In: *Advances in Boundary Integral Methods — Proceedings of the 6th UK Conference on Boundary Integral Methods* (Edited by J. Trevelyan), Durham University Publ., UK, ISBN 978-0-9535558-3-3, 2007, 291-302.

Chapter 31

About Analysis of Some Localized Boundary-Domain Integral Equations for Variable-Coefficient BVPs

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Abstract. *Some direct localized boundary-domain integral equations (LBDIEs) associated with the Dirichlet and Neumann boundary value problems for the "Laplace" linear differential equation with a variable coefficient are formulated. The LBDIEs are based on a parametrix localized by a cut-off function. Applying the theory of pseudo-differential operators, invertibility of the localized volume potentials is proved first. This allows then to prove solvability, solution uniqueness and equivalence of the LBDIEs to the original BVP, and investigate the LBDIE operator invertibility in appropriate Sobolev spaces.*
Keywords. *Partial differential equations, Variable coefficients, Parametrix, Localisation, Boundary-domain integral equations, Pseudo-differential operators.*

31.1 Introduction

Partial Differential Equations (PDEs) with variable coefficients arise naturally in mathematical modelling non-homogeneous linear and/or nonlinear media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows trough porous media, and other areas of physics and engineering.

The Boundary Integral Equation Method/Boundary Element Method (BIEM/BEM) is a well established tool for solution Boundary Value Problems (BVPs) with constant coefficients. The main ingredient for reducing a BVP for a PDE to a BIE is a fundamental solution to the original PDE. However, it is generally not available in an analytical and/or cheaply calculated form for PDEs with variable coefficients. Following Levi and Hilbert, one can use in this case a parametrix (Levi function) as a substitute for the fundamental solution. Parametrix is usually much wider available than a fundamental solution and correctly describes the main part of the fundamental solution although does not have to satisfy the original PDE. This reduces the problem not to boundary but to Boundary-Domain Integral Equation (BDIE), see e.g. [9, 10]. A discretisation of the BDIE leads then to a system of algebraic equations of the similar size as in the FEM, however the matrix of the system is not sparse as in the FEM but dense and thus less efficient for numerical solution.

The Localised Boundary-Domain Integral Equation Method (LBDIEM) emerged recently [14, 15, 12, 11, 6] addressing this deficiency and making the BDIE competitive with the FEM for such problems. The LBDIEM employs specially constructed localized parametrices to reduce linear and non-linear BVPs with variable coefficients to Localised Boundary-Domain Integral or Integro-Differential Equations. After a locally-supported mesh-based or mesh-less discretisation this ends up in sparse systems of algebraic equations. Further advancing the LBDIEM requires a deeper analytical insight on properties of the corresponding integral operators, particularly on LBDIE solvability, uniqueness of solution, equivalence to original BVPs and invertibility of the BDIEs. Analysis of non-localized segregated BDIEs is presented in [1] and of united BDIDEs in [8]. This paper develops analysis of some direct segregated LBDIEs for the Dirichlet and Neumann problems, based on a parametrix localized by multiplying with a cut-off function, [6].

31.2 Formulation of the boundary value problem

Let Ω^+ be a bounded open three–dimensional region of $\mathbb{R}^3$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. For simplicity, we assume that the boundary $\partial\Omega := \partial\Omega^+$ is a simply connected, closed, infinitely smooth surface. Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ for $x \in \mathbb{R}^3$ and $a(x) = const > 0$ for sufficiently large $|x|$. Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

We consider localized boundary-domain integral equations associated with the following scalar elliptic differential equation

$$Lu(x) := L(x, \partial_x)\, u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Big(a(x) \, \frac{\partial u(x)}{\partial x_i} \Big) = f(x), \quad x \in \Omega^+, \tag{31.1}$$

where u is an unknown function and f is a given function in Ω^+.

In what follows $H^s(\Omega^+) = H^s_2(\Omega^+)$, $H^s_{loc}(\Omega^-) = H^s_{2,loc}(\Omega^-)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ denote the Bessel potential spaces (coinciding with the Sobolev–Slobodetski spaces if $s \geq 0$).

From the trace theorem (see, e.g., [4]) for $u \in H^1(\Omega^+)$ $(u \in H^1_{loc}(\Omega^-))$ it follows that $u|^{\pm}_{\partial\Omega} := \tau^{\pm}\, u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\tau^{\pm} = \tau^{\pm}_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from $\Omega^{\pm}$. We will use also notations $u^{\pm}$ for the traces $u|^{\pm}_{\partial\Omega}$, when this will cause no confusion.

For the linear operator L, we introduce the following subspace of $H^s(\Omega)$, c.f. [3, 2, 8], $H^{s,0}(\Omega; L) := \{g :\ g \in H^s(\Omega),\ Lg \in L_2(\Omega)\}$ provided with the norm $\|g\|_{H^{s,0}(\Omega;L)} := \|g\|_{H^s(\Omega)} + \|Lg\|_{L_2(\Omega)}$.

For $u \in H^2(\Omega^+)$, we denote by $T^{\pm}$ the corresponding co–normal differentiation operator on $\partial\Omega$

$$T^{\pm}(x, n(x), \partial_x)\, u(x) := \sum_{i=1}^{3} a(x)\, n_i(x) \left(\frac{\partial u(x)}{\partial x_i} \right)^{\pm} = a(x) \, \left(\frac{\partial u(x)}{\partial n(x)} \right)^{\pm}, \tag{31.2}$$

where $n(x)$ is the unit normal vector at the point $x \in \partial\Omega$ outward to Ω^+, and $\frac{\partial}{\partial n(x)}$ denotes the normal derivative.

