A Methodology for Company Valuation

A Dissertation

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by
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Declaration

I hereby declare that this thesis is my own work and effort and that it has not been submitted anywhere for any award. Where other sources of information have been used, they have been acknowledged.

Oliver Schlueter

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Notations

a	price of replication asset
$a(1), ..., a(\widehat{n})$	prices of replication assets 1 to $\widehat{n}$
$a_1, ..., a_{\widehat{n}}$	prices of replication asset in states 1 to $\widehat{n}$
b	company cash flow
$\widehat{-b}$	maximum negative cash flow barrier
$\widehat{b}_{sum}$	sum of maximum negative cash flows
$\mathbf{B}$	company cash flow matrix
B_0	bond price
$Beta$	relative uncertainty to the overall market portfolio
β_i	payout out of replication portfolio at vertex i
C	call
d	down jump/lower node
df	discount factor
Δ	delta
e	natural logarithm
f	function
$f_i(w)$	value function (e.g. minimum shortfall achievable at all vertices v_j)
F	forward price
g	additional expenditure in the real option setting
$\widehat{g}$	maximum additional expenditure in the real option setting
gf_{add}	additive real option factor
gf_{multi}	multiplicative real option factor
I	investment
irr	cost of capital of the company
$i, j = 1, ..., n$	assets or items numbered from 1 to n
$k_{1...n}$	number of replication assets in states 1 to n or number of items in knapsack
k^*	optimal policy of number of replication assets or items in knapsack
K	strike price
μ	growth rate
m	min. number of upwards moves of S for a call C to finish in-the-money
n	vector of numeraire
$O\left(n^k\right)$	polynomial time computable problems
Ω	omega, number of states
p	probability
q	risk-neutral probability
Q	vector of risk-neutral probabilities
r	risk-free interest
$\mathbb{R}$	real numbers
R^2	correlation coefficient
S	price of underlying/asset price
$S\left(t\right)$	stock price process
σ	volatility

t	time
T	time at expiration
u	up jump/upper node
U	upper bound
v	vertex or state
V_i	corresponding vertex set of all vertices v_j
$\overline{w}$	utilised capacity in the knapsack
w_i	value, wealth or capacity at vertex v_i
$w_{\min}$	minimum wealth
w_{comp}	company value in t_0 after deducting the additional investment g
W_j	set of all possible utilised capacities in the knapsack
$\mathbf{X}$	cash flow matrix of replication portfolio
y	portfolio of traded non-company assets
Y_i	set of all portfolios of traded non-company assets that can be bought in v_i

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Abstract

This thesis presents an approach for company valuation by a replication portfolio of traded assets in discrete time. The model allows us to value companies with an uncertain cash flow stream without having to revert to any discount rates including premia. Modelling of asset values can be achieved in two steps: (i) Choosing a suitable stochastic process and calibrating its parameters to fit the historical asset time series behaviour, and (ii) generating a state space transition graph to implement the stochastic process dynamics in discrete time.

For company valuation, a selected number of "assets" (economic, financial, and other factors) should be captured that may reasonably be assumed to influence future cash flows of the company. Each vertex of the transition graph represents a "state of the world" and is accompanied with a corresponding cash flow caused by the sales (or other company activities) at that vertex. These possible future company cash flows can be "replicated" (without the existence of the company) by investing in a self-financing portfolio of non-company assets at the beginning, and trading this portfolio as the future evolves. The minimum cost of such a self-financing portfolio equals the value of the company. A dynamic programming algorithm for this valuation problem has been derived in discrete time. Due to the fact that an exact duplication is not possible for all cases, the replication strategy will be generated by minimising the deviations in each state to approximate an exact replication.

The company valuation algorithm in discrete time is based on two main ideas: The replication approach for arbitrage-free valuation as it is known for the valuation of contingency claims (Cox, Rubinstein 1985) as well as an optimisation to compute the least value replication portfolio following an approach originally established by Alexander Christofides (Christofides A. 2004). The research results derived in this thesis contribute to the further integration of some methodologies for contingent claim valuation and optimisation techniques. The derived algorithm has been applied for the valuation of companies with a high uncertainty in their expected cash flows (like start-up companies), and gives further insight in the valuation of non-traded companies. With the application of the derived company valuation algorithm, the limitations and shortcomings of determining the companies' weighted average cost of capital (WACC) can be by-passed.

In a further step, the algorithm has been extended to calculate the potential value contribution of the companies' real options. Part of the contribution is the generalisation of the algorithm in the way that decision-making strategies from the capital markets, as well as strategic decisions from inside corporates can be implemented and evaluated. The optimal timing of the additional investment can be computed, and the attached additional value of the "optimal" execution of these investment options is calculated. The implementation of the algorithm is performed in C++.

The extended algorithm has been applied for two high growth companies in the area of Life Sciences, confirming the applicability of the algorithm. Results will be reported for Qiagen and GPC Biotech.

Chapter 1

The Valuation Framework

One of the most basic and often encountered problems in quantitative finance is concerned with the modelling of asset values. Asset values vary stochastically with time, but the nature of the underlying price dynamics depends on the nature of the asset. Stock prices, for example, are reasonably approximated (to a first degree) by geometric brownian motion which is the underlying process used for valuing options according to the Black Scholes pricing formula (Black, Scholes 1973). A direct consequence of assuming geometric brownian motion for stock prices is that returns at any future time t are normally distributed. If the "asset" is an interest rate (or a commodity like oil, steel, electricity), its value may not exhibit growth in the long run, but tends to revert to a long term mean. These dynamics are modelled by a mean-reverting deterministic equation (Vasicek 1977). Yet, mean-reverting processes and stochastic processes which are able to cover the probability of extreme events ("fat-tails"), whose occurrence is underestimated (Campbell, Lo and MacKinlay 1997) are not taken into consideration in the forthcoming work.

Black Scholes assumes that the stock price process $S(t)$ follows a geometric brownian motion, a riskless bond paying interest at a constant rate r is available and that funds may be transferred from bank to stock and vice versa without restrictions or costs. In this described environment, perfect hedging or duplication is possible. A dynamic (time-varying) self-financing portfolio of holding the riskless bond and stocks can be formed which yields a value equal to $\max(S(T) - K, 0)$ with a probability of one at the time of expiration, defined as T (Black, Scholes 1973) where

K is defined as the strike price. Hence, the value of the option at time $t < T$ is the cash value $w(t)$ of the duplication portfolio at that time. Any deviation of the option price $w(t)$ leads to an arbitrage opportunity. The ability to replicate arbitrary contingent claims is described as completeness of the market which is not necessarily given for the valuation of a private company (Karatzas Ioannis 1989). Dynamic programming[1] involves formulating the investment problem in terms of a Hamilton-Jacobi-Bellman equation and computing the value of the asset by backward induction[2] (Bellman 1957). Solving the one period optimisation problem and then moving backwards, ensures the optimal solution for the entire company valuation.

1.1 Basic Idea

The company valuation algorithm in discrete time is based on two main ideas; the duplication approach for arbitrage-free valuation[3] as well as an optimisation algorithm to overcome the incomplete market condition. The algorithm is solved by dynamic programming in discrete time with a recursive structure (Duffie 1996).

The idea based upon the company valuation algorithm is presented in a general setting. In a first step, the replication procedure of cash flows in discrete time will be derived and in the second step, the applied optimisation technique by Alexander Christofides (Christofides A. 2004) will be introduced. As a result of the integration of the two concepts, a company valuation algorithm is derived. It allows to calculate

1 The term was originally used by Richard Bellman to describe the process of solving problems where one needs to find the best decisions one after another.

2 The principle of backward induction describes the concept of solving a problem starting in at the end of the tree in time T and folding it back one step at a time. Backward induction is a general solution concept for games of perfect information.

3 The essence of the technical sense of arbitrage-free is that it should not be possible to guarantee a profit without exposure to risk. Were it possible to do so, the market would not be arbitrage-free anymore.

the least value replication portfolio $w(t)$ of the company's expected future cash flows which is equal to the current worth of the company (Brealey 1991). Throughout the thesis the algorithm will be extended and generalised to allow for real option valuation and identification of its optimal exercise.

1.2 Replication of Contingent Claims

Replication is the strongest assumption for arbitrage-free valuation and is based on the "Law of One Price" [4]. Two general strategies can be considered for buying an underlying S at the time of expiration T. Strategy one is buying a forward which results in no cash flow in t_0 and a payment of $V_T = S_T - F_0(T)$ in $t = T$, where F is defined as the forward price. The convention that the forward contract value V_0 at origin is zero leads to

$$V_0 = V_0(S_T - F_0(T)) \equiv 0 \tag{1.1}$$

Strategy two is buying the underlying S in $t = 0$ and borrowing the same amount $F_0(T)\,B_0(T)$ what results in no cash flow in $t = 0$ again. Here again the forward contract value has to be zero.

$$\begin{aligned} V_0(S_T - F_0(T)) &= S_0 - F_0(T)\,B_0(T) \\ \Leftrightarrow F_0(T) &= S_0 B_0^{-1}(T) \end{aligned} \tag{1.2}$$

Now the forward price $F_0(T)$ is equal to the price of the underlying S_0 in addition to the accrued interest $S_0 B_0^{-1}(T) - S_0$. The equation in T, when the loan will be repaid, will be the difference between the price of the underlying S_T and the costs

[4] In an efficient market, all identical goods must have only one price, independent on how they were created.

for repaying the money including the accrued interest $S_0 B_0^{-1}(T)$: $S_T - S_0 B_0^{-1}(T)$. To eliminate arbitrage opportunities, the Law of One Price must hold which means the forward price fixed in $t = 0$ must be equal to the repayment value $K_0(T) = S_0 B_0^{-1}(T)$.

For a call option C the least price is $C \geq \max\{S - KB_0(T), 0\}$, where K is defined as the strike price. The proof will be outlined in the following: Portfolio A consists of a call $C(T)$ and money in a money account $KB_0(T)$, portfolio B of the underlying S itself. In $t = T$ two cases can be investigated.

$t = T$	Case I: $S_T \leq K$	Case II: $S_T \geq K$
Portfolio A	K	$S_T - K + K$
Portfolio B	S_T	S_T
	A$\geq$ B	A= B

Table 1.1: Replication Portfolio for a Call Option

In t_0 portfolio A has to be more valuable than portfolio B assuming non-negative interest rates $C_0 + KB_0(T) \geq S_0 \Leftrightarrow C_0 \geq S_0 - KB_0(T)$.

The widely applied models for option pricing are the Black Scholes model as well as the binomial model from Cox, Ross and Rubinstein. The latter model has been developed within the discrete time framework. The risk neutrality will be proven by applying a replication approach illustrated by the following example. The price of the underlying $S_0 = 10$ will change to either $S_T = 14$ or $S_T = 6$ in one year from now. The risk-free interest rate is assumed to be $r = 6\%$.

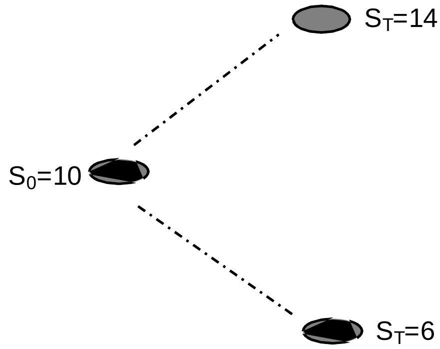


Figure 1.1: Price Path of an Underlying in a Binomial Tree

In the following the price of an European call with a strike of $K = 12$ will be calculated. In the case the share price rises to $S_T = 14$ the call would be worth $C_T = 2$, in the case of a share price decline the call would be worthless $C_T = 0$ (Cox, Rubinstein 1985).

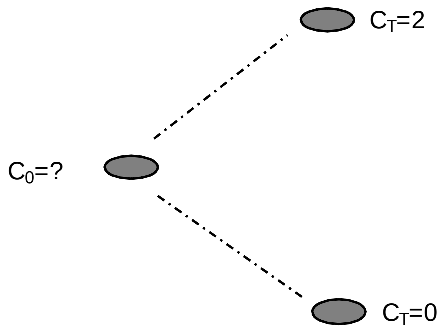


Figure 1.2: Calculation of the Call Price within a Binomial Tree

To solve for C_0 a replication portfolio will be setup which will independently of the state of S in time T earn the risk free interest. Therefore the portfolio exists of a short position of the call $-C$ and a long position of the underlying share ΔS. Due to the risk neutrality condition the portfolio value is independent of the share price development. Therefore the equation will be

$$\begin{aligned} 14 \times \Delta - 2 &= 6 \times \Delta - 0 \qquad (1.3) \\ \Longrightarrow \Delta &= 0.25 \end{aligned}$$

The portfolio now consists of one call short $-C$ and $\Delta = 0.25$ units of the underlying share long. The value of the portfolio in time T will be $14 \times 0.25 - 2 = 6 \times 0.25 - 0 = 1.5$. By discounting this portfolio value V_T with the risk-neutral interest rate r, the portfolio value today will be $V_0 = 1.5e^{-0.06\times 1} = 1.41$. The call C will therefore be worth today the difference between the share price today and the portfolio value V_0.

$$\begin{aligned} \Delta S_0 - C_0 &= V_0 \qquad (1.4) \\ 0.25 \times 10 \times -C_0 &= 1.41 \\ C_0 &= 1.09 \end{aligned}$$

Any deviation of the fair price of the call offers arbitrage opportunities. Generalised the equation changes to (Cox, Rubinstein 1985)

$$(\Delta S_0 u - C_T u)\, e^{-rT} = \Delta S_0 d \times e^{-rT}. \qquad (1.5)$$

It is assumed, that the underlying will be worth S_0 in time t_0 and will be worth either uS with probability q or dS with probability $1 - q$ at the time of in T.

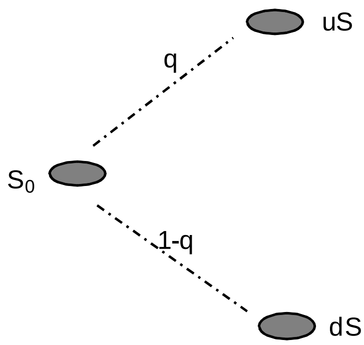


Figure 1.3: Probability Measure in a Binomial Tree

The option behaves identically. At time of expiration the value of the call will be the difference of the price of the underlying asset minus the strike price in the case the difference is positive.

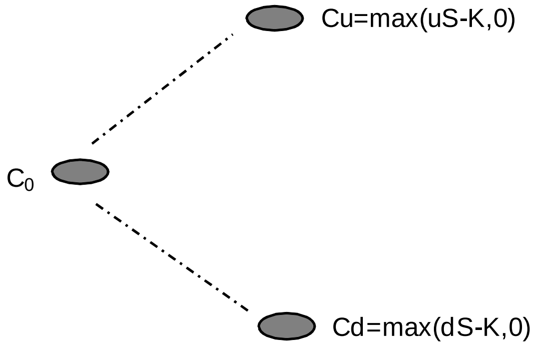


Figure 1.4: Calculation of the Call Price

In the next step a replication portfolio for the call C will be bought, consisting of the underlying asset S and selling a risk free bond B, to finance the purchase price of the underlying asset S.

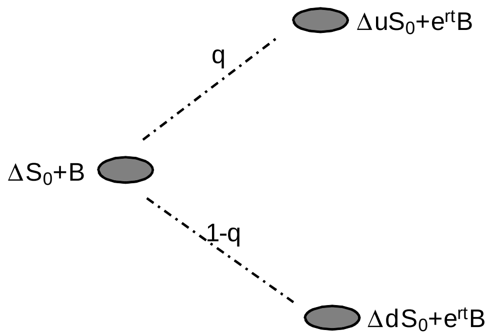


Figure 1.5: Replication Portfolio in a Binomial Tree

By replicating the payout of the call, the price of the call C can be derived. This will be done by equating the call price with the duplication portfolio (Cox, Rubinstein 1985).

$$\begin{aligned} \Delta d S_0 + e^{rT} B &= C_d \\ \Delta u S_0 + e^{rT} B &= C_u \end{aligned} \tag{1.6}$$

By equating the expression for the price increase and the price decline and solving for Δ and B the equation changes to the following:

$$\begin{aligned} \Delta &= \frac{C_u - C_d}{(u-d)\,S_0} \\ B &= \frac{uC_d - dC_u}{(u-d)\,e^{rT}} \end{aligned} \tag{1.7}$$

Due to the no-arbitrage condition the current value of the call C cannot deviate from the value of the replication portfolio $\Delta S_0 + B$, because the generation of riskless profits would be possible. If this would be possible, the generation of a riskless profit would be possible with no net investment. Therefore the following equation has to be true,

$$\begin{aligned} C_0 &= \Delta S_0 + B \\ &= \frac{C_u - C_d}{(u-d)} + \frac{uC_d - dC_u}{(u-d)\,e^{rT}} \\ &= \left[\left(\frac{e^{rT} - d}{u-d}\right) C_u + \left(\frac{u - e^{rT}}{u-d}\right) C_d\right] e^{-rT} \end{aligned} \tag{1.8}$$

which results in a value which is at least as large as $S - K$. The equation can furthermore be simplified by defining probability $p \equiv \left(e^{rT} - d\right) / (u-d)$ which can be endogenously calculated within the model; by transforming, $1-p = \left(u - e^{rT}\right) / (u-d)$ will be derived. Plugging the previous probability equations in equation 1.9 it follows:

$$C_0 = [pC_0 + (1-p)\,C_d]\,e^{-rt} \tag{1.9}$$

It can be seen that the probability p will always be larger or equal to zero and never be larger than one $0 \leq p \leq 1$. p describes the probability of the asset price increase uS_0. Within the given expression the only random variable that impacts the price of the call is the stock price S itself. Based on the assumption of risk-neutrality which means that the sum of the expected prices in T, adjusted for its probability is equal to S_0 in addition to the accrued risk-free interest in the time interval of t_0 and T follows:

$$q\left(uS_0\right)+\left(1-q\right)\left(dS_0\right)=e^{rT}S_0 \tag{1.10}$$

This means that if p has the value of q in equilibrium the investor would be risk neutral (Cox, Rubinstein 1985). Under this assumption $p = \frac{\left(e^{rT}-d\right)}{u-d} = q$ and $1-p = \frac{\left(u-e^{rT}\right)}{u-d} = 1-q$. Finally the price of the call C can be seen as the expectation of its discounted future value in a risk-neutral world. It should be pointed out that the expected return of the call C is different to the return of the risk-free bond B and it will equal the return of the replication portfolio which can be explained by the same payout profile. The return as well as the risk of the call C equals the underlying which has been financed by selling a risk-free bond B and can be interpreted as a „levered long position“ on the underlying. In the following the one time period case should be extended to a two time period model and thereafter generalised for multiple time periods. Due to the recombining nature of the Cox, Ross, Rubinstein binomial model the tree will only have three different prices for the two time period case (Cox, Rubinstein 1985). In the special case of a one time period binomial tree the claim can be replicated by a replication portfolio existing of one asset and a risk-free bond. It has been proven that in the one period setting the economy is arbitrage-free (Cox, Rubinstein 1985). The special case of the economy in one period can be generalised for multi periods which leads to the conclusion that a self-financing strategy can be established where the portfolio is adjusted period by period without considering any cash in or outflows at any time. In a multi period binomial tree the capital market is considered to be incomplete on a static view but dynamically complete. To overcome this hurdle the replication portfolio will be dynamically readjusted without considering any external cash flows. Therefore the dynamic adjustments are called self-financing strategies. The capital market is defined as dynamically complete if each single time period is complete (Schlag 2004).

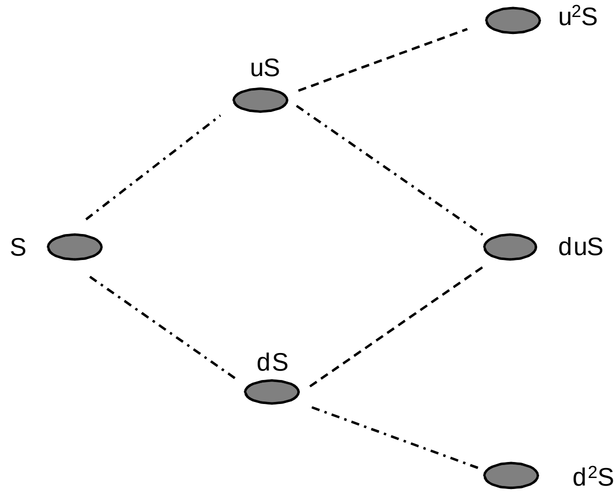


Figure 1.6: Price Development in a Two Time Period Binomial Tree

The same will be for the call C.

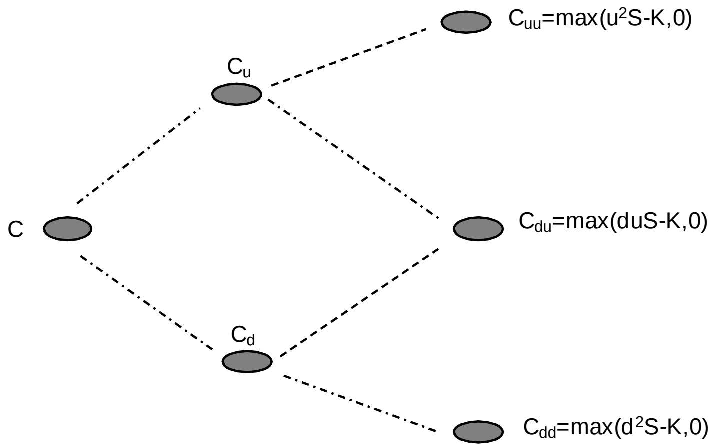


Figure 1.7: Calculation of the Call Price in a Two Time Period Binomial Tree

For example C_{dd} is defined as the value of the call in two time periods from t_0 in the case the stock price moves downwards in each time period the analogues will be true for C_{du} and C_{uu}. Using the equal algorithm as derived so far for the one-period case the calculation will start recursively from T to compute the results in the previous nodes in time t_1.

$$\begin{aligned} C_u &= \left[pC_{uu} + (1-p)\, C_{ud}\right] e^{rT} \qquad (1.11) \\ \text{and } C_d &= \left[pC_{du} + (1-p)\, C_{dd}\right] e^{rT} \end{aligned}$$

The replication portfolio will look the same. After one time period the replication has to be rebalanced which means ΔS and B will be adjusted depending on the state without considering any cash in-/ or outflow. Δ and B will be adjusted in the same way in each period, so the current price of the call in state u and d in time t_1 will be again $C = \left[pC_u + (1-p)\, C_d\right] e^{rT}$. The payout profile for the call price C in time t_0 in a world with positive interest rates and without any dividends will extend to the following

$$\begin{aligned} C_0 &= \left\{p^2 C_{uu} + 2p\,(1-p)\, C_{ud} + (1-p)^2\, C_{dd}\right\} e^{-2rT} \qquad (1.12) \\ &= \left\{ \begin{array}{c} p^2 \max\left[u^2 S - K, 0\right] + 2p\,(1-p) \max\left[d\,\mathrm{S} - K, 0\right] \\ + (1-p)^2 \max\left(d^2 S - K, 0\right) \end{array} \right\} e^{-2rT} \end{aligned}$$

In the last step the two time period case $n = 2$ will be generalised for any number of time periods. The solving procedure will always remain the same by starting at expiration in T and working backwards to t_0. As the price follows a transformed discrete density function $f(i)$ within the binomial model the generalised expression for n is derived according to (Bronstein 2001):

$$\begin{aligned} f(i) &= \binom{n}{i} p\,(1-p)^{n-1} \qquad (1.13) \\ \binom{n}{i} &= \frac{n!}{i!\,(n-1)!} \\ f(i) &= \frac{n!}{i!\,(n-1)!} p\,(1-p)^{n-1} \end{aligned}$$

For the sum of the probabilities in each time step the sum always will be

$$\sum_{i=1}^{n} \left(\frac{n!}{i!\,(n-1)!} \right) p^{i} \, (1-p)^{n-i} = 1 \tag{1.14}$$

Replacing the two-step variables by the derived generalised equation for any n leads in the limits to equation 1.15:

$$C_0 = \left\{ \sum_{i=1}^{n} \left(\frac{n!}{i!\,(n-1)!} \right) p^{i} \, (1-p)^{n-i} \left(u^{i} d^{n-i} S - K \right) \right\} e^{-nrT} \tag{1.15}$$

For calculating the value of the call C only the states have to be considered where the call is in the money which means $S_t - K > 0$ otherwise the call C defaults worthless in T (Cox, Rubinstein 1985). Defining m as the minimum number of upward moves that the stock price must make over the next n periods for the call to finish in the money. According to equation 1.15 all states wherever the call C is in the money can be described by the following inequality:

$$u^{m} d^{n-m} S > K \tag{1.16}$$

By equating the algorithm and taking the natural logarithm

$$m = \frac{\ln\left(\frac{K}{Sd^{n}}\right)}{\ln\left(\frac{u}{d}\right)}. \tag{1.17}$$

For all $i < m$ the price of the call C is zero, for all $i > m$ the price of the call is $C_0 = u^{i} d^{n-i} S - K$ which leads to starting the index at m instead of 0. Therefore,

$$C_0 = \left\{ \sum_{i=m}^{n} \left(\frac{n!}{i!\,(n-1)!} \right) p^{i} \, (1-p)^{n-i} \left(u^{i} d^{n-i} S - K \right) \right\} e^{-nrT}. \tag{1.18}$$

Even in the case it might happen that the stock price moves up in each state and $a > n$. This means that the stock will finish out-of-the-money and the price of the call C must be 0. By splitting the binomial equation into two parts the equation can be further simplified:

$$C_0 = S_0 \left[\sum_{i=m}^{n} \left(\frac{n!}{i!\,(n-1)!} \right) p^i \,(1-p)^{n-i} \left(\frac{u^i d^{n-i}}{e^{rTn}} \right) \right] \tag{1.19}$$

$$-Ke^{-rTn} \left[\sum_{i=m}^{n} \left(\frac{n!}{i!\,(n-1)!} \right) p^i \,(1-p)^{n-i} \right] \tag{1.20}$$

Both parts of the equation can be expressed as a binomial distribution of m, n, p and p' respectively. By replacing the second part of the equation by the complementary binomial distribution function $\Phi\,[m; n; p]$ and the first part of the equation by the complementary binomial distribution function $\Phi\,[m; n; p']$ where p is defined as $p' \equiv \left(ue^{-rT}\right) p$ and $(1-p') = \left(de^{-rT}\right)(1-p)$ the equation will change to

$$p^i \,(1-p)^{n-i} \left(\frac{u^i d^{n-i}}{e^{rTn}} \right) = p' \,(1-p')^{n-i} \tag{1.21}$$

The following equation is the generalised form of the binomial equation from Cox, Ross and Rubinstein (Cox, Ross and Rubinstein, 1979)

$$C_0 = S_0 \Phi\,[m; n; p'] - Ke^{-rTn} \Phi\,[m; n; p] \tag{1.22}$$

Furthermore it is defined $p \equiv \frac{\left(e^{rT}-d\right)}{u-d}$, $p' \equiv ue^{rT}p$ and m is defined as the smallest integer index which is larger than $\log\left(\frac{X}{Sd^n}\right) / \log\left(\frac{u}{d}\right)$ (Cox, Rubinstein 1985).

Finally the calculation of the price of a call C will be shown for a one time period as well as a two time period case. Supposing the following example $S_0 = 10$, $u = 1.4$, $d = 0.6$, $r = 6\%$, $K = 12$ and considering either an expiration in $n = 1$ or respectively in $n = 2$. The calculation of the probability p yields:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.06 \times 1} - 0.06}{1.4 - 0.6} = 0.577 \tag{1.23}$$

By inserting in the basis equation the following call price will be calculated:

$$C_0 = [0.577 \times 2 + (1 - 0.577) \times 0]\, e^{-0.06 \times 1} = 1.087 \tag{1.24}$$

As expected the call price C_0 is equal to the one calculated at the beginning of the derivation of the algorithm. For the two time period case the result changes accordingly:

$$C_0 = \left[0.577^2 \times 7.6 + 2 \times 0.577\,(1-0.577) \times 0 + (1-0.577)^2 \times 0\right] e^{-0.06\times 2} = 2.244 \tag{1.25}$$

By increasing the number of timesteps per period the prices calculated within the discrete binomial model from Cox, Ross and Rubinstein converge to the fair price calculated by the Black and Scholes model in continuous times.

For companies which are either private or exposed to extreme uncertainties, it cannot be assumed that all cash flows can be perfectly duplicated. Therefore the market for the company valuation can be assumed to be incomplete. Knowing that incomplete markets weaken the concept of arbitrage-free valuation, the question arises whether the concept of arbitrage-free valuation can still be applied for those companies, and what costs may arise from the nonexistence of the complete market assumption. At this point the idea of optimising (or in this case minimising the excess value of) the replication portfolio even in an incomplete market is an appealing approach to incorporate an approximate no-arbitrage assumption for the valuation of these companies.

