INTERNALLY 4-CONNECTED BINARY MATROIDS WITH CYCLICALLY SEQUENTIAL ORDERINGS

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Abstract. We characterize all internally 4-connected binary matroids M with the property that the ground set of M can be ordered $(e_0, \ldots, e_{n-1})$ in such a way that $\{e_i, \ldots, e_{i+t}\}$ is 4-separating for all $0 \leq i, t \leq n-1$ (all subscripts are read modulo n). We prove that in this case either $n \leq 7$ or, up to duality, M is isomorphic to the polygon matroid of a cubic or quartic planar ladder, the polygon matroid of a cubic or quartic Möbius ladder, a particular single-element extension of a wheel, or a particular single-element extension of the bond matroid of a cubic ladder.

1. Introduction

We start with a definition: Recall that if M is a matroid on the ground set E, and X is a subset of E, then the connectivity function $\lambda_M(X)$ is defined to be $\mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M)$. We say that $X \subseteq E$ is *k-separating* if $\lambda_M(X) < k$, and a *k-separation* of M is a partition (X_1, X_2) of E such that $|X_1|, |X_2| \geq k$ and both X_1 and X_2 are k-separating. Then M is *n-connected* if it has no k-separations with $k < n$.

A 3-connected matroid M has *path width three* if its ground set can be ordered $(e_0, \ldots, e_{n-1})$ in such a way that $\lambda_M(\{e_0, \ldots, e_i\}) \leq 2$ for all $0 \leq i \leq n-1$. Such a matroid is sometimes said to be *sequential*. The structure of sequential matroids has been studied by Hall, Oxley, and Semple [2], and by Beavers and Oxley [1]. It is natural to generalize sequential matroids, and to consider the 3-connected matroids M whose ground sets can be ordered $(e_0, \ldots, e_{n-1})$ in such a way that $\lambda_M(\{e_i, \ldots, e_{i+t}\}) \leq 2$ for all $0 \leq i, t \leq n-1$, where we read subscripts modulo n. It is not difficult to see that the only matroids satisfying these conditions are the wheels, whirls, lines and colines.

We extend this notion to a higher type of connectivity. In particular, we consider the matroids M such that the ground set of M can be ordered $(e_0, \ldots, e_{n-1})$ so that $\lambda_M(\{e_i, \ldots, e_{i+t}\}) \leq 3$ for all $0 \leq i, t \leq n-1$. We shall say that a matroid with such an ordering of its ground set is *cyclically 4-sequential*, and we will call $(e_0, \ldots, e_{n-1})$ a *cyclically 4-sequential ordering* (or just a *cyclic ordering*).

Date: July 23, 2009.

Let n be a multiple of 3, and consider the following collection of subsets of $\{e_0, \ldots, e_{n-1}\}$ where subscripts are read modulo n:

$$\mathcal{A} = \{\{e_i,\, e_{i+1},\, e_{i+2},\, e_{i+3}\} \mid 0 \leq i \leq n-3,\, i \equiv 0 \pmod 3)\}\} \cup \{\{e_0, \ldots, e_{n-1}\}\}.$$

Then $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering of the transversal matroid $M[\mathcal{A}]$. If $3 \leq i \leq n-3$ and $i \equiv 0 \pmod 3$, then we may arbitrarily declare sets of the form $\{e_x,\, e_y,\, e_i\}$ to be triangles, where $x \in \{i-2,\, i-1\}$ and $y \in \{i+1,\, i+2\}$ (as long as no two such triangles intersect in exactly two elements). In any such matroid $(e_0, \ldots, e_{n-1})$ is a valid cyclically 4-sequential ordering. This seems to indicate that cyclically 4-sequential matroids can be quite diverse. However, if we restrict our attention to cyclically 4-sequential matroids that are also binary and internally 4-connected, we find that there are essentially only four families of examples. Our main result is a characterization of such matroids.

It is easy to see that every matroid on a set of at most seven elements is cyclically 4-sequential. We completely characterize the internally 4-connected binary matroids that are cyclically 4-sequential.

Theorem 1.1. *Let M be an internally 4-connected binary matroid and assume that the ground set of M can be ordered $(e_0, \ldots, e_{n-1})$ in such a way that $\lambda_M(\{e_i, \ldots, e_{i+t}\}) \leq 3$ for all $0 \leq i,\, t \leq n-1$. Then either $n \leq 7$, or one of M or M^* is isomorphic to a matroid in the following list:*

(i) The polygon matroid of a cubic or quartic planar ladder;
(ii) The polygon matroid of a cubic or quartic Möbius ladder;
(iii) A wheel with a tip; or
(iv) A dual cubic ladder with a tip.

The four classes of matroids in Theorem 1.1 will be described in detail in Section 3. They have all been discovered before. For example, the *wheels with tips* were identified by Kingan and Lemos [3] as a family of almost-graphic matroids. Mayhew, Royle and Whittle [4] characterize the internally 4-connected binary matroids that have no minor isomorphic to $M(K_{3,3})$. The basic classes of such matroids include the *triangular Möbius matroids*, which are precisely the dual cubic Möbius ladders with tips, and the *triadic Möbius matroids*, which are duals of wheels with tips.

Our notation follows that of Oxley [5]. A *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. The variable n will typically denote the size of the ground set of a matroid, and $(e_0,\, e_1, \ldots, e_{n-1})$ will be a cyclically 4-sequential ordering of that ground set. Indices are always to be read modulo n. We repeatedly use the fact that in a binary matroid the intersection of a circuit and a cocircuit has even cardinality. For the sake of brevity, we refer to this phenomenon as *orthogonality*. We also make frequent use of the fact that in a binary matroid the symmetric difference of a set of circuits is a disjoint union of circuits, and the symmetric difference of a set of cocircuits is a disjoint union of cocircuits.

2. Preliminaries

Recall that if M is a matroid on the ground set E, then $\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E-X) - \mathrm{r}(M)$ for all subsets $X \subseteq E$. Obviously $\lambda_M(X) = \lambda_M(E-X)$, and it is easy to see that $\lambda_{M^*}(X) = \lambda_M(X)$ for all $X \subseteq E$. Moreover $\lambda_M(X) = \mathrm{r}(X) + \mathrm{r}^*(X) - |X|$. Then $X \subseteq E$ is *k-separating* if $\lambda_M(X) < k$, and it is *exactly k-separating* if $\lambda_M(X) = k-1$. A *k-separation* of M is a partition (X_1, X_2) of E such that $\min\{|X_1|, |X_2|\} \geq k$, and $\lambda_M(X_1) = \lambda_M(X_2) < k$.

The matroid M is *n-connected* if it has no k-separations where $k < n$, and it is *internally 4-connected* if it is 3-connected, and whenever (X_1, X_2) is a 3-separation, either $|X_1| = 3$, or $|X_2| = 3$.

For the sake of completeness, we repeat our principle definition here:

Definition 2.1. A matroid M is *cyclically 4-sequential* if its ground set can be ordered $(e_0, \ldots, e_{n-1})$ in such a way that $\lambda_M(\{e_i, \ldots, e_{i+t}\}) \leq 3$ for all $0 \leq i, t \leq n-1$. Such an ordering is a *cyclically 4-sequential ordering* (or sometimes just a *cyclic ordering*).

It is clear that if $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering for M, then it is also a cyclically 4-sequential ordering for M^*. Thus the property of being cyclically 4-sequential is closed under duality.

A simple argument shows that if a matroid has at most seven elements, then any ordering of its ground set is a cyclically 4-sequential ordering. Thus every matroid on at most seven elements is cyclically 4-sequential. Our next lemma eliminates the possibility of an internally 4-connected binary matroid on eight elements. Recall that the *wheel with r-spokes*, denoted W_r, is the graph obtained by taking a cycle of r vertices, and adding a new vertex that is adjacent to all other vertices.

Lemma 2.2. *No binary matroid on eight elements is internally 4-connected.*

Proof. It is an easy application of the Splitter Theorem (see [5, Theorem 11.1.2]) that the only 3-connected 8-element binary matroids are $M(W_4)$, AG(3, 2) and S_8 (see [5, Exercise 11.2.3]). Since none of these is internally 4-connected the result follows. □

Because of the previous observations, when characterizing the internally 4-connected binary matroids that are cyclically 4-sequential, it suffices to consider matroids on at least nine elements. The rest of the article is devoted to this case.

Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering of the matroid M. We will use S_i to denote the set $\{e_i, \ldots, e_{i+3}\}$ for $0 \leq i \leq n-1$ (remembering that subscripts are read modulo n). Suppose also that M is internally 4-connected and binary, and $n \geq 9$. Then $\lambda_M(S_i) = 3$ for every $0 \leq i \leq n-1$. Therefore $r_M(S_i) + r_{M^*}(S_i) = 7$ and so exactly one the following occurs:

(i) S_i is a circuit;

(ii) S_i is a cocircuit;
(iii) S_i contains a triangle; or,
(iv) S_i contains a triad.

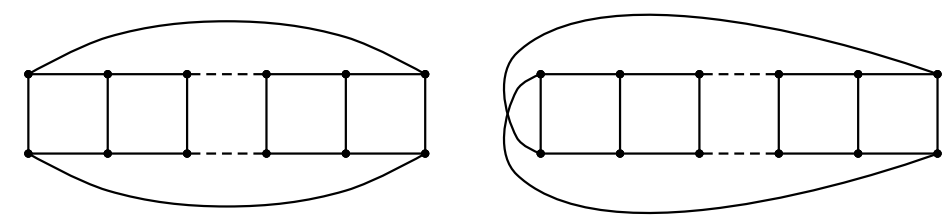
If S_i satisfies (iii) or (iv), then we say that S_i is a *T-set* or a *T^*-set*, respectively.

Consider the sequence $(X_0, \ldots, X_{n-1})$, where the character X_i is either C, C^*, T or T^*, according to whether S_i satisfies statement (i), (ii), (iii) or (iv) above. We shall say that $(X_0, \ldots, X_{n-1})$ is the *label-sequence* corresponding to the cyclic ordering $(e_0, \ldots, e_{n-1})$, and we shall call $(S_0, \ldots, S_{n-1})$ the *set-sequence* of $(e_0, \ldots, e_{n-1})$.

As we shall see, the structure of an internally 4-connected binary matroid that is cyclically 4-sequential is completely determined by its label-sequence. Much of the work of the article is spent eliminating certain subsequences from the label-sequence and classifying the matroid structure that is forced by the remaining label-sequences.

3. Ladders, wheels, and tips

In this section we define the four classes of matroids that appear in Theorem 1.1. The *cubic planar ladders* and the *cubic Möbius ladders* are the families of graphs illustrated by Figure 1. We use CP_n to denote the cubic planar ladder on n vertices, and CM_n to denote the cubic Möbius ladder on n vertices.

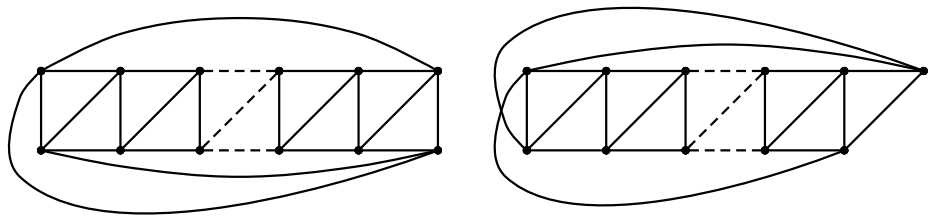


(a) (b)

Figure 1. (a) Cubic planar ladder. (b) Cubic Möbius ladder.

The *quartic planar ladders* and the *quartic Möbius ladders* are illustrated in Figure 2. The quartic planar ladder on n vertices is denoted by QP_n, and the quartic Möbius ladder on n vertices is denoted by QM_n.

We have already defined the wheel W_r to be the graph obtained from the cycle on r vertices by adding a new vertex, adjacent to all other vertices. The new edges are called *spokes*. Let S be the set of spokes. Then S is a basis of $M(W_r)$. Let $M(W_r)^+$ be the binary matroid obtained from $M(W_r)$ by adding a new element x so that $S \cup x$ is a circuit. We shall say that $M(W_r)^+$ is a *wheel with a tip*. The matrix in Figure 3 represents $M(W_r)^+$ over GF(2). It is easy to confirm that $M(W_r)^+$ is internally 4-connected.



(a) (b)

FIGURE 2. (a) Quartic planar ladder. (b) Quartic Möbius ladder.

$$\left[\begin{array}{c|cccccc} & 1 & 0 & 0 & \cdots & 1 & 1 \\ & 1 & 1 & 0 & \cdots & 0 & 1 \\ & 0 & 1 & 1 & \cdots & 0 & 1 \\ I_r & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & 0 & \cdots & 0 & 1 \\ & 0 & 0 & 0 & \cdots & 1 & 1 \end{array}\right]$$

FIGURE 3. A matrix representation of $M(W_r)^+$.

It is clear that $M(W_3)^+$ is F_7, the Fano plane. Moreover $M(W_4)^+$ is $M^*(K_{3,3})$. Note that $M(W_r)^+$ is precisely the matroid F_{r+1}, introduced by Kingan and Lemos [3] in their work on almost-graphic matroids. If r is odd then the dual of $M(W_r)^+$ is Υ_{r+1}, one of the triadic Möbius matroids. This class is one of the fundamental families of internally 4-connected binary matroids with no minor isomorphic to $M(K_{3,3})$ [4].

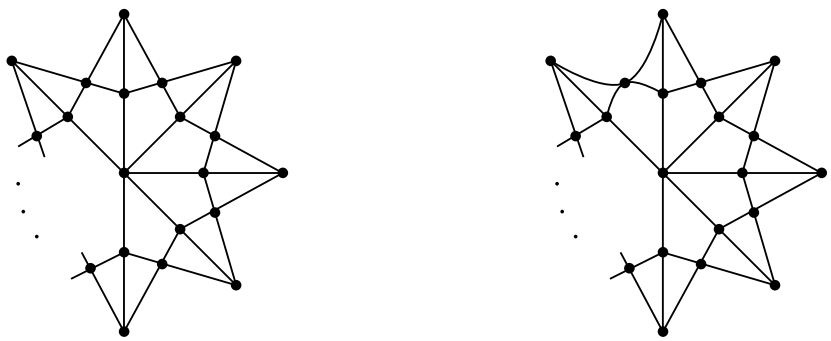
Let $r \geq 3$ be an integer, and consider the matrix $A_r(\alpha)$ over GF(2) displayed in Figure 4. Let e be the element of $M[A_r(\alpha)]$ corresponding to the last column of the identity matrix in $A_r(\alpha)$. If $r \geq 4$ is even and $\alpha = 0$, or if $r \geq 5$ is odd and $\alpha = 1$, then $M[A_r(\alpha)]\backslash e$ is the bond matroid of CP_{2r-2}. In this case we shall denote $M[A_r(\alpha)]$ by $M_r^*(CP)^+$. If $r \geq 4$ is even and $\alpha = 1$, or if $r \geq 5$ is odd and $\alpha = 0$, then $M[A_r(\alpha)]\backslash e$ is the bond matroid of CM_{2r-2}, and we shall denote $M[A_r(\alpha)]$ by $M_r^*(CM)^+$. In either case we shall say that $M[A_r(\alpha)]$ is a *dual cubic ladder with a tip*.

