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Will the boundary (-domain) integral equation method survive?

Preface to the special issue on non-traditional boundary (-domain) integral equation methods

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The Boundary Integral Equation (BIE) method (boundary element method, elastic potential method) has been intensively developed over recent decades both in theory and in engineering applications. Its popularity was due to reducing a Boundary Value Problem (BVP) for a partial differential equation in a domain to an integral equation on the domain boundary, that is, to diminishing the problem dimensionality by one. The main ingredient necessary for the reduction of a BVP to a BIE is a fundamental solution to the original partial differential equation. Employing the fundamental solution in the corresponding Green formula, one can reduce the problem to a boundary integral equation. After an appropriate discretization, this leads to a relatively small system of linear algebraic equations, which can be solved using small computer resources.

In spite of these evident advantages, the popularity of BIE method does not look high nowadays. Although BIEs have their established niche in problems for infinite or semi-infinite domains with constant coefficients, appearing, e.g., in geomechanics, acoustics, fluid mechanics and some other engineering applications, the computational mechanics market is dominated by the Finite Element Method (FEM), at least in solid mechanics. Several reasons for this are listed below.

First, the matrix of the linear algebraic equation system obtained after BIE discretization, is dense, while for FEM it is sparse and moreover, the number of non-zero entries in each of the FEM equations is determined by the element type and is practically independent of the mesh refinement. This outweighs the bigger FEM matrix size for 3D problems.

Second, to be useful for BIE numerical applications, the fundamental solution should be available in an analytical form and/or cheaply calculated. However, such a fundamental solution is generally not available if the coefficients of the original BVP are not constant. One can use, in this case, a parametrix (Levi function), which is usually available, instead of the fundamental solution in the Green formulae. A parametrix is particularly given by the fundamental solution of the corresponding "frozen-coefficient" problem and is much wider available. This allows a reduction of the problem not to boundary but to Boundary-Domain Integral or Integro-Differential Equation, BDI(D)E. For numerical solving the BDI(D)E, one should then discretize not only the domain boundary but also the domain itself, which leads after discretization to a system of linear algebraic equations of about the same size as in the FEM, without any dimension diminution. Unfortunately, the system matrix, unlike FEM, is dense, which prevents application of the economical methods developed for sparse systems. The same problem occurs in the BDIEs of non-linear problems.

Third, the generation of a discrete matrix for BIE or BDI(D)E is rather expensive computationally in comparison with the FEM matrix, unless the fundamental solution is very simple, like, e.g., for the Laplace or homogeneous linear elasticity equations.

Sometimes the higher theoretical complexity of the BIE methods is perceived as another reason prohibiting their wide spread. This may indeed influence the code developers but should be compensated by the code efficiency if it is gained. However, the application of a general commercial package by a designer is not supposed to require a special mathematical qualification anyway. On the other hand, the rigorous mathematical backgrounds of the FEM are not much simpler than those of the BIE method.

Nevertheless, the first three challenges sound pretty serious and it is hard to expect that BIE method will remain a widely used numerical application tool unless they are addressed. That is, some method developments should be aimed at making the BIE matrix sparse; at extending the method effectively to a wider range of PDEs, particularly to the variable-coefficient and non-linear problems; and at decreasing complexity of the matrix generation.

The articles of this special issue deal with some of these challenges.

The paper by Chudinovich, Constanda & Dolberg derives a fundamental solution for linear dynamic coupled thermoelastic equations of shear deformable plates. The paper by Purbolaksono & Aliabadi is devoted to solution of equations of non-linear elastic shear deformable plates by reducing them to nonlinear boundary-domain integral equations solved then iteratively. The paper by Hiptmair & Ostrowski describes formulation, analysis, discretization and numerical implementation of direct boundary integral equations of a steady-state electromagnetic transmission problem. All the three papers address the second challenge, extending the integral equation methods to complicated problems.

The papers by Sladek, Sladek & Zhang; by Mikhailov & Nakhova; and by Mikhailov develop the Localized Boundary-Domain Integral Equation Method emerged recently, which addresses the first, second and (to some extent) the third challenge of the LBDI(D)Es making them competitive with the FEM for variable-coefficient and nonlinear problems. The method employs localized parametrices to reduce linear and non-linear BVPs with variable coefficients to Localized Boundary-Domain Integral or Integro-Differential Equations. After a locally-supported mesh-based or mesh-less discretization this ends up in sparse systems of algebraic equations solved numerically. The parametrices and their localization can be specially chosen to simplify the integral evaluation in the system matrix generation.

The articles presented in this issue do not constitute a survival kit for the boundary(-domain) integral equation method but hopefully provide some elements for reaching that goal.