1.3 Dynamic Programming

Already in 1957, Bellman produced the first algorithm to exactly solve the 0-1 knapsack problem (Bellman 1957). Dantzig was able to contribute an efficient approximation solution by relaxing the exact algorithm and determining an upper bound to the exact solution in 1963 (Dantzig 1963). In the sixties, especially Gilmore and Gomory focused on solving knapsack problems by dynamic programming. In the seventies, the research was dominated by exploring different branch and bound algorithms to allow for solving problems with large numbers of variables; one of the most well known was developed by Horowitz and Sahni. The prototype problem is formulated as follows where $f_n(\overline{w})$ is the value function:

$$f_n(\overline{w}) = \max \sum_{j=1}^{n} a_j k_j \tag{1.26}$$
$$\text{subject to} \sum_{j=1}^{n} h_j k_j \leq w,$$
$$k_j = 0 \text{ or } 1,\ j = 1, ..., n.$$

Within the given algorithm, the different items are numbered from 1 to n. 1 to n translates to the different traded assets within the company valuation algorithm. a is defined as the wealth of the object j or the price of the traded asset within the company valuation algorithm and h_j is defined as the the size or weight of the item j. Finally, w is the overall capacity the knapsack or container can hold and $\overline{w}$ will be defined as the capacity that has been utilised in the knapsack already. Assigned to the company valuation w is comparable to the company value. k_j $(j = 1, ..., n)$ is a vector of binary variables, because you can either choose to select the item j, which means $k_j = 1$ or it will not carried within the knapsack:

$$k_j = \begin{cases} 1 & \text{if object } j \text{ is selected;} \\ 0 & \text{otherwise.} \end{cases} \tag{1.27}$$

The intention of the basis knapsack problem is to maximise the overall value $f_n(\overline{w})$ which can be carried within the knapsack or container, which is equivalent to the optimal solution value $f_n^*(\overline{w})$. Within the developed company valuation algorithm the goal will be to minimise the overall wealth that has to be invested in a replication portfolio which at least replicates the company cash flows in all states.

For the prototype problem the value function would have the following form considering the first item to be carried within the knapsack:

$$f_1(\overline{w}) = \begin{cases} 0 & \text{for } \overline{w} = 0, ..., h_1 - 1; \\ a_1 & \text{for } \overline{w} = h_1, ..., m. \end{cases} \tag{1.28}$$

Generalised, dynamic programming consists of considering n stages one stage for each item and computing at each stage or for each item the values $f_n(\overline{w})$ (for $\overline{w}$ increasing from 0 to w) using the recursion according to (Bellman 1957):

$$f_n(\overline{w}) = \begin{cases} f_{n-1}(\overline{w}) & \text{for } \overline{w} = 0, ..., h_n - 1; \\ \max\ (f_{n-1}(\overline{w}), f_{n-1}(\overline{w} - h_n) + a_n) & \text{for } \overline{w} = h_n, ..., w. \end{cases} \tag{1.29}$$

In the following an exact solution by dynamic programming will be developed which has been applied in the forthcoming company valuation algorithm. It is assumed that profits $a_{j,}$ sizes h_j and capacities w are positive integers. The solution principle of dynamic programming within the context of the knapsack problem will be demonstrated by the following example:

$$f_0(\overline{w}) = max\,(3k_1 + 4k_2 + 2k_3 + 3k_4) \tag{1.30}$$

subject to $3k_1 + 4k_2 + 2k_3 + 3k_4 \leq 9$

$k_j \in \{0, 1\} \qquad j \in 1, ..., 4$

k_j $(j = 1, ..., 4)$ is again defined as a vector of binary variables. At step j will be decided on k_j; $k_j = 1$ means item j will be carried in the knapsack or $k_j = 0$, it will not. The recursion will start recursively by considering if item $j = 4$ will be included in the knapsack or not and will finish by reaching a decision for $j = 1$ with the goal to compute the optimal solution value $f_0(\overline{w})$.

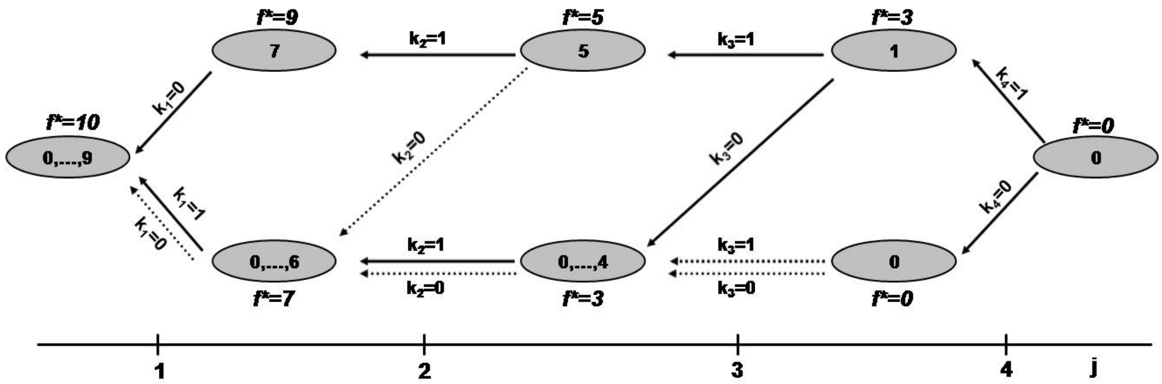


Figure 1.8: Dynamic Programming Solution in a Knapsack Problem

The utilised capacity, will be $\overline{w}_4 = 0$. The goal is to maximise the value of $f_0(\overline{w})$ in this stepwise process which will be reached by maximising $\sum a_j k_j$ for $j \in 1, ..., 4$. At $j = 3$ a capacity of maximum $\overline{w}_3 = 1$ is utilised, corresponding to the set of all possible used capacities at this stage $W_3 = \{0, 1\}$. The following steps yield the utilised capacities in the knapsack $W_2 = \{0, ..., 5\}$, $W_1 = \{0, ..., 7\}$ and $W_0 = \{0, ..., 9\}$, depending if item j has been included in the knapsack or not. Besides the capacity the maximum value has been listed which can be calculated as follows: E.g. for $\overline{w}_2 = 5$ the value $f_2^*(5) = \max\{f_3^*(1), f_2^*(5)\} = \max\{3, 2 + 5\} = 5$. The bold arrows represent the optimal solution policies $k_1^* = k_2^* = 1$, $k_3^* = 0$, $k_4^* = 1$ with the maximum possible value $f_n^*(9) = 10$. In the following tables the recursive solution process starting with step $j = 4$ will be shown. In the first column the possible utilised capacities W_{j-1} are listed, in the second column all potential policies for k_j for item j are listed, in the fifth the overall profit $f_{j-1}(\overline{w}_{j-1})$. The third column holds the utilised capacities W_j in the knapsack for the states that have been considered for the previous item j and in the fourth column the change in profit $f_j(\overline{w}_j) - f_{j-1}(\overline{w}_{j-1})$ is stated.

j=4

M_3	k_4	M_4	$f_3(m_3)-f_4(m_4)$	$f^*_4(m_4)$	$f_3(m_3)$
{0}	0 *	{0}	0	0	0*
{1}	1 *	{0}	3	0	3*

j=3

M_2	k_3	M_3	$f_2(m_2)-f_3(m_3)$	$f^*_3(m_3)$	$f_2(m_2)$
{0,1}	0 *	{0,1}	0	3	3 *
{4}	1	{0}	2	0	2
{1}	0	{1}	0	3	3
{5}	1 *	{1}	2	3	5 *

j=2

M_1	k_2	M_2	$f_1(m_1)-f_2(m_2)$	$f^*_2(m_2)$	$f_1(m_1)$
{0,...,5}	0	{0,...,4}	0	5	5
{0,...,6}	1 *	{0,...,4}	4	3	7 *
{5}	0	{5}	0	5	5
{7}	1 *	{5}	4	5	9 *

j=1

M_0	k_1	M_1	$f_0(m_0)-f_1(m_1)$	$f^*_1(m_1)$	$f_0(m_0)$
{0,...,6}	0	{0,...,6}	0	7	9
{0,...,9}	1 *	{0,...,6}	3	7	10 *
{7}	0 *	{7}	0	9	9 *

Figure 1.9: Recursive Dynamic Programming Solution in a Knapsack Problem

Within the Horowitz and Sahni algorithm the different items are ordered according to decreasing profits per unit size so that:

$$\frac{a_1}{h_1} \geq \frac{a_2}{h_2} \geq ... \geq \frac{a_n}{h_n}. \tag{1.31}$$

For the company valuation algorithm, sorting by decreasing profits is not feasible in the given way and will be demonstrated later on. Horowitz and Sahni have developed a basis principle for more efficient computation of the upper bounds which includes ordering the items according to decreasing profits per unit. The procedure starts with a so-called forward move by inserting the largest possible set of items which do not exceed the capacity w. In a next step a backtracking move follows by removing the last inserted item from the current solution. This iterative procedure will be initiated as follows. Whenever no further forward move without exceeding capacity w can be performed, the upper bound U_1 corresponding to the current optimal policy $\overline{k}_j$ for item j is computed and compared with the best policy so far k_j^*. This procedure will be performed to check if a further forward move could lead to a better solution which means an improved solution value $f_j\left(\overline{w}\right)$. In that case a forward move is performed otherwise a backtracking move follows. After having considered the last item the solution procedure is finished and the optimal solution has been kept in each iteration. The algorithm stops when no further backtracking moves can be performed. In the following the notation of the solution algorithm is summarised (Martello and Toth 1990):

$$\begin{aligned}
\overline{k}_j &= \text{current best policy;} \\
f_j\left(\overline{w}\right) &= \text{current value } \left(=\sum_{j=1}^{n} a_j \overline{k}_j\right); \\
\overline{w} &= \text{utilised capacity in the knapsack } \left(=\sum^{n} h_j \overline{k}_j\right); \\
k_j^* &= \text{best policy computed so far.} \\
f_j^*\left(\overline{w}\right) &= \text{value of the best solution so far } \left(=\sum_{j=1}^{n} a_j k_j\right).
\end{aligned} \tag{1.32}$$

The company valuation algorithm that is going to be developed will have the form of an unbounded knapsack problem. This means an unlimited number of items of each asset is available and can be used to at least replicate the company cash flows. For solving this algorithm dynamic programming has been applied which allows e.g. a recursive solution of a complex problem. Dynamic programming offers a solution procedure to reach an optimal solution for a complex problem by solving a number of consecutive subproblems. The main characteristics lies within the sequential procedure through several steps. In any step only the decision within the subproblem will be considered.

In the following the coding of the algorithm should be explained by an example in some detail applying dynamic programming for an adjusted unbounded knapsack problem:

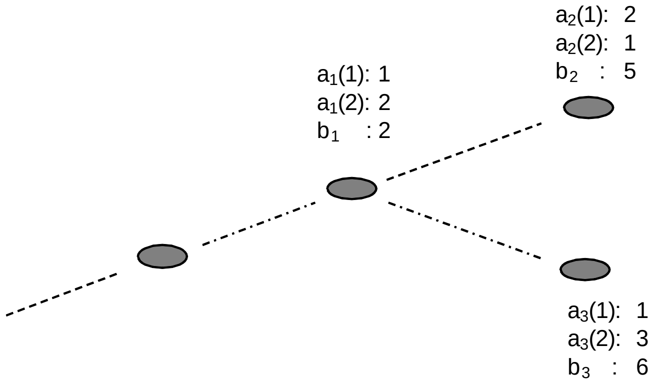


Figure 1.10: Replication Example

The iteration starts by having no w on hand which is defined as the value or the capacity the knapsack can hold. In this situation no asset or item a_1 can be bought or carried. This implies that the function to calculate the deficit in the states equals the overall deficit, which should be completely recovered e.g. $f_2\left(0\right) = 5 - 0$. The maximum of the deficit in both states in the given example is depicted in the last column of the table. By increasing the capacity w of the knapsack by one unit after the other more and more items a_1 with a weight or price of 1 in the initiating node can be carried or bought. For e.g. $w = 3$ the calculation is as follows:

$$max\left(f_2\left(3\right), f_3\left(3\right)\right) = \max\left(5 - 3 \times 2,\ 6 - 3 \times 1\right) = 3.$$

w will be increased to the point that its portfolios consisting only out of a_1 at least replicate the company cash flows in the vertices v_2 and v_3 .

$$max\left(f_2\left(w\right), f_3\left(w\right)\right) = 0 \tag{1.33}$$

With that $w = U_1$. U is defined as an upper bound of the problem, i.e. U_1 is the first investment which at least replicates the company cash flows within this state space. This result is expressed by $f(w) \leq 0$. The number of assets a_1 that can be bought for U_1 will be stored.

w	$a(1)$	$f_2(w)$	$f_3(w)$	$max(f_2(w), f_3(w))$
0	0	5	6	6
1	1	3	5	5
2	2	1	4	4
3	3	-1	3	3
4	4	-3	2	2
5	5	-5	1	1
6	6	-7	0	**0**

Table 1.2: First Step Replication Process

In the second step the iteration starts from the now determined upper bound U_1. The goal of the company valuation algorithm will be to minimise the value, weight or costs w but to reach at least the profit of b_2 as well as b_3 not knowing which root to follow because of being located in state i. Now so many assets a_1 will be sold that at least one asset a_2 can be bought with the remaining wealth $U_1 - ((n_1 - 1) \times a_1) =\triangle w$. The deficit that remains if only one unit of a_2 would be on hand is calculated in column four in the same way as in the first step. This value generation of $f_2\,(6,1) = 1$ will be subtracted from the deficit, which has been calculated for $f_2\,(6-2) = -3$ in the first step of the replication process.

w	$a(1)$	$a(2)$	$f_2(w, a(2))$	$f_2(w)$	$f_2(w, a(2))$	$f_3(w)$	$max(f_2(w), f_3(w))$
0	0	0	0	5	6	6	6
1	0	0	0	5	6	6	6
2	0	1	1	4	3	3	4
3	1	1	1	2	3	2	2
4	2	1	1	0	3	1	1
5	3	1	1	-2	3	0	**0**
6	4	1	1	-4	3	-1	-1

Table 1.3: Second Step Replication Process

The maximum possible deficit at both nodes will be calculated again. In case the replication is now more efficient, a smaller residual value for the cash flow replication in any node will be obtained. The additional investment in a_2 will be calculated. In case $U_2 < U_1$, U_2 will substitute U_1. The related portfolio combination will be stored accordingly. The iteration stops when no item a_2 can be bought anymore for the remaining wealth. Within this result, it can be seen, that a reduction in wealth w can be reached by having replaced 2 items a_1 by 1 item a_2. The minimum costs to replicate the profit in the next nodes has been reduced now to $w = 5$.

w	$a(1)$	$a(2)$	$f_2(w, a(2))$	$f_2(w)$	$f_2(w, a(2))$	$f_3(w)$	$max(f_2(w), f_3(w))$
0	0	0	0	5	6	6	6
1	0	0	0	5	6	6	6
2	0	0	0	5	6	6	6
3	0	0	0	5	6	6	6
4	0	2	2	2	0	-3	2
5	1	2	2	0	0	-4	**0**

Table 1.4: Third Step Replication Process

In a third step , the number of items a_2 will be increased furthermore. In the case a better solution will be found the new solution including its bit pattern of number of each item being bought will be stored. The iteration will run through $a_1, a_2 ... a_n$. As it can be seen, sorting according to its wealth contribution is not feasible for this way of implementation because the wealth contribution is dependent on the company cash flows b_i as well as the prices a_i of the assets in the outvertices.

1.4 Real Option Valuation

The contingent claim valuation assumes the existence of a large set of risky assets so that the stochastic component of the investment project under consideration can exactly be replicated. Through appropriate long and short positions, a riskless portfolio can be constructed consisting of the risky project and traded assets which track the project's uncertainty. In equilibrium with no-arbitrage opportunities, this portfolio must earn the risk-free rate of interest which allows to determine the value of the investment project. The no-arbitrage assumption avoids the necessity of determining the appropriate risk-adjusted discount rate. Schwartz states that the contingent claim approach is the only correct approach to valuing real options: "If what the decision-maker is trying to get is the market value of the project, then, obviously, a subjective discount rate will not do the job" (Schwartz, Trigeorgis 2001). According to the markov property, moving forward from any given state in the tree only depends on the current underlying asset value and not on the history leading up to that state. The geometric brownian motion is one stochastic process holding the markov property. According to the Law of One Price, the state cash flows for these recombined underlying asset price states can be replicated by continuous trading in the underlying asset and the states risk-free cash without additional cash investments. The sequence of portfolios formed by the process are called "replicating portfolios", which are traded according to a "dynamic replication strategy". The relevant conditions, used by Black, Scholes and Merton, are that: First, the term structure of prices to risk-free cash must be known with certainty; second, the dividends from the equity and the uncertainty in its price must depend only on time and that price; and third, the asset price uncertainty must be compact thus continuous replication is feasible.

Assuming that these conditions are met, the five relevant parameters for calculation of contingent claims are:

(1) the strike price K;

(2) the stock price S;

(3) the risk-free interest rate r;

(4) the amount and timing of any dividend to be paid before exercise T; and

(5) the uncertainty (or “volatility”) in the stock price σ.

Within this framework of contingency claim valuation intensified research has been undertaken since the eighties. The initial applications and their extensions have influenced much of the early work in the market based valuation approach for real assets. Plenty research was undertaken by drawing analogies to the Black, Scholes and Merton analysis of equity options. Well known examples of this work include: The analysis of the option to delay initiation of a project (McDonald and Siegel 1986); an empirical analysis of the value of offshore oil field leases (Paddock, Siegel and Smith, 1988) and the analysis of the option for premature abandonment of a project from (McDonald and Siegel, 1985). All of these approaches use the claim to the operating cash flows of the underlying project as the asset analogous to the corporate equity in the stock option analysis. Some of them analyse “timing options” by determining a boundary, possibly dependent on time, between two sets in the underlying asset price. In one of these sets, on one side of the boundary, the managers should act to change the state of the project, and in the other set, on the other side of the boundary, they would wait if allowed to do so, but at least not change the state of the project as they are forced to do on the other side of the boundary. There are major problems with this type of approaches. The claim to the operating cash flows of a real project has complicated dynamics that are difficult to model transparently over time scales. In particular the assumption that the analogy of the flow of stock dividends in the stock option analysis is the operating cash flow itself, is limited. The Black, Scholes and Merton condition with the cash flow at any time depending only on time and the concurrent value of the claim to the whole stream of cash flows is very restrictive. For example, if the cash flow is proportional to the value (Majd and Pindyck, 1987), it cannot be negative. The value of the claim to the cash flow stream should be modelled in terms of the cash flows themselves, if possible, and not the other way around. Furthermore the condition that the uncertainty in the value of the operating cash flows depends only on that value is again very restrictive. After this early work, there was a shift to scenario trees where the scenario tree variables are the determinants of the project cash flows. As a result, the scenarios in the scenario trees are more like the ones that would be used in a standard DCF analysis. This type of scenario trees also provides a much broader modelling environment. The shift had already begun within the valuation by components framework of (Lessard 1979). However, the key step was the use of the copper price as a scenario tree variable by (Brennan and Schwartz 1985) in their real option valuation analysis of copper mines that can be temporarily closed. They use the copper price in almost the same way that (Black, Scholes 1973) and (Merton 1973b) use the underlying asset price in their analysis. Like the underlying asset price in the stock option analysis, it is a variable that evolves continuously. In their model, a complete understanding of the future of the copper market moving forward from any given state in the tree only depends on the current copper price then, and not on the history leading up to that state. By this means, they build on the work of (Cox, Ingersoll and Ross 1985-a) who use the short-term interest rate as a scenario tree variable in a valuation of long-term bonds and bond options. Brennan and Schwartz also use the remaining amount of copper in the mine, and the state of the mine (open or closed) as scenario tree variables. The mine cash flows and the actions of managers depend on these variables as well as the copper price. Although they do not have any independent uncertainty in the Brennan-Schwartz model, these variables do depend on the prior history of copper prices which would have influenced prior opening and closing decisions. Instead of using the whole price history to index states on the tree, Brennan and Schwartz use these "auxiliary" scenario tree variables to represent those aspects of the history that have an influence on future cash flows. They did not include any geological or technological uncertainty and no independent cost uncertainty. They only use the output price. In their model the relevant dynamics of that output price is only influenced by itself. No independent uncertainties of interest rates, dividends or other prices influence the future cash flows. They chose this approach to keep the dynamic programming search for optimal opening and closing policies feasible. However, they allow cash flows to occur continuously and allow decisions about opening or closing the mine to take place at any time. In almost all discounted cash flow (DCF) analyses, cash flows are modelled to occur at discrete (usually annual) time intervals. In most decision tree analyses, decisions and payoffs are also restricted to occur at discrete times. In reality, cash flows occur on an almost continuous basis with some large flows at discrete intervals or random discrete times. Most regular planning decisions occur within the context of a periodic planning cycle, while some extraordinary decisions occur at random times. The real option valuation can be simplified significantly if cash flows and decisions are modelled to occur at discrete intervals as in typical DCF and decision tree analysis (DTA). If these discrete intervals are considered, only a finite number of the characteristics of each scenario on the scenario tree comes into play, as opposed to a continuously infinite set. There are other simplifications that occur in the analysis of timing options and their effects. The errors caused by restricting on discrete cash flows and decisions can be tested by examining the changes in the results that occur as the time interval is decreased. According to the undertaken research, the size of the errors involved are acceptable related to other uncertainties for valuing private companies. The scenario tree variables thus far involve priced risks. Unpriced risks (i.e., risks for which the price of risk is zero) can also be important. Recall that these include local, project-specific risks that do not influence the risk discounting in financial market prices because their effect is not noticeable in the determinants of the welfare of the marginal well-diversified investor. Moreover, the timing of their resolution is frequently influenced by management action which makes them "endogenous" uncertainties.

In the real world, most decisions consist of several consecutive decisions, where the results of the decision in the first stage triggers the possible decisions in the next stage. To solve such consecutive decisions, decision trees are very often used. The frequently encountered technique is the decision tree analysis. The approach has several shortcomings. Two major ones are: The decision tree analysis takes the probabilities into account which can only be estimated, but are generally not known exactly; second, the decision tree is planned from time $t = 0$ until T without taking into account any external influences, what means that the approach is not dynamic.

Nevertheless, by applying the technique of dynamic programming, at least some of the major shortcomings can be resolved. At each state in the scenario tree, the computational aspects of a dynamic programme are: First, the determination of the value of the asset for each action available at that state, where the value is the sum of: a) the cash flow resulting from that action in that state; and b) the value stemming from each follow-on state, in case those is any for that action; and second, the determination of the most valuable action. The scenario trees used thus far in most real option valuation applications are analogues of the trees used by (Black, Scholes 1973) and (Merton 1973b) in their original applications. They partly represent the resolution of the uncertainty in continuous time in a finite set of continuous time series variables involving priced risks. The risk-adjusted movements in these variables are self-contained and follow a markov process. The uncertainty in the movements allows for replication which has been already demonstrated. As a result, at each instant in continuous time, there is a single-dimensional continuum of recombined states, labelled by the variable involved. This variable will be called the "underlying" variable, and their states the "underlying states" of the analysis. For most applications reported so far, the sum over states is done by numerical integration. Accuracy, stability, efficiency and generality are important considerations in choosing a computational method for this purpose. The Cox, Ross and Rubinstein tree (Cox, Ross and Rubinstein, 1979) is a recombining tree which performs state pricing calculations. Cox, Ross and Rubinstein use the state prices to calculate the value of financial assets. The method is based on a discrete approximation to the underlying continuous scenario tree. The focus should be on situations with one underlying variable, where the risk-adjusted process for movement of this variable through time is a single-factor geometric brownian motion. These are the dynamics of the underlying variable in the original work of Black and Scholes as well as Merton which have been applied by Brennan and Schwartz in their price model (Brennan and Schwartz 1985). The time periods in the discrete approximation are small. The continuous uncertainty at any state is represented in each of these small time periods by two branches. As a result, the methods are most often called "binomial tree" methods. Each branch is designed in the way that the magnitude of the underlying variable in the state to which the branch leads is proportional to the magnitude of the underlying variable in the state from which it originates. If the constants of proportionality are time independent, the branching can be designed to recombine. The same number of up and down movements results in the underlying variable having the same magnitude, and thus being in the same recombined state, independent of the order in which the movements occur. Cox, Ross and Rubinstein show how to calculate the risk-adjusted probabilities on this approximate tree which leads to the limit of small time steps to a convergence of continuous time risk-adjusted probabilities. If the asset cash flow in any state only depends on the simultaneous underlying variable, the dynamic programming calculations can then be performed on this discrete approximation using the Cox, Ross and Rubinstein risk-adjusted probabilities. Trigeorgis describes the stability properties of this process and a logarithmic version of it (Trigeorgis 1996). To be stable, both require limits on the size of the time step. In the following a classification of the research on real options and their main contributors according to (Micalizzi and Trigeorgis 1999) is given:

Real Options	Description	Specific applications	References
Deferment or temporary suspension	Option to postpone the investment outlay or to temporarily suspend production while preserving the technical feasibility of the project	Natural resources and oil, real estate and vacant land, launch of new products	Mc Donald & Siegel (1986), Paddock-Siegel-Smith (1988), Ingersoll-Ross (1992), Triegeorgis (1990)
Expansion	Option to expand the scale of the project by investing an additional amount of capital as exercise price	Launch of new products or new versions of the base products, targeting new market niches, entering new geographical markets, strategic alliances	Kester (1984), Mc Donald & Siegel (1985), Trigeorgis (1988) Christofides A, (2004)
Switching	Option to switch among alternative operating modes according to the relative fluctuation of some reference variables	Research and development, geographical diversification, global cost reduction strategy	Kensinger (1988), Kulatilaka (1988), Kulatilaka and Trigeorgis (1993) Margrabe (1978)
Contraction and/or abandonment	Option to reduce the scale of the project, or to abandon it to realise ist scrap value	Altering the R&D process, withdrawing from a market niche, reducing the capital invested in a business unit	Myers & Majd (1990)

Table 1.5: Real Options Classification and Major Authors

Chapter 2

The Company Valuation Algorithm

In this section, the general company valuation algorithm will be developed. The valuation framework underlying the algorithm is based on the work in (Christofides A. 2004) and the general derivative pricing paradigm of (Black, Scholes 1973), (Merton 1973b), (Ross, 1976), (Cox, Ingersoll and Ross 1985-a) and others.

2.1 The Basic Algorithm

The starting point is the assumption that the value of a company is a function of its future cash flows. By far the simplest model in this framework is the discounted cash flow model (DCF) where only expected cash flows are considered (i.e. the uncertainty in the future realised cash flows is ignored) and the valuation function is the discounted value of this expected cash flow using today's (deterministic) yield curve as the discounting function. The DCF method is often used because of its simplicity. It is quite accurate for stable cash flows (and staid) companies. The difficulty to choose the right discount factor arises quite often, which is normally derived by the capital asset pricing model (CAPM). It can be summarised from empirical evidence that the CAPM beta does not completely explain the cross section of expected asset returns[5] cited from (Claessens Dasgupta and Glen 1995). This evidence suggests that at least one further factor may be required to characterise the behaviour of expected returns and naturally leads to the consideration of multifactor pricing models. Additionally, theoretical arguments suggest that more than one factor is re-

[5] After much testing, numerous empirical anomalies about the CAPM cast doubt on the central hypothesis of that theory: that on a cross-sectional basis a positive relationship exists between asset returns and assets' relative riskiness as measured by their β .

quired, since the CAPM will only apply under strong assumptions period by period (Campbell, Lo and MacKinlay 1997). In the forthcoming of the work, a model based on the idea of contingent claim valuation should be developed which by-passes the above mentioned difficulties and shortcomings of the discount factors derived by the CAPM and applied for the DCF valuation.

2.1.1 Derivation of the Algorithm

The required valuation procedure for company cash flows as in option pricing is independent of the real transition probabilities and depends only on risk-neutral probabilities. Assuming company cash flows evolving over time with an increasing magnitude of uncertainty can be captured in a n-dimensional tree; in the simplest case in a non-recombining binomial tree. Each vertex v of the transition graph has a corresponding cash flow b caused by the sales (or other company activities) at that vertex. These possible future cash flows can at least be "replicated", without the existence of the company by investing in portfolios of traded non-company assets $Y_i\left(a\right)$ at the beginning, and trading these portfolios as the future evolves. In general, starting at the root vertex v_0 let V_i be the corresponding vertex set that can be reached from vertex v_i. Furthermore it is defined that $v_j \in V_i$.

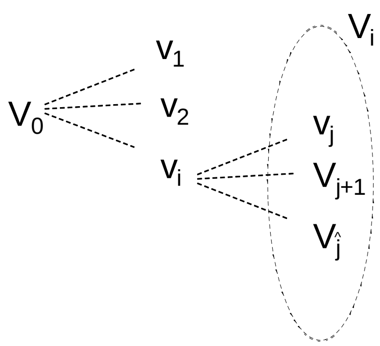


Figure 2.1: Company Valuation Setting

At every vertex v_i that occurs, an amount β_i.will be "paid". The values of β_i.are not given, but it should be assumed that β_i "replicates" the computed company cash flow b_i at that vertex. The remaining wealth $(w_i - \beta_i)$ will be invested optimally in portfolios of assets $Y_i(w_i - \beta_i) = \{y_i(1), y_i(2), ...\}$ in the way that in the vertex set V_i the wealth of the portfolio will be at least equal to the company cash flows b_j. Therefore: $w_j(y_i) \geq b_j$. The minimum wealth w_i that has to be invested at vertex v_0 in portfolios Y_0 after paying out $\beta_0 = b_0$ to replicate the current company cash flow and all potential future cash flows of the company b_i in the vertex set V_i is equal to the current value of the company. Therefore, in each state, the portfolio y_i is traded optimally so that β_i.will be "paid" to "replicate" the computed company cash flow b_i at that vertex.