In Figure 5 we show geometric representations of dual cubic ladders with tips. Note that these are not orthodox geometric representations, as they show matroids with rank greater than four, but they do display the pattern of triangles in these matroids. Assuming that r is odd, Figure 5 (a) shows $M_r^*(CP)^+$, and Figure 5 (b) shows $M_r^*(CM)^+$. If r is even then Figures 5 (a) and 5 (b) depict $M_r^*(CM)^+$ and $M_r^*(CP)^+$ respectively.

We observe that $M_r^*(CP)^+$ is the matroid B_{3r-2}, and $M_r^*(CM)^+$ is S_{3r-2}, where these matroids were used by Kingan and Lemos [3]. It is not difficult to see that $M_r^*(CM)^+$ is Δ_r, a triangular Möbius matroid [4]. It is possible

$$\left[\begin{array}{c|ccccc|ccccc} & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 1 \\ & 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 \\ I_r & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \\ & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & \alpha \end{array}\right]$$

FIGURE 4. The matrix $A_r(\alpha)$.



(a) (b)

FIGURE 5. Geometric representations of $M_r^*(CP)^+$ and $M_r^*(CM)^+$.

to define the duals of quartic ladders with tips, in much the same way that we have defined duals of cubic ladders with tips. The matroids that we obtain in this case are precisely the duals of wheels with tips.

4. Guts and coguts elements

Suppose that M is a matroid on the ground set E, and that $(e_0, \ldots, e_{n-1})$ is a cyclic ordering of E. A subset $P \subseteq E$ is *sequential* if it can be expressed in the form $\{e_i, \ldots, e_{i+t}\}$, for some $0 \leq i$, $t \leq n-1$. Suppose that $(P_0, \ldots, P_{k-1})$ is an ordered partition of E. We say that $(P_0, \ldots, P_{k-1})$ is *displayed* by the ordering $(e_0, \ldots, e_{n-1})$ if every set of the form $P_i \cup \cdots \cup P_{i+t}$ is sequential (in this case $0 \leq i$, $t \leq k-1$, and subscripts are to be read modulo k).

Lemma 4.1. *Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering for the matroid M. Let $(A, \{x\}, B)$ be a partition of $E(M)$ that is displayed by the ordering and suppose that A and B are exactly 4-separating. Then x belongs to exactly one of $\mathrm{cl}(A) \cap \mathrm{cl}(B)$ or $\mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$.*

Proof. Note that $A \cup x$ is exactly 4-separating because B is exactly 4-separating. As A and $A \cup x$ are both exactly 4-separating, it follows that

$$3 = \mathrm{r}(A) + \mathrm{r}^*(A) - |A| = \mathrm{r}(A \cup x) + \mathrm{r}^*(A \cup x) - |A \cup x|$$

and so

$$[\mathrm{r}(A \cup x) - \mathrm{r}(A)] + [\mathrm{r}^*(A \cup x) - \mathrm{r}^*(A)] = 1.$$

That is, A spans x in exactly one of M and M^*. The same argument shows that B spans x in exactly one of M and M^*.

Assume that $x \in \mathrm{cl}(A)$. Then $\mathrm{r}(A) = \mathrm{r}(A \cup x)$. Since A is exactly 4-separating it follows that

$$\mathrm{r}(A) + \mathrm{r}(B \cup x) - \mathrm{r}(M) = 3.$$

Therefore $\mathrm{r}(A \cup x) + \mathrm{r}(B \cup x) - \mathrm{r}(M) = 3$. Similarly, B is exactly 4-separating, so $\mathrm{r}(A \cup x) + \mathrm{r}(B) - \mathrm{r}(M) = 3$. We conclude that $\mathrm{r}(B \cup x) = \mathrm{r}(B)$ and that therefore $x \in \mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)$. The dual argument shows that if $x \in \mathrm{cl}^*(A)$, then $x \in \mathrm{cl}^*(B)$. □

Suppose that M is a matroid on the ground set E. If $(A, \{x\}, B)$ is a partition of E and $x \in \mathrm{cl}(A) \cap \mathrm{cl}(B)$ then we shall say that x is in the *guts* of $(A, \{x\}, B)$. If $x \in \mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$ then we shall say that x is in the *coguts* of $(A, \{x\}, B)$.

Lemma 4.2. *Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering for the matroid M and let x be an element of $E(M)$. Suppose that there is a displayed partition $(A, \{x\}, B)$ of $E(M)$ such that A and B are exactly 4-separating and x is in the guts of $(A, \{x\}, B)$. Then x is in the guts of $(A', \{x\}, B')$ whenever $(A', \{x\}, B')$ is a displayed partition of $E(M)$ such that A' and B' are exactly 4-separating. Similarly, if x is in the coguts of the displayed partition $(A, \{x\}, B)$, where A and B are exactly 4-separating, then x is in the coguts of any displayed partition $(A', \{x\}, B')$ such that A' and B' are exactly 4-separating.*

Proof. Suppose that $x \in \mathrm{cl}(A) \cap \mathrm{cl}(B)$, where $(A, \{x\}, B)$ is a displayed partition, and both A and B are exactly 4-separating. Let $(A', \{x\}, B')$ be another displayed partition such that A' and B' are exactly 4-separating. We lose no generality by assuming that $A \subseteq A'$. Therefore $x \in \mathrm{cl}(A')$, so $x \in \mathrm{cl}(B')$ by Lemma 4.1, and we are done. The result follows from the dual of this argument when $x \in \mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$. □

Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclic ordering of the internally 4-connected matroid M and that $n \geq 9$. Let x be any element of $E(M)$. We can find a displayed partition $(A, \{x\}, B)$ such that $|A|, |B| \geq 4$. Since M is internally 4-connected and A and B are sequential sets, it must be the case that A and B are exactly 4-separating. If x is a guts element of $(A, \{x\}, B)$ then we say that x is a *guts element* of the ordering $(e_0, \ldots, e_{n-1})$, and we label x with a g. If x is a coguts element of $(A, \{x\}, B)$ then we say that x is a *coguts* element of the ordering, and we label x with a c. Lemma 4.1

tells us that every element of $E(M)$ receives a label, and Lemma 4.2 assures us that the labeling is well-defined. We refer to the label of an element as its $(g,\, c)$-*label*.

The next result shows how we can use the $(g,\, c)$-labeling to manipulate the cyclic ordering.

Lemma 4.3. *Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering for the internally 4-connected matroid M, where $n \geq 9$, and suppose that e_i and e_{i+1} have the same $(g,\, c)$-label. Then swapping e_i and e_{i+1} produces another cyclically 4-sequential ordering. Moreover, every element receives the same $(g,\, c)$-label in both cyclic orderings.*

Proof. We may assume by duality that e_i and e_{i+1} are guts elements of the ordering $(e_0, \ldots, e_{n-1})$. We first show that the new sequence

$$(e_0,\, e_1, \ldots, e_{i-1},\, e_{i+1},\, e_i,\, e_{i+2}, \ldots, e_{n-1})$$

is a cyclic ordering for M.

Suppose that the partition $(X_1,\, X_2)$ of $E(M)$ is displayed by the new ordering. If $e_i,\, e_{i+1} \in X_1$ or if $e_i,\, e_{i+1} \in X_2$, then $(X_1,\, X_2)$ is also displayed by the original sequence, so $\lambda(X_1) \leq 3$, as required. Thus, by symmetry, we need only consider the case that $e_{i+1} \in X_1$ and $e_i \in X_2$. Suppose that $\lambda(X_1 - e_{i+1}) \leq 2$. Then it is certainly true that $\lambda(X_1) \leq 3$, as desired. Therefore we will assume that $\lambda(X_1 - e_{i+1}) = 3$. Note that this implies that $|X_1 - e_{i+1}| \geq 3$. By exactly the same argument, we can assume that $\lambda(X_2 - e_i) = 3$, and that therefore $|X_2 - e_i| \geq 3$.

Assume that $(X_1 - e_{i+1}) \cup e_i$ fails to be exactly 4-separating. Since this set is sequential in the original ordering, we conclude that

$$\lambda((X_1 - e_{i+1}) \cup e_i) \leq 2.$$

As M is internally 4-connected this means that either $|(X_1 - e_{i+1}) \cup e_i| \leq 3$ or $|(X_2 - e_i) \cup e_{i+1}| \leq 3$, and in either case we get a contradiction. Therefore $(X_1 - e_{i+1}) \cup e_i$ is exactly 4-separating. Hence $(X_2 - e_i) \cup e_{i+1}$ is also exactly 4-separating.

Now,

$$((X_1 - e_{i+1}) \cup e_i,\, \{e_{i+1}\},\, X_2 - e_i)$$

is a displayed partition in the original ordering. Since e_{i+1} is a guts element in the original ordering, and both $(X_1 - e_{i+1}) \cup e_i$ and $X_2 - e_i$ are exactly 4-separating, it follows that $e_{i+1} \in \mathrm{cl}(X_2 - e_i)$. Next we consider the partition $(X_1 - e_{i+1},\, \{e_i\},\, (X_2 - e_i) \cup e_{i+1})$. This is displayed by the original ordering; and, as e_i is a guts element and both $X_1 - e_{i+1}$ and $(X_2 - e_i) \cup e_{i+1}$ are exactly 4-separating, we conclude that $e_i \in \mathrm{cl}((X_2 - e_i) \cup e_{i+1})$. But $e_{i+1} \in \mathrm{cl}(X_2 - e_i)$, so $e_i \in \mathrm{cl}(X_2 - e_i)$. As $\lambda(X_2 - e_i) = 3$, this implies that $\lambda(X_2) \leq 3$, exactly as desired. Therefore the new ordering is indeed a legitimate cyclic ordering.

To complete the proof, we must show that in the new cyclic ordering, every element keeps the same $(g,\, c)$-label that it had in the original ordering. First suppose that $n \geq 10$. Let $Z_2 = \{e_{i+2}, \ldots, e_{i+5}\}$ and let $Z_1 = E(M) - (Z_2 \cup \{e_i,\, e_{i+1}\})$. Then $|Z_1|,\, |Z_2| \geq 4$. Since $(Z_1 \cup e_i,\, \{e_{i+1}\},\, Z_2)$ is displayed in the original ordering, it follows that $e_{i+1} \in \mathrm{cl}(Z_2)$. Furthermore $(Z_1,\, \{e_i\},\, Z_2 \cup e_{i+1})$ is displayed in the original ordering, so $e_i \in \mathrm{cl}(Z_2 \cup e_{i+1})$, and therefore $e_i \in \mathrm{cl}(Z_2)$. Since $(Z_1 \cup e_{i+1},\, \{e_i\},\, Z_2)$ is displayed in the new ordering, it follows that e_i is a guts element of the new ordering. A symmetric argument shows that the $(g,\, c)$-label of e_{i+1} is unchanged in the new ordering. If $x \in E(M) - \{e_i,\, e_{i+1}\}$ then we can find a displayed partition $(Z_1,\, \{x\},\, Z_2)$ such that $|Z_1|,\, |Z_2| \geq 4$ and either $e_i,\, e_{i+1} \in Z_1$ or $e_i,\, e_{i+1} \in Z_2$. Therefore the $(g,\, c)$-label of x is unchanged.

It remains to consider the case that $n = 9$. It is easily seen that if $x \in E(M) - \{e_i,\, e_{i+1},\, e_{i+5}\}$, then there is again a partition $(Z_1,\, \{x\},\, Z_2)$ such that $|Z_1|,\, |Z_2| \geq 4$ and e_i and e_{i+1} are both contained in either Z_1 or Z_2. Therefore we need only check that the labels are unchanged on e_i, e_{i+1}, and e_{i+5}.

Let $Y_1 = \{e_{i-3},\, e_{i-2},\, e_{i-1}\}$ and let $Y_2 = \{e_{i+2},\, e_{i+3},\, e_{i+4}\}$. The partition $(Y_1 \cup e_i,\, \{e_{i+1}\},\, Y_2 \cup e_{i+5})$ is displayed in the original ordering, so $e_{i+1} \in \mathrm{cl}(Y_2 \cup e_{i+5})$. Similarly, $e_i \in \mathrm{cl}(Y_1 \cup e_{i+5})$ because $e_{i+5} = e_{i-4}$. Suppose that $\lambda(Y_1) = 3$. The partition $(Y_1,\, \{e_i\},\, Y_2 \cup \{e_{i+1},\, e_{i+5}\})$ shows that $e_i \in \mathrm{cl}(Y_2 \cup \{e_{i+1},\, e_{i+5}\})$, and hence $e_i \in \mathrm{cl}(Y_2 \cup e_{i+5})$ as e_{i+1} is a guts element. Since $(Y_1 \cup e_{i+1},\, \{e_i\},\, Y_2 \cup e_{i+5})$ is displayed in the new ordering, it follows that e_i is a guts element in the new ordering. Similarly $(Y_1,\, \{e_{i+1}\},\, Y_2 \cup \{e_i,\, e_{i+5}\})$ is displayed in the new ordering, and as Y_1 is exactly 4-separating and $e_{i+1} \in \mathrm{cl}(Y_2 \cup e_{i+5})$ it follows that e_{i+1} is also a guts element. Finally we note that $(Y_1,\, \{e_{i+5}\},\, Y_2 \cup \{e_i,\, e_{i+1}\})$ is displayed in both orderings, and Y_1 is exactly 4-separating, so the (g, c)-label on e_{i+5} is unchanged. Therefore, in the case where $\lambda(Y_1) = 3$, we are done. A symmetric argument shows that the labels on e_i, e_{i+1}, and e_{i+5} are unchanged if Y_2 is exactly 4-separating. Therefore we will assume that $\lambda(Y_1) \leq 2$ and $\lambda(Y_2) \leq 2$.

If there is some element x in $\mathrm{cl}(Y_1) - Y_1$, then $\lambda(Y_1 \cup x) \leq 2$, and we have a contradiction to internal 4-connectivity. Therefore Y_1, and by symmetry Y_2, is a closed set. We have noted that $e_{i+1} \in \mathrm{cl}(Y_2 \cup e_{i+5})$. Since $e_{i+1} \notin \mathrm{cl}(Y_2)$, this means that $e_{i+5} \in \mathrm{cl}(Y_2 \cup e_{i+1})$. The displayed partition $(Y_1 \cup e_i,\, \{e_{i+5}\},\, Y_2 \cup e_{i+1})$ implies that e_{i+5} is a guts element of the original ordering.

