Asymptotics of a right-angled impedance wedge.

Anthony D Rawlins
Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex, UB8 3PH. U K

Abstract. We shall consider the asymptotics of the wave field scattered by a right angled impedance wedge. By using the the exact complex Sommerfeld type integral solution to the problem of the diffraction of a plane wave by a right angled impedance wedge a new representation for the field is given. This new form is in a form suitable for the asymptotics behaviour to be derived across the specular and shadow boundaries where the usual diffraction coefficient becomes infinite.

Keywords: Diffraction, Impedance wedge, Wedge asymptotics, Absorbing wedge. LaTeX

1. Introduction

In dealing with mobile phone propagation in cities the effect of building corners and their surface cladding is of paramount importance for the signal strength of the phones. One of these ubiquitous corners can be effectively modelled for high frequency propagation by a right angled impedance wedge in two dimensions. To obtain qualitative results for the signal strength when there are multiple diffraction from a number of such corners an effective approach is to use the Keller method of geometrical diffraction. This method requires information about the "diffraction coefficient" which are obtained from the solution of canonical wedge problems. These coefficients need to be uniformly valid in the angular variables in order that the method can be used successfully when considering multiple diffractions at different corners. This work goes some way to addresses this problem.

The first exact solution to the problem of the diffraction of a plane wave by an arbitrary angled impedance wedge was obtained in terms of [illegible] Sommerfeld integrals by (Malyuzhinets, 1958) and later independently by (Williams, 1959). The Sommerfeld integral representation of the field required the solution of complicated difference equations. The two different approaches of Malyuzhinets and Williams revolved about the method of solution of the difference equations. Malyuzhinets introduced a new transcendental function defined by a specific difference equation; Williams used the extant results on the Double Gamma function worked out, in his youth, by the later to be Anglican Bishop of Birmingham, E W Barnes. These research works were a tribute to the ingenuity of these mathematicians. A later paper by (Lebedev, 1963) used the Kontorovich Lebedev transform to derive another equivalent solution. This also involved solving different, but complicated difference equations. A much simpler method to obtain the exact solution to the specific problem of the diffraction of a plane wave by a right angled impedance wedge was later obtained by, (Rawlins, 1990). This method avoided solving the complicated difference equations that arise in the previous methods. We shall use this later result to obtain useful asymptotic results for the field across singular ray directions where the usual diffraction coefficient, used in high frequency methods like Kellers's theory of geometric diffraction, break down. This is because poles and saddle points of the Sommerfeld integrand coalesce. Here, to be specific, we shall consider an electromagnetic E-polarized plane wave that propagates towards and is diffracted by an impedance wedge. We should remark that heuristic diffraction coefficients for arbitrary angled impedance wedges have been considered by various authors, see references in (Nechayev, 2006). These authors use non-rigorous but physically plausible approximations. However they suffer from non-uniformity in the angular variable. The uniform asymptotic far-field, for the perfectly conducting situation, has been carried by using the Kontorovich Lebedev transform by (Jones, 1964).

In section 2 we shall give the mathematical problem that we intend to solve and the complex Sommerfeld integral solution. In section 3 we shall use straightforward asymptotics to give some properties of the diffraction coefficient used to calculate the far field. This diffraction coefficient becomes infinite for particular far field angular directions. In the next section 4 we represent the Sommerfeld integral in terms of two new canonical integral representations $I(\delta), J(\delta)$ that can be asymptotically evaluated for all angular vales. In the remaining parts of the paper we derive uniform asymptotic expressions for these canonical integrals and hence expressions for the far field which is uniformly valid for all angular values.

2. Formulation of the boundary value problem

The right angled wedge is assumed to be defined by the surfaces $y = 0, x > 0$ and $x = 0, y < 0$; and polar coordinates (r, θ) are defined by $x = rcos\theta, y = rsin\theta$. The case when the only component of the incident electric field is that parallel to the z-axis will be considered. We shall assume that the incident field is given by $u_0 = e^{-i[\omega t + krcos(\theta-\theta_0)]}$, $0 \leq \theta_0 \leq 3\pi/2$. If $ue^{-i\omega t}$ denotes the total electric intensity parallel to the z-axis then Maxwell's equation give,

$$(\Delta + k^2)u = 0, \quad 0 < \theta < \frac{3\pi}{2}, \tag{1}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2},$$

and where $k^2 = \epsilon_0 \mu_0 \omega^2$. For a unique solution the total field u must satisfy an edge condition, and the edge diffracted field u_d a radiation condition. The boundary conditions appropriate to the present problem are given by

$$\frac{\partial u(x,0)}{\partial y} - ik\cos\vartheta u(x,0) = 0, \quad (x > 0), \tag{2}$$

$$\frac{\partial u(0,y)}{\partial x} + ik\cos\vartheta u(0,y) = 0, \quad (y < 0). \tag{3}$$