Let $u \in H^{1,0}(\Omega^+; L)$ $(u \in H^{1,0}_{loc}(\Omega^-; L))$, then one can correctly define the generalised (canonical) co–normal derivative $T^{\pm}u = [Tu]^{\pm} \in H^{-\frac{1}{2}}(\partial\Omega)$ with the help of the first Green identity (cf., for example, [2], [5, Lemma 4.3]) ,

$$\big\langle T^{\pm}u \,,\, v^{\pm} \big\rangle_{\partial\Omega} := \pm \int\limits_{\Omega^{\pm}} \big[\, vLu + E(u, v) \,\big]\, dx, \quad \forall\ v \in H^1(\Omega^+) \quad [v \in H^1_{comp}(\overline{\Omega^-})], \tag{31.3}$$

where $E(u,v) := \sum_{i=1}^{3} a(x)[\partial_i u(x)][\partial_j v(x)]$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denote the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ which extend the usual $L_2(\partial\Omega)$ inner product.

We will employ LBDIE approach to find a solution $u \in H^{1,0}(\Omega^+; L)$ of the partial differential equation

$$L\,u = f \quad \text{in} \quad \Omega^+, \tag{31.4}$$

satisfying either the Dirichlet boundary condition

$$u^+ = \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{31.5}$$

or the Neumann boundary condition

$$T^+ u = \psi_0 \quad \text{on} \quad \partial\Omega. \tag{31.6}$$

Equation (31.4) is understood in the distributional sense, condition (31.5) is understood in the trace sense, while equality (31.6) is understood in the functional sense in accordance with (31.3). As usual, we call (31.4), (31.5) the Dirichlet problem, and (31.4), (31.6) the Neumann problem.

We have the following well-known uniqueness theorem

THEOREM 31.2.1 *In $H^{1,0}(\Omega^+; L)$, the homogeneous Dirichlet problem has only the trivial solution, while the homogeneous Neumann problem admits a constant as a general solution.*

Proof. The proof immediately follows from Green's formula (31.3) with $v = u$ as a solution of the corresponding homogeneous boundary value problem. □

31.3 Localized potentials and Green identities

Let us recall the second Green identity for the operator $L(x, \partial_x)$,

$$\int_{\Omega^+} \big[\, v\, L(x,\partial_x)u - u\, L(x,\partial_x)v \,\big]\, dx = \big\langle\, T^+ u \,,\, v^+ \big\rangle_{\partial\Omega} - \big\langle\, T^+ v \,,\, u^+ \,\big\rangle_{\partial\Omega} \,, \tag{31.7}$$

where $u, v \in H^{1,0}(\Omega^+; L)$ are real functions.

Denote by $P(x,y)$ the parametrix (Levi function) of the operator $L(x,\partial_x)$ considered in [6, 1],

$$P(x,y) = -\frac{1}{4\pi\, a(y)\, |x-y|}\,, \quad x, y \in \mathbb{R}^3, \;\; x \neq y, \tag{31.8}$$

with the property

$$L(x,\partial_x)\, P(x,y) = \delta(x-y) + R(x,y), \tag{31.9}$$

where $\delta(\cdot)$ is the Dirac distribution, and the remainder

$$R(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\, a(y)\, |x-y|^3}\, \frac{\partial a(x)}{\partial x_i}\,, \quad x, y \in \mathbb{R}^3, \;\; x \neq y. \tag{31.10}$$

possesses a weak singularity of type $\mathcal{O}(|x-y|^{-2})$ for small $|x-y|$.

Further, let us introduce the class of cut-off functions X_ε^k.

DEFINITION 31.3.1 *Let $k \geq 0$. We say that $\chi \in X_\varepsilon^k$, if*

$$\chi \in C^k(\mathbb{R}^3); \quad \chi(x) \geq 0 \;\; \forall\, x \in \mathbb{R}^3; \quad \chi(0) = 1; \quad \chi(x) = 0 \;\; \textit{for}\; |x| \geq \epsilon > 0\,;$$

and $\chi(x) = \widetilde{\chi}(|x|)$, where $\widetilde{\chi}$ is a non-increasing function on $[0, +\infty)$.

It can be checked that, if $\chi \in X_\varepsilon^k$, $k \geq 1$, then all odd order derivatives of χ up to order k vanish at zero and particularly,

$$|1-\chi(x)| = |1-\widetilde{\chi}(|x|)| \leq c\,|x|^2 \text{ for } |x| < \varepsilon/2, \quad \chi \in X_\varepsilon^2. \tag{31.11}$$

Particular cases for χ are

$$\text{(i)} \quad \chi(x) = \begin{cases} \Big[1 - \dfrac{|x|^2}{\varepsilon^2}\Big]^{k+1} & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \qquad \chi \in X_\varepsilon^k; \tag{31.12}$$

$$\text{(ii)} \quad \chi(x) = \begin{cases} \exp\Big[\dfrac{|x|^2}{|x|^2-\varepsilon^2}\Big] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \qquad \chi \in X_\varepsilon^\infty. \tag{31.13}$$

Now we define a localized parametrix

$$P_\chi(x,y) := \chi(x-y)P(x,y), \quad x,y \in \mathbb{R}^3, \tag{31.14}$$

with $\chi \in X_\varepsilon^k$, where $k \geq 2$. Evidently,

$$L(x,\partial_x)\, P_\chi(x,y) = \delta(x-y) + R_\chi(x,y), \tag{31.15}$$

where

$$\begin{aligned} R_\chi(x,y) = -\frac{1}{4\pi\, a(y)} \Big\{ a(x) \Big[\frac{\Delta_x\, \chi(x-y)}{|x-y|} + 2\sum_{j=1}^{3} \frac{\partial \chi(x-y)}{\partial x_j}\, \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \Big] \\ + \sum_{j=1}^{3} \frac{\partial a(x)}{\partial x_j}\, \frac{\partial}{\partial x_j} \frac{\chi(x-y)}{|x-y|} \Big\}, \quad x,y \in \mathbb{R}^3. \end{aligned} \tag{31.16}$$

We see that the function $R_\chi(x,y)$ possesses a weak singularity $\mathcal{O}(|x-y|^{-2})$ as $x \to y$.

