Quaternionic Hyperbolic Function Theory

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Abstract We are studying hyperbolic function theory in the skew-field of quaternions. This theory is connected to k-hyperbolic harmonic functions that are harmonic with respect to the hyperbolic Riemannian metric

$$ds_k^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^k}$$

in the upper half space $\mathbb{R}_+^4 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_3 > 0\}$. In the case $k = 2$, the metric is the hyperbolic metric of the Poincaré upper half-space. Hempfling and Leutwiler started to study this case and noticed that the quaternionic power function $x^m\,(m \in \mathbb{Z})$, is a conjugate gradient of a 2-hyperbolic harmonic function. They researched polynomial solutions. We find fundamental k-hyperbolic harmonic functions depending only on the hyperbolic distance and x_3. Using these functions we are able to verify a Cauchy type integral formula. Earlier these results have been verified for quaternionic functions depending only on reduced variables (x_0, x_1, x_2). Our functions are depending on four variables.

1 Introduction

We study hyperbolic function theory in the skew- field of quaternions, denoted by $\mathbb{H}$. This theory was initiated by Thomas Hempfling and Heinz Leutwiler in [15]. They studied quaternion valued twice continuous differentiable functions $f(x)$ defined in

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the full space $\mathbb{R}^4$ satisfying the following modified Cauchy-Riemann system

$$x_3\left(\frac{\partial f_0}{\partial x_0}-\frac{\partial f_1}{\partial x_1}-\frac{\partial f_2}{\partial x_2}-\frac{\partial f_3}{\partial x_3}\right)+2f_3=0,$$
$$\frac{\partial f_0}{\partial x_i}=-\frac{\partial f_i}{\partial x_0} \text{ for all } i=1,2,3,$$
$$\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i} \text{ for all } i,j=1,2,3.$$

In [17] Leutwiler noticed that the power function x^m, where $m\in\mathbb{Z}$, calculated using quaternions, is a conjugate gradient of a hyperbolic harmonic function h which satisfies the equation

$$\Delta_2 h = x_3^2\Delta h - 2x_3\frac{\partial h}{\partial x_3}=0$$

where as usual

$$\Delta h=\frac{\partial^2 h}{\partial x_0^2}+\frac{\partial^2 h}{\partial x_1^2}+\frac{\partial^2 h}{\partial x_2^2}+\frac{\partial^2 h}{\partial x_3^2}.$$

The operator Δ_2 is the hyperbolic Laplace-Beltrami operator with respect to the Poincaré hyperbolic metric

$$ds^2=\frac{dx_0^2+dx_1^2+dx_2^2+dx_3^2}{x_3^2}.$$

These functions are called conjugate gradients of real hyperbolic harmonic functions.

Leutwiler and the first author in [7] studied the total Clifford algebra valued functions, called hypermonogenic functions. Their Cauchy-type formula was proved in [6] and the key ideas are the relations between k and $-k$-hypermonogenic functions, introduced in [3]. An introduction to the theory is given in [18] and in more recent paper [8].

In this paper, we verify the Cauchy type theorems for quaternionic valued fuctions called k-hyperregular. Our Cauchy type theorems are not directly following from the theory of quaternionic valued hypermonogenic functions, which are depending only on three variables. Our functions are depending on four variables and k is an arbitrary real coefficient. However, it is possible to deduce some results from the theory of paravector valued k-hypermonogenic funcions (see [9]) which domain of the definition is an open subset of $\mathbb{R}^4$ and the values are in the Clifford algebra $\mathscr{C}\ell_{0,3}$. These methods are rather complicated in case of quaternions and we prefer the direct methods.

2 Preliminaries

The space of quaternions $\mathbb{H}$ is four dimensional associative division algebra over reals with an identity $\mathbf{1}$ and generated by the elements $\mathbf{1}$, e_1, e_2 and e_3 satisfying the relations

$$e_3 = e_1 e_2$$

and

$$e_i e_j + e_j e_i = -2\delta_{ij}\mathbf{1},$$

where δ_{ij} is the usual Kronecker delta. The elements $\alpha\mathbf{1}$ and α may be identified.

We denote the coefficients of the components of a quaternion x with respect to the base $\{1, e_0, e_1, e_2\}$ by x_0, x_1, x_2 and x_3, that is

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

where x_0, x_1, x_2 and x_3 are real numbers. The spaces $\mathbb{R}^4$ and $\mathbb{H}$ may be identified as vector spaces.

We denote the upper half space by

$$\mathbb{H}_+ = \{x \mid x_i \in \mathbb{R},\ i = 0,1,2,3 \text{ and } x_3 > 0\}$$

and the lower half space by

$$\mathbb{H}_- = \{x \mid x_i \in \mathbb{R}\ i = 0,1,2,3 \text{ and } x_3 < 0\}.$$

The hyperbolic distance $d_h(x,a)$ between the points x and a in $\mathbb{H}_+$ may be computed from the formula $d_h(x,a) = \operatorname{arcosh}\lambda(\mathrm{x},\mathrm{a})$, where

$$\begin{aligned}
\lambda(x,a) &= \frac{(x_0-a_0)^2 + (x_1-a_1)^2 + (x_2-a_2)^2 + x_3^2 + a_3^2}{2x_3a_3} \\
&= \frac{\|x-a\|^2 + \|x-a^*\|^2}{4x_3a_3} \\
&= \frac{\|x-a\|^2}{2x_3a_3} + 1 = \frac{\|x-a^*\|^2}{2x_3a_3} - 1,
\end{aligned}$$

putting $a^* = a_0 + a_1e_1 + a_2e_2 - a_3e_3$ and the distance

$$\|x-a\| = \sqrt{(x_0-a_0)^2 + (x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2}$$

is the usual Euclidean distance (see the proof for example in [18]). Similarly, we may compute the hyperbolic distance between the points x and a in $\mathbb{H}_-$. Notice that if both x and a belong to $\mathbb{H}_+$ or in $\mathbb{H}_-$ then

$$d_h(x,a) = d_h(x^*,a^*).$$

We recall the following simple calculation rules

$$\|x-a\|^2 = 2x_3a_3\left(\lambda(x,a)-1\right), \tag{1}$$
$$\|x-a^*\|^2 = 2x_3a_3\left(\lambda(x,a)+1\right), \tag{2}$$
$$\frac{\|x-a\|^2}{\|x-a^*\|^2} = \frac{\lambda(x,a)-1}{\lambda(x,a)+1} = \tanh^2\left(\frac{d_h(x,a)}{2}\right). \tag{3}$$

We remind that hyperbolic balls are also Euclidean balls with a shifted center given by the next result.

Proposition 1. *The hyperbolic ball $B_h(a,r_h)$ with the hyperbolic center a in $\mathbb{H}_+$ and the radius r_h is the same as the Euclidean ball with the Euclidean center*

$$c_a(r_h) = a_0 + a_1e_1 + a_2e_2 + a_3\cosh r_h e_3$$

and the Euclidean radius $r_e = a_3\sinh r_h$. Conversely, if $b = (b_0,b_1,b_2,b_3)$ is a point in $\mathbb{H}_+$ and $r_e < b_3$ then the Euclidean ball $B_e(b,r_e)$ is the same as the hyperbolic ball with the hyperbolic radius

$$r_h = \operatorname{artanh}\left(\frac{r_e}{b_3}\right)$$

and the hyperbolic center

$$a = \left(b_0,b_1,b_2,\frac{b_3}{\cosh r_h}\right).$$

Corollary 1. *The hyperbolic metric in $\mathbb{H}_+$ (resp. in $\mathbb{H}_-$) is equivalent with the Euclidean metric in $\mathbb{H}_+$ (resp. in $\mathbb{H}_-$), that is they generate the same topology.*

We may extend the hyperbolic topology to the whole space. Indeed, if $U \subset \mathbb{H}$ and the set $U \cap \{x \in \mathbb{H} \mid x_3 = 0\}$ is non-empty then we call the set U open if it is open with respect to usual Euclidean topology. The inner product $\langle x,y\rangle$ in $\mathbb{H}$ is defined by

$$\langle x,y\rangle = \sum_{i=0}^{3} x_iy_i$$

similarly as in the Euclidean space $\mathbb{R}^4$.