If $e_i \notin \mathrm{cl}(Y_2 \cup \{e_{i+1},\, e_{i+5}\})$ then $\lambda(Y_1 \cup e_i) \leq 2$, and we again have a contradiction to internal 4-connectivity. Therefore e_i is in $\mathrm{cl}(Y_2 \cup \{e_{i+1},\, e_{i+5}\})$, and as $e_{i+1} \in \mathrm{cl}(Y_2 \cup e_{i+5})$, it follows that $e_i \in \mathrm{cl}(Y_2 \cup e_{i+5})$. Since $(Y_1 \cup e_{i+1},\, \{e_i\},\, Y_2 \cup e_{i+5})$ is displayed in the new ordering, it follows that e_i is a guts element in the new ordering. Moreover $e_i \in \mathrm{cl}(Y_2 \cup e_{i+5})$, but $e_i \notin \mathrm{cl}(Y_2)$ implies that $e_{i+5} \in \mathrm{cl}(Y_2 \cup e_i)$, and now the displayed partition $(Y_1 \cup e_{i+1},\, \{e_{i+5}\},\, Y_2 \cup e_i)$ implies that e_{i+5} is a guts element in the new ordering, as desired. Finally, we have noted that $e_{i+5} \in \mathrm{cl}(Y_2 \cup e_i)$, so

$e_{i+1} \in \mathrm{cl}(Y_2 \cup e_i)$. Now, the partition $(Y_1 \cup e_{i+5},\ \{e_{i+1}\},\ Y_2 \cup e_i)$ implies that e_{i+1} is a guts element in the new ordering, completing the proof. □

If $(e_0, \ldots, e_{n-1})$ is a cyclic ordering in which e_i and e_{i+1} share the same (g, c)-label, then we shall say that

$$(e_0,\, e_1, \ldots, e_{i-1},\, e_{i+1},\, e_i,\, e_{i+2}, \ldots, e_{n-1})$$

is obtained by *switching* e_i and e_{i+1}. After we perform a switching in the cyclic ordering, we denote the new set-sequence by $(S'_0, S'_1, \ldots, S'_{n-1})$.

Recall that if $(e_0, \ldots, e_{n-1})$ is a cyclic ordering of the internally 4-connected matroid M, then $S_i = \{e_i,\, e_{i+1},\, e_{i+2},\, e_{i+3}\}$ is 4-separating for $0 \le i \le n-1$. It is easy to see that if S_i is a T-set or a circuit, then $\mathrm{r}(S_i) = 3$. On the other hand, if S_i is a cocircuit or a T^*-set, then it must be the case that $\mathrm{r}(S_i) = 4$. The next lemma shows that when consecutive sets S_i and S_{i+1} have the same rank we can deduce information about the $(g,\, c)$-labeling.

Lemma 4.4. *Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering for the matroid M and that $n \ge 9$. Let $S_i = \{e_i,\, e_{i+1},\, e_{i+2},\, e_{i+3}\}$ for all $0 \le i \le n-1$. Suppose that S_i and S_{i+1} have the same rank. Then either:*

(i) both e_i and e_{i+4} are guts elements; or,
(ii) both e_i and e_{i+4} are coguts elements.

In particular, if $\{e_{i+1},\, e_{i+2},\, e_{i+3}\}$ is a triangle then e_i and e_{i+4} receive the same $(g,\, c)$-label.

Proof. Taking the dual when necessary, we may assume that e_i is a guts element of the cyclic ordering. Then e_i is in the closure of S_{i+1}, so $\mathrm{r}(S_{i+1} \cup e_i) = \mathrm{r}(S_{i+1}) = \mathrm{r}(S_i)$. This implies that S_i spans $S_{i+1} \cup e_i = S_i \cup e_{i+4}$. Thus $e_{i+4} \in \mathrm{cl}(S_i)$ and it follows that e_{i+4} is a guts element. □

Suppose that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering of a matroid and that $n \ge 9$. If $S \subseteq S_i$ for some i, then let $P = \{e_j, \ldots, e_{j+t}\}$ be the smallest possible sequential set that contains S, where $0 \le j,\, t \le n-1$. Note that P is well defined as $n \ge 9$ and $S \subseteq S_i$. We say that e_j and e_{j+t} are the *endpoints* of S. The following useful observation is easily checked.

Lemma 4.5. *Let $(e_0, \ldots, e_{n-1})$ be a cyclically 4-sequential ordering of the matroid M, where $n \ge 9$. Then*

(i) if S_i contains a cocircuit C^*, then the endpoints of C^* are coguts elements; and,
(ii) if S_i contains a circuit C, then the endpoints of C are guts elements.

5. Characterizing label-sequences

This section is devoted to characterizing label-sequences and their associated matroid substructures together since the two topics are intimately related. We shall see that when we have knowledge of some sequential part of a label-sequence, then we also have much information on the structure of

that part of the matroid. The results of this section are therefore of independent interest to situations where some 4-separator of a binary internally 4-connected matroid behaves like part of a cyclic ordering. While we haven't explored it yet, we believe that the relationship will be similar to that of wheels, whirls and fans in 3-connected matroids.

We recall some definitions: if $(e_0, \ldots, e_{n-1})$ is a cyclic ordering for a matroid M, then a set S_i is a T- or T^*-set if it contains a triangle or triad respectively. The set-sequence corresponding to the ordering $(e_0, \ldots, e_{n-1})$ is $(S_0, \ldots, S_{n-1})$, and the corresponding label-sequence is the sequence $(X_0, \ldots, X_{n-1})$, where X_i is a character from the set $\{C, C^*, T, T^*\}$, depending on whether S_i is a circuit, cocircuit, T-set, or T^*-set. Throughout this section, M will be an internally 4-connected binary matroid on at least nine elements, and $(e_0, \ldots, e_{n-1})$ will be a cyclically 4-sequential ordering for M.

Lemma 5.1. *If S_i is a T-set, then S_{i+1} is not a T^*-set.*

Proof. This is true simply because no triangle meets a triad in an internally 4-connected binary matroid with at least nine elements. □

Lemma 5.2. *If S_i is a circuit or a cocircuit, then neither S_{i-1} nor S_{i+1} is a circuit or a cocircuit.*

Proof. Suppose that S_i is a circuit. Then S_{i+1} cannot be a cocircuit, for this would violate orthogonality (note that e_i and e_{i+4} are distinct elements as $n \geq 9$). If S_{i+1} is a circuit, then $S_i \Delta S_{i+1}$, the symmetric difference of S_i and S_{i+1}, is a disjoint union of circuits. As $S_i \Delta S_{i+1} = \{e_i, e_{i+4}\}$, and M has no parallel elements, this cannot occur. It follows that if S_i is a circuit then S_{i+1} is neither a circuit nor a cocircuit. The result follows now by applying duality and symmetry. □

Lemma 5.3. *If S_i is a circuit, then neither S_{i-4} nor S_{i+4} is a circuit, and if S_i is a cocircuit, then neither S_{i-4} nor S_{i+4} is a cocircuit.*

Proof. Assume that S_i is a circuit. We first show that S_{i+4} is not a circuit. By cyclically shifting labels as necessary, we can assume that $i = 0$. Assume that the lemma fails, so that S_0 and S_4 are both circuits.

Note that $e_0, \ldots, e_7$ are distinct elements as $n \geq 9$. The set S_1 cannot be a circuit or a cocircuit by Lemma 5.2. Moreover, S_1 cannot be a T^*-set, since a triad containing e_4 would contradict orthogonality with the circuit S_4, and if $\{e_1, e_2, e_3\}$ were a triad then we would have a violation of orthogonality with the circuit S_0. Therefore S_1 must be a T-set. A symmetric argument shows that S_3 is also a T-set. Also, the triangle T_1 contained in S_1 must contain e_4, and the triangle T_3 contained in S_3 must contain e_3.

Note that both triangles T_1 and $T_1 \Delta S_0$ contain e_4 and just one of them contains e_3. Let T_1' be the triangle contained in $S_0 \cup e_4$ that contains $\{e_3, e_4\}$. By symmetry, there is a triangle T_3' in $S_4 \cup e_3$ such that $\{e_3, e_4\} \subseteq T_3'$. Now $T_1' \Delta T_3'$ is a 2-element set that is the union of circuits of M; a contradiction.

The lemma follows by duality and symmetry. □

Lemma 5.4. *By switching elements, we can assume that either:*

(i) every set S_i is a circuit or a T^*-set; or,
(ii) every set S_i is a cocircuit or a T-set.

Proof. We begin with a sublemma.

5.4.1. Suppose that S_i is a circuit and that S_{i+1} is a T-set. Then we can switch a pair of elements in such a way that the label-sequence is unchanged, except that in the new ordering S'_i is a T-set.

Proof. By cyclically shifting labels as necessary we will assume that $i = 0$. Since no triangle is contained in $\{e_0, e_1, e_2, e_3\}$, the triangle in S_1 contains e_4. Lemma 4.5 implies that e_3 and e_4 are guts elements of the ordering. We therefore apply Lemma 4.3 and deduce that

$$(e_0, e_1, e_2, e_4, e_3, e_5, \ldots, e_{n-1})$$

is a valid cyclic ordering and that all elements retain their (g, c)-label. Let $(S'_0, \ldots, S'_{n-1})$ and $(X'_0, \ldots, X'_{n-1})$ respectively be the set-sequence and label-sequence of this new ordering. Clearly $S'_j = S_j$ if $j \notin \{0, 4\}$, so in this case $X'_j = X_j$. We will show that S'_0 is a T-set and that $X'_4 = X_4$.

There is a partition $\{A, B\}$ of S_0 such that $A \cup e_4$ and $B \cup e_4$ are triangles of M. Note that one of them is contained in $\{e_0, e_1, e_2, e_4\}$, thus S'_0 is a T-set.

It remains only to show that $X'_4 = X_4$. Lemma 5.3 tells us that S_4 is not a circuit. Also, triangle $A \cup e_4$ tells us that S_4 is not a cocircuit. If S_4 is a T^*-set, then $\{e_5, e_6, e_7\}$ must be a triad by orthogonality with S_0. Then $S'_4 = \{e_3, e_5, e_6, e_7\}$ is also a T^*-set, so we are done. Therefore we will assume that S_4 is a T-set. If the triangle in S_4 does not contain e_4, then S'_4 is also a T-set, so we assume that it does contain e_4. Let $\{x, y, z\} = \{e_5, e_6, e_7\}$, and suppose that the triangle in S_4 is $\{x, y, e_4\}$. By orthogonality with S_0 the set $S'_4 = \{e_3, e_5, e_6, e_7\}$ does not contain a cocircuit using e_3, so S'_4 is not a cocircuit. Moreover, no triad can meet $\{x, y\}$, as these elements are contained in a triangle. Therefore S'_4 is not a T^*-set. If $X'_4 \neq X_4$ then it must be the case that S'_4 is a circuit. Then

$$\{x, y, z, e_3\} \Delta \{x, y, e_4\} = \{z, e_3, e_4\}.$$

is a triangle, and as $\{e_3, e_4\}$ is contained either in triangle $A \cup e_4$ or in triangle $B \cup e_4$, we conclude that z is parallel with an element in S_0, a contradiction. Thus $X'_4 = X_4$. □

Sublemma 5.4.1 above shows that whenever the subsequence (C, T) appears in the label-sequence, we can switch a pair of elements in such a way that we remove the C and replace it with a T, leaving every other character in the label-sequence unchanged. By performing this operation wherever possible, and using symmetry, we can assume that if S_i is a circuit, then neither S_{i-1} nor S_{i+1} are T-sets. Lemma 5.2 now implies that if S_i is a

circuit, then S_{i-1} and S_{i+1} are T^*-sets. Moreover, by using duality, we can also assume that if S_i is a cocircuit, then S_{i-1} and S_{i+1} are T-sets.

Suppose that S_i is a circuit. Let t be the least positive integer such that S_{i+t} is not a T^*-set. Since S_{i+t-1} is a T^*-set it follows that S_{i+t} cannot be a cocircuit by our earlier assumption. Nor can S_{i+t} be a T-set by the dual of Lemma 5.1. Thus S_{i+t} is a circuit, so S_{i+t+1} is a T^*-set. Continuing in this way we see that every set S_j is either a circuit or a T^*-set. Similarly, if S_i is a cocircuit, we can show that every set S_j is either a cocircuit or a T-set. Therefore we will assume that no set S_i is a circuit or cocircuit. Lemma 5.1 shows that in this case either every set S_j is a T-set, or every set S_j is a T^*-set. This completes the proof of Lemma 5.4. □

By virtue of Lemma 5.4, and by using duality, we will henceforth assume that $(e_0, \ldots, e_{n-1})$ is a cyclic ordering of the internally 4-connected binary matroid M, where $n \geq 9$, and that every set S_i is either a cocircuit or a T-set.

Lemma 5.5. *The subsequence*

$$(T, T, T, C^*, T, T, T)$$

does not occur in the label-sequence of $(e_0, \ldots, e_{n-1})$.

Proof. Suppose that the lemma fails. By cyclically shifting labels as necessary we may assume that S_i is a T-set for all $i \in \{0, 1, 2, 4, 5, 6\}$, and that S_3 is a cocircuit. The triangle in S_6 must be $\{e_7, e_8, e_9\}$ by orthogonality with S_3. The same argument shows that $\{e_0, e_1, e_2\}$ is a triangle. Lemma 4.5 applied to $\{e_7, e_8, e_9\}$ asserts that e_9 is a guts element. Since S_5 and S_6 are both T-sets it follows that they both have rank 3, so Lemma 4.4 implies that e_5 is also a guts element.

Therefore $e_5 \in \mathrm{cl}(S_1)$. We also know that $e_0 \in \mathrm{cl}(S_1)$, because $\{e_0, e_1, e_2\}$ is a triangle. As S_1 is a T-set, and hence has rank 3, it follows that $\mathrm{r}(\{e_0, \ldots, e_5\}) = 3$. Therefore M restricted to $\{e_0, \ldots, e_5\}$ is isomorphic to $M(K_4)$. As $\{e_0, e_1, e_2\}$ is a triangle, there must be a triangle that contains $\{e_4, e_5\}$ and an element from $\{e_0, e_1, e_2\}$. But symmetric arguments show that M restricted to $\{e_4, \ldots, e_9\}$ is isomorphic to $M(K_4)$ and that e_4 and e_5 are in a triangle with an element of $\{e_7, e_8, e_9\}$. This leads to a parallel pair and a contradiction unless $n = 9$ and $\{e_4, e_5, e_0\} = \{e_4, e_5, e_9\}$ is a triangle. In this case, S_3 is a cocircuit of rank 4 and $\{e_0, \ldots, e_5\}$ and $\{e_4, \ldots, e_9\}$ are isomorphic to $M(K_4)$, therefore $E(M) \subseteq \mathrm{cl}(S_3)$ and $\mathrm{r}(M) = 4$. Now, we observe that

$$\lambda(\{e_0, \ldots, e_4\}) = \mathrm{r}(\{e_0, \ldots, e_4\}) + \mathrm{r}(\{e_5, \ldots, e_9\}) - \mathrm{r}(M) \leq 3 + 3 - 4 = 2;$$

a contradiction since M is internally 4-connected. This completes the proof. □

Lemma 5.6. *The subsequence*

$$(C^*, T, T, T, T, C^*)$$

does not occur in the label-sequence of $(e_0, \ldots, e_{n-1})$.