The complex angle ϑ represents the impedance properties of the wedge surfaces. For absorbing surfaces it is necessary that $\pi < \Re\vartheta \leq 3\pi/2$. A special case of this diffraction problem has been dealt with in (Rawlins, 1990) and from this work a convenient expression for the diffracted far field is given by :

$$\begin{aligned} u_d(r,\theta) = & \frac{(\cos\frac{4\theta_0}{3} - \cos\frac{4(\pi+\vartheta)}{3})}{\sqrt{3}\pi i(\cos\theta_0 + \cos\vartheta)(\sin\theta_0 - \cos\vartheta)} \\ \times & \int_{S(\theta)} \frac{\sin\frac{2\theta_0}{3}\sin\frac{2\gamma}{3}e^{ikr\cos(\gamma-\theta)}}{(\cos\frac{2(\gamma-\theta_0)}{3} + \frac{1}{2})(\cos\frac{2(\gamma+\theta_0)}{3} + \frac{1}{2})} \\ \times & \frac{(\cos\gamma - \cos\vartheta)(\sin\gamma + \cos\vartheta)(2\cos\frac{2\theta_0}{3}\cos\frac{2\gamma}{3} + \frac{1}{2} - \cos\frac{4(\pi+\vartheta)}{3})d\gamma}{(\cos\frac{4(\gamma-\pi-\vartheta)}{3} + \frac{1}{2})(\cos\frac{4(\gamma+\pi+\vartheta)}{3} + \frac{1}{2})}, \end{aligned} \tag{4}$$

where the path of integration $S(\theta)$ is the path of steepest descent through θ. A straight forward application of the method of steepest descent applied to the above integral gives, assuming that no poles of the integrand occur near the saddle point $\gamma = \theta$,

$$u_d(r,\theta,\theta_0) = D(\theta,\theta_0)\frac{e^{ikr}}{\sqrt{r}} + O((kr)^{-\frac{3}{2}}), \tag{5}$$

where the "diffraction coefficient" $D(\theta,\theta_0)$ is given by

$$\begin{aligned} D(\theta,\theta_0) = & \frac{2e^{i\frac{\pi}{4}}(\cos\theta - \cos\vartheta)(\sin\theta + \cos\vartheta)(\cos\frac{4\theta_0}{3} - \cos\frac{4(\pi+\vartheta)}{3})}{\sqrt{6\pi k}(\cos\theta_0 + \cos\vartheta)(\sin\theta_0 - \cos\vartheta)(\cos\frac{4(\theta-\pi-\vartheta)}{3} + \frac{1}{2})} \\ & \frac{(2\cos\frac{2\theta_0}{3}\cos\frac{2\theta}{3} + \frac{1}{2} - \cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta}{3}\sin\frac{2\theta_0}{3}}{(\cos\frac{4(\theta+\pi+\vartheta)}{3} + \frac{1}{2})(\cos\frac{2(\theta-\theta_0)}{3} + \frac{1}{2})(\cos\frac{2(\theta+\theta_0)}{3} + \frac{1}{2})}. \end{aligned} \tag{6}$$

The results above can become infinite for certain combinations of the angular variables θ, θ_0. To overcome this drawback we need to apply a more sophisticated asymptotic analysis to the integral (4). Before we can do this we need to represent this Sommerfeld integral in terms of two new integrals.

3. Alternative integral representations for the integral (4)

Rewriting the integral (4) in a more convenient form

$$u_d(r,\theta) = \frac{1}{2\pi i\sqrt{3}} \int_{S(\theta)} \mathbf{D}(\gamma,\theta_0) \left(\frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\theta_0)}{3}} - \frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\theta_0)}{3}} \right) e^{ikr\cos(\gamma-\theta)} d\gamma, \tag{7}$$

where

$$\mathbf{D}(\gamma,\theta_0) = \frac{(\cos\gamma - \cos\vartheta)(\sin\gamma + \cos\vartheta)}{(\cos\theta_0 + \cos\vartheta)(\sin\theta_0 - \cos\vartheta)} \times \frac{(\cos\frac{4\theta_0}{3} - \cos\frac{4(\pi+\vartheta)}{3})(2\cos\frac{2\theta_0}{3}\cos\frac{2\gamma}{3} + \frac{1}{2} - \cos\frac{4(\pi+\vartheta)}{3})}{(\cos\frac{4(\gamma-\pi-\vartheta)}{3} + \frac{1}{2})(\cos\frac{4(\gamma+\pi+\vartheta)}{3} + \frac{1}{2})}. \tag{8}$$

The reason for writing $u_d(r,\theta)$ in this form, is that, complex poles involving ϑ only appear in $\mathbf{D}(\gamma,\theta_0)$. These poles never occur in the range of γ for which we shall be concerned with provided $\Re(\sqrt{1-\cos^2\vartheta}) > 1$. The physical reason for this is that no surface waves are excited or supported on the wedge faces under the present conditions, see, (Williams, 1960) and (Bowman, 1967). For certain values of θ and θ_0, the poles of the integrand in (7) approach near to, and can exist at, the saddle point. In such a situation the normal saddle point method breaks down, and the expression (6) obtained for the diffraction coefficient is no longer valid. The situation described above corresponds to the physical situation when the field is observed near to the geometrical optics boundaries. We shall now use the method of (Oberhettinger, 1959) to derive expressions for the diffracted field which are uniformly valid near the geometrical optics boundaries. Before applying the method of Oberhettinger the diffracted field integral representation (7) must be represented in terms of Laplace type integrals. Letting $\theta + iw = \gamma$ in the equation (7) gives

$$u_d(r,\theta) = \frac{1}{2\pi\sqrt{3}} \int_{-\infty}^{\infty} \mathbf{D}(\theta + iw, \theta_0) \tag{9}$$

$$\left(\frac{1}{\cos\frac{2\pi}{3}-\cos\frac{2(\theta+iw-\theta_0)}{3}}-\frac{1}{\cos\frac{2\pi}{3}-\cos\frac{2(\theta+iw+\theta_0)}{3}}\right)e^{ikr\cosh w}dw.$$

