Computer Methods in Biomechanics and Biomedical Engineering - 4 (Ed.: J. Middleton, N. G. Shrive & M. L. Jones), University of Wales College of Medicine, ISBN 1 903847095, 2002

HOMOGENIZATION IN STRENGTH AND DURABILITY ANALYSIS OF REINFORCED TOOTH FILLING

Sergey E. Mikhailov[1], Julia Orlik[2]

1. ABSTRACT

An asymptotic homogenization procedure is employed to obtain effective elastic properties of the composite tooth filling, a homogenized macro– stress field and a first approximation to the micro-stress field, from properties of the components and applied macro–loads. Using the approximate micro–stress field, a non–local initial strength and fatigue durability macro–conditions for the composite filling material is expressed in terms of the homogenized macro–stresses. An illustrative example with the stress singularity on the tooth–filling interface is presented showing the need in the non-local analysis. Effective elastic properties of the tooth filling is numerically simulated for some size distributions of the reinforcing particles.

2. INTRODUCTION AND MOTIVATION

Tooth filling materials are considered consisting of a polymer matrix filled with glass particles. The volume fraction, size, shape and distribution of the particles influence mechanical properties of the filling material. Since 70-es, the homogenization techniques were widely used for obtaining effective elastic properties of the composite and the homogenized macro-stress field. The simplest averaging theories are mixture theory and Representative Volume Element (RVE) method (Hashin, 1983).

In [1], the approximation to the micro-stress-field is derived from properties of the components, micro–geometry of the composite and applied macro–loads, using a

Keywords: Asymptotic homogenization, Strength, Fatigue, Singularity, Non-local conditions

[1]Professor, Dept. of Mathematics, Glasgow Caledonian University, Cowcaddens Road, Glasgow, G4 0BA, U.K.

[2]Doctor, Fraunhofer Institut Techno- und Wirtschaftsmathematik, Gottlieb-Daimler-Str. 49, 67663 Kaiserslautern, Germany

formal asymptotic expansion. Convergence of the micro–solution, as the structure period tends to zero, is proved in [2] by the two–scale homogenization technique. The present work is based on the fact that this limit for the micro–stress field is exactly the first term of its asymptotic expansion, which is a product of the homogenized macro–stress tensor and the so–called stress concentration tensor, related only to the micro–geometry and stiffness tensors of the composite components.

Fatigue caused by normal chewing and temperature changes seems to be the main reason of tooth filling fracture. We use here the Wöhler durability functions and the linear damage accumulation rule to analyze durability under fatigue. It is also merged with the non-local approach of [3]. This generalizes to fatigue some application of the homogenization to micro-strength analysis presented in [4]. As a result, the initial micro-strength and durability conditions (for each component of the composite) are presented in terms of the elastic, strength and durability micro-properties of the composite components and of the macro-stresses.

3. ELEMENTS OF STRENGTH AND FATIGUE DURABILITY ANALYSIS

For a bounded stress field $\sigma_{ij}(y)$, any local strength condition at a point $y \in \Omega$ can be written in the form $\Lambda\,(\sigma(y), y) < 1$, where Λ is a normalised equivalent stress function, a material characteristic. For some materials Λ is associated with the von Mises equivalent stress

$$\Lambda_M(\sigma(y), y) = \sqrt{[(\sigma_1(y) - \sigma_2(y))^2 + (\sigma_2(y) - \sigma_3(y))^2 + (\sigma_3(y) - \sigma_1(y))^2]/(2\sigma_c^2(y))},$$

or with the Tresca equivalent stress $\Lambda_T(\sigma(y), y) = \max_{k,m} |\sigma_k(y) - \sigma_m(y)|/\sigma_c(y)$, where $\sigma_1,\ \sigma_2,\ \sigma_3$ are the principal stresses and σ_c is a known uniaxial strength of material. The local strength conditions, however, are generally not applicable to unbounded (singular) stresses since the conditions will predict fracture under virtually any singular stress field.