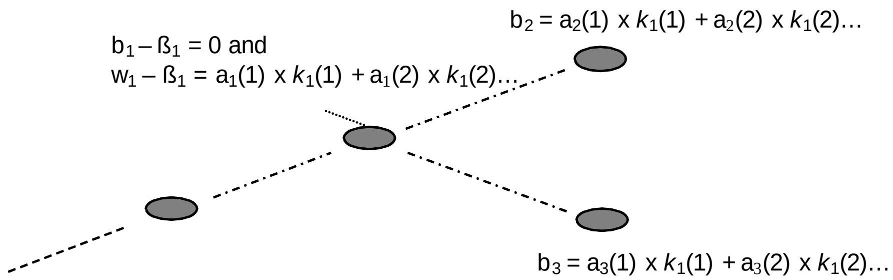


Figure 2.2: Company Valuation Algorithm

The solution procedure that is shown graphically can be transferred in a matrix form. The asset price is denominated by a and the number of assets being bought within one chosen portfolio y is denominated by k. Both variables are labelled by $n = 1, ..., \widehat{n}$ for the different assets used within the replication portfolio.

In comparison to building the self-financing portfolio within the Cox, Ross and Rubinstein binomial model which could be bought that exactly "duplicates" the call C here the situation is slightly different. As stated earlier at least two aspects are missing for a perfect "duplication" of the company cash flows b_i; firstly, the market is incomplete, and secondly, endogenous company risks which are, e.g. presented by extreme company cash flows, so-called outlier cash flows. Therefore, it is only possible to replicate as well as possible by searching for the least amount of w_i which finally minimally "replicates" the company cash flows. This would mean, the optimal solution of the replication is an upper boundary for the company value, but under some considerations almost equal to the perfect "duplication". The goal can be achieved by trying to minimise any excess replication. In particular, $\max[b_i - \beta_i, 0]$, the "shortfall" at vertex v_i should be 0 at every vertex $v_{i.}$, which results in a replication of the company cash flow b_i. In the vertices v_j, the wealth of the portfolio $w_j\left(y_i\right)$ should be at least as large as the company cash flow b_j and, therefore, $\max[b_j - w_j\left(y_i\right), 0]$, the "shortfall" at vertex v_j should be 0 as well.

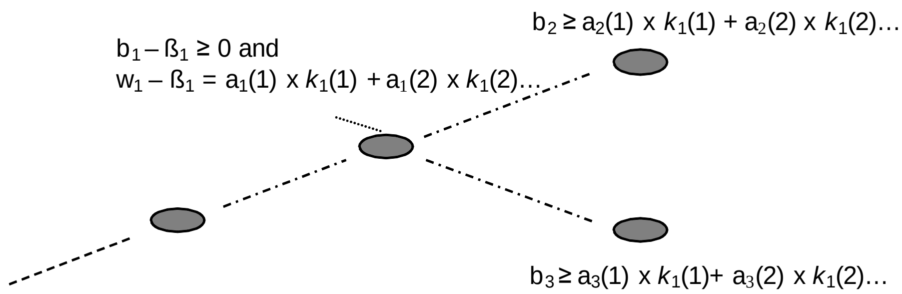


Figure 2.3: Company Valuation Algorithm Adjusted

Combining both requirements for vertex v_i as well as for the vertices v_j leads to a function $f_i\left(w\right)$ which minimises the shortfall achievable at all vertices $v_j \in V_i$ starting with wealth w at vertex v_i, before rebalancing. The following algorithm by (Christofides A. 2004) for one time period will be derived,

$$f_i\left(w\right) = \min_{0 \leq \beta_i \leq w} \left[\max\left(\max\left(b_i - \beta_i, 0\right), \min_{y_i \epsilon Y_i\left(w - \beta_i\right)} \left(\max_{v_j \epsilon V_i} \left[f_j\left(w_j\left(y_i\right)\right)\right]\right)\right)\right] \tag{2.1}$$

where $Y_i(w_i - \beta_i)$ is written for the set of all portfolios that can be bought for an amount of money $w_i - \beta_i$ at vertex v_i, and $w_j(y)$ is the value of one specific portfolio y at vertex v_j. The initialisation of the above recursion is given by

$$f_j(w) = \max(b_j - w, 0), \qquad \forall\, v_j \epsilon V_i. \tag{2.2}$$

An overview of equation (2.1) that provides some insight into the recursion is as follows:

(a) Starting with wealth w at vertex v_i implies that we have the choice of "allocating" an amount of money β_i (where $0 \leq \beta_i \leq w$) to the process of "replicating" the payoff of the company at that vertex. The measure of how good this replication is, is given by $(b_i - \beta_i, 0)$ i.e. if $\beta_i \geq b_i$ the company payoff at v_i is fully replicated , but if $\beta_i < b_i$ then there is a shortfall in replication of the company payoff at vertex v_i.

(b) The wealth remaining at vertex v_i after the "allocation" of β_i is $w - \beta_i$. The portfolios that can be bought at v_i with this amount of money are given by the set $Y_i(w - \beta_i)$. Suppose that portfolio $y_i \in Y_i(w - b_i)$ is bought at v_i. The value of this portfolio at state $v_j \in V_i$ (if state v_j occurs after v_i) is $w_j(y_i)$ and obviously depends on the asset prices at v_j. The minimum shortfall in the replication of the company payoffs at all vertices v_k following vertex v_j (i.e. all vertices in the subtree rooted at v_j) is - by definition - $f_j(w_j(y_i))$. This shortfall clearly depends on the choice of portfolio y_i that was chosen at v_i. For a given y_i at v_i, the best best replication possible is $\max\limits_{v_j \in V_i}[f_j(w_j(y_i))]$ and hence the best replication at all vertices following vertex v_i is

$$\min_{y_i \epsilon Y_i(w - \beta_i)} \left(\max_{v_j \epsilon V_i} [f_j(w_j(y_i))] \right) \tag{2.3}$$

as shown by the last term of equation (2.1).

(c) The only arbitrary decision remaining is the amount β_i allocated at v_i and this determines if the vertex with the worst replication shortfall is v_i or another vertex following v_i hence for a given β_i, the worst shortfall is

$$\max\left(\max\left(b_i-\beta_i,0\right),\min_{y_i \epsilon Y_i(w-\beta_i)}\left(\max_{v_j \epsilon V_i}\left[f_j\left(w_j\left(y_i\right)\right)\right]\right)\right). \tag{2.4}$$

Hence $f_i(w)$ is the minimum value above for all possible choices $0 \leq \beta_i \leq w$, which is the recursion (2.1). Note that, in fact, β_i is never optimal for $\beta_i > b_i$ and we can write $0 \leq \beta_i \leq \min(w, b_i)$ instead of $0 \leq \beta_i \leq w$.

For solving the state space graph, all vertices will be solved recursively starting in T. The calculated result of the company value of the one time period setting will then substitute b_i at vertex v_i in $T-1$. It has to be pointed out that the market represented by the state transition graph is almost certainly incomplete. Would the market be complete, the valuations had been exact because of the no-arbitrage condition (i.e. there should be no risk-free profits buying/selling the company and buying/selling other non company assets). Exact replication would then also be possible. With the market being incomplete, exact replication is impossible and the above expressions are not more than upper bounds of the company valuation. Due to the degree of control over the dimensionality of the state space graph, the out-degree of the vertices can be kept small which avoids high levels of incompleteness and produces bounds very close to the company value.

2.1.2 First Example of the Basic Company Valuation Algorithm

In the following, a numerical example will be demonstrated using only integers as input factors in the basic company valuation algorithm. The company cash flows evolving over time with an increasing magnitude of uncertainty will be captured in a non-recombining binomial tree in the simplest example. Each vertex v_i of the transition graph has a corresponding company cash flow b_i. These possible future cash flows will be "replicated" by a set of portfolios Y_i investing in traded non-company assets a_i at the beginning and trading these portfolios as the future evolves. The minimum cost $w_{\min}$ which enables to replicate the company cash flows b_i is equal to the value of the company as shown in the following example for a single time period.

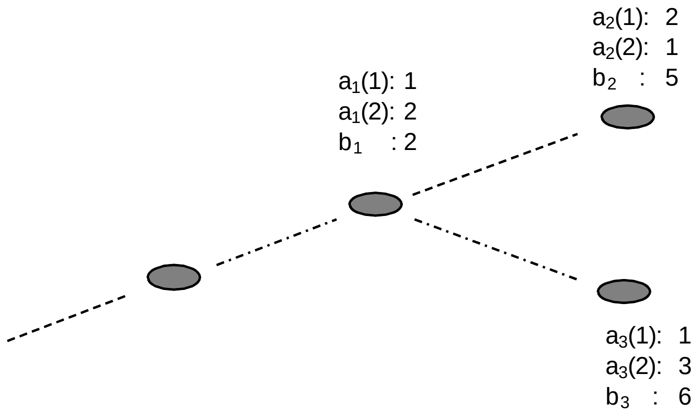


Figure 2.4: Example Valuation One Period

It is known that the least amount of wealth w_i to replicate the company cash flows has to fulfill the boundary condition $b_j - w_j \geq 0$. Therefore, wealth w_j at time T has to be at least the amount b_j at time T. $w_1 = 1$ will be chosen as a starting value in $v_{1.}$ This wealth w_1 will be split in an investment in replication portfolios $Y_1\,(w_1 - \beta_1)$ and a payout β_1. The value of β_1 "replicates" the computed company cash flow b_1. Therefore, $w_1 - \beta_1 = 1 - 2 = -1$ not even replicates the company cash flow b_1. The remaining wealth, if there is any, could be invested in portfolios $Y_1\,(w_1 - \beta_1)$. As an educated guess, $w_1 = 7$ will be chosen. In a first step, the company cash flow b_1 will be replicated by a cash-outflow β_1 which leads to $\max[b_1 - \beta_1, 0] =$

$\max[2-2,0]=0$. Now the remaining wealth $w_1-\beta_1=7-2=5$ will be invested in portfolios $Y_1(w_1-\beta_1)$ with the following asset prices $a_1(1), a_1(2)$ in the chosen example. This yields the following portfolio combinations considering only investment in integers of an asset, i.e. $w_1-\beta_1=5=y_1=\binom{0}{2.5}=k_1(1)\times a_1(1)+k_1(2)\times a_1(2)=0\times 1+2.5\times 2$. All other portfolio combinations are listed as followed:

$$Y_1(w_1-\beta_1)=5=\left\{\begin{pmatrix}0\\2.5\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1.5\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}4\\0.5\end{pmatrix},\begin{pmatrix}5\\0\end{pmatrix}\right\} \tag{2.5}$$

The first portfolio combination $y_i=\binom{0}{2.5}$ leads to the following result at vertex v_2:

$$\begin{aligned}(w_2(y_1)) &= (0\times 2+2.5\times 1)=2.5 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2-w_2,0) \\ &= \max(5-2.5,0) \\ &= 2.5\end{aligned} \tag{2.6}$$

and at v_3:

$$\begin{aligned}(w_3(y_1)) &= (0\times 1+2.5\times 3)=7.5 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3-w_3,0) \\ &= \max(6-7.5,0) \\ &= 0\end{aligned} \tag{2.7}$$

therefore:

$$\begin{aligned}f_1(w_1(y_1)) &= \max[f_2(w_2(y_1)), f_3(w_3(y_1))] \\ &= \max[2.5,0] \\ &= 2.5\end{aligned} \tag{2.8}$$

This calculation will be conducted with all six portfolio combinations listed above. The second portfolio combination $y_i = \binom{1}{2}$ leads to the following result at vertex v_2:

$$\begin{aligned}(w_2(y_1)) &= (1 \times 2 + 2 \times 1) = 3 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(5 - 3) \\ &= 2\end{aligned} \tag{2.9}$$

and at v_3 :

$$\begin{aligned}(w_3(y_1)) &= (1 \times 1 + 2 \times 3) = 7 \text{ and} \\ f_3(w_3(y1)) &= \max(b_3 - w_3, 0) \\ &= \max(6 - 7) \\ &= 0\end{aligned} \tag{2.10}$$

therefore:

$$\begin{aligned}f_1(w_1(y_1)) &= \max[f_2(w_2(y_1)), f_3(w_3(y_1))] \\ &= \max[2, 0] \\ &= 2\end{aligned} \tag{2.11}$$

The third portfolio combination $y_1 = \binom{2}{1.5}$ leads to the following result at vertex v_2:

$$\begin{aligned}(w_2(y_1)) &= (2 \times 2 + 1.5 \times 1) = 5.5 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(5 - 5.5, 0) \\ &= 0\end{aligned} \tag{2.12}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (2 \times 1 + 1.5 \times 3) = 6.5 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(6 - 6.5, 0) \\ &= 0 \end{aligned} \tag{2.13}$$

therefore:

$$\begin{aligned} f_1(w_1(y_1)) &= \max[f_2(w_2(y_1)), f_3(w_3(y_1))] \\ &= \max[0, 0] \\ &= 0 \end{aligned} \tag{2.14}$$

The portfolio combination $y_1 = \binom{3}{1}$ leads to the following result at vertex v_2:

$$\begin{aligned} (w_2(y_1)) &= (3 \times 2 + 1 \times 1) = 7 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(5 - 7, 0) \\ &= 0 \end{aligned} \tag{2.15}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (3 \times 1 + 1 \times 3) = 6 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(6 - 6, 0) \\ &= 0 \end{aligned} \tag{2.16}$$

therefore:

$$\begin{aligned} f_1\left(w_{V_1}\left(y_1\right)\right) &= \max\left[f_2\left(w_2\left(y_1\right)\right), f_3\left(w_3\left(y_1\right)\right)\right] \\ &= \max\left[0, 0\right] \\ &= 0 \end{aligned} \tag{2.17}$$

The fifth portfolio combination $y_1 = \binom{4}{0.5}$ leads to the following result at vertex v_2:

$$\begin{aligned} \left(w_2\left(y_1\right)\right) &= (4 \times 2 + 0.5 \times 1) = 8.5 \text{ and} \\ f_2\left(w_2\left(y_1\right)\right) &= \max(b_2 - w_2, 0) \\ &= \max(5 - 8.5, 0) \\ &= 0 \end{aligned} \tag{2.18}$$

and at v_3:

$$\begin{aligned} \left(w_3\left(y_1\right)\right) &= (4 \times 1 + 0.5 \times 3) = 5.5 \text{ and} \\ f_3\left(w_3\left(y_1\right)\right) &= \max(b_3 - w_3, 0) \\ &= \max\left(6 - 5.5, 0\right) \\ &= 0.5 \end{aligned} \tag{2.19}$$

therefore:

$$\begin{aligned} f_1\left(w_1\left(y_1\right)\right) &= \max\left[f_2\left(w_2\left(y_1\right)\right), f_3\left(w_3\left(y_1\right)\right)\right] \\ &= \max\left[0, 0.5\right] \\ &= 0.5 \end{aligned} \tag{2.20}$$

The sixth portfolio combination $y_1 = \binom{5}{0}$ leads to the following result at vertex v_2:

$$\begin{aligned} (w_2(y_1)) &= (5 \times 2 + 0 \times 1) = 10 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(5 - 10, 0) \\ &= 0 \end{aligned} \tag{2.21}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (5 \times 1 + 0 \times 3) = 5 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(6 - 5, 0) \\ &= 1 \end{aligned} \tag{2.22}$$

therefore:

$$\begin{aligned} f_1(w_1(y_1)) &= \max\left[f_2(w_2(y_1)), f_3(w_3(y_1))\right] \\ &= \max[0, 1] \\ &= 1 \end{aligned} \tag{2.23}$$

For an invested wealth of $w_1 = 7$, all company cash flows in the vertices v_2 and v_3, for the up as well as the down movement can be replicated by investing in certain portfolio combinations. Therefore, the minimum shortfall that can be achieved with at least one portfolio combination with an investment $w_1 - \beta_1 = 7 - 2 = 5$ is $\min\limits_{y_i \epsilon Y_i(w-\beta_i)} \left(\max\limits_{v_j \epsilon V_i} \left[f_i(w_j(Y_i))\right] \right) = 0$. The results for the maximum shortfall $f_i(w_j(y_i))$ for all investigated portfolio combination with an invested wealth of

$w_i - \beta_i = 5$ are:

$$y_i \epsilon Y\left(5\right) = \left\{2.5, 2, 0, 0, 0.5, 1\right\} \tag{2.24}$$

This shows that two portfolio combinations $y_1 = \binom{2}{1.5}, \binom{3}{1}$ with an invested wealth $w_1 = 7$ are sufficient to replicate the company cash flows b in the vertices v_2 and v_3. Since two portfolio combinations are sufficient to replicate the company cash flows b, these are the only optimal and, therefore, the upper bound of the company value in the case of considering integers. With an interpolation, it is possible to approximate the optimal solution for the company value w_1 which will be in the range between six and seven.

2.1.3 Comparison to Other Approaches

In a first step the company valuation algorithm should be validated by some examples calculated within the DCF as well as the partial differential equation (PDE) framework. Due to the fact that the established model is based on the Law of One Price, the discount rate for the comparable DCF calculations will always be performed with the risk-free interest rate r unless anything else is specifically mentioned.

Considering a simple generic example:

- Opportunity to invest in a project with a duration of $T = 10$ years
- In $t = 0$ the project will generate a positive cash flow of $b_0 = 100$
- After one year the project will generate either a subsequent cash flow of $b_2 = 143$ if the market moves up or $b_3 = 78$ if the market moves down
- The project cash flow b follows a multiplicative binomial process

- The probability that the project moves up is $p = 0.53$ and that the project moves down with $1 - p = 0.47$
- The risk-free rate is $r = 6\%$
- The volatility is $\sigma = 0.3$
- Assume at least one traded asset exists which has the same risk characteristics as the given project

Variable	Value
b_0	100.00
σ	0.30
r	0.06
T	10
$e^{(-r\times\Delta t)}$	0.94
Δt	1.00
u	1.35
d	0.74
p	0.53
$1-p$	0.47

Table 2.1: Input Parameters and Preliminary Calculations

First proposition: No WACC is needed for valuing this particular uncertain cash flow streams. If the cash flows are modelled to occur at periodic intervals as in the given case, the valuation exactly parallels a scenario DCF calculation (Salahor 1998) and can be resolved in the following way. In these situations, there is a single sum over states of the product of the state cash flow and state price. The state price determination is equivalent to the computation of discount factors in the DCF analysis and the probabilities of the scenarios (Bradley 1998). Therefore, a state pricing analysis is not more computationally intensive than the typical Net Present Value analysis anymore, and can be managed in a similar way. The derived cash flow stream throughout the ten year time of the project is given in the following table:

Year 0	1	2	3	4	5	6	7	8	9
100.00	134.99	182.21	245.96	332.01	448.17	604.96	816.62	1102.32	1487.97
	74.08	100.00	134.99	182.21	245.96	332.01	448.17	604.96	816.62
		54.88	74.08	100.00	134.99	182.21	245.96	332.01	448.17
			40.66	54.88	74.08	100.00	134.99	182.21	245.96
				30.12	40.66	54.88	74.08	100.00	134.99
					22.31	30.12	40.66	54.88	74.08
						16.53	22.31	30.12	40.66
							12.25	16.53	22.31
								9.07	12.25
									6.72

Table 2.2: Cash Flow Stream

In the case of a cash flow stream which is in some way uncertain, three different methods are usually applied to calculate the Net Present Value. Firstly one cash flow stream can be modelled which will then be discounted by the WACC, deriving the extent of uncertainty relative to the overall market portfolio which is expressed by Beta. The shortcomings of this broadly applied method are commonly known. Secondly, the uncertainty is expressed in a scenario analysis by discounting several expected scenarios of the cash flow stream. The discounting will be undertaken by the same WACC, and in addition be multiplied by its estimated probability. Thirdly, the potential cash flow scenarios can be generated stochastically by using the expected grade of uncertainty which is expressed by the volatility σ. By applying a stochastic process, all generated cash flow streams will be discounted by the WACC again multiplied by its probability. Using the method in discrete time, the binomial tree is favoured to calculate the wealth of the overall cash flow stream. The third method within an arbitrage-free setting is closest to the approach developed in the previous chapters of the thesis. Bradley discusses how financial data can be used in the determination of state prices in some simple situations where the scenario tree is constructed so that portfolios of assets with known prices can be dynamically traded to replicate the consequences of holding each of the relevant state assets (Bradley

1998). Further Baker elaborates on Bradley's discussion (Baker Gibbons and Murphy 1998) and a detailed description is given by Laughton (Laughton 1998). In the following results from applying method two will be presented. The table shows the results of the expected project cash flows which have been generated by a stochastic process.

Year 0	1	2	3	4	5	6	7	8	9
840.79	941.73	1041.91	1134.88	1211.07	1256.65	1251.96	1169.48	971.19	604.96
	516.83	571.81	622.83	664.65	689.67	687.09	641.83	533.00	332.01
		313.82	341.82	364.77	378.50	377.08	352.24	292.52	182.21
			187.59	200.19	207.72	206.95	193.31	160.54	100.00
				109.87	114.00	113.58	106.09	88.10	54.88
					62.57	62.33	58.23	48.35	30.12
						34.21	31.95	26.54	16.53
							17.54	14.56	9.07
								7.99	4.98
									2.73

Table 2.3: Value by Applying a WACC of Ten Percent

The project value will be calculated by discounting with the WACC as well as multiplying by their probability. The traditional method results in a value of $w_0 = 840,79$.

Year 0	1	2	3	4	5	6	7	8	9
1000.00	1144.12	1292.86	1438.10	1567.02	1660.06	1688.28	1609.67	1364.19	867.11
	627.91	709.54	789.25	860.00	911.06	926.55	883.40	748.68	475.88
		389.40	433.15	471.98	500.00	508.50	484.82	410.89	261.17
			237.72	259.03	274.41	279.07	266.08	225.50	143.33
				142.16	150.60	153.16	146.03	123.76	78.66
					82.65	84.05	80.14	67.92	43.17
						46.13	43.98	37.27	23.69
							24.14	20.46	13.00
								11.23	7.14
									3.92

Table 2.4: Value by Applying the Risk-Free Interest Rate of Six Percent

As already mentioned, the company valuation approach has been developed within the framework of arbitrage-free setting and, therefore, discounting with the risk-free interest rate would be closest to the approach used in the company valuation algorithm. Discounting with the risk-free interest rate yields a value of $w_0 = 1000$. The result of the programmed basis company valuation algorithm yields for the same parameters a value of $w_0 = 1078$, if at least three replicating assets are available which do have volatilities within the range of $\sigma = 25\%$ and $\sigma = 35\%$. The deviation in the result arises from the incomplete replication as well as the restriction of the model to integers. It can be assumed at this point that the model overstates the project wealth by around 8% compared to a contingent claim valuation according to Cox, Ross and Rubinstein.

2.2 The Real Option Extension

In the following step, the established algorithm will be extended to the possibility of valuing real options. This implies that the management can actively decide to invest additional money to enhance the overall company value. The question is whether additional money spent increases the company value. Therefore these real options can be compared to financial options in some ways. With financial options, the holder of a call option on an underlying asset has the right to buy the underlying asset for an agreed exercise price until the expiration date. In this case the more precise question is, what amount of money additionally invested at what time and in which scenario will enhance the company's value most. Depending on the company itself and its industry, the additional investment can be made e.g. to fund further marketing expenditures or to spent on further R&D.

Due to the holders right, the maximum loss is limited to the price that the option holder has paid to obtain the option. The value of the option implies the so-called asymmetric distribution. In analogy to the financial options, the management for example, has the possibility to decide whether to invest in a further production plant right away or to delay the decision. Holding the option to delay the investment decision is described as a deferral option. Exercising the option amounts to paying for the investment e.g. the new production plant and exchanging the option value for the value of the free cash flows the project generates in the future as part of the companies operations. Like a financial call, the value of a deferral option is highly dependent on the variability/volatility of the expected payoffs or free cash flows expected to be generated by the additional investment.

So far, the company valuation algorithm "solely" considered the present value w_o of the expected cash flows in an uncertain environment. In the next step, the potential additional value of an investment project which can be undertaken some time in the future should be considered. Therefore, the value of the option to invest in some projects with a positive can sometimes add further value to the company compared to the same company without these further investment opportunities which have been calculated already within the basic valuation algorithm. In the following, the algorithm will be extended to incorporate these operating options. The additional value of these options stems from the free cash flows that the investment generates, if exercised, and the value to defer the investment (deferral options) which saves the investment in case it is not optimal to exercise. Options to abandon or to shut down some of the existing operations should not be considered in this work and would be of interest for further research within this model framework.

As the underlying model is based on a non-recombining binomial tree, the call options can be seen as leveraging good vertices and put options as cancelling out vertices which would not recover the investment. Therefore, the evaluated operating options can mainly mitigate capital investment risk and be seen as expansion options. One further important aspect of operating options is that they are almost every time only internally existing and that they are not or only exclusively tradeable with high transaction costs. To determine the companies operating options and the necessary input parameters, the following steps should be considered:

(1) Evaluate the main investment opportunities the company has

(2) Consider potential overlaps or synergistic acting investment opportunities

(3) Estimate the necessary investment to exercise these investment opportunities

(4) Derive a linear algorithm to calculate the new expected cash flow upon exercising the option for all future time steps

(5) Evaluate the impact of the additional investment for different evolving scenarios.

(6) Define the maximum additional investment volume

The real option value of the potential investment projects then is the value of the replication portfolio for the adjusted company cash flows minus the basis value for the non-adjusted company cash flows. This calculation follows the assumption of perfect capital markets what means that the price of the new asset has to equal the value of the least cost replication portfolio. In general, it is assumed that the investment projects can be postponed indefinitely. In the given valuation model, this postponement is limited to $T - 1$. Therefore, the contribution to the company value decreases in the case of postponing the investment decision because there will be fewer future additional cash flows from the investment projects contributing to the company value.

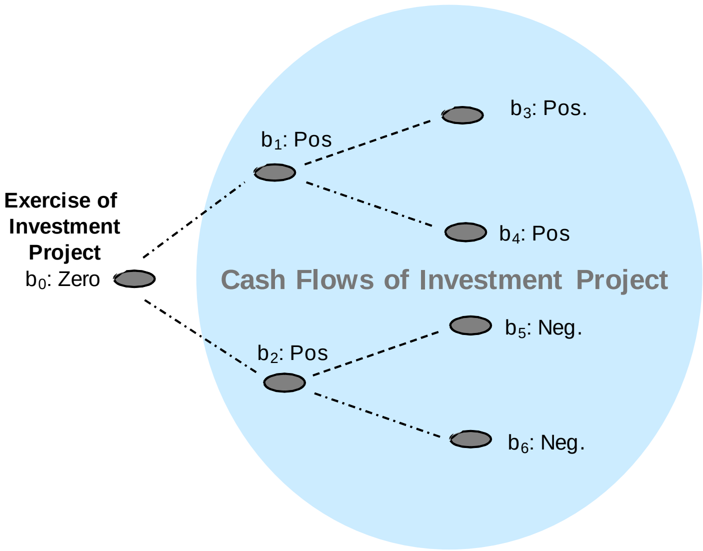


Figure 2.5: Real Option Impact of Deferral Options I

Economically spoken, waiting to invest mitigates the capital investment risk which is depicted in the following figure, but on the opposite side by waiting near term positive cash flow contributions are resigned and will be lost forever.

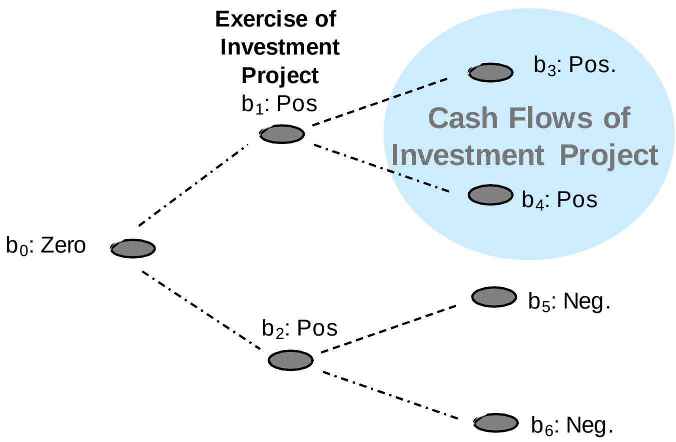


Figure 2.6: Real Option Impact of Deferral Options II

The variance or the uncertainty for the operating options are assumed to be the same as the one of the company's cash flows itself in the multiplicative case, and independent in the additive case. The probability distribution for the future cash flows depends only on the current cash flow of the process. The changes in two different time intervals are independent of each other. It is furthermore assumed that the impact on the future cash flows is linear to the amount invested. For the additive case the contribution is constant in any future node, for the multiplicative case the contribution is linear to the company cash flows itself. The cash flows from the additional investment projects are considered in perpetuity, respectively, until $t = T$. Therefore, the expected cash flows are exposed to the same source of uncertainty. The possibility that projects are exposed to different sources of uncertainty requires a further increase in complexity and could be of interest to further research.