The elements

$$x = x_0 + x_1e_1 + x_2e_2$$

are called *reduced quaternions* if $x_0, x_1$and x_2 are real numbers. The set of reduced quaternions is identified with $\mathbb{R}^3$.

We recall that the *prime involution* in $\mathbb{H}$ is the mapping $x \to x'$ defined by

$$x' = x_0 - x_1e_1 - x_2e_2 + x_3e_3.$$

Similarly, the *reversion* in $\mathbb{H}$ is the mapping $x \to x^*$ defined by

$$x^* = x_0 + x_1 e_1 + x_2 e_2 - x_3 e_3.$$

The *conjugation* in $\mathbb{H}$ is the mapping $x \to \bar{x}$ defined by $\bar{x} = (x')^* = (x^*)'$, that is

$$\bar{x} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3.$$

These involutions satisfy the following product rules

$$(xy)' = x'y',$$
$$(xy)^* = y^* x^*$$

and

$$\overline{xy} = \bar{y}\,\bar{x}$$

for all $x, y \in \mathbb{H}$.

The prime involution may be characterized also as

$$xe_3 = e_3 x'$$

for all quaternions x.

The real part of a quaternion x is defined by

$$\operatorname{Re} x = x_0$$

and the vector part by

$$\operatorname{Vec} x = x_1 e_1 + x_2 e_2 + x_3 e_3.$$

We recall the product rule

$$xy = -\langle x, y \rangle + x \times y$$

if $\operatorname{Re} x = \operatorname{Re} y = 0$, where $\times$ is the usual cross product in $\mathbb{R}^3$.

We define the mappings $S : \mathbb{H} \to \mathbb{R}^3$ and $T : \mathbb{H} \to \mathbb{R}$ by

$$Sa = a_0 + a_1 e_1 + a_2 e_2$$

and

$$Ta = a_3$$

for $a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}$. Using the reversion, we compute the formulas

$$Sa = \frac{1}{2}\left(a + a^*\right), \tag{4}$$

$$Ta = -\frac{1}{2}\left(a - a^*\right) e_3. \tag{5}$$

We recall the identities

$$ab + ba = 2a\text{Re } b + 2b\text{Re } a - 2\langle a, b\rangle \tag{6}$$

and

$$\frac{1}{2}\left(a\overline{b}c + c\overline{b}a\right) = \langle b, c\rangle a - [a, b, c] \tag{7}$$

valid for all quaternions a, b and c . The term $[a, b, c]$ is called a *triple product* and is defined by

$$[a, b, c] = \langle a, c\rangle b - \langle a, b\rangle c.$$

If a, b and c are quaternions with $\text{Re } a = \text{Re } b = \text{Re } c = 0$, then (cf. [14])

$$[a, b, c] = a \times (b \times c).$$

3 Hyperregular functions

We use the following hyperbolic modifications H_k^l and H_k^r of the Cauchy-Riemann operators

$$H_k^l f(x) = D_l f(x) + k\frac{f_3}{x_3}, \quad \overline{H}_k^l f(x) = \overline{D}_l f(x) - k\frac{f_3}{x_3},$$
$$H_k^r f(x) = D_r f(x) + k\frac{f_3}{x_3}, \quad \overline{H}_k^r f(x) = \overline{D}_r f(x) - k\frac{f_3}{x_3},$$

where the parameter $k \in \mathbb{R}$ and the generalized Cauchy-Riemann operators are defined by

$$D_l f = \sum_{i=0}^{3} e_i \frac{\partial f}{\partial x_i}, \qquad \overline{D}_l f = \sum_{i=0}^{3} \overline{e_i} \frac{\partial f}{\partial x_i},$$
$$D_r f = \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} e_i, \qquad \overline{D}_r f = \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} \overline{e_i}.$$

We also abbreviate $D_l f$ by Df and H_k^l by H_k.

Definition 1. Let $\Omega \subset \mathbb{H}$ be open. A function $f : \Omega \to \mathbb{H}$ is called *k-hyperregular*, if $f \in \mathscr{C}^1(\Omega)$ and

$$H_k^l f(x) = H_k^r f(x) = 0.$$

for any $x \in \Omega \setminus \{x_3 = 0\}$.

We may simply compute the components of the operators H_k^l and H_k^r as follows.

Lemma 1. *Let $\Omega \subset \mathbb{H}$ be open. If a function $f : \Omega \to \mathbb{H}$ is differentiable then the coordinate functions of H_k^l and H_k^r are given by*

$$\left(H_k^l f\right)_0 = \tfrac{\partial f_0}{\partial x_0} - \tfrac{\partial f_1}{\partial x_1} - \tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_3}{\partial x_3} + k\tfrac{f_3}{x_3}, \qquad \left(H_k^r f\right)_0 = \left(H_k^l f\right)_0,$$
$$\left(H_k^l f\right)_1 = \tfrac{\partial f_0}{\partial x_1} + \tfrac{\partial f_1}{\partial x_0} - \tfrac{\partial f_2}{\partial x_3} + \tfrac{\partial f_3}{\partial x_2}, \quad \left(H_k^r f\right)_1 = \tfrac{\partial f_0}{\partial x_1} + \tfrac{\partial f_1}{\partial x_0} + \tfrac{\partial f_2}{\partial x_3} - \tfrac{\partial f_3}{\partial x_2},$$
$$\left(H_k^l f\right)_2 = \tfrac{\partial f_0}{\partial x_2} + \tfrac{\partial f_2}{\partial x_0} + \tfrac{\partial f_1}{\partial x_3} - \tfrac{\partial f_3}{\partial x_1}, \quad \left(H_k^r f\right)_2 = \tfrac{\partial f_0}{\partial x_2} + \tfrac{\partial f_2}{\partial x_0} - \tfrac{\partial f_1}{\partial x_3} + \tfrac{\partial f_3}{\partial x_1},$$
$$\left(H_k^l f\right)_3 = \tfrac{\partial f_0}{\partial x_3} + \tfrac{\partial f_3}{\partial x_0} - \tfrac{\partial f_1}{\partial x_2} + \tfrac{\partial f_2}{\partial x_1}, \quad \left(H_k^r f\right)_3 = \tfrac{\partial f_0}{\partial x_3} + \tfrac{\partial f_3}{\partial x_0} + \tfrac{\partial f_1}{\partial x_2} - \tfrac{\partial f_2}{\partial x_1},$$

where $(\cdot)_j$ denotes the real coefficient of the element e_j for each $j = 0,1,2,3$.

We obtain immediately the following result.

Proposition 2. *Let $\Omega \subset \mathbb{H}$ be open and a function $f : \Omega \to \mathbb{H}$ continuously differentiable. A function f is $k-$hyperregular in Ω if and only if*

$$\tfrac{\partial f_0}{\partial x_0} - \tfrac{\partial f_1}{\partial x_1} - \tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_3}{\partial x_3} + k\tfrac{f_3}{x_3} = 0, \text{ if } x_3 \neq 0,$$
$$\tfrac{\partial f_0}{\partial x_i} = -\tfrac{\partial f_i}{\partial x_0} \text{ for all } i = 1,2,3,$$
$$\tfrac{\partial f_i}{\partial x_j} = \tfrac{\partial f_j}{\partial x_i} \text{ for all } i,j = 1,2,3.$$

Our operators are connected to the hyperbolic metric via the hyperbolic Laplace operator as follows.