For $v(x) := P_\chi(x,y)$ with $\chi \in X_\varepsilon^2$ and $u \in H^{1,0}(\Omega^+;L)$, we obtain from (31.7) and (31.9) by the standard limiting procedures (see, e.g., [9]) the third Green identity,

$$u(y) + \mathcal{R}_\chi u(y) - V_\chi T^+ u(y) + W_\chi u^+(y) = \mathcal{P}_\chi L u(y), \quad y \in \Omega^+, \tag{31.17}$$

where

$$V_\chi g(y) := -\int_{\partial\Omega} P_\chi(x,y)\, g(x)\, dS_x, \tag{31.18}$$

$$W_\chi\, g(y) := -\int_{\partial\Omega} \big[T(x,n(x),\partial_x))\, P_\chi(x,y) \big]\; g(x)\, dS_x, \tag{31.19}$$

$$\mathcal{P}_\chi\, g(y) := \int_{\Omega^+} P_\chi(x,y)\, g(x)\, dx, \tag{31.20}$$

$$\mathcal{R}_\chi g(y) := \int_{\Omega^+} R_\chi(x,y)\, g(x)\, dx. \tag{31.21}$$

Let us also define the corresponding boundary operators

$$\mathcal{V}_\chi\, g(y) := -\int\limits_{\partial\Omega} P_\chi(x,y)\, g(x)\, dS_x, \tag{31.22}$$

$$\mathcal{W}_\chi\, g(y) := -\int\limits_{\partial\Omega} \big[\, T(x,n(x),\partial_x))\, P_\chi(x,y)\,\big]\; g(x)\, dS_x, \tag{31.23}$$

$$\mathcal{W}'_\chi\, g(y) := -\int\limits_{\partial\Omega} \big[\, T(y,n(y),\partial_y))\, P_\chi(x,y)\,\big]\, g(x)\, dS_x, \tag{31.24}$$

$$\mathcal{L}^{\pm}_\chi g(y) := [T(y,n(y),\partial_y))\, W_\chi g(y)]^{\pm}. \tag{31.25}$$

We remark that from (31.8), (31.14), (31.18)-(31.25), we have,

$$\mathcal{P}_\chi g = \frac{1}{a}\mathcal{P}_{\chi\Delta} g, \quad V_\chi g = \frac{1}{a} V_{\chi\Delta} g, \quad W_\chi g = \frac{1}{a} W_{\chi\Delta}(ag), \tag{31.26}$$

$$\mathcal{V}_\chi g = \frac{1}{a}\mathcal{V}_{\chi\Delta} g, \quad \mathcal{W}_\chi g = \frac{1}{a}\mathcal{W}_{\chi\Delta}(ag), \tag{31.27}$$

$$\mathcal{W}'_\chi g = \mathcal{W}'_{\chi\Delta}(g) - \frac{1}{a}\left[\frac{\partial a}{\partial n}\right]\mathcal{V}_{\chi\Delta}, \quad \mathcal{L}^{\pm}_\chi g = \mathcal{L}^{\pm}_{\chi\Delta}(g) - \frac{1}{a}\left[\frac{\partial a}{\partial n}\right] W^{\pm}_{\chi\Delta}, \tag{31.28}$$

where the localized potentials $\mathcal{P}_{\chi\Delta}$, $V_{\chi\Delta}$, $W_{\chi\Delta}$, $\mathcal{V}_{\chi\Delta}$, $\mathcal{W}_{\chi\Delta}$, $\mathcal{W}'_{\chi\Delta}$, $\mathcal{L}^{\pm}_{\chi\Delta}$ are associated with the operator L for $a = 1$, i.e., with the Laplace operator Δ.

The localized potentials introduced above enjoy the same mapping and jump properties as their non-localized counterparts, i.e. with $\chi = 1$ in $\mathbb{R}^3$, described in [1], and we will essentially use this in the paper.

If $u \in H^{1,0}(\Omega^+; L)$, then using the mapping and jump properties of the localized potentials we derive from (31.17)

$$u + \mathcal{R}_\chi u - V_\chi T^+ u + W_\chi u^+ = \mathcal{P}_\chi f \qquad \text{in} \quad \Omega^+, \tag{31.29}$$

$$\frac{1}{2}\, u^+ + \mathcal{R}^+_\chi u - \mathcal{V}_\chi T^+ u + \mathcal{W}_\chi u^+ = \mathcal{P}^+_\chi f^+ \qquad \text{on} \quad \partial\Omega, \tag{31.30}$$

$$\frac{1}{2}\, T^+ u + T^+\mathcal{R}_\chi u - \mathcal{W}'_\chi T^+ u + \mathcal{L}^+ u^+ = T^+\mathcal{P}_\chi f \qquad \text{on} \quad \partial\Omega. \tag{31.31}$$

Here $\mathcal{R}^+_\chi u := (\,\mathcal{R}_\chi u\,)^+$, $\quad \mathcal{P}^+_\chi f := (\,\mathcal{P}_\chi f\,)^+$.

