,\mathbf{t}}}{T[w_{\epsilon,\mathbf{t}}](t_i)}\phi_{\epsilon,\mathbf{t}}\left(\frac{T[w_{\epsilon,\mathbf{t}}](t_i) - T[w_{\epsilon,\mathbf{t}}]}{T[w_{\epsilon,\mathbf{t}}]}\right)dx \\ &\quad +\xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega(\mu(t_i)-\mu(x))\phi_{\epsilon,\mathbf{t}}\frac{d\tilde{w}_i}{dx}\,dx - \xi_\epsilon^{-1}\epsilon^{-1}\int_\Omega\frac{w_{\epsilon,\mathbf{t}}^2}{(T[w_{\epsilon,\mathbf{t}}])^2}(T'[w_{\epsilon,\mathbf{t}}]\phi_{\epsilon,\mathbf{t}})\frac{d\tilde{w}_i}{dx}\,dx + O(\epsilon^2) = O(\epsilon), \end{aligned}$$

by (4.5), (4.11), (4.12) since

$$|\mu'(t_i) - \mu(x)| = O(\epsilon|y|), \quad \|\phi_{\epsilon,\mathbf{t}}\|_{H^2(\Omega_\epsilon)} = O(\epsilon).$$

Combining I_1, I_2 and I_3, we have

$$W_{\epsilon,i}(\mathbf{t}) = \mu_i^{3/2}\left[c_i\mu'(t_i) - d_{ii}\nabla_{t_i}H_D(t_i,t_i) + \sum_{j\neq i} d_{ij}\nabla_{t_i}G_D(t_i,t_j)\right] + O(\epsilon), \tag{6.5}$$

where

$$c_i = 3\hat{\xi}_i, \quad d_{ij} = 2.4\mu_i\hat{\xi}_j^2\mu_j^{3/2}.$$

Recall from (2.13) that

$$F(\mathbf{t}) = (F_1(\mathbf{t}),\ldots,F_N(\mathbf{t})),$$

where

$$W_{\epsilon,i}(\mathbf{t}) = 2.4\mu_i^{5/2}F_i(\mathbf{t}) + O(\epsilon), \quad i = 1,\ldots,N.$$

By assumption **(H3)**, we have $F(\mathbf{t}_0) = 0$ and

$$\det\left(D_{\mathbf{t}_0}F\left(\mathbf{t}_0\right)\right) \neq 0.$$

Therefore the vector field $W_\epsilon(\mathbf{t}) = (W_{\epsilon,1}(\mathbf{t}),\ldots,W_{\epsilon,N}(\mathbf{t}))$ satisfies $W_\epsilon(\mathbf{t}) = D_{\mathbf{t}_0}F(\mathbf{t}_0)(\mathbf{t}-\mathbf{t}_0) + O(\epsilon)$.

Thus for ϵ small enough $F(\mathbf{t})$ has exactly one zero in $B_{\epsilon^{3/4}}(\mathbf{t}^0)$ and we compute the mapping degree of $W_\epsilon(\mathbf{t})$ for the set $B_{\epsilon^{3/4}}(\mathbf{t}^0)$ and the value 0 as follows:

$$\deg(W_\epsilon, 0, B_{\epsilon^{3/4}}(\mathbf{t}^0)) = \text{sign }\det\left(D_{\mathbf{t}_0}F\left(\mathbf{t}_0\right)\right) \neq 0.$$

Therefore, standard degree theory implies that, for ϵ small enough, there exists a $\mathbf{t}_\epsilon \in B_{\epsilon^{3/4}}(\mathbf{t}_0)$ such that $W_\epsilon(\mathbf{t}_\epsilon) = 0$ and $\mathbf{t}_\epsilon \to \mathbf{t}_0$ as $\epsilon \to 0$.

Thus we have proved the following proposition.

Proposition 6.1. *For ϵ sufficiently small there exists a point $\mathbf{t}_\epsilon \in B_{\epsilon^{3/4}}(\mathbf{t}_0)$ with $\mathbf{t}_\epsilon \to \mathbf{t}_0$ such that $W_\epsilon(\mathbf{t}_\epsilon) = 0$.*

Finally, we prove Theorem 2.1.

Proof of Theorem 2.1: We sketch the main arguments. By Proposition 6.1, there exists a $\mathbf{t}_\epsilon \in B_{\epsilon^{3/4}}(\mathbf{t}_0)$ such that $\mathbf{t}_\epsilon \to \mathbf{t}^0$ and $W_\epsilon(\mathbf{t}_\epsilon) = 0$. In other words, $S_\epsilon[w_{\epsilon,\mathbf{t}_\epsilon} + \phi_{\epsilon,\mathbf{t}_\epsilon}] = 0$. Let $w_\epsilon = w_{\epsilon,\mathbf{t}_\epsilon} + \phi_{\epsilon,\mathbf{t}_\epsilon}$. By the Maximum Principle, $w_\epsilon > 0$. Moreover, by its construction, w_ϵ has all the properties required in Theorem 2.1. The proof is finished.

□

7. Stability Analysis I: Large Eigenvalues

In this section, we study the eigenvalues with $\lambda_\epsilon \to \lambda_0 \neq 0$ as $\epsilon \to 0$. The key ingredient is Theorem 3.1.

We need to analyze the following eigenvalue problem

$$\tilde{L}_{\epsilon,\mathbf{t}_\epsilon}\phi_\epsilon = \epsilon^2 \nabla^2 \phi_\epsilon - \mu(x)\phi_\epsilon + \frac{2A\phi_\epsilon}{T[A]} - \frac{A^2}{(T[A])^2}(T'[A]\phi_\epsilon) = \lambda_\epsilon \phi_\epsilon, \tag{7.1}$$

where λ_ϵ is some complex number, $A = w_{\epsilon,\mathbf{t}_\epsilon} + \phi_{\epsilon,\mathbf{t}_\epsilon}$ with $\mathbf{t}_\epsilon \in B_{\epsilon^{3/4}}(\mathbf{t}^0)$ determined in the previous section and

$$\phi_\epsilon \in H^2_N(\Omega). \tag{7.2}$$

(Recall that $T'[A]$ was defined in (5.1).)

Because we study the large eigenvalues there exists some small $c > 0$ such that $|\lambda_\epsilon| \geq c > 0$ for ϵ sufficiently small. We are looking for a condition under which Re $(\lambda_\epsilon) < 0$ for all eigenvalues λ_ϵ of (7.1), (7.2) if ϵ is sufficiently small. If $\mathrm{Re}(\lambda_\epsilon) \leq -c$, then λ_ϵ is a stable large eigenvalue. Therefore for the rest of this section we assume that $\mathrm{Re}(\lambda_\epsilon) \geq -c$ and study the stability properties of such eigenvalues.

In (7.1), (7.2) it is assumed that $\tau = 0$. By a straight-forward perturbation argument all the results also hold true for $\tau > 0$ sufficiently small.

We first rigorously derive the limiting problem of (7.1), (7.2) as $\epsilon \to 0$ which will be given by a system of NLEPs. Let us assume that

$$\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1.$$

We cut off ϕ_ϵ as follows: Introduce

$$\phi_{\epsilon,j}(y) = \phi_\epsilon(y)\chi_{\epsilon,P_j^\epsilon}(\epsilon y), \tag{7.3}$$

where $y = (x - t_j)/\epsilon$ for $x \in \Omega$.

From (7.1), (7.2), using $\mathrm{Re}(\lambda_\epsilon) \geq -c$ and $\|\phi_{\epsilon,\mathbf{t}_\epsilon}\|_{H^2(\Omega_\epsilon)} = O(\epsilon)$, it follows that

$$\phi_\epsilon = \sum_{j=1}^N \phi_{\epsilon,j} + O(\epsilon^2) \quad \text{in } H^2(\Omega_\epsilon). \tag{7.4}$$

Then by a standard procedure we extend $\phi_{\epsilon,j}$ to a function defined on $\mathbb{R}$ such that

$$\|\phi_{\epsilon,j}\|_{H^2(\mathbb{R})} \leq C\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)}, \qquad j = 1, \ldots, N.$$

Since $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$, $\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)} \leq C$. By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,j} \to \phi_j$ as $\epsilon \to 0$ in $H^1(\mathbb{R})$ for $j = 1, \ldots, N$.

Sending $\epsilon \to 0$ with $\lambda_\epsilon \to \lambda_0$, (7.1) for $x \in B_{\epsilon^{3/4}}(t_i)$ can be written as

$$\nabla_y^2 \phi_i - \mu_i \phi_i + 2\tilde{w}_i \phi_i$$

$$-2\left(\sum_{k=1}^{N} G_D(t_i^0, t_k^0) \int_{\mathbb{R}} \hat{\xi}_k^0 \tilde{w}_k \phi_k \, dy\right)\left(\sum_{k=1}^{N} G_D(t_i^0, t_k^0) \int_{\mathbb{R}} \left(\hat{\xi}_k^0 \tilde{w}_k\right)^2 \, dy\right)^{-1} \tilde{w}_i^2 = \lambda_0 \phi_i.$$

Rewriting this as a system, using the transformation $\tilde{y} = \sqrt{\mu} y$, this implies (after dropping the tilde)

$$L\Phi = \nabla^2 \Phi - \Phi + 2w\Phi - 2\left(\int_{\mathbb{R}} w \mathcal{B} \Phi \, dy\right)\left(\int_{\mathbb{R}} w^2 \, dy\right)^{-1} w^2 = \lambda_0 \Phi, \tag{7.5}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(\mathbb{R}))^N$$

and (2.3), **(H1)** have been used.

Then we have

Theorem 7.1. *Let λ_ϵ be an eigenvalue of (7.1) and (7.2) such that $Re(\lambda_\epsilon) > -c$ for some $c > 0$.*

(1) Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the problem (NLEP) given in (7.5).

(2) Let $\lambda_0 \neq 0$ with $Re(\lambda_0) > 0$ be an eigenvalue of the problem (NLEP) given in (7.5). Then for ϵ sufficiently small, there is an eigenvalue λ_ϵ of (7.1) and (7.2) with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.

Proof:

(1) of Theorem 7.1 follows by asymptotic analysis similar to Section 5.