Proof. Suppose that the lemma is false. We can assume that S_i is a T-set for $i \in \{1, \ldots, 4\}$, while S_0 and S_5 are cocircuits. By orthogonality with S_5, the triangle in S_2 must be $\{e_2, e_3, e_4\}$, and the cocircuit S_0 implies that the triangle in S_3 is $\{e_4, e_5, e_6\}$.

By applying Lemma 4.5 to S_0 and S_5 we see that e_3 and e_5 are coguts elements. Thus $e_5 \in \mathrm{cl}^*(S_1)$, so there is a cocircuit C_1^* in $S_1 \cup e_5$ that contains e_5. Similarly, there is a cocircuit $C_2^* \subseteq S_4 \cup e_3$ and this cocircuit contains e_3. Furthermore, C_1^* is properly contained in $\{e_1, \ldots, e_5\}$ by orthogonality with the triangle $\{e_2, e_3, e_4\}$, and C_2^* is properly contained in $\{e_3, \ldots, e_7\}$. It cannot be the case that C_1^* is a triad, for then it would meet the triangle $\{e_2, e_3, e_4\}$. The triangle $\{e_4, e_5, e_6\}$ means that C_2^* is not a triad. It follows that both C_1^* and C_2^* have cardinality four. As C_1^* contains e_5, orthogonality with $\{e_4, e_5, e_6\}$ means that it contains e_4. Similarly, orthogonality with $\{e_2, e_3, e_4\}$ means that $e_4 \in C_2^*$. Thus C_1^* is either $\{e_1, e_2, e_4, e_5\}$ or $\{e_1, e_3, e_4, e_5\}$, and C_2^* is either $\{e_3, e_4, e_5, e_7\}$ or $\{e_3, e_4, e_6, e_7\}$. This gives us four cases to consider.

First suppose that $C_1^* = \{e_1, e_3, e_4, e_5\}$. If $C_2^* = \{e_3, e_4, e_5, e_7\}$, then e_1 and e_7 are in series, a contradiction. If $C_2^* = \{e_3, e_4, e_6, e_7\}$ then

$$C_1^* \Delta C_2^* \Delta S_5 = \{e_1, e_8\}$$

is a cocircuit, so e_1 and e_8 are in series. This is again a contradiction.

Hence $C_1^* = \{e_1, e_2, e_4, e_5\}$. By symmetry, $C_2^* = \{e_3, e_4, e_6, e_7\}$. Then

$$S_0 \Delta C_1^* \Delta C_2^* \Delta S_5 = \{e_0, e_8\}$$

is a series pair and we have a contradiction. □

Lemma 5.7. *The subsequence*

$$(C^*, T, T, C^*, T, C^*)$$

does not occur in the label-sequence of $(e_0, \ldots, e_{n-1})$.

Proof. Suppose that the lemma fails. By shifting labels we can assume that S_0, S_3 and S_5 are cocircuits, and that S_1, S_2 and S_4 are T-sets. By orthogonality with S_5 we deduce that the triangle in S_2 is $\{e_2, e_3, e_4\}$. Lemma 4.5 implies that e_5 is a coguts element. Therefore there is a cocircuit $C^* \subseteq S_1 \cup e_5$ that contains e_5. Since $\{e_2, e_3, e_4\}$ is a triangle, $C^* \neq \{e_1, \ldots, e_5\}$ and C^* is not a triad. Therefore $|C^*| = 4$. Suppose that C^* contains $\{e_1, e_2, e_3\}$. Then $C^* \Delta S_0 = \{e_0, e_5\}$, a contradiction. Therefore $e_4 \in C^*$. Now $e_3 \notin C^*$, for otherwise $|C^* \Delta S_3| = 2$. Hence $C^* = \{e_1, e_2, e_4, e_5\}$ and

$$S_0 \Delta C^* \Delta S_3 = \{e_0, e_6\},$$

so M contains a series pair. This contradiction completes the proof. □

Lemma 5.8. *If* $\{e_i, e_{i+1}, e_{i+2}\}$ *is a triangle and* S_{i+1}, S_{i+2}, *and* S_{i+3} *are* T*-sets, then* $\{e_{i+2}, e_{i+3}, e_{i+4}\}$ *is also a triangle.*

Proof. We will assume that $i = 0$, so that $\{e_0, e_1, e_2\}$ is a triangle, and S_1, S_2, and S_3 are T-sets. Assume that $\{e_2, e_3, e_4\}$ is not a triangle. Since the triangle in S_1 has at least two elements not in $\{e_0, e_1, e_2\}$, it must contain e_3 and e_4. Therefore $\{e_1, e_3, e_4\}$ is a triangle. The triangles in S_2 and S_3 must contain two elements not in $\{e_1, e_3, e_4\}$, so the triangle in S_2 contains e_2 and e_5 and the triangle in S_3 contains e_5 and e_6. These two triangles also do not share two elements, so they are $\{x, e_2, e_5\}$ and $\{y, e_5, e_6\}$, where $\{x, y\} = \{e_3, e_4\}$. Now

$$\{e_0, e_1, e_2\}\Delta\{e_1, e_3, e_4\}\Delta\{x, e_2, e_5\}\Delta\{y, e_5, e_6\} = \{e_0, e_6\}$$

so M has a parallel pair, a contradiction. □

Recall that a set is sequential if it can be expressed in the form $\{e_i, \ldots, e_{i+t}\}$, where $0 \leq i, t \leq n-1$.

Lemma 5.9. *Suppose that $S_i, \ldots, S_{i+4}$ are T-sets. Then one of them contains a sequential triangle.*

Proof. Suppose that none of the sets $S_i, \ldots, S_{i+4}$ contains a sequential triangle. We will assume that $i = 0$. Now $\mathrm{r}(S_0) = 3$, as S_0 is a T-set. The triangle in S_1 must contain e_4, so $e_4 \in \mathrm{cl}(S_0)$. The same argument shows that $e_5, e_6, e_7 \in \mathrm{cl}(S_0)$. As $n \geq 9$ the elements $e_0, \ldots, e_7$ are distinct. Therefore $\mathrm{cl}(S_0)$ is a rank-3 flat containing at least eight elements, a contradiction. □

Lemma 5.10. *Suppose that $S_{i+1}, \ldots, S_{i+t}$ are distinct T-sets, where $t < n$ and that S_i and S_{i+t+1} are cocircuits. Then either $t = 1$ or $t = 2k$ for some positive integer $k \neq 2$.*

Proof. We first assume that $t < n$. It follows from Lemma 5.6 that t is not equal to four. Suppose that t is an odd number greater than one. We can assume that S_0 and S_{t+1} are cocircuits (not necessarily distinct), and that S_j is a T-set for all $j \in \{1, \ldots, t\}$. By orthogonality with S_0, the triangle in S_3 must be $\{e_4, e_5, e_6\}$. Thus S_4 is a T-set, so $t \geq 5$. If $t = 5$ then we have a contradiction to orthogonality between $\{e_4, e_5, e_6\}$ and S_6, so $t \geq 7$. By repeatedly applying Lemma 5.8, we see that $\{e_j, e_{j+1}, e_{j+2}\}$ is a triangle if $j \in \{4, \ldots, t-1\}$ is an even integer. In particular, $\{e_{t-1}, e_t, e_{t+1}\}$ is a triangle that meets the cocircuit S_{t+1} in a single element. This contradiction proves that the lemma holds. □

Lemma 5.11. *Suppose that $\{a, e_{i+1}, e_{i+2}\}$, $\{e_{i+2}, e_{i+3}, e_{i+4}\}$, and $\{e_{i+4}, e_{i+5}, b\}$ are triangles for some $a, b \notin \{e_{i+1}, e_{i+2}, \ldots, e_{i+5}\}$, and that $e_{i+5} \notin \mathrm{cl}(S_{i+1})$. Then $\{e_{i+1}, e_{i+2}, e_{i+4}, e_{i+5}\}$ is a cocircuit of M.*

Proof. Since $e_{i+5} \notin \mathrm{cl}(S_{i+1})$, we have $\mathrm{r}(S_{i+1} \cup e_{i+5}) = 4$, and since $S_{i+1} \cup e_{i+5}$ is exactly 4-separating it must contain a cocircuit C^*. No triad of M meets a triangle, so C^* has at least four elements, and therefore must meet $\{a, e_{i+1}, e_{i+2}\}$, $\{e_{i+2}, e_{i+3}, e_{i+4}\}$, and $\{e_{i+4}, e_{i+5}, b\}$. By orthogonality with these triangles, C^* must be $\{e_{i+1}, e_{i+2}, e_{i+4}, e_{i+5}\}$. □

Lemma 5.12. *Suppose that S_i is a T-set for all $i \in \{0, \dots, n-1\}$. Then n is even and up to a cyclic shift of the ordering, for every even integer $k \in \{0, \dots, n-1\}$, $\{e_k, e_{k+1}, e_{k+2}\}$ is a triangle and $\{e_{k+1}, e_{k+2}, e_{k+4}, e_{k+5}\}$ is a cocircuit.*

Proof. Suppose that n is odd. Lemma 5.9 tells us that there is a sequential triangle. By shifting labels we can assume that $\{e_0, e_1, e_2\}$ is a triangle. By repeatedly applying Lemma 5.8 we see that $\{e_j, e_{j+1}, e_{j+2}\}$ is a triangle if $j \in \{0, \dots, n-1\}$ is even. In particular, $\{e_{n-1}, e_n, e_{n+1}\} = \{e_{n-1}, e_0, e_1\}$ is a triangle. The symmetric difference of this triangle with $\{e_0, e_1, e_2\}$ produces a parallel pair, a contradiction. We conclude that n is even.

Again, by Lemma 5.9, we may assume, after a possible cyclic shift of the ordering, that $\{e_0, e_1, e_2\}$ is a sequential triangle. Now, repeatedly applying Lemma 5.8 implies that $\{e_k, e_{k+1}, e_{k+2}\}$ is a triangle if $k \in \{0, \dots, n-1\}$ is an even integer.

Suppose that e_5 is a guts element. Note that $e_4 \in \mathrm{cl}(S_0)$ because of the triangle $\{e_2, e_3, e_4\}$. Now, $e_5 \in \mathrm{cl}(S_1)$ because it is a guts element, and $e_6 \in \mathrm{cl}(S_2)$ because $\{e_4, e_5, e_6\}$ is a triangle. Thus the restriction of M to $\mathrm{cl}(S_0)$ contains the seven elements $\{e_0, \dots, e_6\}$ and the disjoint triangles $\{e_0, e_1, e_2\}$ and $\{e_4, e_5, e_6\}$. As $\mathrm{r}(S_0) = 3$, this is a contradiction. Therefore e_5 is a coguts element. By symmetry, we see that e_{k+1} is a coguts element for every even integer $k \in \{0, \dots, n-1\}$. We now apply Lemma 5.11 and see that $\{e_{k+1}, e_{k+2}, e_{k+4}, e_{k+5}\}$ is a cocircuit for every even integer k, as desired. □

Lemma 5.13. *Suppose that S_i is a cocircuit and that S_{i+1}, S_{i+2}, and S_{i+3} are T-sets. Then S_{i-1} is a T-set. Moreover*

(i) if e_{i+1} is a guts element then S_{i-2} is a T-set and S_{i-3} is a cocircuit; and,
(ii) if e_{i+1} is a coguts element then S_{i-2} is a cocircuit.

Proof. We see from Lemma 5.2 that S_{i-1} must be a T-set. Note that if S_{i-2} is a T-set, then S_{i-3} must be a cocircuit by Lemma 5.5.

It remains to show that e_{i+1} is a guts element if and only if S_{i-2} is a T-set. If S_{i-2} is a cocircuit then e_{i+1} is a coguts element by Lemma 4.5. For the converse, suppose S_{i-2} is a T-set. By orthogonality with cocircuit S_i, the triangle in S_{i-2} contains e_i and e_{i+1}, hence we apply Lemma 4.5 and conclude that e_{i+1} is a guts element. □

Lemma 5.14. *Suppose that $S_{i+1}, \dots, S_{i+t}$ are distinct T-sets for some $t > 6$, and that S_i and S_{i+t+1} are cocircuits. Then $\{e_{i+j}, e_{i+j+1}, e_{i+j+2}\}$ is a triangle if $j \in \{2, \dots, t\}$ is an even integer. Furthermore, for $j \in \{1, \dots, t+3\}$, e_{i+j} is a guts element if j is even, and e_{i+j} is a coguts element if j is odd.*

Proof. We will assume by shifting labels that $i = 0$, so that S_0 and S_{t+1} are cocircuits, while $S_1, \dots, S_t$ are T-sets. Lemma 5.10 implies that t is

even. Orthogonality with S_0 implies that the triangle in S_3 is $\{e_4, e_5, e_6\}$. By repeatedly applying Lemma 5.8, we deduce that $\{e_j, e_{j+1}, e_{j+2}\}$ is a triangle if $j \in \{4, \ldots, t-2\}$ is an even integer. It follows from Lemma 4.5 that e_j is a guts element if $j \in \{4, \ldots, t\}$ is even. As both S_2 and S_3 are T-sets, it follows that $\mathrm{r}(S_2) = \mathrm{r}(S_3) = 3$. Now Lemma 4.4 implies that e_2 and e_6 receive the same (g, c)-labeling, so e_2 is a guts element. Similarly, $\mathrm{r}(S_{t-2}) = \mathrm{r}(S_{t-1}) = 3$, so e_{t+2} is also a guts element.

By applying Lemma 4.5 to S_0 we see that e_3 is a coguts element. Suppose that e_5 is a guts element. Since $\{e_6, e_7, e_8\}$ is a triangle, Lemma 4.4 implies that e_9 is also a guts element. It cannot be the case that $t = 8$, for in that case S_9 would be a cocircuit, and therefore e_9 would be a coguts element by Lemma 4.5. Both e_8 and e_{10} are guts elements, so $e_8 \in \mathrm{cl}(S_4)$, $e_9 \in \mathrm{cl}(S_5)$, and $e_{10} \in \mathrm{cl}(S_6)$. We deduce that $\mathrm{cl}(S_4)$ contains $\{e_4, \ldots, e_{10}\}$. But $\mathrm{r}(S_4) = 3$, and as $t > 8$ we see that $\{e_4, \ldots, e_{10}\}$ contains two disjoint triangles: $\{e_4, e_5, e_6\}$ and $\{e_8, e_9, e_{10}\}$. This is a contradiction, so e_5 is a coguts element.

