

CONTRACTING AN ELEMENT FROM A COCIRCUIT.

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ABSTRACT. We consider the situation that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N -minor for some $x_0 \in C^*$. We show that either there is an element $x \in C^*$ such that $\text{si}(M/x)$ or $\text{co}(\text{si}(M/x))$ is 3-connected with an N -minor, or there is a four-element fan of M that contains two elements of C^* and an element x such that $\text{si}(M/x)$ is 3-connected with an N -minor.

1. INTRODUCTION

There are a number of tools in matroid theory that tell us when we can remove an element or elements from a matroid, while maintaining both the presence of a minor and a certain type of connectivity. Some recent results are of this type, but have the additional restriction that the element(s) must have a certain relation to a given substructure in the matroid. For example, Oxley, Semple, and Whittle [9], consider a given basis of a matroid and consider either contracting elements that are in the basis, or deleting elements that are not in the basis. Hall [3] has investigated when it is possible to contract an element from a given hyperplane in a 3-connected matroid and remain 3-connected (up to parallel pairs).

We make a contribution to this collection of tools by investigating the circumstances under which we can contract an element from a cocircuit while maintaining both the presence of a minor and 3-connectivity (up to parallel pairs), and the structures which prevent us from doing so. Our result has been employed by Geelen, Gerards, and Whittle [2] in their characterization of when three elements in a matroid lie in a common circuit.

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Theorem 1.1. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N -minor for some $x_0 \in C^*$. Then either:*

- (i) *there is an element $x \in C^*$ such that $\text{si}(M/x)$ is 3-connected and has an N -minor;*
- (ii) *there is an element $x \in C^*$ such that $\text{co}(\text{si}(M/x))$ is 3-connected and has an N -minor; or,*
- (iii) *there is a sequence of elements (x_1, x_2, x_3, x_4) from $E(M)$ such that $\{x_1, x_2, x_3\}$ is a circuit, $\{x_2, x_3, x_4\}$ is a cocircuit, $x_1, x_3 \in C^*$, and $\text{si}(M/x_2)$ is 3-connected with an N -minor.*

The next example shows that statement (ii) of Theorem 1.1 is necessary.

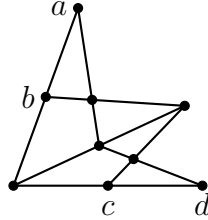


FIGURE 1. The graphic matroid $M(K_5 \setminus e)$.

Consider the rank-4 matroid M whose geometric representation is shown in Figure 1. Note that $M \cong M(K_5 \setminus e)$. The set $C = \{a, b, c, d\}$ is a circuit of M , and hence a cocircuit of M^* . Moreover M^*/x has a minor isomorphic to $M(K_4)$ for any element $x \in C$. However $\text{co}(M \setminus x)$ is not 3-connected, as it contains a parallel pair, so $\text{si}(M^*/x)$ is not 3-connected. On the other hand $\text{co}(\text{si}(M^*/x))$ is 3-connected, and has a minor isomorphic to $M(K_4)$.

More generally we suppose that r is an integer greater than two. Consider a basis $A = \{a_1, \dots, a_r\}$ in the projective space $\text{PG}(r-1, \mathbb{R})$. Let l be a line of $\text{PG}(r-1, \mathbb{R})$ that is freely placed relative to A , and for all $i \in \{1, \dots, r\}$ let b_i be the point that is in both l and the hyperplane of $\text{PG}(r-1, \mathbb{R})$ spanned by $A - a_i$. Let $B = \{b_1, \dots, b_r\}$. We will use Θ_r to denote the restriction of $\text{PG}(r-1, \mathbb{R})$ to $A \cup B$.

Suppose that Θ'_r is an isomorphic copy of Θ_r with $\{a'_1, \dots, a'_r\} \cup B$ as its ground set. Assume also that the isomorphism from Θ_r to Θ'_r acts as the identity on B and takes a_i to a'_i for all $i \in \{1, \dots, r\}$. Let M be the generalized parallel connection of Θ_r and Θ'_r . That is, M is a matroid on the ground set $A \cup A' \cup B$ and the flats of M are exactly the sets F such that $F \cap (A \cup B)$ is a flat of Θ_r and $F \cap (A' \cup B)$ is a

flat of Θ'_r . Note that if $r = 3$ then M is isomorphic to $M(K_5 \setminus e)$, the matroid illustrated in Figure 1.

It is easy to see that Θ_r is self-dual and that $C = (A - a_1) \cup (A' - a'_1)$ is a circuit of M , and hence a cocircuit of M^* . Moreover M^*/x has an isomorphic copy of Θ_r as a minor for every element $x \in C$. We note that every three-element subset of A is a circuit of M^* . Thus $A - x$ is a parallel class of M^*/x for every $x \in C \cap A$. However the simplification of M^*/x contains a unique series pair, and is therefore not 3-connected. On the other hand $\text{co}(\text{si}(M^*/x))$ is 3-connected, and has a minor isomorphic to Θ_r .

The structure described in the last example has been discovered before. The matroid Θ_r is a fundamental object in the generalized Δ - Y operation of Oxley, Semple, and Vertigan [7]. Furthermore this construction is an example of a ‘crocodile’, as described by Hall, Oxley, and Semple [4].

To see that statement (iii) of Theorem 1.1 is necessary consider the graph G shown in Figure 2. Let C^* be the cocircuit of $M = M(G)$ comprising the edges incident with the vertex a . It is easy to see that if x is any edge between a and a vertex in $\{b, c, d, e, f\}$ then M/x has a minor isomorphic to $M(K_6)$, and that these are the only edges in C^* with this property. But in this case neither $\text{si}(M/x)$ nor $\text{co}(\text{si}(M/x))$ is 3-connected. On the other hand, if we let x_1 be the edge ad , x_2 be cd , x_3 be ac , and x_4 be bc , then (x_1, x_2, x_3, x_4) is a sequence of the type described in statement (iii) of Theorem 1.1.

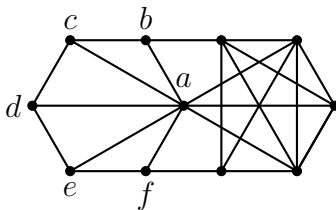


FIGURE 2. The graph G .

Our main result shows that there are essentially only two structures that prevent us from finding an element $x \in C^*$ such that $\text{si}(M/x)$ is 3-connected with an N -minor. These structures are named ‘segment-cosegment pairs’ and ‘four-element fans’. The dual of the matroid in Figure 1 contains a segment-cosegment pair, and the graph in Figure 2 contains a four-element fan. Before describing our result in detail we fix some terminology. Suppose that M is a matroid. Recall that a *triangle* of M is a three-element circuit, and a *triad* is a three-element

cocircuit. A *four-element fan* of M is a sequence (x_1, x_2, x_3, x_4) of distinct elements from $E(M)$ such that $\{x_1, x_2, x_3\}$ is a triangle and $\{x_2, x_3, x_4\}$ is a triad. A *segment* of M is a set L such that $|L| \geq 3$ and every three-element subset of M is a triangle, and a *cosegment* of M is a segment of M^* . We say that (L, L^*) is a *segment-cosegment pair* if $L = \{x_1, \dots, x_t\}$ is a segment of M , and $L^* = \{y_1, \dots, y_t\}$ is a set such that $L \cap L^* = \emptyset$ and for every $x_i \in L$ the set $(\text{cl}(L) - x_i) \cup y_i$ is a cocircuit. Segment-cosegment pairs will be considered in detail in Section 3. A *spore* is a pair (P, s) such that P is a rank-one flat, and $P \cup s$ is a cocircuit. A matroid M is *3-connected up to a unique spore* if M contains a single spore (P, s) , and whenever (X, Y) is a k -separation of M for some $k < 3$ then either $X \subseteq P \cup s$ or $Y \subseteq P \cup s$. Theorem 1.1 follows from the next result. It gives a more detailed analysis of the structures we encounter.

Theorem 1.2. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N -minor for some $x_0 \in C^*$. Then either:*

- (i) *there is an element $x \in C^*$ such that $\text{si}(M/x)$ is 3-connected and has an N -minor;*
- (ii) *there is a four-element fan (x_1, x_2, x_3, x_4) of M such that $x_1, x_3 \in C^*$, and $\text{si}(M/x_2)$ is 3-connected with an N -minor;*
- (iii) *there is a segment-cosegment pair (L, L^*) such that $L \subseteq C^*$, and $\text{cl}(L) - L$ contains a single element e . In this case $e \notin C^*$ and $\text{si}(M/e)$ is 3-connected with an N -minor. Moreover $M/\text{cl}(L)$ is 3-connected with an N -minor, and if $x_i \in L$ then M/x_i is 3-connected up to a unique spore $(\text{cl}(L) - x_i, y_i)$; or,*
- (iv) *there is a segment-cosegment pair (L, L^*) such that L is a flat and $|L - C^*| \leq 1$. In this case M/L is 3-connected with an N -minor, and if $x_i \in L$ then M/x_i is 3-connected up to a unique spore $(L - x_i, y_i)$.*

We note that if (L, L^*) is a segment-cosegment pair of the matroid M , and $M/\text{cl}(L)$ has an N -minor, then $|E(M) - \text{cl}(L)| \geq 4$. Under these hypotheses Proposition 3.6 tells us that $M/\text{cl}(L)$ is isomorphic to $\text{co}(\text{si}(M/x_i))$ for any element $x_i \in L$. Therefore Theorem 1.1 does indeed follow from Theorem 1.2.

By dualizing we immediately obtain the following corollary of Theorem 1.1.

Theorem 1.3. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C is a circuit of M with the property that $M \setminus x_0$ has an N -minor for some $x_0 \in C$. Then either:*

- (i) there is an element $x \in C$ such that $\text{co}(M \setminus x)$ is 3-connected and has an N -minor;
- (ii) there is an element $x \in C$ such that $\text{si}(\text{co}(M \setminus x))$ is 3-connected and has an N -minor; or,
- (iii) there is a four-element fan (x_1, x_2, x_3, x_4) in M such that $x_2, x_4 \in C$, and $\text{co}(M \setminus x_3)$ is 3-connected with an N -minor.

We note that Lemos [5] has considered the situation that a 3-connected matroid M contains a circuit C with the property that $M \setminus x$ is not 3-connected for any element $x \in C$. He shows that in this case C meets at least two triads of M .