Proposition 3. *Let $f : \Omega \to \mathbb{H}$ be twice continuously differentiable. Then*

$$\begin{aligned} H_k^l \overline{H}_k^l f =& \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{kf_3}{x_3^2}e_3 + \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) \\ &+ \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right)e_2 \\ =& \overline{H}_k^l H_k^l f \end{aligned}$$

and

$$\begin{aligned} H_k^r \overline{H}_k^r f =& \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{kf_3e_3}{x_3^2} + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \\ &+ \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right)e_2 \\ =& \overline{H}_k^r H_k^r f. \end{aligned}$$

Proof. We just compute

$$\begin{aligned} D_l\overline{H_k^l} f &= D_l\overline{D}_l f - k\frac{Df_3}{x_3} + \frac{kf_3e_3}{x_3^2} \\ &= \Delta f - k\frac{\frac{\partial f_3}{\partial x_0} + \frac{\partial f_3}{\partial x_1}e_1 + \frac{\partial f_3}{\partial x_2}e_2 + \frac{\partial f_3}{\partial x_3}e_3}{x_3} + \frac{kf_3e_3}{x_3^2} \end{aligned}$$

and

$$\left(\overline{H}_k^l f\right)_3 = \left(\overline{D}_l f\right)_3 = -\frac{\partial f_0}{\partial x_3} + \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0}.$$

Hence we obtain

$$\begin{aligned} H_k^l \overline{H}_k^l f =& \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3}{x_3^2} e_3 + \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) \\ & + \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right) e_2. \end{aligned}$$

Similarly, we compute

$$\begin{aligned} D_r \overline{H}_k^r f &= D_r \overline{D}_r f - k\frac{D_r f_3}{x_3} + \frac{k f_3 e_3}{x_3^2} \\ &= \Delta f - k\frac{\frac{\partial f_3}{\partial x_0} + \frac{\partial f_3}{\partial x_1} e_1 + \frac{\partial f_3}{\partial x_2} e_2 + \frac{\partial f_3}{\partial x_3} e_3}{x_3} + \frac{k f_3 e_3}{x_3^2} \end{aligned}$$

and

$$\left(\overline{H}_k^r f\right)_3 = \left(\overline{D}_r f\right)_3 = -\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0}.$$

Hence we have

$$\begin{aligned} H_k^r \overline{H}_k^r f =& \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3 e_3}{x_3^2} + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \\ & + \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right) e_2. \end{aligned}$$

Moreover, we easily deduce that $\overline{H}_k^l H_k^l f = H_k^l \overline{H}_k^l f$ and $\overline{H}_k^r H_k^r f = H_k^r \overline{H}_k^r f$.

We immediately obtain two corollaries.

Corollary 2. *If $f : \Omega \to \mathbb{H}$ is twice continuously differentiable and $k \neq 0$ then*

$$H_k^l \overline{H}_k^l f = H_k^r \overline{H}_k^r f = \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3 e_3}{x_3^2}$$

if and only if $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i, j = 1, 2, 3$.

Corollary 3. *If $f : \Omega \to \mathbb{R}$ is real vakued and twice continuously differentiable then*

$$x_3^k H_k^l \overline{H}_k^l f = x_3^k H_k^r \overline{H}_k^r f = \Delta_k f,$$

where the operator

$$\Delta_k = x_3^k \left(\Delta - \frac{k}{x_3}\frac{\partial}{\partial x_3}\right)$$

is the Laplace-Beltrami operator (see [19]) with respect to the Riemannian metric

$$ds_k^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^k}. \tag{8}$$

Differentiating the first equation of Proposition 2 with respect to x_i and applying the rest of the equations of Proposition 2 we obtain the following result.

Proposition 4. *Let $\Omega \subset \mathbb{H}$ be open and a function $f : \Omega \to \mathbb{H}$ twice continuously differentiable. If f is k-hyperregular then*

$$x_3^k H_k^l \overline{H}_k^l f = x_3^k H_k^r \overline{H}_k^r f = \Delta_k f + x_3^{k-2} k f_3 e_3 = 0.$$

The previous results motivate the following definition.

Definition 2. Let $\Omega \subset \mathbb{H}$ be open. A twice continuously differentiable function $f : \Omega \to \mathbb{H}$ is called *k-hyperbolic*, if

$$\Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3 e_3}{x_3^2} = 0.$$

There exists a characterization of k-hyperregular functions in terms of $k-$hyperbolic functions.

Theorem 1. *Let $\Omega \subset \mathbb{H}$ be open. A twice continuously differentiable hyperbolic harmonic function $f : \Omega \to \mathbb{H}$ is k-hyperregular if and only if the functions f and $xf + fx$ are k-hyperbolic and $H_k^l f = H_k^r f$.*

Proof. In order to abbreviate notations, we denote $g = xf + fx$. Using the standard formulas $\Delta(xf) = x\Delta f + 2D_l f$ and $\Delta(fx) = (\Delta f)x + 2D_r f$ we obtain by virtue of Proposition 4, that

$$\begin{aligned}
x_3^2 \Delta g - k x_3 \frac{\partial g}{\partial x_3} + k g_3 e_3 =& x_3^2 x H_k^l \overline{H}_k^l f + x_3^2 \left(H_k^l \overline{H}_k^l f\right) x + 2x_3^2 H_k^l f + 2x_3^2 H_k^r f \\
& - 4k x_3 f_3 - k x_3 (e_3 f + f e_3) + 2k (x_0 f_3 + x_3 f_0) e_3 + \\
& - 2k f_3 (x_0 e_3 - x_3) \\
=& x_3^2 x H_k^l \overline{H}_k^l f + x_3^2 \left(H_k^l \overline{H}_k^l f\right) x \\
& + 2x_3^2 H_k^l f + 2x_3^2 H_k^r f.
\end{aligned}$$

If f is k-hyperregular then

$$x_3^2 H_k^l \overline{H}_k^l f = x_3^2 \Delta f - k x_3 \frac{\partial f}{\partial x_3} + k f_3 e_3 = 0$$

and $H_k^l f = H_k^r f = 0$ which implies that g is $k-$hyperbolic. Conversely, if g and f are k-hyperbolic and $H_k^l f = H_k^r f$ then

$$H_k^l f + H_k^r f = 0.$$

Hence f is k-hyperregular.

Real valued k-hyperbolic functions are especially important, since they produce k-hyperregular functions.

Theorem 2. *Let Ω be an open subset of $\mathbb{H}$. If h is real valued k-hyperbolic on Ω then the function $f = \overline{D}h$ is k-hyperregular on Ω. Conversely, if f is k-hyperregular on Ω, there exists locally a real valued k-hyperbolic function h satisfying $f = \overline{D}h$.*

Proof. Let h be real k- hyperbolic on Ω and denote $f = \overline{D}h$. Applying Proposition 3 we obtain

$$H_k^l f = H_k^l \overline{H}_k^l h = \Delta h - \frac{k}{x_3}\frac{\partial h}{\partial x_3} = 0 = H_k^r \overline{H}_k^r h = H_k^r f.$$

Hence f is k-hyperregular. The converse statement is verified similarly as in [7].

We use the following transformation property proved in [5].

Lemma 2. *Let Ω be an open set contained in $\mathbb{H}_+$ or in $\mathbb{H}_-$. A function $f : \Omega \to \mathbb{R}$ is k-hyperbolic harmonic if and only if the function $g(x) = x_3^{\frac{2-k}{2}} f(x)$ satisfies the equation*

$$\Delta_2 Sg + \frac{1}{4}\left(9 - (k+1)^2\right) Sg = 0. \tag{9}$$

4 Cauchy type integral formulas

We first recall the quaternionic version of the Stokes theorem verified for example in [14] as follows. If Ω is an open subset of $\mathbb{H}$, K a 3-chain satisfying $\overline{K} \subset \Omega$ and $f, g \in \mathscr{C}^1(\Omega, \mathbb{H})$, then

$$\int_{\partial K} g \nu f d\sigma = \int_K (D_r g f + g D_l f)\, dm \tag{10}$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ is the outer normal, $d\sigma$ the surface element and dm is the usual Lebesgue volume element in $\mathbb{R}^4$ identified with $\mathbb{H}$ as a vector space.

