An orthogonality relation for a class of problems with high-order boundary conditions; applications in sound/structure interaction

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May 1998

Abstract

There are numerous interesting physical problems, in the fields of elasticity, acoustics and electromagnetism etc., involving the propagation of waves in ducts or pipes. Often the problems consist of pipes or ducts with abrupt changes of material properties or geometry. For example, in car silencer design, where there is a sudden change in cross-sectional area, or when the bounding wall is lagged. As the wavenumber spectrum in such problems is usually discrete, the wave-field is representable by a superposition of travelling or evanescent wave modes in each region of constant duct properties. The solution to the reflection or transmission of waves in ducts is therefore most frequently obtained by mode-matching across the interface at the discontinuities in duct properties. This is easy to do if the eigenfunctions in each region form a complete orthogonal set of basis functions; therefore, orthogonality relations allow the eigenfunction coefficients to be determined by solving a simple system of linear algebraic equations.

The objective of this paper is to examine a class of problems in which the boundary conditions at the duct walls are not of Dirichlet, Neumann or of impedance type, but involve second or higher derivatives of the dependent variable. Such wall conditions are found in models of fluid/structural interaction, for example membrane or plate boundaries, and in electromagnetic wave propagation. In these models the eigenfunctions are not orthogonal, and also extra edge conditions, imposed at the points of discontinuity, must be included when mode matching. This article presents a new orthogonality relation, involving eigenfunctions and their derivatives, for the general class of problems involving a scalar wave equation and high-order boundary conditions. It also discusses the procedure for incorporating the necessary edge conditions. Via two specific examples from structural acoustics, both of which have exact solutions obtainable by other techniques, it is shown that the orthogonality relation allows mode matching to follow through in the same manner as for simpler boundary conditions. That is, it yields coupled algebraic systems for the eigenfunction expansions which are easily solvable, and by which means more complicated cases, such as that illustrated in figure 1, are tractable.

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1 Introduction

There is a vast number of physical situations that can be modelled in terms of the propagation and scattering of acoustic waves in a waveguide with high-order boundary conditions. The waveguide may have discontinuities in the material properties of its boundaries (two or multi-part boundary conditions) but be continuous in geometry, in which case the problem may be solved by recourse to the classical Wiener-Hopf technique. If, however, the waveguide is discontinuous in geometry (i.e. abrupt changes in width) then alternative solution methods must be sought (see figure 1). Mode-matching of the eigenfunction expansions of the velocity potential at the interface of each discontinuity provides an attractive way forward. Not only are eigenfunction expansions conceptually simple but they often lead to numerically efficient (albeit infinite) systems of equations to solve as opposed to the rather more daunting Fourier integral representations that are obtained via application of the Wiener-Hopf technique or Green's function methods. However, unless the eigenfunctions form an orthogonal basis set, or at least the algebraic system has a matrix which is diagonally dominant, the eigenfunction approach is rendered ineffectual. In other words, if for a general solution there are many eigenfunctions of similar weight required so that their contributions do not reduce much in significance as the mode number increases, then not only will it be difficult to invert such systems but their use as an approximating expansion will be curtailed. In such cases the only viable way forward is to establish an orthogonality relation by which the Fourier coefficients of the eigenfunction expansion can be isolated and expressed in terms of known boundary data.

The complexity of an orthogonality relation depends both on the type of boundary that forms the surface of the waveguide and the order of the field equation. For problems in which the waveguide walls comprise of soft, hard or impedance (Robin's condition) surfaces and the field equation is no higher than second order, the solution can be found, using separation of variables techniques, in terms of an eigenfunction expansion. The resulting eigen-sub-system will be of Sturm-Liouville type and, thus, the eigenfunctions satisfy well-known, simple orthogonality relations. Hence, for problems of mode-matching across the interface between two semi-infinite (or finite) regions, the orthogonality relation permits the problem to be reduced to that of solving a well-behaved infinite system of linear algebraic equations. This approach has been utilized by many authors, in an enormous variety of subject disciplines, enabling them to solve a wide range of problems involving complicated discontinuous geometric structures. In the fields of water waves, acoustics and electromagnetic theory etc., orthonormal basis functions have been, and continue to be extremely useful, as can be seen in, for example, Lebedev et al. (1979), Evans & Linton (1991), Peat (1991), Evans & Porter (1995), and Dalrymple & Martin (1996).