31.4 LBDIE approach to the Dirichlet problem

For the Dirichlet problem (31.4), (31.5), where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $f \in H^0(\Omega^+)$, denoting the unknown canonical [7] co-normal derivative T^+u as a new function ψ, we obtain from (31.29), (31.30) the system of direct segregated LBDIEs,

$$u + \mathcal{R}_\chi u - V_\chi \psi = \mathcal{P}_\chi f - W_\chi \varphi_0 \qquad \text{in} \quad \Omega^+, \tag{31.32}$$

$$\mathcal{R}^+_\chi u - \mathcal{V}_\chi \psi = \mathcal{P}^+_\chi f - \frac{1}{2}\,\varphi_0 - \mathcal{W}_\chi \varphi_0 \qquad \text{on} \quad \partial\Omega, \tag{31.33}$$

where u and ψ are unknown functions.

Our goal is to prove the following equivalence and invertibility theorems.

THEOREM 31.4.1 *Let $\chi \in X^3_\varepsilon$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $f \in H^0(\Omega^+)$. The Dirichlet problem* (31.4), (31.5) *is equivalent to LBDIEs* (31.32)-(31.33) *in the following sense: if a function $u \in H^1(\Omega^+)$ solves the Dirichlet problem* (31.4), (31.5) *then the pair (u, ψ) with $\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ solves the LBDIEs* (31.32)-(31.33), *and vice versa, if a pair $(u, \psi) \in H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves the LBDIEs* (31.32)-(31.33), *then u solves the Dirichlet problem* (31.4), (31.5), *and $T^+u = \psi$.*

THEOREM 31.4.2 *Let $\chi \in X^3_\varepsilon$. The localized boundary-domain integral operator generated by the left hand side expressions in* (31.32)-(31.33),

$$\mathcal{A}^D_\chi := \begin{bmatrix} I + \mathcal{R}_\chi & -V_\chi \\ \mathcal{R}^+_\chi & -\mathcal{V}_\chi \end{bmatrix} \; : \; H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \tag{31.34}$$

is continuously invertible.

To prove these theorems we need the following auxiliary material.

Let

$$\lambda(\xi) \equiv \lambda_\chi(\xi) := \mathcal{F}_{x\to\xi}\Big[-\frac{1}{4\,\pi}\,\frac{\chi(x)}{|x|}\Big], \tag{31.35}$$

where $\chi \in X^k_\varepsilon$. Here and in what follows $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalised Fourier transform operators. Note that λ is a Fourier transform of the localized parametrix $P_{\chi_\Delta}(x)$ for the Laplace operator, thus corresponding to the case $a(x) = 1$ (see (31.8) and (31.14)).

By standard simple manipulations we arrive at the equality

$$\lambda(\xi) \;\;=\;\; -\frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3}\frac{\chi(x)}{|x|}\,e^{i\,x\cdot\xi}\,dx = -\frac{1}{|\xi|}\int\limits_0^\varepsilon \widetilde{\chi}(\varrho)\;\sin(\varrho\,|\xi|)\,d\varrho\,. \tag{31.36}$$

Integrating (31.36) by parts and making estimates of the integrals, one can prove the following main lemma crucial in our further analysis.

LEMMA 31.4.3 *Let $\chi \in X^k_\varepsilon$ with $k \geq 0$ and λ be defined by* (31.35). *Then $\lambda \in C^\infty(\mathbb{R}^3)$,*

$$\lambda(\xi) < 0 \;\; \textit{for all} \;\; \xi \in \mathbb{R}^3, \tag{31.37}$$

and for $\xi \neq 0$ the following equality holds

$$\lambda(\xi) = \sum_{m=0}^{k^*}\frac{(-1)^{m+1}}{|\xi|^{2m+2}}\,\partial^{2m}_\varrho\widetilde{\chi}(0) - \frac{1}{|\xi|^{k+1}}\int\limits_0^\varepsilon \sin\Big(|\xi|\varrho + \frac{k\pi}{2}\Big)\,\partial^k_\varrho\widetilde{\chi}(\varrho)\,d\varrho\,, \tag{31.38}$$

where k^* is the integer part of $(k-1)/2$ and $\partial_\varrho := \partial/\partial\varrho$.

From this lemma it follows that there are positive constants c_1 and c_2 such that if $\chi \in X^1_\varepsilon$, then

$$c_1\,(1+|\xi|)^{-2} \leq |\lambda(\xi)| \leq c_2\,(1+|\xi|)^{-2} \;\text{ for all }\; \xi \in \mathbb{R}^3. \tag{31.39}$$

Denote

$$m(\xi) := \frac{1}{\lambda(\xi)}\,. \tag{31.40}$$

By (31.39)

$$c_2^{-1}\,(1+|\xi|)^2 \leq |m(\xi)| \leq c_1^{-1}\,(1+|\xi|)^2 \;\text{ for all }\; \xi \in \mathbb{R}^3. \tag{31.41}$$

Denote by $\mathcal{M}$ the pseudodifferential operator with symbol $m(\xi)$,

$$\mathcal{M}\, v := \mathcal{F}^{-1}\,[\, m(\xi)\,\mathcal{F}v\,]\,. \tag{31.42}$$

It is evident that $\mathcal{M}$ is a pseudodifferential operator of order 2, i.e.,

$$\mathcal{M}\ :\ H^t(\mathbb{R}^3) \to H^{t-2}(\mathbb{R}^3)\ \text{ for arbitrary }\ t \in \mathbb{R}. \tag{31.43}$$

Moreover, due to (31.41) the operator (31.43) is invertible for arbitrary $t \in \mathbb{R}$.