In Section 2 we introduce essential notions of matroid connectivity. Section 3 contains a detailed discussion of one of the structures we uncover: segment-cosegment pairs. In Section 4 we collect some preliminary lemmas, and in Section 5 we complete the proof of Theorem 1.2. Notation and terminology generally follows that of Oxley [6], except that the simple (respectively cosimple) matroid associated with the matroid M is denoted $\text{si}(M)$ (respectively $\text{co}(M)$). We consistently write z instead of $\{z\}$ for the set containing the single element z .

2. ESSENTIALS

This section collects some elementary results on matroid connectivity. Let M be a matroid on the ground set E . The *connectivity function* of M , denoted by λ_M (or λ when there is no ambiguity), takes subsets of E to $\mathbb{Z}^+ \cup \{0\}$. It is defined so that

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r(M)$$

for any subset $X \subseteq E$. Note that $\lambda(X) = \lambda(E - X)$ and $\lambda_{M^*}(X) = \lambda_M(X)$ for any subset $X \subseteq E$. It is well known, and easy to verify, that the connectivity function of M is submodular. That is, for all $X, Y \subseteq E$, the inequality

$$\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$$

is satisfied.

We say that a subset $X \subseteq E$ is *k-separating* or a *k-separator* of M if $\lambda(X) < k$, and we say that a partition $(X, E - X)$ is a *k-separation* of M if X is *k-separating* and $|X|, |E - X| \geq k$. A *k-separator* X or a *k-separation* $(X, E - X)$ is *exact* if $\lambda(X) = k - 1$. A matroid M is *n-connected* if M has no *k-separation* for any $k < n$. We define a *k-partition* of M to be a partition (X_1, X_2, \dots, X_n) of E such that X_i is *k-separating* for all $1 \leq i \leq n$. We say that the *k-partition* (X_1, X_2, \dots, X_n) is *exact* if each *k-separator* X_i is exact.

The next result is easy.

Proposition 2.1. *Let N be a minor of the matroid M and let X be a subset of $E(M)$. Then $\lambda_N(E(N) \cap X) \leq \lambda_M(X)$.*

Proposition 2.2. *Suppose that M is a matroid and that (X, Y, z) is a partition of $E(M)$. If $\lambda(X) = \lambda(Y)$ then z is in $\text{cl}(X) \cap \text{cl}(Y)$ or in $\text{cl}^*(X) \cap \text{cl}^*(Y)$, but not both.*

Proof. Since

$$\lambda(X) = r(X) + r(Y \cup z) - r(M) = r(X \cup z) + r(Y) - r(M) = \lambda(Y)$$

it follows that $r(Y \cup z) - r(Y) = r(X \cup z) - r(X)$. Therefore, $z \in \text{cl}(X)$ if and only if $z \in \text{cl}(Y)$. In the case that $z \notin \text{cl}(X)$ and $z \notin \text{cl}(Y)$ then

$$\begin{aligned} r^*(Y \cup z) - r^*(Y) &= (|Y \cup z| + r(X) - r(M)) \\ &\quad - (|Y| + r(X \cup z) - r(M)) = 1 + r(X) - r(X \cup z) = 0. \end{aligned}$$

Thus $z \in \text{cl}^*(Y)$. The same argument shows that $z \in \text{cl}^*(X)$.

Finally we note that $z \in \text{cl}^*(X)$ if and only if $z \notin \text{cl}(Y)$. Thus $\text{cl}(X) \cap \text{cl}(Y)$ and $\text{cl}^*(X) \cap \text{cl}^*(Y)$ are disjoint. \square

The next result is well known, and follows without difficulty from the dual of [8, Lemma 2.5].

Proposition 2.3. *Suppose that X is an exactly 3-separating set of the 3-connected matroid M . Suppose also that $A \subseteq E(M) - X$. If $|A| \geq 3$ and $A \subseteq \text{cl}^*(X)$ then A is a cosegment of M .*

Definition 2.4. Suppose that M is a matroid and that $x \in E(M)$. Let (X_1, X_2) be a partition of $E(M) - x$ such that there is a positive integer k with the property that:

- (i) $\lambda(X_1) = \lambda(X_2) = k - 1$;
- (ii) $r(X_1), r(X_2) \geq k$; and,
- (iii) $x \in \text{cl}(X_1) \cap \text{cl}(X_2)$.

In this case (X_1, X_2, x) is a *vertical k -partition* of M .

The next result is well known and easy to prove.

Proposition 2.5. *Let M be a 3-connected matroid and suppose that $\text{si}(M/x)$ is not 3-connected for some $x \in E(M)$. Then there exists a vertical 3-partition (X_1, X_2, x) of M .*

Proposition 2.6. *Suppose that (X_1, X_2, x) is vertical k -partition of the k -connected matroid M . Let A be a subset of $\text{cl}(X_2 \cup x)$. Then $(X_1 - A, (X_2 \cup A) - x, x)$ is also a vertical k -partition of M .*

Proof. Suppose that z is some element in $X_1 \cap A$. Then $\lambda(X_1 - z)$ is either $k - 2$ or $k - 1$. If $\lambda(X_1 - z) = k - 2$ then $(X_1 - z, X_2 \cup \{x, z\})$ is a $(k - 1)$ -separation of M , a contradiction. Hence $\lambda(X_1 - z) = k - 1$ which implies that $r(X_1 - z) = r(X_1)$. Thus $\text{cl}(X_1 - z) = \text{cl}(X_1)$, and hence $x \in \text{cl}(X_1 - z)$. It follows that $(X_1 - z, X_2 \cup z, x)$ is a vertical k -partition of M . By continuing to transfer elements in $X_1 \cap A$ from X_1 into X_2 we eventually conclude that $(X_2 - A, (X_2 \cup A) - x, x)$ is a vertical k -partition of M , as desired. \square

Suppose that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$. Then we can define the *parallel connection* of M_1 and M_2 , denoted by $P(M_1, M_2)$. The ground set of $P(M_1, M_2)$ is $E(M_1) \cup E(M_2)$. If p is a loop in neither M_1 nor M_2 then the circuits of $P(M_1, M_2)$ are exactly the circuits of M_1 , the circuits of M_2 , and sets of the form $(C_1 - p) \cup (C_2 - p)$, where C_i is a circuit of M_i such that $p \in C_i$ for $i = 1, 2$. If p is a loop in M_1 then $P(M_1, M_2)$ is defined to be the direct sum of M_1 and M_2/p . Similarly, if p is a loop in M_2 then $P(M_1, M_2)$ is defined to be the direct sum of M_1/p and M_2 . We say that p is the *basepoint* of the parallel connection. It is clear that $P(M_1, M_2) = P(M_2, M_1)$.

The next result follows from [6, Proposition 7.1.15 (v)].

Proposition 2.7. *Suppose that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$. If $e \in E(M_1) - p$ then $P(M_1, M_2) \setminus e = P(M_1 \setminus e, M_2)$ and $P(M_1, M_2)/e = P(M_1/e, M_2)$.*

Assume that M_1 and M_2 are matroids such that $E(M_1) \cap E(M_2) = \{p\}$. If p is not a loop or a coloop in either M_1 or M_2 then $P(M_1, M_2) \setminus p$ is the 2-sum of M_1 and M_2 , denoted by $M_1 \oplus_2 M_2$. We say that p is the *basepoint* of the 2-sum.

The next result follows from [10, (2.6)].

Proposition 2.8. *If (X_1, X_2) is an exact 2-separation of a matroid M then there exist matroids M_1 and M_2 on the ground sets $X_1 \cup p$ and $X_2 \cup p$ respectively, where p is in neither X_1 nor X_2 , such that M is equal to $M_1 \oplus_2 M_2$.*

Proposition 2.9. *Suppose that N is a 3-connected matroid. Let M be a matroid with a vertical 3-partition (X_1, X_2, x) such that N is a minor of M/x . Then either $|E(N) \cap X_1| \leq 1$, or $|E(N) \cap X_2| \leq 1$.*

Proof. Since (X_1, X_2) is a 2-separation of M/x the result follows immediately from Proposition 2.1. \square

Lemma 2.10. *Suppose that N is a 3-connected matroid such that $|E(N)| \geq 2$. Let M be a matroid with a vertical 3-partition (X_1, X_2, x)*

such that N is a minor of M/x . If $|E(N) \cap X_1| \leq 1$ then $M/x/e$ has an N -minor for every element $e \in X_1 - \text{cl}_M(X_2)$.

Proof. Since (X_1, X_2) is an exact 2-separation of M/x , it follows from Proposition 2.8 that M/x is the 2-sum of matroids M_1 and M_2 along the basepoint p , where $E(M_1) = X_1 \cup p$ and $E(M_2) = X_2 \cup p$. Thus $M/x = P(M_1, M_2) \setminus p$.

Suppose that $E(N) \cap X_1 = \emptyset$. Then there is a partition (A, B) of X_1 such that N is a minor of $M/x/A \setminus B$. Suppose that p is a loop in $M_1/A \setminus B$. Proposition 2.7 implies that

$$M/x/A \setminus B = P(M_1/A \setminus B, M_2) \setminus p.$$

Now the definition of parallel connection implies that $M/x/A \setminus B$ is isomorphic to M_2/p . It is easily seen that if $e \in X_1$ then there is a minor M' of M_1/e such that $E(M') = \{p\}$ and p is a loop of M' . Proposition 2.7 implies that $P(M', M_2) \setminus p$ is a minor of $M/x/e$. But $P(M', M_2) \setminus p$ is isomorphic to M_2/p , so $M/x/e$ has an N -minor.

Next we suppose that p is a coloop of $M_1/A \setminus B$. Then, by definition of the parallel connection, $M/x/A \setminus B$ is isomorphic to $M_2 \setminus p$. Suppose that $e \in X_1 - \text{cl}(X_2)$. Since p is not a coloop of M_2 it follows easily that $p \in \text{cl}_M(X_2)$. Thus e is not parallel to p in M_1 . Therefore there is a minor M' of M_1/e such that $E(M') = \{p\}$ and p is a coloop of M' . Again using Proposition 2.7 we see that $P(M', M_2) \setminus p$ is a minor of $M/x/e$. But since $P(M', M_2) \setminus p$ is isomorphic to $M_2 \setminus p$ we deduce that $M/x/e$ has an N -minor.

Now we assume that $|E(N) \cap X_1| = 1$ and that z is the unique element in $E(N) \cap X_1$. There is a partition (A, B) of $X_1 - z$ such that N is a minor of $M/x/A \setminus B$. It follows from Proposition 2.7 that $P(M_1/A \setminus B, M_2) \setminus p$ has an N -minor. Consider the matroid $M_1/A \setminus B$. If $\{z, p\}$ is not a parallel pair in this matroid then z must be a loop or coloop in $P(M_1/A \setminus B, M_2) \setminus p$. This implies that z is a loop or coloop in N , a contradiction as N is 3-connected and $|E(N)| \geq 2$. Therefore z and p are parallel in $M_1/A \setminus B$, and therefore $P(M_1/A \setminus B, M_2) \setminus p$ is isomorphic to M_2 . Thus M_2 has an N -minor.