To illustrate that such singular stress fileds are relevant to the tooth filling simulation, a finite element calculation was done for a simple modell of a half-spherical tooth (radius $a = 5mm$) with a half-spherical filling (radius $a/2$) presented on figures 1 and 2. It is supposed that the tooth with filling is loaded by a uniform pressure

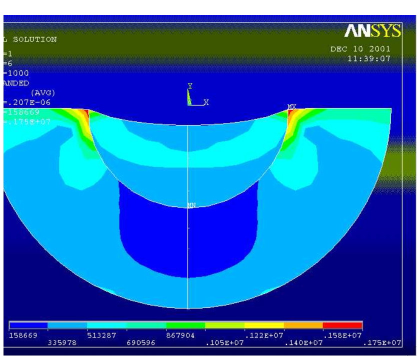


Figure 1: Equivalent von Mises stresses

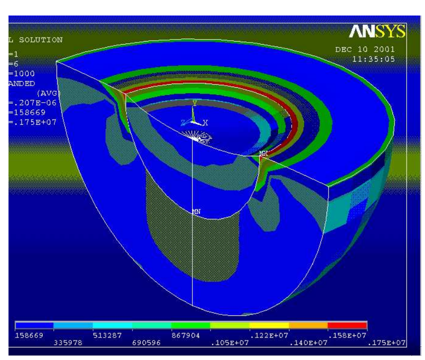


Figure 2: Equivalent von Mises stresses

1MP from above and interaction of the tooth with the jaw was described by the

Winkler contact condition. The Young modulus and the Poisson ratio are taken $E = 90GPa$, $\nu = 0.3$ for the tooth tissue and $E = 8GPa$, $\nu = 0.3$ for the filling material. One can observe that the equivalent von Mises stresses in the filling and tooth grow sharply near the singular line, which is the intersection of the tooth-filling interface with the external boundary. Applying the techniques described in [5], it was shown that the stresses in the both materials have indeed a singular asymptoics $\sigma_{ij}(\rho,\theta) = KF_{ij}(\theta)\rho^{-\gamma} + \bar{\sigma}_{ij}(\rho,\theta)$, where ρ, θ are local polar coordinates with the origin in a singular point, constant K is a stress intensity factor, $F_{ij}(\theta)$ is a bounded function of the local polar angle θ, $\bar{\sigma}_{ij}(\rho,\theta)$ is a smaller term. Solving the corresponding transcendental equation given in [5], we obtain the singularity exponent $\gamma \approx 0.207$ for the particular materials and geometry.

For general, especially singular stress fields, e.g. belonging to $L_2(\Omega)$, a (point) non-local strength condition $\underline{\Lambda}(\sigma, y) < 1$ can be applied. Here $\underline{\Lambda}(\sigma, y)$ is a normalised equivalent stress functional, see [3]. Particularly, $\underline{\Lambda}$ can be connected with some kind of weighted averaging of $\sigma_{ij}(x)$, $x \in \Omega$ in some surrounding of the point y,

$$\underline{\Lambda}(\sigma, y) = \Lambda\left(\tilde{\sigma}(y), y\right), \quad \tilde{\sigma}_{ij}(y) = \int_\Omega \varphi_{ijkl}(x,y)\sigma_{kl}(x)dx, \tag{1}$$

where $\tilde{\sigma}_{ij}$ are components of an auxiliary non-local stress tensor, and $\varphi(x,y)$ is a material characteristic, such as $\int_\Omega \varphi_{ijkl}(x,y)dx = \delta_{ik}\delta_{jl}$, and Λ is a function, as above. Then the strength condition for the whole body is

$$\underline{\Lambda}_\Omega\left(\sigma\right) := \sup_{y\in\Omega} \Lambda\left(\tilde{\sigma}, y\right) < 1, \tag{2}$$

where $\underline{\Lambda}_\Omega(\sigma)$ is the *body* normalized equivalent stress functional.

Example (i) If $\varphi_{ijkl}(x,y) = \delta_{ik}\delta_{jl}\varphi(x,y)$, $\varphi(x,y) = \begin{cases} \frac{3}{4\pi d^3} & |x-y| < d \\ 0 & |x-y| \geq d \end{cases}$ for 3D, where d is a material constant, then $\tilde{\sigma}(y) = \frac{3}{4\pi d^3}\int\int\int_{|x-y|<d}\sigma(x)dx$.
Example (ii) If $\varphi(x,y)$ is the Dirac-function, then $\tilde{\sigma}(y) = \sigma(y)$, and the non-local strength-condition coincides with the local one.