To prove (2) of Theorem 7.1, we follow the argument given in Section 2 of [3], where the following eigenvalue problem was studied:

$$\begin{cases} \epsilon^2 \nabla^2 h - h + p u_\epsilon^{p-1} h - \frac{qr}{s+1+\tau\lambda_\epsilon} \frac{\int_\Omega u_\epsilon^{r-1} h}{\int_\Omega u_\epsilon^r} u_\epsilon^p = \lambda_\epsilon h \text{ in } \Omega, \\ h = 0 \text{ on } \partial\Omega, \end{cases} \tag{7.6}$$

where u_ϵ is a solution of the single equation

$$\begin{cases} \epsilon^2 \nabla^2 u_\epsilon - u_\epsilon + u_\epsilon^p = 0 \text{ in } \Omega, \\ u_\epsilon > 0 \text{ in } \Omega, \; u_\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Here $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$ and $1 < p < +\infty$ if $n = 1, 2$, $\frac{qr}{(s+1)(p-1)} > 1$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. If u_ϵ is a single interior peak solution, then it can be shown ([51]) that the limiting eigenvalue problem is a NLEP

$$\nabla^2 \phi - \phi + p w^{p-1} \phi - \frac{qr}{s+1+\tau\lambda_0} \frac{\int_{\mathbb{R}^N} w^{r-1} \phi}{\int_{\mathbb{R}^N} w^r} w^p = \lambda_0 \phi \tag{7.7}$$

where w is the corresponding ground state solution in $\mathbb{R}^n$:

$$\nabla^2 w - w + w^p = 0, \quad w > 0 \text{ in } \mathbb{R}^n, \quad w = w(|y|) \in H^1(\mathbb{R}^n).$$

Dancer in [3] showed that if $\lambda_0 \neq 0$, $\mathrm{Re}(\lambda_0) > 0$ is an unstable eigenvalue of (7.7), then there exists an eigenvalue λ_ϵ of (7.6) such that $\lambda_\epsilon \to \lambda_0$.

We now follow his idea. Let $\lambda_0 \neq 0$ be an eigenvalue of problem (7.5) with $\mathrm{Re}(\lambda_0) > 0$. We first note that by(5.1) we can express $T'[A]\phi_\epsilon$ in terms of ϕ_ϵ by the Green's function. Then we rewrite (7.1) as follows:

$$\phi_\epsilon = -R_\epsilon(\lambda_\epsilon)\left[\frac{2A\phi_\epsilon}{T[A]} - \frac{A^2}{T[A]}T'[A]\phi_\epsilon\right], \tag{7.8}$$

where $R_\epsilon(\lambda_\epsilon)$ is the inverse of $-\nabla^2 + (\mu(x) + \lambda_\epsilon)$ in $H^2(\mathbb{R})$ (which exists if $\mathrm{Re}(\lambda_\epsilon) > -\min_{x\in\mathbb{R}} \mu(x)$ or $\mathrm{Im}(\lambda_\epsilon) \neq 0$). The important thing is that $R_\epsilon(\lambda_\epsilon)$ is a compact operator if ϵ is sufficiently small. The rest of the argument follows in the same way as in [3]. For the sake of limited space, we omit the details here.

□

We now study the stability of (7.1), (7.2) for large eigenvalues explicitly and prove (2.19) and (2.21) of Theorem 2.2.

Let σ_i, $i = 1, \ldots, N$ be the eigenvalues of the matrix $\mathcal{B}$. These eigenvalues are real, see Remark 2.1.

Then the system (7.5) can be re-written as

$$L\phi_i = \nabla^2\phi_i - \phi_i + 2w\phi_i - 2\sigma_i\left(\int_{\mathbb{R}} w\phi_i\,dy\right)\left(\int_{\mathbb{R}} w^2\,dy\right)^{-1} w^2 = \lambda_0\phi_i, \quad i = 1, \ldots, N, \tag{7.9}$$

where

$$\phi_i \in H^2(\mathbb{R}), \quad i = 1, \ldots, N.$$

Suppose that we have

$$\min_{\sigma\in\sigma(\mathcal{B})} \sigma < \frac{1}{2}, \tag{7.10}$$

by Theorem 3.1 (1), there exists a positive eigenvalue of (7.9) and so also of (7.5). By Theorem 7.1 (2), for ϵ sufficiently small, there exists an eigenvalue λ_ϵ of (7.1) and (7.2) such that $\mathrm{Re}(\lambda_\epsilon) > c_0$ for some positive number $c_0 > 0$. This implies that $A = w_{\epsilon,\mathbf{t}_\epsilon} + \phi_{\epsilon,\mathbf{t}_\epsilon}$ is (linearly) unstable.

Suppose now that

$$\min_{\sigma\in\sigma(\mathcal{B})} \sigma > \frac{1}{2}, \tag{7.11}$$

is satisfied, then by Theorem 3.1 (2), we know that for any nonzero eigenvalue λ_0 in (7.9) and so also in (7.5) we have

$$\mathrm{Re}(\lambda_0) \leq c_0 < 0 \qquad \text{for some } c_0 > 0.$$

So by Theorem 7.1 (1), for ϵ sufficiently small, all nonzero large eigenvalues of (7.1), (7.2) all have strictly negative real parts. We conclude that in this case all eigenvalues λ_ϵ of (7.1), (7.2), for which $|\lambda_\epsilon| \geq c > 0$ holds, satisfy $\mathrm{Re}(\lambda_\epsilon) \leq -c < 0$ for ϵ sufficiently small. This implies that $A = w_{\epsilon,\mathbf{t}_\epsilon} + \phi_{\epsilon,\mathbf{t}_\epsilon}$ is stable.

□

In conclusion, we have finished the study of large eigenvalues and derived results on their stability properties. It remains to study small eigenvalues which will be done in the next section.

8. Stability Analysis II: Small Eigenvalues

Now we study (7.1), (7.2) for small eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$.

Let

$$\bar{w}_\epsilon = \xi_\epsilon^{-1}\left[w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}\right], \quad \bar{H}_\epsilon = T[w_{\epsilon,\mathbf{t}^\epsilon} + \phi_{\epsilon,\mathbf{t}^\epsilon}], \tag{8.1}$$

where $\mathbf{t}^\epsilon = (t_1^\epsilon, \ldots, t_N^\epsilon)$.

After re-scaling, the eigenvalue problem (7.1), (7.2) becomes

$$\epsilon^2\nabla^2\phi_\epsilon - \mu(x)\phi_\epsilon + 2\frac{\bar{w}_\epsilon}{\bar{H}_\epsilon}\phi_\epsilon - \frac{\bar{w}_\epsilon^2}{\bar{H}_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \tag{8.2}$$

$$D\nabla^2\psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon\bar{w}_\epsilon\phi_\epsilon = \lambda_\epsilon\tau\psi_\epsilon. \tag{8.3}$$

where ξ_ϵ is given by (2.4).

Our basic idea is the following: the eigenfunction ϕ_ϵ can be expanded as

$$\sum_{j=1}^N a_j\frac{\partial}{\partial t_j}(w_{\epsilon,\mathbf{t}}).$$

Note that $w_{\epsilon,\mathbf{t}} \sim \sum_{j=1}^N \xi_j(\mathbf{t})w_j(x)$. So when we differentiate $w_{\epsilon,\mathbf{t}}$ with respect to t_j, we also need to differentiate ξ_j and $\mu(t_j)$ with respect to t_j. Thus we have to expand ϕ_ϵ up to $O(\epsilon^2)$.

Let us define

$$\tilde{w}_{\epsilon,j}(x) = \chi\left(\frac{x-t_j^\epsilon}{r_0}\right)\bar{w}_\epsilon(x), \quad j = 1, ..., N, \tag{8.4}$$

where r_0 and $\chi(x)$ are given in (4.2) and (4.3). Similarly as in Section 5, we define

$$\mathcal{K}_{\epsilon,\mathbf{t}^\epsilon}^{new} := \text{span}\,\{\tilde{w}_{\epsilon,j}^{'}|j = 1, \ldots, N\} \subset H^2(\Omega_\epsilon),$$

$$\mathcal{C}_{\epsilon,\mathbf{t}^\epsilon}^{new} := \text{span}\,\{\tilde{w}_{\epsilon,j}^{'}|j = 1, \ldots, N\} \subset L^2(\Omega_\epsilon).$$

Then it is easy to see that

$$\bar{w}_\epsilon(x) = \sum_{j=1}^N \tilde{w}_{\epsilon,j}(x) + \text{e.s.t.} \tag{8.5}$$

and

$$\bar{H}_\epsilon^{'}(t_l) = \xi_\epsilon\int_{-1}^1 \nabla_{t_l}G_D(t_l;z)\bar{w}_\epsilon^2\,dz$$

$$= \sum_{k=1}^N \nabla_{t_l}G_D(t_l,t_k)\hat{\xi}_k^2\mu_k^{3/2} + O(\epsilon) = -\frac{5}{4}\hat{\xi}_l\mu_l^{-1}\mu_l^{'} + O(\epsilon) \tag{8.6}$$

by (**H3**).

Note that $\tilde{w}_{\epsilon,j}(x) = \hat{\xi}_j w_j\left(\frac{x-t_j^\epsilon}{\epsilon}\right) + O(\epsilon)$ in $H^2_{loc}(-1,1)$ and $\tilde{w}_{\epsilon,j}$ satisfies

$$\epsilon^2\nabla^2\tilde{w}_{\epsilon,j} - \mu(x)\tilde{w}_{\epsilon,j} + \frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon} + \text{e.s.t.} = 0$$

Thus $\tilde{w}'_{\epsilon,j} := \frac{d\tilde{w}_{\epsilon,j}}{dx}$ satisfies

$$\epsilon^2\nabla^2\tilde{w}'_{\epsilon,j} - \mu(x)\tilde{w}'_{\epsilon,j} + \frac{2\tilde{w}_{\epsilon,j}}{\bar{H}_\epsilon}\tilde{w}'_{\epsilon,j} - \frac{\tilde{w}^2_{\epsilon,j}}{(\bar{H}_\epsilon)^2}\bar{H}'_\epsilon - \mu'(x)\tilde{w}_{\epsilon,j} + \text{e.s.t.} = 0. \tag{8.7}$$

Let us now decompose

$$\phi_\epsilon = \epsilon\sum_{j=1}^N a_j^\epsilon\tilde{w}'_{\epsilon,j} + \phi_\epsilon^\perp \tag{8.8}$$

with complex numbers a_j^ϵ, (the scaling factor ϵ is introduced to ensure $\phi_\epsilon = O(1)$ in $H^2_{loc}(\Omega_\epsilon)$), where $\phi_\epsilon^\perp \perp \mathcal{K}^{new}_{\epsilon,\mathbf{t}^\epsilon}$.