The two integrals in the expression (9) are of the general form

$$P(r,\theta)=\frac{1}{2\pi\sqrt{3}}\int_{-\infty}^{\infty}\frac{\mathbf{D}(\theta+iw,\theta_0)e^{ikr\cosh w}dw}{(\cos\frac{2\pi}{3}-\cos\frac{2(\psi+iw)}{3})},\qquad(10)$$

where $\psi=\theta\pm\theta_0$. Some manipulation, see appendix, enables one to rewrite (10) in the form

$$\begin{aligned}P(r,\theta)\;=\;&-\frac{1}{6\pi}\sin\frac{2(\pi-\psi)}{3}\int_0^{\infty}\frac{\mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})}\\&-\frac{1}{6\pi}\sin\frac{2(\pi+\psi)}{3}\int_0^{\infty}\frac{\mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}\\&-\frac{i}{6\pi}\int_0^{\infty}\frac{\mathbf{D}_o(\theta,\theta_0,w)\sinh\frac{2w}{3}e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})}\\&+\frac{i}{6\pi}\int_0^{\infty}\frac{\mathbf{D}_o(\theta,\theta_0,w)\sinh\frac{2w}{3}e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})},\end{aligned}\qquad(11)$$

where

$$\begin{aligned}\mathbf{D}_e(\theta,\theta_0,w)&=\mathbf{D}(\theta+iw,\theta_0)+\mathbf{D}(\theta-iw,\theta_0),\\\mathbf{D}_o(\theta,\theta_0,w)&=\mathbf{D}(\theta+iw,\theta_0)-\mathbf{D}(\theta-iw,\theta_0).\end{aligned}\qquad(12)$$

Thus the field $u_d(r,\theta)$ can now be written in the form

$$\begin{aligned}u_d(r,\theta)=&-\{I(\pi-\theta+\theta_0)+J(\pi-\theta+\theta_0)\}\\&-\{I(\pi+\theta-\theta_0)-J(\pi+\theta-\theta_0)\}\\&+\{I(\pi-\theta-\theta_0)+J(\pi-\theta-\theta_0)\}\\&+\{I(\pi+\theta+\theta_0)-J(\pi+\theta+\theta_0)\},\end{aligned}\qquad(13)$$

where

$$I(\delta)=\frac{\sin\frac{2\delta}{3}}{6\pi}\int_0^{\infty}\frac{\mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2\delta}{3})},\qquad(14)$$

$$J(\delta)=\frac{i}{6\pi}\int_0^{\infty}\frac{\mathbf{D}_o(\theta,\theta_0,w)\sinh\frac{2w}{3}e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2\delta}{3})}.\qquad(15)$$

$$\begin{aligned} I(\delta) &= -I(-\delta), \quad J(\delta) = J(-\delta), \\ I(\delta + 3\pi n) &= I(\delta), \quad J(\delta + 3\pi m) = J(\delta), |n| = 0, 1, 2, \ldots \end{aligned} \tag{16}$$

To evaluate (14) and (15) asymptotically for large kr it is necessary to expand the numerator, excluding the exponential, in terms of $\sinh \frac{w}{2}$. This is achieved by means of Burman's theorem, (Whittaker and Watson, 1962), which gives

$$\mathbf{D}(\theta + iw, \theta_0) = \sum_{n=0}^{\infty} B_n(\theta, \theta_0)(\sinh \frac{w}{2})^n, \tag{17}$$

where $B_0(\theta, \theta_0) = \mathbf{D}(\theta, \theta_0)$, $B_1(\theta, \theta_0) = 2i\mathbf{D}_1(\theta, \theta_0)$; $\mathbf{D}_1(\theta, \theta_0) \equiv \frac{\partial \mathbf{D}(\theta, \theta_0)}{\partial \theta}$. The remaining coefficients are given by

$$B_n(\theta, \theta_0) = \frac{1}{n!} \frac{d^{n-1}}{dw^{n-1}} \left[\frac{d\, \mathbf{D}(\theta + iw, \theta_0)}{dw} \left(\frac{w}{\sinh \frac{w}{2}} \right)^n \right]_{w=0}, n = 2, 3, \ldots$$

It is clear that B_n is a function of θ, θ_0 but we shall drop the explicit representation of this dependence in the rest of the paper. From the relationship between $\mathbf{D}(\theta - iw, \theta_0)$, $\mathbf{D}_o(\theta, \theta_0, w)$ and $\mathbf{D}_e(\theta, \theta_0, w)$ with the expansion,(Erdelyi et al, 1958)

$$\sinh \nu w = 2\nu \sinh \frac{w}{2} {}_2F_1(\nu + \frac{1}{2}, \frac{1}{2} - \nu; \frac{3}{2}; -\sinh^2 \frac{w}{2}),$$

we obtain

$$\mathbf{D}_e(\theta, \theta_0, w) = 2 \sum_{n=0}^{\infty} B_{2n}(\sinh \frac{w}{2})^{2n}, \tag{18}$$

$$\mathbf{D}_o(\theta, \theta_0, w) = 2 \sum_{n=0}^{\infty} B_{2n+1}(\sinh \frac{w}{2})^{2n+1}. \tag{19}$$