The Wöhler diagram (Wöhler function) for a material under a regular uniaxial cycling with constant stress range $\Delta\sigma = \sigma_{max} - \sigma_{min}$ and mean stress $\sigma_m = (\sigma_{max} + \sigma_{min})/2$, describes the critical number of cycles $N^*(\Delta\sigma, \sigma_m)$ needed to reach rapture. For an in-fase multiaxial cycling, we consider $\Delta\sigma = \Delta\sigma_{ij}$ and $\sigma_m = \sigma_{mij}$ $(i,j = 1,2,3)$ as tensors. For simplicity, suppose further that $\sigma_{mij} = 0$. If the material fatigue properties and/or stress field vary with the coordinate, one can write for a body Ω an (initial) durability condition in the form $N < \inf_{y\in\Omega} N^*(\Delta\sigma(y), y)$ where $N^* := N^*(\Delta\sigma, y)$ is the Wöhler curve for a homogeneous material with the fatigue properties as at the point y, under the cycling $\Delta\sigma_{ij}$ homogeneous in space coordinates. However the local fatigue durability condition is generally not applicable to singular stress fields. For the more general classes of stress fields, a non-local in space fatigue durability condition can be applied, e.g.,

$$N < \inf_{y\in\Omega} \underline{N}^*(\Delta\sigma; y), \quad \underline{N}^*(\Delta\sigma; y) = N^*(\Delta\tilde{\sigma}(y), y), \tag{3}$$

$$\Delta\tilde{\sigma}_{ij}(y) = \int_\Omega \varphi_{ijkl}(x,y)\Delta\sigma_{kl}(x)dx \tag{4}$$

where $\varphi(x,y)$ is as above.

4. ELEMENTS OF HOMOGENISATION TECHNIQUE

Consider the elasticity problem for the composite material with a large number of periodically distributed inclusions,

$$\frac{\partial \sigma^{\varepsilon}_{ij}(x)}{\partial x_j} = f_i(x) \quad x \in \Omega, \quad \sigma^{\varepsilon}_{ij}(x) = a_{ijkl}(\frac{x}{\varepsilon})\frac{\partial u^{\varepsilon}_k(x)}{\partial x_l} \ (i,j=1,...,3) \qquad (5)$$
$$+ \text{ boundary and transmission conditions}$$

Here, $(u^{\varepsilon}_i)_3$, $(\sigma^{\varepsilon}_{ij})_{3\times 3}$ and $(a_{ijkl})_{3\times3\times3\times3}$ are displacements, stresses and elastic coefficients respectively, ε is a small parameter related to the periodicity cell εY of the composite. Let $\mathcal{N}_q = (\mathcal{N}_{pjq})_{3\times3\times3} \in H_{per[0]}(Y)$ be a solution to the auxiliary periodic weak problem of elasticity:

$$\int_Y a_{ihjk}(\xi)\frac{\partial}{\partial \xi_k}(\mathcal{N}_{pjq}(\xi) + \xi_p\delta_{jq})\frac{\partial v_i(\xi)}{\partial \xi_h}d\xi = 0 \quad \forall v_i \in H_{per[0]}(Y) \qquad (6)$$

which depends only on the micro-structure of the composite, stiffness of its components and is independent of the boundary conditions. The homogenized displacement and stress fields, u^0_i, $\hat{\sigma}_{ij}$ are given by a solution to the uniquely solvable homogenized problem coinciding with (5) after replacement there the variable elastic coefficients a_{ihjk} by their homogenized counterparts: $\hat{a}_{ihjk} = \frac{1}{|Y|}\int_Y a_{ihqp}(\xi)\left[\delta_{jq}\delta_{kp} + \frac{\partial}{\partial \xi_p}\mathcal{N}_{kqj}(\xi)\right]d\xi$. Using results of [2], one can show that as the structure period ε tends to zero , the sequence of stress fields $\sigma^{\varepsilon}(x)$ contains a subsequence, which two-scale converges to a function $\sigma^0(x, x/\varepsilon)$,

$$\text{where} \quad \sigma^0_{ih}(x,\xi) = a_{ihjk}(\xi)\frac{\partial}{\partial \xi_k}(\mathcal{N}_{jpq}(\xi) + \xi_q\delta_{jp})\frac{\partial u^0_p(x)}{\partial x_q} =: B_{ihjk}(\xi)\hat{\sigma}_{jk}(x). \qquad (7)$$