Suppose that $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$. Then $|a_j^\epsilon| \le C$.

The decomposition of ϕ_ϵ implies the following decomposition of ψ_ϵ:

$$\psi_\epsilon = \epsilon\sum_{j=1}^N a_j^\epsilon\psi_{\epsilon,j} + \psi_\epsilon^\perp, \tag{8.9}$$

where $\psi_{\epsilon,j}$ satisfies

$$D\nabla^2\psi_{\epsilon,j} - \psi_{\epsilon,j} + 2\xi_\epsilon\bar{w}_\epsilon\tilde{w}'_{\epsilon,j} = 0, \tag{8.10}$$

$\psi_\epsilon^\perp$ satisfies

$$D\nabla^2\psi_\epsilon^\perp - \psi_\epsilon^\perp + 2\xi_\epsilon\bar{w}_\epsilon\phi_\epsilon^\perp = 0. \tag{8.11}$$

and both (8.10) and (8.11) are solved with Neumann boundary conditions.

Throughout this section, we denote

$$\mu_j = \mu(t_j^\epsilon), \quad \mu'_j = \mu'(t_j^\epsilon), \quad \mu''_j = \mu''(t_j^\epsilon).$$

Substituting the decompositions of ϕ_ϵ and ψ_ϵ into (8.2) we have, using (8.7)

$$\begin{aligned}&\epsilon\sum_{j=1}^N a_j^\epsilon\left(\frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon^2}\bar{H}'_\epsilon - \frac{(\bar{w}_\epsilon)^2}{\bar{H}_\epsilon^2}\psi_{\epsilon,j}\right) + \epsilon\sum_{j=1}^N a_j^\epsilon\mu'(x)\tilde{w}'_{\epsilon,j}\\ &+\epsilon^2\nabla^2\phi_\epsilon^\perp - \mu_(x)\phi_\epsilon^\perp + 2\frac{\bar{w}_\epsilon}{\bar{H}_\epsilon}\phi_\epsilon^\perp - \frac{\bar{w}_\epsilon^2}{\bar{H}_\epsilon^2}\psi_\epsilon^\perp - \lambda_\epsilon\phi_\epsilon^\perp + \text{e.s.t.} \quad = \quad \lambda_\epsilon\left(\epsilon\sum_{j=1}^N a_j^\epsilon\tilde{w}'_{\epsilon,j}\right).\end{aligned} \tag{8.12}$$

Let us first compute

$$\begin{aligned} I_4 &:= \epsilon \sum_{j=1}^N a_j^\epsilon \left(\frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon^2} \bar{H}_\epsilon' - \frac{(\bar{w}_\epsilon)^2}{\bar{H}_\epsilon^2} \psi_{\epsilon,j} \right) \\ &= \epsilon \sum_{j=1}^N a_j^\epsilon \left(\frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon^2} (\bar{H}_\epsilon' - \psi_{\epsilon,j}) \right) - \epsilon \sum_{j=1}^N a_j^\epsilon \psi_{\epsilon,j} \sum_{k\neq j} \frac{(\tilde{w}_{\epsilon,k})^2}{\bar{H}_\epsilon^2} + \text{e.s.t.} \\ &= \epsilon \sum_{j=1}^N a_j^\epsilon \frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon^2} \left[-\psi_{\epsilon,j} + \bar{H}_\epsilon' \right] - \epsilon \sum_{j=1}^N \sum_{k\neq j} a_k^\epsilon \psi_{\epsilon,k} \frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon^2} + \text{e.s.t.}. \end{aligned}$$

We can rewrite I_4 as follows

$$I_4 = -\epsilon \sum_{j=1}^N \sum_{k=1}^N a_k^\epsilon \frac{(\tilde{w}_{\epsilon,j})^2}{\bar{H}_\epsilon^2} \left(\psi_{\epsilon,k} - \bar{H}_\epsilon' \delta_{jk} \right) + \text{e.s.t.} \tag{8.13}$$

Let us also put

$$\tilde{L}_\epsilon \phi_\epsilon^\perp := \epsilon^2 \nabla^2 \phi_\epsilon^\perp - \mu(x) \phi_\epsilon^\perp + \frac{2\bar{w}_\epsilon}{\bar{H}_\epsilon} \phi_\epsilon^\perp - \frac{\bar{w}_\epsilon^2}{\bar{H}_\epsilon^2} \psi_\epsilon^\perp \tag{8.14}$$

and

$$\mathbf{a}^\epsilon := (a_1^\epsilon, ..., a_N^\epsilon)^T. \tag{8.15}$$

Multiplying both sides of (8.12) by $\tilde{w}_{\epsilon,l}'$ and integrating over $(-1,1)$, we obtain, using (2.3),

$$\begin{aligned} \text{r.h.s.} &= \epsilon \lambda_\epsilon \sum_{j=1}^N a_j^\epsilon \int_{-1}^1 \tilde{w}_{\epsilon,j}' \tilde{w}_{\epsilon,l}' \, dx \\ &= \lambda_\epsilon a_l^\epsilon \hat{\xi}_l^2 \int_{\mathbb{R}} (w_l'(y))^2 \, dy \; (1 + O(\epsilon)) \qquad (8.16) \\ &= \lambda_\epsilon a_l^\epsilon \hat{\xi}_l^2 \mu_l^{5/2} \int_{\mathbb{R}} (w'(z))^2 \, dz \; (1 + O(\epsilon)) \qquad (8.17) \end{aligned}$$

and, using (8.13),

$$\begin{aligned} \text{l.h.s.} &= -\epsilon \sum_{j=1}^N \sum_{k=1}^N a_k^\epsilon \int_{-1}^1 \frac{\tilde{w}_{\epsilon,j}^2}{\bar{H}_\epsilon^2} \left(\psi_{\epsilon,k} - \bar{H}_\epsilon' \delta_{jk} \right) \tilde{w}_{\epsilon,l}' \, dx + \int_{-1}^1 \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2} (\bar{H}_\epsilon' \phi_\epsilon^\perp) \, dx \\ &\quad + \epsilon \sum_{j=1}^N a_j^\epsilon \int_{-1}^1 \mu' \tilde{w}_{\epsilon,j} \tilde{w}_{\epsilon,l}' \, dx - \int_{-1}^1 \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2} (\psi_\epsilon^\perp w_{\epsilon,l}') \, dx + \int_{-1}^1 \mu' \phi_\epsilon^\perp w_{\epsilon,l} \, dx \\ &= (J_{1,l} + J_{2,l} + J_{3,l} + J_{4,l} + J_{5,l})(1 + O(\epsilon)), \end{aligned}$$

where $J_{i,l}$, $i = 1,2,3,4,5$ are defined by the last equality.

For $J_{3,l}$, integrating by parts gives

$$\begin{aligned} \epsilon \sum_{j=1}^N a_j^\epsilon \int_{-1}^1 \mu' \tilde{w}_{\epsilon,j} \tilde{w}_{\epsilon,l}' \, dx &= -\frac{\epsilon a_l^\epsilon}{2} \int_{-1}^1 \mu'' \tilde{w}_{\epsilon,l}^2 \, dx + o(\epsilon^2) \\ &= -\frac{\epsilon^2 a_l^\epsilon}{2} \xi_{\epsilon,l}^2 \mu_l^{3/2} \mu_l'' \int_{\mathbb{R}} w^2 \, dy + o(\epsilon^2). \end{aligned}$$

For $J_{4,l}$, we decompose

$$J_{4,l} = J_{6,l} + J_{7,l},$$

where

$$J_{6,l} = -\int_{-1}^{1} \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2}(\psi_\epsilon^\perp(t_l^\epsilon) w_{\epsilon,l}')\,dx \tag{8.18}$$

$$J_{7,l} = -\int_{-1}^{1} \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2}(\psi_\epsilon^\perp(x) - \psi_\epsilon^\perp(t_l^\epsilon)) w_{\epsilon,l}'\,dx. \tag{8.19}$$

We define the vectors

$$\mathbf{J}_i = (J_{i,1}, ..., J_{i,N})^T, \quad i = 1, 2, 3, 4, 5, 6, 7. \tag{8.20}$$

The following is the key lemma.

Lemma 8.1. *We have*

$$\begin{aligned}
(8.21)\quad \mathbf{J}_1 &= -\epsilon^2 \left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}\mu^{5/2} \left[(\nabla^2 \mathcal{G}_D)\mathcal{H}^2\mu^{3/2} - \mathcal{Q}\mathcal{H}^2\mu^{3/2}\right]\mathbf{a}^0 \\
&\quad -\epsilon^2 \left(\frac{5}{6}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}\mu^{5/2} \left[(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mu^{1/2}\mu' - \frac{5}{4}\mathcal{H}\mu^{-2}(\mu')^2\right]\mathbf{a}^0 + o(\epsilon^2), \\
(8.22)\quad \mathbf{J}_2 &= \epsilon^2 \left(\frac{5}{4}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}\mu^{5/2} \left[(\nabla\xi)\mu^{-1}\mu' + \frac{5}{6}\mathcal{H}\mu^{-2}(\mu')^2\right]\mathbf{a}^0 + o(\epsilon^2), \\
(8.23)\quad \mathbf{J}_3 &= -\epsilon^2 \left(\frac{5}{12}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}^2\mu^{3/2}\mu''\mathbf{a}^0 + o(\epsilon^2), \\
(8.24)\quad \mathbf{J}_5 &= -\epsilon^2 \left(\frac{5}{6}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}\mu^{5/2} \int_{\mathbb{R}} w^2 \left[(\nabla\xi)\mu^{-1}\mu' + \frac{3}{4}\mathcal{H}\mu^{-2}(\mu')^2\right]\mathbf{a}^0 + o(\epsilon^2), \\
(8.25)\quad \mathbf{J}_6 &= -\epsilon^2 \left(\frac{5}{6}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}\mu^{5/2} \left[2\mathcal{G}_D\mathcal{H}\mu^{3/2}(\nabla\xi)\mu^{-1}\mu' + \frac{3}{2}\mathcal{G}_D\mathcal{H}\mu^{-1/2}(\mu')^2\right]\mathbf{a}^0 + o(\epsilon^2), \\
(8.26)\quad \mathbf{J}_7 &= -\epsilon^2 \left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right) \mathcal{H}\mu^{5/2} \left[2(\nabla\mathcal{G}_D)\mathcal{H}(\nabla\xi)\mu^{3/2} + \frac{3}{2}(\nabla\mathcal{G}_D)\mathcal{H}^2\mu^{1/2}\mu'\right]\mathbf{a}^0 + o(\epsilon^2),
\end{aligned}$$