Since both e_3 and e_5 are coguts elements and the sets $S_1, \ldots, S_t$ all have rank 3, by repeatedly using Lemma 4.4, we can easily see that e_k is a coguts element if $k \in \{1, \ldots, t+3\}$ is odd.

The fact that $\{e_2, e_3, e_4\}$ is a triangle follows easily from Lemma 4.5 because S_1 is a T-set and e_1 is a coguts element. A similar argument shows that $\{e_t, e_{t+1}, e_{t+2}\}$ is a triangle. □

Lemma 5.15. *Suppose that $S_{i+1}, \ldots, S_{i+6}$ are T-sets, and that S_i and S_{i+7} are cocircuits. By switching consecutive guts elements we can assume that $\{e_{i+2}, e_{i+3}, e_{i+4}\}$, $\{e_{i+4}, e_{i+5}, e_{i+6}\}$ and $\{e_{i+6}, e_{i+7}, e_{i+8}\}$ are triangles, and that one of the following cases holds:*

(i) e_{i+j} is a guts element if $j \in \{2, 4, 6, 8\}$, and a coguts element if $j \in \{1, 3, 5, 7, 9\}$; or

(ii) e_{i+j} is a guts element if $j \in \{1, 2, 4, 5, 6, 8, 9\}$ and a coguts element if $j \in \{3, 7\}$. Also $\{e_{i+1}, e_{i+2}, e_{i+5}\}$, $\{e_{i+1}, e_{i+3}, e_{i+6}\}$, $\{e_{i+4}, e_{i+7}, e_{i+9}\}$ and $\{e_{i+5}, e_{i+8}, e_{i+9}\}$ are triangles.

Moreover the label-sequence is unchanged by this switching.

Proof. We will assume by shifting labels that $i = 0$, so that S_0 and S_7 are cocircuits while $S_1, \ldots, S_6$ are T-sets. Orthogonality with S_0 implies that the triangle in S_3 is $\{e_4, e_5, e_6\}$. Thus e_4 and e_6 are guts elements by Lemma 4.5. Since every set in $S_1, \ldots, S_6$ has rank 3, we can apply Lemma 4.4 and deduce that e_2 and e_8 are also guts elements. Applying Lemma 4.5 to S_0 and S_7, we see that e_3 and e_7 are coguts elements.

Suppose that e_5 is a coguts element. Lemma 4.4 implies that e_1 and e_9 are also coguts elements. Therefore the triangle in S_1 cannot contain e_1 by Lemma 4.5, so $\{e_2, e_3, e_4\}$ is a triangle. A symmetric argument shows that $\{e_6, e_7, e_8\}$ is a triangle. Therefore the lemma holds (as statement (i) is true) without any switching in the case that e_5 is a coguts element.

Therefore we assume that e_5 is a guts element. Lemma 4.4 implies that e_1 and e_9 are guts elements also. Therefore $e_1 \in \mathrm{cl}(S_2)$. Note that $e_6 \in \mathrm{cl}(S_2)$ because of the triangle $\{e_4, e_5, e_6\}$. Thus $\mathrm{cl}(S_2)$ contains $\{e_1, \ldots, e_6\}$. As $\mathrm{r}(S_2) = 3$, this means that M restricted to $\{e_1, \ldots, e_6\}$ is isomorphic to $M(K_4)$. A similar argument shows that $M|\{e_4, \ldots, e_9\} \cong M(K_4)$. Note that Lemma 5.13 shows that S_{n-2} is a T-set.

As e_4, e_5, and e_6 are consecutive guts elements, Lemma 4.3 implies that any reordering of these three elements produces a valid cyclic ordering for M. Since $\{e_4, e_5, e_6\}$ is a triangle of the $M(K_4)$-restriction $\{e_4, \ldots, e_9\}$, there are elements $x, y \in \{e_4, e_5, e_6\}$ such that $\{x, e_7, e_8\}$ and $\{y, e_8, e_9\}$ are triangles. We switch e_4, e_5, and e_6 so that $\{e_5, e_8, e_9\}$ and $\{e_6, e_7, e_8\}$ are triangles. Lemma 4.3 asserts that this reordering does not change the (g, c)-label of any element.

Since $\{e_4, e_5, e_6\}$ is a triangle it follows that $\mathrm{r}(S_3) = 3$. We have already stated that e_7 is a coguts element, so $e_7 \notin \mathrm{cl}(S_3)$. Therefore $\mathrm{r}(S_3 \cup e_7) = 4$. As $S_3 \cup e_7$ is exactly 4-separating it follows that $S_3 \cup e_7$ contains a cocircuit C^*. It cannot be the case that C^* is a triad, for then C^* would meet the triangle $\{e_4, e_5, e_6\}$. Nor can C^* contain e_5, because of orthogonality with the triangle $\{e_5, e_8, e_9\}$. Therefore $C^* = \{e_3, e_4, e_6, e_7\}$.

As $\{e_4, e_5, e_6\}$ is a triangle of the $M(K_4)$-restriction $\{e_1, \ldots, e_6\}$, there is an element $z \in \{e_4, e_5, e_6\}$ such that $\{z, e_1, e_2\}$ is a triangle. By orthogonality with C^*, it must be the case that $z = e_5$. Now, it follows that either $\{e_1, e_3, e_4\}$ or $\{e_2, e_3, e_4\}$ is a triangle. We have already deduced that e_1 and e_2 are guts elements. Therefore Lemma 4.3 implies that we can switch e_1 and e_2 if necessary and assume that $\{e_2, e_3, e_4\}$ is a triangle. Then $\{e_1, e_3, e_6\}$ is also a triangle.

We now have that $\{e_2, e_3, e_4\}$, $\{e_4, e_5, e_6\}$ and $\{e_6, e_7, e_8\}$ are all triangles, as desired. Moreover statement (ii) of the lemma holds. It remains to show that the label-sequence has not been changed by this switching. Before performing any switching, the sets S_1, S_2, S_5, and S_6 were all T-sets. After switching, $\{e_2, e_3, e_4\}$, $\{e_4, e_5, e_6\}$ and $\{e_6, e_7, e_8\}$ are triangles, so S'_1, S'_2, S'_5 and S'_6 remain T-sets. Furthermore, after switching, S'_0 is a cocircuit while S'_1, S'_2 and S'_3 are T-sets, and e_1 is a guts element. Therefore S'_{n-2} is a T-set by Lemma 5.13. We had already noted that S_{n-2} was a T-set before switching. As S'_{n-2}, S'_1, S'_2, S'_5 and S'_6 were the only sets whose labels could have been changed by our switching, the proof is complete. □

Lemma 5.16. *Suppose that $S_{i+1}, \ldots, S_{i+t}$ are distinct T-sets, where $t \geq 6$, and that S_i and S_{i+t+1} are cocircuits. By switching, we can assume that either:*

(i) for $j \in \{1, \ldots, t+3\}$, element e_{i+j} is a guts element if j is even, and a coguts element if j is odd. In this case S_{i-2} and S_{i+t+3} are cocircuits while S_{i-1} and S_{i+t+2} are T-sets; or,

(ii) $t = 6$ *and the elements* $e_{i+1}, e_{i+2}, \ldots, e_{i+t+3}$ *do not alternate between guts and coguts. Moreover,* S_{i-3} *and* S_{i+t+4} *are cocircuits while* S_{i-2}, S_{i-1}, S_{i+t+2} *and* S_{i+t+3} *are* T*-sets.*

Proof. Suppose that $t > 6$. Then Lemma 5.14 implies that e_{i+j} is a guts element if $j \in \{1, \ldots, t+3\}$ is even, and a coguts element otherwise. Then S_{i-2} is a cocircuit and S_{i-1} is a T-set by Lemma 5.13. Symmetry shows that S_{i+t+3} is a cocircuit and that S_{i+t+2} is a T-set.

Now we will suppose that $t = 6$. We will assume, by switching if necessary, that Lemma 5.15 holds. If statement (i) of that lemma is true, then again by applying Lemma 5.13 we can see that statement (i) of the current lemma is true. Assume that statement (ii) of Lemma 5.15 holds. Lemma 5.13 implies that S_{i-3} is a cocircuit, and that S_{i-2} and S_{i-1} are T-sets. By symmetry we see that S_{i+t+4} is a cocircuit, and that S_{i+t+2} and S_{i+t+3} are T-sets. □

Lemma 5.17. *Suppose that* $S_{i+1}, \ldots, S_{i+t}$ *are* T*-sets, where* $t \geq 6$*, and that* S_i *and* S_{i+t+1} *are cocircuits. Assume also that if* $j \in \{1, \ldots, t+3\}$ *is even, then* e_{i+j} *is a guts element, and otherwise* e_{i+j} *is a coguts element. Then* $n = t+3$*, and* S_{i-2} *is a cocircuit, while* S_{i-1} *is a* T*-set. Furthermore, up to switching elements in such a way as to leave the label-sequence unchanged, we can assume that* $\{e_{i-1}, e_{i+1}, e_{i+2}\}$ *is a triangle and that* $\{e_i, e_{i+j}, e_{i+j+1}, e_{i+j+2}\}$ *is a cocircuit if* $j \in \{3, \ldots, t-1\}$ *is odd.*

Proof. We can assume that $i = 0$. Lemma 5.10 implies that t is even, and since the elements e_{i+j} for $j \in \{1, 2, \ldots, t+3\}$ alternate between guts and coguts, Lemma 5.16 implies that S_{n-2} and S_{t+3} are cocircuits while S_{n-1} and S_{t+2} are T-sets. Lemmas 5.14 and 5.15 assert that $\{e_j, e_{j+1}, e_{j+2}\}$ is a triangle for all even integers $j \in \{2, \ldots, t\}$. Orthogonality with the cocircuits S_0 and S_{n-2} implies that the triangle in S_{n-1} contains both e_{n-1} and e_2, so either $\{e_{n-1}, e_0, e_2\}$ or $\{e_{n-1}, e_1, e_2\}$ is a triangle. We will show that we can switch e_0 and e_1 if necessary, and assume the latter. Certainly both e_0 and e_1 are coguts elements, by virtue of Lemma 4.5 applied to S_{n-2} and S_0. Thus, by Lemma 4.3, switching e_0 and e_1 produces a valid cyclic ordering. Before this switching, S_1 is a T-set. After switching, S'_1 still contains the triangle $\{e_2, e_3, e_4\}$ and is a T-set. Since S_{n-2} is a cocircuit, S_{n-3} must be a T-set by Lemma 5.2. The triangle in S_{n-3} cannot contain e_0 by orthogonality with S_0, so $\{e_{n-3}, e_{n-2}, e_{n-1}\}$ is a triangle. Therefore, after switching, S'_{n-3} still contains a triangle. As S'_{n-3} and S'_1 are the only members of the set-sequence that are changed by switching e_0 and e_1, it now follows that this switching leaves the label-sequence unaltered. We will henceforth assume that $\{e_{n-1}, e_1, e_2\}$ is a triangle. We continue with the following sublemma.

5.17.1. $\{e_0, e_j, e_{j+1}, e_{j+2}\}$ is a cocircuit if $j \in \{3, \ldots, t-1\}$ is an odd integer.

Proof. Let j be any odd integer in $\{3, \ldots, t-3\}$. We use C_j^* to denote the set $\{e_j, e_{j+1}, e_{j+3}, e_{j+4}\}$. Now $\{e_{j-1}, e_j, e_{j+1}\}$ and $\{e_{j+1}, e_{j+2}, e_{j+3}\}$ are triangles, and e_{j+4} is a coguts element, so $e_{j+4} \notin \mathrm{cl}(S_j)$. Therefore, Lemma 5.11 implies that C_j^* is a cocircuit.

Now if j is any odd integer in $\{3, \ldots, t-3\}$, then

$$\{e_0,\, e_3,\, e_4,\, e_5\}\Delta C_3^*\Delta C_5^*\Delta \cdots \Delta C_j^* = \{e_0,\, e_{j+2},\, e_{j+3},\, e_{j+4}\}$$

must be a cocircuit. This completes the proof of the sublemma. □

It remains to show only that $n = t+3$. Suppose first that $n \leq t+5$. It cannot be the case that $n \leq t+2$, for in that case S_{n-2} would be both a cocircuit and a T-set. Suppose that $n = t+4$. Then $S_{n-2} = S_{t+2}$ is both a cocircuit, and a T-set, a contradiction. Finally we suppose that $n = t+5$. Then S_{t+1} and $S_{t+5} = S_0$ are cocircuits, and this violates Lemma 5.3. Thus in the case that $n \leq t+5$ it follows that $n = t+3$, so the result holds.

We now assume that $n > t+5$. Consider the triangle in the T-set S_{t+2}. It cannot be contained in either of the cocircuits S_{t+1} or S_{t+3}, so it contains e_{t+2} and e_{t+5}. Therefore either $\{e_{t+2},\, e_{t+3},\, e_{t+5}\}$ or $\{e_{t+2},\, e_{t+4},\, e_{t+5}\}$ is a triangle. We will switch e_{t+3} and e_{t+4} if necessary so as to assume that $\{e_{t+2},\, e_{t+3},\, e_{t+5}\}$ is a triangle. Note that both e_{t+3} and e_{t+4} are coguts elements, by virtue of applying Lemma 4.5 to S_{t+1} and S_{t+3}. Therefore this switch produces a valid cyclic ordering. Moreover, the only sets in the set-sequence that are changed by this switch are S_t and S_{t+4}. The set S_t contains the triangle $\{e_t,\, e_{t+1},\, e_{t+2}\}$, and is therefore a T-set both before and after the switch. Before the switch, S_{t+4} must be a T-set, as S_{t+3} is a cocircuit. The triangle in S_{t+4} must be $\{e_{t+5},\, e_{t+6},\, e_{t+7}\}$, by orthogonality with S_{t+1}. Now we can see that the label-sequence is unchanged by switching e_{t+3} and e_{t+4}. Moreover, as we have assumed that $n > t+5$, switching e_{t+3} and e_{t+4} does not affect our assumption that $\{e_{n-1},\, e_1,\, e_2\}$ is a triangle. Nor does it affect the claim made in 5.17.1.

As e_{t+3} is a coguts element, $e_{t+3} \notin \mathrm{cl}(S_{t-1})$, so $S_{t-1} \cup e_{t+3}$ has rank 4. Now, as $\{e_{t-2},\, e_{t-1},\, e_t\}$, $\{e_t,\, e_{t+1},\, e_{t+2}\}$, and $\{e_{t+2},\, e_{t+3},\, e_{t+5}\}$ are triangles, we apply Lemma 5.11 and deduce that $C_{t-1}^* = \{e_{t-1},\, e_t,\, e_{t+2},\, e_{t+3}\}$ is a cocircuit. By taking the symmetric difference of this set with S_{t+1}, we discover that $\{e_{t-1},\, e_t,\, e_{t+1},\, e_{t+4}\}$ is a cocircuit. However, $\{e_0,\, e_{t-1},\, e_t,\, e_{t+1}\}$ is also a cocircuit by 5.17.1. By taking the symmetric difference of these two cocircuits we see that $\{e_0,\, e_{t+4}\}$ is a union of cocircuits. But e_0 and e_{t+4} are distinct elements as $n > t+5$, so we have a contradiction. □

Suppose that t is a positive integer. We say that $(S_i, \ldots, S_{i+3t})$ is a *C^*TTC^*-sequence* if, for $j \in \{0, \ldots, 3t\}$, set S_{i+j} is a cocircuit if j is a multiple of 3 and a T-set otherwise.