Similar to [1], Chap.9, Sec.4, we call $B_{ih\gamma\delta}(\xi) = a_{ihjk}(\xi)\frac{\partial}{\partial \xi_k}(\mathcal{N}_{jpq}(\xi) + \xi_q\delta_{jp})\hat{a}^{-1}_{pq\gamma\delta}$ the stress concentration tensors. Here $(\hat{a}^{-1}_{pq\gamma\delta})_{3\times3}$ is the homogenized compliance tensor, which is the inverse to the homogenized stiffness tensor $(\hat{a}_{\gamma\delta\alpha\beta})_{3\times3}$.

5. HOMOGENIZATION OF NORMALISED EQUIVALENT STRESS AND WÖHLER FUNCTION

In a periodic medium, all becomes dependent on the period ε. Functions $\Lambda(\sigma(y), y)$ and $N^*(\Delta\sigma(y), y)$ in the local strength and durability conditions become $\Lambda^{\varepsilon}(\sigma^{\varepsilon}(y); y)$ and $N^{*\varepsilon}(\Delta\sigma^{\varepsilon}(y); y)$ respectively; particular non-local representations (1), (4) become $\underline{\Lambda}^{\varepsilon}(\sigma^{\varepsilon}; y) = \Lambda^{\varepsilon}(\tilde{\sigma}^{\varepsilon}(y); y)$, $\underline{N}^{*\varepsilon}(\Delta\sigma^{\varepsilon}; y) = N^{*\varepsilon}(\Delta\tilde{\sigma}^{\varepsilon}(y); y)$, $\tilde{\sigma}^{\varepsilon}_{ij}(y) = \int_\Omega \varphi^{\varepsilon}_{ijkl}(x,y)\sigma^{\varepsilon}_{kl}(x)dx$, for all $y \in \Omega$. Suppose $\Lambda^{\varepsilon}(\tilde{\sigma}^{\varepsilon}(y); y) = \Lambda(\tilde{\sigma}^{\varepsilon}(y); \frac{y}{\varepsilon})$, $N^{*\varepsilon}(\Delta\tilde{\sigma}^{\varepsilon}(y); y) = N^*(\Delta\tilde{\sigma}^{\varepsilon}(y); \frac{y}{\varepsilon})$,

$$\tilde{\sigma}^{\varepsilon}_{ij}(y) := \int_\Omega \varphi_{ijkl}(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon})\sigma^{\varepsilon}_{kl}(x)dx, \quad \Delta\tilde{\sigma}^{\varepsilon}_{ij}(y) := \int_\Omega \varphi_{ijkl}(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon})\Delta\sigma^{\varepsilon}_{kl}(x)dx, \ (8)$$

Our aim is to derive initial macro-strength and macro-durability conditions similar to (2) and (3), i.e., the homogenized normalized equivalent stress function $\hat{\Lambda}$ and the Wöhler function $\hat{N}^*$ for the composite from the micro-strength and micro-durability conditions. This will allow to estimate the macro-strength/durability in terms of averaged stresses and material micro-characteristics. One can prove the following