Since p is not a loop or coloop of M_1 there is a circuit of size at least two in M_1 that contains p . Suppose that $e \in X_1 - \text{cl}_M(X_2)$. Then e cannot be parallel to p in M_1 , so M_1/e has a circuit of size at least two that contains p . Hence there is a minor M' of M_1/e such that $p \in E(M')$ and M' consists of a parallel pair. Proposition 2.7 implies that $P(M', M_2) \setminus p$ is a minor of $M/x/e$. But $P(M', M_2) \setminus p$ is isomorphic to M_2 , so $M/x/e$ has an N -minor. \square

Definition 2.11. Suppose that M is a matroid and that A and B are subsets of $E(M)$. The *local connectivity* between A and B , denoted by $\sqcap(A, B)$, is defined to be $r(A) + r(B) - r(A \cup B)$. Equivalently, $\sqcap(A, B)$ is equal to $\lambda_{M|(A \cup B)}(A)$.

Proposition 2.12. [8, Lemma 2.4 (iv)] *Let M be a matroid and let (A, B, C) be a partition of $E(M)$. Then $\sqcap(A, B) + \lambda(C) = \sqcap(A, C) + \lambda(B)$. Hence $\sqcap(A, B) = \sqcap(A, C)$ if and only if $\lambda(B) = \lambda(C)$.*

Corollary 2.13. *Let (X, Y, Z) be an exact 3-partition of the 3-connected matroid M . Then $\sqcap(X, Y) = \sqcap(X, Z) = \sqcap(Y, Z)$.*

Proposition 2.14. *Suppose that M is a matroid and that X and Y are disjoint subsets of $E(M)$ such that $\sqcap(X, Y) = 1$. If $x, y \in X \cap \text{cl}(Y)$ then $r(\{x, y\}) \leq 1$.*

Proof. Assume that $r(\{x, y\}) = 2$. Let $X' = \text{cl}(X)$ and $Y' = \text{cl}(Y)$. It is easy to see that $r(X' \cup Y') = r(X \cup Y)$. However

$$r(X' \cup Y') \leq r(X') + r(Y') - r(X' \cap Y') \leq r(X) + r(Y) - 2 = r(X \cup Y) - 1.$$

This contradiction completes the proof. \square

We conclude this section by stating a fundamental tool in the study of 3-connected matroids, due to Bixby [1].

Theorem 2.15 (Bixby's Lemma). *Let M be a 3-connected matroid and suppose that x is an element of $E(M)$. Then either $\text{si}(M/x)$ or $\text{co}(M \setminus x)$ is 3-connected.*

3. SEGMENT-COSEGMENT PAIRS

Suppose that M is a matroid. Recall that L is a segment of M if $|L| \geq 3$ and every three-element subset of L is a circuit of M , and that L^* is a cosegment of M if $|L^*| \geq 3$ and every three-element subset of L^* is a cocircuit. We restate the definition of segment-cosegment pairs given in Section 1.

Definition 3.1. Suppose that $L = \{x_1, \dots, x_t\}$ is a segment of the matroid M and there is a set $L^* = \{y_1, \dots, y_t\}$ with the property that $L \cap L^* = \emptyset$ and $(\text{cl}(L) - x_i) \cup y_i$ is a cocircuit of M for all $i \in \{1, \dots, t\}$. In this case we say that (L, L^*) is a *segment-cosegment pair* of M .

In a 3-connected matroid a segment-cosegment pair is an example of a ‘crocodile’, a structure that provides a collection of equivalent 3-separations. ‘Crocodiles’ were considered by Hall, Oxley, and Semple [4]. The next result explains the name segment-cosegment pair.

Proposition 3.2. *Suppose that (L, L^*) is a segment-cosegment pair of the 3-connected matroid M . Then L^* is a cosegment of M .*

Proof. Suppose that $y_i \in L^*$. The definition of a segment-cosegment pair means that $y_i \in \text{cl}^*(\text{cl}(L))$. Thus $L^* \subseteq \text{cl}^*(\text{cl}(L))$. Moreover $\text{cl}(L)$ is exactly 3-separating in M . The result follows by Proposition 2.3. \square

Proposition 3.3. *Suppose that (L, L^*) is a segment-cosegment pair of the 3-connected matroid M . Then $M/\text{cl}(L)$ is 3-connected.*

Proof. Suppose that $L = \{x_1, \dots, x_t\}$ and $L^* = \{y_1, \dots, y_t\}$. Assume that $M/\text{cl}(L)$ is not 3-connected, so that (X_1, X_2) is a k -separation of $M/\text{cl}(L)$ for some $k \leq 2$. Let $L_0 = \text{cl}(L)$. Note that for $i \in \{1, 2\}$ we have

$$r_{M/L_0}(X_i) = r_M(X_i \cup L_0) - r_M(L_0) = r_M(X_i) - \square_M(X_i, L_0),$$

so $r_M(X_i) = r_{M/L_0}(X_i) + \square_M(X_i, L_0)$.

Suppose that $\square_M(X_1, L_0) = 0$. Then $r_M(X_1) = r_{M/L_0}(X_1)$ and $r_M(X_2 \cup L_0) = r_{M/L_0}(X_2) + 2$, so

$$\begin{aligned} \lambda_M(X_1) &= r_{M/L_0}(X_1) + (r_{M/L_0}(X_2) + 2) - (r(M/L_0) + 2) \\ &= \lambda_{M/L_0}(X_1) < k. \end{aligned}$$

This is a contradiction as M is 3-connected. By using a symmetric argument we can conclude that $\square_M(X_i, L_0) > 0$ for all $i \in \{1, 2\}$.

Suppose that $x_i \in \text{cl}_M(X_1)$ for some $i \in \{1, \dots, t\}$. Then there is a circuit $C_1 \subseteq X_1 \cup x_i$ such that $x_i \in C_1$. For all $k \in \{1, \dots, t\} - i$ the set $(L_0 - x_k) \cup y_k$ is a cocircuit. It cannot be the case that C_1 meets this cocircuit in a single element, so $y_k \in X_1$ for all $k \in \{1, \dots, t\} - i$.

Now suppose that $x_j \in \text{cl}_M(X_2)$ for some $j \in \{1, \dots, t\}$. By using the same arguments as above we can conclude that $L^* - y_j \subseteq X_2$. As $L^* - y_i$ and $L^* - y_j$ have a non-empty intersection this is a contradiction. Therefore $\text{cl}_M(X_2) \cap L = \emptyset$. Note that $\square(X_2, L_0) \leq 2$ because $r(L_0) = 2$. If $\square(X_2, L_0)$ were two, it would follow that $L_0 \subseteq \text{cl}(X_2)$. Hence $\square(X_2, L_0) = 1$.

Let j be an element of $\{1, \dots, t\} - i$. Then $L_0 \subseteq \text{cl}_M(X_2 \cup x_j)$, and there must be a circuit $C_2 \subseteq X_2 \cup \{x_i, x_j\}$ such that $\{x_i, x_j\} \subseteq C_2$. But then C_2 meets the cocircuit $(L_0 - x_j) \cup y_j$ in a single element, x_i . From this contradiction we conclude that $\text{cl}_M(X_1) \cap L = \emptyset$, and by symmetry $\text{cl}_M(X_2) \cap L = \emptyset$. This means that

$$\square_M(X_1, L_0) = \square_M(X_2, L_0) = 1.$$

It must be the case that $x_2 \in \text{cl}_M(X_1 \cup x_1)$, and there is a circuit $C_3 \subseteq X_1 \cup \{x_1, x_2\}$ such that $\{x_1, x_2\} \subseteq C_3$. Since $(L_0 - x_1) \cup y_1$ is a cocircuit we conclude that $y_1 \in X_1$. But we can use an identical

argument to show that $y_1 \in X_2$. This contradiction completes the proof. \square

We now restate the definition of a spore.

Definition 3.4. Suppose that P is a rank-one flat of a matroid M and that s is an element of $E(M)$ such that $P \cup s$ is a cocircuit. Then we say that (P, s) is a *spore*.

Recall from Section 1 that a matroid M is 3-connected up to a unique spore if it contains a single spore (P, s) , and whenever (X, Y) is a k -separation of M for some $k < 3$ then either $X \subseteq P \cup s$ or $Y \subseteq P \cup s$.

Lemma 3.5. *Suppose that (L, L^*) is a segment-cosegment pair of the 3-connected matroid M where $|E(M) - \text{cl}(L)| \geq 4$. Let $L = \{x_1, \dots, x_t\}$ and $L^* = \{y_1, \dots, y_t\}$. Then M/x_i is 3-connected up to a unique spore $(\text{cl}(L) - x_i, y_i)$, for all $i \in \{1, \dots, t\}$.*

Proof. Let E be the ground set of M and let $L_0 = \text{cl}(L)$. We will show that M/x_i is 3-connected up to the unique spore $(L_0 - x_i, y_i)$. Certainly $(L_0 - x_i, y_i)$ is a spore of M/x_i . Suppose that (P, s) is a spore of M/x_i that is distinct from $(L_0 - x_i, y_i)$.

We initially assume that $L_0 - x_i = P$. Thus $s \neq y_i$. As $(L_0 - x_i) \cup s$ and $(L_0 - x_i) \cup y_i$ are both cocircuits of M/x_i it follows that $E - (L_0 \cup \{s, y_i\})$ is the intersection of two hyperplanes of M/x_i . Thus

$$r_{M/x_i}(E - (L_0 \cup \{s, y_i\})) \leq r(M/x_i) - 2.$$

and therefore

$$r_{M/L_0}(E - (L_0 \cup \{s, y_i\})) \leq r(M/x_i) - 2 = r(M/L_0) - 1.$$

Hence $\{s, y_i\}$ contains a cocircuit in M/L_0 . Therefore M/L_0 contains a cocircuit of size at most two, a contradiction as M/L_0 is 3-connected by Proposition 3.3, and $|E(M/L_0)| \geq 4$.