Lemma 5.18. *Let $(S_i, \ldots, S_{i+3t})$ be a C^*TTC^*-sequence for some integer $t \geq 1$. For $j \in \{0, \ldots, 3t+3\}$, if j is a multiple of 3 then e_{i+j} is a coguts element, otherwise e_{i+j} is a guts element. Furthermore, if $j \in \{0, \ldots, 3t-3\}$ is a multiple of 3, then either:*

(i) $\{e_{j+1}, e_{j+3}, e_{j+4}\}$ *and* $\{e_{j+2}, e_{j+3}, e_{j+5}\}$ *are triangles; or,*
(ii) $\{e_{j+1}, e_{j+3}, e_{j+5}\}$ *and* $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ *are triangles.*

Proof. We start by proving that either statement (i) or (ii) is true. We assume by shifting labels that $i = 0$. Let $j \in \{0, \dots, 3t-3\}$ be a multiple of 3. Then S_j and S_{j+3} are cocircuits, while S_{j+1} and S_{j+2} are T-sets. Suppose that $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ is not a triangle, and let T_1 and T_2 be the triangles contained in S_{j+1} and S_{j+2} respectively. Since T_1 is not contained in the cocircuit S_j, it must be the case that e_{j+4} is in T_1. Orthogonality with S_{j+3} shows that $e_{j+3} \in T_1$, so $T_1 = \{e_{j+1}, e_{j+3}, e_{j+4}\}$. A similar argument shows that $T_2 = \{e_{j+2}, e_{j+3}, e_{j+5}\}$. Therefore statement (i) of the lemma holds.

Next we suppose that $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ is a triangle. Suppose that $e_{j+1} \notin \mathrm{cl}(S_{j+2})$, so that $\mathrm{r}(S_{j+2} \cup e_{j+1}) = 4$. As $S_{j+2} \cup e_{j+1}$ is a sequential set it must therefore contain a cocircuit C^*. It cannot be the case that C^* is a triad, for then C^* would meet the triangle $\{e_{j+2}, e_{j+3}, e_{j+4}\}$. Nor can C^* have cardinality five by orthogonality with the same triangle. Therefore $|C^*| = 4$. If C^* were to meet S_j in three elements, then the symmetric difference of these two sets would have cardinality two and be a union of cocircuits. This is impossible, so $|C^* \cap S_j| \neq 3$. The same argument shows that $|C^* \cap S_{j+3}| \neq 3$. It now follows easily that $C^* = \{e_{j+1}, e_{j+2}, e_{j+4}, e_{j+5}\}$. However this means that

$$S_j \Delta C^* \Delta S_{j+3} = \{e_j, e_{j+6}\},$$

and therefore M contains a cocircuit of size at most two. This contradiction implies that $e_{j+1} \in \mathrm{cl}(S_{j+2})$.

As S_{j+2} contains the triangle $\{e_{j+2}, e_{j+3}, e_{j+4}\}$, it follows that $\mathrm{r}(S_{j+2} \cup e_{j+1}) = 3$, thus there is a circuit in $S_{j+2} \cup e_{j+1}$ that contains e_{j+1}. Since the symmetric difference of this circuit with $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ is also a circuit, the cardinalities of this circuit and this symmetric difference are both three. Thus, there is a triangle in $S_{j+2} \cup e_{j+1}$ that contains e_{j+1} and e_{j+5}. This triangle must contain exactly two elements from each of S_j and S_{j+3}. Therefore $\{e_{j+1}, e_{j+3}, e_{j+5}\}$ is a triangle. Hence statement (ii) of the lemma holds.

It remains to show that if $j \in \{0, \dots, 3t+3\}$ is a multiple of 3, then e_{i+j} is a coguts element, and that e_{i+j} is a guts element otherwise. Suppose that $j \in \{0, \dots, 3t\}$ is a multiple of 3. Then S_j is a cocircuit, so e_j and e_{j+3} are coguts elements by Lemma 4.5. By applying Lemma 4.5 to the triangles in (i) and (ii) of the lemma, we see that e_{j+1} and e_{j+2} are guts elements. This completes the proof. □

Lemma 5.19. *Suppose that* $S_{i+1}, \dots, S_{i+6}$ *are* T*-sets,* S_i *and* S_{i+7} *are cocircuits, and that* e_{i+j} *is a guts element if* $j \in \{1, 2, 4, 5, 6, 8, 9\}$*. Then* $n \equiv 1 \pmod 3$*, and if* $j \in \{7, 8, \dots, n\}$ *is equivalent to* $1 \pmod 3$*, then* S_{i+j} *is a cocircuit. Otherwise,* S_{i+j} *is a* T*-set. Moreover, up to switching consecutive guts elements, we may assume that if* $j \in \{7, 8, \dots, n-1\}$ *and* $j \equiv 1 \pmod 3$*, then* $\{e_{i+5}, e_{i+j+1}, e_{i+j+2}\}$ *is a triangle and either:*

(i) $\{e_{j+1},\, e_{j+3},\, e_{j+4}\}$ *and* $\{e_{j+2},\, e_{j+3},\, e_{j+5}\}$ *are triangles; or,*
(ii) $\{e_{j+1},\, e_{j+3},\, e_{j+5}\}$ *and* $\{e_{j+2},\, e_{j+3},\, e_{j+4}\}$ *are triangles.*

Proof. We assume that $i = 0$, so that S_0 and S_7 are cocircuits. Lemma 5.2 implies that S_8 is a T-set. If S_9 were a cocircuit then e_9 would be a coguts element by Lemma 4.5, contrary to our hypothesis. Thus S_9 is a T-set. Consider the maximal string of T-sets that contains S_8 and S_9. Lemma 5.5 implies that this string cannot have length greater than two, so S_{10} must be a cocircuit. Therefore $(S_7,\, S_8,\, S_9,\, S_{10})$ is a C^*TTC^*-sequence. Let $(S_7, \ldots, S_{7+3t})$ be a maximal C^*TTC^*-sequence for some integer $t \geq 1$. Then by Lemma 5.2, S_{8+3t} is a T-set. Consider the maximal string of T-sets that contains S_{8+3t}. This string cannot have length one, by Lemma 5.7. It cannot have length two, for that would contradict the maximality of $(S_7, \ldots, S_{7+3t})$. Now, Lemma 5.10 tells us that its length is even and at least six. Furthermore, since S_{5+3t} and S_{6+3t} are T-sets, Lemma 5.16 tells us that the string of T-sets containing S_{8+3t} has length exactly six. It follows that $S_{8+3t},\, S_{9+3t}, \ldots, S_{13+3t}$ are all T-sets. Moreover, as S_{7+3t} is a cocircuit and S_{5+3t} is a T-set, Lemma 5.13 (i) implies that e_{8+3t} is a guts element. Now, by Lemma 5.15 (ii), we may assume $\{e_{8+3t}, e_{9+3t}, \ldots, e_{16+3t}\}$ are ordered such that $e_{8+3t},\, e_{9+3t},\, e_{11+3t},\, e_{12+3t},\, e_{13+3t},\, e_{15+3t}$ and e_{16+3t} are guts elements and $\{e_{8+3t}, e_{9+3t}, e_{12+3t}\}$ is a triangle. Note that any switching of elements from $\{e_{8+3t}, e_{9+3t}, \ldots, e_{16+3t}\}$ that was necessary to make this assumption has not altered the label-sequence or any properties of our sequence established so far. For example, $S_7, S_8, \ldots, S_{7+3t}$ is still a C^*TTC^*-sequence and $\{e_1, \ldots, e_9\}$ still contains the same set of guts elements.

Now, we may further assume by Lemma 5.15 that $\{e_1,\, e_2,\, e_5\}$ and $\{e_5,\, e_8,\, e_9\}$ are triangles. We assert that any switching that is necessary to ensure these are triangles, does not alter the label-sequence or any properties of our sequence established so far. To see this, it suffices to check the case where $\{e_1, \ldots, e_9\} = \{e_{8+3t}, \ldots, e_{16+3t}\}$. In this case we would have $\{e_1,\, e_2,\, e_5\} = \{e_{8+3t},\, e_{9+3t},\, e_{12+3t}\}$ is a triangle, and $\{e_{8+3t}, e_{9+3t}, e_{11+3t}, e_{12+3t}, e_{13+3t}, e_{15+3t}, e_{16+3t}\} = \{e_1, e_2, e_4, e_5, e_6, e_8, e_9\}$ are guts elements, as required.

We now show by induction that $\{e_5,\, e_{j+1},\, e_{j+2}\}$ is a triangle if $j \in \{7, 8, \ldots, 7 + 3t\}$ and $j \equiv 1 \pmod 3$. This is certainly true if $j = 7$. Take $j \in \{10, 11, \ldots, 7 + 3t\}$ such that $j \equiv 1 \pmod 3$ and such that $\{e_5, e_{j-2}, e_{j-1}\}$ is a triangle of M. Then $(S_{j-3}, \ldots, S_j)$ is a C^*TTC^*-sequence, so by Lemma 5.18, e_{j+1} and e_{j+2} are guts elements. Thus $e_{j+2} \in \mathrm{cl}(S_{j-2})$ and as $\{e_5, e_{j-2}, e_{j-1}\}$ is a triangle, we also have $e_5 \in \mathrm{cl}(S_{j-2})$. Then $S_{j-2} \cup \{e_{j+2}, e_5\}$ has rank 3 and is an $M(K_4)$-restriction of M in which $\{e_5, e_{j-2}, e_{j-1}\}$ is a triangle. Also, by Lemma 5.18, either $\{e_{j-2}, e_j, e_{j+2}\}$ and $\{e_{j-1}, e_j, e_{j+1}\}$ are triangles, or $\{e_{j-2}, e_j, e_{j+1}\}$ and $\{e_{j-1}, e_j, e_{j+2}\}$ are triangles. In either case, the fourth triangle of the $M(K_4)$-restriction must be $\{e_5, e_{j+1}, e_{j+2}\}$, as required.

In particular, $\{e_5, e_{8+3t}, e_{9+3t}\}$ is a triangle. But we have already seen earlier in the proof that $\{e_{8+3t}, e_{9+3t}, e_{12+3t}\}$ is a triangle of M. Since M has no parallel pairs, it follows that $e_5 = e_{12+3t}$, therefore $n = 7 + 3t$ and $(S_7, S_8, \ldots, S_0)$ is a C^*TTC^*-sequence in which $\{e_5, e_{j+1}, e_{j+2}\}$ is a triangle if $j \in \{7, 8, \ldots, n\}$ and $j \equiv 1 \pmod 3$.

Furthermore, by Lemma 5.18, either $\{e_{j+1}, e_{j+3}, e_{j+5}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ are triangles or $\{e_{j+1}, e_{j+3}, e_{j+4}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+5}\}$ are triangles if $j \in \{7, 8, \ldots, n-3\}$ is such that $j \equiv 1 \pmod 3$, as required. □

The following lemma is the core of our argument. It shows that every possible label-sequence falls into one of only four families. In the next section, we will see that the label-sequence is enough to determine the structure of the corresponding cyclically sequential matroid.

Lemma 5.20. *Suppose that M is an internally 4-connected binary matroid, and that $(e_0, \ldots, e_{n-1})$ is a cyclically 4-sequential ordering for M, where $n \geq 9$. By switching the order of elements in the cyclic ordering, cyclically shifting the ordering and exploiting duality, we can assume that one of the following statements is true:*

(i) n is even and S_j is a T-set for every $j \in \{0, 1, \ldots, n-1\}$. Moreover, for every even integer $j \in \{0, 1, \ldots, n-1\}$, $\{e_j, e_{j+1}, e_{j+2}\}$ is a triangle and $\{e_{j+1}, e_{j+2}, e_{j+4}, e_{j+5}\}$ is a cocircuit.

(ii) n is odd, S_{n-2} and S_0 are cocircuits, while S_i is a T-set if $i \in \{1, 2, \ldots, n-3, n-1\}$. Furthermore, $\{e_{n-1}, e_1, e_2\}$ is a triangle, and if $j \in \{2, 3, \ldots, n-3\}$ is even, then $\{e_j, e_{j+1}, e_{j+2}\}$ is a triangle and $\{e_0, e_{j-1}, e_j, e_{j+1}\}$ is a cocircuit.

(iii) n is a multiple of 3. For $j \in \{0, 1, \ldots, n-1\}$, set S_j is a cocircuit if j is a multiple of 3, otherwise S_j is a T-set. Furthermore, if j is a multiple of 3, then either $\{e_{j+1}, e_{j+3}, e_{j+5}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ are triangles, or $\{e_{j+1}, e_{j+3}, e_{j+4}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+5}\}$ are triangles.

(iv) $n \equiv 1 \pmod 3$. Set S_j is a cocircuit if $j = 0$ or $j \in \{7, 8, \ldots, n-1\}$ and $j \equiv 1 \pmod 3$, and a T-set otherwise. Also, $\{e_2, e_3, e_4\}$, $\{e_4, e_5, e_6\}$, $\{e_6, e_7, e_8\}$, $\{e_1, e_2, e_5\}$, $\{e_5, e_8, e_9\}$, $\{e_1, e_3, e_6\}$ and $\{e_4, e_7, e_9\}$ are triangles. Furthermore, if $j \in \{7, 8, \ldots, n-1\}$ and $j \equiv 1 \pmod 3$, then $\{e_5, e_{j+1}, e_{j+2}\}$ is a triangle and either $\{e_{j+1}, e_{j+3}, e_{j+5}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ are triangles, or $\{e_{j+1}, e_{j+3}, e_{j+4}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+5}\}$ are triangles.

Proof. By Lemma 5.4 we can assume that S_i is either a T-set or a cocircuit for all $i \in \{0, \ldots, n-1\}$. By Lemma 5.10, a consecutive sequence of T-sets has length either 1, 2 or an even integer greater than 4. If every S_i is a T-set then (i) follows from Lemma 5.12.