The T-part and S-part play a strong role in our operator H_k. We have therefore two versions of the Stokes theorem. The first version deals with T-part and the second one with S-part.

Theorem 3. *Let Ω be an open subset of $\mathbb{H} \setminus \{x_3 = 0\}$ and K a 3-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathscr{C}^1(\Omega, \mathbb{H})$, then*

$$\int_{\partial K} g \nu f d\sigma = \int_K \left(\left(H_{-k}^r g\right) f + g H_k^l f + \frac{k}{x_3}\left((g_3) Sf - Sg f_3\right) \right) dm$$

and therefore

$$T\left(\int_{\partial K} g\nu f d\sigma\right) = \int_K T\left(\left(H^r_{-k}g\right)f + gH^l_k f\right)dm$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ *is the outer normal,* $d\sigma$ *the surface element and* dm *is the usual Lebesgue volume element in* $\mathbb{R}^4$.

Proof. Since $D_r g = H^r_{-k}g + k\frac{g_3}{x_3}$ and $D_l f = H^l_k f - k\frac{f_3}{x_3}$ we deduce using (10) that

$$\begin{aligned}\int_{\partial K}(g d\sigma f) &= \int_K\left(\left(H^r_{-k}g\right)f + gH^l_k f + \frac{k}{x_3}\left((g_3)f - gf_3\right)\right)dm\\ &= \int_K\left(\left(H^r_{-k}g\right)f + gH^l_k f + \frac{k}{x_3}\left((g_3)Sf - Sgf_3\right)\right)dm,\end{aligned}$$

completing the proof.

We may also prove

Theorem 4. *Let* Ω *be an open subset of* $\mathbb{H}^4\setminus\{x_3=0\}$ *and* K *a* 3*-chain satisfying* $\overline{K}\subset\Omega$. *If* $f,g\in\mathscr{C}^1(\Omega,\mathbb{H})$, *then*

$$\int_{\partial K} f\nu g d\sigma = \int_K\left(\left(H^r_k f\right)g + fH^l_{-k}g + \frac{k}{x_3}\left((g_3)Sf - Sgf_3\right)\right)dm$$

and therefore

$$T\left(\int_{\partial K} f\nu g d\sigma\right) = \int_K T\left(\left(H^r_k f\right)g + fH^l_{-k}g\right)dm,$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ *is the outer normal,* $d\sigma$ *the surface element and* dm *is the usual Lebesgue volume element in* $\mathbb{R}^4$.

Proof. Since $D_l g = H^l_{-k}g + k\frac{g_3}{x_3}$ and $D_r f = H^r_k f - k\frac{f_3}{x_3}$ we deduce using (10) that

$$\begin{aligned}\int_{\partial K}(g\nu f)d\sigma &= \int_K\left(\left(H^r f\right)g + fH^l_{-k}g + \frac{k}{x_3}\left(fg_3 - f_3 g\right)\right)dm\\ &= \int_K\left(\left(H^r f\right)f + gH^l_{-k}g + \frac{k}{x_3}\left((g_3)Sf - Sgf_3\right)\right)dm,\end{aligned}$$

completing the proof.

Combining previous results we conclude the result

Theorem 5. *Let* Ω *be an open subset of* $\mathbb{R}^4\setminus\{x_3=0\}$ *and* K *a* 3*-chain satisfying* $\overline{K}\subset\Omega$. *If* $f,g\in\mathscr{C}^1(\Omega,\mathbb{H})$, *then*

$$\int_{\partial K} T\left(g\nu f + f\nu g\right)d\sigma = \int_K T\left(H^r_{-k}gf + gH^l_k f + H^r_k fg + fH^l_{-k}g\right)dm,$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ *is the outer normal,* $d\sigma$ *the surface element and* dm *is the usual Lebesgue volume element in* $\mathbb{R}^4$.

Theorem 6. *Let Ω be an open subset of $\mathbb{R}^4 \setminus \{x_3 = 0\}$ and K a 3-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathscr{C}^1(\Omega, \mathbb{H})$, then*

$$\int_{\partial K} S(g\nu f + f\nu g)\frac{d\sigma}{x_3^k} = \int_K S\left(H_k^r g f + g H_k^l f + H_k^r f g + f H_k^l g\right)\frac{dm}{x_3^k},$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ is the outer normal, $d\sigma$ the surface element and dm is the usual Lebesgue volume element in $\mathbb{R}^4$.

Proof. Applying (10), we deduce

$$\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} = \int_K \left(D_r g f + g D_l f - k\frac{g e_3 f}{x_3}\right)\frac{dm}{x_3^k}.$$

Since $H_k^r g = D_r g + \frac{k g_3}{x_3}$ and $H_k^l f = D_l g + \frac{k f_3}{x_3}$, we infer

$$\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} = \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_3 f + g f_3 + g e_3 f}{x_3}\right)\frac{dm}{x_3^k}.$$

Using the formula $g e_3 f = g e_3 S f - g f_3$, we obtain

$$\begin{aligned}\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} &= \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_3 f + g e_3 S f}{x_3}\right)\frac{dm}{x_3^k} \\ &= \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_3 f_3 e_3 + S g e_3 S f}{x_3}\right)\frac{dm}{x_3^k}.\end{aligned}$$

If we compute the coordinates of $S g e_3 S f$, we have

$$\begin{aligned}\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} =& \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_0 f_0 + g_1 f_1 + g_2 f_2 + g_3 f_3}{x_3} e_3\right)\frac{dm}{x_3^k} \\ & - \int_K k\frac{g_1 f_2 - g_2 f_1 + (g_2 f_0 - g_0 f_2) e_1 + (g_0 f_1 - g_1 f_0) e_2}{x_3^{k+1}} dm.\end{aligned}$$

If we interchange the roles of f and g, we infer

$$\begin{aligned}\int_{\partial K} f\nu g \frac{d\sigma}{x_3^k} =& \int_K \left(H_k^r f g + f H_k^l g - k\frac{g_0 f_0 + g_1 f_1 + g_1 f_1 + g_3 f_3}{x_3} e_3\right)\frac{dm}{x_3^k} \\ & - \int_K k\frac{f_1 g_2 - f_2 g_1 + (f_2 g_0 - f_0 g_2) e_1 + (f_0 g_1 - f_1 g_0) e_2}{x_3^{k+1}} dm\end{aligned}$$

Hence

$$\int_{\partial K}(g\nu f+f\nu g)\frac{d\sigma}{x_3^k}=\int_K\left(H_k^r g f+gH_k^l f+H_k^r f g+fH_k^l g\right)\frac{dm}{x_3^k}$$
$$-2ke_3\int_K\frac{g_0f_0+g_1f_1+g_1f_1+g_3f_3}{x_3}\frac{dm}{x_3^k}$$

and therefore

$$\int_{\partial K}S(g\nu f+f\nu g)\frac{d\sigma}{x_3^k}=\int_K S\left(H_k^r g f+gH_k^l f+H_k^r f g+fH_k^l g\right)\frac{dm}{x_3^k}.$$

The hyperbolic Laplace operator of functions depending on λ is computed in [5] as follows.