Let us introduce the distributions $\psi\,\delta_{\partial\Omega}$ and $-\partial_n(\varphi\,\delta_{\partial\Omega})$ defined by the relations

$$\langle \psi\,\delta_{\partial\Omega}\,,\, h\rangle := \langle \psi\,,\, h\rangle_{\partial\Omega}, \quad \langle -\partial_n(\varphi\,\delta_{\partial\Omega})\,,\, h\rangle := \Big\langle \varphi\,,\, \frac{\partial h}{\partial n}\Big\rangle_{\partial\Omega} \ \text{ for all }\ h \in \mathcal{D}(\mathbb{R}^3), \tag{31.44}$$

where $\mathcal{D}(\mathbb{R}^3)$ denotes the totality of infinitely differentiable functions with compact support. Note that, for "good" functions, e.g., $\psi \in C(\partial\Omega)$ and $\varphi \in C(\partial\Omega)$,

$$\langle \psi\,\delta_{\partial\Omega}\,,\, h\rangle = \int\limits_{\partial\Omega} \psi\, h\, dS\,, \qquad \langle -\partial_n(\varphi\,\delta_{\partial\Omega})\,,\, h\rangle = \int\limits_{\partial\Omega} \varphi\,\frac{\partial h}{\partial n}\, dS\,, \tag{31.45}$$

Evidently,

$$\operatorname{supp}(\psi\delta_{\partial\Omega}) \subset \partial\Omega \quad\text{and}\quad \operatorname{supp}[-\partial_n(\varphi\,\delta_{\partial\Omega})] \subset \partial\Omega, \tag{31.46}$$

which shows that the distributions introduced above are compactly supported.

Denote by $\mathcal{P}_{0,\lambda}$ the pseudodifferential operator with the symbol λ,

$$\mathcal{P}_{0,\lambda}\, v := \mathcal{F}^{-1}\,[\,\lambda(\xi)\,\mathcal{F}v\,]\,, \quad v \in \mathcal{S}'(\mathbb{R}^3), \tag{31.47}$$

where $\mathcal{S}'(\mathbb{R}^3)$ is the space of tempered distributions (Schwartz space).

It is well-known that the single layer, double layer and volume potentials can be represented as convolutions (see, e.g., [13] for harmonic potentials):

$$V_{\chi\Delta}\,\psi = \frac{1}{4\pi}\int\limits_{\partial\Omega} \frac{\chi(x-y)}{|x-y|}\,\psi(x)\,dS_x = \frac{1}{4\,\pi}\,\Big[\,\frac{\chi(x)}{|x|}\ *\ (\psi\,\delta_{\partial\Omega})\Big](y), \tag{31.48}$$

$$W_{\chi\Delta}\,\varphi = \frac{1}{4\pi}\int\limits_{\partial\Omega} \frac{\partial}{\partial n(x)}\frac{\chi(x-y)}{|x-y|}\ \varphi(x)\,dS_x = \frac{1}{4\,\pi}\,\Big[\,\frac{\chi(x)}{|x|}\ *\ [-\partial_n(\varphi\,\delta_{\partial\Omega})]\Big](y), \tag{31.49}$$

$$\mathcal{P}_{0,\lambda}\, v = -\frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3} \frac{\chi(x-y)}{|x-y|}\, v(x)\,dx = -\frac{1}{4\,\pi}\,\Big[\,\frac{\chi(x)}{|x|}\ *\ v\,\Big](y), \tag{31.50}$$

where the symbol $*$ denotes the generalised convolution operation in $\mathbb{R}^3$.

Evidently, $\mathcal{P}_{0,\lambda}v = \mathcal{P}_{\chi\Delta}v$ for $v \in \widetilde{H}^s(\Omega^+)$, $\ s \geq 0$. In particular,

$$\mathcal{P}_{0,\lambda}\widetilde{f} = \mathcal{P}_{\chi\Delta}f \text{ for } f \in H^0(\Omega^+), \tag{31.51}$$

where $\widetilde{f}$ is the extension of f by zero from Ω^+ onto the whole of $\mathbb{R}^3$.

From these representations immediately follows that the potentials can be written as pseudodifferential operators

$$V_{\chi\Delta}\,\psi = -\mathcal{F}^{-1}\,\{\,\lambda(\xi)\ \mathcal{F}(\psi\,\delta_{\partial\Omega})\,\} = -\mathcal{P}_{0,\lambda}(\psi\,\delta_{\partial\Omega})\,, \tag{31.52}$$

$$W_{\chi\Delta}\,\varphi = -\mathcal{F}^{-1}\,\{\,\lambda(\xi)\ \mathcal{F}[\,-\partial_n(\varphi\,\delta_{\partial\Omega})\,]\} = -\mathcal{P}_{0,\lambda}[\,-\partial_n(\varphi\,\delta_{\partial\Omega})\,], \tag{31.53}$$

$$\mathcal{P}_{\chi\Delta}\, f = \mathcal{F}^{-1}\,\{\,\lambda(\xi)\ \mathcal{F}\widetilde{f}\,\}. \tag{31.54}$$

Now we can prove the following assertions for the localized potentials associated with the Laplace operator, cf. (31.26).