In contrast, for higher-order field equations, separation of variables will lead to eigenfunction expansions for which the eigen-sub-system is usually not Sturm-Liouville – even for simple boundary conditions. For example, in the field of elasticity, Folk & Herczynski (1986) and Herczynski & Folk (1989) consider a system in which separation of variables reduces the governing equations to a pair of coupled or uncoupled second-order ordinary differential equations which are solved subject to impedance type boundary conditions. The resulting eigensystem is not Sturm-Liouville but nevertheless they are able to derive an orthogonality relation via which the problem in question can be solved. This orthogonality relation is, in fact, equivalent to one formerly derived by Fama (1972) in the context of the elastostatic response of a circular cylinder. Other examples of non Sturm-Liouville systems which possess orthogonality relations can be found in the contexts of fluid flows (Orr-Sommerfeld equation) in Drazin & Reid (1981); viscoelastic motions in Shen & Mote (1992); fluid-loaded elastic structures in Murphy et al. (1994) and Zheng-Dong & Hagiwara (1991); electromagnetism in Seligson (1988); and porous media in Scandrett & Frenzen (1995). As a further example of a system governed by a fourth-

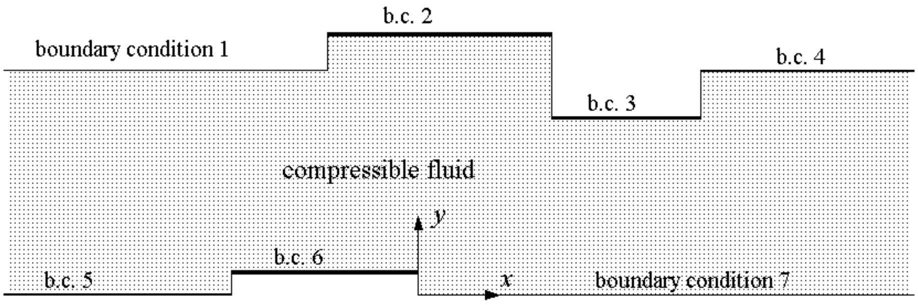

FIG. 1. The general class of boundary value problems for which an orthogonality relation is derived. The governing equation is the scalar wave equation and the boundary conditions can have high-order derivatives.

order ordinary differential equation, Rao & Rao (1988) obtained an orthogonality relation for the *in vacuo* eigenmodes of a thin elastic plate. High-order field equations, as in the examples cited above, usually lead to orthogonality relations that are not simple. These often consist of finite integrals of a linear combination of eigenfunctions and their derivatives.

Even when applied in conjunction with a separable second-order field equation, high-order boundary conditions, such as those that describe the fluid-coupled motion of a membrane or elastic plate (Junger & Feit, 1986), give rise to non-Sturm-Liouville eigen-sub-systems. The eigenvalues are now defined as the roots of a complicated dispersion relation and, as a consequence, the eigenfunctions are not usually orthogonal even with respect to a weight function. With the wealth of literature available on the subject of mode-matching, it seems remarkable that boundary value problems involving waveguides with high-order boundary conditions have not been extensively studied with a view to establishing the relevant orthogonality relations. Wu et al. (1995) examine the wave-induced response of an elastic floating plate using modal expansions of the structural motion, but their approach by-passes the issue to some extent by considering the forced response of the plate rather than the fluid-coupled motion. There may be a number of reasons for the limited literature in this area. Firstly, as in the case of high-order field equations, the orthogonality relations are likely to be non-simple in form. Secondly, for problems involving high-order boundary conditions on semi-infinite or finite domains, there is inevitably the question of how to impose the edge conditions at the junction of discontinuity. Thus, derivation of an appropriate orthogonality relation is not, in itself, sufficient to enable this class of boundary value problems to be solved. A practical and convenient means of imposing the edge conditions is also required. The aim of this paper is to address both these points: a general orthogonality relation for a class of boundary value problems with high-order boundary conditions is derived and simple procedures are demonstrated by which appropriate edge conditions can be incorporated. The general results derived in this article have already been successfully applied to solve a problem involving the scattering of sound waves in a waveguide with discontinuities in both geometry and material property (Warren & Lawrie, 1996), that is, the two-dimensional duct changes both its height and boundary conditions at some point (for example the origin shown in figure 1). It is anticipated that the eigenfunction expansion method proposed herein will enable many more complicated problems of this class to be solved.