Now we must assume that $L_0 - x_i \neq P$. Hence $P \cup x_i$ is a rank-two flat of M that meets L_0 in exactly one element, x_i . Suppose that P contains a single element p . Then $\{p, s\}$ is a cocircuit of M , a contradiction. Therefore $P \cup x_i$ contains at least one triangle. Suppose that P does not contain y_j , where $j \neq i$. Then there is a triangle in $P \cup x_i$ that meets the cocircuit $(L_0 - x_j) \cup y_j$ in exactly one element, x_i . This contradiction shows that $L^* - y_i \subseteq P$.

Assume that $t > 3$. As L^* is a cosegment there is a triad of M contained in $L^* - y_i$. However this triad is also contained in the segment $P \cup x_i$, and is therefore a triangle. But $|E(M)| > 4$ and a 3-connected matroid with at least five elements cannot contain a triangle that is also a triad. This contradiction shows that $t = 3$.

Suppose $j \in \{1, 2, 3\}$ and that $j \neq i$. If $|P| > 2$ then there is a triangle contained in P that contains y_j . However this triangle would meet the cocircuit $(L_0 - x_j) \cup y_j$ in exactly one element. Thus $|P| = 2$, and $P = L^* - y_i$.

Suppose that $j, k \in \{1, 2, 3\}$ and neither j nor k is equal to i . Then $L_0 \cup P$ contains the two cocircuits $(L_0 - x_j) \cup y_j$ and $(L_0 - x_k) \cup y_k$. Hence $r_M(E - (L_0 \cup P)) \leq r(M) - 2$. However it is easy to see that $r_M(L_0 \cup P) = 3$. As $|P| = 2$ it follows that $E - (L_0 \cup P)$ contains at least two elements. Thus $(L_0 \cup P, E - (L_0 \cup P))$ is a 2-separation of M , a contradiction.

We have shown that $(L_0 - x_i, y_i)$ is the unique spore of M/x_i . Next we show that M/x_i is 3-connected up to this spore. Suppose that (X, Y) is a k -separation of M/x_i for some $k < 3$. By relabeling if necessary we will assume that $y_i \in X$. Assume that the result is false, so that neither X nor Y is contained in $(L_0 - x_i) \cup y_i$. Therefore X contains at least one element from $E - (L_0 \cup y_i)$. As M/L_0 is 3-connected by Proposition 3.3 we deduce from Proposition 2.1 that either $X - L_0$ or $Y - L_0$ contains at most one element. We have already concluded that $X - L_0$ contains at least two elements (as $y_i \in X$), so $Y - L_0$ contains precisely one element. As M is 3-connected it contains no parallel pairs, so M/x_i contains no loops. Therefore $r_{M/x_i}(Y) = 2$, and hence $r_{M/x_i}(X) \leq r(M/x_i) - 1$. Thus Y contains a cocircuit of M/x_i . As M/x_i has no coloops, and any cocircuit that meets a parallel class contains that parallel class it follows that $L_0 - x_i \subseteq Y$. Let s be the single element in $Y - L_0$. It cannot be the case that Y is a cocircuit in M/x_i , for that would imply that $(L_0 - x_i, s)$ is a spore of M/x_i that differs from $(L_0 - x_i, y_i)$, contradicting our earlier conclusion. Now we see that $Y - s = L_0 - x_i$ must be a cocircuit of M/x_i , but this is a contradiction as $L_0 - x_i$ is properly contained in the cocircuit $(L_0 - x_i) \cup y_i$. This completes the proof. \square

The next result shows that Theorem 1.1 is a consequence of Theorem 1.2.

Proposition 3.6. *Suppose that (L, L^*) is a segment-cosegment pair of a matroid M , and that $M/\text{cl}(L)$ is 3-connected and $|E(M) - \text{cl}(L)| \geq 4$. Let $L = \{x_1, \dots, x_t\}$ and $L^* = \{y_1, \dots, y_t\}$. Then $\text{co}(\text{si}(M/x_i)) \cong M/\text{cl}(L)$ for any element $x_i \in L$.*

Proof. Let $L_0 = \text{cl}(L)$ and let $x_j \neq x_i$ be an element of L . Suppose that P and S are disjoint subsets of $E(M) - x_i$ chosen so that $\text{co}(\text{si}(M/x_i)) \cong M/x_i \setminus P/S$. As $L_0 - x_i$ is a parallel class in M/x_i we may assume that $L_0 - \{x_i, x_j\} \subseteq P$ and that $x_j \notin P$. We may assume that $y_i \notin P$,

and hence $\{x_j, y_i\}$ is a union of cocircuits in $M/x_i \setminus P$. Therefore we may assume $x_j \in S$. Since the elements in $L_0 - \{x_i, x_j\}$ are loops in $M/x_i/x_j$ it follows that

$$M/x_i \setminus P/S = M/x_i/x_j/(L_0 - \{x_i, x_j\}) \setminus (P - (L_0 - \{x_i, x_j\})) / (S - x_j).$$

This last matroid is equal to $M/L_0 \setminus (P - (L_0 - \{x_i, x_j\})) / (S - x_j)$. Since M/L_0 is 3-connected and the elements in $P - (L_0 - \{x_i, x_j\})$ are either loops or parallel elements in M/L_0 it follows that $P = L_0 - \{x_i, x_j\}$. Thus $M/x_i \setminus P/S = M/L_0 / (S - x_j)$. But M/L_0 is 3-connected, so $S - x_j$ must be empty. Thus $M/L_0 \cong \text{co}(\text{si}(M/x_i))$, as desired. \square

4. PRELIMINARY LEMMAS

Proposition 4.1. *Suppose that C^* is a cocircuit of the 3-connected matroid M . Assume that (X_1, X_2, x) is a vertical 3-partition of M such that $x \in C^*$. Then $C^* \cap (X_1 - \text{cl}(X_2)) \neq \emptyset$ and $C^* \cap (X_2 - \text{cl}(X_1)) \neq \emptyset$.*

Proof. Note that $r(X_1), r(X_2) \geq 3$ implies that $|E(M)| \geq 4$, so every circuit and cocircuit of M contains at least three elements. Let X be $X_1 - \text{cl}(X_2)$. The fact that $r(X_1) \geq 3$ implies that X contains a cocircuit, so $|X| \geq 3$. Suppose that x is not in $\text{cl}(X)$. Then $r(X) < r(X_1)$. Since $|X| \geq 3$ this implies that $(X, \text{cl}(X_2))$ is a 2-separation of M , a contradiction.

Now suppose that $C^* \subseteq \text{cl}(X_2)$. Then as $x \in \text{cl}(X)$ and $x \in C^*$ there is a circuit in M that meets C^* in exactly one element, x . This is a contradiction. The same argument shows that $C^* \cap (X_2 - \text{cl}(X_1)) \neq \emptyset$, so the proposition holds. \square

Definition 4.2. Suppose that M is a 3-connected matroid and that A is a subset of $E(M)$. A *minimal partition* with respect to A is a vertical 3-partition (X_1, X_2, x) of M that satisfies the following properties:

- (i) $x \in A$;
- (ii) if (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap (X_1 \cup x)$ and $X_2 \cap Y_1 = \emptyset$, then $(Y_1, Y_2, y) = (X_1, X_2, x)$; and,
- (iii) if (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap (X_1 \cup x)$ and $X_2 \cap Y_2 = \emptyset$ then $(Y_2, Y_1, y) = (X_1, X_2, x)$.

If there is no ambiguity we will refer to a minimal partition with respect to A as a *minimal partition*.

Lemma 4.3. *Suppose that M is a 3-connected matroid and that A is a subset of $E(M)$. Suppose that for some element $z \in A$ there is a vertical 3-partition (Z_1, Z_2, z) of M . Let $Z = Z_1 - \text{cl}(Z_2)$. Then there is a minimal partition (X_1, X_2, x) with respect to A such that $X_1 \subseteq Z$ and $x \in A \cap (Z \cup z)$.*

Proof. Let \mathcal{Z} be the family of vertical 3-partitions (S_1, S_2, z) with the property that $S_1 \subseteq Z_1$. Choose (Z'_1, Z'_2, z) from \mathcal{Z} so that if (S_1, S_2, z) is in \mathcal{Z} , then S_1 is not properly contained in Z'_1 . Observe that Proposition 2.6 implies that $Z'_1 \subseteq Z$.

Let \mathcal{S} be the family of vertical 3-partitions (S_1, S_2, s) with $s \in A \cap (Z'_1 \cup z)$. Let \mathcal{S}_0 be the set of vertical 3-partitions (S_1, S_2, s) in \mathcal{S} with the property that either $S_1 \subseteq Z'_1$ or $S_2 \subseteq Z'_1$. Without loss of generality we will assume that if (S_1, S_2, s) is in \mathcal{S}_0 then $S_1 \subseteq Z'_1$. Suppose that (S_1, S_2, z) is a member of \mathcal{S}_0 . Then our choice of (Z'_1, Z'_2, z) means that $S_1 = Z'_1$ and $S_2 = Z'_2$. If (Z'_1, Z'_2, z) is the only member of \mathcal{S}_0 then we can set (X_1, X_2, x) to be (Z'_1, Z'_2, z) , and we will be done. Therefore we will assume that there is at least one vertical 3-partition (S_1, S_2, s) in \mathcal{S}_0 such that $s \neq z$. Let \mathcal{S}_1 be the collection of such partitions.

We now let (X_1, X_2, x) be a vertical 3-partition in \mathcal{S}_1 chosen so that if $(S_1, S_2, s) \in \mathcal{S}_1$, then $S_1 \cup s$ is not properly contained in $X_1 \cup x$. We will prove that (X_1, X_2, x) is the desired vertical 3-partition.

It is certainly true that $X_1 \subseteq Z$. If there is some element e in $X_1 \cap \text{cl}(X_2 \cup x)$ then $(X_1 - e, X_2 \cup e, x)$ is a vertical 3-partition by Proposition 2.6. However this contradicts our choice of (X_1, X_2, x) . Therefore $X_2 \cup x$ is a flat. We assume that (Y_1, Y_2, y) is a vertical 3-partition and that $y \in A \cap (X_1 \cup x)$. As $X_1 \subseteq Z'_1$ it follows that $y \in A \cap Z'_1$. Our assumption on (X_1, X_2, x) means that neither $Y_1 \cup y$ nor $Y_2 \cup y$ can be properly contained in $X_1 \cup x$.