LEMMA 31.4.4 *Let* $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $f \in H^0(\Omega^+)$. *Then*
(i) $V_{\chi_\Delta}\,\psi$ *has a compact support and*

$$\operatorname{supp} \mathcal{M}\, V_{\chi_\Delta}\,\psi \subset \partial\Omega; \tag{31.55}$$

(ii) $W_{\chi_\Delta}\,\varphi$ *has a compact support and*

$$\operatorname{supp} \mathcal{M}\, W_{\chi_\Delta}\,\varphi \subset \partial\Omega; \tag{31.56}$$

(iii) $\mathcal{P}_{\chi_\Delta}\, f$ *has a compact support and*

$$\operatorname{supp} \mathcal{M}\, \mathcal{P}_{\chi_\Delta}\, f \subset \overline{\Omega^+}. \tag{31.57}$$

Proof. Taking into consideration (31.40) and (31.52) we get

$$\begin{aligned}\mathcal{M}\, V_{\chi_\Delta}\,\psi &= \mathcal{F}^{-1}\,\{\, m(\xi)\, \mathcal{F}(V_{\chi_\Delta}\,\psi)\,\} = -\mathcal{F}^{-1}\,\{\, m(\xi)\ \lambda(\xi)\, \mathcal{F}(\psi\,\delta_{\partial\Omega})\,\} \\ &= -\mathcal{F}^{-1}\,\{\, \mathcal{F}(\psi\,\delta_{\partial\Omega})\,\} = -(\psi\,\delta_{\partial\Omega}).\end{aligned} \tag{31.58}$$

Quite similarly we derive

$$\mathcal{M}\, W_{\chi_\Delta}\,\varphi = \partial_n(\varphi\,\delta_{\partial\Omega}), \tag{31.59}$$

$$\mathcal{M}\, \mathcal{P}_{\chi_\Delta}\, f = \widetilde{f}. \tag{31.60}$$

From equalities (31.58), (31.59) and (31.60) the inclusions (31.55), (31.56) and (31.57) follow immediately.

Finally, let us remark that due to the localized character of the kernel functions of the potentials it follows that the surface potentials are compactly supported in the ε neighbourhood of the surface $\partial\Omega$, while the volume potential is supported in the ε neighbourhood of the domain Ω^+. □

REMARK 31.4.5 *From* (31.58), (31.59) *and* (31.60) *it follows that*

$$\langle \mathcal{M}\, V_{\chi_\Delta}\,\psi\,,\, h\rangle = -\langle \psi\,,\, h\rangle_{\partial\Omega}, \tag{31.61}$$

$$\langle \mathcal{M}\, W_{\chi_\Delta}\,\varphi\,,\, h\rangle = -\langle \varphi\,,\, \frac{\partial h}{\partial n}\rangle_{\partial\Omega}, \tag{31.62}$$

$$\langle \mathcal{M}\, \mathcal{P}_{\chi_\Delta}\, f\,,\, h\rangle = \langle f\,,\, h\rangle_{\Omega^+} \tag{31.63}$$

for arbitrary $h \in \mathcal{D}(\mathbb{R}^3)$.

REMARK 31.4.6 *From the above analysis it follows that* $\mathcal{P}_{0,\lambda}$ *is the inverse to the operator* $\mathcal{M}$ *in the space of rapidly decreasing functions* $\mathcal{S}(\mathbb{R}^3)$ *and in the space of tempered distributions* $\mathcal{S}'(\mathbb{R}^3)$. *Thus, the operators*

$$\begin{array}{llcll} \mathcal{M} & : H^t(\mathbb{R}^3) & \to & H^{t-2}(\mathbb{R}^3) & \textit{for arbitrary} \ \ t \in \mathbb{R}, \\ & \ \ \mathcal{S}(\mathbb{R}^3) & \to & \mathcal{S}(\mathbb{R}^3), & \\ & \ \ \mathcal{S}'(\mathbb{R}^3) & \to & \mathcal{S}'(\mathbb{R}^3), & \\[1ex] \mathcal{P}_{0,\lambda} & : H^t(\mathbb{R}^3) & \to & H^{t+2}(\mathbb{R}^3) & \textit{for arbitrary} \ \ t \in \mathbb{R}, \\ & \ \ \mathcal{S}(\mathbb{R}^3) & \to & \mathcal{S}(\mathbb{R}^3), & \\ & \ \ \mathcal{S}'(\mathbb{R}^3) & \to & \mathcal{S}'(\mathbb{R}^3), & \end{array}$$

are invertible operators.

LEMMA 31.4.7 *Let* $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $f \in H^0(\Omega^+)$. *If* $V_{\chi_\Delta}\,\psi + \mathcal{P}_{\chi_\Delta}\, f = 0$ *in* Ω^+, *then* $\psi = 0$ *on* $\partial\Omega$ *and* $f = 0$ *in* Ω^+.

Proof. Denote $U := V_{\chi_\Delta}\,\psi + \mathcal{P}_{\chi_\Delta}\,f$ in $\mathbb{R}^3$. Let us show that U is zero in $\mathbb{R}^3$. To this end, let us note that $U \in \widetilde{H}^1(\Omega^-_\varepsilon)$, where Ω^-_ε is a one-sided (exterior) ε neighbourhood of $\partial\Omega$, i.e., $\Omega^-_\varepsilon := \{x \in \Omega^- \,:\, \rho(x,\partial\Omega) < \varepsilon\}$, where $\rho(x,\partial\Omega)$ is the distance from the reference point x to the surface $\partial\Omega$. Therefore, there exists a sequence $U_n \in \mathcal{D}(\Omega^-_\varepsilon)$, $n = \overline{1,\infty}$, converging to U in the space $\widetilde{H}^1(\Omega^-_\varepsilon)$, i.e., $\lim\limits_{n\to\infty} ||U - U_n||_{H^1(\mathbb{R}^3)} = 0$. Due to (31.58) and (31.60) it follows that $\mathcal{M}\,U$ is a distribution with compact support,

$$\mathcal{M}\,U = \widetilde{f} - \psi\,\delta_{\partial\Omega}, \tag{31.64}$$

where $\widetilde{f}$ is the extension by zero of the function f from Ω^+ onto the whole of $\mathbb{R}^3$. Therefore, $\mathcal{M}\,U = 0$ in Ω^- in the distributional sense, i.e., $\langle \mathcal{M}\,U\,,\,v\rangle = 0 \quad \forall\, v \in \mathcal{D}(\Omega^-)$. In particular, $\langle \mathcal{M}\,U\,,\,U_n\rangle = 0, \quad n = \overline{1,\infty}$. Then we have