In section 2 the class of boundary value problems is described and a detailed derivation of the general orthogonality relation is given. Following, in sections 3 and 4, the application of the orthogonality relation is demonstrated by two specific examples. These illustrate the two approaches by which appropriate physical edge conditions can be enforced. Finally, section 5 briefly summarizes the results and techniques presented in this article.

2 The orthogonality relation

In this section an orthogonality relation for a general class of boundary value problems is derived. This group of problems is one that occurs in a number of fields of applied mathematics, for example, water-waves, elasticity, acoustics and electromagnetic theory. Common to all these areas are problems which involve the propagation of waves along waveguides in which one or both of the boundaries is described by the Robin's or a higher-order condition. It is with the high-order boundary conditions that this section, and indeed this article, is concerned.

2.1 The generalized boundary value problem

For ease of exposition, the general boundary value problem is posed in terms of Helmholtz' (or the reduced wave) equation, that is

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 1 \right\} \phi(x,y) = 0 \tag{2.1}$$

in which x and y are the usual Cartesian coordinates but are non-dimensionalized with respect to k^{-1} where k is the fluid wavenumber. The advantage of a non-dimensionalized analysis is to omit as much mathematical clutter as possible and to present the minimum number of free (non-dimensional) parameters in the equations. It is a simple matter to return to the physical variables and confirm they have the correct dimensional form. More complicated governing equations to that in (2.1) can easily be tackled, for example Herczynski & Folk (1989) consider a system in which the governing equations comprise two coupled second-order ordinary differential equations but with Robin's boundary conditions. In this article the field equation holds in a strip of finite height $0 \le y \le a$ and infinite length $-\infty < x < \infty$ which is bounded by walls with high-order boundary conditions. The mathematical statement of the most general, physically relevant, pair of boundary conditions consists of

$$\mathcal{L}_a\left(\frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial y} + \mathcal{M}_a\left(\frac{\partial}{\partial x}\right)\phi = 0, \quad y = a, \quad -\infty < x < \infty, \tag{2.2}$$

on the upper waveguide surface together with

$$\mathcal{L}_0\left(\frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial y} + \mathcal{M}_0\left(\frac{\partial}{\partial x}\right)\phi = 0, \quad y = 0, \quad -\infty < x < \infty, \tag{2.3}$$

on the lower surface. Here $\mathcal{L}_p(\frac{\partial}{\partial x})$ and $\mathcal{M}_p(\frac{\partial}{\partial x})$, $p = a$,0, are differential operators of the form

$$\mathcal{L}_p\left(\frac{\partial}{\partial x}\right) = \sum_{k=0}^{K_p} c_k^p \frac{\partial^{2k}}{\partial x^{2k}}, \quad \mathcal{M}_p\left(\frac{\partial}{\partial x}\right) = \sum_{j=0}^{J_p} d_j^p \frac{\partial^{2j}}{\partial x^{2j}}, \tag{2.4}$$

where c_k^0, c_k^a, d_j^0, d_j^a are constant coefficients. Note that, for physical reasons, only even derivatives in x are included. Higher derivatives in y are easily removed by recourse to equation (2.1), hence the absence of such terms in (2.2) and (2.3).