Letting

$$i \sinh \frac{2w}{3} \mathbf{D}_o(\theta, \theta_0, w) = 4 \sinh^2 \frac{w}{2} \mathbf{D}_o'(\theta, \theta_0, w) \tag{20}$$

where

$$\mathbf{D}_o'(\theta, \theta_0, w) = \sum_{n=0}^{\infty} B_{2n}'(\sinh \frac{w}{2})^{2n}, \tag{21}$$

gives a relationship between the coefficients B_{2n}' and B_{2n+1}, for $n = 0, 1, 2, \ldots$; the first few of which are given by:

$$B_0' = \frac{2i}{3} B_1, \quad B_2' = \frac{2i}{3} B_3 + \frac{7i}{27} B_1, \quad B_4' = \frac{2i}{3} B_5 + \frac{7i}{27} B_3 - \frac{91i}{486} B_1, \ldots$$

The above expansions are valid in the region $|\sinh\frac{w}{2}| < 1$,and, since the saddle point of the integrals (14) and (15) occur at $w = 0$, the integrands are in a suitable form for the standard asymptotic evaluation of the integrals. We exclude the situation where $\Re\vartheta \approx 3\pi/2$ and $\Im\vartheta \approx 0$ since the non captured poles of $\mathbf{D}(\theta + iw, \theta_0)$ will approach the saddle point and the modified saddle point method will also have to be applied in this case. Since these poles do not exist when $\vartheta = 3\pi/2$ we shall not carry out this special situation here.

4. Uniform asymptotic expansion for $\mathbf{I}(\delta)$

We shall now obtain the asymptotic expansion for the integral (14) for arbitrary δ and kr large. Let $\cosh w = 1 + t$ in (14) so that

$$I(\delta) = e^{ikr} \int_0^\infty K(t) e^{ikrt} \frac{dt}{\sqrt{t}}, \tag{22}$$

where

$$K(t) = \sin\frac{2\delta}{3} \cdot \frac{(t+2)^{-\frac{1}{2}}}{3\pi} \cdot \frac{\mathbf{D}_e(\theta, \theta_0, \cosh^{-1}(1+t))}{\left[\cosh\frac{2\ln(1+t+(t^2+2t)^{\frac{1}{2}})}{3} - \cos\frac{2\delta}{3})\right]}, \tag{23}$$

and from (18)

$$\mathbf{D}_e(\theta, \theta_0, \cosh^{-1}(1+t)) = 2\sum_{n=0}^{\infty} B_{2n}(\frac{t}{2})^n. \tag{24}$$

The expression (23) can be expanded in an infinite power series in t, about the origin, and the function to which the series converges uniformly to, will be analytic in the region $|t| < |t_0|$, where $t_0 (= -2\sin^2\frac{\delta}{2})$ is the nearest pole to the origin. Thus

$$\begin{aligned} K(t) &= \sum_{n=0}^{\infty} c_n t^n, \qquad |t| < |t_0|, \\ c_0 &= \frac{\mathbf{D}(\theta, \theta_0)}{3\pi\sqrt{2}} \cot\frac{\delta}{3}, \qquad c_n = \frac{1}{n!}\left[\frac{d^n}{dt^n} K(t)\right]_{t=0}, n = 1, 2, ... \end{aligned} \tag{25}$$

To obtain a uniform asymptotic expansion for $I(\delta)$, which is valid for $\delta \to 0$, it is necessary to remove the pole $t = t_0$ from the expression for $K(t)$. Thus we define $K^*(t)$ by

$$K^*(t) = K(t) - \frac{b_{-1}}{(t - t_0)}, \tag{26}$$

where $K^*(t)$ is analytic at $t = t_0$, and from (23)

$$b_{-1} = \lim_{t \to t_0} (t - t_0) K(t) = \frac{\mathbf{D}_e(\theta, \theta_0, -2\sin^2 \frac{\delta}{2})}{\sqrt{2\pi}}. \qquad (27)$$

From (26) it can be seen that $K^*(t)$ has a larger radius of analyticity than $K(t)$. This is because we have removed from $K(t)$ the singularity, t_0, nearest to the origin. Denoting by t_1, the singularity of $K^*(t)$ nearest to the origin, then $|t_0| < |t_1|$; and thus we can expand $K^*(t)$ about the origin, the expansion being analytic for $|t| < |t_1|$. Thus

$$\begin{aligned} K^*(t) &= \sum_{n=0}^{\infty} d_n t^n, \qquad |t| < |t_1|, \\ &= d_n = \frac{1}{n!} \left[\frac{d^n}{dt^n} K^*(t) \right]_{t=0}, n = 1, 2, ... \end{aligned} \qquad (28)$$

Hence substituting $K(t)$ given by (26) into (22) we obtain

$$I(\delta) = e^{ikr} b_{-1} \int_0^{\infty} e^{ikrt} \frac{dt}{\sqrt{t}(t - t_0)} + e^{ikr} \int_0^{\infty} K^*(t) e^{ikrt} \frac{dt}{\sqrt{t}}. \qquad (29)$$