$$\begin{aligned}
0 &= \langle \mathcal{M}\,U\,,\,U_n\rangle = \langle \mathcal{F}^{-1}\,[m(\xi)\,\mathcal{F}U\,]\,,\,U_n\rangle = (2\pi)^{-3}\,\langle m(\xi)\,\mathcal{F}U\,,\,\overline{\mathcal{F}U_n}\,\rangle \\
&= (2\pi)^{-3}\int\limits_{\mathbb{R}^3} m(\xi)\,\mathcal{F}U\;\overline{\mathcal{F}U_n}\,d\xi \\
&= (2\pi)^{-3}\int\limits_{\mathbb{R}^3} m(\xi)\,|\mathcal{F}U|^2 d\xi + (2\pi)^{-3}\int\limits_{\mathbb{R}^3} m(\xi)\,\mathcal{F}U\;\overline{[\mathcal{F}U_n - \mathcal{F}U]}\,d\xi.
\end{aligned} \tag{31.65}$$

By (31.41), $|m(\xi)| \le C\,(1+|\xi|^2)$ with C independent of ξ. Therefore, from (31.65) we get

$$\begin{aligned}
\left|\,\int\limits_{\mathbb{R}^3} m(\xi)\,|\mathcal{F}U|^2 d\xi\right| &\le C\int\limits_{\mathbb{R}^3}(1+|\xi|^2)\,|\mathcal{F}U|\;|\overline{\mathcal{F}(U_n - U)}|\,d\xi \\
&\le C||U||_{H^1(\mathbb{R}^3)}\;||U_n - U||_{H^1(\mathbb{R}^3)} \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Thus $\int\limits_{\mathbb{R}^3} m(\xi)\,|\mathcal{F}U|^2 d\xi = 0$, whence $\mathcal{F}U = 0$ due to the inequality (31.41) and negativity of m, see (31.40), (31.37). Consequently, $U = V_{\chi_\Delta}\,\psi + \mathcal{P}_{\chi_\Delta}\,f = 0$ in $\mathbb{R}^3$.

Now, from (31.64) it follows that $\widetilde{f} - \psi\,\delta_{\partial\Omega} = 0$ in the distributional sense in $\mathbb{R}^3$, which implies $f = 0$ in Ω^+ and $\psi = 0$ on $\partial\Omega$. ☐

REMARK 31.4.8 *Remark that the right hand side expressions in* (31.32) *and* (31.33) *vanish if and only if $f = 0$ in Ω^+ and $\varphi_0 = 0$ on $\partial\Omega$. Indeed, the equalities*

$$\mathcal{P}_\chi f - W_\chi \varphi_0 = 0 \;\text{ in }\; \Omega^+, \tag{31.66}$$

$$\mathcal{P}^+_\chi f - \frac{1}{2}\,\varphi_0 - \mathcal{W}_\chi \varphi_0 = 0 \;\text{ on }\; \partial\Omega, \tag{31.67}$$

imply $\varphi_0 = 0$ on $\partial\Omega$ if we take the trace of (31.66) *and subtract from* (31.67) *taking into account the boundary properties of the volume and double layer localized potentials. Therefore, we get $\mathcal{P}_\chi f = 0$ in Ω^+, which due to Lemma* 31.4.7 *gives $f = 0$ in Ω^+.*

Proof of Theorem 31.4.1.

Let $u \in H^1(\Omega^+)$ be a solution of the Dirichlet problem (31.4), (31.5). Then $u \in H^{1,0}(\Omega^+; L)$ since $f \in H^0(\Omega+)$, and by (31.29) and (31.30) we see that the pair (u,ψ) with $\psi = T^+u$ solves the LBDIEs (31.32)-(31.33).

Now, let a pair $(u,\psi) \in H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve LBDIEs (31.32)-(31.33). From mapping properties of the operators participating in LBDIEs (31.32)-(31.33) it follows that $u \in$

$H^{1,0}(\Omega^+;L)$. Taking trace of (31.32) on $\partial\Omega$ and comparing the result with (31.33), we easily derive that $u^+ = \varphi_0$ on $\partial\Omega$. Therefore, from Green's identity (31.17) for the function $u \in H^{1,0}(\Omega^+;L)$ we have

$$u + \mathcal{R}_\chi u - V_\chi T^+ u = \mathcal{P}_\chi Lu - W_\chi \varphi_0, \quad y \in \Omega^+. \tag{31.68}$$

Subtract (31.68) from (31.32) to obtain

$$\mathcal{P}_\chi(Lu - f) + V_\chi(T^+u - \psi) = 0 \text{ in } \Omega^+. \tag{31.69}$$

Due to Lemma 31.4.7 then it follows that $Lu - f = 0$ in Ω^+ and $T^+u - \psi = 0$ on $\partial\Omega$, which completes the proof. □

Proof of Theorem 31.4.2.

Denote by $\mathcal{A}^D$ the operator

$$\mathcal{A}^D := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix} : H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \tag{31.70}$$

where V and $\mathcal{V}$ are the (non-localized) operators defined by (31.18) and (31.22) with $P(x,y)$ instead of $P_\chi(x,y)$.