The optimal investment decision, in respect when exercising operating options, should be undertaken in the states where it is most value enhancing. This is the case if exercising is more valuable than waiting and even more valuable than exercising at any other state. The current expected probabilities of the contingent cash flows are replaced by risk-neutral probabilities and the risk-free discount rate can be used again. For real options which are generally not traded, the question can be raised whether these assumptions also hold. For financial options on traded assets, the no-arbitrage condition enforces risk-neutral probabilities and the risk-free discount rate. According to Amran and Kulatilaka, this issue becomes less critical as a result of increasingly more investment risks being traded by the financial markets (Amran, Kulatilaka 1999).

2.2.1 Derivation of the Real Option Company Valuation Algorithm

Often the impact of exercising an investment opportunity can be divided into two main categories. In the first category, an exercise enhances the cash flows in all out-vertices by a certain amount or a certain percentage of the basis cash flows e.g. the investment in a sales or marketing campaign for an already marketed product. In the second category, the cash flows will be significantly increased in only one or maximum at a very limited number of outvertices e.g. for the investment in additional R&D expenditure. For R&D, the impacted nodes represent the outcome of a successful development which leads to a marketable product with significant sales and respectively free-cash flows. All other vertices would be unaffected by the additional R&D expenditure. The additional R&D expenditure does not lead to positive development outcome in these other nodes, which results in a termination of the project and, therefore no positive cash flow generation. The additional real option expenditure g is considered to be separated to w.

It is assumed that an extra expenditure g leads to a company cash flow $b_j(g)$ at least as large as the company cash flow b_j. The maximum additional expenditure is limited to $\widehat{g}$. It is considered that for $g_1 < g_2$ the company cash flows are $b_j(g_1) \leq b_j(g_2)$. Now the minimum shortfall $f_i(w,g)$ achievable at all vertices $v_j \in V_i$ is dependent on the wealth w as well as the discretionary amount g. The recursion from (Christofides A. 2004) derived by having considered the previous assumptions leads to:

$$f_i(w,g) = \min_{0 \leq \beta_i \leq w} \left[\max \left(\begin{array}{c} \max \left(b_i\left(\widehat{g} - g\right) - \beta_i \right), \\ \max\limits_{0 \leq g_i \leq g} \left[\min\limits_{y_i \epsilon Y_i(w-\beta_i)} \left(\max\limits_{v_j \epsilon V_i} \left[f_j\left(w_j\left(y_i\right), g - g_i\right) \right] \right) \right] \end{array} \right) \right] \tag{2.25}$$

The initialisation of the above recursion is given by:

$$f_j(w,g) = \max\left(b_j(g) - w, 0\right), \qquad \forall v_j \epsilon V_{i,} \quad 0 \leq g \leq \widehat{g} \tag{2.26}$$

An overview of equation (2.25) that provides insight into that recursion closely follows the overview of equation (2.1). We will, therefore deal only with the differences between the two recursions. In this case, in addition to the wealth w available (at vertex v_i) to "replicate" the company payoff, we also have a separate "budget" for enhancing the company payoffs themselves. The objective is to maximally enhance these payoffs so that the replication wealth required (and hence the value of the company) is maximised.

In the first instance, we use $b_j\left(g\right)$ to be the company payoff at any vertex v_j in the subtree rooted at v_i, when the amount g is expended at v_i to enhance the company payoffs. Note that if g is available at v_i, it means that $\widehat{g}-g$ must have been expended prior to v_i and hence the term $\max\left(b_i\left(\widehat{g}-g\right)-\beta_i\right)$ in equation (2.25).

If an amount g_i is expended at v_i for payoff enhancement then $g-g_i$ is left for expanding at vertices $v_j \in V_i$, hence the last term in equation (2.25). Finally, a decision as to what amount g_i should be expended at v_i must be made; hence the operator $\max\limits_{0\leq g_i\leq g}[\cdot]$ in equation (2.25). Note, that although conceptually the recursion (2.1) and (2.25) are similar, recursion (2.25) is much more computationally demanding because of (i) the extra optimisation (over g_i) as mentioned above, and (ii) because the memory requirement increases drastically due to the increase (by 1) in the dimensionality of the state space. Even if $\widehat{g}$ is discretised in just 10 timesteps, the size of the state space is increased 10 fold and the computational effort (in computing time) increases a lot more than that.

It should be pointed out, that the cost for raising the additional money g which has been invested has to be subtracted from the computed company value w_0. It is assumed that the impact of the discretionary expenditure on the company cash flows occurs with a time lag of one period. The boundary condition, which means the value between postponing the investment decision or exercising right away, has to be determined again in any node. Therefore, the calculation of the aggregate value from the adjusted cash flows should be considered because of being more precise in an incomplete market setting. Many other approaches assume that the decision to invest can be postponed by infinity either without having any costs or taking any penalty into account. The first approach lacks the applicability in most of the cases and the second accepts further complexity. In the derived algorithm, the advantage of not considering any perpetuity is that by further postponing the investment decision, the company gives away cash flows from the investment project in the states prior exercising it. In the given model, the valuation is limited to countable time periods which, nevertheless, does not necessarily restrict the attractiveness of the approach. On the one hand, the main differences that can be justified occur within a forecast period of less than ten years and on the other hand, the residual value in the case of companies with long-term stable cash flows can be added by applying traditional well established valuation techniques like the DCF, if required. According to internal research of DZ BANK, the contribution of cash flows beyond 25 years in the future accounts for on average less than 20% of the current DCF valuation, if applied to the German DAX companies. The number is still high in comparison to other asset classes as a result of the expected long term growth rate above the risk free interest rate r. This adds one further argument that there is only limited necessity for considering a terminal value in case a reasonably long forecasting period can be considered. To find an optimal dynamic investment policy, the value of the option to invest has to be computed for every project, in every state, in every point of time.

2.2.2 1. Example of the Real Option Company Valuation Algorithm

In a first example for the extended real option company valuation algorithm, the valuation process should be clarified. It should be assumed that an additional expenditure of $g = 2$ which is made in vertex v_i adds 1 unit of cash flow in the outvertices to all company cash flows b_j in all future nodes. Therefore, the adjusted company cash flows b_j lead to:

$$\begin{aligned} b_j(g) &= b_j + gf_{add} \times g \\ b_j(2) &= 5 + 0.5 \times 2 \\ b_j(2) &= 6 \end{aligned} \tag{2.27}$$

in the upper node and:

$$\begin{aligned} b_j(2) &= 6 + 0.5 \times 2 \\ b_j(2) &= 7 \end{aligned} \tag{2.28}$$

in the lower node for example. In this case, the previous example changes accordingly:

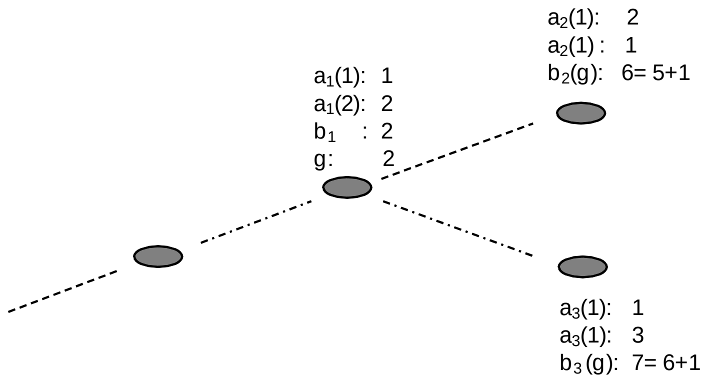


Figure 2.7: Example Real Option Company Valuation in One Period

A wealth $w_1 = 7$ in v_1 should be chosen as a starting point which has been the solution in the previous basic company valuation example. This wealth w_i will be split in an investment in the replication portfolios $Y_i(w_i - \beta_i)$ and a payout β_i. The value of β_i."replicates" the computed company cash flow b_i at that vertex. Therefore the payout β_1 is equal to $b_1 = 2$ in the given example. The remaining wealth $w_1 - \beta_1 = 7 - 2 = 5$ will be invested in portfolios $Y_1(w_1 - \beta_1)$ with the following asset prices $a_1(1), a_1(2)$ in the chosen example. This yields the following portfolio combinations considering only investments in integer units in the assets i.e. $w_1 - \beta_1 = y_1 = 5 = \binom{0}{2.5} = k_1(1) \times a_1(1) + k_1(2) \times a_1(2) = 0 \times 1 + 2.5 \times 2$. All other portfolio combinations are listed as followed:

$$Y_i(w_i - \beta_i) = 5 = \left\{ \begin{pmatrix} 0 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\} \tag{2.29}$$

The first portfolio combination $y_1 = \binom{0}{2.5}$ leads to the following result at vertex v_2:

$$\begin{aligned} (w_2(y_1)) &= (0 \times 2 + 2.5 \times 1) = 2.5 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(6 - 2.5, 0) \\ &= 3.5 \end{aligned} \tag{2.30}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (0 \times 1 + 2.5 \times 3) = 7.5 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(7 - 7.5, 0) \\ &= 0 \end{aligned} \tag{2.31}$$

therefore:

$$\begin{aligned} f_1\left(w_1\left(y_1\right)\right) &= \max\left[f_2\left(w_2\left(y_1\right)\right), f_3\left(w_3\left(y_1\right)\right)\right] \\ &= \max\left[3.5, 0\right] \\ &= 3.5 \end{aligned} \tag{2.32}$$

The second portfolio combination $y_1 = \binom{1}{2}$ leads to the following result at vertex v_2:

$$\begin{aligned} \left(w_2\left(y_1\right)\right) &= \left(1 \times 2 + 2 \times 1\right) = 4 \text{ and} \\ f_2\left(w_2\left(y_1\right)\right) &= \max(b_2 - w_2, 0) \\ &= \max(6 - 4, 0) \\ &= 2 \end{aligned} \tag{2.33}$$

and at v_3:

$$\begin{aligned} \left(w_3\left(y_1\right)\right) &= \left(1 \times 1 + 2 \times 3\right) = 7 \text{ and} \\ f_3\left(w_3\left(y_1\right)\right) &= \max(b_3 - w_3, 0) \\ &= \max\left(7 - 7, 0\right) \\ &= 0 \end{aligned} \tag{2.34}$$

therefore:

$$\begin{aligned} f_1\left(w_1\left(y_1\right)\right) &= \max\left[f_2\left(w_2\left(y_1\right)\right), f_3\left(w_3\left(y_1\right)\right)\right] \\ &= \max\left[2, 0\right] \\ &= 2 \end{aligned} \tag{2.35}$$

The third portfolio combination $y_1 = \binom{2}{1.5}$ leads to the following result at vertex v_2:

$$\begin{aligned} (w_2(y_1)) &= (2 \times 2 + 1.5 \times 1) = 5.5 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(6 - 5.5, 0) \\ &= 0.5 \end{aligned} \tag{2.36}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (2 \times 1 + 1.5 \times 3) = 6.5 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(7 - 6.5, 0) \\ &= 0.5 \end{aligned} \tag{2.37}$$

therefore:

$$\begin{aligned} f_1(w_1(y_1)) &= \max[f_2(w_2(y_1)), f_3(w_3(y_1))] \\ &= \max[0.5, 0.5] \\ &= 0.5 \end{aligned} \tag{2.38}$$

The fourth portfolio combination $y_1 = \binom{3}{1}$ leads to the following result at vertex v_2:

$$\begin{aligned} (w_2(y_1)) &= (3 \times 2 + 1 \times 1) = 7 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(6 - 7, 0) \\ &= 1 \end{aligned} \tag{2.39}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (3 \times 1 + 1 \times 3) = 6 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(7 - 6, 0) \\ &= 1 \end{aligned} \tag{2.40}$$

therefore:

$$\begin{aligned} f_1(w_1(y_1)) &= \max[f_2(w_2(y_1)), f_3(w_3(y_1))] \\ &= \max[1, 1] \\ &= 1 \end{aligned} \tag{2.41}$$

The fifth portfolio combination $y_1 = \binom{4}{0.5}$ leads to the following result at vertex v_2:

$$\begin{aligned} (w_2(y_1)) &= (4 \times 2 + 0.5 \times 1) = 8.5 \text{ and} \\ f_2(w_2(y_1)) &= \max(b_2 - w_2, 0) \\ &= \max(6 - 8.5, 0) \\ &= 0 \end{aligned} \tag{2.42}$$

and at v_3:

$$\begin{aligned} (w_3(y_1)) &= (4 \times 1 + 0.5 \times 3) = 5.5 \text{ and} \\ f_3(w_3(y_1)) &= \max(b_3 - w_3, 0) \\ &= \max(7 - 5.5, 0) \\ &= 1.5 \end{aligned} \tag{2.43}$$

therefore:

$$\begin{aligned} f_1\left(w_1\left(y_1\right)\right) &= \max\left[f_2\left(w_2\left(y_1\right)\right), f_3\left(w_3\left(y_1\right)\right)\right] \\ &= \max\left[0, 1.5\right] \\ &= 1.5 \end{aligned} \tag{2.44}$$

The sixth portfolio combination $y_1 = \binom{5}{0}$ leads to the following result at vertex v_2:

$$\begin{aligned} \left(w_2\left(y_1\right)\right) &= \left(5 \times 2 + 0 \times 1\right) = 10 \text{ and} \\ f_2\left(w_2\left(y_1\right)\right) &= \max(b_2 - w_2, 0) \\ &= \max(6 - 10, 0) \\ &= 0 \end{aligned} \tag{2.45}$$

and at v_3:

$$\begin{aligned} \left(w_3\left(y_1\right)\right) &= \left(5 \times 1 + 0 \times 3\right) = 5 \text{ and} \\ f_3\left(w_3\left(y_1\right)\right) &= \max(b_3 - w_3, 0) \\ &= \max\left(7 - 5, 0\right) \\ &= 2 \end{aligned} \tag{2.46}$$

therefore:

$$\begin{aligned} f_1\left(w_1\left(y_1\right)\right) &= \max\left[f_2\left(w_2\left(y_1\right)\right), f_3\left(w_3\left(y_1\right)\right)\right] \\ &= \max\left[0, 2\right] \\ &= 2 \end{aligned} \tag{2.47}$$

The results for the maximum shortfall $f_1\left(w_1\left(y_1\right)\right)$ for all investigated portfolio combinations with an invested wealth of $w_1 - \beta_1 = 5$ are:

$$y_i \epsilon Y\left(5\right) = \{3.5; 2; 0.5; 1; 1.5; 2\} \tag{2.48}$$

This shows that an invested wealth $w_1 = 6$ is not sufficient to replicate the company cash flows in the vertices v_2 and v_3. Therefore, the invested wealth w_1 in v_1 will be increased by one integer to 8. Now the remaining wealth $w_1 - \beta_1 = 8 - 2 = 6$ will be invested in portfolios $Y_1\left(w_1 - \beta_1\right)$ again with the asset prices $a_1(1)$ and $a_1\left(2\right)$. This yields the following portfolio combinations considering only investment in integers in the first asset:

$$Y_1\left(w_1 - \beta_1\right) = 5 = \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \end{pmatrix} \right\} \tag{2.49}$$

The results for the maximum shortfall $f_1\left(w_1\left(y_1\right)\right)$ for all investigated portfolio combinations V_i with an invested wealth of $w_1 - \beta_1 = 6$ are:

$$y_i \epsilon Y\left(6\right) = \left\{3, 1.5, 0, 0, 0, 0.5, 1\right\} \tag{2.50}$$

For an invested wealth of $w_0 = 8$, all company cash flows in the vertices v_2 and v_3 can be replicated by investing in at least one portfolio combination. Therefore, the minimum shortfall that can be achieved with at least one portfolio combination with an investment

$$w_1 - \beta_1 = 8 - 2 = 6 \text{ is } \min_{y_i \epsilon Y_i\left(w - \beta_i\right)} \left(\max_{v_j \epsilon V_i} \left[f_i\left(w_j\left(Y_i\right)\right)\right] \right) = 0. \tag{2.51}$$

At this point, the iterative replication procedure could be either refined to invested wealth w_0 between 7 and 8 in fractions of an integer, or the optimal wealth w_0 could be determined by interpolation. The same result will be computed by the algorithm implemented in C++. Nevertheless, one more step has to be considered until having the final result for the company value with an additional expenditure of $g = 2$. This expenditure has to be deducted from the computed value $w_.$ Therefore the final company value is defined as $w_{comp} \equiv w_0 - g = 8 - 2 = 6$ in the given example. In this example it is only considered that one time step is impacted by the additional amount of money of $g = 2$ at time $t = 0$ the company value w_0 would decrease by one unit compared to the basic example after deducting the additional expenditure. As a result it is not beneficial to have the previous option in hand and even decremental to exercise it.

2.3 Extension to Multiple Time Steps

So far, the algorithm was only applied for one time step examples. For solving multi period scenarios, all vertices will be solved recursively starting with the vertices in T. The calculated result of the company value w_0 of the one time period setting will then substitute b_i at vertex v_i in $T - 1$.

2.3.1 2. Example of the Real Option Company Valuation Algorithm

With the following company cash flows b and asset prices a, the company value w_0 should be calculated.

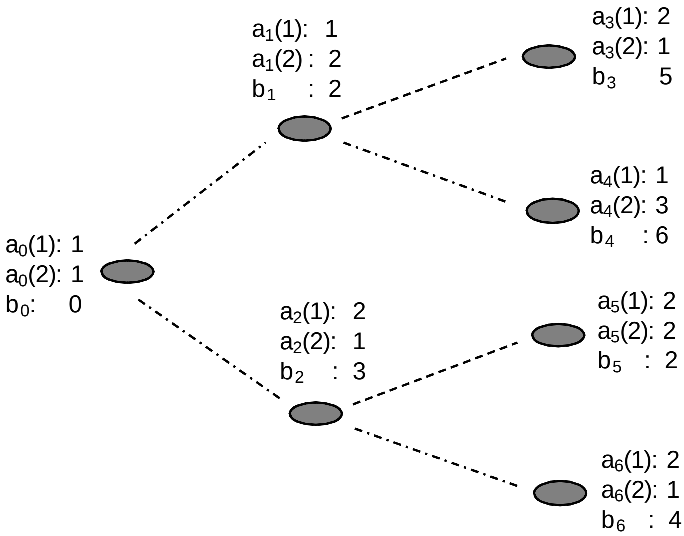


Figure 2.8: First Step - Two Period Real Option Company Valuation

The same calculation procedure as in the previous examples will be applied for the vertices ending in T. For vertex v_1 a wealth of $w_1 = 7$ is the least wealth replicating all company cash flows in the vertex v_1 and its outvertices V_i with at least one portfolio combination. For v_2, the least wealth that can be calculated to be sufficient to replicate company cash flows is $w_2 = 7$ as well. As a result of the recursive structure, the company cash flows b_2 and b_3 are now going to be replaced by the calculated least wealth $w_1 = 7$ and $w_2 = 7$.

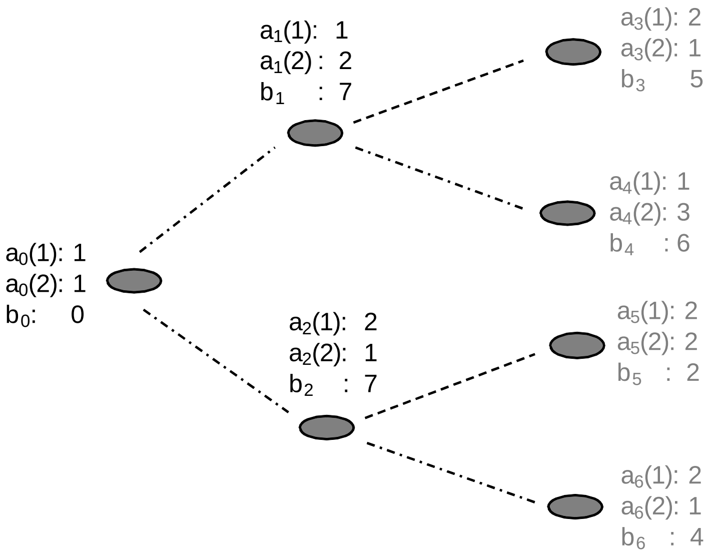


Figure 2.9: Second Step - Two Period Real Option Company Valuation

In the next step, the wealth that has to be invested at least to replicate the company cash flows b_1 and b_2 will be calculated. Now $w_0 = 5$ as an educated guess should be chosen as a starting point. The remaining wealth $w_0 - \beta_0 = 5 - 0 = 5$ will be invested in portfolios $Y_0 (w_0 - \beta_0)$ consisting of two assets with prices $a_0 (1)$ and $a_0 (2)$ again.

This yields the following portfolio combinations considering only investment in integers in asset 1: $w_0 - \beta_0 = 5 = \binom{0}{5} = k_0 (1) \times a_0 (1) + k_0 (2) \times a_0 (2) = 0 \times 1 + 5 \times 1$. All other portfolio combinations are listed as followed:

$$Y_0 (w_0 - \beta_0) = 5 = \left\{ \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\} \tag{2.52}$$

The first portfolio combination $y_0 = \binom{0}{5}$ leads to the following result at vertex v_0:

$$\begin{aligned}(w_1(y_0)) &= (0 \times 1 + 5 \times 2) = 10 \text{ and} \\ f_1(w_1(y_0)) &= \max(b_1 - w_1, 0) \\ &= \max(7 - 10, 0) \\ &= 0\end{aligned} \tag{2.53}$$

and at v_2:

$$\begin{aligned}(w_2(y_0)) &= (0 \times 2 + 5 \times 1) = 5 \text{ and} \\ f_2(w_2(y0)) &= \max(b_2 - w_2, 0) \\ &= \max(7 - 5, 0) \\ &= 2\end{aligned} \tag{2.54}$$

therefore:

$$\begin{aligned}f_0(w_0(y_0)) &= \max[f_1(w_1(y_0)), f_2(w_2(y_0))] \\ &= \max[0, 2] \\ &= 2\end{aligned} \tag{2.55}$$

The second portfolio combination $y_0 = \binom{1}{4}$ leads to the following result at vertex v_1:

$$\begin{aligned}(w_1(y_0)) &= (1 \times 1 + 4 \times 2) = 9 \text{ and} \\ f_1(w_1(y_0)) &= \max(b_1 - w_1, 0) \\ &= \max(7 - 9, 0) \\ &= 0\end{aligned} \tag{2.56}$$

and at v_2:

$$\begin{aligned} (w_2(y_0)) &= (1 \times 2 + 4 \times 1) = 6 \text{ and} \\ f_2(w_2(y_0)) &= \max(b_2 - w_2, 0) \\ &= \max(7 - 6, 0) \\ &= 1 \end{aligned} \tag{2.57}$$

therefore:

$$\begin{aligned} f_0(w_0(y_0)) &= \max[f_1(w_1(y_0)), f_2(w_2(y_0))] \\ &= \max[0, 1] \\ &= 1 \end{aligned} \tag{2.58}$$

The third portfolio combination $y_0 = \binom{2}{3}$ leads to the following result at vertex v_1:

$$\begin{aligned} (w_1(y_0)) &= (2 \times 1 + 3 \times 2) = 8 \text{ and} \\ f_1(w_1(y_0)) &= \max(b_1 - w_1, 0) \\ &= \max(7 - 8, 0) \\ &= 0 \end{aligned} \tag{2.59}$$

and at v_2:

$$\begin{aligned} (w_2(y_0)) &= (2 \times 2 + 3 \times 1) = 7 \text{ and} \\ f_2(w_2(y_0)) &= \max(b_2 - w_2, 0) \\ &= \max(7 - 7, 0) \\ &= 0 \end{aligned} \tag{2.60}$$

therefore:

$$
\begin{aligned}
f_0\left(w_0\left(y_0\right)\right) &= \max\left[f_1\left(w_1\left(y_0\right)\right), f_2\left(w_2\left(y_0\right)\right)\right] \\
&= \max\left[0, 0\right] \\
&= 0
\end{aligned}
\tag{2.61}
$$

The fourth portfolio combination $y_0 = \binom{3}{2}$ leads to the following result at vertex v_1:

$$
\begin{aligned}
\left(w_1\left(y_0\right)\right) &= \left(3 \times 1 + 2 \times 2\right) = 7 \text{ and} \\
f_1\left(w_1\left(y_0\right)\right) &= \max(b_1 - w_1, 0) \\
&= \max(7 - 7, 0) \\
&= 0
\end{aligned}
\tag{2.62}
$$

and at v_2:

$$
\begin{aligned}
\left(w_2\left(y_0\right)\right) &= \left(3 \times 2 + 2 \times 1\right) = 8 \text{ and} \\
f_2\left(w_2\left(y_0\right)\right) &= \max(b_2 - w_2, 0) \\
&= \max\left(7 - 8, 0\right) \\
&= 0
\end{aligned}
\tag{2.63}
$$

therefore:

$$
\begin{aligned}
f_0\left(w_0\left(y_0\right)\right) &= \max\left[f_1\left(w_1\left(y_0\right)\right), f_2\left(w_2\left(y_0\right)\right)\right] \\
&= \max\left[0, 0\right] \\
&= 0
\end{aligned}
\tag{2.64}
$$

The fifth portfolio combination $y_0 = \binom{4}{1}$ leads to the following result at vertex v_1:

$$\begin{aligned} (w_1(y_0)) &= (4 \times 1 + 1 \times 2) = 6 \text{ and} \\ f_1(w_1(y_0)) &= \max(b_1 - w_1, 0) \\ &= \max(7 - 6, 0) \\ &= 1 \end{aligned} \tag{2.65}$$

and at v_2:

$$\begin{aligned} (w_2(y_0)) &= (4 \times 2 + 1 \times 1) = 9 \text{ and} \\ f_2(w_2(y_0)) &= \max(b_2 - w_2, 0) \\ &= \max(7 - 9, 0) \\ &= 0 \end{aligned} \tag{2.66}$$

therefore:

$$\begin{aligned} f_0(w_0(y_0)) &= \max\left[f_1(w_1(y_0)), f_2(w_2(y_0))\right] \\ &= \max[1, 0] \\ &= 1 \end{aligned} \tag{2.67}$$

The sixth portfolio combination $y_0 = \binom{5}{0}$ leads to the following result at vertex v_1:

$$\begin{aligned} (w_1(y_0)) &= (5 \times 1 + 0 \times 2) = 5 \text{ and} \\ f_1(w_1(y_0)) &= \max(b_1 - w_1, 0) \\ &= \max(7 - 5, 0) \\ &= 2 \end{aligned} \tag{2.68}$$

and at v_2:

$$\begin{aligned} (w_2(y_0)) &= (5 \times 2 + 0 \times 1) = 10 \text{ and} \\ f_2(w_2(y_0)) &= \max(b_2 - w_2, 0) \\ &= \max(7 - 10, 0) \\ &= 0 \end{aligned} \tag{2.69}$$

therefore:

$$\begin{aligned} f_0(w_0(y_0)) &= \max[f_1(w_1(y_0)), f_2(w_2(y_0))] \\ &= \max[2, 0] \\ &= 2 \end{aligned} \tag{2.70}$$

The amount invested in the asset portfolio is sufficient to replicate the company cash flows. Solving for the minimum wealth w_0 which at least replicates the company cash flows $b_1 = 7$ and $b_2 = 7$, $w_0 = 5$, will be derived. For an invested wealth of $w_0 = 5$ all company cash flows in the vertices v_1 and v_2 can be replicated by investing in the optimal portfolio combination. Therefore the minimum shortfall that can be achieved with at least one portfolio combinations with an investment $w_0 - \beta_0 = 5 - 0 = 5$ is $\min\limits_{y_i \epsilon Y_i(w-\beta_i)} \left(\max\limits_{v_j \epsilon V_i} [f_i(w_j(Y_i))] \right) = 0$. The results for the maximum shortfall $f_i(w_j(y_i))$ for all six portfolio combinations with an invested wealth of $w_0 - \beta_0 = 5$ are:

$$y_i \epsilon Y(5) = \{2; 1; 0; 0; 1; 2\} \tag{2.71}$$

Finally, the company value is $w_{comp} = 5$.

Assuming that an additional expenditure of $g = 2$ is made in vertex v_0 in $t = 0$

again. This investment project now leverages the company cash flow in a multiplicative way by increasing every following node $b_j\,(g)$ by 40% per unit invested. Therefore:

$$b_j\,(2) = (1 + (g \times gf_{multi})) \times b_j \tag{2.72}$$

for $t = 2$. Finalising the calculation of the adjusted company cash flows b leads to

$$\begin{aligned} b_3\,(2) &= (1 + (2 \times 0,40)) \times 5 \\ b_3\,(2) &= 1,8 \times 5 = 9 \end{aligned} \tag{2.73}$$

in vertex v_3,

$$\begin{aligned} b_4\,(2) &= (1 + (2 \times 0.4)) \times 6 \\ b_4\,(2) &= 1.8 \times 6 \approx 11 \end{aligned} \tag{2.74}$$

vertex v_4 and

$$\begin{aligned} b_5\,(2) &= (1 + (2 \times 0.4)) \times 2 \\ b_5\,(2) &= 1.8 \times 2 \approx 4 \end{aligned} \tag{2.75}$$

in vertex v_5 and so on.