Lemma 3. *Let x and y be poins in the upper half space. If f is twice continuously differentiable depending only on $\lambda=\lambda(x,y)$, then*

$$\Delta_h f(x)=\left(\lambda^2-1\right)\frac{\partial^2 f}{\partial\lambda^2}+4\lambda\frac{\partial f}{\partial\lambda}.$$

We recall the definition of the associated Legendre function of the second kind

$$Q_\nu^\mu(\lambda)=C\left(\lambda^2-1\right)^{\frac{\mu}{2}}\lambda^{-\nu-\mu-1}\,{}_2F_1\left(\frac{\nu+\mu+2}{2},\frac{\mu+\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\lambda^2}\right)$$

where

$$C=-\frac{\sqrt{\pi}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}.$$

and the hypergeometric function is defined by

$${}_2F_1(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!},$$

converging in the usual sense at least for x satisfying $|x|<1$. Associated Legendre functions satisfies the differential equation (see [20])

$$(\lambda^2-1)u''(\lambda)+2\lambda u'(\lambda)-\left(\nu(\nu+1)-\frac{\mu^2}{1-\lambda^2}\right)u(\lambda)=0. \qquad (11)$$

We are looking for solutions of the equation

$$\Delta_h f(\lambda)+\gamma f(\lambda)=0$$

in the form

$$f(\lambda)=\left(\lambda^2-1\right)^\alpha g(\lambda).$$

We just compute that

$$\left(\lambda^2-1\right)g''(\lambda)+(4\alpha+4)\lambda g'(\lambda)+\left(4\alpha^2+6\alpha+\gamma+\frac{2\alpha(2+2\alpha)}{\lambda^2-1}\right)g(\lambda)=0.$$

In order to compute the solutions using Legendre functions, we compare this equation with (11) and first we set $4\alpha+4=2$ and therefore $\alpha=-\frac{1}{2}$. Then we have the equation

$$\left(\lambda^2-1\right)g''(\lambda)+2\lambda g'(\lambda)+\left(-2+\gamma-\frac{1}{1-\lambda^2}\right)g(\lambda)=0$$

and again comparing with (11), we obtain equations

$$\nu(\nu+1)=2-\gamma,$$
$$\mu^2=\frac{(n-1)^2}{4}.$$

Hence $\mu=\pm1$ and $\nu=\frac{\sqrt{9-4\gamma}-1}{2}$. Setting $-\gamma=\frac{1}{4}\left((k+1)^2-9\right)$, we obtain

$$\nu=\frac{\pm|k+1|-1}{2}.$$

Consequently, we found a solution $\left(\lambda^2-1\right)^{-\frac{1}{2}}Q^1_{\frac{|k+1|-1}{2}}(\lambda)$. Note that $Q^1_{\frac{|k+1|-1}{2}}(\lambda)$ is well defined since $\lambda>1$ and $\frac{|k+1|-1}{2}>-1$.

Denote $\nu=\frac{|k+1|-1}{2}$. Applying [20, S.2.9-4.] and the definition of $Q^1_\nu(\lambda)$, we obtain

$$\begin{aligned}Q^1_\nu(\lambda)&=-\frac{\nu+1}{2^{\nu+1}}\frac{\int_0^\pi(\lambda+\cos\alpha)^{-\nu}\sin^{2\nu+1}\alpha\,d\alpha}{(\lambda^2-1)^{\frac{1}{2}}}\\&=-\frac{\sqrt{\pi}\Gamma(\nu+2)\lambda^{-\nu}\,{}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\lambda^2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)(\lambda^2-1)}.\end{aligned}$$

We recall that the volume measure of the Riemannian metric ds_k defined in (8) is

$$dm_k=y_3^{-2k}dm$$

where dm is the usual Lebesgue measure. Its surface element is defined by $d\sigma_{(k)}=y_3^{-\frac{3k}{2}}d\sigma$. The outer normal in $\partial B_h(x,R_h)$ is denoted by n_e and the outer normal derivative is defined by $\frac{\partial u}{\partial n^k}=y_3^{\frac{k}{2}}\frac{\partial u}{\partial n_e}$.

We prove that the function

$$F_k(x,y)=-\frac{x_3^{\frac{k-2}{2}}y_3^{\frac{k-2}{2}}Q^1_\nu(\cosh d_h(x,y))}{\omega_3\sinh d_h(x,y)}$$

is the fundamental k-hyperbolic harmonic function at the point x (symmetrically y), that is $-\Delta_k F_k = \delta_x$ in the distributional sense with respect to the volume measure of the Riemannian metric ds_k and $\omega_3 = 2\pi^2$ is the Euclidean surface area of the unit ball in $\mathbb{H}$. We also remind that the fundamental k-harmonic function is unique up to the k-hyperbolic harmonic function.

We first verify the following crucial result.

Lemma 4. *Let x be a point in the upper half space and denote $\nu = \frac{|k+1|-1}{2}$. The function*

$$g_k\left(d_h(x,y)\right) = \frac{\nu+1}{2^{\nu+1}} \int_0^{\pi} \left(\cosh d_h(x,y) + \cos\alpha\right)^{-\nu} \sin^{2\nu+1}\alpha \, d\alpha$$
$$= \frac{\sqrt{\pi}\Gamma(\nu+2)\lambda^{-\nu} \, {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{\cosh^2 d_h(x,y)}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}$$

is positive and continuous for any $y \in \mathbb{H}_+$ and

$$g_k(0) = 1.$$

Proof. Applying properties of hypergeometric functions (see for example [2]) and the Gamma function, we infer that

$${}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; 1\right) = \frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(1)}{\Gamma\left(\frac{\nu+3}{2}\right)\Gamma\left(\frac{\nu+2}{2}\right)} = \frac{\Gamma\left(\nu+\frac{3}{2}\right)2^{\nu+1}}{\sqrt{\pi}\Gamma(\nu+2)}.$$

Hence $g_k(0) = 1$.

Next we prove that $F_k(x,y)$ is integrable in the hyperbolic ball $B_h(a,R_h)$ with respect to the Riemannian volume measure dm_k.

Lemma 5. *The function $F_k(x,y)$ is integrable in the hyperbolic ball $B_h(x,R_h)$with respect to the volume measure dm_k in the hyperbolic ball $B_h(x,R_h)$ and*

$$\int_{B_h(x,R_h)} F_k\left(d_h(y,x)\right) dm_k(y) \le 2^{-\frac{3k+4}{2}} M e^{\frac{|3k+2|}{2}} x_3^{-k} \sinh^2 R_h,$$

where $M = \max_{y \in \overline{B_h(x,R_h)}} \left(g_k(y,x)\right) \ge 1$.

Proof. Using Proposition 1 we infer that the hyperbolic ball $B_h(x,R_h)$ is an Euchlidean ball with the Euchlidean center $c_x(R_h) = x_0 + x_1 e_1 + x_2 e_2 + x_2 \cosh R_h$ and the Euchlidean radius $R_e = x_3 \sinh R_h$. Hence we deduce

$$\frac{g_k\left(d_h(x,y)\right)}{x_3^2 \sinh^2 d_h(y,x)} = \frac{g_k\left(d_h(x,y)\right)}{\|y - c_x(R_h)\|^2}$$

and in $B_h(x,R_h)$

$$2x_3e^{-R_h} = x_3(\cosh R_h - \sinh R_h) \leq y_3 \leq x_3(\cosh R_h + \sinh R_h) = 2x_3e^{R_h}$$

for all $y \in B_h(x, R_h)$. Since $g_k(d_h(x,y))$ is a continuous function, it attains its maximum in the closure of the ball $B_h(x, R_h)$. Since

$$\begin{aligned}\int_{B_h(x,R_h)} x_3^{-2}\sinh^{-2} d_h(y,x)\, dm(y) &= \int_{B_e(c_x(R_h), x_3 \sinh R_h)} \frac{dm(y)}{\|y - c_x(R_h)\|^2} \\ &= \int_0^{x_3 \sinh R_h} r \int_{\partial B_h(c_x(r_h),1)} dS dr \\ &= \frac{\omega_3 x_3^2 \sinh^2 R_h}{2}\end{aligned}$$

we conclude

$$\int_{B_h(x,R_h)} F_k(y,x)\, dm_k(y) \leq 2^{-\frac{3k+4}{2}} M e^{\frac{|3k+2|}{2}} x_3^{-k} \sinh^2 R_h.$$

We also need the result

Lemma 6. *Let $\Omega \subset \mathbb{H}_+$ be open and $\overline{B_h(x,R_h)} \subset \Omega$. Let u be a continuous real valued function in Ω. Then*

$$\lim_{R_h \to 0} \int_{\partial B_h(x,R_h)} u \frac{\partial F_k(x,y)}{\partial n^k} d\sigma_{(k)}(y) = -u(x).$$