It is evident that the operator

$$\mathcal{A}^D_\chi - \mathcal{A}^D : H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$$

is compact due the compactness of the operators $\mathcal{V}_\chi - \mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$,, $\mathcal{R}_\chi : H^1(\Omega^+) \to H^1(\Omega^+)$ and $\mathcal{R}^+_\chi : H^1(\Omega^+) \to H^{\frac{1}{2}}(\partial\Omega)$, which follow from (31.11) and (31.16). Note that the operator

$$\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

is invertible (see [1], Remark 3.9). Therefore, we easily conclude that the operator (31.70) is invertible.

Thus, the operator (31.34) is Fredholm with zero index. It remains to show that the operator (31.34) is injective, that is, the homogeneous system

$$u + \mathcal{R}_\chi u - V_\chi \psi = 0 \quad \text{in} \quad \Omega^+, \tag{31.71}$$

$$\mathcal{R}^+_\chi u - \mathcal{V}_\chi \psi = 0 \quad \text{on} \quad \partial\Omega, \tag{31.72}$$

has only the trivial solution. The latter, in turn, follows from Remark 31.4.8, uniqueness Theorem 31.2.1 and equivalence Theorem 31.4.1.

Thus operator (31.34) is invertible. □

31.5 LBDIE approach for the Neumann problem

In this subsection we consider the LBDIEs corresponding to the Neumann problem (31.4), (31.6), where $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in H^0(\Omega^+)$. Denoting the unknown trace u^+ as a new function φ, we obtain from (31.29), (31.30) the direct segregated LBDIE system

$$u + \mathcal{R}_\chi u + W_\chi \varphi = \mathcal{P}_\chi f + V_\chi \psi_0 \quad \text{in} \quad \Omega^+, \tag{31.73}$$

$$\mathcal{R}^+_\chi u + \frac{1}{2}\varphi + \mathcal{W}_\chi \varphi = \mathcal{P}^+_\chi f + \mathcal{V}_\chi \psi_0 \quad \text{on} \quad \partial\Omega, \tag{31.74}$$

with the unknowns u and φ.

First we prove the following equivalence theorem.

THEOREM 31.5.1 *Let $\chi \in X^3_\varepsilon$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in H^0(\Omega^+)$. The Neumann problem* (31.4), (31.6) *is equivalent to LBDIEs* (31.73)-(31.74) *in the following sense: if a function $u \in H^1(\Omega^+)$ solves the Neumann problem* (31.4), (31.6) *then the pair (u, φ) with $\varphi = u^+ \in H^{\frac{1}{2}}(\partial\Omega)$ solves the LBDIEs* (31.73)-(31.74)*, and vice versa, if a pair $(u, \psi) \in H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$ solves LBDIEs* (31.73)-(31.74)*, then u solves the Neumann problem* (31.4), (31.6) *and $u^+ = \varphi$.*

Proof. Let $u \in H^1(\Omega^+)$ be a solution of the Neumann problem (31.4), (31.6). Then $u \in H^{1,0}(\Omega^+; L)$ since $f \in H^0(\Omega+)$, and by (31.29) and (31.31) we see that the pair (u, φ) with $\varphi = u^+$ solves the LBDIEs (31.73)-(31.74).

Now, let a pair $(u, \varphi) \in H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$ solve the LBDIEs (31.73)-(31.74). From mapping properties of the operators participating in LBDIEs (31.73)-(31.74) it follows that $u \in H^{1,0}(\Omega^+; L)$. Further, taking the trace of (31.73) on $\partial\Omega$ and comparing the result with (31.74), we easily derive that $u^+ = \varphi$ on $\partial\Omega$. Therefore, from Green's identity (31.17) for the function u we have

$$u + \mathcal{R}_\chi u - V_\chi T^+ u + W_\chi \varphi = \mathcal{P}_\chi L u \ \text{ in } \ \Omega^+. \tag{31.75}$$

Taking the difference of the equations (31.73) and (31.75) we arrive at the relation

$$\mathcal{P}_\chi (f - Lu) + V_\chi(\psi_0 - T^+ u) = 0 \ \text{ in } \ \Omega^+. \tag{31.76}$$

By Lemma 31.4.7 then it follows that $Lu = f$ in Ω^+ and $T^+ u = \psi_0$ on $\partial\Omega$, i.e., u solves the Neumann problem (31.4), (31.6). □

As a consequence we have the following theorem.

THEOREM 31.5.2 *The localized boundary-domain integral operator generated by the left hand side expressions in* (31.73)-(31.74)*,*

$$\mathcal{A}^N_\chi =: \begin{bmatrix} I + \mathcal{R}_\chi & W_\chi \\ \mathcal{R}^+_\chi & \frac{1}{2} I + \mathcal{W}_\chi \end{bmatrix} : \ H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \tag{31.77}$$

is Fredholm with zero index and has one dimensional null-space, $\ker \mathcal{A}^N_\chi$. Moreover, the pair $u = 1$ in Ω^+ and $\varphi = 1$ on $\partial\Omega$ is the only linearly independent element of $\ker \mathcal{A}^N_\chi$.

The equality

$$\langle f \,,\, 1 \rangle_{\Omega^+} - \langle \psi \,,\, 1 \rangle_{\partial\Omega} = 0 \tag{31.78}$$

is necessary and sufficient for solvability of LBDIEs (31.73)-(31.74).

Proof. Condition (31.78) is necessary and sufficient for solvability of the Neumann BVP (see, e.g., [4]). Therefore, the proof of the theorem directly follows from the compactness properties of the operators $\mathcal{W}_\chi$, $\mathcal{R}_\chi$ and $\mathcal{R}^+_\chi$ and Theorem 31.5.1, similar to the proof of Theorem 31.4.2. □

Acknowledgements

This work was prepared under International Joint Project Grant - 2005/R4 "Boundary-Domain Integral Equations: Formulation, Analysis, Localisation " of the Royal Society, UK.

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