By considering the integral

$$\oint \frac{e^{iz} dz}{\sqrt{z}(z + a)},$$

over a quarter circle of infinite radius indented at the origin in the first quadrant, it can be shown that for $z = x + iy$

$$\begin{aligned} \int_0^{\infty} \frac{e^{ix} dx}{\sqrt{x}(x + a)} &= e^{-i\frac{\pi}{4}} \int_0^{\infty} \frac{e^{-y} dy}{\sqrt{y}(y - ia)}, \\ &= 2\sqrt{2} e^{-i\frac{\pi}{4}} \frac{e^{ia}}{\sqrt{a}} F(\sqrt{a}), \end{aligned}$$

where $|\arg a| < \pi$, and $F(v)$ is the Fresnel integral

$$F(v) = \int_v^{\infty} e^{iu^2} du.$$

Thus the first integral of the expression (29) can be evaluated by using the result:

$$\int_0^{\infty} \frac{e^{iax} dx}{\sqrt{x}(x + b)} = 2\sqrt{\frac{\pi}{b}} e^{-i(ab + \pi/4)} F(\sqrt{ab}), \qquad (30)$$

which is valid for $|\arg b| < \pi$, $a > 0$. The second integral of the expression (29) can be evaluated by a straightforward application of Watson's Lemma; and noting that

$$\begin{aligned} d_n &= \frac{1}{n!}\left[\frac{d^n}{dt^n}K^*(t)\right]_{t=0} = d_n = \frac{1}{n!}\left[\frac{d^n}{dt^n}(K(t) - \frac{b_{-1}}{(t-t_0)})\right]_{t=0}, \\ &= \frac{1}{n!}\left[\frac{d^n}{dt^n}K(t)\right]_{t=0} - \frac{b_{-1}(-1)^n}{(-t_0)^{n+1}}, \\ &= c_n - \frac{(-1)^n 2^{-n-\frac{1}{2}}\mathbf{D}_e(\theta,\theta_0,-2\sin^2\frac{\delta}{2})}{2\pi(\sin\frac{\delta}{2})^{2n+1}}, \end{aligned} \tag{31}$$

we obtain

$$\begin{aligned} I(\delta) &= \frac{\mathbf{D}_e(\theta,\theta_0,-2\sin^2\frac{\delta}{2})e^{i(kr\cos\delta-\frac{\pi}{4}\mathrm{sgn}\delta)}}{\sqrt{\pi}}F(\sqrt{2kr}|\sin\frac{\delta}{2}|) \\ &+ \frac{e^{i(kr+\frac{\pi}{4})}}{\sqrt{kr}}\sum_{n=0}^{\infty}\left[c_n - \frac{(-1)^n 2^{-n-\frac{1}{2}}\mathbf{D}_e(\theta,\theta_0,-2\sin^2\frac{\delta}{2})}{2\pi(\sin\frac{\delta}{2})^{2n+1}}\right]\frac{i^n\Gamma(n+\frac{1}{2})}{(kr)^n}, \end{aligned} \tag{32}$$

which is the uniform asymptotic series for $I(\delta)$.

5. Uniform asymptotic expansion for $\mathbf{J}(\delta)$

Making the change of variable $\cosh w = 1 + t$ in the integral representation (15) gives

$$J(\delta) = e^{ikr}\int_0^{\infty} t^{\frac{1}{2}}M(t)e^{ikrt}dt, \tag{33}$$

where

$$M(t) = \frac{(t+2)^{-\frac{1}{2}}\mathbf{D}_o'(\theta,\theta_0,\cosh^{-1}(1+t))}{3\pi\left[\cosh\frac{2\ln(1+t+(t^2+2t)^{\frac{1}{2}})}{3} - \cos\frac{2\delta}{3})\right]}, \tag{34}$$

and from (21)

$$\mathbf{D}_o'(\theta,\theta_0,\cosh^{-1}(1+t)) = \sum_{n=0}^{\infty}B_{2n}'(\frac{t}{2})^n.$$

The expression (34) can be expanded in an infinite power series of t, (about the origin), which will be analytic in the region $|t| < |t_0|$, where

$t_0(= -2\sin^2\frac{\delta}{2})$ is the nearest pole to the origin. Thus

$$M(t) = \sum_{n=0}^{\infty} c_n' t^n, \qquad |t| < |t_0|,$$
$$c_0' = \frac{-2\mathbf{D}_1(\theta,\theta_0)}{9\pi\sqrt{2}(\sin\frac{\delta}{3})^2}, \qquad c_n' = \frac{1}{n!}\left[\frac{d^n}{dt^n}M(t)\right]_{t=0}, n = 1, 2, ... \tag{35}$$

To obtain a uniformly valid asymptotic expansion for $J(\delta)$ it is necessary to isolate the pole t_0 by defining a new function $M^*(t)$ by

$$M(t) = M^*(t) + \frac{b_{-1}'}{(t-t_0)}, \tag{36}$$

where

$$b_{-1}' = \lim_{t\to t_0}(t-t_0)M(t) = \frac{\mathbf{D}_o'(\theta,\theta_0,-2\sin^2\frac{\delta}{2})\sin\frac{\delta}{2}}{\sqrt{2}\pi\sin^2\frac{\delta}{3}}. \tag{37}$$