Suppose that $X_2 \cap Y_1 = \emptyset$. Then $Y_1 \cup y$ must be equal to $X_1 \cup x$. If $y \neq x$ then the fact that $y \in \text{cl}(Y_2)$ and $Y_2 = X_2$ means that $y \in \text{cl}(X_2)$, which is a contradiction as $X_2 \cup x$ is a flat. Therefore $y = x$, so (Y_1, Y_2, y) is equal to (X_1, X_2, x) . The same argument shows that if $X_2 \cap Y_2 = \emptyset$ then $(Y_1, Y_2, y) = (X_2, X_1, x)$. Thus (X_1, X_2, x) is the desired minimal partition. \square

Proposition 4.4. *Suppose that M is a matroid and that $A \subseteq E(M)$. Suppose that (X_1, X_2, x) is a minimal partition with respect to A . Then $X_2 \cup x$ is a flat of M .*

Proof. Suppose that there is some element $z \in X_1 \cap \text{cl}(X_2 \cup x)$. Then $(X_1 - z, X_2 \cup z, x)$ is a vertical 3-partition of M by Proposition 2.6. This contradicts the fact that (X_1, X_2, x) is a minimal partition. \square

Lemma 4.5. *Suppose that M is a 3-connected matroid and that $A \subseteq E(M)$. Suppose that (X_1, X_2, x) is a minimal partition with respect to A . Suppose also that (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap X_1$ and $x \in Y_1$. Then the following statements hold:*

- (i) $X_i \cap Y_j \neq \emptyset$ for all $i, j \in \{1, 2\}$;
- (ii) Each of $X_1 \cap Y_2$, $(X_1 \cap Y_2) \cup y$, $X_2 \cap Y_1$, $(X_2 \cap Y_1) \cup x$, and $X_2 \cap Y_2$ is 3-separating in M ;
- (iii) $(X_1 \cap Y_1) \cup \{x, y\}$ is 4-separating in M ;
- (iv) Neither $X_1 \cap Y_1$ nor $X_1 \cap Y_2$ is contained in $\text{cl}(X_2)$, $X_1 \cap Y_1 \not\subseteq \text{cl}(Y_2)$, and $X_1 \cap Y_2 \not\subseteq \text{cl}(Y_1)$;
- (v) $r((X_1 \cap Y_2) \cup y) = 2$; and,
- (vi) If $(X_1 \cap Y_1) \cup \{x, y\}$ is 3-separating in M , then $r((X_1 \cap Y_1) \cup \{x, y\}) = 2$.

Proof. We start by proving (i). Since $y \neq x$ the definition of a minimal partition means that $X_2 \cap Y_1 \neq \emptyset$ and $X_2 \cap Y_2 \neq \emptyset$. Moreover $X_2 \cup x$ is a flat of M by Proposition 4.4, and $y \in X_1$, so $y \notin \text{cl}(X_2 \cup x)$. However $y \in \text{cl}(Y_1) \cap \text{cl}(Y_2)$. It follows that neither Y_1 nor Y_2 can be contained in $X_2 \cup x$. Thus both Y_1 and Y_2 meet X_1 .

Next we prove (ii). Consider $X_1 \cap Y_2$. Since $\lambda(X_1) = 2$ and $\lambda(Y_2) = 2$ the submodularity of the connectivity function implies that $\lambda(X_1 \cap Y_2) + \lambda(X_1 \cup Y_2) \leq 4$. If $X_1 \cap Y_2$ is not 3-separating then $\lambda(X_1 \cup Y_2) \leq 1$. However $|X_1 \cup Y_2| \geq 2$ and the complement of $X_1 \cup Y_2$ certainly contains at least two elements, since it contains x , and $X_2 \cap Y_1$ is non-empty. Thus M has a 2-separation, a contradiction. This shows that $X_1 \cap Y_2$ is 3-separating.

Since X_1 and $Y_2 \cup y$ are both 3-separating the same argument shows that $(X_1 \cap Y_2) \cup y$ is 3-separating. Since the complement of $X_2 \cup Y_1$ contains both y and at least one element in $X_1 \cap Y_2$, we can also show that $X_2 \cap Y_1$ and $(X_2 \cap Y_1) \cup x$ are both 3-separating. The same argument shows that $X_2 \cap Y_2$ is 3-separating.

Consider (iii). The submodularity of the connectivity function shows that

$$\lambda((X_1 \cap Y_1) \cup \{x, y\}) + \lambda(X_1 \cup Y_1) \leq 4.$$

Thus if $(X_1 \cap Y_1) \cup \{x, y\}$ is not 4-separating then $\lambda(X_1 \cup Y_1) = 0$. But this cannot occur as $X_1 \cup Y_1$ is non-empty, and its complement contains $X_2 \cap Y_2$, which is non-empty.

Next we move to (iv). Since $X_2 \cup x$ is a flat of M it follows that $\text{cl}(X_2)$ does not meet X_1 . Therefore $\text{cl}(X_2)$ cannot contain $X_1 \cap Y_1$ or $X_1 \cap Y_2$.

Suppose that $X_1 \cap Y_1$ is contained in $\text{cl}(Y_2)$. Then $Y_1 - \text{cl}(Y_2)$ is contained in $X_2 \cup x$. However Proposition 2.6 says that

$$(Y_1 - \text{cl}(Y_2), \text{cl}(Y_2) - y, y)$$

is a vertical 3-partition of M . Thus y is in the closure of $Y_1 - \text{cl}(Y_2)$, which means that $y \in \text{cl}(X_2 \cup x)$. But this is a contradiction as $y \in X_1$,

and $X_2 \cup x$ is a flat of M . The same argument shows that $X_1 \cap Y_2$ is not contained in $\text{cl}(Y_1)$.

To prove (v) we suppose that $r((X_1 \cap Y_2) \cup y) \geq 3$. Consider the partition $(X_1 \cap Y_2, X_2 \cup Y_1, y)$ of $E(M)$. It follows from (ii) that

$$\lambda((X_1 \cap Y_2) \cup y) = \lambda(X_1 \cap Y_2) = 2,$$

so $\lambda(X_2 \cup Y_1) = 2$. Furthermore $y \in \text{cl}(Y_1)$, so y is in the closure of $X_2 \cup Y_1$. Proposition 2.2 shows that $y \in \text{cl}(X_1 \cap Y_2)$, so $r(X_1 \cap Y_2) \geq 3$. Now it is easy to see that

$$(X_1 \cap Y_2, X_2 \cup Y_1, y)$$

is a vertical 3-partition of M . However $y \in A \cap X_1$ and $X_1 \cap Y_2$ does not meet X_2 , so we have a contradiction to the fact that (X_1, X_2, x) is a minimal partition.

We conclude by proving (vi). Suppose that $\lambda((X_1 \cap Y_1) \cup \{x, y\}) = 2$. This implies that $\lambda(X_2 \cup Y_2) = 2$. Since $y \in \text{cl}(Y_2)$ it follows easily that $\lambda((X_1 \cap Y_1) \cup x) = 2$. Consider the partition

$$((X_1 \cap Y_1) \cup x, X_2 \cup Y_2, y)$$

of $E(M)$. Since $y \in \text{cl}(Y_2)$ it follows from Proposition 2.2 that y is in the closure of $(X_1 \cap Y_1) \cup x$. Thus if $r((X_1 \cap Y_1) \cup \{x, y\}) \geq 3$ it follows that $r((X_1 \cap Y_1) \cup x) \geq 3$. In this case

$$((X_1 \cap Y_1) \cup x, X_2 \cup Y_2, y)$$

is vertical 3-partition of M that violates the fact that (X_1, X_2, x) is a minimal partition. This completes the proof of the lemma. \square

Proposition 4.6. *Suppose that (X_1, X_2, x) is a minimal partition of the 3-connected matroid M with respect to the set $A \subseteq E(M)$. Assume that (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap X_1$ and $x \in Y_1$. If $|X_1 \cap Y_2| \geq 2$ then*

$$\sqcap((X_1 \cap Y_1) \cup \{x, y\}, X_1 \cap Y_2) = \sqcap((X_1 \cap Y_1) \cup y, X_1 \cap Y_2) = 1.$$

Proof. The hypotheses imply that $|E(M)| \geq 4$, so every circuit or cocircuit of M contains at least three elements. Let $\pi = \sqcap((X_1 \cap Y_1) \cup \{x, y\}, X_1 \cap Y_2)$. We know from Lemma 4.5 (v) that $r(X_1 \cap Y_2) \leq 2$. Therefore $\pi \leq 2$. On the other hand, since $|X_1 \cap Y_2| \geq 2$, the fact that $r((X_1 \cap Y_2) \cup y) \leq 2$ implies that $y \in \text{cl}(X_1 \cap Y_2)$. This in turn implies that $\pi \geq 1$.

Assume that $\pi = 2$. Then $X_1 \cap Y_2 \subseteq \text{cl}((X_1 \cap Y_1) \cup \{x, y\})$. Since $x, y \in \text{cl}(Y_1)$ this means that $X_1 \cap Y_2 \subseteq \text{cl}(Y_1)$. But this contradicts (iv) of Lemma 4.5. Exactly the same argument shows that $\sqcap((X_1 \cap Y_1) \cup y, X_1 \cap Y_2) = 1$. \square

Lemma 4.7. *Suppose that (X_1, X_2, x) is a minimal partition of the 3-connected matroid M with respect to the set $A \subseteq E(M)$. Assume that (Y_1, Y_2, y) is a vertical 3-partition of M such that $y \in A \cap X_1$ and $x \in Y_1$. If $|X_1 \cap Y_2| \geq 2$ then $y \in \text{cl}((X_1 \cap Y_1) \cup x)$.*

Proof. The hypotheses imply that every circuit of M contains at least three elements. Since $|X_1 \cap Y_2| \geq 2$ it follows from Lemma 4.5 (v) implies that $y \in \text{cl}(X_1 \cap Y_2)$. We assume that $y \notin \text{cl}((X_1 \cap Y_1) \cup x)$. Since $X_1 \cap Y_1$ is non-empty by Lemma 4.5 (i) it follows that $|(X_1 \cap Y_1) \cup x| \geq 2$, so $\lambda((X_1 \cap Y_1) \cup x) \geq 2$. Furthermore $\lambda((X_1 \cap Y_1) \cup \{x, y\}) \leq 3$ by (iii) of Lemma 4.5. As $y \in \text{cl}(Y_2)$ we deduce that

$$2 \leq \lambda((X_1 \cap Y_1) \cup x) < \lambda((X_1 \cap Y_1) \cup \{x, y\}) \leq 3.$$

Thus $\lambda((X_1 \cap Y_1) \cup x) = 2$. Moreover it follows from (ii) in Lemma 4.5 that $\lambda((X_1 \cap Y_2) \cup y) = 2$. Therefore

$$((X_1 \cap Y_1) \cup x, (X_1 \cap Y_2) \cup y, X_2)$$

is an exact 3-partition.