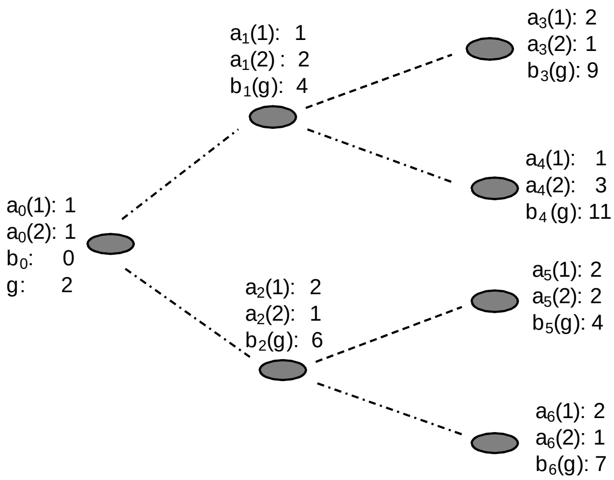


Figure 2.10: Third Step - Two Period Real Option Company Valuation

In this example the company cash flows b_i in $t = 1$ as well as in $t = 2$ have been increased as a result of the real option expenditure already in $t = 0$. The same solution procedure as for the previous example without real options will apply. Within the recursive solution procedure the company wealth in t_1 in vertex v_1 yields $w_1 = 13$ which replicates all company cash flows in vertex v_1 and its outvertices V_1 with at least one portfolio combination. For v_2, the least wealth that can be calculated being sufficient to replicate company cash flows is $w_2 = 13$. As a result of the recursive structure, the company cash flows b_1 and b_2 are now going to be replaced by the calculated least wealth $w_1 = 13$ and $w_2 = 13$.

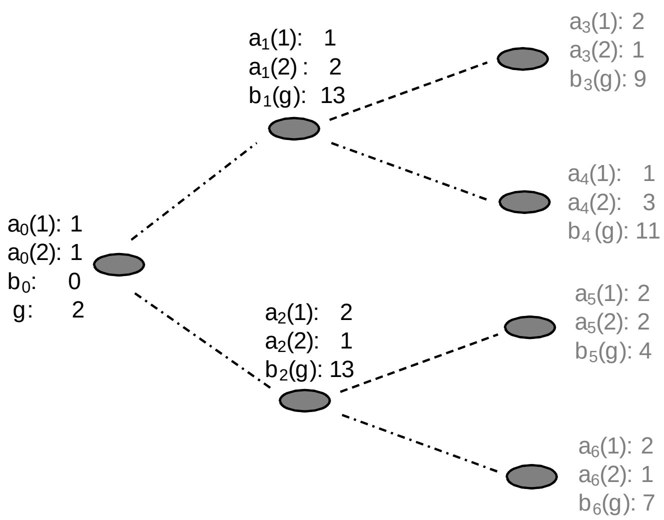


Figure 2.11: Fourth Step - Two Period Real Option Company Valuation

In the next step, the wealth that has to be invested at least to replicate the company cash flows b_1 and b_2 will be calculated. Now $w_0 = 9$ should be chosen as a starting point. The remaining wealth $w_0 - \beta_0 = 9 - 0 = 9$ will be invested in portfolios $Y_0 (w_0 - \beta_0)$ holding again only two assets. This yields the following portfolio combinations considering only investment in integers in asset 1, i.e. $w_0 - \beta_0 = 11 = \binom{0}{9} = k_0 (1) \times a_0 (1) + k_0 (2) \times a_0 (2) = 0 \times 1 + 9 \times 1$. All other portfolio combinations are listed as followed:

$$Y_0\left(w_0-\beta_0\right)=7=\left\{\begin{pmatrix}0\\9\end{pmatrix},\begin{pmatrix}1\\8\end{pmatrix},\begin{pmatrix}2\\7\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix},\begin{pmatrix}4\\5\end{pmatrix},\begin{pmatrix}5\\4\end{pmatrix},\begin{pmatrix}6\\3\end{pmatrix},\begin{pmatrix}7\\2\end{pmatrix},\begin{pmatrix}8\\1\end{pmatrix},\begin{pmatrix}9\\0\end{pmatrix}\right\} \tag{2.76}$$

The results for the maximum shortfall $f_0\left(w_0\left(y_0\right)\right)$ for all three portfolio combination with an invested wealth of $w_0-\beta_0=9$ are:

$$y_i\epsilon Y\left(5\right)=\{4;3;2;1;0;0;1;2;3;4\} \tag{2.77}$$

This shows that two portfolio combinations Y_0 with an invested wealth $w_0-\beta_0=9$ are sufficient to replicate the company cash flows in the vertices v_1 and v_2. The wealth $w_0=9$ invested in the asset portfolio is sufficient to replicate the company cash flows. With this invested wealth w_0 all company cash flows in the vertices v_1 and v_2 can be replicated by investing in the optimal portfolio combinations. Therefore the minimum shortfall that can be achieved with at least one portfolio combination with an investment $w_0-\beta_0=9-0=9$ is $\min\limits_{y_i\epsilon Y_i(w-\beta_i)}\left(\max\limits_{v_j\epsilon V_i}\left[f_i\left(w_j\left(Y_i\right)\right)\right]\right)=0$. The company value in the case of exercising the investment opportunity would finally be $w_{comp}=w_0-g=9-2=7$. Therefore, the company value finally changes to $w_{comp}=\max(w_{comp;}w_{comp}(g))=\max(6;7)=7$, because the positive impact on future cash flows of the additional investment project outweighs its investment costs of $g=2$. Hence, the real option of the company adds 1 unit additional wealth to the basis company value. In the forthcoming work, some examples will be investigated where an early investment is value enhancing as well as a potentially later investment is more valuable than the investment in vertex v_0 at t_0. In these situations hurdle rates can be determined at which levels it is feasible and value creating to invest additional money.

Chapter 3
Further Additions to the Company Valuation Algorithm

As an extension to the results shown so far, firstly a self assembling tree with stochastic asset price generation will be implemented and secondly a generalisation should allow for incorporating negative company cash flows. Thirdly the process to determine the optimal investment decision should be derived including some examples.

3.1 Generic Tree Generation

For the asset price evolution, a self assembling tree with asset values has been implemented according to Cox, Ross and Rubinstein (Cox, Ross and Rubinstein, 1979). The binomial model assumes that the asset prices can only move upwards or downwards in the next discrete time step, therefore, the distribution is called binomial. The derivation of building the riskless replication or hedging portfolio has been explained in chapter 1. The one step view is extended to several steps with a decreasing time span between each time step. It is still assumed that in each time step the stock can only move upwards or downwards. The next asset values are denoted Su and Sd. The probability for an up movement will be denoted by p and of a down movement by $1 - p$. As a reminder of the "risk-neutral-valuation" principle: The probability p is such that, the expected return of the traded assets is the risk-free interest rate, and future cash flows can be valued by being discounted with the risk-free interest rate. First of all, the input parameters p, u, and d are calculated in a risk-neutral setting which means:

$$\begin{aligned} Se^{r\Delta t} &= pSu + (1-p)\,Sd \\ e^{r\Delta t} &= pu + (1-p)\,d \end{aligned} \tag{3.1}$$

The percentage change of the asset price in one time step is $\sigma\Delta t$. The variance of a variable Q is defined as $E\left(Q^2\right) - \left[E\left(Q\right)\right]^2$, therefore:

$$pu^2 + (1-p)\,d^2 - \left[pu + (1-p)\,d^2\right] = \sigma^2\Delta t \tag{3.2}$$

Substituting for p from the previous equation leads to:

$$e^{r\Delta t}\left(u+d\right) - ud - e^{2r\Delta t} = \sigma^2\Delta t \tag{3.3}$$

Furthermore, Cox, Ross and Rubinstein state the condition

$$u = \frac{1}{d} \tag{3.4}$$

to recombine the tree. This leads to the conditions for p, u, and d:

$$\begin{aligned} p &= \frac{a-d}{u-d} \\ u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \end{aligned} \tag{3.5}$$

where:

$$a \equiv e^{r\Delta t} \tag{3.6}$$

For the implementation, this means that any asset price can be calculated by forward iteration

$$Su^j d^{i-j} \text{ for } j = 0, 1, ..., i \tag{3.7}$$

In case of the implemented company valuation model, the relationship

$$u = \frac{1}{d} \tag{3.8}$$

could be relaxed due to the fact that we are working with a non-recombining tree.

If the number of subnodes in each state is increased to three the generalisation of the binomial tree according to (Clewlow and Strickland 1998) has been applied.

The case of considering 4 subnodes in each state according to (Clewlow and Strickland 1998) is as shown in the following figure (3.1):

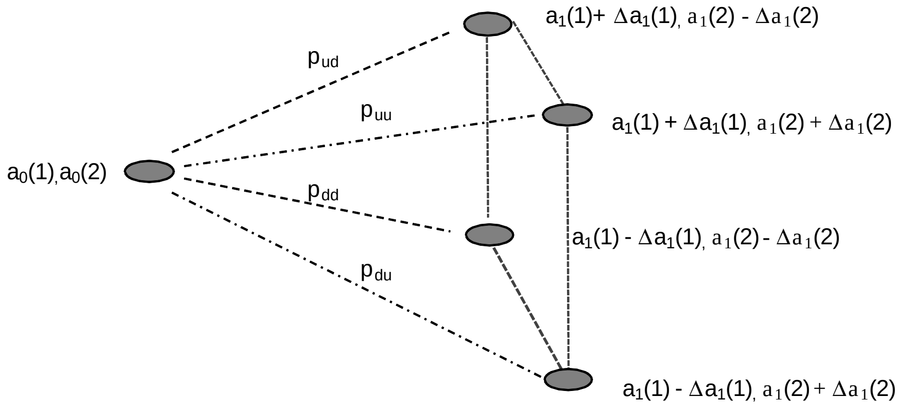


Figure 3.1: Potential Price Process in a Four Subnodes Szenario

3.2 Incorporation of Negative Cash Flows and Their Impact

In the meantime the algorithm has been generalised to allow for stochastic tree generation. The given algorithm does only work with positive numbers so far. This limits the use to positive company cash flow processes. Especially for high growth start-up companies, it is required to include negative cash flows at least in the initiating phase. Therefore, the stochastic price process has been adopted. For a given $b^* > 0$, the quantity $\widetilde{b} = b + b^*$ is modelled by the binomial tree, thus $\widetilde{b} > 0$, i.e. $b > \widehat{-b}$. Here $\widehat{-b}$ is the maximum level of negative company cash flows. Instead of

$b_0 > 0$, now $b_0 > \widehat{-b}$ is required, thus initial allowing negative cash flows. This approach even allows to model increasing losses in the vertices going downwards. In the given example, a starting cash flow $b_0 = -5$ has been assumed having a volatility of $\sigma = 0.2$ and a growth rate of $r = 0.06$. The maximum negative cash flow barrier has been set on the left side of the figure at $\widehat{-b} = -10$ and on the right side at $\widehat{-b} = -15$. Higher $\widehat{-b}$ relative to the negative starting cash flow b_0 applying the same growth rate μ as well as volatility σ leads to higher absolute values of the generated cash flows in the outvertices. This should be kept in mind, applying this model for e.g. valuing a start-up company.

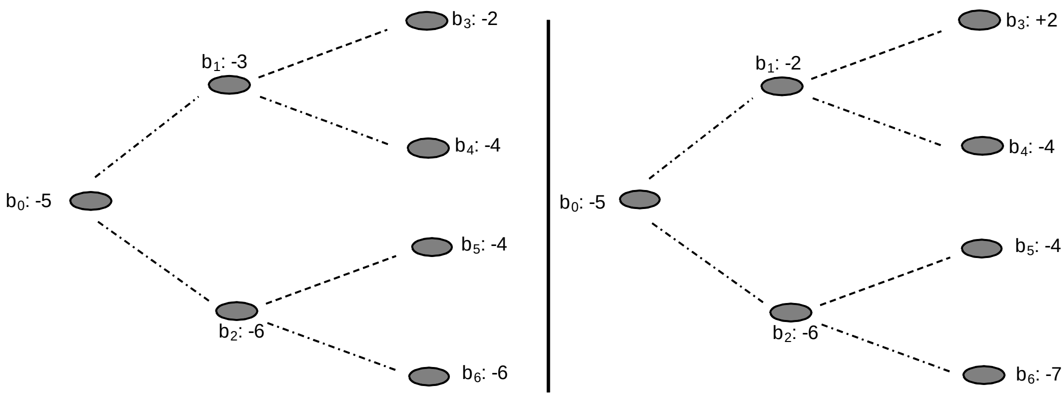


Figure 3.2: Adapted Stochastic Cash Flow Process Allowing Negative Values

With this procedure also negative company cash flows can be considered without the necessity to adopt a different stochastic process. For application to real case scenario this implies, the negative cash flows could potentially sum up to an amount of money no investor is willing to pay for the investment scenario. As a result a sum of maximum negative cash flow $\widehat{b}_{sum}$ is defined which limits the aggregation of losses to a certain amount the investor is willing to carry. The overall value of the company w_0 then is only calculated for the branches which do not exceed $\widehat{b}_{sum}$. This additional assumption which has been implemented in the algorithm strongly improves its applicability for scenarios which do include a time span with negative cash flows as well. In the following a scenario of a company which allows unlimited negative cash flow aggregation should be presented:

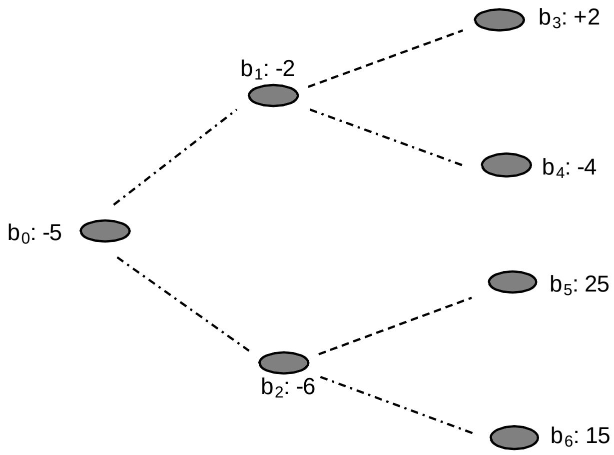


Figure 3.3: Company Cash Flow with Unlimited Negative Cash Flow Aggregation

Having computed the value of this scenario by a generic replication setting, the company is worth $w_0 = 6$. The well established investment theory considers to invest in such a project. Nevertheless the question remains if the investor is willing to bear the risk of pouring additional 6 units money in the investment project, if vertex v_2 will be reached after having already invested 5 units in t_0. The investor could decide to do so on back of the potential, that the company will be sold for a high premium in vertices v_5 and v_6. The overall maximum investment the investor has to provide for this scenarios would be $\widehat{b}_{sum} = 11$. In the case the the investor has only free liquidity in the extent of e.g. $\widehat{b}_{sum} = 10$, the lower branch would be cancelled out and the overall company value $w_0 << 0$ would be strongly negative.

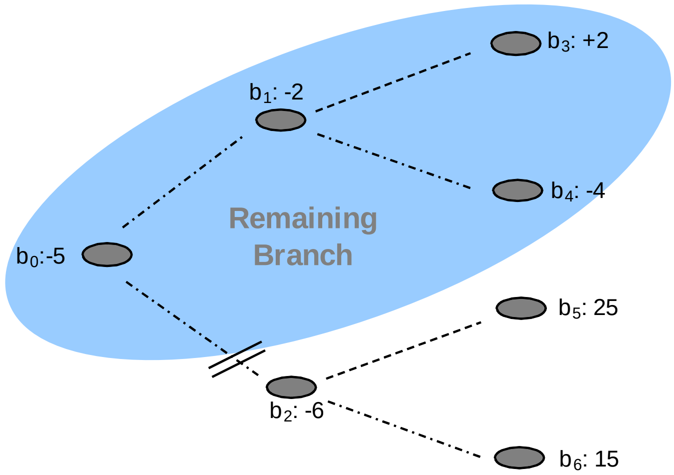


Figure 3.4: Company Cash Flow Process with Limited Negative Cash Flow Aggregation

The opposite case could happen as well; by limiting $\widehat{b}_{sum}$ for the basis company cash flow scenario, the investor improves his return by cancelling out loss making branches, which will probably happen more often.

3.3 Determination of the Optimal Investment Strategy

So far the algorithm determines the real option value if an additional investment increases the company value in general. Nevertheless, the practical application has not been solved, showing the point in time and scenario that is most value enhancing, in case an additional investment enhances the company value at all. Calculating the results again within a recursive structure, at least two company values replicating all future cash flows have to be stored instead of one in every state: The so-called basic company cash flow $b_i\,(0)$ will be replaced by the company value w_i which is calculated throughout the recursion as shown in chapter 3. Therefore $b_i\,(0)$ is equal to b_i which is already known from the previous chapter. For any real option scenario the company value $b_i\,(g)$ will be calculated for the situation where the project is exercised in v_i and where the investment project has been exercised in any previous vertex. In almost all cases, the company value where the investment project has been exercised in any earlier state yields a higher value because $b_i(\widehat{g}) \geq b_i(g)$ and larger than $b_i(0)$ as well. To find the dominating scenario the company value for the scenario of an exercise in v_i will be compared to $b_i(0)$. The investment scenario $b_i(g)$ which exceeds $b_i(0)$ most, will be stored.

In the chosen setting, the maximum additional investment $\widehat{g} = 3$ will be limited to three units and it is assumed that each additional project requires an investment of one unit. It is supposed as described in chapter 3 that the projects can be exercised in any node except in T. As a result in vertex v_1 four investment possibilities are evaluated which are $g = 0, 1, ..., \widehat{g}$. The value of the company in the case of an exercise of the investment g in v_1 is depicted in the second column named "*inv*".

Furthermore the so far optimal scenario is stored which would be $b_1(2)$ in the given example all other results in column *inv.* are discarded. Storing, e.g. of $b_1(1)$ in column *inv.* is not required because the company wealth potentially generated is dominated by the company wealth generated by $b_1(2)$. $b_1(3)$ in column *inv.* is also not stored because it does not deliver any additional value and reduces the investment flexibility in previous nodes. In v_2 no further result in column *inv.* will be stored because they failed to exceed $b_0(0)$. In the root node v_0 finally all combinations of the so far dominating scenarios are evaluated and the scenario yielding the highest company value $w_{comp} = b_0(g) - g$ which is the optimal investment strategy will be determined.

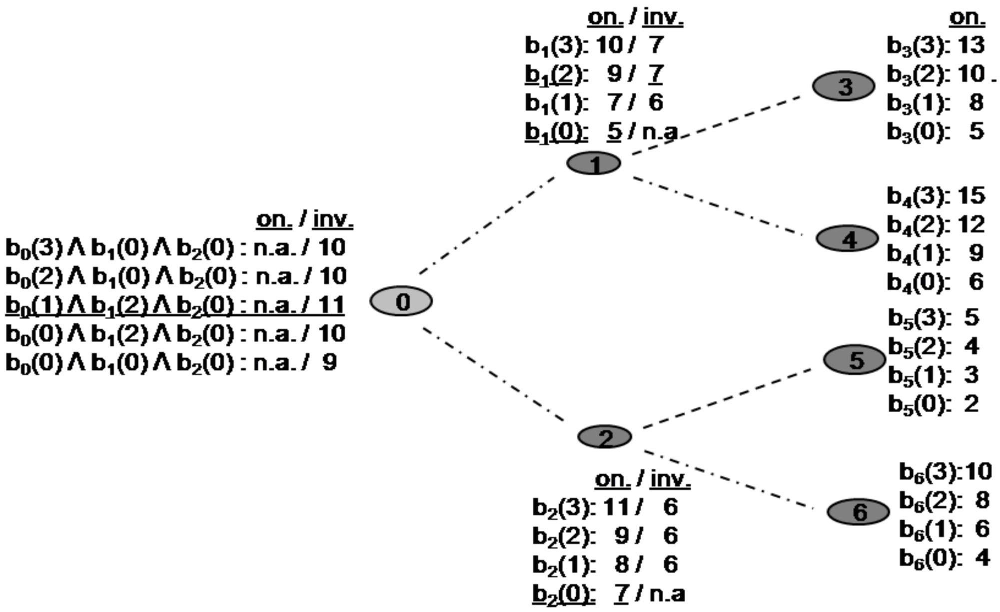


Figure 3.5: Optimal Exercise and Dominating States

In the given example the company value would reach $w_{comp} = 11$ in the case that the company will follow its a priori determined optimal investment which will be exercising the first project right away in the root node expressed by $b_0\,(1)$ and two project in the vertex v_1.

In the following the impact on the optimal exercise of different assumptions on the real option investments should be computed for some examples. It is assumed that the company cash flow depicted in the graph are given as a basis assumption.

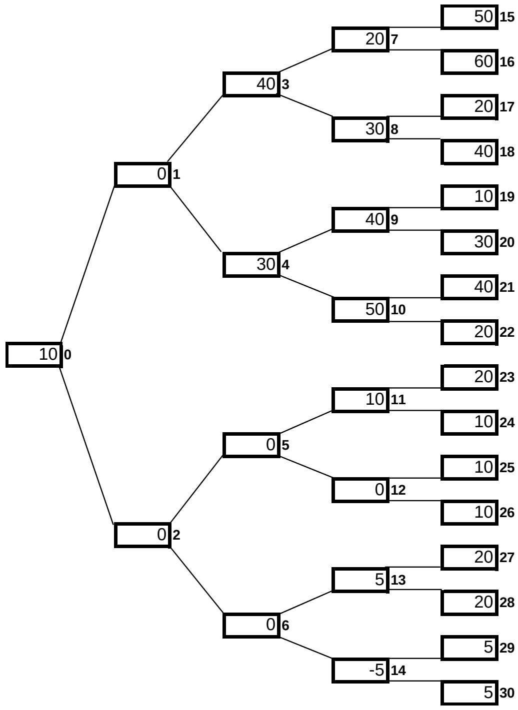


Figure 3.6: Assumed Basis Company Cash Flows

These cash flows will be replicated in a next step by three replication assets. Their price process should be approximated by a GBM with the following parameters:

Parameter	Asset 1	Asset 2	Asset 3
Price a	5	6	7
Interest Rate r	6%	6%	0%
Volatility σ	20%	30%	10%

Table 3.1: Input Parameter of Replication Assets

The developed company valuation algorithm yields the following results for the basis values w_i of the company in the equivalent nodes:

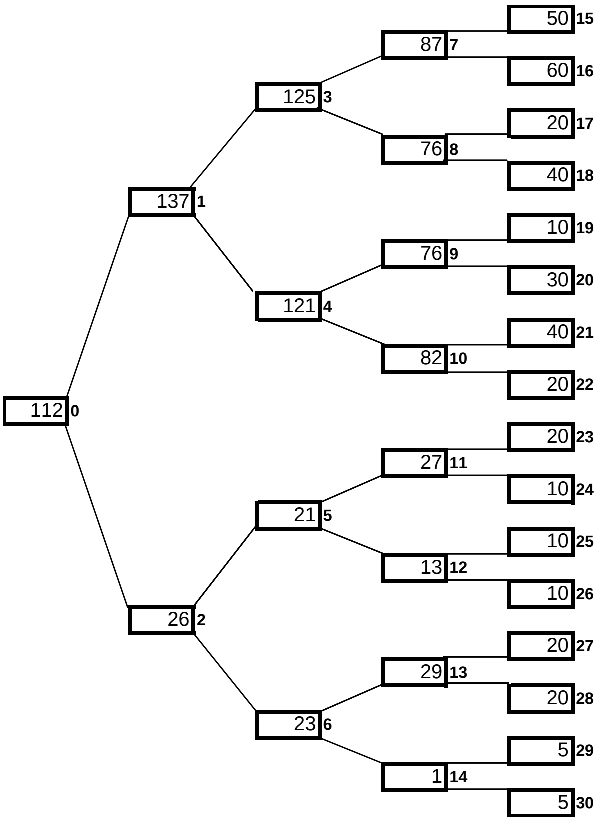


Figure 3.7: Computed Basis Company Value

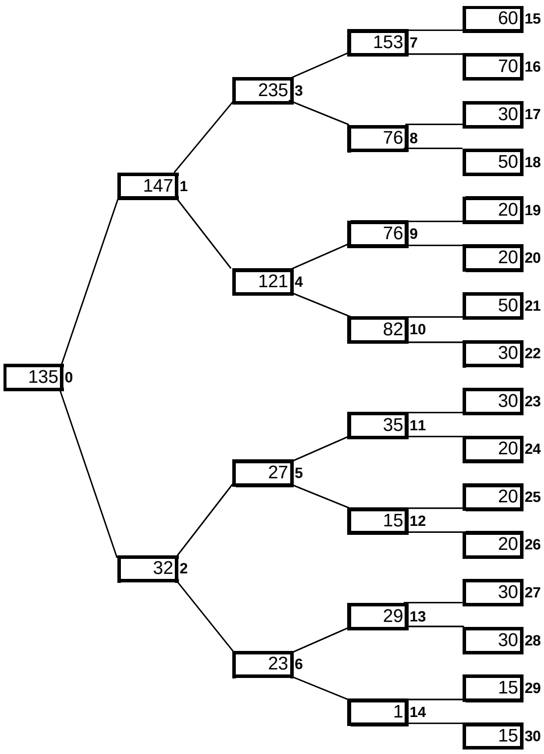
The expected cash flows in the different scenarios translate to the computed basis values of the company in each node. Figure 3.7 clearly indicates that the main value accumulation of this e.g. start-up company scenario occurs in the upper nodes. In the next step the company investigates some further investment projects which can be exercised anytime between t_0 and $T-1$. In a first scenario it is assumed that the investments increase the company basis cash flow by the additive factor gf_{add} as described in chapter 2.2. The maximum additional expenditure is limited to $\widehat{g} = 20$ which can be invested two times in a project that costs $g = 10$ and increases the company cash flows $b\left(g\right)_j$ after exercising the investment project by 50% per invested unit in all future nodes. Therefore, the adjusted company cash flows b_T in the upper node would lead to equation 3.9:

$$b_j(\widehat{g}) = b_j + g f_{add} \times g \tag{3.9}$$

$$b_T(20) = 50 + 0.5 \times 20$$

$$b_T(2) = 60$$

The following results are shown to give the reader the full picture of the computation of the algorithm. The only company value which is directly comparable to the basis company values is the one in node v_0 in t_0; in all other nodes the costs for investing in the additional projects have not been deducted as seen in figure 3.5. If the project is value creating its investment will be deducted in the root node v_0.



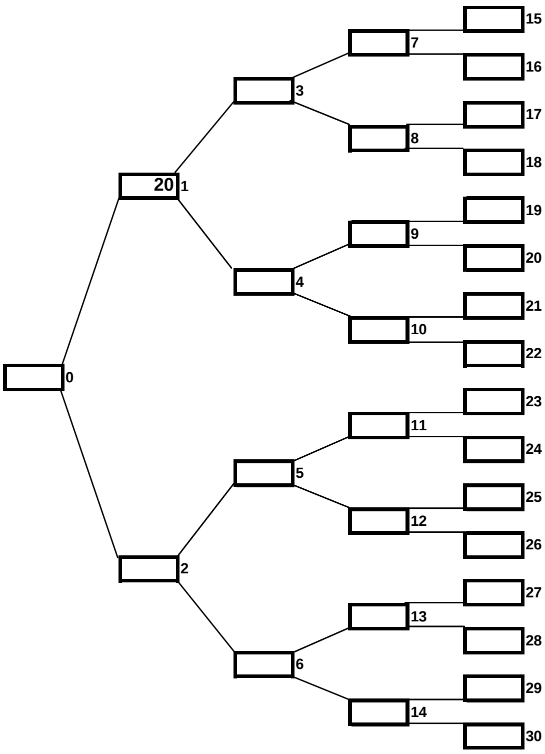


Figure 3.8: Additive Real Option Company Value Scenario and Optimal Exercise

The result shows that an additional amount of money $\widehat{g} = 20$ is sufficient to increase the company value by 20.5% in the case the investment projects are exercised optimal. On the right hand of the graph the optimal exercise of the real option investments is computed. For the given example it would be optimal to invest two times in the investment project in node v_1.