The general solution of the boundary value problem described by (2.1)–(2.4) can be expressed as a separable eigenfunction expansion of the form

$$\phi(x,y) = \sum_{n=0}^{\infty} \sigma_n Y_n(y) e^{\pm i\nu_n x}. \tag{2.5}$$

Here σ_n are arbitrary constants, $Y_n(y)$ satisfies the eigensystem

$$Y_n''(y) = \gamma_n^2 Y_n(y), \quad \gamma_n = (\nu_n^2 - 1)^{1/2} \tag{2.6}$$

where the primes denote differentiation with respect to y and the eigenvalues ν_n, $n = 0, 1, 2, \ldots$ are defined as the roots of the coupled equations

$$\begin{aligned} P_a(\nu_n)Y_n'(a) + Q_a(\nu_n)Y_n(a) &= 0, \quad (2.7)\\ P_0(\nu_n)Y_n'(0) + Q_0(\nu_n)Y_n(0) &= 0. \quad (2.8) \end{aligned}$$

The functions $P_p(\nu_n)$ and $Q_p(\nu_n)$, $p = a, 0$, are characteristic polynomials and correspond to the action of the operators $\mathcal{L}_p(\frac{\partial}{\partial x})$ and $\mathcal{M}_p(\frac{\partial}{\partial x})$ on the eigen-expansion (2.5), i.e. $P_p(\nu_n) \equiv \mathcal{L}_p(i\nu_n)$, $Q_p(\nu_n) \equiv \mathcal{M}_p(i\nu_n)$.

It is a simple matter to show that (2.6)–(2.8) imply an explicit form for $Y_n(y)$:

$$\begin{aligned} Y_n(y) &\propto P_0(\nu_n)\cosh(\gamma_n y) - \frac{1}{\gamma_n} Q_0(\nu_n)\sinh(\gamma_n y) \\ &\propto P_a(\nu_n)\cosh(\gamma_n(a-y)) + \frac{1}{\gamma_n} Q_a(\nu_n)\sinh(\gamma_n(a-y)) \quad (2.9) \end{aligned}$$

from which we immediately deduce the dispersion relation, or consistency condition for (2.9), which the eigenvalues ν_n must satisfy. This is

$$K(\nu) = \left[Q_0(\nu)Q_a(\nu) - \gamma^2 P_0(\nu)P_a(\nu)\right]\frac{\sinh(\gamma a)}{\gamma} + \left[P_a(\nu)Q_0(\nu) - Q_a(\nu)P_0(\nu)\right]\cosh(\gamma a) = 0, \quad (2.10)$$

where $\gamma^2 = \nu^2 - 1$. The fact that the operators (2.4) contain only even derivatives in x ensures that the characteristic polynomials are functions of ν^2. Thus, the dispersion relation can be expressed as a function of even powers of γ, and it is a straightforward matter to prove the following for the general case when $P_p(\nu)$, $Q_p(\nu)$, $p = 0, a$, contain real coefficients:
(i) for every root ν there is another root $-\nu$;
(ii) there is a finite number of real roots, the particular number depending on the number of real zeros of the polynomials in the square brackets in (2.10), located on $|\nu| > 1$;
(iii) there is an infinite number of roots located on the imaginary axis of γ, or equivalently a finite number on $|\Re(\nu)| < 1$, $\Im(\nu) = 0$, and an infinite number on $\Re(\nu) = 0$;
(iv) there is a finite number of roots, ν, with non-zero real and imaginary parts.
A convention can be employed that the $+\nu_n$ roots have either

$$\Re(\nu_n) > 0 \quad \text{or} \quad \Re(\nu_n) = 0, \Im(\nu_n) > 0. \quad (2.11)$$

The main focus of this article is an orthogonality relation and thus the actual location of the roots of $K(\nu) = 0$ is immaterial; all that is required is that $Y_n(y) : n = 0, 1, 2, \ldots$, where the $Y_n(y)$ are given in (2.9), be complete. Here it is asserted that the summation in (2.5) is over a complete set of eigenfunctions in view of the fact that there is no continuous spectrum of eigenfrequencies, ν, arising from the dispersion relation (2.10), that is, no branch-cut contributions. One procedure by which this may be proved involves deriving the Green's function solution for an infinite duct with boundary conditions (2.2), (2.3) by Fourier transform methods; a solution for any particular forcing can be obtained by convolving the Green's function with the forcing data. Since the Green's function is meromorphic (with, in general, only simple poles) for this class of problems, this solution is also free of branch-cuts. Therefore, deformation of the inverse Fourier integral contour allows the solution to be expressed as a superposition of discrete eigenfunctions of the form (2.5) (cf. expansion (3.17) in section 3).