where we recall that $\mathcal{G}_D$ are $\mathcal{H}$ are introduced in (2.5) and (2.12), respectively, $\mathbf{a}_\epsilon$ is given in (8.15)and

$$\mathbf{a}^0 = \lim_{\epsilon\to 0} \mathbf{a}^\epsilon. \tag{8.27}$$

By Lemma 8.1, Theorem 2.3 can be proved. Indeed, using the identity

$$(\nabla\mathcal{G}_D)^T\mathcal{H}^2\mu^{1/2}\mu' - (\nabla\xi)\mu^{-1}\mu' - \frac{5}{4}\mathcal{H}\mu^{-2}(\mu')^2 + 2\mathcal{G}_D\mathcal{H}\mu^{3/2}(\nabla\xi)\mu^{-1}\mu' + \frac{3}{2}\mathcal{G}_D\mathcal{H}\mu^{-1/2}(\mu')^2 = 0$$

which follows from (2.29), we have

$$\begin{aligned}\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_6 &= -\epsilon^2\left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mathcal{H}\mu^{5/2}\left[(\nabla^2\mathcal{G}_D)\mathcal{H}^2\mu^{3/2}-\mathcal{Q}\mathcal{H}^2\mu^{3/2}\right]\mathbf{a}^0\\ &\quad +\epsilon^2\frac{5}{4}\left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mathcal{H}\mu^{5/2}\left[(\nabla\xi)\mathcal{H}\mu^{-1}\mu'+\frac{5}{2}\mathcal{H}\mu^{-2}(\mu')^2\right]\mathbf{a}^0+o(\epsilon^2).\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_6+\mathbf{J}_5 &= -\epsilon^2\left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mathcal{H}\mu^{5/2}\left[(\nabla^2\mathcal{G}_D)\mathcal{H}^2\mu^{3/2}-\mathcal{Q}\mathcal{H}^2\mu^{3/2}\right]\mathbf{a}^0\\ &\quad -\epsilon^2\frac{5}{4}\left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mathcal{H}\mu^{5/2}\left[(\nabla\xi)\mathcal{H}\mu^{-1}\mu'-\mathcal{H}\mu^{-2}(\mu')^2\right]\mathbf{a}^0+o(\epsilon^2).\end{aligned}$$

Combining the above estimate with those of $\mathbf{J}_3$ and $\mathbf{J}_7$, and using (2.3), we have

$$\begin{aligned}\text{l.h.s.} &= \mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3+\mathbf{J}_5+\mathbf{J}_6+\mathbf{J}_7\\ &= -\epsilon^2\left(\frac{1}{3}\int_{\mathbb{R}} w^3\,dy\right)\mathcal{H}\mu^{5/2}\Bigg[((\nabla^2\mathcal{G}_D)-\mathcal{Q})\mathcal{H}^2\mu^{3/2}+2(\nabla\mathcal{G}_D)\mathcal{H}(\nabla\xi)\mu^{3/2}\\ &\quad +\frac{5}{4}(\nabla\hat{\xi})\mu^{-1}\mu'+\frac{5}{4}\mathcal{H}\left[\mu^{-1}\mu''-\mu^{-2}(\mu')^2\right]+\frac{3}{2}(\nabla\mathcal{G}_D)\mathcal{H}^2\mu^{1/2}\mu'\Bigg]+o(\epsilon^2).\end{aligned}$$

Comparing with r.h.s. and recalling the computation of $\mathcal{M}(\mathbf{t}^0)$ at (2.28), we obtain

$$-2.4\epsilon^2\mathcal{H}\mu^{5/2}\mathcal{M}(\mathbf{t}^\epsilon)\mathbf{a}_\epsilon+o(\epsilon^2)=\lambda_\epsilon\mu^{5/2}\mathcal{H}^2\mathbf{a}_\epsilon\int_{\mathbb{R}}(w'(y))^2\,dy\,(1+O(\epsilon)), \tag{8.28}$$

using (1.7). Equation (8.28) shows that the small eigenvalues λ_ϵ of (8.2) are

$$\lambda_\epsilon\sim -2\epsilon^2\sigma\left(\mathcal{H}^{-1}\mathcal{M}(\mathbf{t}^0)\right)$$

by (1.7).

Arguing as in Theorem 7.1, this shows that if all the eigenvalues of $\mathcal{M}(\mathbf{t}^0)$ are positive, then the small eigenvalues are stable. On the other hand, if $\mathcal{M}(\mathbf{t}^0)$ has a negative eigenvalue, then we can construct eigenfunctions and eigenvalues to make the system unstable.

This proves Theorem 2.3.

□

Lemma 8.1 follows from the following series of lemmas.

We first study the asymptotic behavior of $\psi_{\epsilon,j}$.

Lemma 8.2. *We have*

$$((\psi_{\epsilon,k}-\bar{H}_\epsilon'\delta_{kl})(t_l^\epsilon)) = -\mathcal{H}^2\mu^{3/2}(\nabla\mathcal{G}_D)+\frac{5}{4}\mathcal{H}\mu^{-1}\mu'+O(\epsilon). \tag{8.29}$$

Proof:

Note that for $l \neq k$, we have

$$\begin{aligned}(\psi_{\epsilon,k} - \bar{H}_\epsilon^{'}\delta_{kl})(t_l^\epsilon) &= \psi_{\epsilon,k}(t_l^\epsilon) \\ &= 2\xi_\epsilon \int_{-1}^1 G_D(t_l^\epsilon, z)\bar{w}_\epsilon \tilde{w}_{\epsilon,k}^{'}\, dz \\ &= -\nabla_{t_k^\epsilon} G_D(t_k^\epsilon, t_l^\epsilon)\hat{\xi}_k^2 \mu_k^{3/2}. \end{aligned} \tag{8.30}$$

Next we compute $\psi_{\epsilon,l} - \bar{H}_\epsilon^{'}$ near t_l^ϵ:

$$\begin{aligned}\bar{H}_\epsilon(x) &= \xi_\epsilon \int_{-1}^1 G_D(x,z)\bar{w}_\epsilon^2\, dz \\ &= \xi_\epsilon \int_{-\infty}^{+\infty} K_D(|z|)\tilde{w}_{\epsilon,l}^2(x+z)dz - \xi_\epsilon \int_{-1}^1 H_D(x,z)\tilde{w}_{\epsilon,l}^2\, dz + \xi_\epsilon \sum_{k\neq l}\int_{-1}^1 G_D(x,z)\tilde{w}_{\epsilon,k}^2\, dz.\end{aligned}$$

So

$$\begin{aligned}\bar{H}_\epsilon^{'}(x) &= \xi_\epsilon \int_{-\infty}^{+\infty} K_D(|z|)(2\tilde{w}_{\epsilon,l}(x+z)\tilde{w}_{\epsilon,l}^{'}(x+z))\, dz - \xi_\epsilon \int_{-1}^1 \nabla_x H_D(x,z)\tilde{w}_{\epsilon,l}^2\, dz \\ &\quad + \xi_\epsilon \sum_{k\neq l}\int_{-1}^1 \nabla_x G_D(x,z)\tilde{w}_{\epsilon,k}^2\, dz.\end{aligned}$$

Thus

$$\begin{aligned}\bar{H}_\epsilon^{'} - \psi_{\epsilon,l} &= -\xi_\epsilon \int_{-1}^1 \nabla_x H_D(x,z)\tilde{w}_{\epsilon,l}^2\, dz + \xi_\epsilon \sum_{k\neq l}\int_{-1}^1 \nabla_x G_D(x,z)\tilde{w}_{\epsilon,k}^2\, dz \\ &\quad - \left(-2\xi_\epsilon \int_{-1}^1 H_D(x,z)\tilde{w}_{\epsilon,l}\tilde{w}_{\epsilon,l}^{'}\, dz\right). \end{aligned} \tag{8.31}$$

Therefore we have,

$$\begin{aligned}\bar{H}_\epsilon^{'}(t_l^\epsilon) - \psi_{\epsilon,l}(t_l^\epsilon) &= -\xi_\epsilon \int_{-1}^1 \nabla_{t_l^\epsilon} H(t_l^\epsilon, z)\tilde{w}_{\epsilon,l}^2 + \xi_\epsilon \sum_{k\neq l}\int_{-1}^1 \nabla_{t_l^\epsilon} G(t_l^\epsilon, z)\tilde{w}_{\epsilon,k}^2 \\ &\quad -\nabla_{t_l^\epsilon} H_D(t_l^\epsilon, t_l^\epsilon)\hat{\xi}_l^2\mu_l^{3/2} + O(\epsilon) \\ &= \sum_{k=1}^N \nabla_{t_l^\epsilon} G_D(t_l^\epsilon, t_k^\epsilon)\hat{\xi}_k^2 \mu_k^{3/2} - \nabla_{t_l^\epsilon} H_D(t_l^\epsilon, t_l^\epsilon)\hat{\xi}_l^2\mu_l^{3/2} + O(\epsilon). \\ &= -\frac{5}{4}\hat{\xi}_l \mu_l^{-1}\mu_l^{'} - \nabla_{t_l^\epsilon} H_D(t_l^\epsilon, t_l^\epsilon)\hat{\xi}_l^2\mu_l^{3/2} + O(\epsilon). \end{aligned} \tag{8.32}$$

Combining (8.30) and (8.32), we have (8.29).

□

Similar to the proof of Lemma 8.2, the following result is derived.