If there is no consecutive sequence of more than two T-sets, then every consecutive sequence of T-sets has length one or two, while Lemma 5.2

implies that every occurrence of a cocircuit in the label-sequence is immediately followed and preceded by a T-set. Lemma 5.7 now tells us that either every maximal consecutive sequence of T-sets has length one or every maximal consecutive sequence of T-sets has length two, while Lemma 5.3 rules out the first of these two possibilities. Thus, n is a multiple of 3, and by cyclically shifting labels as necessary, we can assume that S_i is a cocircuit whenever $i \in \{0, 1, \ldots, n-1\}$ is a multiple of 3, while S_i is a T-set for all other $i \in \{0, 1, \ldots, n-1\}$ (that is $(S_0, S_1, \ldots, S_{n-1}, S_0)$ is a C^*TTC^*-sequence). The fact that either $\{e_{j+1}, e_{j+3}, e_{j+5}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+4}\}$ are triangles or $\{e_{j+1}, e_{j+3}, e_{j+4}\}$ and $\{e_{j+2}, e_{j+3}, e_{j+5}\}$ are triangles follows immediately from Lemma 5.18. Therefore (iii) must hold.

Finally suppose that some S_i is a cocircuit and there is a consecutive sequence of six or more T-sets. We may assume that $S_1, \ldots, S_t$ is a maximal consecutive sequence of T-sets with S_0 and S_{t+1} being cocircuits. Lemma 5.16 tells us that by possibly switching, we can assume that either the elements $\{e_1, \ldots, e_{t+3}\}$ alternate between guts and coguts, or that $t = 6$ and the guts elements of $\{e_1, \ldots, e_9\}$ are $\{e_1, e_2, e_4, e_5, e_6, e_8, e_9\}$. For the first of these cases, we apply Lemma 5.17 which tells us that $n = t + 3$ and provides us with the list of triangles and cocircuits described in (ii). Thus, (ii) follows from the first of our two cases. We now consider the second case. We may apply Lemma 5.19 here to deduce that $n \equiv 1 \pmod 3$ and that $(S_7, \ldots, S_{n-1}, S_0)$ is a C^*TTC^*-sequence. We obtain the list of triangles of (iv) from Lemmas 5.15 and 5.19. We conclude that (iv) follows from the second of our two cases.

The result now follows. □

Proposition 5.21. *Let $(S_i, \ldots, S_{i+3t})$ be a C^*TTC^*-sequence. Then $\{e_{i+1}, e_{i+2}, e_{i+j+1}, e_{i+j+2}\}$ is a circuit if $j \in \{3, \ldots, 3t\}$ is a multiple of 3 and $j < n$.*

Proof. Note that the restrictions on j ensure that e_{i+1}, e_{i+2}, e_{i+j+1} and e_{i+j+2} are distinct elements. We assume that $i = 0$. Lemma 5.18 tells us that either $\{e_1, e_3, e_4\}$ and $\{e_2, e_3, e_5\}$ are triangles, or $\{e_1, e_3, e_5\}$ and $\{e_2, e_3, e_4\}$ are triangles. In either case, we take the symmetric difference of the two triangles and find that $\{e_1, e_2, e_4, e_5\}$ is a circuit. This provides the base case of an inductive argument.

Suppose that $j \in \{6, \ldots, 3t\}$ is a multiple of 3, $j < n$, and that the lemma holds for $j - 3$ so that $\{e_1, e_2, e_{j-2}, e_{j-1}\}$ is a circuit. By applying Lemma 5.18 to the C^*TTC^*-sequence $(S_{j-3}, S_{j-2}, S_{j-1}, S_j)$, we see that either $\{e_{j-2}, e_j, e_{j+1}\}$ and $\{e_{j-1}, e_j, e_{j+2}\}$ are triangles, or $\{e_{j-2}, e_j, e_{j+2}\}$ and $\{e_{j-1}, e_j, e_{j+1}\}$ are triangles. In either case, by taking the symmetric difference of these two triangles with $\{e_1, e_2, e_{j-2}, e_{j-1}\}$, we see that $\{e_1, e_2, e_{j+1}, e_{j+2}\}$ is a circuit. This completes the proof. □

Proposition 5.22. *Let $(S_i, \ldots, S_{i+3t})$ be a C^*TTC^*-sequence for some $6 \leq 3t \leq n-3$. Let $k \in \{6, \ldots, 3t\}$ be a multiple of 3, and let $a \in \{e_{i+k-2}, e_{i+k-1}\}$*

and $b \in \{e_{i+k+1}, e_{i+k+2}\}$ *be such that* $\{e_{i+k}, a, b\}$ *is not a triangle. Then* $\{e_{i+1}, e_{i+2}, e_{i+k}, a, b\}$ *is a circuit of* M.

Proof. Note that the restrictions on k and t ensure that e_{i+1}, e_{i+2}, e_{i+k}, a and b are distinct elements. We may assume by a cyclic shift that $i = 0$. Let $a' \in \{e_{k-2}, e_{k-1}\} - \{a\}$. We know that $\{e_k, a', b\}$ is a triangle by Lemma 5.18, and that $\{e_1, e_2, a, a'\}$ is a circuit by Proposition 5.21. The symmetric difference $\{e_k, a', b\}\Delta\{e_1, e_2, a, a'\} = \{e_1, e_2, e_k, a, b\}$ is a union of disjoint circuits. By the connectivity of M, it cannot contain more than one circuit. Therefore $\{e_1, e_2, e_k, a, b\}$ is a circuit, as required. □

6. Proof of the main theorem

In the last section we characterized the possible label-sequences that can arise from a cyclically 4-sequential ordering of an internally 4-connected binary matroid and we built up a description of many of the small circuits and cocircuits of such a matroid. In this section we show that each of the four possible sequences mentioned in Lemma 5.20 leads to a matroid in one of our basic classes described in Section 3.

Note that it is well known that if two binary matroids have the same ground set, and they share a common basis for which the fundamental circuits are the same, then the matroids have the same representation over GF(2) and are therefore equal. We will use this to show that our cyclically sequential binary matroids are indeed isomorphic to matroids of the basic classes from Section 3. Note also, that traditionally the term "fundamental circuit" has been used with regards to bases only. We extend its definition to encompass all independent sets as follows. Let I be an independent set of a matroid M, and let $e \in E(M) - I$. If $I \cup e$ contains a circuit C, then C is the *fundamental circuit of* e *with respect to* I. Otherwise e has no fundamental circuit with respect to I.

Throughout this section, M will be an internally 4-connected binary matroid and $(e_0, \ldots, e_{n-1})$ will be a cyclically 4-sequential ordering, where $n \geq 9$. By switching and applying duality we will assume that one of the four statements in Lemma 5.20 holds.

Lemma 6.1. *Suppose that statement (i) of Lemma 5.20 holds. Then* M *is the polygon matroid of a quartic planar ladder, or the polygon matroid of a quartic Möbius ladder.*

Proof. We construct a basis B of M and show that it has the same collection of fundamental circuits as a corresponding basis of the quartic planar or Möbius ladder. Let $B = \{e_0, e_2, e_4, \ldots, e_{n-4}\}$. We show that B is indeed a basis of M by first showing that it is independent. Clearly $\{e_0, e_2\}$ is independent by the size and connectivity of M. Now let $2 \leq i \leq n-6$ be an even integer such that $\{e_0, e_2, e_4, \ldots, e_i\}$ is independent. Then by Lemma 5.20, $\{e_{i+1}, e_{i+2}, e_{i+4}, e_{i+5}\}$ is a cocircuit avoiding $\{e_0, e_2, e_4, \ldots, e_i\}$, thus $\{e_0, e_2, e_4, \ldots, e_i, e_{i+2}\}$ is independent. It follows

that B is independent. We must now show that B spans $E(M)$. Since $\{e_j,\, e_{j+1},\, e_{j+2}\}$ is a triangle for all even j, we see that $\{e_1, e_3, e_5, \ldots, e_{n-5}\} \subseteq \mathrm{cl}(B)$. By the connectivity of M, if $\{e_{n-3},\, e_{n-2},\, e_{n-1}\}$ were not contained in $\mathrm{cl}(B)$, then it would be a triad of M meeting the triangle $\{e_{n-4},\, e_{n-3},\, e_{n-2}\}$, a contradiction to the fact that no triangle meets any triad in M. Hence $\{e_{n-3},\, e_{n-2},\, e_{n-1}\} \subseteq \mathrm{cl}(B)$ and B spans $E(M)$, thus B is indeed a basis of M.

We now find the fundamental circuits of M with respect to B. Let $1 \leq i \leq n-5$ be an odd integer. Then by Lemma 5.20, $\{e_{i-1},\, e_i,\, e_{i+1}\}$ is a triangle, and hence the fundamental circuit for e_i with respect to B. Now consider the fundamental circuit for e_{n-2}. Clearly $\{e_0, e_2, e_4, \ldots, e_{n-2}\}$ is dependent. Let $0 \leq i \leq n-2$ and $2 \leq j \leq n-4$ be even integers such that $\{e_{i+2}, e_{i+4}, \ldots, e_{i+j}\}$ is independent. Then by Lemma 5.20, $\{e_{i+j+1},\, e_{i+j+2},\, e_{i+j+4},\, e_{i+j+5}\}$ is a cocircuit avoiding $\{e_{i+2}, e_{i+4}, \ldots, e_{i+j}\}$, and hence $\{e_{i+2}, e_{i+4}, \ldots, e_{i+j}, e_{i+j+2}\}$ is also independent. It follows that for all even integers $0 \leq i \leq n-2$, $\{e_0, e_2, e_4, \ldots, e_{n-2}\} - \{e_i\}$ is independent, and therefore $\{e_0, e_2, e_4, \ldots, e_{n-2}\}$ is a circuit, namely the fundamental circuit of e_{n-2} with respect to B.

Now consider the fundamental circuit of e_{n-3} with respect to B. We know that $\{e_0, e_2, e_4, \ldots, e_{n-2}\}$ and $\{e_{n-4},\, e_{n-3},\, e_{n-2}\}$ are circuits of M, and that the symmetric difference $\{e_0, e_2, e_4 \ldots, e_{n-6}\} \cup \{e_{n-3}\}$ is a union of disjoint circuits of M. We have already established that $\{e_0, e_2, e_4 \ldots, e_{n-6}\}$ is independent, thus $\{e_0, e_2, e_4 \ldots, e_{n-6}\} \cup \{e_{n-3}\}$ is a circuit, namely the fundamental circuit of e_{n-3} with respect to B. A symmetric argument shows that the fundamental circuit of e_{n-1} with respect to B is $\{e_2, e_4, e_6 \ldots, e_{n-4}\} \cup \{e_{n-1}\}$.

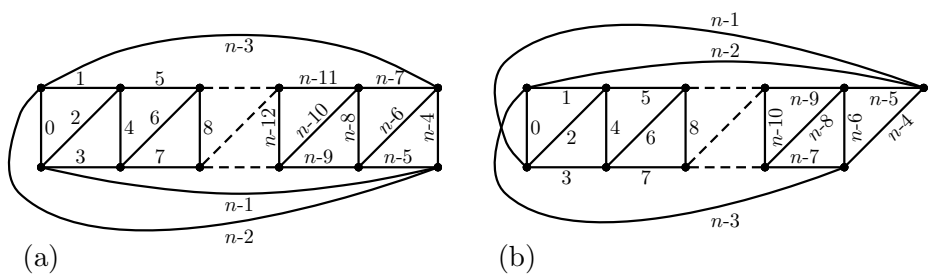


FIGURE 6. (a) Quartic planar ladder. (b) Quartic Möbius ladder.

Now consider the quartic planar and Möbius ladders of Figure 6. It is easily checked that $B' = \{0, 2, 4, \ldots, n-4\}$ is a basis and that the fundamental circuits with respect to this basis are $\{i-1, i, i+1\}$ for all odd integers $1 \leq i \leq n-5$; $\{0, 2, 4, \ldots, n-2\}$; $\{0, 2, 4, \ldots, n-6\} \cup \{n-3\}$ and $\{2, 4, 6, \ldots, n-4\} \cup \{n-1\}$. It follows that M is isomorphic to the cycle matroid of a quartic planar or Möbius ladder. Note that the quartic ladder underlying M is planar when n is a multiple of 4, and Möbius otherwise. $\square$

Lemma 6.2. *Suppose that statement (ii) of Lemma 5.20 holds. Then M is a wheel with a tip.*

Proof. Consider the set $B = \{e_2, e_4, \ldots, e_{n-1}\}$. Any circuit in this set would violate orthogonality with one of the cocircuits S_{n-2} or $\{e_0, e_j, e_{j+1}, e_{j+2}\}$ for odd integers $1 \leq j \leq n-4$. Therefore B is independent. Considering the list of triangles in the statement of Lemma 5.20(ii), clearly every element, except for possibly e_0, is spanned by B. As M has no coloops it follows that B is a basis of M. The unique circuit in $B \cup e_0$ must be $B \cup e_0$, for otherwise there is a violation of orthogonality with one of the cocircuits S_{n-2} or $\{e_0, e_j, e_{j+1}, e_{j+2}\}$, where $1 \leq j \leq n-4$ is odd.

If $j \in \{3, \ldots, n-2\}$ is an odd integer, then the fundamental circuit of e_j with respect to B is $\{e_{j-1}, e_j, e_{j+1}\}$. The fundamental circuit of e_1 is $\{e_{n-1}, e_1, e_2\}$, and the fundamental circuit of e_0 is $B \cup e_0$. It follows immediately that M is represented over GF(2) by a matrix of the type shown in Figure 3. Therefore M is isomorphic to a wheel with a tip. □

Lemma 6.3. *Suppose that statement (iii) of Lemma 5.20 holds. Then M is the bond matroid of a cubic planar ladder, or the bond matroid of a cubic Möbius ladder.*

Proof. We would like to increase the number of sequential triangles in the ordering as much as possible. Let us say that two cyclic orderings of $E(M)$ are *switching-equivalent* if the corresponding label-sequences are identical, and one can be obtained from the other by switching adjacent elements of the same (g, c)-label. Let our cyclic ordering be $(x_0, x_1, \ldots, x_{n-1})$, and consider the set of all cyclic orderings that are switching-equivalent to $(x_0, \ldots, x_{n-1})$. Suppose that $(e_0, \ldots, e_{n-1})$ is such a cyclic ordering. There is an index, $i+1$, such that if $j < i+1$ and S_j is a T-set, then S_j contains a sequential triangle, and $i+1$ is as large as possible subject to this property. Let us suppose that $(e_0, \ldots, e_{n-1})$ has been chosen so that this index is as large as possible.

We will show by contradiction that $i+1 \geq n-2$. Assume that $i+1 < n-2$. Note that if S_j contains a sequential triangle, then so does either S_{j-1} or S_{j+1}. Now our choice of $i+1$ means that i is a multiple of 3, S_i is a cocircuit, and S_{i-2}, S_{i-1}, S_{i+1}, and S_{i+2} are all T-sets. Since i is a multiple of 3 and $i+1 < n-2$, it follows that $i+1 \leq n-5$. As S_{i+1} does not contain a sequential triangle it follows from Lemma 5.20 (iii) that $\{e_{i+1}, e_{i+3}, e_{i+4}\}$ and $\{e_{i+2}, e_{i+3}, e_{i+5}\}$ are triangles. Moreover, e_{i+4} and e_{i+5} are guts elements by Lemma 4.5, so we can switch e_{i+4} and e_{i+5} and produce a new cyclic ordering by Lemma 4.3.