We wish to employ different eigenfunction expansions, of the form (2.5), in each of the uniform regions of the duct between the discontinuities, such as are shown in figure 1. In each segment the eigenfunctions and eigenvalues will be known, but the coefficients σ_n must be found by mode-matching across the interfaces at the location of the discontinuities. Essential to this procedure is the existence of an orthogonality relation for the eigensystem. The following theorem states the appropriate orthogonality relation for eigensystems of the class described by (2.6) – (2.8).

2.2 The orthogonality relation

Theorem

The eigenfunctions $Y_n(y)$ described by (2.6)–(2.8) have the orthogonality property

$$\int_0^a \{(\gamma_n^2 Y_n Y_m' + \gamma_m^2 Y_m Y_n') + \frac{B_{mn}}{A_{mn}}(\gamma_n^2 Y_n Y_m + Y_m' Y_n') - \frac{B_{nm}}{A_{mn}}(\gamma_m^2 Y_m Y_n + Y_m' Y_n')\}\, dy = \begin{cases} 0, & m \neq n \\ C_n, & n = m \end{cases} \quad (2.12)$$

or equivalently, by (2.6)

$$\int_0^a \{(Y_n'' Y_m' + Y_m'' Y_n') + \frac{B_{mn}}{A_{mn}}(Y_n'' Y_m + Y_m' Y_n') - \frac{B_{nm}}{A_{mn}}(Y_m'' Y_n + Y_m' Y_n')\}\, dy = \begin{cases} 0, & m \neq n \\ C_n, & n = m \end{cases} \quad (2.13)$$

which may be written

$$-(\gamma_m^2 - \gamma_n^2)\frac{B_{nm}}{A_{mn}}\int_0^a Y_n Y_m\, dy + Y_n'(a)Y_m'(a) + \frac{(B_{mn} - B_{nm})}{A_{mn}}Y_n'(a)Y_m(a) - Y_n'(0)Y_m'(0) - \frac{(B_{mn} - B_{nm})}{A_{mn}}Y_n'(0)Y_m(0) = \begin{cases} 0, & m \neq n \\ C_n, & n = m \end{cases} \quad (2.14)$$

where

$$A_{mn} = Q_a(\nu_m)Q_0(\nu_n)P_a(\nu_n)P_0(\nu_m) - Q_a(\nu_n)Q_0(\nu_m)P_a(\nu_m)P_0(\nu_n), \quad (2.15)$$

$$B_{mn} = Q_a(\nu_m)Q_0(\nu_m)\{Q_0(\nu_n)P_a(\nu_n) - Q_a(\nu_n)P_0(\nu_n)\}, \quad (2.16)$$

and the non-zero constant C_n is defined by

$$C_n = \Big[(Y_n')^2 + D_n Y_n' Y_n\Big]_0^a - E_n \int_0^a Y_n^2\, dy \quad (2.17)$$

with

$$D_n = \frac{Q_0^2(\nu_n)\{Q_a'(\nu_n)P_a(\nu_n) - Q_a(\nu_n)P_a'(\nu_n)\} - Q_a^2(\nu_n)\{Q_0'(\nu_n)P_0(\nu_n) - Q_0(\nu_n)P_0'(\nu_n)\}}{Q_0(\nu_n)P_a(\nu_n)\{Q_a(s)P_0(s)\}'|_{s=\nu_n} - Q_a(\nu_n)P_0(\nu_n)\{Q_0(s)P_a(s)\}'|_{s=\nu_n}}, \quad (2.18)$$

$$E_n = \frac{2\nu_n B_{nn}}{Q_0(\nu_n)P_a(\nu_n)\{Q_a(s)P_0(s)\}'|_{s=\nu_n} - Q_a(\nu_n)P_0(\nu_n)\{Q_0(s)P_a(s)\}'|_{s=\nu_n}}. \quad (2.19)$$