As $x \in \text{cl}(X_2)$ it follows that $\Pi((X_1 \cap Y_1) \cup x, X_2) \geq 1$. Now Corollary 2.13 implies that $\Pi((X_1 \cap Y_2) \cup y, X_2) \geq 1$. But (iv) and (v) of Lemma 4.5 imply that $X_1 \cap Y_2 \not\subseteq \text{cl}(X_2)$ and that $r((X_1 \cap Y_2) \cup y) = 2$. We deduce that $\Pi((X_1 \cap Y_2) \cup y, X_2) = 1$. Again using Corollary 2.13 we see that

$$\Pi((X_1 \cap Y_1) \cup x, (X_1 \cap Y_2) \cup y) = 1.$$

Proposition 4.6 tells us that

$$\Pi((X_1 \cap Y_1) \cup \{x, y\}, X_1 \cap Y_2) = 1.$$

Since $y \in \text{cl}(X_1 \cap Y_2)$ we can easily deduce that $y \in \text{cl}((X_1 \cap Y_1) \cup x)$, contrary to our initial assumption. \square

Lemma 4.8. *Suppose that C^* is a cocircuit of the 3-connected matroid M . Suppose that (X_1, X_2, x) is a minimal partition of M with respect to C^* . Assume that $\text{si}(M/x_0)$ is not 3-connected for any element $x_0 \in C^* \cap X_1$. Let (Y_1, Y_2, y) be a vertical 3-partition of M such that $y \in C^* \cap X_1$, and assume that $x \in Y_1$. Then $|X_1 \cap Y_2| = 1$.*

Proof. The hypotheses of the lemma imply that every circuit and co-circuit of M contains at least three elements. Let us assume that the lemma fails, so that $|X_1 \cap Y_2| \geq 2$. Now (v) of Lemma 4.5 implies that $(X_1 \cap Y_2) \cup y$ contains a triangle of M that contains y . Since C^* meets this triangle in y , there must be an element $z \in X_1 \cap Y_2$ such that $z \in C^*$.

By assumption $\text{si}(M/z)$ is not 3-connected so Proposition 2.5 implies that there is vertical 3-partition (Z'_1, Z'_2, z) . Let us assume that $x \in Z'_1$.

Suppose that $y \in Z'_i$, where $\{i, j\} = \{1, 2\}$. Since $r((X_1 \cap Y_2) \cup y) = 2$ and $z \in \text{cl}(Z'_i)$ it follows that $(X_1 \cap Y_2) \cup y \subseteq \text{cl}(Z'_i)$, as $y \neq z$ and $z \in X_1 \cap Y_2$. Let $Z_i = Z'_i \cup (X_1 \cap Y_2) \cup y$ and let $Z_j = Z'_j - Z_i$. Then Proposition 2.6 implies that (Z_1, Z_2, z) is a vertical 3-partition. Note that $x \in Z_1$, whether i is equal to 1 or 2.

Suppose that $i = 2$. Then $(X_1 \cap Y_2) \cup y \subseteq Z_2 \cup z$. This means that $(X_1 \cap Z_1) \cup x \subseteq (X_1 \cap Y_1) \cup \{x, y\}$. Lemma 4.7 says that $z \in \text{cl}((X_1 \cap Z_1) \cup x)$. Therefore $z \in \text{cl}((X_1 \cap Y_1) \cup \{x, y\})$. But since $\{y, z\}$ spans $(X_1 \cap Y_2) \cup y$ this implies that $(X_1 \cap Y_1) \cup \{x, y\}$ spans $X_1 \cap Y_2$. As $x, y \in \text{cl}(Y_1)$ it now follows that Y_1 spans $X_1 \cap Y_2$, in contradiction to Lemma 4.5 (iv). Therefore $i = 1$, so $(X_1 \cap Y_2) \cup y \subseteq Z_1 \cup z$.

We conclude that $X_1 \cap Z_2 \subseteq (X_1 \cap Y_1) \cup \{x, y\}$. Suppose that $|X_1 \cap Z_2| \geq 2$. It follows from (v) of Lemma 4.5 that $r((X_1 \cap Z_2) \cup z) = 2$. Therefore z is in $\text{cl}(X_1 \cap Z_2)$, and hence in $\text{cl}((X_1 \cap Y_1) \cup \{x, y\})$. Exactly as before, we conclude that Y_1 spans $X_1 \cap Y_2$, a contradiction. Therefore $|X_1 \cap Z_2| \leq 1$.

As $r(Z_2) \geq 3$ we deduce that $|X_2 \cap Z_2| \geq 2$. But $\lambda(X_2 \cap Z_2) \leq 2$ by (ii) of Lemma 4.5, so it follows that $\lambda(X_2 \cap Z_2) = 2$, and hence $\lambda(X_1 \cup Z_1) = 2$. Now $\lambda(X_1 \cup x) + \lambda(Z_1 \cup z) = 4$, so the submodularity of the connectivity function implies that

$$\lambda((X_1 \cap Z_1) \cup \{x, z\}) + \lambda(X_1 \cup Z_1) \leq 4.$$

We now conclude that $\lambda((X_1 \cap Z_1) \cup \{x, z\}) \leq 2$. It follows from (vi) of Lemma 4.5 that $r((X_1 \cap Z_1) \cup \{x, z\}) = 2$.

We have already deduced that $(X_1 \cap Y_2) \cup y \subseteq Z_1 \cup z$, so $X_1 \cap Y_2 \subseteq (X_1 \cap Z_1) \cup z$. But $|X_1 \cap Y_2| \geq 2$, and $r((X_1 \cap Z_1) \cup \{x, z\}) = 2$. Therefore $x \in \text{cl}(X_1 \cap Y_2)$. We also know that $y \in \text{cl}(X_1 \cap Y_2)$. Proposition 4.6 asserts that

$$\cap((X_1 \cap Y_1) \cup \{x, y\}, X_1 \cap Y_2) = 1.$$

Since $x, y \in \text{cl}(X_1 \cap Y_2)$ it follows from Proposition 2.14 that $r(\{x, y\}) \leq 1$, a contradiction as M is 3-connected. This completes the proof of the lemma. \square

5. PROOF OF THE MAIN RESULT

We restate Theorem 1.2 here.

Theorem 5.1. *Suppose that M and N are 3-connected matroids such that $|E(N)| \geq 4$ and C^* is a cocircuit of M with the property that M/x_0 has an N -minor for some $x_0 \in C^*$. Then either:*

- (i) *there is an element $x \in C^*$ such that $\text{si}(M/x)$ is 3-connected and has an N -minor;*

- (ii) there is a four-element fan (x_1, x_2, x_3, x_4) of M such that $x_1, x_3 \in C^*$, and $\text{si}(M/x_2)$ is 3-connected with an N -minor;
- (iii) there is a segment-cosegment pair (L, L^*) such that $L \subseteq C^*$, and $\text{cl}(L) - L$ contains a single element e . In this case $e \notin C^*$ and $\text{si}(M/e)$ is 3-connected with an N -minor. Moreover $M/\text{cl}(L)$ is 3-connected with an N -minor, and if $x_i \in L$ then M/x_i is 3-connected up to a unique spore $(\text{cl}(L) - x_i, y_i)$; or,
- (iv) there is a segment-cosegment pair (L, L^*) such that L is a flat and $|L - C^*| \leq 1$. In this case M/L is 3-connected with an N -minor, and if $x_i \in L$ then M/x_i is 3-connected up to a unique spore $(L - x_i, y_i)$.

Proof. Assume that M is a counterexample to the theorem. Let x_0 be an element of C^* such that N is a minor of M/x_0 . By hypothesis $\text{si}(M/x_0)$ is not 3-connected, so Proposition 2.5 implies there is a vertical 3-partition (Z_1, Z_2, x_0) . It follows easily that $|E(M)| \geq 7$. By Proposition 2.9 we will assume, relabeling as necessary, that $|E(N) \cap Z_1| \leq 1$. Let $Z = Z_1 - \text{cl}(Z_2)$. Lemma 2.10 implies that M/e has an N -minor for every element $e \in Z$, and Lemma 4.3 implies that there is a minimal partition (X_1, X_2, x) with respect to C^* such that $x \in C^* \cap (Z \cup x_0)$, and $X_1 \subseteq Z$.

Proposition 4.1 implies that C^* has a non-empty intersection with $X_1 - \text{cl}(X_2)$. If $s \in C^* \cap (X_1 - \text{cl}(X_2))$ then $\text{si}(M/s)$ is not 3-connected by hypothesis. Therefore there is a vertical 3-partition (S_1, S_2, s) .

5.1.1. *Suppose that $s \in C^*$ is contained in $X_1 - \text{cl}(X_2)$ and that (S_1, S_2, s) is a vertical 3-partition such that $x \in S_1$. Then $|X_1 \cap S_1| \geq 2$ and $(X_1 \cap S_1) \cup \{s, x\}$ is a segment of M .*

Proof. Lemma 4.8 tells us that $|X_1 \cap S_2| = 1$. By Lemma 4.5 (i) we know that $|X_1 \cap S_1| \geq 1$. Assume that $|X_1 \cap S_1| = 1$. Then X_1 contains exactly three elements: the unique element in $X_1 \cap S_2$, the unique element in $X_1 \cap S_1$, and s . By the definition of a vertical 3-partition it follows that $r(X_1) = 3$ and that X_1 is a triad of M . As $x \in \text{cl}(X_1)$ it follows that there is a circuit $C \subseteq X_1 \cup x$ that contains x . It cannot be the case that the single element in $X_1 \cap S_2$ is in C , for that would imply that $X_1 \cap S_2 \subseteq \text{cl}(S_1)$, contradicting Lemma 4.5 (iv). As C does not meet the triad X_1 in a single element it follows that $(X_1 \cap S_1) \cup \{x, s\}$ is a triangle.

If we let x_2 be the unique element in $X_1 \cap S_1$, let x_4 be the unique element in $X_1 \cap S_2$, and let $x_1 = x$ and $x_3 = s$, then (x_1, x_2, x_3, x_4) is a four-element fan of M . If $\text{si}(M/x_2)$ is 3-connected then statement (ii)

of Theorem 5.1 holds, which is a contradiction as M is a counterexample to the theorem. Therefore we will assume that $\text{si}(M/x_2)$ is not 3-connected.