Proposition 1 *(Homogenization of non-local strength and fatigue durability). Let $\sigma^\varepsilon \in L^2(\Omega)$ be a sequence of solutions to (5). Suppose the body is macro-homogeneous in strength and fatigue durability and its point micro-strength and micro-durability conditions are $\underline{\Lambda}^\varepsilon(\sigma^\varepsilon; y) = \Lambda\left(\tilde{\sigma}^\varepsilon; \frac{y}{\varepsilon}\right) < 1$, $N < \underline{N}^{*\varepsilon}(\Delta\sigma^\varepsilon; y) := N^*\left(\Delta\tilde{\sigma}^\varepsilon; \frac{y}{\varepsilon}\right)$, where $\Lambda(\tilde{\sigma}^\varepsilon; \zeta)$ and $N^{*-1}(\Delta\tilde{\sigma}^\varepsilon; \zeta)$ are continuous in the first arguments and are periodic and bounded in the second argument; $\tilde{\sigma}^\varepsilon$ and $\Delta\tilde{\sigma}^\varepsilon$ have form (8). Then the limit (as $\varepsilon \to 0$) initial sufficient non-local macro-strength and macro-durability conditions*

are
$$\sup_{y\in\Omega} \hat{\underline{\Lambda}}(\hat{\sigma}; y) < 1, \quad N < \inf_{y\in\Omega} \hat{\underline{N}}^*(\Delta\hat{\sigma}; y), \tag{9}$$

where
$$\hat{\underline{\Lambda}}(\hat{\sigma}; y) := \sup_{\zeta\in Y} \Lambda\left(\int_\Omega \hat{\varphi}_{ijkl}(y,\zeta,x)\hat{\sigma}_{kl}(x)dx; \zeta\right), \tag{10}$$

$$\hat{\underline{N}}^*(\Delta\hat{\sigma}; y) := \inf_{\zeta\in Y} N^*\left(\int_\Omega \hat{\varphi}_{ijkl}(y,\zeta,x)\Delta\hat{\sigma}_{kl}(x)dx; \zeta\right), \tag{11}$$

$$\hat{\varphi}_{ijkl}(y,\zeta,x) = \frac{1}{|Y|}\int_Y \varphi(x,y,\xi,\zeta)B_{ijkl}(\xi)d\xi.$$

For a finite dimension of the periodicity cell ($\varepsilon > 0$), the strength and durability conditions (9) can be considered as approximate.

Example: In the particular case when the non-local weight function is independent of the cell characteristics, i.e. $\varphi^\varepsilon(x,y) = \varphi(x,y)$, we have $\hat{\varphi}_{ih\gamma\delta}(x,y,\zeta) = \varphi(x,y)$ and $\hat{\underline{\Lambda}}(\hat{\sigma}; y) = \sup_{\zeta\in Y} \Lambda\left(\int_\Omega \varphi(x,y)\hat{\sigma}(x)dx, \zeta\right)$, $\hat{\underline{N}}^*(\Delta\hat{\sigma}; y) = \inf_{\zeta\in Y} N^*\left(\int_\Omega \varphi(x,y)\Delta\hat{\sigma}(x)dx, \zeta\right)$. That is, the cell stress concentration does not influence the composite strength and durability for sufficiently small cells obeying the non-local strength/durability condition.

Let us consider now an in-face multi-axial loading, homogeneous in space coordinates, with *varying* cycle parameters such that closed loops can be always identified. Let $n = 1, 2, \ldots$ be the number of a closed loop with the stress range $\Delta\sigma_{ij}(n)$ in the loading history. Let $N^*(\Delta\sigma)$ be the Wöhler function for the considered material. The discrete linear damage accumulation rule gives the durability condition in the form $\sum_{n=1}^{N} \frac{1}{N^*[\Delta\sigma(n)]} < 1$. Taking into account that the function $\Delta\sigma_{ij}(n)$ can change only at integer values of the cycle number n, the linear damage accumulation rule can be rewritten also in the integral form $\int_0^N \frac{dn}{N^*[\Delta\sigma(n)]} < 1$.
Note that a more general phenomenological concept based on normalized equivalent stress functionals and allowing non-linear damage accumulation is introduced in [6] to estimate strength and durability under creep or fatigue.

For a periodically inhomogeneous composite and varying macro-stress, one can prove the following
Proposition 2 *Let $\Delta\sigma^\varepsilon_{ij}(y,n)$ be a sequence of solutions to (5), which belong to $L^2(\Omega)$ for any loop number $n < N$. Suppose the body micro-durability condition is $\sup_{y\in\Omega}\int_0^N \frac{dn}{\underline{N}^{*\varepsilon}(\Delta\sigma(.,n); y)} < 1$, where $\underline{N}^{*\varepsilon}(\Delta\sigma; y)$ is as in Proposition 1. Then the limit (as $\varepsilon \to 0$) initial sufficient non-local macro-durability condition is $\sup_{y\in\Omega}\int_0^N \frac{dn}{\hat{\underline{N}}^*(\Delta\hat{\sigma}(.,N); y)} < 1$, where $\hat{\underline{N}}^*$ is given by (11).*
For $\varepsilon > 0$, the last durability condition can be considered as approximate.