Proof

First consider the case in which $n \neq m$. Expression A_{mn} times (2.12) may be rewritten as

$$\int_0^a \{A_{mn}\frac{d}{dy}(Y_n' Y_m') + B_{mn}\frac{d}{dy}(Y_n' Y_m) - B_{nm}\frac{d}{dy}(Y_m' Y_n)\}\, dy = 0, \quad n \neq m. \quad (2.20)$$

It clearly follows that

$$[A_{mn} Y_n' Y_m' + B_{mn} Y_n' Y_m - B_{nm} Y_n Y_m']_0^a = 0. \quad (2.21)$$

With the choice of A_{mn} and B_{mn} given in (2.15) – (2.16), it is straightforward but tedious to show that (2.21) may be re-expressed in the form

$$\begin{aligned}[Q_a(\nu_m)Q_0(\nu_n)\{P_a(\nu_n)Y_n' &+ Q_a(\nu_n)Y_n\}\{P_0(\nu_m)Y_m' + Q_0(\nu_m)Y_m\} \\ - Q_a(\nu_n)Q_0(\nu_m)\{P_a(\nu_m)Y_m' &+ Q_a(\nu_m)Y_m\}\{P_0(\nu_n)Y_n' + Q_0(\nu_n)Y_n\}]_0^a = 0.\end{aligned} \tag{2.22}$$

This is obviously true from boundary conditions (2.7) and (2.8).

By recourse to (2.6), the integral expression in (2.12) may be written as

$$\int_0^a \{\frac{d}{dy}(Y_n'Y_m') + \frac{B_{mn} - B_{nm}}{A_{mn}}\frac{d}{dy}(Y_n'Y_m) - \frac{(\gamma_m^2 - \gamma_n^2)B_{nm}}{A_{mn}}Y_nY_m\}\, dy. \tag{2.23}$$

Taking the limit $m \to n$ this reduces to

$$\int_0^a \{\frac{d}{dy}(Y_n')^2 + D_n\frac{d}{dy}(Y_n'Y_n) - E_nY_n^2\}\, dy \tag{2.24}$$

where

$$\begin{aligned} D_n &= \lim_{m\to n}\frac{B_{mn} - B_{nm}}{A_{mn}} && (2.25)\\ E_n &= \lim_{m\to n}\frac{(\gamma_m^2 - \gamma_n^2)B_{nm}}{A_{mn}}. && (2.26)\end{aligned}$$

Integrating the first two terms in (2.24) proves the form of C_n given in (2.17), and the value of the coefficients D_n, E_n are easily shown, by L'Hospital's rule, to reduce to (2.18) and (2.19). □

Note that C_n is non-zero as long as ν_n is a simple zero of the dispersion relation (2.10), and may be written in the form

$$\begin{aligned} C_n = \Big[\lim_{m\to n}\{Q_a(\nu_m)Q_0(\nu_n)\{P_a(\nu_n)Y_n' &+ Q_a(\nu_n)Y_n\}\{P_0(\nu_m)Y_m' + Q_0(\nu_m)Y_m\} \\ -Q_a(\nu_n)Q_0(\nu_m)\{P_a(\nu_m)Y_m' &+ Q_a(\nu_m)Y_m\}\{P_0(\nu_n)Y_n' + Q_0(\nu_n)Y_n\}\}/A_{mn}\Big]_0^a.\end{aligned} \tag{2.27}$$

Further, A_{mn} is assumed to be non-zero in the above, and in fact it vanishes only for trivial cases, such as Robin's boundary conditions on both duct walls. In the latter situation standard Sturm-Liouville theory is appropriate.