Lemma 8.3. *We have*

$$\begin{aligned}&(\psi_{\epsilon,k} - \bar{H}_\epsilon^{'}\delta_{lk})(t_l^\epsilon + \epsilon y) - (\psi_{\epsilon,k} - \bar{H}_\epsilon^{'}\delta_{lk})(t_l^\epsilon) \\ &= -\epsilon y \nabla_{t_l^\epsilon}\nabla_{t_k^\epsilon} G_D(t_l^\epsilon, t_k^\epsilon)\hat{\xi}_k^2\mu_k^{3/2} + O(\epsilon^2 y^2), \quad \text{for } l \neq k \end{aligned} \tag{8.33}$$

and

$$(\psi_{\epsilon,k} - \bar{H}_\epsilon^{'}\delta_{lk})(t_l^\epsilon + \epsilon y) - (\psi_{\epsilon,k} - \bar{H}_\epsilon^{'}\delta_{lk})(t_l^\epsilon) \tag{8.34}$$

$$= -\epsilon y \sum_{m=1}^{N} \nabla^2_{t_l^\epsilon} G_D(t_l^\epsilon, t_m^\epsilon)\hat{\xi}_m^2 \mu_m^{3/2} + O(\epsilon^2 y^2), \quad for\ l = k.$$

Next we study the asymptotic expansion of $\phi_\epsilon^\perp$. Let us first denote

$$\begin{aligned} \phi^1_{\epsilon,j}(x) &= -\sum_{l=1}^{N} \left((\nabla_{t_j^\epsilon}\hat{\xi}_l)\tilde{w}_{\epsilon,l}(x) \right) - \hat{\xi}_j \left(\mu_j' w_j(\sqrt{\mu_j}x) + \frac{1}{2}\mu_j^{1/2}\mu_j' x w_j'(\sqrt{\mu_j}x) \right) \\ &= -\sum_{l=1}^{N} \left((\nabla_{t_j^\epsilon}\hat{\xi}_l)\tilde{w}_{\epsilon,l}(x) \right) - \hat{\xi}_j \mu_j^{-1}\mu_j' \left(\tilde{w}_{\epsilon,j}(x) + \frac{1}{2} x \tilde{w}_{\epsilon,j}'(x) \right), \quad \phi_\epsilon^1 := \epsilon \sum_{j=1}^{N} a_j^\epsilon \phi^1_{\epsilon,j}. \end{aligned} \tag{8.35}$$

Now we derive

Lemma 8.4. *For ϵ sufficiently small, we have*

$$\|\phi_\epsilon^\perp - \epsilon\phi_\epsilon^1\|_{H^2(-1/\epsilon,1/\epsilon)} = O(\epsilon^2). \tag{8.36}$$

Proof: As the first step in the proof of Lemma 8.4, we obtain a relation between $\psi_\epsilon^\perp$ and $\phi_\epsilon^\perp$. Note that similar to the proof of Proposition 5.1, $\tilde{L}_\epsilon$ is invertible from $(\mathcal{K}_\epsilon^{new})^\perp$ to $(\mathcal{C}_\epsilon^{new})^\perp$ with uniformly bounded inverse for ϵ small enough. By (8.12), (8.13), Lemma 8.2 and the fact that $\tilde{L}_\epsilon$ is uniformly invertible, we deduce that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = O(\epsilon). \tag{8.37}$$

Let us decompose

$$\tilde{\phi}_{\epsilon,j} = \frac{\phi_\epsilon^\perp}{\epsilon}\chi\left(\frac{x - t_j^\epsilon}{r_0}\right). \tag{8.38}$$

Then

$$\phi_\epsilon^\perp = \epsilon \sum_{j=1}^{N} \tilde{\phi}_{\epsilon,j} + O(\epsilon^2)$$

Suppose that

$$\tilde{\phi}_{\epsilon,j} \to \hat{\phi}_j \ \text{ in } H^1(\Omega_\epsilon). \tag{8.39}$$

Let us also define

$$\hat{\phi}_j(y) = \mu_j \phi_j(\sqrt{\mu_j} y).$$

Set

$$\Phi_0 = (\phi_1, ..., \phi_N)^T.$$

Then we have by the equation for $\psi_\epsilon^\perp$:

$$\begin{aligned} \psi_\epsilon^\perp(t_j^\epsilon) &= 2\epsilon \sum_{k=1}^{N} \xi_\epsilon \int_{-1}^{1} G_D(t_j^\epsilon, z)\bar{w}_\epsilon \tilde{\phi}_{\epsilon,k}\, dz \\ &= 2\epsilon \sum_{k=1}^{N} G_D(t_j^\epsilon, t_k^\epsilon)\hat{\xi}_k \mu_k^{3/2} \frac{\int_{\mathbb{R}} w\phi_k\, dy}{\int_{\mathbb{R}} w^2\, dy} + O(\epsilon^2). \end{aligned}$$

Hence

$$(\psi_\epsilon^\perp(t_1^\epsilon), ..., \psi_\epsilon^\perp(t_N^\epsilon))^T = 2\epsilon \mathcal{G}_D \mathcal{H} \mu^{3/2} \frac{\int_{\mathbb{R}} w \Phi_0 \, dy}{\int_{\mathbb{R}} w^2 \, dy} + O(\epsilon^2). \tag{8.40}$$

This relation between $\psi_\epsilon^\perp$ and Φ_0 will be important for the rest of the proof.

Now we substitute (8.40) into (8.12) and using Lemma 8.2, we have that the limit Φ_0 satisfies

$$\nabla^2 \Phi_0 - \Phi_0 + 2w\Phi_0 - 2\mathcal{G}_D \mathcal{H} \mu^{3/2} \frac{\int_{\mathbb{R}} w\Phi_0}{\int_{\mathbb{R}} w^2} w^2$$

$$+(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mu^{3/2} \mathbf{a}^0 w^2 - \frac{5}{4} \mathcal{H} \mu^{-1} \mu' \mathbf{a}^0 w^2 + \mathcal{H} \mu^{-1} \mu' \mathbf{a}^0 w = 0.$$

Hence, using the relations

$$L_0^{-1} w^2 = w, \quad L_0^{-1} w = \frac{1}{2} y w' + w,$$

by (2.23), (2.24) we have

$$\begin{aligned} \Phi_0 &= -\mathcal{P}\left[(\nabla \mathcal{G}_D)^T \mathcal{H}^2 \mu^{3/2} \mathbf{a}^0 - \frac{5}{4} \mathcal{H} \mu^{-1} \mu' \mathbf{a}^0 + \frac{3}{2} \mathcal{G}_D \mathcal{H}^2 \mu^{1/2} \mu' \mathbf{a}^0 \right] w - \mathcal{H} \mu^{-1} \mu' \mathbf{a}^0 L_0^{-1} w \\ &= -(\nabla \xi) \mathbf{a}^0 w - \mathcal{H} \mu^{-1} \mu' \mathbf{a}^0 \left(w + \frac{1}{2} y w' \right). \end{aligned} \tag{8.41}$$

Now we compare Φ_0 with ϕ_ϵ^1. By definition

$$\begin{aligned} \phi_\epsilon^1 &= -\epsilon \sum_{l=1}^N a_l^\epsilon \sum_{j=1}^N \left((\nabla_{t_l^\epsilon} \hat{\xi}_j) \tilde{w}_{\epsilon,j} \right) - \epsilon \sum_{j=1}^N a_j^\epsilon \hat{\xi}_j \mu_j^{-1} \mu_j' \left(\tilde{w}_{\epsilon,j}(x) + \frac{1}{2} x \tilde{w}_{\epsilon,j}'(x) \right) \\ &= -\epsilon \sum_{j=1}^N \left[\sum_{l=1}^N \left(\nabla_{t_l^\epsilon} \hat{\xi}_j \right) a_l^\epsilon \right] \tilde{w}_{\epsilon,j} - \epsilon \sum_{j=1}^N a_j^\epsilon \hat{\xi}_j \mu_j^{-1} \mu_j' \left(\tilde{w}_{\epsilon,j}(x) + \frac{1}{2} x \tilde{w}_{\epsilon,j}'(x) \right). \end{aligned} \tag{8.42}$$

On the other hand

$$\phi_\epsilon^\perp = \epsilon \sum_{j=1}^N \tilde{\phi}_{\epsilon,j} + O(\epsilon^2) = \epsilon \sum_{j=1}^N \phi_j \left(\frac{x - t_j^\epsilon}{\epsilon} \right) + O(\epsilon^2). \tag{8.43}$$

Using (8.41) and comparing (8.42) with (8.43), we obtain (8.36).

□

From Lemma 8.4 and (8.40), we have that

$$(\psi_\epsilon^\perp(t_1^\epsilon), ..., \psi_\epsilon^\perp(t_N^\epsilon))^T = -2\epsilon \mathcal{G}_D \mathcal{H} \mu^{3/2} \left[\nabla \xi + \frac{3}{4} \mathcal{H} \mu^{-1} \mu' \right] \mathbf{a}^0 + O(\epsilon^2). \tag{8.44}$$

Further,

$$\psi_\epsilon^\perp(t_j^\epsilon + \epsilon y) - \psi_\epsilon^\perp(t_j^\epsilon) = 2\epsilon^2 y \sum_{k=1}^N \nabla_{t_j^\epsilon} G_D(t_j^\epsilon, t_k^\epsilon) \hat{\xi}_k \mu_k^{3/2} \frac{\int_{\mathbb{R}} w \phi_k \, dy}{\int_{\mathbb{R}} w^2 \, dy} + O(\epsilon^3) \tag{8.45}$$

$$(\psi_\epsilon^\perp(t_1^\epsilon + \epsilon y) - \psi_\epsilon^\perp(t_1^\epsilon), ..., \psi_\epsilon^\perp(t_N^\epsilon + \epsilon y) - \psi_\epsilon^\perp(t_N^\epsilon))^T = -2\epsilon^2 (\nabla \mathcal{G}_D) \mathcal{H} \mu^{3/2} \left[\nabla \xi + \frac{3}{4} \mathcal{H} \mu^{-1} \mu' \right] \mathbf{a}^0 + O(\epsilon^3).$$

Finally we prove the key lemma – Lemma 8.1.