Thus (15) can be written in the form

$$J(\delta) = e^{ikr}b_{-1}'\int_0^\infty \sqrt{t}e^{ikrt}\frac{dt}{(t-t_0)} + e^{ikr}\int_0^\infty M^*(t)e^{ikrt}\frac{dt}{\sqrt{t}}, \tag{38}$$

where,

$$M^*(t) = \sum_{n=0}^{\infty} d_n' t^n, \qquad |t| < |t_1'|,$$
$$= d_n' = \frac{1}{n!}\left[\frac{d^n}{dt^n}M^*(t)\right]_{t=0}, n = 1, 2, ... \tag{39}$$

The first integral in the expression (38) can be evaluated by noting that

$$\int_0^\infty \frac{x^{\frac{1}{2}}e^{iax}}{(x+b)}dx = \int_0^\infty \frac{e^{iax}}{x^{\frac{1}{2}}}dx - b\int_0^\infty \frac{e^{iax}}{x^{\frac{1}{2}}(x+b)}dx \tag{40}$$

and using the result (30) to give

$$\int_0^\infty \frac{x^{\frac{1}{2}}e^{iax}}{(x+b)}dx = \sqrt{\frac{\pi}{a}}e^{i\frac{\pi}{4}} - 2\sqrt{\pi b}e^{-iab-i\frac{\pi}{4}}F(\sqrt{ab}). \tag{41}$$

The second integral of the expression (38) is evaluated by a straightforward application of Watson's Lemma. Thus

$$
\begin{aligned}
J(\delta) &= \frac{\mathbf{D}_o'(\theta,\theta_0,-2\sin^2\frac{\delta}{2})\sin\frac{\delta}{2}}{\sqrt{2\pi kr}\sin\frac{2\delta}{3}} && (42)\\
&- \frac{2\mathbf{D}_o'(\theta,\theta_0,-2\sin^2\frac{\delta}{2})(\sin\frac{\delta}{2})^2\mathrm{sgn}\delta}{\sqrt{\pi}\sin\frac{2\delta}{3}} F(\sqrt{2kr}|\sin\frac{\delta}{2}|)\\
&+ \frac{e^{i(kr+\frac{3\pi}{4})}}{(kr)^{\frac{3}{2}}}\cdot\sum_{n=0}^{\infty}\left[c_n' - \frac{(-1)^n\mathbf{D}_o'(\theta,\theta_0,-2\sin^2\frac{\delta}{2})2^{-n-\frac{1}{2}}}{2\pi(\sin\frac{\delta}{2})^{2n+1}\sin\frac{2\delta}{3}}\right]\frac{i^n\Gamma(n+\frac{3}{2})}{(kr)^n}\\
& && (43)
\end{aligned}
$$

Having determined the uniform asymptotic expansions for $I(\delta)$ and $J(\delta)$ we shall now determine the expressions for particular values of δ.

6. δ not near zero

In this situation the arguments of the Fresnel integrals are large and we can use the asymptotic expansion for these integrals, viz.

$$
F(|x|) \approx \frac{e^{ix^2}e^{i\pi/2}}{2\sqrt{\pi}|x|}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})(-i)^n}{x^{2n}}, \quad |x|\to\infty. \qquad (44)
$$

Thus the expression (32) becomes after using (44)

$$
\begin{aligned}
I(\delta) &\approx \frac{e^{i(kr+\pi/4)}}{\sqrt{kr}}\sum_{n=0}^{\infty}\frac{c_n\Gamma(n+\frac{1}{2})i^n}{(kr)^n},\\
&= \frac{\mathbf{D}(\theta,\theta_0)}{3\sqrt{3kr}}e^{i(kr+\pi/4)}\cot\frac{\delta}{3} + O((kr)^{-3/2}). \qquad (45)
\end{aligned}
$$

Rewriting (44) as

$$
F(|x|) \approx \frac{e^{ix^2}e^{i\pi/2}}{2\sqrt{\pi}|x|} - \frac{e^{ix^2}e^{i\pi/2}}{2\sqrt{\pi}|x|^{3/2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{2})(-i)^n}{x^{2n}}, \quad |x|\to\infty. \quad (46)
$$

and substituting this expression into (42) yields

$$
\begin{aligned}
J(\delta) &\approx \frac{e^{i(kr+3\pi/4)}}{(kr)^{3/2}}\sum_{n=0}^{\infty}\frac{c_n'\Gamma(n+\frac{3}{2})i^n}{(kr)^n},\\
&= 0 + O((kr)^{-3/2}). \qquad (47)
\end{aligned}
$$

Substituting the expression (45) and (47) into the expression (13) gives the expression (5), where $D(\theta, \theta_0)$ is given by (6), in accordance with expectation.