Proof. Applying Proposition 1 we obtain that the outer normal at $y \in \partial B_h(x,R_h)$ is

$$n_e = (n_0, n_1, n_2, n_3) = \frac{(y_0 - x_0, y_1 - x_1, y_2 - x_2, y_3 - x_3 \cosh R_h)}{x_3 \sinh R_h}$$

In order to abbreviate the notations, we denote briefly $r_h = d_h(y,x)$. We compute the outer normal derivative by

$$\begin{aligned}\frac{\partial F_k(x,y)}{\partial n^k} =& y_3^{\frac{k}{2}} \frac{\partial F_k(x,y)}{\partial n_e} = y_3^{\frac{k}{2}} \langle n_e, \operatorname{grad} F_k(x,y)\rangle \\ =& y_3^{k-1} x_3^{\frac{k-2}{2}} \frac{\partial \frac{g_k(r_h)}{\sinh^2 r_h}}{\partial r_h} \sum_{i=0}^{3} n_i \frac{\partial r_h}{\partial y_i} \\ &+ \frac{k-2}{2} y_3^{\frac{k-2}{2}} n_3 F_k(x,y).\end{aligned}$$

Since $r_h = \operatorname{arcos} \lambda(\mathrm{y}, \mathrm{x})$ we deduce

$$\frac{\partial r_h}{\partial y_i} = \frac{\partial \arccos \lambda(y,x)}{\partial y_i} = \frac{y_i - x_i - x_3(\cosh r_h - 1)\delta_{i3}}{y_3 x_3 \sinh r_h}$$

and therefore the identity

$$\sum_{i=0}^{3} n_i \frac{\partial r_h}{\partial y_i} = \frac{1}{y_3}$$

holds. Hence we compute further

$$\begin{aligned}\frac{\partial F_k(x,y)}{\partial n^k} =& \frac{y_3^{k-2} x_3^{\frac{k-2}{2}}}{\omega_3 \sinh^2 r_h} \frac{\partial g_k(r_h)}{\partial r_h} + \frac{k-2}{2\omega_3} y_3^{k-2} n_3 F_k(x,y) \\ & - \frac{y_3^{k-2} x_3^{\frac{k-2}{2}} g_k(r_h) \cosh r_h}{\omega_3 \sinh^3 r_h}.\end{aligned}$$

Since $B_h(x,R_h) = B(c_x(R_h), x_3 \sin R_h)$ for $c_x(R_h) = x_0 + x_1 e_1 + x_2 e_2 + x_2 \cosh R_h$ we infer that

$$\lim_{R_h \to 0} \frac{x_3^{\frac{k-4}{2}}}{\omega_3 x_3^3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} \sinh R_h y_3^{k-2} \frac{\partial g_k}{\partial r_h}(R_h)\, d\sigma_{(k)} = 0.$$

Similarly, we compute that

$$\lim_{R_h \to 0} \frac{(k-2)\, x_3^{\frac{k-6}{2}}}{2\omega_3 x_3^3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} y_3^{k-2} (y_3 - x_3 \cosh R_h)\, g_k(R_h)\, d\sigma_{(k)} = 0.$$

Finally, manipulating the last integral, we obtain

$$\begin{aligned}&\lim_{R_h \to 0} -\frac{g_k(R_h) \cosh R_h}{\omega_3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} y_3^{k-2} x_3^{\frac{k-2}{2}}\, d\sigma_{(k)} \\ &= \lim_{r_h \to 0} -\frac{x_3^{\frac{k+4}{2}} \cosh R_h g_k(R_h)}{\omega_3 x_3^3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} y_3^{-\frac{k+4}{2}}\, d\sigma \\ &= -u(x),\end{aligned}$$

completing the proof.

Theorem 7. *Let $\Omega \subset \mathbb{H}_+$ be open and $B_h(a,\rho)$ a hyperbolic ball with a center a and the hyperbolic radius ρ satisfying $\overline{B_h(a,\rho)} \subset \Omega$. If u is a twice continuously differentiable functions in Ω and $x \in B_h(a,\rho)$ then*

$$\begin{aligned}u(x) =& \int_{\partial B_h(a,\rho)} \left(F_k(y,x) \frac{\partial u(y)}{\partial n^k} - u(y) \frac{\partial F_k(y,x)}{\partial n^k} \right) d\sigma_{(k)}(y) \\ & - \int_{B_h(a,\rho)} \Delta_k u(y) F_k(y,x)\, dm_k(y),\end{aligned}$$

where $dm_k = y_3^{-2k} dx$, $d\sigma_{(k)} = y_n^{-\frac{3k}{2}} d\sigma$ and the outer normal $\frac{\partial u}{\partial n^k} = y_3^{\frac{k}{2}} \frac{\partial u}{\partial n_e}$.

Proof. Denote $B_h(a,\rho) = B$ and pick a hyperbolic ball such that $\overline{B_h(x,R_h)} \subset B$. Denote $R = B \backslash \overline{B_h(x,R_h)}$. Since F_k is k-hyperbolic harmonic in R, we may apply the

Green's formula

$$\int_R (u\Delta_k v - v\Delta_k u)\, dm_k = \int_{\partial R} \left(u\frac{\partial v}{\partial n^k} - v\frac{\partial u}{\partial n^k} \right) d\sigma_{(k)}$$

of the Laplace-Beltrami operator

$$\Delta_k = x_3^k \left(\Delta - \frac{k}{x_3} \frac{\partial}{\partial x_3} \right)$$

with respect to the Riemannian metric ds_k^2 (see [1]) and obtain

$$\int_R F_k(y,x)\, \Delta_k u dx_k = \int_{\partial B} \left(F_k(y,x) \frac{\partial u}{\partial n^k} - u\frac{\partial F_k(y,x)}{\partial n^k} \right) d\sigma\,({}_k)$$
$$- \int_{\partial B_h(x,R_h)} (F_k(y,x) \frac{\partial u}{\partial n^k} - u\frac{\partial F_k(y,x)}{\partial n^k}) d\sigma_{(k)}.$$

Since $\frac{\partial u}{\partial n^k}$ and $y_3^{-\frac{2k+2}{2}} x_3^{\frac{k-2}{2}} g_k(d_h(x,y))$ are bounded we obtain

$$\int_{\partial B_h(x,R_h)} |F_k(y,x) \frac{\partial u}{\partial n^k}| d\sigma_{(k)}(y) \leq \frac{M}{\sinh^2 R} \int_{\partial B_h(x,R_h)} d\sigma = M \sinh R_h$$

and therefore

$$\lim_{R_h \to 0} \int_{\partial B_h(x,R_h)} |F_k(y,x) \frac{\partial u}{\partial n^k}| d\sigma_{(k)}(y) = 0.$$

Moreover, since $F_k(x,y)$ is integrable and u is bounded on $\overline{B}$ we infer

$$\int_{B_h(a,\rho)} \Delta_k u(y)\, F_k(y,x)\, dm_k = \lim_{R_h \to 0} \int_{R_h} F_k(y,x)\, \Delta_k u\; dm_k.$$

Then applying the previous result we conclude the result.

Using the standard methods, we deduce that

$$\phi(x) = -\int \Delta_k \phi(y)\, F_k(y,x)\, dm_k$$

for all $\phi \in \mathscr{C}_0^\infty(\mathbb{H}_+)$. Hence we have reached our main result.