As mentioned above, $P_p(\nu_n)$ and $Q_p(\nu_n)$, $p = a$, 0, are polynomials in ν_n^2 and thus may *always* be expressed as polynomials in $\gamma_n^2 = \nu_n^2 - 1$. It follows that the coefficients A_{mn} and B_{mn} in (2.12) are themselves polynomials in the two variables γ_n^2 and γ_m^2. However, any power of γ_m^2 multiplying Y_m or its derivative in (2.12) can, from (2.6), be re-expressed as a higher derivative of Y_m. Therefore, the orthogonality relation (in any of the forms (2.12)–(2.14)) may be expressed in terms of higher derivatives of Y_m, with the associated coefficients independent of m, which we can write as the inner product of eigenfunctions

$$(Y_m, Y_n) = C_n\delta_{mn} \tag{2.28}$$

where δ_{mn} is the usual Kronecker delta. Hence, this allows for straightforward application of the orthogonality property, in particular by enabling the definition of an inner-product relation:

$$(f, Y_n) = \sigma_nC_n, \tag{2.29}$$

where

$$f(y) = \sum_{m=0}^{\infty}\sigma_mY_m(y) \tag{2.30}$$

is a sufficiently differentiable function over the interval $0 \le y \le a$ for the inner product to exist.

As a final point, it should be noted that the above orthogonality relation is derived for the general case in which both boundaries of the waveguide are described by high-order conditions. This is not always the case; very often one surface comprises an acoustically soft or hard surface. To obtain the appropriate result for either a soft or hard condition on the lower surface, the high-order condition is replaced simply with

$$\beta\frac{\partial\phi}{\partial y} + \alpha\phi = 0. \tag{2.31}$$

Then

$$P_0(\nu_n) = \beta, \quad Q_0(\nu_n) = \alpha \tag{2.32}$$

and it follows that

$$A_{mn} = \alpha\beta\{Q_a(\nu_m)P_a(\nu_n) - Q_a(\nu_n)P_a(\nu_m)\}, \tag{2.33}$$
$$B_{mn} = \alpha Q_a(\nu_m)\{\alpha P_a(\nu_n) - \beta Q_a(\nu_n)\}. \tag{2.34}$$

After cancelling α throughout, the orthogonality relations for a soft (hard) lower surface follow directly from (2.12) by putting $\beta = 0$ ($\alpha = 0$).

3 Application to a problem involving the scattering of structural waves by a corner

In this section the first of two illustrative problems is considered. Both involve high-order boundary conditions over a semi-infinite domain and thus both require the application of edge conditions to achieve a unique solution. The model problems have been chosen to demonstrate the applicability of orthogonality relation (2.12) (or equivalently (2.14)). However, they can also be solved by recourse to other standard techniques which thus provide essential checks for the eigenfunction expansion technique propounded in section 2. In both this and the following sections, a detailed discussion regarding the physical models and the alternative solution methods is omitted because it is beyond the scope of this article. These boundary value problems, whilst perhaps their particular solutions do not appear in the literature, belong to a very well studied class and so the reader is directed elsewhere for other specific examples (e.g. Leppington (1976), Abrahams (1982), Papanikolaou (1997)).

The problem is to determine the reflection of, and the scattered sound field generated by, a fluid-coupled membrane wave of radian frequency ω incident in the negative x-direction along a semi-infinite membrane (see figure 2). The equation of motion of the membrane, for two-dimensional disturbances, is

$$-T\frac{\partial^2\eta}{\partial x^2} + \rho_m\frac{\partial^2\eta}{\partial t^2} = p \tag{3.1}$$

where η is the membrane deflection, p is the fluid pressure inside the duct, T is the membrane tension per unit span, and ρ_m is the membrane mass per unit area. This flexible structure forms the upper boundary of a two-dimensional semi-infinite duct. The lower surface of the duct and the vertical end face comprise acoustically hard surfaces, the interior region contains a compressible fluid of sound speed c and density ρ whilst the region exterior to the duct is *in vacuo*. In terms of non-dimensional Cartesian coordinates (scaled on the acoustic wavelength) the duct occupies the region $0 \le y \le a$, $x > 0$; see figure 2. Further, time is non-dimensionalized by scaling on the inverse of ω. It is convenient to express the total non-dimensional fluid velocity potential, $\phi_{tot}(x, y)$ in terms of an incident and a scattered field, both of which are assumed to have harmonic time dependence. Thus, we can write

$$\phi_{tot}(x, y, t) = \Re\left[\{\phi_{inc}(x, y) + \phi(x, y)\}e^{-it}\right] \tag{3.2}$$