In the following scenarios it is assumed that exercising the two identical investment projects increases the current company cash flows b_j by a certain percentage gf_{multi}. The more successful the company is in terms of cash flow generation, the more valuable is the further investment project that has been exercised in any previous node. In the first scenario each unit invested money g increases all future company cash flows b_j by $gf_{multi} = 25\%$.

$$
\begin{aligned}
b\left(\widehat{g}\right)_j &= \left(1 + \left(g \times gf_{multi}\right)\right) \times b_j \qquad (3.10)\\
b\left(20\right)_T &= \left(1 + \left(20 \times 0.25\right)\right) \times b_T \\
300 &= 6 \times 50
\end{aligned}
$$

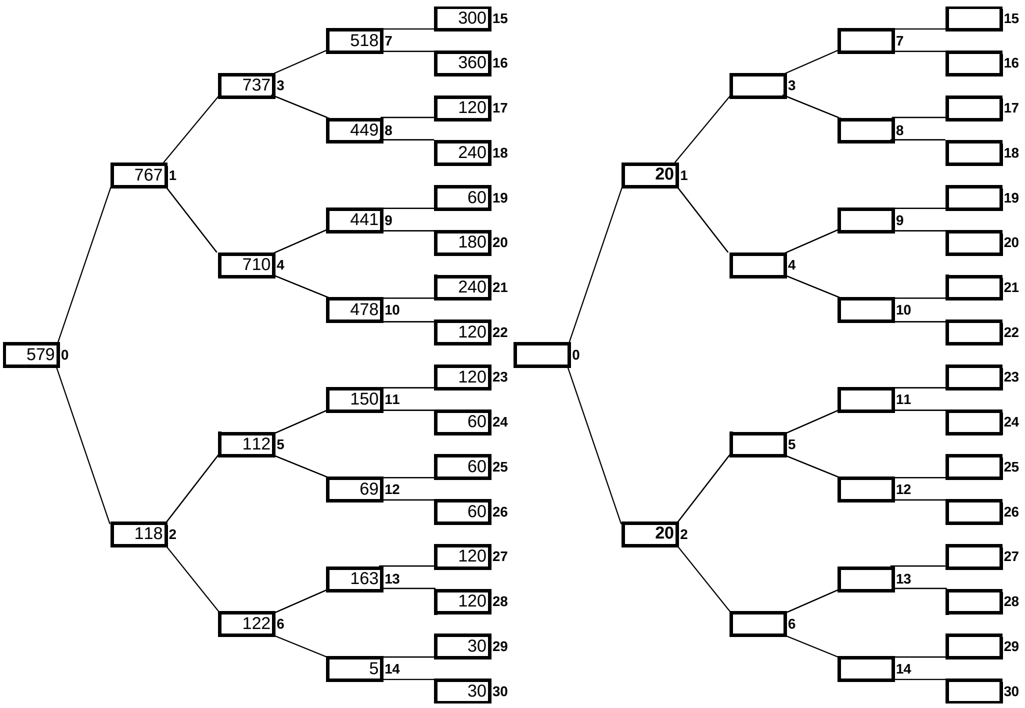


Figure 3.9: 1. Multiplicative Scenario: Real Option Company Value and Optimal Exercise

As expected the overall company value increases strongly due to the high leverage of the additional investment opportunity. As it can be seen the availability of further $\widehat{g} = 20$ units of cash within this quite optimistic and artificial setting would multiply the company value by a factor of almost 5 in the case the investment options would be exercised optimal; which is in the given case again in t_1 independent of the branch the company follows and therefore in vertex v_1 as well as v_2.

In the meantime e.g. the economic environment has slowed down and the additional investment projects were reevaluated again. It has to be stated that the first assumptions were overly optimistic and the new expectations only leads to an increase of the basis cash flow b_j by 10% per invested unit in all future nodes. All other assumptions including the basis company cash flows are unchanged:

$$\begin{aligned} b\,(20)_T &= (1 + (20 \times 0.10)) \times b_T \\ 150 &= 3 \times 50 \end{aligned} \tag{3.11}$$

The computation changes accordingly:

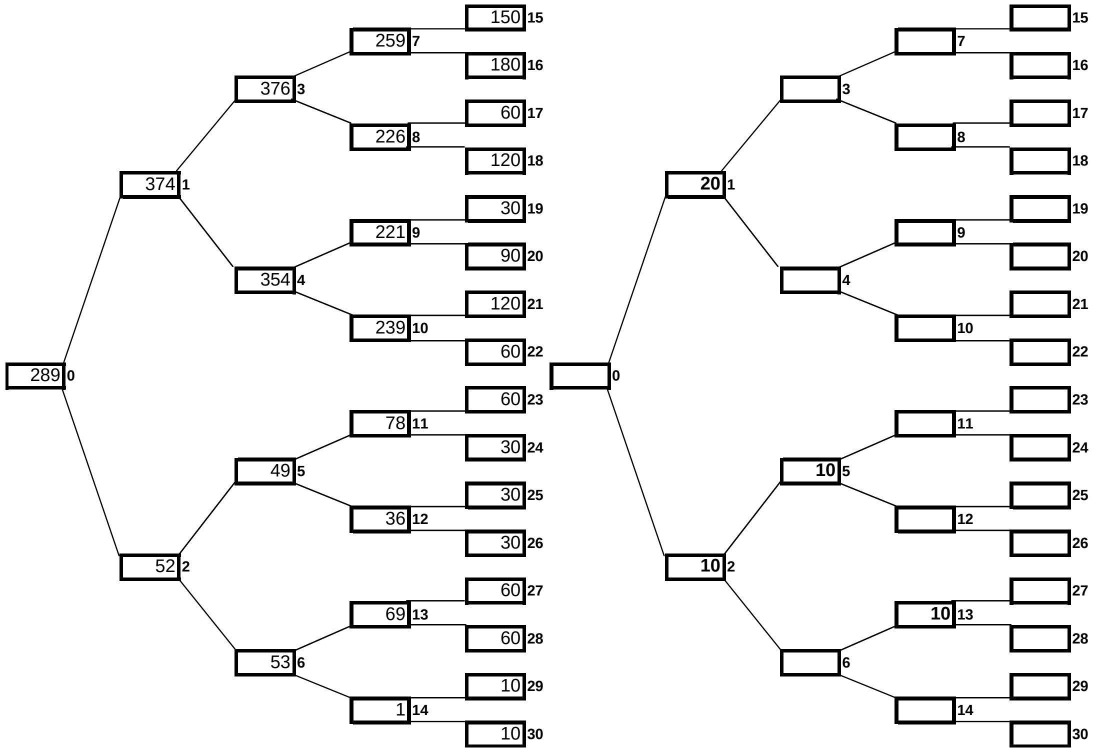


Figure 3.10: 2. Multiplicative Scenario: Real Option Company Value and Optimal Exercise

The overall real option company value now yields $w_0 = 289$ in the case of optimal exercise of the investment projects. Interesting is, that it is not optimal at this stage to exercise the investment projects in t_1 independent of the state. Now it is optimal to invest in both projects in t_1 in vertex v_1 and only in one project in the v_2. In t_2 it is value enhancing to invest additionally in one project in v_5 and in t_3 it is furthermore optimal to exercise the investment project in v_{13}. For an investor, especially in young start-up companies like a venture capitalist, the risk mitigation effect of knowing about the optimal exercise of the real options will be invaluable. The plain value aspect yields 6 units more in company value by exercising optimal instead of investing $g = 20$ right away in t_0.

Further investigation brings up e.g. that another competitor plans to establish the same additional technology which the company planned to invest in by its two investment projects, which leads to even more conservative assumptions regarding these additional investment opportunities:

$$b\,(20)_T = (1 + (20 \times 0.04)) \times b_T \tag{3.12}$$
$$90 = 1.8 \times 50$$

The value of the further two investment opportunities in t_0 is now even more reduced and delivers "only" 54% additional value to the company if both investment projects are going to be exercised in vertex v_1.

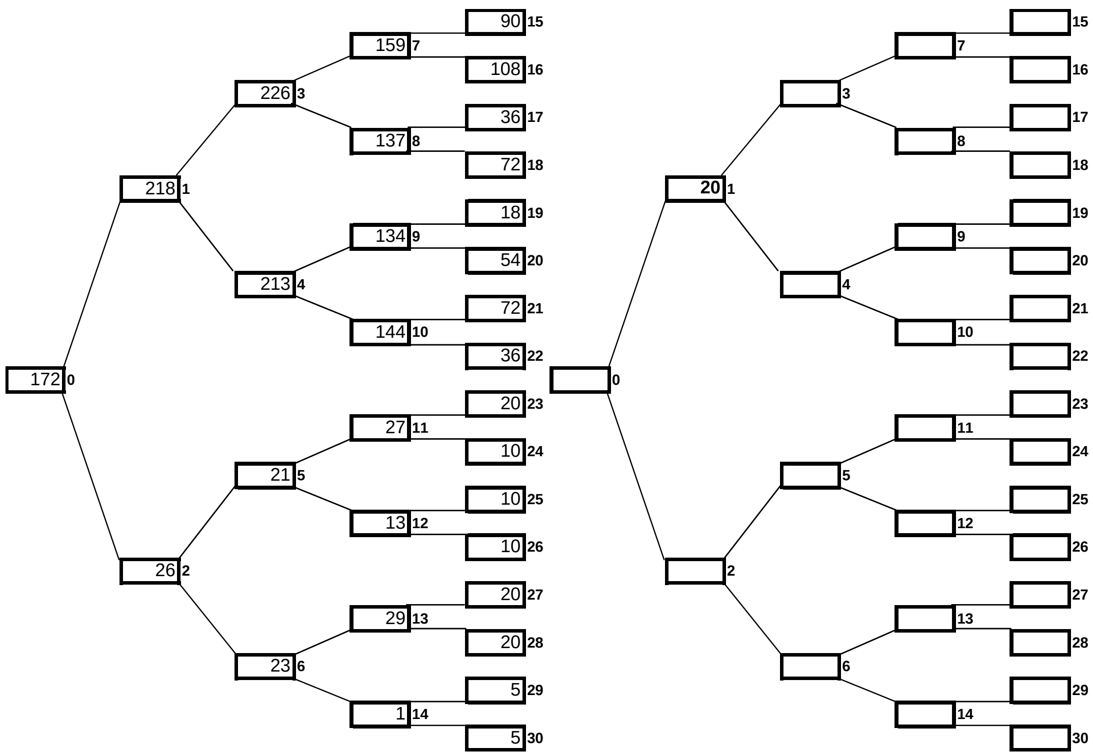


Figure 3.11: 3. Multiplicative Scenario: Real Option Company Value and Optimal Exercise

Assuming the company would have followed the lower branch and an investment in v_2 would have been made, this results in a 4 unit lower company value, which equals around 8% of the additional real option value.

In a last example some more insight in the situation of e.g. a strongly R&D related investment should be given, where the additional spent money only increases the company cash flows in one single branch. Such a scenario applies if the additional R&D investment solely increases the cash flows if the underlying project is successful at all, which happens in one single branch only. It is assumed that each unit invested money g increases the future company cash flows b_j by 10% only in the best branch.

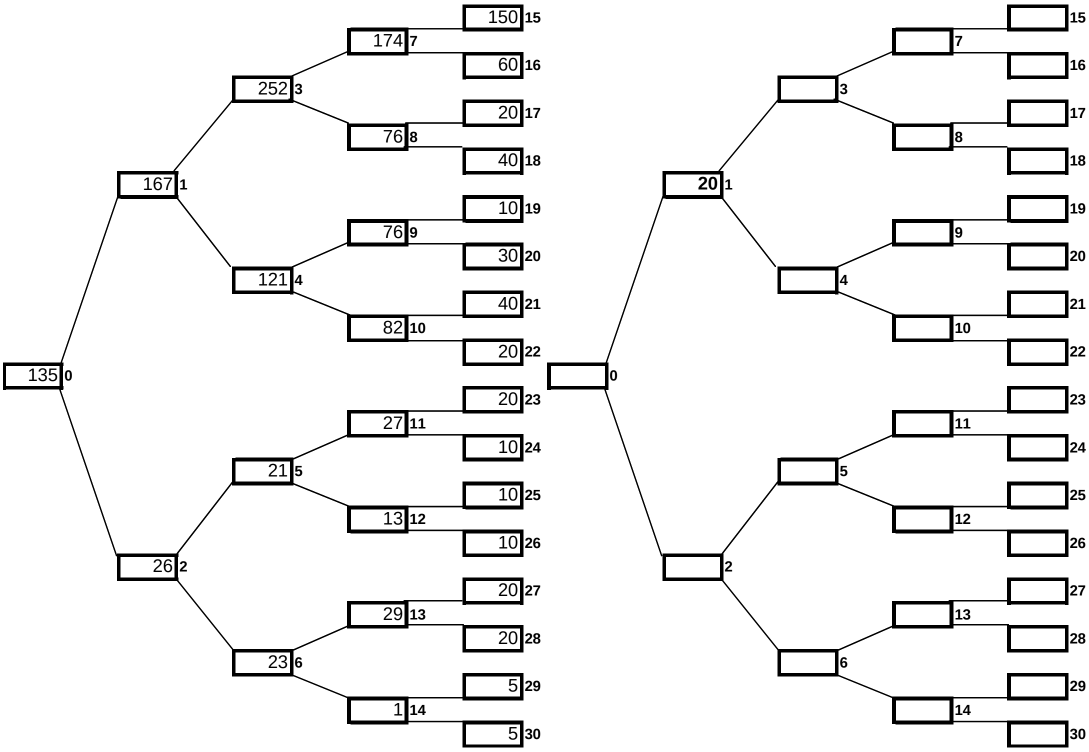


Figure 3.12: 4. Multiplicative Scenario: Real Option Company Value and Optimal Exercise

On the one hand the overall value contribution is strongly reduced which is not a surprise at all. On the other hand it is now only value enhancing to exercise the investment project in v_1. By these simple examples it has be shown how different optimal exercises of investment projects can be, only by differing investment project assumptions. Furthermore the algorithm computes these optimal exercises in any given scenario and therefore facilitates decision makers as well as investors. By this a priori certain hurdle rates for further milestones in a project can be determined and scenarios where it is more valuable to stop a further investment. This will be reached by computing the value proposition of the project as well as the optimal states to exercise the project a priori.

3.4 Comparison to Other Real Option Valuation Approaches

According to Hartmann's (Hartmann 2006) findings, real option valuation tools are still not widely accepted by practitioners in financial service firms and the industry itself. The majority of practitioners see real option valuation tools as an auxiliary tool in addition to the DCF approach.

According to his research, most applied real option models fall in the two categories: Black Scholes and binomial lattices. As it is well known, by using a constant discount rate contingent claim valuation and dynamic programming will yield different results. Explanations are provided by Ingersoll (Ingersoll 1987) and Trigeorgis (Trigeorgis 1996). Trigeorgis proofs that using a constant risk-adjusted discount rate implies that the market risk born per period is constant or, in other words, the total risk increases at a constant rate through time. This will not usually be the case for real option problems which involve multiple embedded options that can be exercised at different points in time. Fama (Fama 1977) shows that correctly risk-adjusted discount rates implied by the CAPM model will not be constant in general, but must evolve deterministically through time. However, the use of a constant risk-adjusted discount rate may be a reasonable approximation in certain situations. This is not the likely case when a management has to make the decision of, e.g. to delay, expand, or contract an investment. It appears that dynamic programming using a constant discount rate and the contingent claim valuation will not yield the same answers for the investment project, except under certain restrictive assumptions. When using the two approaches the optimal exercising nodes can differ significantly. Both approaches are based on the idea to look at the options payoff with a known dependence on a large number of uncertain variables where the timing is optional. Longstaff and Schwartz (Longstaff and Schwartz 1999) work backward in time discretely along a random sample of scenarios to find out when the option should be exercised in each scenario. The value of waiting at each time in each scenario is determined by regressing across the sample of the scenarios. The regression will be performed between the future value of waiting against a finite set of basis functions for an infinite dimensional space of functions on the underlying state space. One of the first real option valuation applications in the upstream petroleum industry, was the analysis of U.S. offshore leases by Paddock, Siegel and Smith (Paddock, Siegel and Smith, 1988). They use the value of the developed field as an underlying variable. Their argument that the volatility in the value of the developed fields is the same as that for crude oil. They calculate the rate-of-return shortfall for the developed reserves by assuming that there are no leverages or other differences in the benefits and costs of holding oil above or below ground. There is no single reversion in their valuation model. It has been presumed that development begins immediately upon the end of exploration and that production begins after a predetermined lag once development has begun. The exploration/development decision may be made at any time during the lease, and the owner may walk away from the lease. Pickles and Smith (Pickles and Smith 1993) continue in this vein by allowing an option in the timing of development rather than exploration. They use a binomial lattice method for performing valuations. Bjerksund and Ekern (Bjerksund and Ekern 1990) use the Brennan-Schwartz price model (Brennan and Schwartz 1985) to build simple models of: First, fixed-time options to develop a field for which they use the Black, Scholes formula; and second, indefinite leases for which they use the Merton formula for an indefinite call on a dividend-paying stock. Smith and McCardle compare the results obtained from an real option evaluation and an equivalent decision tree analysis (DTA) where the cash flows are discounted using DCF methods with a single discount rate. The exploration and delineation leases are also more valuable with increased geological uncertainty if actions can be taken to respond to the resolution of that uncertainty. The simplest method used to price real options is the DTA. This method exhibits two major drawbacks. First, the decision tree structure becomes complicated even in simple real world situations, second the decision tree analysis uses real probabilities and risk-adjusted interest rates. According to Trigeorgis (Trigeorgis 1996) DTA can be seen as an advanced version of DCF. Furthermore, he adds that the asymmetry resulting from operating flexibility and other strategic aspects can, nevertheless, be properly analysed by thinking of discretionary investment opportunities as options on real assets through the technique of contingent claim analysis. These main drawbacks are eliminated in the contingent claim analysis by using risk-neutral probabilities as well as the risk-free interest rate.

3.5 Sensitivity Analysis of the Real Option Company Valuation Algorithm

The contingent claim analysis is based, as already mentioned, on the replication idea which means the replication of the payoff structure of the project/company and its inherent options via traded assets and financial transactions within this replication portfolio. According to Trigeorgis the Cox, Ross and Rubinstein binomial tree is one broadly applied model within the contingent claim analysis. In the following the Cox, Ross, and Rubinstein (CRR) model should be applied for the case of an expansion option and compared to the developed company valuation algorithm:

$S =$	104	price of the traded asset that is almost perfectly correlated with w
$r =$	8%	risk-free interest rate
$I =$	104	equates the original investment opportunity as the initial-scale project plus a call option on a future opportunity
$g =$	40	is the firm's option to invest an additional 40 outlay one year after the initial investment
$T =$	1	Time
p		risk-neutral probability for an up movement per period
u		multiplicative factor for an up movement per period
d		multiplicative factor for an down movement per period
w		total value of the project

Table 3.2: Input Parameter for Model Comparison

The total value of the project can be calculated by applying the following algorithm, which was already derived in chapter 1:

$$w_0 = (p \times b_{iu} + (1 - p) \times b_{id})) \times df - I \tag{3.13}$$

The given input factors lead to the following parameters: $u = 1.73$, $d = 0.58$, $S_{1u} = 180$, $S_{1d} = 60$, $df = exp(-r \times T) = 0.923$, $p = 0.44$, $1 - p = 0.56$, and the wealth $w_0 = 0$ without considering the growth option. The following graph depicts the structure within the Cox, Ross, Rubinstein framework for an option to expand (growth option), which means the right to make follow-on investments.

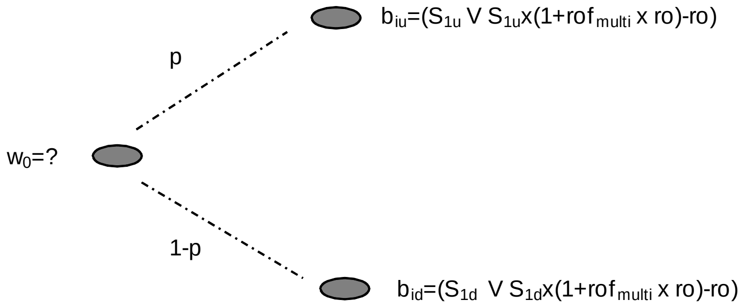


Figure 3.13: Valuation of an Expansion Option within the CRR Framework

Interpret the original investment opportunity as the initial-scale project plus a call option on a future opportunity, and suppose the firm has the option to invest an additional $g = 40$ outlay one year after the initial investment which adds a range of percentage points $(0\% - 100\%)$ to the scale and value of the project.

$$\begin{aligned} w_0 &= \max(((p \times b_{iu} + (1-p) \times b_{id})) \times df - I; ((p \times b\,(g)_{iu} \\ &\quad + (1-p) \times b\,(g)_{id})) \times df - (I + a) \end{aligned}$$

The results show that the newly developed company valuation algorithm generates robust results which converge to the results generated by the Cox, Ross, Rubinstein model with increasing option values. For low option values, the deviation is the result of the optimisation by integers and by cutting the fractions of an integer in the programming instead of rounding.

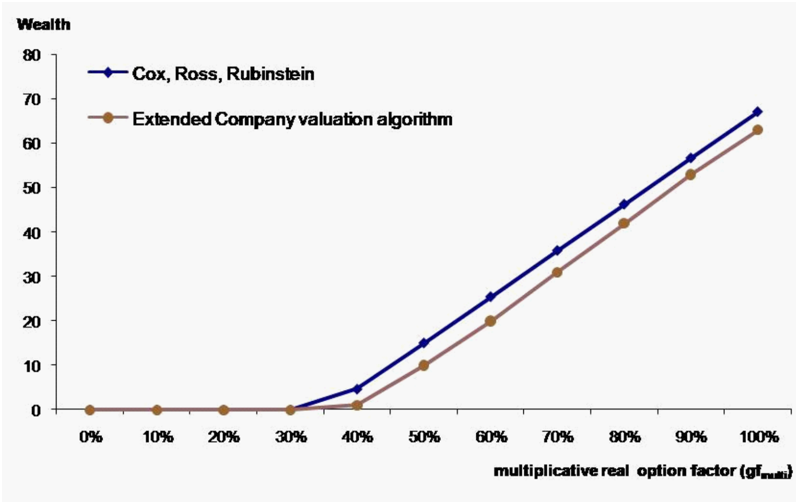


Figure 3.14: Model Comparison for One Step Expansion Option

Within the developed model expansion options for multiple time steps are not directly comparable due to the nature of the developed model. In the developed model, the value of the overall cash flows throughout the valuation period will be computed instead of taking the value of the project in $t = T$ as the basis for calculating the wealth of the option. Deferable options are also not directly comparable to established other real option valuation models. The same argument is valid as mentioned above in addition to the fact that there is always a delay of one time period between investment and the improved value of the investment or its underlying cash flow stream. This assumption, even though not directly comparable to other applications at this point, takes a more realistic approach into account.

In the following, a sensitivity analysis for a generic expansion option for multiple time steps should be performed. The analysis for the given generic example shows that any changes in r and σ do not have an impact on the option value of the project or company.

As expected, an increase in b is linearly correlated and an increase in g is nega-

Variable	Value
Company cash flows	
b_o	10
r	6%
T	$1-10$
σ	20%
Real options	
g	10
gf_{multi}	50%
Replications assets	
r	6%
σ	15%/20%/25%

Table 3.3: Input Parameter for Sensitivity Analysis

tively correlated to the wealth of the projects expansion option.

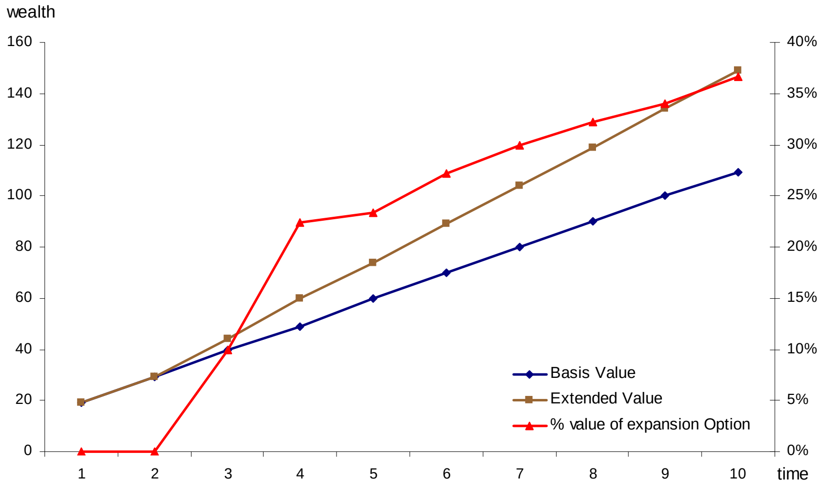


Figure 3.15: Sensitivity of the Expansion Option Value by Time

For the multiplicative case, it can be shown that by increasing the number of time steps the proportion of the option value increases, but with a decreasing relative magnitude.

Chapter 4

Evaluation of the Developed Algorithm to Two Case Studies

In the following, two real-world companies are analysed and valued. Both companies are listed, high growth Life Science companies. In this work, the focus should be on applying and furthermore validating the valuation algorithm which can be shown easier within the framework of traded assets compared to non-traded assets. This experience can then be transferred to the application of the company valuation algorithm on non-traded private companies. Life Science companies were mainly chosen because of three reasons: First, their R&D process is comparably costly and complex which makes the application of real option valuation techniques most valuable. Second, young publicly start-up companies are existing, and third, my personal academic as well as working experience is leveraged heavily towards the valuation of life science companies. The last aspects should help to reduce pitfalls as a result of a lack of industrial experience.

4.1 Generic Marketing Example

4.1.1 Description

Qiagen is one of the market leaders for supplying purification technologies in the laboratory medicine market and has a strong presence in molecular diagnostics. The company has established a high market penetration within diagnostic laboratories as well as blood banks. Qiagen has acquired the American firm Digene in June 2007 for around USD 1.6 billion (55% in cash and 45% in shares). Digene is a molecular diagnostics company, with the only human papilloma virus (HPV) test approved by the FDA. The HPV test is used to detect the risk of cervical cancer in women. The acquisition will increase Qiagen's sales of diagnostic tests by USD 270 million in 2008 boosting the company's total sales by almost 50%. Digene's sales from HPV tests should reach around USD 180 million in full year 2006/07 which translates to an organic growth rate of 42% year over year. The market share of HPV testing of Digene is estimated at around 90% since the only significant competitor is Roche Diagnostics having one product on hand which is only approved in Europe so far. According to analyst estimates, the total market for the HPV test should reach around USD 200 million, which equates to a compounded annual growth rate (CAGR) of around 30% for the next five years. In the U.S., the HPV test is recommended as part of the first-line diagnostics, in Europe the HPV test is only an additional out of the pocket testing option for cancer prevention. With the acquisition of Digene, Qiagen will achieve around 45% of its sales in the more attractive market segment of molecular diagnostics, due to higher pre-tax margins and higher expected growth rates.

4.1.2 Forecasting Company Cash Flows

The free cash flow is that part of a company's cash flow that is earned in excess to the cash flow that is required to carry out its own business plan including the required expansion costs. Thus, the free cash flow can be withdrawn by the investors throughout the life span of the company. According to the well established DCF valuation, the free cash flow expectations discounted with the appropriate WACC to the present time is equivalent to the value of a company. In the derived company valuation model, the same free cash flow will be used as a basis for the calculation of the company value. In general the free cash flow will not be explicitly estimated by analysts, but it can be derived by different profit and loss figures like the EBIT. The estimates most widely available in I/B/E/S or Datastream are sales, EBIT, and net income. Therefore, a robust choice will be to use the net income as a basis for calculating the free cash flow of Qiagen.

	Net Income
+	Depreciation/Amortisation
-	Change in Working Capital
-	Capital Expenditure
=	Free Cash Flow

Table 4.1: Free Cash Flow Derivation

For further expanding business from its current status, no significant further investments above the magnitude of the depreciation/amortisation should be required. The change in working capital should be in the area of around $5 - 15\%$ of the sales increase and, therefore, reduces the free cash flow in the initiating node but will offset this effect partly by a slightly higher growth rate of the free cash flow compared to not taking the change in working capital into account. This effect results of increasing margins throughout the estimate period. The average of the net income estimates, being provided by ten different analysts, shows a net income CAGR of 22% for the next four years. The standard deviation or volatility of the estimates counts for an average of $\sigma = 20\%$ per year. The volatility on the historical share price is $\sigma = 24\%$ for a one year period calculated on 260 trading days.

For the coming twelve months, the share price of Qiagen will mainly be driven by three factors. Firstly, the progress on integrating the Digene acquisition, secondly the expected margin improvements in their core business laboratory purification technologies, and thirdly the overall sentiment for European high growth equity investments.

	Year	2007e	2008e	2009e	2010e	2011e
	Net Income estimates	$90,102$	$112,444$	$148,568$	$174,244$	$201,066$
+	Depreciation/Amortisation	$34,000$	$34,500$	$35,000$	$35,500$	$36,100$
-	Change in Working Capital	$6,000$	$6,600$	$7,260$	$7,986$	$8,785$
-	Capital Expenditure	$34,000$	$34,500$	$35,000$	$35,500$	$36,100$
=	**Free Cash Flow**	$\mathbf{84,102}$	$\mathbf{105,844}$	$\mathbf{141,308}$	$\mathbf{166,258}$	$\mathbf{192,282}$
	CAGR	$\mathbf{23.0\%}$				
	Volatility	$\mathbf{19.7\%}$				

Table 4.2: Qiagen Free Cash Flow Derivation

4.1.3 Replication Assets

For the replication portfolio, traded assets which are closely linked to the performance of the company should be chosen. The attributes of a good predictor variable are: It is closely related to the dependent variable and is not highly related to other independent predictor variables which then would be referred to as collinearity. Therefore, the collinearity of the replication assets will be taken into account as well. Finally, the intercorrelation of the selected assets should be limited but should explain most of the value of the company w that has to be calculated by

$$w = k_0 + k_1 a_1 + k_2 a_2 + k_2 a_3 ... + \epsilon \tag{4.1}$$

where a is defined as the asset price value, $k_{1...n}$ are the optimal weights, and ϵ is the error or residual term. It should be searched for a replication portfolio which minimises ϵ to achieve a result which is closest to a complete market setting.