Proof of Lemma 8.1: The computation of J_1 follows from Lemmas 8.2 and 8.3: In fact,

$$\begin{aligned} J_{1,l} &= -\epsilon \sum_{k=1}^{N} a_k^\epsilon \int_{-1}^{1} \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2} \left(\psi_{\epsilon,k} - \bar{H}_\epsilon^{'} \delta_{lk}\right) \tilde{w}_{\epsilon,l}^{'}\, dx + o(\epsilon^2) \\ &= -\epsilon \sum_{k=1}^{N} a_k^\epsilon \int_{-1}^{1} \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2} \left(\psi_{\epsilon,k}(t_l) - \bar{H}_\epsilon^{'}(t_l) \delta_{lk}\right) \tilde{w}_{\epsilon,l}^{'}\, dx + o(\epsilon^2) \\ &\quad -\epsilon \sum_{k=1}^{N} a_k^\epsilon \int_{-1}^{1} \frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2} \left([\psi_{\epsilon,k}(x) - \bar{H}_\epsilon^{'}(x)\delta_{lk}] - [\psi_{\epsilon,k}(t_l) - \bar{H}_\epsilon^{'}(t_l)\delta_{lk}]\right) \tilde{w}_{\epsilon,l}^{'}\, dx + o(\epsilon^2) \\ &= J_{8,l} + J_{9,l}. \end{aligned}$$

For $J_{8,l}$, we use Lemma 8.2 to obtain

$$J_{8,l} = -\frac{2}{3}\epsilon \sum_{k=1}^{N} a_k^\epsilon \int_{-1}^{1} \frac{\tilde{w}_{\epsilon,l}^3}{\bar{H}_\epsilon^3} \bar{H}_\epsilon^{'} \left(\psi_{\epsilon,k}(t_l) - \bar{H}_\epsilon^{'}(t_l)\delta_{lk}\right)\, dx + o(\epsilon^2) \tag{8.46}$$

$$= -\frac{2}{3}\epsilon^2 \sum_{k=1}^{N} a_k^\epsilon \left(\int_{\mathbb{R}} w_l^3\, dy\right) \bar{H}_\epsilon^{'}(t_l^\epsilon) \left(\psi_{\epsilon,k}(t_l) - \bar{H}_\epsilon^{'}(t_l)\delta_{lk}\right) + o(\epsilon^2). \tag{8.47}$$

Similarly,

$$J_{9,l} = \epsilon^2 \hat{\xi}_l \int_{\mathbb{R}} \left(y w_l^2 w_l^{'}(y)\right)\, dy \sum_{k=1}^{N} \nabla_{t_l^\epsilon} \nabla_{t_k^\epsilon} G_D(t_l^\epsilon, t_k^\epsilon) \hat{\xi}_k^2 \mu_k^{3/2} a_k^\epsilon + o(\epsilon^2)$$

$$= -\epsilon^2 \left(\frac{1}{3}\int_{\mathbb{R}} w^3\, dy\right) \mu_l^{5/2} \hat{\xi}_l \sum_{k=1}^{N} (\nabla_{t_l^\epsilon} \nabla_{t_k^\epsilon} G_D(t_l^\epsilon, t_k^\epsilon) - q_{lk}\delta_{lk}) \hat{\xi}_k^2 \mu_k^{3/2} a_k^\epsilon + o(\epsilon^2). \tag{8.48}$$

Combining (8.47) and (8.48), and using (8.6), Lemma 8.2, Lemma 8.3, (2.26) and (2.27), we obtain (8.21).

For $J_{2,l}$, we have by Lemma 8.4

$$J_{2,l} = \epsilon \bar{H}_\epsilon^{'}(t_l) \int_{-1}^{1} w_l^2 \phi_\epsilon^1\, dx + o(\epsilon^2),$$

where

$$\bar{H}_\epsilon^{'}(t_l) = \sum_{k=1}^{N} \nabla_{t_l} \mathcal{G}_D(t_l, t_k) \hat{\xi}_k^2 \mu_k^{3/2} = -\frac{5}{4} \hat{\xi}_l \mu_l^{-1} \mu_l^{'} + O(\epsilon) \tag{8.49}$$

by (8.6), and

$$\int_{-1}^{1} w_l^2 \phi_\epsilon^1\, dx = -\epsilon \mu_l^{5/2} \left(\int_{\mathbb{R}} w^3\, dy\right) \left[\sum_{j=1}^{N} (\nabla_{t_j^\epsilon} \hat{\xi}_l) a_j^\epsilon + \frac{5}{6} \hat{\xi}_l \mu_l^{-1} \mu_l^{'}\right] + O(\epsilon^2)$$

by (8.41), (8.43), using

$$\int_{\mathbb{R}} w^2 (L_0^{-1} w)\, dy = \int_{\mathbb{R}} \left(w^3 + \frac{1}{2} y w^2 w^{'}\right)\, dy = \frac{5}{6} \int_{\mathbb{R}} w^3\, dy,$$

which proves (8.22).

For (8.25), we have

$$
\begin{aligned}
J_{6,l} &= -\psi_\epsilon^\perp(t_l^\epsilon)\int_{-1}^{1}\frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2}\tilde{w}_{\epsilon,l}'\,dx \\
&= -\frac{2}{3}\psi_\epsilon^\perp(t_l^\epsilon)\int_{-1}^{1}\frac{\tilde{w}_{\epsilon,l}^3}{\bar{H}_\epsilon^3}\bar{H}_\epsilon'\,dx + o(\epsilon^2) \\
&= -\epsilon\frac{2}{3}\bar{H}_\epsilon'(t_l^\epsilon)\psi_\epsilon^\perp(t_l^\epsilon)\mu_l^{5/2}\left(\int_{\mathbb{R}} w^3\,dy\right) + o(\epsilon^2).
\end{aligned}
\tag{8.50}
$$

Then (8.25) follows from (8.6) and (8.44).

For J_7 we have

$$
J_7 = -\int_{-1}^{1}\frac{\tilde{w}_{\epsilon,l}^2}{\bar{H}_\epsilon^2}(\psi_\epsilon^\perp(x)-\psi_\epsilon^\perp(t_l^\epsilon))w_{\epsilon,l}'\,dx + o(\epsilon^2).
\tag{8.51}
$$

Now (8.26) follows from (8.41), (8.44) and (8.46).

(8.24) follows from Lemma 8.4.

□

9. Numerical Simulations

We now show some numerical simulations for effects of precursors in the behavior of system (1.2). We choose $\Omega = (-1, 1)$, $\tau = 0.1$ and varying diffusion constants (first $\epsilon^2 = 0.001$, $D = 0.1$ and second $\epsilon^2 = 0.0001$, $D = 0.01$).

In each situation we always present the final state (for $t = 10^5$) which in all cases is numerically stable (long-time limit). Always A is shown on the left, H on the right.

We first consider the system without precursor $\mu(x) \equiv 1$, $\epsilon^2 = 0.001$, $D = 0.1$.

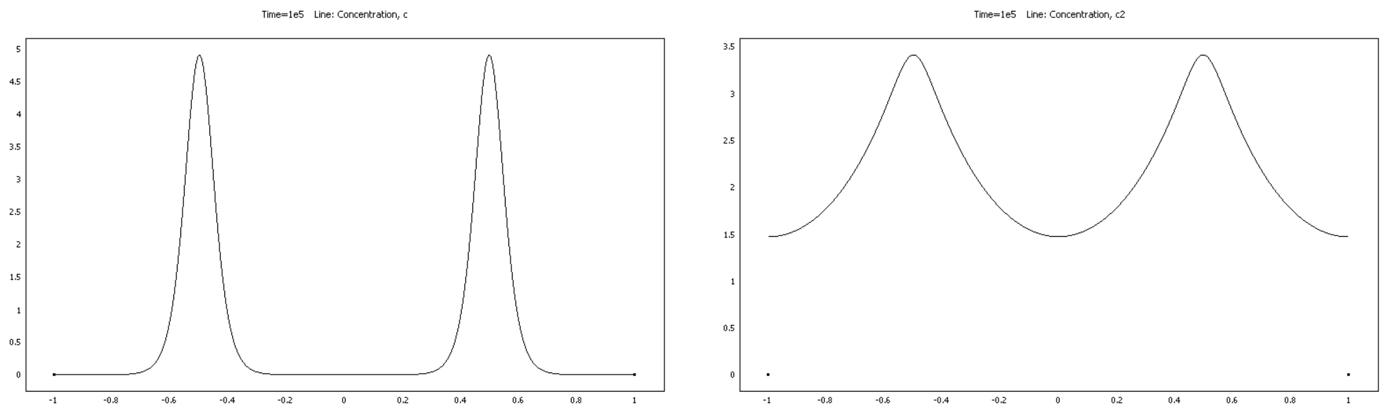


Figure 1. Two Spikes for (1.2) with $\epsilon^2 = 0.001$, $D = 0.1$, $\mu \equiv 1$ (i.e. no precursor). The two spikes are symmetric: They have the same amplitude and the spacing is regular.

Choosing a precursor with linear gradient we have the following picture.