Theorem 8. *Let x and y be poins in the upper half space and denote* $\nu = \frac{|k+1|-1}{2}$. *The fundamental k-hyperbolic harmonic function is*

$$
\begin{aligned}
F_k(x,y) &= -\frac{x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}} Q_\nu^1(\lambda(x,y))}{2^{\nu+1}\omega_3 \left(\lambda(x,y)^2-1\right)^{\frac{1}{2}}} \\
&= \frac{(\nu+1) x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}} \int_0^\pi (\lambda(x,y)+\cos\alpha)^{-\nu} \sin^{2\nu+1}\alpha\, d\alpha}{2^{\nu+1}\omega_3\left(\lambda(x,y)^2-1\right)} \\
&= \frac{\sqrt{\pi}\Gamma(\nu+2) x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}-1} \lambda^{-\nu}\, {}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\lambda^2}\right)}{2^{\nu+1}\omega_3\Gamma\left(\nu+\frac{3}{2}\right)\left(\lambda(x,y)^2-1\right)}.
\end{aligned}
$$

Corollary 4. *Let x and y be points in the upper half-space $\mathbb{H}_+$. Then*

$$F_k(x,y) = x_3^{k+1} y_3^{k+1} F_{-k-2}(x,y).$$

The previous result follows also from the correspondence principle of Weinstein (see [21]).

Lemma 7. *If we denote*

$$K_k(f) = \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3}$$

then

$$K_k(f) = x_3^{k+1} K_{-k-2}\left(x_3^{-k-1} f\right).$$

A kind of fundamental k-hyperbolic harmonic function has also been computed by GowriSankaram and Singman in [13] using more technical deductions. In order to compare the results, we first verify the following lemma.

Lemma 8. *Let $\lambda > 1$ and $\nu > -1$. Then*

$$\int_0^\pi (\lambda-\cos\alpha)^{-\nu-1}\sin^{2\nu+1}\alpha\, d\alpha = 2^{\nu+1} Q_\nu^0(\lambda)$$

and therefore

$$
\begin{aligned}
\left(\lambda^2-1\right)^{-\frac{1}{2}} Q_\nu^1(\lambda) &= -(\nu+1)2^{-\nu-1}\int_0^\pi (\lambda-\cos\alpha)^{-\nu-2}\sin^{2\nu+1}\alpha\, d\alpha \\
&= -2(\nu+1) x_2^{\nu+2} y_3^{\nu+2} \int_0^\pi \left(\|x-y\|^2 + 2x_3y_3(1-\cos\alpha)\right)^{-\nu-2} \sin^{2\nu+1}\alpha\, d\alpha.
\end{aligned}
$$

Proof. Appying [20, S.2.9-4.] and using complex numbers in computations, we obtain

$$
\begin{aligned}
Q_\nu^0(\lambda) &= e^{i(\nu+1)\pi} Q_\nu^0(-\lambda) = e^{i(\nu+1)\pi} 2^{-(\nu+1)} \int_0^\pi (-\lambda+\cos\alpha)^{-\nu-1}\sin^{2\nu+1}\alpha\, d\alpha \\
&= 2^{-(\nu+1)}\int_0^\pi (\lambda-\cos\alpha)^{-\nu-1}\sin^{2\nu+1}\alpha\, d\alpha
\end{aligned}
$$

Recalling the known formula

$$Q_\nu^1(\lambda) = \left(\lambda^2-1\right)^{\frac{1}{2}} \frac{d}{d\lambda} Q_\nu^0(\lambda)$$

we obtain the first equality. The second one follows from it when we substitute $\lambda = \frac{\|x-y\|^2+2x_3y_3}{2x_3y_3}$.

Theorem 9. *Let x and y be poins in the upper half space and denote* $\nu = \frac{|k+1|-1}{2}$. *The fundamental k-hyperbolic harmonic function is*

$$\begin{aligned}
\omega_3 F_k(x,y) &= \frac{(\nu+1)x_3^{\frac{k-2}{2}} y^{\frac{k-2}{2}} \int_0^\pi (\lambda-\cos\alpha)^{-\nu-2} \sin^{2\nu+1}\alpha\, d\alpha}{2^{\nu+1}} \\
&= 2(\nu+1)x_3^{\frac{k-2}{2}+\nu+2} y_3^{\frac{k-2}{2}+\nu+2} \int_0^\pi \left(\|x-y\|^2+2x_3y_3(1-\cos\alpha)\right)^{-\nu-2} \sin^{2\nu+1}\alpha\, d\alpha \\
&= \begin{cases} -\frac{k\int_0^\pi \left(\|x-y\|^2+2x_3y_3(1-\cos\alpha)\right)^{\frac{k-2}{2}} \sin^{-k-1}\alpha\, d\alpha,}{\omega_3}, & \text{if } k \le -1, \\ \frac{(k+2)x_3^{k+1}y_3^{k+1}\int_0^\pi \left(\|x-y\|^2+2x_3y_3(1-\cos\alpha)\right)^{-\frac{k+4}{2}} \sin^{k+1}\alpha\, d\alpha}{\omega_3}, & \text{if } k \ge -1. \end{cases}
\end{aligned}$$

We may compute the following special cases.

1. Let $k=0$. Then

$$\begin{aligned}
\phi(x,y) &= \frac{1}{2\omega_3 x_3 y_3}\left(\frac{1}{\lambda-1} - \frac{1}{\lambda+1}\right) \\
&\frac{1}{\omega_3}\left(\frac{1}{\|x-y\|^2} - \frac{1}{\|x-y^*\|^2}\right)
\end{aligned}$$

2. Let $k=-2$. Then

$$\begin{aligned}
F_{-2}(x,y) &= \frac{1}{2\omega_3 x_3^2 y_3^2}\int_0^\pi \left(\cosh d_h(x,y) - \cos\alpha\right)^{-2} \sin\alpha d\alpha \\
&= \frac{1}{2\omega_3 x_3^2 y_3^2}\left(\frac{1}{\lambda-1} - \frac{1}{\lambda+1}\right) \\
&= \frac{1}{\omega_3 x_3^2 y_3^2 (\lambda^2-1)} \\
&= \frac{1}{2\omega_3 x_3 y_3}\left(\frac{1}{\|x-y\|^2} - \frac{1}{|x-y^*|^2}\right) \\
&= \frac{4}{\omega_3 \|x-y\|^2 \|x-y^*\|^2}.
\end{aligned}$$

3. Let $k=2$, then

$$
\begin{aligned}
2\omega_3^{-1}F_2(x,y) &= \int_0^{\pi}(\cosh d_h(x,y)-\cos\alpha)^{-3}\sin^3\alpha d\alpha \\
&= \left[-2^{-1}(\cosh d_h(x,y)-\cos\alpha)^{-2}\sin^2\alpha\right]_0^{\pi} \\
&\quad + \int_0^{\pi}(\cosh d_h(x,y)-\cos\alpha)^{-2}\sin\alpha\cos\alpha d\alpha \\
&= -\left[(\cosh d_h(x,y)-\cos\alpha)^{-1}\cos\alpha\right]_0^{\pi} \\
&\quad - \int_0^{\pi}(\cosh d_h(x,y)-\cos\alpha)^{-1}\sin\alpha d\alpha \\
&= \frac{1}{\lambda-1}+\frac{1}{\lambda+1}-(\log(\lambda+1)-\log(\lambda-1)) \\
&= \frac{2\lambda}{\lambda^2-1}-\log(\lambda+1)+\log(\lambda-1).
\end{aligned}
$$

Comparing this function with the kernel function computed in [12], we obtain

$$
\begin{aligned}
-\int_{\frac{\|a-x\|}{\|x-a^*\|}}^{1}\frac{\left(1-s^2\right)^2}{s^3}ds &= -\int_{\frac{\|a-x\|}{\|x-a^*\|}}^{1}\left(s^{-3}-2s^{-1}+s\right)ds \\
&= \frac{|x-a^*|^2}{2\|a-x\|^2}+2\log\frac{\|a-x\|}{\|x-a^*\|}-\frac{1}{2}\frac{\|a-x\|^2}{\|x-a^*\|^2}.
\end{aligned}
$$

Applying the properties (1) and (2), we infer that

$$
-\frac{1}{4}\int_{\frac{\|a-x\|}{\|x-a^*\|}}^{1}\frac{\left(1-s^2\right)^2}{s^3}ds = \frac{\lambda}{\lambda^2-1}-\frac{\log(\lambda+1)}{2}+\frac{\log(\lambda-1)}{2}
$$

In order to compute the kernel function for k-hyperregular functions, we need the following lemma (see [12]).