7. δ near zero

For small δ the arguments of the Fresnel integrals are also small,and a suitable asymptotic expansion for $F(|x|)$ must be found. For $|x| \to 0$,

$$F(|x|) = \int_{|x|}^{\infty} e^{it^2} dt = \int_0^{\infty} e^{it^2} dt - \int_0^{|x|} e^{it^2} dt = \frac{\sqrt{\pi}}{2} e^{\frac{i\pi}{4}} - \int_0^{|x|} e^{it^2} dt. \tag{48}$$

For $|x| << 1$, the series expansion for e^{it^2} is uniformly convergent for all $t << |x|$, and we may therefore integrate term by term the series expansion for e^{it^2} in (48). Thus

$$F(|x|) = \frac{\sqrt{\pi}}{2} e^{\frac{i\pi}{4}} - |x| \sum_{n=0}^{\infty} \frac{(ix^2)^n}{n!(2n+1)}, \quad |x| \to 0. \tag{49}$$

By means of the following formulae

$$\begin{aligned}
\csc \frac{2\delta}{3} &= \frac{3}{4} \csc \frac{\delta}{2} + \frac{21}{216} \sin \frac{\delta}{2} + O(\sin^3 \frac{\delta}{2}), \\
\cos \frac{\delta}{3} &= 1 - \frac{2}{9} \sin^2 \frac{\delta}{2} - \frac{16}{243} \sin^4 \frac{\delta}{2} + O(\sin^6 \frac{\delta}{2}), \\
\sin \frac{\delta}{3} &= \frac{2}{3} \sin \frac{\delta}{2} + \frac{10}{162} \sin^3 \frac{\delta}{2} + \frac{154}{5832} \sin^4 \frac{\delta}{2} + O(\sin^6 \frac{\delta}{2}), \\
\csc \frac{\delta}{3} &= \frac{3}{2} \csc \frac{\delta}{2} - \frac{15}{108} \sin \frac{\delta}{2} - \frac{543}{11664} \sin^3 \frac{\delta}{2} + O(\sin^5 \frac{\delta}{2}), \\
\cot \frac{\delta}{3} &= \frac{3}{2} \csc \frac{\delta}{2} - \frac{51}{108} \sin \frac{\delta}{2} - \frac{1335}{11664} \sin^3 \frac{\delta}{2} + O(\sin^5 \frac{\delta}{2}),
\end{aligned}$$

which are obtained from (Erdelyi et al, 1958),

$$\begin{aligned}
\sin \nu z &= 2\nu \sin \frac{z}{2} {}_2F_1(\frac{1}{2} + \nu, \frac{1}{2} - \nu; \frac{3}{2}; \sin^2 \frac{z}{2}), \\
\cos \nu z &= 2 {}_2F_1(\nu, -\nu; \frac{1}{2}; \sin^2 \frac{z}{2}),
\end{aligned}$$

and the expression (49), we may expand (32) and (42) in terms of $\sin \frac{\delta}{2}$. The manipulations have to be carried out with care, and this is a rather tedious task. The parts which would be singular if $\delta = 0$ cancel each other out and we finally obtain

$$\begin{aligned} I(\delta) &= \mathrm{sgn}\delta \mathbf{D}(\theta,\theta_0)e^{i(kr-\pi/4)} \\ &\times \left[\frac{\sqrt{\pi}}{2}e^{i\pi/4} - \sqrt{2kr}|\sin\frac{\delta}{2}| - \frac{i}{3}(\sqrt{2kr}|\sin\frac{\delta}{2}|)^3 - \ldots\right] \\ &+ \frac{e^{i(kr+\pi/4)}}{2\sqrt{2\pi kr}}\left(B_2 - \frac{17}{54}\mathbf{D}(\theta,\theta_0)\right)\sin\frac{\delta}{2} \\ &+ \frac{e^{i(kr+3\pi/4)}}{8\sqrt{2\pi}(kr)^{\frac{3}{2}}}\left(B_4 - \frac{17}{54}B_2 + \frac{7}{54}\mathbf{D}(\theta,\theta_0)\right)\sin\frac{\delta}{2} \\ &+ O(\frac{\sin\frac{\delta}{2}}{(kr)^{\frac{5}{2}}}) + O(\sin^2\frac{\delta}{2}). \end{aligned} \tag{50}$$

Similarly

$$\begin{aligned} J(\delta) &= \frac{3B_0' e^{i(kr+\pi/4)}}{4\sqrt{2\pi kr}} - \mathrm{sgn}\delta\frac{3B_0'}{2\sqrt{\pi}}\sin\frac{\delta}{2}e^{i(kr-\pi/4)} \\ &\times \left[\frac{\sqrt{\pi}}{2}e^{i\pi/4} - \sqrt{2kr}|\sin\frac{\delta}{2}| - \frac{i}{3}(\sqrt{2kr}|\sin\frac{\delta}{2}|)^3 - \ldots\right] \\ &+ \frac{e^{i(kr-\pi/4)}}{2\sqrt{2\pi kr}}\left(\frac{3}{8}B_2' - \frac{17}{144}B_0'\right) \\ &+ O(\frac{\sin\frac{\delta}{2}}{(kr)^{\frac{5}{2}}}) + O(\sin^3\frac{\delta}{2}). \end{aligned} \tag{51}$$