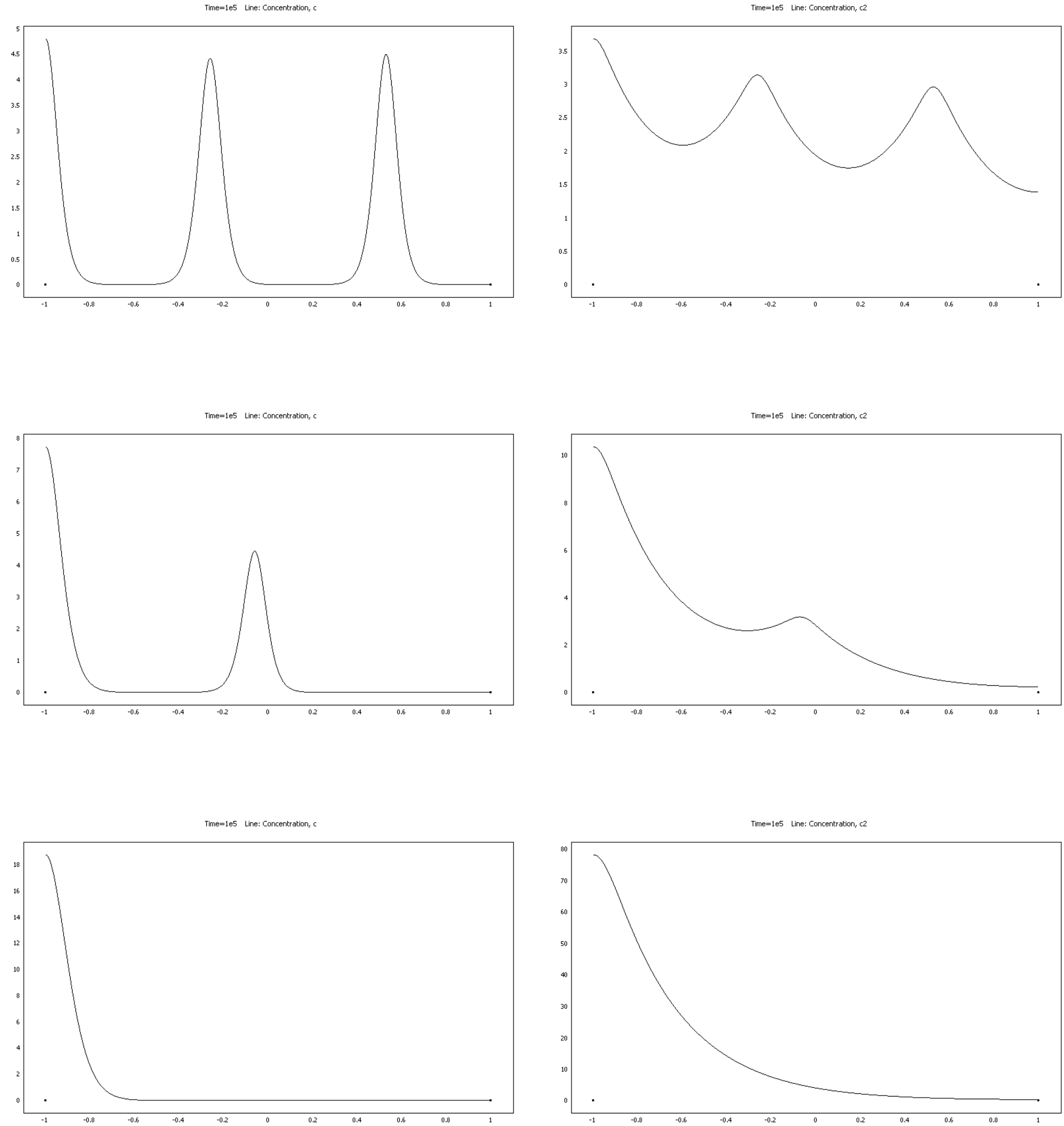


Figure 2. Multiple spikes in the precursor case for (1.2) with $\epsilon^2 = 0.001$, $D = 0.1$ and $\mu = 1 + 0.1x$ (top row), $\mu = 1+0.5x$ (middle row), $\mu = 1+0.9x$ (bottom row). Note that the number of spikes changes depending on the strength of the precursor: As the precursor becomes more pronounced the number of spikes decreases

and they move closer to the left side of the interval where the precursor is smaller. The spikes are asymmetric: They have different amplitudes and the spacing is irregular.

Choosing a precursor with general gradient we have the following picture. The profile has the shape of a cosine function which in leading order gives a quadratic profile near the maxima and minima. Again the spikes move to values of smaller μ.

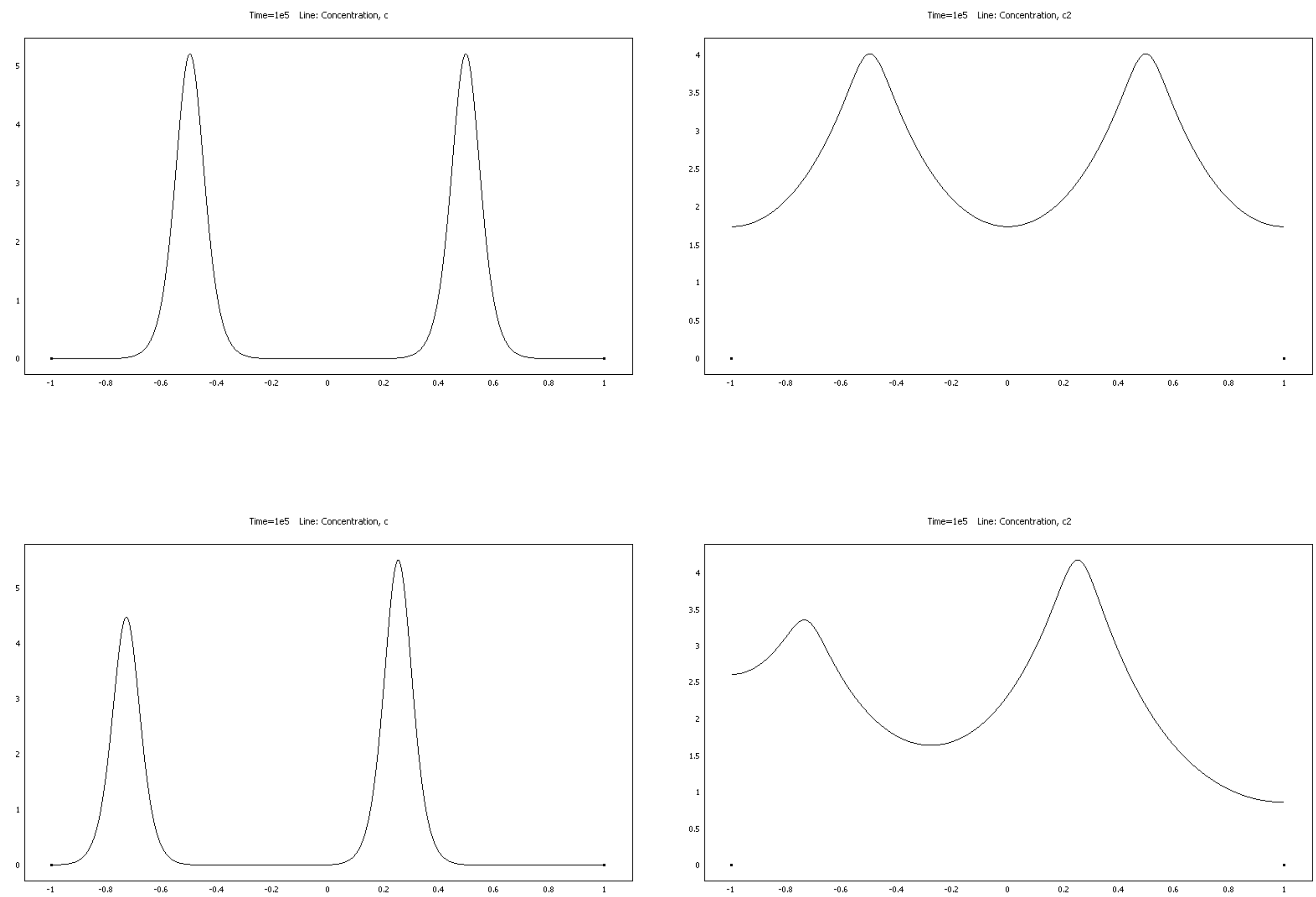


Figure 3. We simulate (1.2) with $\epsilon^2 = 0.001$, $D = 0.1$ and $\mu(x) = 1 + 0.1\cos(2\pi x)$ (top row) or $\mu(x) = 1 + 0.1\cos(4\pi x)$ (bottom row). In the first case we have two spikes pushed away from the middle (compared to Figure 1), in the second case we have an asymmetric pattern of spikes with two different amplitudes.

Finally we do the simulations again with smaller diffusion constants which results in a higher number of spikes. For the rest of the figures we choose $\epsilon^2 = 0.0001$, $D = 0.01$.

We first consider the system without precursor $\mu(x) \equiv 1$, $\epsilon^2 = 0.0001$, $D = 0.01$.