Being one of the leading suppliers in laboratory purification technologies as well as a meaningful player in the molecular diagnostics market, the first idea is to include the closest comparator of Qiagen in both segments which could be Applied Biosystems (ABI) as well as Cepheid (CPHD). Because Qiagen (QIA) is not directly comparable to any other listed company on the market two indices created by

DZ BANK, the Supply [6] index as well as the Diagnostics [7] index have been investigated. As Qiagen's operations mainly focus on these two segments it looks attractive to replicate Qiagen's cash flows by the performance of the indices. In 4.3 some indices like the Nasdaq Biotechnology Index (NBI), and the FTSE World Pharmaceutical index (F3PHRM) have been tested due to the sector relation of Qiagen and the dependence on the same approval authorities as well as reimbursement schemes for diagnostics and new drugs. The FTSE World Pharmaceutical index consists of the 25 largest pharmaceutical companies with an overall market capitalisation of well above USD 1.000 billion, holding by far the largest market share of the worldwide pharmaceutical market. This market had sales of USD 570 billion in 2006 and is expected to grow at a rate of around 7% per year for the coming five years. Due to Qiagen's focus on German high growth companies, the TecDax (TDXP) index has been evaluated for inclusion as well.

The analysis of their correlation yields results between 0.417 to 0.847 as shown in the correlation matrix (4.3) for the chosen eight potential replication assets. The assets suited best for replication at a first glance would be the German TecDax Index and the DZ BANK Supply Index.

[6] The DZ BANK Biotech Supply index has the following members: Applied Biosystems Group-Applera Corp, Bruker Corp., Invitrogen Corp., Millipore Corp, Tecan, Techne Corp., Waters Corp., Affymetrix, Caliper Life Sciences Inc., Mettler Toledo Intl, Perkinelmer Inc, Sigma Aldrich Corp, Qiagen, Sartorius AG, Illumina, Luminex Corp., Nanogen Inc, Geneart AG, Thermo Fisher Scientific Inc., Abcam, Tepnel Life Sciences, Bio Rad Laboratories

[7] The DZ BANK Biotech Diagnostics index has the following members: Biosite Inc, Cytyc Corp., Digene Corp., Diagnostic Products, EXACT Sciences Corp., Innogenetics, Pharmanetics, Quidel Corp., Qiagen, Medtox Scientific, Trinity Biotech, Third Wave Tech Inc, Ventana Medical Syst Inc, Affymetrix, Sequenom, Techne Corp., Idexx Laboratories Inc, Given Imaging Ltd., Orasure Technologies Inc, Bioveris Corp., Nanogen Inc, Neogen Corporation, Quinton Cardiology Systems Inc, Cholestech Corporation, Biotest AG, Diagnostic Medical Systems, Cytogen Corp, GenProbe Incorporated, Applied Imaging Corp., Clarient Inc, MZT Holdings, Cepheid, Diagnocure, Immunicon, Epigenomics AG, Biomerieux Inc, Beckman Coulter, Dade Behring Inc, Genomic Health Inc., Inverness Medical Innovations, Immucor Inc, Meridian Bioscience Inc

Assets/Ind.	QIA	NBI	F3PHRM	TDXP	CPHD	ABI	EUR12M	JITX	Supply
NBI	0.428								
F3PHRM	0.065	−0.261							
TDXP	0.847	0.542	−0.213						
CPHD	0.417	0.395	−0.341	0.724					
ABI	0.461	0.168	0.352	0.367	0.158				
EUR12M	0.788	0.252	0.093	0.832	0.572	0.731			
JITX	0.812	0.336	0.083	0.828	0.446	0.674	0.944		
Supply	0.841	0.460	−0.137	0.934	0.644	0.597	0.943	0.934	
Diagn.	0.748	0.489	−0.376	0.940	0.801	0.334	0.811	0.762	0.927

Table 4.3: Correlation Matrix for Two Years Historical Prices

One major difficulty as already mentioned by e.g. choosing these indices for replication is the high collinearity. Therefore, the best replication with two assets will be reached by the German TecDax and the FTSE World Pharmaceutical index offering the highest explained contribution to the historical share prices of Qiagen with an overall $R^2 = 78.1\%$. In a second trial, the best three replicating assets will be chosen which would be the two previous mentioned assets and Cepheid explaining $R^2 = 83.1\%$ on historical prices. In case of choosing four assets the optimal replication portfolio would hold the German TecDax, Cepheid, the FTSE World Pharma Index and the DZ BANK Diagnostic Index explaining an overall $R^2 = 84.1\%$ to the historical share price of Qiagen. It is assumed that all selected assets/indices follow a GBM. It can be seen that the additional explanation of the third and fourth asset/index being included as an replication asset is relatively small.

4.1.4 Results Base Case

The current market capitalisation of Qiagen accounts for USD 2.5 billion as of August 1st 2007. For calculating the value of Qiagen without considering further real options, a CAGR of 23% in free cash flows over the valuation period was considered. The applied volatility range is geared to the expected volatility that can be derived, taking all analyst earnings expectations for the coming years into account, with $\sigma =$

20% and the historical volatility for the last two years at $\sigma = 24\%$ per year. In the following the overview of the chosen input parameters is given.

Parameter	QIA	TDXP	F3PHRM
adj. FCF/Price	$84,104$	921	$8,271$
CAGR/Interest Rate	23%	6%	6%
Volatility	$15\% - 30\%$	18.77%	16.53%

Table 4.4: Overview of Qiagen Input Parameter

The table outlines that the results are highly sensitive regarding the chosen volatility σ. The rise in value on the high side of the cash flow volatility is a result of an increasing excess replication in some nodes and can be viewed as a high incompleteness of the market. The excess replication value would even further increase by assuming that the replication assets versus the company cash flow are not perfectly correlated.

Volatility	15%	20%	25%	30%
10 timesteps	$1,368$	$1,572$	$1,875$	$2,313$
9 timesteps	$1,191$	$1,353$	$1,581$	$1,902$
8 timesteps	$1,032$	$1,155$	$1,320$	$1,554$
7 timesteps	882	969	$1,095$	$1,254$
6 timesteps	744	807	891	999
5 timesteps	612	657	708	783

Table 4.5: Qiagen Base Case Company Valuation with Two Replication Assets

The lowest replication values are going to be achieved at the lowest volatility of $\sigma = 15\%$ per year. In the following chart, it can be seen that the sensitivity on a change in σ increases by adding further time periods because the excess replication value accumulates over time.

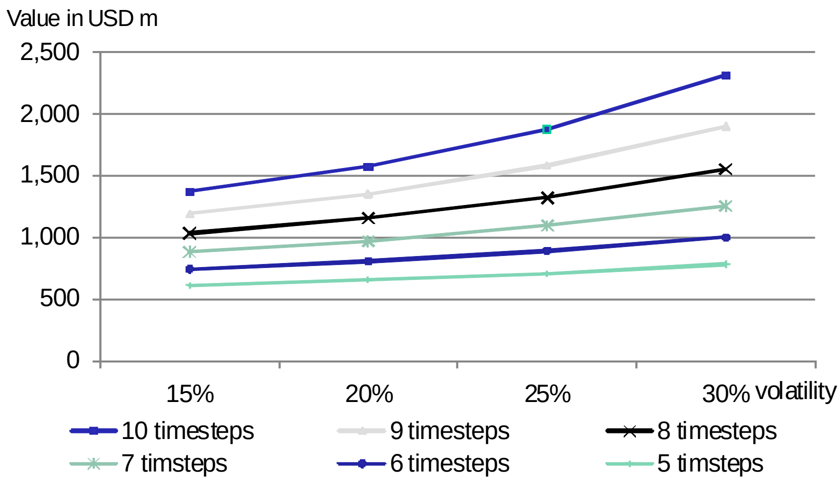


Figure 4.1: Qiagen Base Case Valuation Sensitivity Chart on Volatility Changes

The added value for further time periods increases in absolute terms for each additional time period. The relative contribution ranges from 15% to more than 25% by adding a further timestep. The sensitivity regarding the addition of further timesteps decreases for higher number of timesteps which was expected because cash flows far out in the future do contribute less to the company than near term cash flows. In this example the sensitivity of the value change is higher by assuming higher volatility for Qiagen. This effect is mainly attributed from a low volatility replication portfolio which can be seen later on.

Volatility	15%	20%	25%	30%
10 timesteps	15%	16%	19%	22%
9 timesteps	15%	17%	20%	22%
8 timesteps	17%	19%	21%	24%
7 timesteps	19%	20%	23%	26%
6 timesteps	22%	23%	26%	28%

Table 4.6: Qiagen Base Case Val.: Relative Contribution for Increasing N.o. of Time Periods

The given results were based on two replicating assets/indices; in a next step, it should be investigated whether the excess replication can be reduced by adding the third selected replication asset, Cepheid, to the replication portfolio. With a historical volatility of $\sigma = 44.53\%$ this index is expected to reduce the increase in the replication value for the scenarios with expected volatilities on the high side. Nevertheless, it should be pointed out that the additional explanatory value for the Qiagen share price is very limited, as the R^2 only increases by around 5% percent points to $R^2 = 83\%$ as mentioned before.

Parameter	QIA	TDXP	F3PHRM	CPHD
adj. FCF/Price	$84,104$	921	8271	14.75
CAGR/Interest Rate	23%	6%	6%	6%
Volatility	$15\% - 30$	18.77%	16.53%	44.53%

Table 4.7: Overview of Qiagen Input Parameter with Three Assets

The following graph shows that including the third replication asset with a high volatility decreases the replication values significantly for examples assuming high volatilities for the Qiagen's future cash flows.

Volatility	15%	20%	25%	30%
10 timesteps	766	792	832	880
9 timesteps	701	730	764	814
8 timesteps	642	671	708	749
7 timesteps	585	609	643	669
6 timesteps	528	555	581	596
5 timesteps	472	490	516	523

Table 4.8: Qiagen Base Case Company Valuation with Three Replication Assets

The derived company values are almost constant independent of the underlying volatility of the company cash flows ranging between $\sigma = 15\%$ and $\sigma = 30\%$. This result can again only be achieved under the assumption of perfect correlation of asset price and cash flow development.

Volatility	15%	20%	25%	30%
10 timesteps	9%	9%	9%	8%
9 timesteps	9%	9%	8%	9%
8 timesteps	10%	10%	10%	12%
7 timesteps	11%	10%	11%	12%
6 timesteps	12%	13%	13%	14%

Table 4.9: Qiagen Base Case Company Relative Value Increase with Three Replication Assets

It remains open what is the best scenario the investor should use to value Qiagen for the base case. The expected volatility does not have an impact on the company value within a broad volatility range especially if the number of replicating assets will be increased in each state.

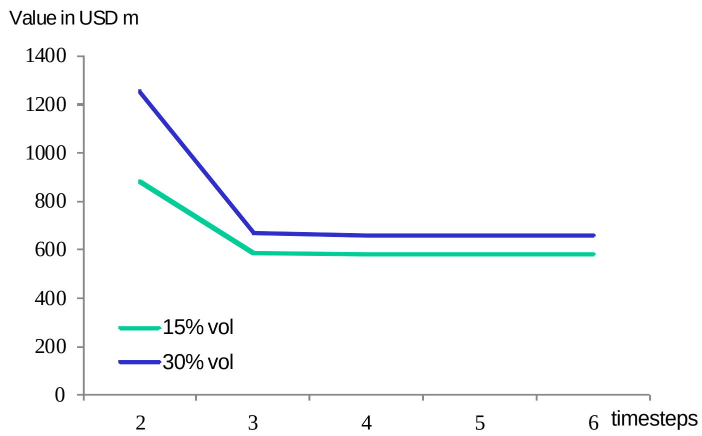


Figure 4.2: Reduced Excess Replication by Increased Number of Replication Assets

An increased number of subnodes by itself does not change the company value if the price path of the replicating assets as well as the company cash flows are 100% correlated. In the case where more than two basis scenarios for the company cash flows are expected in every state the number of subnodes should be increased accordingly.

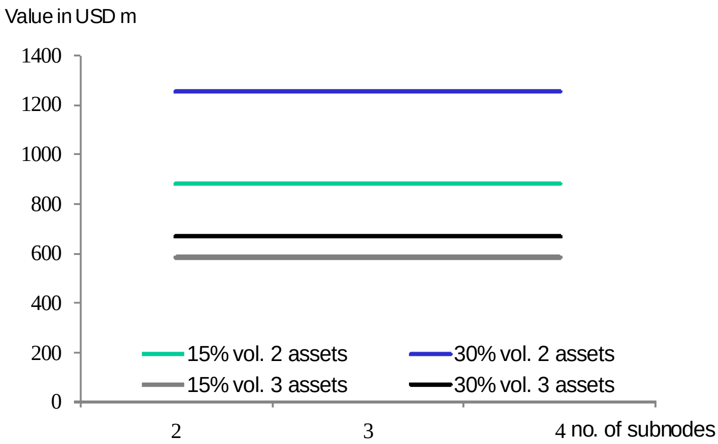


Figure 4.3: Company Value for Increasing Number of Subnodes

The company value, therefore, is only highly sensitive and strongly correlated to an increasing number of time steps. By having assumed an above market growth rate of the company's cash flows, the result is obvious. The answer has been investigated in several econometric models and is valid for this company valuation model, too. The above market growth rate of the company being considered for valuation will deteriorate and converge to the market growth rate over time. It seems to be reasonable to take an above market growth rate for $5 - 25$ years into account, depending on the company and its industry. It would be optimal to include a smoothing factor for the company growth rate which would lead to the most realistic scenario. The market capitalisation of Qiagen on the 1st of August 2007 was USD 2.5 billion and therefore about 33% higher than the computed company value with $\sigma = 25\%$ and a growth rate of $\mu = 24\%$ p.a. for ten time periods with 2 replication assets and 2 subnodes for each state. The difference dramatically increases if the number of replication assets will be increased but not if only the number of subnodes for each state will increase. The second argument is highly dependent on the assumed underlying price path. Some value difference between the market and the model can potentially be explained by the companies real options.

4.1.5 Real Option Assumptions and Evaluation

So far, the value for Qiagen has been calculated on a set of scenarios mainly taking the organic growth into account. Firstly, the company's main investment opportunities beyond the organic growth, should be evaluated. This can be achieved either by raising further capital and, e.g. buying another competitor like Digene, or heavily expanding the product base through inhouse research which could be sold through their sales force. Furthermore the management could consider to broaden its franchise by buying further market players in the field of molecular diagnostics in the US or Europe which would enable the company to capture an additional market share in this high margin market segment. Therefore, one real option the company could consider would be to buy either Gen-Probe, a large US molecular diagnostics company with a current market capitalisation of USD 3.2 billion, or some smaller molecular diagnostics companies. Although most of them are still loss-making, an acquisition could be attractive due to their innovative diagnostics technologies. Almost all of these companies are currently valued at around USD 100 million and generating sales of USD $20-35$ million. With such an acquisition, Qiagen would not only be able to add these sales and generate some cost synergies, but most importantly, the company could broaden the product portfolio in molecular diagnostics of their sales forces in the US as well as Europe. Based on the valuation parameters, it can reasonably be assumed that Qiagen's free cash flow b will improve by around 9% through an acquisition for USD 100 million.

Taking a brief look on the balance sheet, the company was almost net debt free prior to the Digene acquisition. It seems to be reasonable to raise further capital through a secondary in the extent of minimum 10%, which would give Qiagen at least the potential for further investments of USD 300 million. Raising this 10% capital would be possible for the Qiagen management even in market situations which do look far less favorable than nowadays.

The second feasible opportunity Qiagen has, is to reach an approval for its multi-respiratory disease diagnostic and sell this new product throughout their US as well as European sales force. Industry experts expect Qiagen to be able to market its diagnostic test to simultaneously detect several respiratory diseases in the U.S. within the next two to three years from now. The test is based on Qiagen's own technology platform and the market penetration can be improved significantly by combining Qiagen's and the newly acquired Digene sales forces. The required investment would stay below USD 50 million for the respiratory diagnostics, but could add 4.5 percent points in free cash flow. Both options should be considered and valued regarding its potential value contribution to Qiagen.

Even though this procedure is quite generic and limits the input parameters to a few, it gives the opportunity to rate different investment opportunities regarding the attractiveness in value terms. Furthermore the computation of the optimal exercise based on an outlay of information the executive management has agreed upon offers further insight into the implications of preferring one investment opportunity over another.

4.1.6 Results Real Option Valuation

In the following, the two evaluated strategic options are being considered and their potential value contribution will be calculated. In a first scenario the calculation will performed in a binomial setting as a starting point as well. The replication will performed again by the German TecDax and the FTSE World Pharmaceutical index. Changes in real option values by increasing the number of replication assets and the number of subnodes per state to 3 or 4 are evaluated at the end of chapter 4.

Volatility	15%	20%	25%	30%
10 timesteps	$1,461$	$1,716$	$2,106$	$2,649$
9 timesteps	$1,242$	$1,437$	$1,731$	$2,136$
8 timesteps	$1,035$	$1,185$	$1,404$	$1,686$
7 timesteps	882	969	$1,107$	$1,308$
6 timesteps	744	807	891	999
5 timesteps	612	657	708	783

Table 4.10: Qiagen Extended Real Option Valuation

The results do show an increasing impact of the real options with an increasing number of timesteps but do not differ from the basis company values if the valuation is limited to 5 and 6 timesteps. By limiting the valuation to 5 or 6 timesteps the real options expire worthless in any scenario.

Volatility	15%	20%	25%	30%
10 timesteps	7%	9%	12%	15%
9 timesteps	4%	6%	9%	12%
8 timesteps	0%	3%	6%	8%
7 timesteps	0%	0%	1%	4%
6 timesteps	0%	0%	0%	0%
5 timesteps	0%	0%	0%	0%

Table 4.11: Qiagen Extended Real Option Relative Value Contribution

Exercising the first option up to three times and considering up to 10 timesteps adds up to 15% to the company's basis value w_{comp}. The calculation yields the result that the optimal exercise of the expansion option to acquire other small competitors is only value enhancing in some better developing nodes. Two results are depicted in the following graph: On the left side the scenario with 7 timesteps and a assumed volatility of $\sigma = 30\%$ and on the right side of the figure for 8 timesteps and a volatility of the company's cash flows of $\sigma = 25\%$.

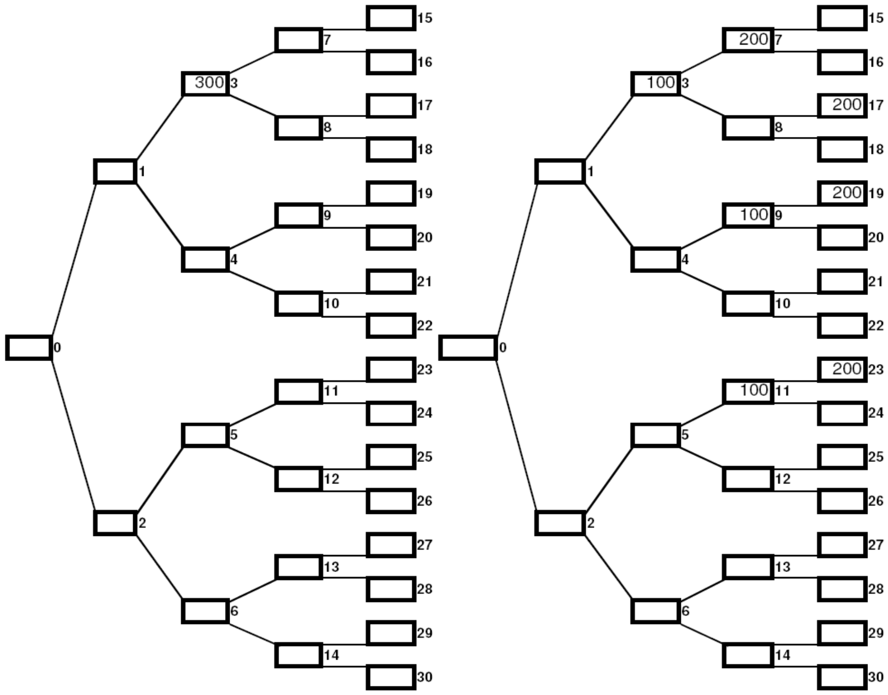


Figure 4.4: Qiagen Optimal Exercise

All other results for optimal exercise are listed in the appendix. Sticking to a valuation by considering ten time periods, it can be seen that the maximum company value including both real options increases to USD 2.1 billion which would only be 16% below the market capitalisation of USD 2.5 billion end of July 2007. It can be assumed, that either the investor expects the company to carry out these real options near term or the time period with above market growth is expected to be sustainable over a time period of more than ten years. In the meantime the Qiagen market capitalisation has even increased to USD 3.5 billion until the 15th of March 2008 within an overall weak market environment for equities. The performance was mainly driven by better prospect for the integrated company as well as an upbeat market sentiment for diagnostics companies with strong M&A activity at high multiples.

Finally it has been determined that even up to 10 timesteps with 3 and 4 subnodes for each state applying the same correlated price process as for the basis Qiagen valuation no additional value has been generated. This is due to the multiplicative assumption of the real option investment projects.

4.1.7 Computational Performance

The computational performance of the previous valuation of Qiagen is presented in the following tables. The computation was performed on a machine with one Intel Pentium with a 1.86 GHz processor. The results always have been displayed in seconds.

The computational time for the basis valuation of Qiagen with 2 replication assets and 2 subnodes in each state is listed in the following table:

Volatility	15%	20%	25%	30%
10 timesteps	161.94	237.94	299.03	452.66
9 timesteps	67.42	87.92	120.66	156.74
8 timesteps	24.06	30.41	44.17	61.98
7 timesteps	10.00	12.03	15.09	22.92
6 timesteps	4.08	5.11	8.00	7.16
5 timesteps	1.67	1.88	2.16	2.72

Table 4.12: Qiagen Base Case Valuation: Computational Time in Seconds

By adding one further timestep the time for computing the algorithm increases by on average 167%, which clearly confirms the better than exponential increase in computing time for the algorithm implemented by applying dynamic programming.

Volatility	15%	20%	25%	30%
10 timesteps	140%	171%	148%	189%
9 timesteps	180%	189%	173%	153%
8 timesteps	141%	153%	193%	170%
7 timesteps	145%	135%	89%	220%
6 timesteps	144%	173%	271%	163%

Table 4.13: Qiagen Base Case Valuation: Relative Change in Computational Time

The following table depicts the computational time for the real option setting which is coded by sequentially computing the basis scenario and then the real option scenario which is much more computationally demanding. The two main reasons are (i) the extra optimisation (over g_i), and (ii) because the memory requirement increases drastically due to the increase (by 1) in the dimensionality of the state space. Even if $\widehat{g}$ is discretised in just 10 timesteps, the size of the state space is increased 10 fold and the computational effort (in computing time) increases a lot more than that. As the basis scenario 2 replication assets and 2 subnodes in each states are being considered.

Volatility	15%	20%	25%	30%
10 timesteps	547.33	624.49	972.39	1,494.88
9 timesteps	180.83	263.81	384.44	546.37
8 timesteps	77.25	101.53	124.92	172.25
7 timesteps	35.47	39.67	46.05	65.55
6 timesteps	15.98	15.63	20.36	24.09
5 timesteps	4.83	6.17	7.53	8.50

Table 4.14: Qiagen Real Option Valuation: Computational Time for 2 Assets and 2 Subnodes

The comparison to the basis scenario shows an increase of only up to 292% in computational time which is far better than expected and is mainly result of focusing on dominating results within the dynamic programming algorithm.

volatility	15%	20%	25%	30%
10 timesteps	238%	162%	225%	230%
9 timesteps	168%	200%	219%	249%
8 timesteps	221%	234%	183%	178%
7 timesteps	255%	230%	205%	186%
6 timesteps	292%	206%	154%	237%
5 timesteps	189%	229%	249%	213%

Table 4.15: Qiagen Real Option Valuation: Increase in Computational Time to Basis Scenario

The next table depicts the computational time for the real option setting again in a scenario with 3 replication assets and 2 subnodes in each states.

Volatility	15%	20%	25%	30%
10 timesteps	308,531	352,024	548,138	842,664
9 timesteps	116,050	122,416	189,631	280,152
8 timesteps	48,354	76,589	91,282	128,987
7 timesteps	41,047	45,912	53,290	79,234
6 timesteps	15,606	13,206	17,920	21,784
5 timesteps	6,745	8,623	10,523	11,875

Table 4.16: Qiagen Real Option Valuation: Computational Time for 3 Assets and 2 Subnodes

The following table shows the computational performance of the scenario with 2 replication assets and 3 subnodes in each state. The computational time increases on average by a factor of more than 2 which clearly indicates further improvement potential regarding the computational performance of the implementation of the algorithm.

Volatility	15%	20%	25%	30%
10 timesteps	960,020	1,095,352	1,705,581	2,622,024
9 timesteps	473,664	691,037	1,007,005	1,431,165
8 timesteps	353,266	464,303	571,271	787,703
7 timesteps	159,203	178,073	206,688	294,212
6 timesteps	48,088	47,007	61,250	72,483
5 timesteps	18,495	23,644	28,854	32,562

Table 4.17: Qiagen Real Option Valuation: Computational Time for 2 Assets and 3 Subnodes

The last table depicts the computational time for the real option setting in an scenario with 3 replication assets and 3 subnodes in each state.

Volatility	15%	20%	25%	30%
10 timesteps	3,521,246	4,017,630	6,255,880	9,617,292
9 timesteps	1,163,359	1,697,243	2,473,285	3,515,057
8 timesteps	496,989	653,201	803,687	1,108,172
7 timesteps	220,416	246,541	286,159	407,335
6 timesteps	95,340	93,199	121,435	143,708
5 timesteps	27,576	35,253	43,021	48,550

Table 4.18: Qiagen Real Option Valuation: Computational Time for 3 Assets and 3 Subnodes

4.2 Case Study in the R&D Arena

In a second case study, a company heavily driven by R&D should be evaluated and the differences regarding the company's real options should be elaborated.

4.2.1 Description

GPC Biotech is a publicly traded biopharmaceutical company focused on discovering, developing, and commercializing new anticancer drugs. GPC Biotech's lead product candidate Satraplatin is currently in a phase 3 registrational trial for second-line hormone-refractory prostate cancer. Beside its lead compound, the company is also developing a monoclonal antibody with a novel mechanism-of-action against a variety of lymphoid tumors, currently in clinical phase 1 development. The company does not own technologies beside its two drug candidates which currently contribute significantly to the company value. The company planned to submit its phase 3 registrational data of Satraplatin to the U.S. Food and Drug Administration (FDA) in July 2007. The data which was published prior to registration showed a significant effect on its primary clinical endpoint progression free survival (PFS) compared to the standard treatment arm of the trial (GPC Biotech). Nevertheless, the Oncologic Drugs Advisory Committee (ODAC) which holds an important stake within the approval process of oncology drugs in the U.S. recommended the FDA to wait with an approval decision until the final overall survival (OS) data of the phase 3 registration trial will be released (FDA Briefing Documents for ODAC). This outcome unexpected by most of the investors so far, led to a slump in the company value by more than 63% from EUR 843 million on the 19th July 2007 to its lowest level on the 27th of July at a market capitalisation of EUR 301 million. The reason for the decline in company value was the reduced expectation of the investors regarding an approval of Satraplatin. Satraplatin was the only product expected to generate positive cash flows for GPC Biotech near term. Furthermore, the company was burdened by being sued in an U.S. class action lawsuit on behalf of some investors. The complaint alleges GPC Biotech made false public statements relating to the prospects of Satraplatin, and thereby artificially inflating the company value.