Since $\text{si}(M/x_3)$ is not 3-connected Theorem 2.15 asserts that $\text{co}(M \setminus x_3)$ is 3-connected. Assume that every triad of M that contains x_3 also contains x_2 . Then $\text{co}(M \setminus x_3) \cong M \setminus x_3/x_2$. However x_3 is contained in a parallel pair in M/x_2 , so $\text{si}(M/x_2)$ is obtained from $M \setminus x_3/x_2$ by possibly deleting parallel elements. As $M \setminus x_3/x_2$ is 3-connected it follows that $\text{si}(M/x_2)$ is 3-connected, contrary to hypothesis.

Therefore there is a triad T^* of M that contains x_3 but not x_2 . Now T^* cannot meet the triangle $\{x_1, x_2, x_3\}$ in exactly one element, and therefore $x_1 \in T^*$. Let y_2 be the unique element in $T^* - \{x_1, x_3\}$. Since every triad that contains x_3 must contain either x_1 or x_2 , and since both $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are contained in triads of M it follows that $\text{co}(M \setminus x_3) \cong M \setminus x_3/x_1/x_2$. Note that x_3 is a loop of $M/x_1/x_2$, so $M \setminus x_3/x_1/x_2 = M/x_3/x_1/x_2$.

As $\text{si}(M/x_3)$ is not 3-connected there is a vertical 3-partition (Z_1, Z_2, x_3) of M . By relabeling as necessary we may assume that $x_1 \in Z_2$. Hence $x_2 \in \text{cl}(Z_2 \cup x_3)$, so by Proposition 2.6 we may assume that $x_2 \in Z_2$. Now (Z_1, Z_2) is an exact 2-separation of M/x_3 , but $M/x_3/x_1/x_2$ is 3-connected. By Proposition 2.1 we see that $Z_2 - \{x_1, x_2\}$ must contain at most one element. If $Z_2 = \{x_1, x_2\}$ then $r(Z_2) \leq 2$, a contradiction. Therefore $Z_2 - \{x_1, x_2\}$ contains exactly one element. Let this element be y_3 . It is easy to see that Z_2 must be a triad of M .

We relabel x_4 with y_1 . Let $L = \{x_1, x_2, x_3\}$ and let $L^* = \{y_1, y_2, y_3\}$. Now L is a segment of M . Proposition 4.4 implies $X_2 \cup x_1$ is a hyperplane, and as $\{x_1, x_2, x_3\}$ is a triangle it is easy to see that $\cap(X_2 \cup x_1, \{x_2, x_3\}) = 1$. If there were some element e in $\text{cl}(L) - L$ then Proposition 2.14 would imply that $r(\{e, x_1\}) \leq 1$, a contradiction. Therefore L is a flat of M . Moreover $(L - x_i) \cup y_i$ is a cocircuit of M for all $i \in \{1, 2, 3\}$, so (L, L^*) is a segment-cosegment pair of M .

By applying Proposition 3.3 and Lemma 3.5 we see that M/L is 3-connected, and that M/x_i is 3-connected up to a unique spore $(L - x_i, y_i)$ for all $i \in \{1, 2, 3\}$. We know that M/x_3 has an N -minor. However $\{x_1, x_2\}$ is a parallel pair in M/x_3 , so $M/x_3 \setminus x_1$ has an N -minor. Furthermore $\{x_2, y_3\}$ is a series pair of $M/x_3 \setminus x_1$, so $M/x_3 \setminus x_1/x_2$, and hence M/L , has an N -minor. Thus statement (iv) of Theorem 5.1 holds, a contradiction. We conclude that $|X_1 \cap S_1| \geq 2$.

Since $\lambda(X_1 \cup x) = \lambda(S_1 \cup s) = 2$ it follows that

$$\lambda((X_1 \cap S_1) \cup \{s, x\}) + \lambda(X_1 \cup S_1) \leq 4.$$

Suppose that $\lambda((X_1 \cap S_1) \cup \{s, x\}) \geq 3$. Then $\lambda(X_1 \cup S_1) \leq 1$, so $\lambda(X_2 \cap S_2) \leq 1$. However, as $|X_1 \cap S_2| = 1$ it follows that $|X_2 \cap S_2| \geq 2$, so M contains a 2-separation, a contradiction. Thus $\lambda((X_1 \cap S_1) \cup \{s, x\}) \leq 2$ and it follows from Lemma 4.5 (vi) that $(X_1 \cap S_1) \cup \{s, x\}$ is a segment. \square

5.1.2. *The rank of $X_1 \cup x$ is three. Moreover, X_1 is a cocircuit of M .*

Proof. Let $s \in C^*$ be an element in $X_1 - \text{cl}(X_2)$ and suppose that (S_1, S_2, s) is a vertical 3-partition such that $x \in S_1$. Then $r((X_1 \cap S_1) \cup \{s, x\}) = 2$ by 5.1.1, and as $|X_1 \cap S_2| = 1$, Lemma 4.5 (iv) implies that $r(X_1 \cup x) = 3$.

Proposition 4.4 asserts that $X_2 \cup x$ is a flat of M , so X_1 is a cocircuit. \square

5.1.3. *Suppose that y and z are elements in $C^* \cap X_1$, and (Y_1, Y_2, y) and (Z_1, Z_2, z) are vertical 3-partitions such that $x \in Y_1 \cap Z_1$. Then*

$$|X_1 \cap Y_2| = |X_1 \cap Z_2| = 1 \quad \text{and} \quad X_1 \cap Y_2 = X_1 \cap Z_2.$$

Moreover

$$(X_1 \cap Y_1) \cup \{x, y\} = (X_1 \cap Z_1) \cup \{x, z\}.$$

Proof. Let x' be the unique element in $X_1 \cap Y_2$. From 5.1.1 we see that $(X_1 \cap Y_1) \cup \{x, y\}$ is a segment. The only element of X_1 not in $(X_1 \cap Y_1) \cup \{x, y\}$ is x' . It cannot be the case that $x' \in \text{cl}((X_1 \cap Y_1) \cup \{x, y\})$ by Lemma 4.5 (vi). The same arguments shows that $(X_1 \cap Z_1) \cup \{x, z\}$ is a segment, and the only element of X_1 not in this segment is x' . Now the result follows easily. \square

5.1.4. *Let $y \in C^*$ be an element in X_1 and suppose that (Y_1, Y_2, y) is a vertical 3-partition such that $x \in Y_1$. Then $|X_2 \cap Y_1| = 1$.*

Proof. We know by 5.1.1 that $(X_1 \cap Y_1) \cup \{x, y\}$ is a segment. Let $L' = (X_1 \cap Y_1) \cup \{x, y\}$ and let x' be the unique element in $X_1 \cap Y_2$. Since the complement of C^* is a flat of M which does not contain the segment L' it follows that at most one element of L' is not contained in C^* . As $|X_1 \cap Y_1| \geq 2$ we can find an element $z \in (X_1 \cap Y_1) \cap C^*$. There must be a vertical 3-partition (Z_1, Z_2, z) such that $x \in Z_1$. From 5.1.3 we see that the unique element in $X_1 \cap Z_2$ is x' , and that $(X_1 \cap Z_1) \cup \{x, z\} = L'$.

Let Y'_i and Z'_i denote $X_2 \cap Y_i$ and $X_2 \cap Z_i$ respectively for $i = 1, 2$. As (X_1, X_2, x) is a minimal partition it follows that Y'_i and Z'_i are non-empty for all $i \in \{1, 2\}$. Henceforth we will assume that $|Y'_1| > 1$ in order to obtain a contradiction.

5.1.5. $x \in \text{cl}(Y'_1)$.

Proof. We know that $\lambda(Y'_1 \cup x) \leq 2$ by Lemma 4.5 (ii). Since $|Y'_1| \geq 2$ it follows that $\lambda(Y'_1 \cup x) = 2$ and hence $\lambda(X_1 \cup Y_2) = 2$. Since $x \in \text{cl}(X_1 \cup Y_2)$ it follows that $\lambda(Y'_1) = 2$, so Lemma 2.2 implies that $x \in \text{cl}(Y'_1)$. \square

5.1.6. *Neither $Y'_1 \cap Z'_1$ nor $Y'_2 \cap Z'_2$ is empty.*

Proof. We know from 5.1.5 that $x \in \text{cl}(Y'_1)$. Since $z \in \text{cl}(Z_2)$ but $(X_1 \cap Z_1) \not\subseteq \text{cl}(Z_2)$, we deduce that $x \notin \text{cl}(Z_2)$ as L' is a segment containing both x and z . Thus $x \notin \text{cl}(Z'_2 \cup x')$. Hence $Y'_1 - Z'_2 \neq \emptyset$ so $Y'_1 \cap Z'_1 \neq \emptyset$.

Note that z is in the closure of $Z_2 = Z'_2 \cup x'$, but $z \notin \text{cl}(Z'_2)$ as X_1 is a cocircuit by 5.1.2. This observation means that $x' \in \text{cl}(Z'_2 \cup z)$. However $z \in Y_1$, and $x' \notin \text{cl}(Y_1)$ by Lemma 4.5 (iv). Thus $x' \notin \text{cl}(Y'_1 \cup z)$. It follows that $Z'_2 - Y'_1 \neq \emptyset$, so $Z'_2 \cap Y'_2 \neq \emptyset$. \square

5.1.7. *$(L' \cup (Y'_1 \cap Z'_1), Y_2 \cup Z_2)$ is a 3-separation of M .*

Proof. Note that $\lambda(Y_2) = \lambda(Z_2) = 2$, so $\lambda(Y_2 \cap Z_2) + \lambda(Y_2 \cup Z_2) \leq 4$. From 5.1.6 we see that $Y'_2 \cap Z'_2 \neq \emptyset$. Moreover $x' \in (Y_2 \cap Z_2) - (Y'_2 \cap Z'_2)$, which implies that $|Y_2 \cap Z_2| \geq 2$. Thus $\lambda(Y_2 \cap Z_2) \geq 2$, so $\lambda(Y_2 \cup Z_2) \leq 2$. As both $L' \cup (Y'_1 \cap Z'_1)$ and $Y_2 \cup Z_2$ have cardinality at least three the claim follows. \square

Note that $y, z \in \text{cl}(Y_2 \cup Z_2)$. As y and z are contained in the segment L' it follows that $L' \subseteq \text{cl}(Y_2 \cup Z_2)$. If $|Y'_1 \cap Z'_1| \geq 2$ then it must be the case that $L' \subseteq \text{cl}(Y'_1 \cap Z'_1)$, for otherwise $(Y'_1 \cap Z'_1, (Y_2 \cup Z_2) \cup L')$ is a 2-separation of M . But $L' \subseteq \text{cl}(Y'_1 \cap Z'_1)$ implies that $X_1 \cap Y_1 \subseteq \text{cl}(X_2)$, a contradiction.