Lemma 9. *If $a\in\mathbb{R}_+^{n+1}$ and $c_a(d_h(x,a)) = a_0+a_1e_1+a_2e_2+a_3\cosh d_h(x,a)e_3$ then*

$$
\overline{D}^x\lambda(x,a) = \frac{\overline{x-c_a(d_h(x,a))}}{x_3a_3}.
$$

Theorem 10. *Denote $r_h = d_h(x,y)$, $t=\frac{k-2}{2}$, $\nu=\frac{|k+1|-1}{2}$ and define as earlier*

$$
g_k(d_h(x,y)) = \frac{|k+1|+1}{2^{\nu+2}}\int_0^{\pi}(\cosh d_h(x,y)+\cos\alpha)^{-\nu}\sin^{2\nu+1}\alpha\, d\alpha.
$$

The k-hyperregular kernel is the function

$$\begin{aligned}
h_k(x,y) &= \overline{D}^x\left(F_k(x,y)\right) \\
&= -x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}} \left(\frac{tg(r_h)\,e_3}{2x_3\sinh^2 r_h} + \left(\frac{\sinh r_h g'(r_h) - 2g(r_h)\cosh r_h}{2\sinh^4 r_h} \right) \frac{\overline{x - c_y(r_h)}}{x_3 y_3} \right) \\
&= x_3^{\frac{k-2}{2}} y_3^{\frac{k+4}{2}} w_k(x,y)\, p(x,y) \\
&= x_3^{\frac{k-2}{2}} y_3^{\frac{k+4}{2}} p(x,y)\, v_k(x,y)
\end{aligned}$$

where

$$w_k(x,y) = -te_3 g_k(r_h)\frac{x-Py}{y_3} + \sinh r_h g_k'(r_h) - (t+2)\, g_k(r_h)\cosh r_h$$

and

$$p(x,y) = \frac{(x - c_y(r_h))^{-1}}{2x_3 \|x - c_y(r_h)\|^2}$$

is 2-hyperregular with respect to x.

Proof. The function $F_k(x,y)$ is k-hyperbolic and therefore the function $h_k = \overline{D}^x F_k(x,y)$ is k-hyperregular outside y and y^*. Denoting $t = \frac{k-2}{2}$ and $\lambda(x,y) = \cosh r_h$, we compute as follows

$$\frac{h_k(x,y)}{x_3^{\frac{k-2}{2}} y_3^{\frac{k+4}{2}}} = -\frac{te_3 g(r_h)}{2x_3 y_3^3 \sinh^2 r_h} + \left(\frac{\sinh r_h g'(r_h) - 2g(r_h)\cosh r_h}{2y_3^3 \sinh^3 r_h} \right) \overline{D}^x r_h.$$

Applying [12] we obtain

$$\overline{D}^x r_h = \frac{\overline{x - c_y(r_h)}}{x_3 y_3 \sinh r_h}$$

and

$$\frac{x_3 \overline{D}^x r_h}{y_3^3 \sinh^3 r_h} = \frac{\overline{x - c_y(r_h)}}{\|x - c_y(r_h)\|^4} = \frac{(x - c_y(r_h))^{-1}}{\|x - c_y(r_h)\|^2}.$$

Since

$$\begin{aligned}
\frac{x - c_y(r_h)}{x_3 y_3} \frac{(x - c_y(r_h))^{-1}}{\|x - c_y(r_h)\|^2} &= \frac{1}{x_3 y_3 \|x - c_y(r_h)\|^2} \\
&= \frac{1}{x_3 y_3^3 \sinh^2 r_h}.
\end{aligned}$$

Hence we obtain

$$\frac{h_k(x,y)}{y_3^{t+3} x_3^t} = w_k(x,y)\frac{(x - c_y(r_h))^{-1}}{2x_3 \|x - c_y(r_h)\|^2},$$

where

$$w_k(x,y) = -te_3 g_k(r_h)\frac{x-Py}{y_3} + \sinh r_h g_k'(r_h) - (t+2)\, g_k(r_h)\cosh r_h.$$

Using the similar deductions as in [4] we may prove the formula for S and T-parts.

Theorem 11. *Let Ω and be an open subsets of $\mathbb{H}_+$ (or $\mathbb{H}_{-}$). Assume that K is an open subset of Ω and $\overline{K} \subset \Omega$ is a compact set with the smooth boundary whose outer unit normal field is denoted by ν. If f is $k-$hyperregular in Ω and $a \in K$, then*

$$\begin{aligned} Sf(a) &= -\frac{1}{2}\int_{\partial K} S\left(h_k(y,a)\,\nu f(y) + f(y)\,\nu h_k(y,a)\right)\frac{d\sigma}{y_3^k} \\ &= \frac{1}{2}\int_{\partial K} S\left[h_k(y,a), \overline{\nu(y)}, f(y)\right]\frac{d\sigma}{y_3^k} - \frac{1}{2}\int_{\partial K} Sh_k(y,a)\left\langle \overline{\nu(y)}, f(y)\right\rangle \frac{d\sigma}{y_3^k}. \end{aligned}$$

Proof. Let $a \in K$. Denote $R = K \backslash B_h(a, r_h)$ and

$$A = \int_{\partial K} S\left(h_k(y,a)\,\nu f(y) + f(y)\,\nu h_k(y,a)\right)\frac{d\sigma}{y_3^k}.$$

Then we obtain

$$\begin{aligned} 0 &= \int_{\partial R} S\left(h_k(y,a)\,\nu f(y) + f(y)\,\nu h_k(y,a)\right)\frac{d\sigma}{y_3^k} \\ &= A - \int_{\partial B_h(a,r_h)} S\left(h_k(y,a)\,\nu(y) f(y) + f(y)\,\nu(y)\,h_k(y,a)\right)\frac{d\sigma}{y_3^k}. \end{aligned}$$

By virtue of Proposition 1, we deduce that

$$\nu(y) = \frac{y - c_a(r_h)}{\|y - c_a(r_h)\|}.$$

Hence we obtain

$$\begin{aligned} A &= -\lim_{r_h} \frac{a_3^{\frac{k-4}{2}}}{2\omega_3 \|a - c_a(r_h)\|^3} \int_{\partial B_h(a,r_h)} S\left(w_k(y,a) f(y) + f(y)\, v_k(y,a)\right)\frac{d\sigma}{y_3^{\frac{k-4}{2}}} \\ &= -2f(a) \end{aligned}$$

The last formula follows from (7) and the definition of the triple product.

Similarly we may verify the result for the T-part. The main difference is that we use the surface measure $d\sigma$, not $y_3 d\sigma$.

Theorem 12. *Let Ω be an open subsets of $\mathbb{H}_+$ (or $\mathbb{H}_{-}$). Assume that K is an open subset of Ω and $\overline{K} \subset \Omega$ is a compact set with the smooth boundary whose outer unit normal field at y is denoted by ν. If f is $k-$hyperregular in Ω and $a \in K$,*

$$
\begin{aligned}
Tf(a) &= -\frac{a_3^k}{2}\int_{\partial K} T\left(h_{-k}(y,a)\,\nu(y)\,f(y)+f(y)\,\nu(y)\,h_{-k}(y,a)\right)d\sigma \\
&= \frac{a_3^k}{2}\left(\int_{\partial K} T\left[h_{-k}(y,a),\overline{\nu(y)},f(y)\right]d\sigma-\int_{\partial K} Th_{-k}(y,a)\left\langle\overline{\nu(y)},f(y)\right\rangle d\sigma\right).
\end{aligned}
$$

5 Conclusion

Our main results produce integral formulas for the T- and S-parts of k-hyperregular functions. An interesting problem is to research integral operators produced by these formulas. However, these results requires much computations and therefore they are left to the consequent publications.

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