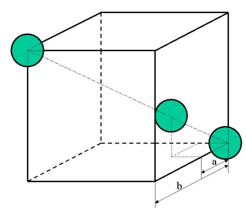
Computer Methods in Biomechanics and Biomedical Engineering - 4 (Ed.: J. Middleton, N. G. Shrive & M. L. Jones), University of Wales College of Medicine, ISBN 1 903847095, 2002

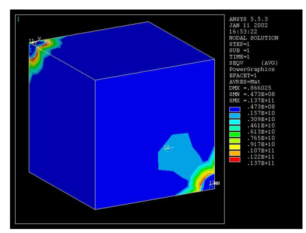
6. NUMERICAL EXAMPLE

To illustrate the influence of the micro–geometry on the macro–stiffness and macro–strength, a numerical example with the following characteristics was considered: bulk modulus, k, for cement is $7.2 \cdot 10^9$Pa, for glass is $78.7 \cdot 10^9$Pa; shear modulus, μ, for cement is $3.3 \cdot 10^9$Pa, for glass is $32.2 \cdot 10^9$Pa; uniaxial strength, σ_c, for cement is $50 \cdot 10^6$Pa, for glass is $200 \cdot 10^6$Pa.

The averaged hydrostatic strength is determined from the solution to an auxiliary problem for a unit-cell hydrostatically extended from all sides by the normal displacements $\pm 1/2$ with zero shear tractions. For the averaged uniaxial strength, an auxiliary problem with uniaxial displacements $\pm 1/2$ on two opposite sides of the unit cell and zero normal displacement on other sides with zero shear tractions should be solved. The Finite Element code ANSYS was used to solve the auxiliary periodic problems (6). We suppose that the periodicity cell is symmetric and will consider 1/8–th of the cell, structured as in the picture below: two glass spheres at the opposite corners and another glass sphere on the cube diagonal between them. Inclusion radii are $0.124b$, $a = b/4$, then the volume fraction of inclusions is 1.597%. The calculated composite properties are presented in the table.

Bulk Mod.,[Pa]	$7.23 \cdot 10^9$
Shear Mod.,[Pa]	$3.53 \cdot 10^9$
Uniax. str.,[Pa]	$43.91 \cdot 10^6$
Hydr. str.,[Pa]	$95.89 \cdot 10^6$





Maximal von Mises stress arises for both hydrostatic and uniaxial tensions on the surface of the middle glass ball at the point nearest to the corner ball. Nevertheless, the most dangerous appears to be the neighboring point of the matrix, due to the large difference in the strengths of the matrix and inclusion. Note that, unlike to the components strength, the homogenized hydrostatic strength for the composite is not infinite. The composite tensile initial strength is lower than strengths of the both composite components.

7. REFERENCES

1. Pobedrya, B. E., Mechanics of composite materials, Moscow State University Publishing, 1984.
2. Allaire, G., Homogenization and Two-Scale Convergence, SIAM J. Math. Anal., 1992, Vol. 23, 1482–1518.
3. Mikhailov, S. E., A functional approach to non-local strength conditions and fracture criteria: I&II, Eng. Fract. Mech., 1995, Vol. 52, No.4, 731–754.
4. Mikhailov, S. E. and Orlik, J., Homogenization Methods and Macro-Strength of Composites, Proceedings of the annual meeting GAMM 2001.
5. Bogy, D.B., Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. Trans. ASME, Ser. E: J. Appl. Mech., 1971, Vol. 38, 377–386.
6. Mikhailov, S.E., Theoretical Backgrounds of Durability Analysis by Normalised Equivalent Stress Functionals, Preprint PP/MAT/SEM/00-122, Glasgow Caledonian University, 2000.