The only sets in the set-sequence that are changed by this switch are S_{i+1} and S_{i+5}. After the switch, we have $S'_{i+1} = \{e_{i+1}, e_{i+2}, e_{i+3}, e_{i+5}\}$, which contains the triangle $\{e_{i+2}, e_{i+3}, e_{i+5}\}$, so it is a T-set of the new cyclic order. Moreover by Lemma 5.20, either $\{e_{i+4}, e_{i+6}, e_{i+7}\}$ or $\{e_{i+4}, e_{i+6}, e_{i+8}\}$ is a triangle, hence $S'_{i+5} = \{e_{i+4}, e_{i+6}, e_{i+7}, e_{i+8}\}$ is a T-set of the new cyclic ordering. Thus the new cyclic ordering is indeed switching-equivalent to $(x_0, \ldots, x_{n-1})$. Note also that $i+1 \leq n-5$ implies that $i+5 < n$, so

any T-set in $S_0, \ldots, S_i$ that contains a sequential triangle in the original cyclic order also contains a sequential triangle in the new cyclic ordering. Furthermore, S'_{i+1} contains the sequential triangle $\{e_{i+2},\, e_{i+3},\, e_{i+5}\}$ of the new ordering, contradicting our choice of $(e_0, \ldots, e_{n-1})$.

Therefore we will assume that whenever S_j is a T-set and $j \in \{0, \ldots, n-3\}$ then S_j contains a sequential triangle. In particular, $\{e_j,\, e_{j+1},\, e_{j+2}\}$ is a triangle for all integers $j \in \{0, \ldots, n-3\}$ such that $j \equiv 2 \pmod 3$.

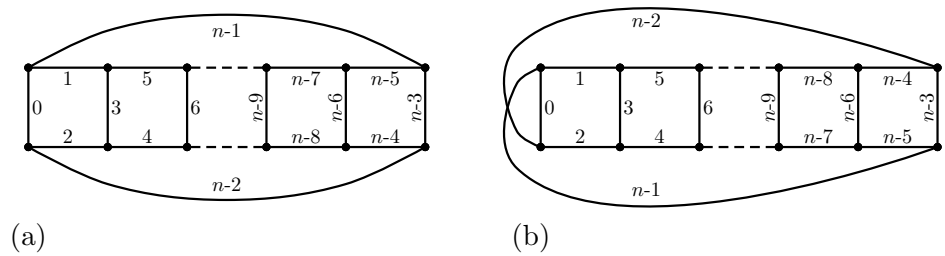
Note that by Lemma 5.20, either $\{e_{n-2},\, e_0,\, e_2\}$ and $\{e_{n-1},\, e_0,\, e_1\}$ are triangles or $\{e_{n-2},\, e_0,\, e_1\}$ and $\{e_{n-1},\, e_0,\, e_2\}$ are triangles. We now construct a basis of M. Let $B = \{e_1, e_4, e_7, \ldots, e_{n-2}\} \cup \{e_2\}$. We first show that B is independent. Suppose that B contains a circuit. Then that circuit cannot contain any element of $e_4, e_7, \ldots, e_{n-2}$ for that would violate orthogonality with a cocircuit S_j where $j \in \{3, 6, \ldots, n-3\}$. Therefore, if B contains a circuit then it is a subset of $\{e_1, e_2\}$, a contradiction to connectivity. Hence B is independent.

We now show that every element not in B has a fundamental circuit with respect to B, which will immediately imply that B is spanning, and hence a basis. First consider e_i where $i \equiv 2 \pmod 3$ and $i \neq 2$. Then by Proposition 5.21, $\{e_1,\, e_2,\, e_{i-1},\, e_i\}$ is a circuit in which $\{e_1,\, e_2,\, e_{i-1}\} \subseteq B$, and this is the fundamental circuit for e_i. The fundamental circuit for e_3 is $\{e_2,\, e_3,\, e_4\}$, while for e_0 it is either $\{e_{n-2},\, e_0,\, e_1\}$ or $\{e_{n-2},\, e_0,\, e_2\}$ depending on whether the triangle of S_{n-2} is sequential. Now consider e_i, where $i \notin \{0, 3\}$ is a multiple of 3. Then by Proposition 5.22, $\{e_1, e_2, e_{i-2}, e_i, e_{i+1}\}$ is a circuit in which $\{e_1,\, e_2,\, e_{i-2},\, e_{i+1}\} \subseteq B$ and e_1, e_2, e_{i-2} and e_{i+1} are distinct elements (note that when applying Proposition 5.22 here, a and b are e_{i-2} and e_{i+1} respectively). Therefore, $\{e_1, e_2, e_{i-2}, e_i, e_{i+1}\}$ is the fundamental circuit for e_i with respect to B.

We have now found all fundamental circuits with respect to B, and as each element not in B has such a fundamental circuit, B is indeed a basis. We must now demonstrate that these fundamental circuits correspond to those of the bond matroid of a cubic planar or cubic Möbius ladder, and under which conditions each is obtained.

First consider the cubic planar ladder in the case where n is even, and the cubic Möbius ladder in the case where n is odd. See Figure 7 for the cyclic ordering of elements. In these cases, the associated bond matroids have sequential triangles in all T-sets. Clearly $B' = \{1, 4, 7, \ldots, n-2\} \cup \{2\}$ is a basis for these bond matroids with the same collection of fundamental circuits as the basis B has for our matroid M. Therefore, in the case where $\{e_{n-1},\, e_0,\, e_1\}$ is a sequential triangle of M, we see that when n is even, M is isomorphic to the bond matroid of a cubic planar ladder, and when n is odd, M is isomorphic to the bond matroid of a cubic Möbius ladder.

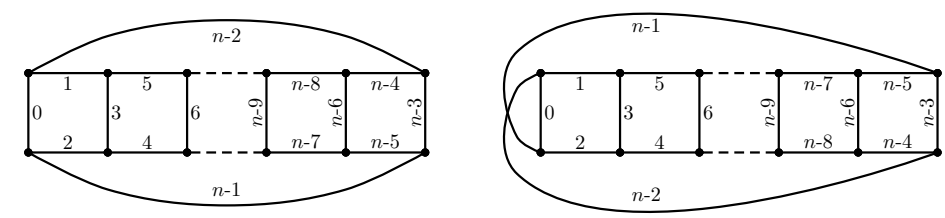
Now consider the cubic planar ladder in the case where n is odd, and the cubic Möbius ladder in the case where n is even, see Figure 8. Here, the bond matroids of these graphs have sequential triangles in all T-sets except for S_{n-2} and S_{n-1}. It is easily checked that $B' = \{1, 4, 7, \ldots, n-2\} \cup \{2\}$



(a) (b)

FIGURE 7. (a) Cubic planar ladder, for n even. (b) Cubic Möbius ladder, for n odd.

is a basis for these matroids, and it has the same collection of fundamental circuits as the basis B has for our matroid M. Therefore, in the case where $\{e_{n-2}, e_0, e_1\}$ and $\{e_{n-1}, e_0, e_2\}$ are triangles of M, we see that when n is odd, M is isomorphic to the bond matroid of a cubic planar ladder, and when n is even, M is isomorphic to the bond matroid of a cubic Möbius ladder. This completes the proof.



(a) (b)

FIGURE 8. (a) Cubic planar ladder, for n odd. (b) Cubic Möbius ladder, for n even.

□

Lemma 6.4. *Suppose that statement (iv) of Lemma 5.20 holds. Then M is a dual cubic ladder with a tip.*

Proof. Note that as $n \equiv 1 \pmod 3$, it follows that $n \geq 10$. We next increase the number of sequential triangles in the ordering of $\{e_7, \ldots, e_{n-1}\}$ as much as possible. Suppose that S_8 does not contain a sequential triangle and assume that $n > 10$. Then by Lemma 5.20, $\{e_8, e_{10}, e_{11}\}$ and $\{e_9, e_{10}, e_{12}\}$ are triangles, while e_{11} and e_{12} are both guts elements (because they are triangle endpoints), so switching these elements produces a valid cyclic ordering. Note that $S'_8 = \{e_8, e_9, e_{10}, e_{12}\}$ contains the triangle $\{e_9, e_{10}, e_{12}\}$, so in our new ordering, S'_8 will be a T-set with a sequential triangle. Moreover, $S'_{12} = \{e_{11}, e_{13}, e_{14}, e_{15}\}$ cannot be a circuit or a cocircuit because of

its intersection with the cocircuit S_{13}. It cannot contain a triad because S_{11} contains a triangle. Therefore, $\{e_{11}, e_{13}, e_{14}, e_{15}\}$ contains a triangle and is thus a T-set of the new cyclic ordering. Therefore switching e_{11} and e_{12} has not changed the label-sequence.

Now suppose that $8 \leq i \leq n-3$, and that $i \equiv 2 \pmod 3$. We will inductively assume that if $8 \leq j < i$ and $j \equiv 2 \pmod 3$, then S_j and S_{j+1} contain the sequential triangle $\{e_{j+1}, e_{j+2}, e_{j+3}\}$. If S_i does not contain a sequential triangle, then $\{e_i, e_{i+2}, e_{i+3}\}$ and $\{e_{i+1}, e_{i+2}, e_{i+4}\}$ are triangles by Lemma 5.20, and we can switch the consecutive guts elements e_{i+3} and e_{i+4}. Note that $S'_i = \{e_i, e_{i+1}, e_{i+2}, e_{i+4}\}$ contains the triangle $\{e_{i+1}, e_{i+2}, e_{i+4}\}$. Now, $S'_{i+4} = \{e_{i+3}, e_{i+5}, e_{i+6}, e_{i+7}\}$ cannot be a circuit or cocircuit because of its intersection with the cocircuit S_{i+5}. It cannot contain a triad because $S_{i+3} = \{e_{i+3}, e_{i+4}, e_{i+5}, e_{i+6}\}$ contains a triangle, so in this case the label-sequence is unchanged.

In summary, by possibly switching consecutive guts elements, we can assume that for all $i \in \{8, 9, \ldots, n-3\}$ such that $i \equiv 2 \pmod 3$, $\{e_i, e_{i+2}, e_{i+4}\}$ and $\{e_{i+1}, e_{i+2}, e_{i+3}\}$ are triangles, and that either $\{e_{n-2}, e_0, e_1\}$ and $\{e_{n-1}, e_0, e_2\}$ are triangles or $\{e_{n-2}, e_0, e_2\}$ and $\{e_{n-1}, e_0, e_1\}$ are triangles.

By Lemma 5.20, $\{e_5, e_{i+1}, e_{i+2}\}$ is a triangle for all $i \in \{7, \ldots, n-3\}$ such that $i \equiv 1 \pmod 3$. We also know that $\{e_2, e_3, e_4\}$, $\{e_4, e_5, e_6\}$, $\{e_6, e_7, e_8\}$, $\{e_1, e_2, e_5\}$, $\{e_5, e_8, e_9\}$, $\{e_1, e_3, e_6\}$ and $\{e_4, e_7, e_9\}$ are triangles.

We now construct a basis of M. Consider the set $B = \{e_1, e_4\} \cup \{e_5, e_8, e_{11}, \ldots, e_{n-2}\}$. We first show that B is independent. Any circuit in B cannot contain an element from $\{e_8, e_{11}, e_{14}, \ldots, e_{n-2}, e_1\}$ by orthogonality with the cocircuits $S_7, S_{10}, S_{13}, \ldots, S_{n-3}$ and S_0. As $\{e_4, e_5\}$ cannot contain a circuit, it follows that B is independent.

We now find all fundamental circuits of B to show that every element not in B has a fundamental circuit with respect to B. This will imply that B is spanning, and hence a basis. First consider e_i where $i \in \{8, 9, \ldots, n-1\}$ such that $i \equiv 0 \pmod 3$. Then $\{e_5, e_{i-1}, e_i\}$ is a triangle, hence it is the fundamental circuit for e_i with respect to B. Now consider e_i where $i \in \{8, 9, \ldots, n-1\}$ such that $i \equiv 1 \pmod 3$. Then $\{e_5, e_{i-2}, e_{i-1}\}$ and $\{e_{i-1}, e_i, e_{i+1}\}$ are triangles of M, thus $\{e_5, e_{i-2}, e_i, e_{i+1}\} = \{e_5, e_{i-2}, e_{i-1}\}\Delta\{e_{i-1}, e_i, e_{i+1}\}$ is a circuit of M, the fundamental circuit for e_i with respect to B. Now consider e_2, e_3, e_6, and e_7. Using our knowledge of the triangles of M, we see that the fundamental circuit for e_2 is $\{e_1, e_2, e_5\}$; for e_3 it is $\{e_1, e_3, e_4, e_5\} = \{e_1, e_2, e_5\}\Delta\{e_2, e_3, e_4\}$; for e_6 it is $\{e_4, e_5, e_6\}$; and for e_7 it is $\{e_4, e_5, e_7, e_8\} = \{e_4, e_5, e_6\}\Delta\{e_6, e_7, e_8\}$. Finally, we consider the fundamental circuit for e_0. In the case where $\{e_{n-1}, e_0, e_1\}$ is a triangle, the fundamental circuit for e_0 is $\{e_5, e_{n-2}, e_0, e_1\} = \{e_{n-1}, e_0, e_1\}\Delta\{e_5, e_{n-2}, e_{n-1}\}$. In the case where $\{e_{n-1}, e_0, e_1\}$ is not a triangle, $\{e_{n-2}, e_0, e_1\}$ is a triangle by Lemma 5.20, and is the fundamental circuit for e_0 with respect to B.

$$
\begin{array}{c}
\begin{array}{cccccccccccccccc}
e_1 & e_4 & e_8 & e_{11} & \cdots & e_{n-2} & e_5 & e_2 & e_6 & e_9 & \cdots & e_{n-1} & e_3 & e_7 & e_{10} & \cdots & e_0
\end{array} \\
\left[\begin{array}{c|ccccc|ccccc}
 & & & & & & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 1 \\
 & & & & & & 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\
 & & & & & & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 \\
 & I_{(n+2)/3} & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & & & & & & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \\
 & & & & & & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & \alpha
\end{array}\right]
\end{array}
$$

FIGURE 9. A matrix representation of M.

Having found fundamental circuits with respect to B for all elements not in B, it now follows that B is spanning and therefore a basis. It also follows that M is represented over GF(2) by the matrix in Figure 9, where $\alpha = 0$ if and only if $\{e_{n-1}, e_0, e_1\}$ is a triangle. Therefore M is a dual cubic ladder with a tip, as desired. □

Theorem 1.1 follows from Lemmas 2.2, 5.20, 6.1, 6.2, 6.3, and 6.4.

ACKNOWLEDGEMENTS

We thank James Oxley for suggesting the problem, and for helpful discussions.

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