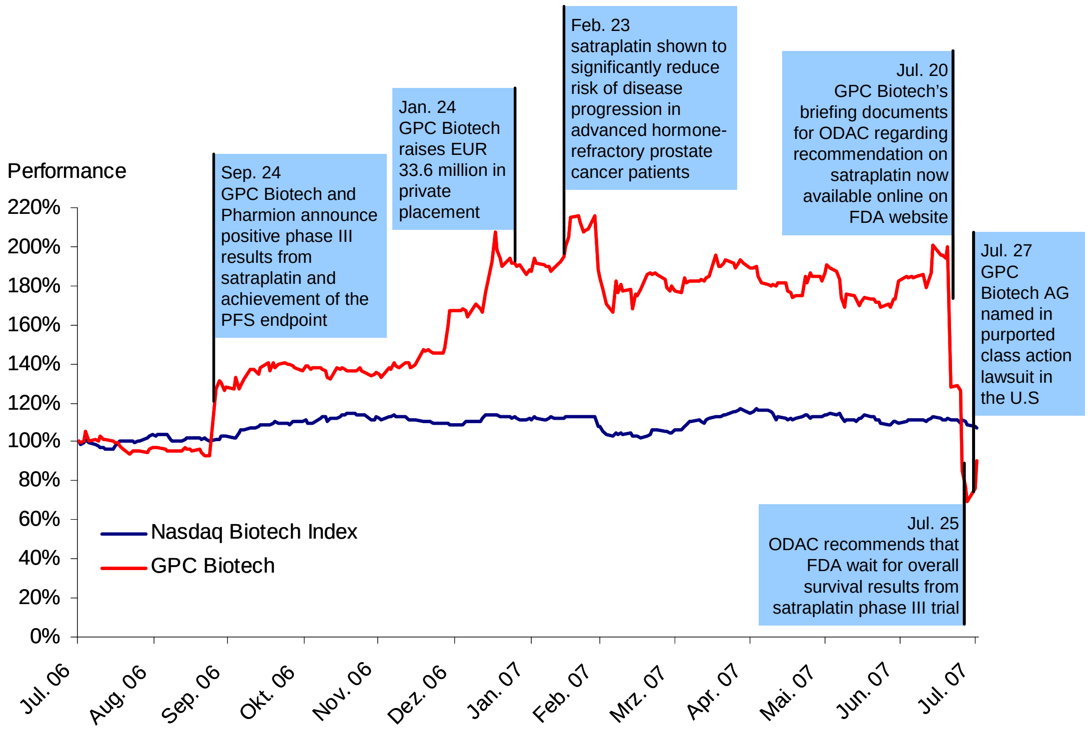


Figure 4.5: GPC Biotech Performance Chart

4.2.2 Forecasting Company Cash Flows

As already used for the Qiagen case, the most robust figure to get to the free cash flow is to use the I/B/E/S net income analyst expectations. For further expanding its business from its current status, GPC Biotech neither requires a significant further investment nor allows for strong reduction of further investments and therefore leads to investments in the magnitude of the depreciation/amortisation. The change in working capital is estimated to be at around 10% of the sales increase which leads to a reduction of the growth in free cash flow compared to the expected net income growth by on average 20 percent points. The average of the net income estimates, being provided by seven different analysts, states an increased net loss for 2007 compared to 2006 as a result of higher operating costs for the phase 3 study, pre-marketing expenditures, and the delayed approval of Satraplatin (Bloomberg 2007). For 2008 and the following years the variance between the different estimates is significant. This is the result of the high uncertainty regarding the outcome of Satraplatin showing a significant survival benefit over the current treatment standard (standard of care) and the subsequent uncertainty about the approval or non-approval decision from the FDA in the US and the European Medicines Agency (EMEA). Finally, uncertainty arises of the time period of the exclusivity rights of the product and the validity of the intellectual property (IP) position in case a generic version potentially infringes some IP of Satraplatin for the treatment of hormone-refractory prostate cancer. The average of the analysts estimates an increase of the free cash flow from minus EUR 83 million in 2007 to EUR 22 million in 2010. The standard deviation or volatility of the estimates for the next four years is in average 43% per year for sales. The one year historical share price volatility amounts to 80%. For the coming twelve month, the share price of GPC Biotech will mainly be driven by two factors. Firstly, the outcome of the phase 3 overall survival benefit with the subsequent reaction of the approval authorities and secondly, whether claims regarding the class action law sued will be discovered or the lawsuit will be settled.

Regarding the input parameters, the usual GBM will be used as an underlying cash flow process again. To solve this aspect the lognormal cash flow distribution will be shifted by a constant factor $-b$, in our case EUR 100 million for the free cash flow. Now the growth rate μ can be calculated by plugging in the real estimates as well. Finally, this leads to a $\mu = 36\%$ for the adjusted free cash flow figure for the time period 2006 until 2010 and an adjusted volatility of $\sigma = 30\%$ based on the analyst estimates. As the adjusted free cash flow in 2006 EUR 28.6 million were calculated.

4.2.3 Replication Assets

At a first glance, the drivers for the future expected cash flows are fairly independent from the performance of other industries or assets. This aspect is underpinned by an on average lower correlation to some industry specific indices compared to the correlation dataset of Qiagen in the first case study.

Assets/Ind.	GPC	NBI	F3PHRM	TDXP	DAX	EUR003M	JITXE5XU
GPC	1.000						
NBI	0.633	1.000					
F3PHRM	0.356	0.498	1.000				
TDXP	0.835	0.730	0.573	1.000			
DAX	0.746	0.644	0.713	0.937	1.000		
EUR003M	0.790	0.542	0.634	0.895	0.952	1.000	
JITXE5XU	0.838	0.680	0.684	0.890	0.940	0.930	1.000

Table 4.19: Correlation Matrix for Historical Prices Since January 2004

By selecting only two replication assets, the combination of TecDAX and the iTraxx Crossover 5-Year Performance Index (JITXE5XU) yields the highest explained contribution to the historical share prices of GPC Biotech with an overall $R^2 = 75.2\%$. Interesting is that for GPC Biotech the two asset combinations which explain most of the historical share price include an index that expresses the price for default risk of traded fixed income products. Two reasons can be drawn from this result: Firstly, coincidentally the iTraxx significantly fell as a result of the subprime crisis in the US in the same time period when GPC Biotech plunged, or secondly, the ability to raise money for a biotech-company like GPC Biotech is highly correlated to the price differential for non-investment grade loans to "riskless" government bonds. Analysing the time period prior to the events which have taken place in July 2007 leads to the result, that the iTraxx Crossover 5-Year Performance Index is not anymore the replication asset having the second highest explanatory value to the GPC Biotech share price. In case the 3-month money market performance would be the index correlated second highest to the replication asset. In general, equity financing is easier in a low interest environment compared to a high one. Nevertheless, for the given case study the iTraxx Crossover 5-Year Performance Index will be chosen as a second replication asset. In a second trial, the computation will be performed by including three replication assets explaining maximum $R^2 = 80.6\%$ of the GPC Biotech historical share price. In addition to the TecDAX and the iTraxx Index, the FTSE World Pharma Index was included. The additional explanation of the third asset being included as an replication asset only yields an additional R^2 of around 5% to the share price of GPC Biotech throughout its history. The main reason for including the third asset is due to its tracking error reducing effect. It is assumed that all selected indices follow a GBM. In comparison to the TecDAX and the FTSE World Pharma Index, the iTraxx Index is assumed not to encounter any growth rate μ.

4.2.4 Results Base Case

In the following the overview of the chosen input parameters is given.

Parameter	GPC Biotech	F3PHRM	TDXP	JITXE5XU
adj. FCF/Price	29	8271	921	115
CAGR/Interest Rate	36.0%	6.0%	6.0%	0.0%
Volatility	0.0%/29.6%/ 50.0%/78.8%	15.8%	18.8%	36.6%

Table 4.20: Overview of GPC Biotech Input Parameter

For calculating the value of GPC Biotech without considering further real options an adjusted CAGR of $\mu = 36\%$ in free cash flow over the valuation period was considered. Regarding the volatility the potential range lies between $\sigma = 79\%$ for the historical volatility for the last twelve month and $\sigma = 30\%$ per year for the adjusted volatility that can be derived by taking all analyst earnings expectations for the coming years into account. The strong deviation between the historical volatility and the expected is hard to justify. At a first glance it could be assumed that the shaky history is not expected to continue for the future.

	Historical:		Estimated:		
Volatility	79%	50%	30%	20%	0%
10 timesteps	4,987	661	165	69	30
9 timesteps	3,225	481	109	33	0
8 timesteps	2,169	325	49	0	0
7 timesteps	1,299	198	9	0	0
6 timesteps	735	97	0	0	0
5 timesteps	373	13	0	0	0

Table 4.21: GPC Biotech Preliminary Basic Company Values

GPC Biotech had a market capitalisation of EUR 381 million on the 1 August, 2007. The results do not match the reality because the assumptions on the CAGR $\mu = 36\%$ are being derived from the analyst estimates for the next five years as well as the high volatility range of $\sigma = 30\%$ to 79% .

This leads to a potential net company cash flows of $b_T = 1135$ for $\sigma = 10\%$ and $b_T = 6.826$ already for $\sigma = 30\%$ in the last year of a ten year planning period. As it can be seen a volatility of $\sigma = 30\%$ or above, returns unreasonable net cash flow projections at least. The available time period for analyst estimates is the period when GPC Biotech will either be able to successfully market Satraplatin after having received an approval, or will go bankrupt. For the time period of five to ten years ahead, GPC Biotech will try to maximise the return of the then approved product until generic competition will deteriorate their margins on the product. Even the most optimistic peak sales estimates from analysts for Satraplatin did not significantly exceed EUR 1 billion per year. In the following, the potential maximum net cash flow which can be generated from the drug will be calculated. For the EU, Satraplatin has been licensed to Pharmion in exchange to a royalty rate of between $26 - 34\%$ of net sales. In the U.S., GPC Biotech initially planned to sell the product through their own sales force which yields higher net cash flow margins.

Markets	EU in %	EU in EUR m	US in %	US in EUR m	Worldwide in EUR m
Maximum Peak-Sales Estimates	100%	400	100%	600	1000
Costs of Goods Sold	−7%	−28	−7%	−42	−70
Royalties from Pharmion (26 − 34%)	30%	120	0%	0	120
Royalties to Spectrum Pharmaceuticals	−8%	−32	−8%	−48	−80
Marketing and Sales Costs	0%	0	−30%	−180	−180
EBT	15%	60	55%	330	390
Working Capital	0%	0	−10%	−60	−60
Taxes	−30%	−18	−30%	−99	−117
Free Cash flow (FCF)	**11%**	**42**	**29%**	**171**	**213**

Table 4.22: Calculation of Best Case Free Cash Flow Scenario of Satraplatin

The net cash flow GPC Biotech could generate through a successful launch of Satraplatin is limited to EUR 213 million in the most optimistic case. Corporate costs for, e.g. administration and other development projects which are still loss making were not taken into account in this simplified calculation. Therefore this number should be seen as the maximum net cash flow the company could generate in the best case scenario. This number impressively shows that the derived CAGR as well as the volatility derived from the analyst estimates for the coming five years is not a good estimate for the ten year period. To reach this figure in the upper node of the tree either the CAGR μ or the volatility σ can be decreased. Therefore, the following results are being calculated with different combinations of μ and σ which all yield a free cash flow of around EUR 213 million in $t = 10$ in the upper node.

Maximum cumulated negative cash flows		Extrapolation of analyst estimates	Combinations based on best case scenario for Satraplatin		
	μ	36.0%	25.0%	10.0%	0.0%
	σ	30.0%	0.0%	13.0%	22.0%
143	10 timesteps	99	0	0	0
160	10 timesteps	99	0	0	0
180	10 timesteps	165	0	0	0
unlimited	10 timesteps	165	0	0	0

Table 4.23: Company Valuation Results of Best Case Sceanrio of Satraplatin FCF Expectations

Any combination of volatility σ and expected growth rate μ which would not exceed the most optimistic free cash flow scenario in $t = 10$ yields a positive company value. Furthermore the results show that the potential amount of cash available to carry out a risky investment influences the overall wealth w of the company as well, which can be seen in the first column. With restricted amount of cash available, the losses are limited but sometimes cut out branches which would contribute to the company value later on. The free liquidity of GPC Biotech end of July 2007 was estimated at EUR 60 million for the end of 2007. Therefore, EUR 143 million is the amount the company can still spent before going bankrupt at most, which is the sum of the expected negative net cash flow of EUR 83 million in 2007 plus the remaining cash position. Another source of funds could be money inflow from a capital increase. Assuming the capital markets to be rational, the investor would invest further money in the company only if the company generates a potential excess return. This implies, an investor will only invest further money if the company follows a good branch (approval of Satraplatin) and will not invest in any "bad" branches (non-approval of Satraplatin). Only an additional investment of at least EUR 17 million through a capital increase would increase the company value to $w = 165$ million Euro for the unrealistic case of a growth rate of $\mu = 36\%$ and a volatility of $\sigma = 30\%$. The likelihood for an approval of Satraplatin is expected at the beginning of August 2007 at around 50% which is the prerequisite of the company to generate positive net cash flows. Interpreting the results clearly states that the wealth of GPC Biotech is zero, assuming the current market estimates to be reasonable. In the following, the company cash flows will be modelled directly focusing only on two simplified scenarios which have a high likelihood according to analyst reports.

Scenarios	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Best Case	-71.0	-83.2	-22.1	-7.6	21.8	31.9	46.6	68.1	99.6	145.7	213.0
Worst Case	-71.0	-117.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 4.24: Simplified Discrete Modelled GPC Biotech Cash Flow Scenarios

Therefore, GPC Biotech has been valued in a second way by considering only two branches which do have the highest likelihood: The best case scenario was modelled according to the analyst estimates and the inherent volatility for the next ten years. The second branch is the worst case scenario which assumes GPC Biotech will not get an approval for Satraplatin from the FDA and the EMEA until end of 2008. In this case, the company will go bankrupt by the end of 2008 and, therefore no further negative or positive cash flows in this branch have been considered. The company valuation algorithm yields a company value of $w_0 = 98$ million Euro in this case assuming the same replication assets as in the stochastic process. Applying a simplified risk-adjusted DCF valuation with a probability of 50% for the best case branch, 50% for the worst case branch and a risk-free discount rate of $r = 6\%$ yields a company value of $w_0 = 91$ million Euro. In the case of applying a WACC of $irr = 10\%$, the company wealth would decline to $w_0 = 44$ million Euro. In comparison to the DCF valuation technique, the new company valuation algorithm by applying the risk-free interest rate of $r = 6\%$ and using the above shown company cash flows as two branches, yields with 8% deviation, almost the same result. This furthermore indicates the good applicability of the developed algorithm. The results for the company valuation algorithm based on stochastically modelled net company cash flows are significantly lower applying reasonable assumptions. This is mainly due to the assumed constant volatility over all future time steps which is not consistent to the expectations which would apply in reality. A stochastic process with, e.g. a mean-reverting volatility would lead to a better applicability in cases of highly changing volatility scenarios like in maturing high growth companies or turn-around situations.

4.2.5 Real Option Assumptions and Evaluation

So far, the values for GPC Biotech have been calculated on a set of scenarios taking the current setting into account. But beside these mainly digital expectations, the question remains whether there are further investment opportunities. Two scenarios can be considered: The options in case of an approval of Satraplatin and the options in the case of a non-approval. If Satraplatin will be approved until end of 2008, GPC Biotech could try to start another clinical trial in a cancer indication with Satraplatin, where the company expects to hold a solid IP position which at least does not expire before 2018. Secondly, the opportunity could be generated to speed up the development of its phase 1 monoclonal antibody by assigning more resources to its development. This option is not valid for generating sustainable cash flows within the next ten year time horizon. Thirdly, GPC Biotech could acquire another marketed product in a cancer indication to generate synergies by a higher utilisation of its already established U.S. sales force.

Real Options if Satraplatin Gets Approved	Prerequirements	Investment (in EUR m)	Free Cash Flow Contribution p.a. (in EUR m)	Additive/ Multiplicative	Comment
Phase 2 trial in further cancer indication	Solid IP on line extensions for satraplatin valid >2018	40	59	multi	potentially
Accelerate development	Good safety profile in phase I	50	150	add	takes to long
Acquisition of marketed product	Successful large capital increase	220	30	add	potentially

Table 4.25: Potential Real Options in the Case of an Approval of Satraplatin

In case Satraplatin will not get an approval, the company will not have many options remaining because it will run out of money in 2008. Starting a new phase 3 trial for Satraplatin is not a valid option anymore, even if the company could raise the required financing. The limited time for an exclusive marketing of the drug on basis of valid IP claims protecting the product from generic competition is simply to short to recover a further investment. On a stand-alone basis GPC Biotech would not have any substantial options left. Even to lay off most of their 280 employees and selling the remaining assets would not change much. The opportunity to raise a significant amount of money would have been foregone. Therefore, it can be assumed that in case of a non-approval decision, GPC Biotech will have to leave the market forever. A different scenario could be considered if a strategic investor views GPC Biotech as a platform to establish a midsize specialty pharmaceutical company. In this case, the question will be if any value can be generated within GPC Biotech by further investing in the firm. A potential strategy could be to bring in a well-established specialty cancer drug, which has not been promoted to that extent by its prior owner, and where GPC Biotech is considered to promote the drug more effectively. Therefore, this strategy could be reduced to the question if a bargain buy is feasible and what additional cash flows can be squeezed out by stronger marketing efforts utilising the recently built up GPC Biotech US sales force. A second factor could potentially influence the investment decision positively; GPC Biotech's carry forward losses. By year end 2006, the testified losses carried forward amounted to EUR 92 million. By the end of 2007, losses carried forward in the range of around EUR 150 million are assumed.

Real Options if Straplatin Won't Get Approved	Prerequirements	Investment (in EUR m)	Free Cash Flow Contribution p.a. (in EUR m)	Additive/ Multi-plicative	Comment
Acquisition of marketed product	Pos. decision of strategic investors plus large capital increase	220	30	add	potentially

Table 4.26: Potential Real Options in the Case of a Non-Approval of Satraplatin

4.2.6 Results Real Option Valuation

The option for a further phase 2 trial is multiplicative connected to the basis cash flows of GPC Biotech, which means the potential positive effect only occurs when Satraplatin already generates positive net cash flows for GPC Biotech, but not in case of a non-approval of the product. The acquisition of a marketed product is mainly independent of the approval of Satraplatin, thus its cash flows contributions have been considered to be independent and, therefore, additive to the basis cash flow of GPC Biotech. The real option results were calculated for the discrete cash flow example for GPC Biotech. The real option results were calculated for the discrete cash flow example because the stochastically generated cash flows do not meet the real world requirements close enough in this specific example.

Real Options	Prerequirements	Investment (in EUR m)	Free Cash Flow Contribution p.a. (in EUR m)	Additive/ Multiplicative	Company Value	Value contribution
Phase 2 trial in further cancer indication	Solid IP on line extensions for satraplatin valid >2018	40	59	multi	136	39%
Accelerate development	Good safety profile in phase I	50	150	add	-	-
Acquisition of marketed product	Successful large capital increase	220	30	add	261	166%

Table 4.27: GPC Biotech Company Value Considering Real Options

The computed results confirm the expectations that even in case of a high return of the additional cancer trial, the impact is by far smaller than the contribution of the acquisition of the marketed product which is independent of the Satraplatin approval. Furthermore, the result derives the optimal decision scenario which states that for the second cancer trial the investment should be carried out only for the company following the best case branch in time step $t = 3$ which equates to the year 2009. For the acquisition of the marketed product, the investment would be optimal to carry out right away. This real option would rescue the company and generate an additional company wealth following the best case branch as well as the worst case branch.

4.2.7 Retrospective View

On October the 31st 2007, GPC Biotech announced the final results from its ongoing phase 3 trial. The results did not show any statistically significant improvement of the overall survival between the group treated with Satraplatin and the standard care group. The share price dropped furthermore by 60% from EUR 7.96 to EUR 3.22 per share. The overall market capitalisation was EUR 119 million at the closing of the markets of October 31. The result at least partly confirm the previous calculations with the newly developed company valuation approach. The remaining market capitalisation can be interpreted as the hope of some investors of a rescue plan to utilise the losses carried forward in the range of around EUR 150 million at the end of 2007 as well as generating some value of the sales of the remaining assets, i.e. mainly the antibody project in clinical phase 1.

In the meantime even this hope diminished and the only news from the company side is the announcement of another restructuring program. The market capitalisation has reached a new low at EUR 85 million on 6 of March 2008 and the company therefore is trading at around EUR 20 million above cash.

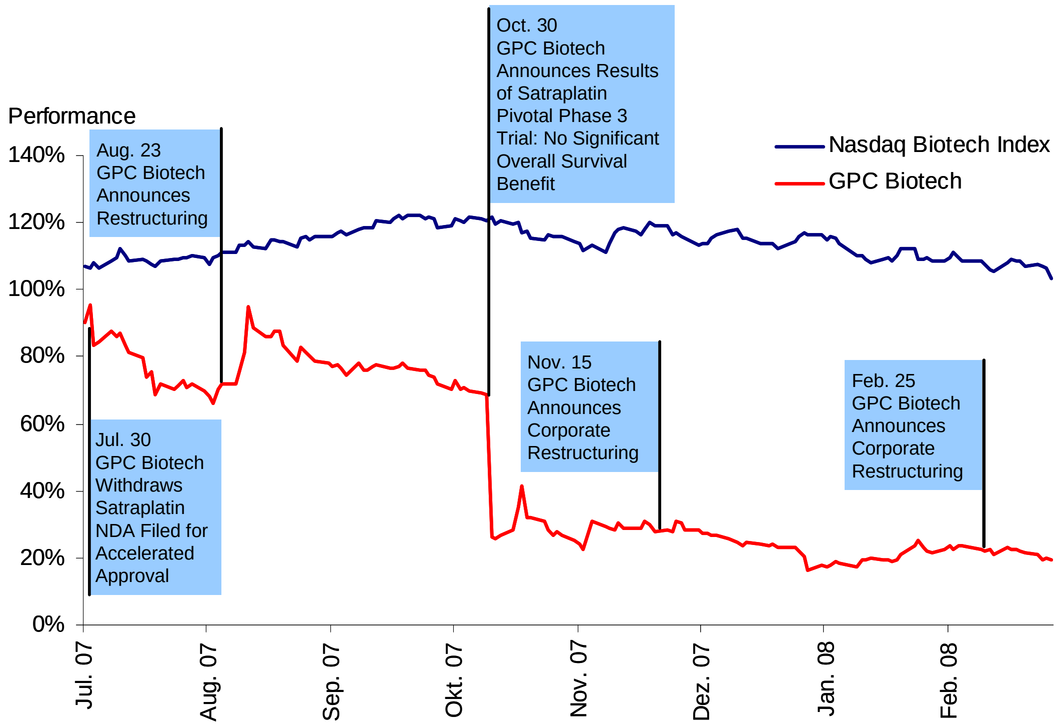


Figure 4.6: Retrospective View on GPC Biotech Performance

Chapter 5

Conclusions

This thesis expands the literature on the integration of option pricing theory and optimisation in the area of operations research. In a further step, this thesis contributes by the application of real options theory to reduce its complexity of calculating the value preposition and determining the optimal exercise based on a generic set of input parameters. Previous research primarily focused on the motivation of real options theory in general and its application to specific situations and different techniques for evaluation of real options.

The model integrates the valuation of predictable cash flow scenarios within almost any project only by applying appropriate stochastic price processes of traded assets which are able to replicate the cash flow scenarios at least to a certain extent. In a second step additional investment scenarios can be valued and the ones being most value enhancing within the additional investment budget can be chosen. One difficulty facing organisations is the question of the optimal exercise. Within the established algorithm the optimal exercise of the additional investments will computed. And last the amount of additional investment for these additional investments can be chosen and therefore the amount of additional risks the company or the investor is willing to carry. The algorithm contributes to some further answers to the quite generic question: "What is the optimal investment strategy within the framework of budgeting constraints and different future scenarios?" The developed algorithm therefore enables decision makers, analysts, and other interested parties to determine the optimal time and scenario to undertake the investment project as well as its value contribution a priori. Depending on the branch the company follows throughout the time, the input parameters can be updated and the optimal exercise/no-exercise decisions can be adjusted to enhance the company value throughout the evolving time. Focusing on the few most valuable real options allows for an easier communication of the underlying assumptions and its value contribution. Nevertheless, a thoroughly evaluation of the underlying options is prerequisite to generate valuable outcomes. The algorithm therefore supports organisations as well as investors to mitigate investment risks. At this stage milestones already trigger further investments in a variety of projects, even though these milestones are fixed for reaching certain goals within one project. Supported by the newly developed algorithm the value of reaching a variety of various possible future milestones could be calculated and therefore the related payment could be fixed more precisely.

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Appendix

Estimates on Qiagen:

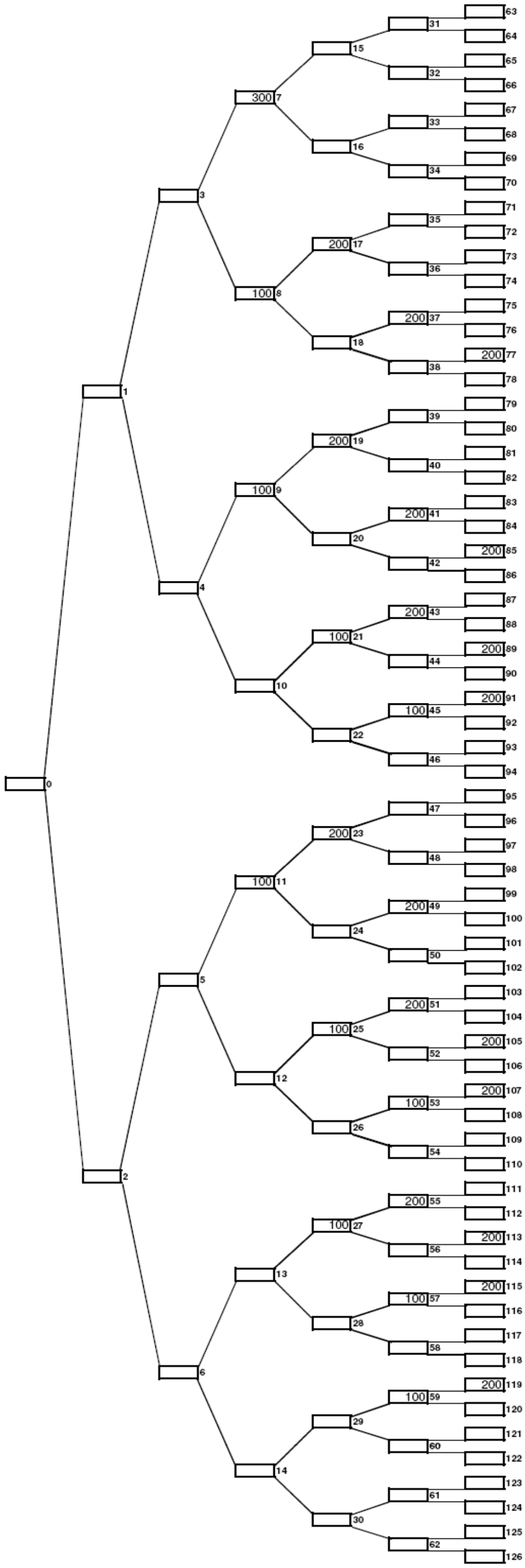
Year	2007	2008	2009	2010	2011
Analyst 1	92,240	113,240	146,070		
Analyst 2	116,806	141,354	176,520	214,500	214,500
Analyst 3	75,300	172,600	216,300		
Analyst 4	88,090	113,010	134,240	160,500	
Analyst 5	74,120	90,430	104,510	203,513	238,065
Analyst 6	81,300	10,400	123,900	138,700	153,300
Analyst 7	89,328	118,009		144,694	
Analyst 8	95,431	114,956		191,000	
Analyst 9	86,916	120,484	133,005	166,800	198,400
Analyst 10	96,992	118,400			
Analyst 11	94,600	124,000	154,000		
Mean	**90,102**	**112,444**	**148,568**	**174,244**	**201,066**
Volatility	**12%**	**34%**	**22%**	**16%**	**15%**

Table 5.1: I/B/E/S Net Income Estimates for Qiagen from 30th of July 2007

Optimal Exercises of Qiagen Real Option Scenario:

Volatility	15%	20%	25%	30%
10 timesteps	300 in 7	300 in 7	see	300 in 7
	300 in 17	300 in 17	graph	300 in 17
	300 in 19	300 in 19		300 in 19
	300 in 23	300 in 23		300 in 23
	300 in 37	300 in 37		300 in 37
	300 in 41	300 in 41		300 in 41
	300 in 43	300 in 43		300 in 43
	300 in 49	300 in 49		300 in 49
	300 in 51	300 in 51		300 in 51
	300 in 55	300 in 55		300 in 55
9 timesteps	300 in 7	300 in 7	100 in 3	300 in 3
	300 in 17	300 in 17	200 in 7	300 in 11
	300 in 19	300 in 19	100 in 9	300 in 27
	300 in 23	300 in 23	100 in 11	
			200 in 17	
			200 in 19	
			100 in 21	
			200 in 23	
			100 in 25	
			100 in 27	
			200 in 37	
			200 in 41	
			200 in 43	
			200 in 49	
			200 in 51	
			200 in 55	
8 timeteps	300 in 7	300 in 7	100 in 3	300 in 3
			200 in 7	300 in 9
			100 in 9	300 in 11
			100 in 11	
			200 in 17	
			200 in 19	
			200 in 23	
7 timesteps	0	0	100 in 3	300 in 3
			200 in 7	
6 timesteps	0	0	0	0
5 timesteps	0	0	0	0

Table 5.2: Qiagen Extended Real Option Optimal Exercise



Qiagen Optimal Exercise

Estimates on GPC Biotech:

Year	2007	2008	2009	2010
Analyst 1	-89.30	-70.10	-30.40	
Analyst 2	-76.10	-17.60		
Analyst 3	-92.66	-15.17	18.60	
Analyst 4	-89.80	30.00		
Analyst 5	-69.90	-8.90		
Analyst 6	-82.50	-28.90	-9.60	28.10
Analyst 7	-72.40	-41.10	-8.10	9.70
Analyst 8	-93.20	-68.00	-10.10	27.50
Mean	**-83.23**	**-27.47**	**-7.92**	**21.77**

Table 5.3: I/B/E/S Net Income Estimates for GPC Biotech from 30th of July 2007