Therefore $|Y'_1 \cap Z'_1| \leq 1$. We know from 5.1.6 that $Y'_1 \cap Z'_1$ is not empty. Let e be the unique element in $Y'_1 \cap Z'_1$. Suppose that $e \in \text{cl}(L')$. As $X_2 \cup x$ is a hyperplane and L' is a segment we see that $\cap(X_2 \cup x, L' - x) = 1$. As $e, x \in \text{cl}(L' - x)$ it follows from Proposition 2.14 that $r(\{e, x\}) \leq 1$. We deduce from this contradiction that $e \notin \text{cl}(L')$.

Hence $r(L' \cup e) = 3$, so $r(Y_2 \cup Z_2) = r(M) - 1$ by 5.1.7. Thus the complement of $\text{cl}(Y_2 \cup Z_2)$ is a cocircuit. However $L' \subseteq \text{cl}(Y_2 \cup Z_2)$, so e is a coloop of M , a contradiction.

Our assumption that $|X_2 \cap Y_1| \geq 2$ has lead to an impossibility. Since $X_2 \cap Y_1$ is non-empty by Lemma 4.5 (i) we conclude that 5.1.4 is true. \square

Now we are in a position to complete the proof of Theorem 5.1. Let $x_1 = x$, and let x_2 be some element in $C^* \cap X_1$. There is a vertical 3-partition (Y_1^2, Y_2^2, x_2) such that $x_1 \in Y_1^2$. Lemma 4.8 tells us that $|X_1 \cap Y_2^2| = 1$. Let y_1 be the unique element in $X_1 \cap Y_2^2$.

We know that $|X_1 \cap Y_1^2| \geq 2$ and $(X_1 \cap Y_1^2) \cup \{x_1, x_2\}$ is a segment by 5.1.1. It follows from Proposition 2.14, and the fact that $(X_1 \cap Y_1^2) \cup x_2$ is a segment while $X_2 \cup x_1$ is a hyperplane, that $(X_1 \cap Y_1^2) \cup \{x_1, x_2\}$ is a flat. The complement of C^* can contain at most one element of $(X_1 \cap Y_1^2) \cup \{x_1, x_2\}$. Let $L = C^* \cap ((X_1 \cap Y_1^2) \cup \{x_1, x_2\})$. Then $\text{cl}(L) = (X_1 \cap Y_1^2) \cup \{x_1, x_2\}$, and $\text{cl}(L) - L$ contains at most one element.

Suppose that $L = \{x_1, \dots, x_t\}$. We know that $t \geq 3$. Let i be a member of $\{2, \dots, t\}$. As $x_i \in C^*$ the fact that M is a counterexample to the theorem means that $\text{si}(M/x_i)$ is not 3-connected, so there is a vertical 3-partition (Y_1^i, Y_2^i, x_i) such that $x_1 \in Y_1^i$. Then

$$(X_1 \cap Y_1^i) \cup \{x_1, x_i\} = (X_1 \cap Y_1^2) \cup \{x_1, x_2\}$$

by 5.1.3, and 5.1.4 implies that there is a unique element in $X_2 \cap Y_1^i$. Let y_i be this element.

Define L^* to be $\{y_1, \dots, y_t\}$. Note that $L \cap L^* = \emptyset$. We already know that $(\text{cl}(L) - x_1) \cup y_1 = X_1$ is a cocircuit. Suppose that $i \in \{2, \dots, t\}$. Then $(\text{cl}(L) - x_i) \cup y_i$ is Y_1^i . As Y_1^i contains only one element that is not in the segment $\text{cl}(L)$ it follows that $r(Y_1^i) = 3$. Thus $r(Y_2^i \cup x_i) = r(M) - 1$. Furthermore $Y_2^i \cup x_i$ is a flat, for otherwise the complement of $\text{cl}(Y_2^i \cup x_i)$ is a cocircuit of rank at most two, which cannot occur since M is 3-connected. Hence $(\text{cl}(L) - x_i) \cup y_i$ is a cocircuit.

We have shown that (L, L^*) is a segment-cosegment pair. Proposition 3.3 says that $M/\text{cl}(L)$ is 3-connected. It is easy to see that the hypotheses of Lemma 3.5 are satisfied, so M/x_i is 3-connected up to the unique spore $(\text{cl}(L) - x_i, y_i)$, for all $i \in \{1, \dots, t\}$. We know that M/x_2 has an N -minor, but as $\text{cl}(L) - x_2$ is a parallel class of M/x_2 it follows that $M/x_2 \setminus (\text{cl}(L) - \{x_1, x_2\})$ has an N -minor. Since $\{x_1, y_2\}$ is a series pair of $M/x_2 \setminus (\text{cl}(L) - \{x_1, x_2\})$ it follows that $M/x_2 \setminus (\text{cl}(L) - \{x_1, x_2\})/x_1$, and hence $M/\text{cl}(L)$, has an N -minor.

Suppose that $|\text{cl}(L) - C^*| = 0$. Then $L = \text{cl}(L)$, and statement (iv) of Theorem 5.1 holds. Therefore we must assume that there is a single element e in $\text{cl}(L) - L$. Lemma 2.10 tells us that M/e has an N -minor. If $\text{si}(M/e)$ is 3-connected, then statement (iii) holds. Therefore we must assume $\text{si}(M/e)$ is not 3-connected.

Let $x_{t+1} = e$. There must be a vertical 3-partition $(Y_1^{t+1}, Y_2^{t+1}, x_{t+1})$. We assume that $x_1 \in Y_1^{t+1}$. Since $\text{cl}(Y_1^{t+1})$ contains x_1 and x_{t+1} it follows that $\text{cl}(L) \subseteq \text{cl}(Y_1^{t+1})$. By Proposition 2.6 we may assume that Y_1^{t+1} contains $\text{cl}(L) - x_{t+1} = L$.

As $X_2 \cup x_1$ is a flat it follows that $x_{t+1} \notin \text{cl}(X_2)$. However $x_{t+1} \in \text{cl}(Y_2^{t+1})$, so $X_1 \cap Y_2^{t+1} \neq \emptyset$. We know that $X_1 = (L \cup \{x_{t+1}, y_1\}) - x_1$, and as $L \subseteq Y_1^{t+1}$ it follows that $X_1 \cap Y_2^{t+1} = \{y_1\}$.

Since $x_{t+1} \in \text{cl}(Y_2^{t+1})$, there is a circuit $C_1 \subseteq Y_2^{t+1} \cup x_{t+1}$ such that $x_{t+1} \in C_1$. But $Y_1^2 = (L \cup \{x_{t+1}, y_2\}) - x_2$ is a cocircuit of M and C_1 must meet this cocircuit in more than one element. The only element of $Y_1^2 - x_{t+1}$ that can be in C_1 is y_2 . Thus $y_2 \in Y_2^{t+1}$.

Since (X_1, X_2, x) is a minimal partition it follows that $X_2 \cap Y_1^{t+1}$ is non-empty. Assume that $|X_2 \cap Y_1^{t+1}| \geq 2$. As $\lambda(X_1) + \lambda(Y_2^{t+1} \cup x_{t+1}) = 4$, it follows that

$$\lambda((X_1 \cap Y_2^{t+1}) \cup x_{t+1}) + \lambda(X_1 \cup Y_2^{t+1}) \leq 4.$$

Furthermore $\lambda(X_1 \cup x_1) + \lambda(Y_2^{t+1} \cup x_{t+1}) = 4$, so

$$\lambda((X_1 \cap Y_2^{t+1}) \cup x_{t+1}) + \lambda(X_1 \cup Y_2^{t+1} \cup x_1) \leq 4.$$

As $(X_1 \cap Y_2^{t+1}) \cup x_{t+1} = \{x_{t+1}, y_1\}$ we deduce that $\lambda((X_1 \cap Y_2^{t+1}) \cup x_{t+1}) = 2$. Thus

$$(1) \quad \lambda(X_1 \cup Y_2^{t+1}), \lambda(X_1 \cup Y_2^{t+1} \cup x_1) \leq 2.$$

Both of the sets in Equation (1) contain at least two elements, and by assumption $|X_2 \cap Y_1^{t+1}| \geq 2$. Therefore $X_2 \cap Y_1^{t+1}$ and $(X_2 \cap Y_1^{t+1}) \cup x_1$ are exactly 3-separating. Since $x_1 \in \text{cl}(X_1)$ we see from Lemma 2.2 that $x_1 \in \text{cl}(X_2 \cap Y_1^{t+1})$. Thus there is a circuit $C_2 \subseteq (X_2 \cap Y_1^{t+1}) \cup x_1$ such that $x_1 \subseteq C_2$. We have already noted that Y_1^2 is a cocircuit, and as $x_1 \in Y_1^2$ it follows that $|C_2 \cap Y_1^2| \geq 2$. As $C_2 - x_1 \subseteq X_2$ the only element other than x_1 that can be in $C_2 \cap Y_1^2$ is y_2 . Hence $y_2 \in C_2 \subseteq Y_1^{t+1}$, a contradiction as we have already deduced that $y_2 \in Y_2^{t+1}$.

We are forced to conclude that $X_2 \cap Y_1^{t+1}$ contains a unique element. Let this element be y_{t+1} . Therefore $Y_1^{t+1} = L \cup y_{t+1}$. Thus $r(Y_1^{t+1}) = 3$, so $r(Y_2^{t+1}) = r(M) - 1$. If $Y_2^{t+1} \cup x_{t+1}$ is not a hyperplane, then the complement of $\text{cl}(Y_2^{t+1} \cup x_{t+1})$ is a cocircuit of rank at most two, a contradiction. Therefore $(\text{cl}(L) - x_{t+1}) \cup y_{t+1} = Y_1^{t+1}$ is a cocircuit.

Let $L_0 = \{x_1, \dots, x_{t+1}\}$ and let $L_0^* = \{y_1, \dots, y_{t+1}\}$. Note that $L_0 = \text{cl}(L)$, so L_0 is a flat. We have shown that (L_0, L_0^*) is a segment-cosegment pair. Moreover, M/x_{t+1} is 3-connected up to a unique spore $(L_0 - x_{t+1}, y_{t+1})$, by Lemma 3.5. By relabeling L_0 and L_0^* as L and L^* respectively we see that statement (iv) of Theorem 5.1 holds. Hence M is not a counterexample, and this contradiction completes the proof of Theorem 5.1. \square

6. ACKNOWLEDGEMENTS

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