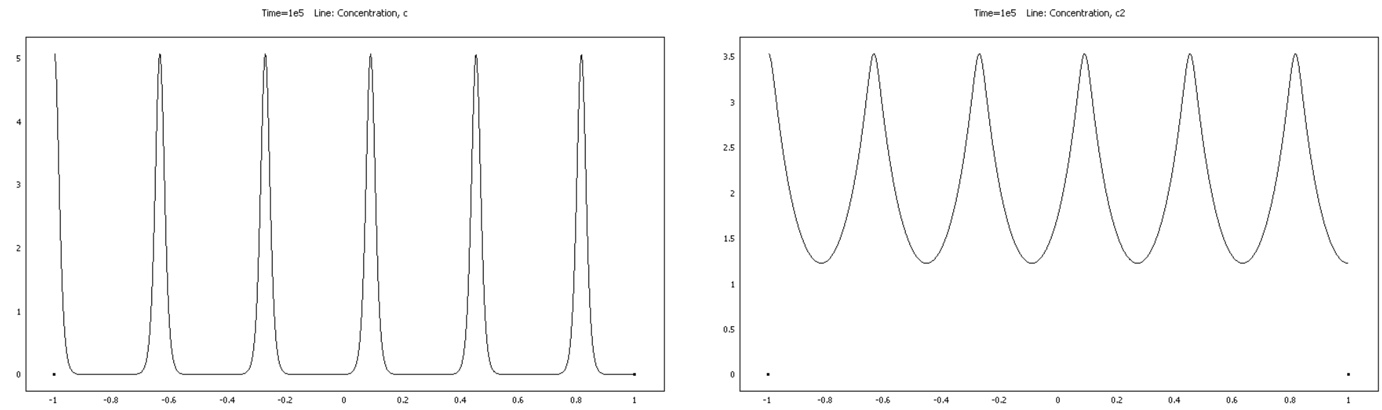
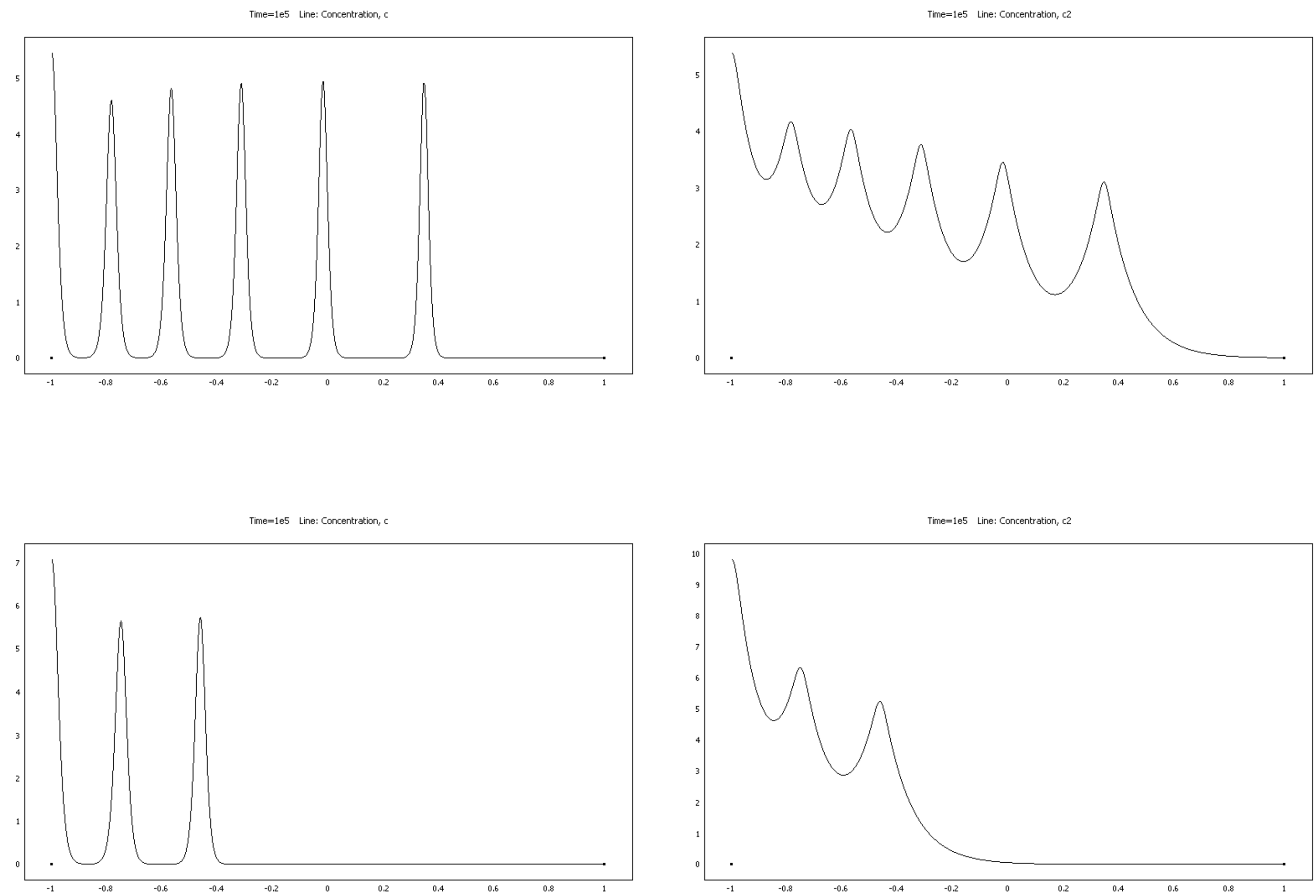


Figure 4. Two spikes for (1.2) with $\epsilon^2 = 0.0001$, $D = 0.01$, $\mu \equiv 1$ (i.e. no precursor). The spikes are symmetric: They have the same amplitude and the spacing is regular.

Choosing a precursor with linear gradient we have the following picture.



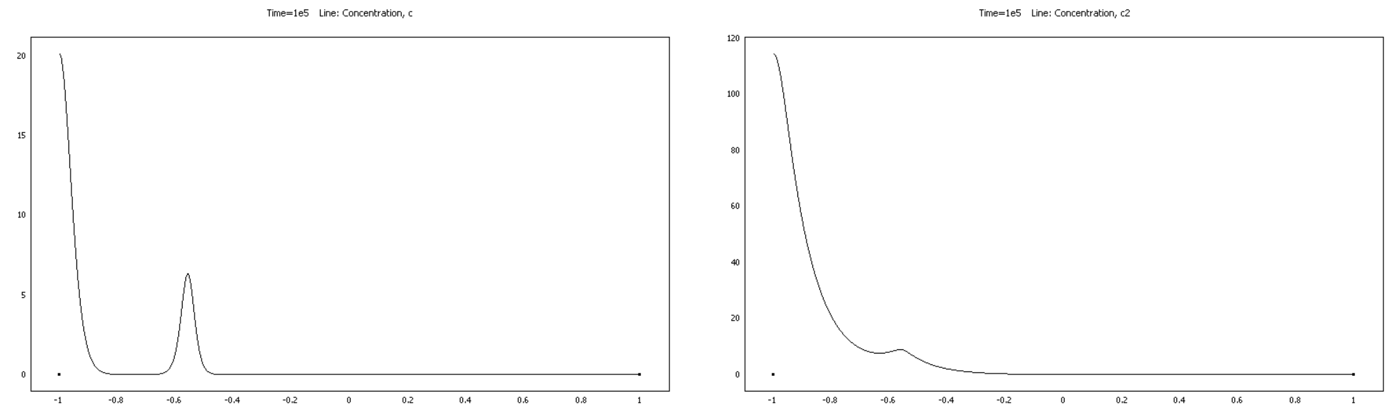
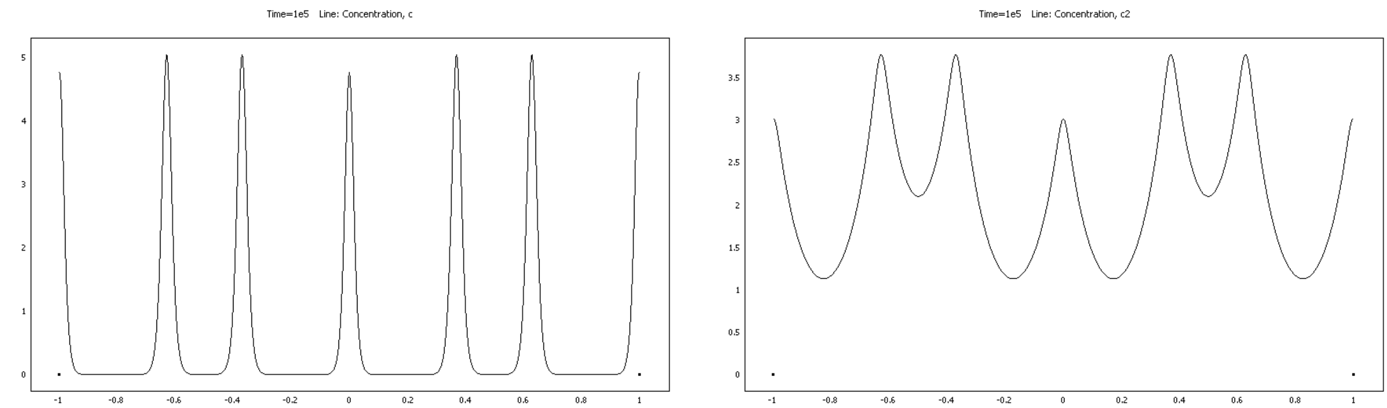


Figure 5. Multiple spikes in the precursor case for (1.2) with $\epsilon^2 = 0.0001$, $D = 0.01$ and $\mu = 1 + 0.1x$ (top row), $\mu = 1 + 0.5x$ (middle row), $\mu = 1 + 0.9x$ (bottom row). Note that the number of spikes changes depending on the strength of the precursor: As the precursor becomes more pronounced the number of spikes decreases and they move closer to the left side of the interval where the precursor is smaller. The spikes are asymmetric: They have different amplitudes and the spacing is irregular.

Choosing a precursor with general gradient we have the following picture. The profile has the shape of a cosine function which in leading order gives a quadratic profile. Again the spikes move to values of smaller μ.



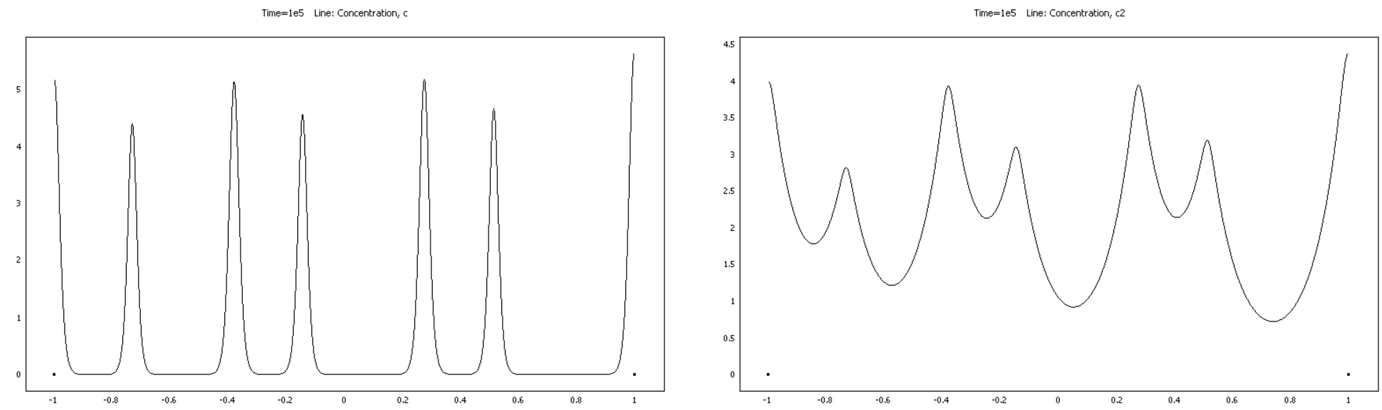


Figure 6. We simulate (1.2) with $\epsilon^2 = 0.0001$, $D = 0.01$ and $\mu(x) = 1 + 0.1\cos(2\pi x)$ (top row) or $\mu(x) = 1 + 0.1\cos(3\pi x)$ (bottom row). In both cases we have an asymmetric pattern of spikes with two different amplitudes.

The effects of precursors on spiky solutions explored in this paper such as asymmetric positions or amplitudes of spikes or movement of spikes to positions with small precursor values play an important role in a variety of biological models such as animal skin patterns, formation of head structure in hydra, segmentation in *Drosophila melanogaster* or ecology. We plan to shed more light on these issues in the future, in particular in the higher-dimensional case, combining analysis with simulation and applying the outcomes to biological observations and experiments.

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