8. $\delta = \psi + \delta'$ where δ' is small and ψ is a fixed angle

The arguments of the Fresnel integrals will be large and we can therefore substitute (44) into (42) and (47) and setting $\delta = \psi + \delta'$ we obtain after some manipulations of the resulting expressions

$$\begin{aligned} I(\psi \pm \delta') &= \frac{\mathbf{D}(\theta,\theta_0)e^{i(kr+\pi/4)}}{3\sqrt{2\pi kr}}\cot\frac{\psi}{3} \mp \frac{2\mathbf{D}(\theta,\theta_0)e^{i(kr+\pi/4)}\sin\frac{\delta'}{2}}{9\sqrt{2\pi}(kr)^{\frac{3}{2}}\sin^2\frac{\delta'}{3}} \\ &- \frac{i\mathbf{D}(\theta,\theta_0)e^{i(kr+\pi/4)}}{24\sqrt{2\pi}(kr)^{\frac{3}{2}}}\left(1 + \frac{8}{9\sin^2\frac{\delta'}{3}}\right)\cot\frac{\psi}{3} \end{aligned}$$

$$+ \frac{iB_2 e^{i(kr+\pi/4)}}{12\sqrt{2\pi}(kr)^{\frac{3}{2}}} \cot\frac{\psi}{3} + O(\frac{\sin\frac{\delta'}{2}}{(kr)^{\frac{3}{2}}}). \tag{52}$$

$$J(\delta') = -\frac{B_0' e^{i(kr-\pi/4)}}{12\sqrt{2\pi}(kr)^{\frac{3}{2}}} (\csc\frac{\psi}{3})^2 + O(\frac{\sin\frac{\delta'}{2}}{(kr)^{\frac{5}{2}}}). \tag{53}$$

9. Conclusions

We have derived some new results for the uniform asymptotic expansions for the electromagnetic field produced when an E-polarized plane wave is diffracted by a right angled impedance wedge. These results are of use in applying the Keller method of geometrical diffraction theory to problems which involve diffraction by right angled corners, for instance, the problem of the high frequency diffraction by a rectangular cylinder (a model for a sky scraper) has been successfully solved by using these results. The methods used here can also be used *mutatis mutandis* for an arbitrary angled imperfectly conducting wedge, other polarizations, and the effect of surface waves.

Appendix

Here we shall derive in detail the result (11) from the expression (10).

$$\begin{aligned} P(r,\theta) &= \frac{1}{2\pi\sqrt{3}} \int_{-\infty}^{\infty} \frac{\mathbf{D}(\theta+iw,\theta_0)e^{ikr\cosh w}dw}{(\cos\frac{2\pi}{3} - \cos\frac{2(\psi+iw)}{3})}, \\ &= \frac{1}{4\pi\sqrt{3}} \int_{-\infty}^{\infty} (\ \mathbf{D}_e(\theta,\theta_0,w) + \mathbf{D}_o(\theta,\theta_0,w))e^{ikr\cosh w} \\ &\times \frac{(\cos\frac{2\pi}{3} - \cosh\frac{2w}{3}\cos\frac{2\psi}{3} - i\sinh\frac{2w}{3}\sin\frac{2\psi}{3})dw}{(\cos\frac{2\pi}{3} - \cosh\frac{2w}{3}\cos\frac{2\psi}{3})^2 + (\sinh\frac{2w}{3}\sin\frac{2\psi}{3})^2}. \end{aligned}$$

Now by using the evenness and oddness of the integrand and noting that $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ we get

$$P(r,\theta) = \frac{1}{3\pi} \int_0^{\infty} \mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}$$

$$
\begin{aligned}
&\times \frac{(\cos\frac{2\pi}{3}\sin\frac{2\pi}{3}-\cosh\frac{2w}{3}\cos\frac{2\psi}{3}\sin\frac{2\pi}{3})dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}\\
&- \frac{i}{3\pi}\int_0^\infty \mathbf{D}_o(\theta,\theta_0,w)e^{ikr\cosh w}\\
&\times \frac{(\sinh\frac{2w}{3}\sin\frac{2\psi}{3}\sin\frac{2\pi}{3})dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}.
\end{aligned}
$$

$$
\begin{aligned}
P(r,\theta) &= \frac{1}{6\pi}\int_0^\infty \frac{\mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}\\
&\times [-(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})\sin\frac{2(\pi+\psi)}{3}\\
&- (\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})\sin\frac{2(\pi-\psi)}{3}]dw\\
&- \frac{i}{6\pi}\int_0^\infty \frac{\mathbf{D}_o(\theta,\theta_0,w)e^{ikr\cosh w}}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}\\
&\times \sinh\frac{2w}{3}[(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})\\
&- (\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})]dw.
\end{aligned}
$$

$$
\begin{aligned}
P(r,\theta) &= -\frac{1}{6\pi}\sin\frac{2(\pi+\psi)}{3}\int_0^\infty \frac{\mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}\\
&- \frac{1}{6\pi}\sin\frac{2(\pi-\psi)}{3}\int_0^\infty \frac{\mathbf{D}_e(\theta,\theta_0,w)e^{ikr\cosh w}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})}\\
&- \frac{i}{6\pi}\int_0^\infty \frac{\mathbf{D}_o(\theta,\theta_0,w)e^{ikr\cosh w}\sinh\frac{2w}{3}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi-\psi)}{3})}\\
&+ \frac{i}{6\pi}\int_0^\infty \frac{\mathbf{D}_o(\theta,\theta_0,w)e^{ikr\cosh w}\sinh\frac{2w}{3}dw}{(\cosh\frac{2w}{3}-\cos\frac{2(\pi+\psi)}{3})}.
\end{aligned}
$$

Which is the result (11).

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