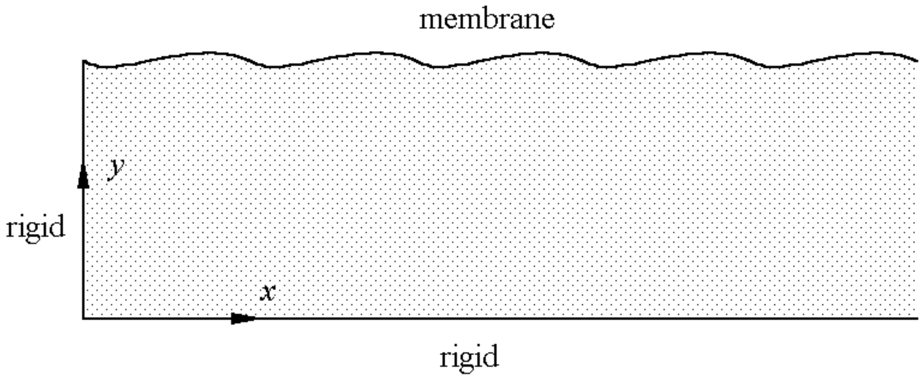

FIG. 2. A semi-infinite duct containing a compressible fluid. On $y = 0$, $x > 0$ and on $x = 0$, $0 < y < a$ the walls are hard; on $y = a$, $x > 0$ the duct boundary is a thin membrane with condition (3.8).

where the incident field contains an incident structural wave together with a reflected wave of equal amplitude, the latter being included purely for algebraic convenience. Hence,

$$\phi_{inc}(x,y) = A\{e^{-i\nu_0 x} + e^{i\nu_0 x}\}\cosh(\gamma_0 y), \tag{3.3}$$

where A is an arbitrary constant amplitude and the positive real quantities γ_0 and ν_0 will be defined later in the text. Note that the fluid pressure and membrane displacement written in (3.1) are related to the velocity potential through

$$p(x,y,t) = -\rho\frac{\partial\phi_{tot}}{\partial t}(x,y,t), \quad \frac{\partial\eta}{\partial t}(x,t) = \frac{\partial\phi_{tot}}{\partial y}(x,a,t). \tag{3.4}$$

The boundary value problem may be formulated in terms of the scattered field. This quantity satisfies Helmholtz' equation, that is

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 1\right\}\phi(x,y) = 0 \tag{3.5}$$

and on the two rigid duct surfaces the normal component of fluid velocity vanishes:

$$\frac{\partial\phi}{\partial y} = 0, \quad y = 0, \quad x \geq 0 \tag{3.6}$$

$$\frac{\partial\phi}{\partial x} = 0, \quad x = 0, \quad 0 \leq y \leq a. \tag{3.7}$$

The upper duct surface comprises the membrane, for which the non-dimensional form of the equation of motion (3.1) is most conveniently expressed as

$$\left\{\frac{\partial^2}{\partial x^2} + \mu^2\right\}\frac{\partial\phi}{\partial y} + \alpha\phi = 0, \quad y = a, \quad x > 0. \tag{3.8}$$

Here μ and α are the non-dimensional *in vacuo* wavenumber and fluid loading parameter respectively, which are defined in terms of the physical variables as

$$\mu = \sqrt{\rho_m c^2/T}, \qquad \alpha = c^3\rho/(\omega T). \tag{3.9}$$

The derivation of (3.8) can be found in Leppington (1978). The edge of the membrane meets the vertical face of the duct at the point $y = a$, $x = 0$ and the membrane displacement is taken to be zero here. In terms of $\phi(x,y)$ the edge condition is

$$\frac{\partial\phi}{\partial y}(0,a) = -\frac{\partial\phi_{inc}}{\partial y}(0,a). \tag{3.10}$$

It remains only to state that the scattered field must consist only of outward going or decaying waves, that is, waves propagating or evanescing in the positive x-direction.

Equations (3.5)–(3.10) comprise the boundary value problem for the scattered potential, $\phi(x,y)$. Further, in view of the boundary condition (3.7) we can seek solutions even in x and hence deal with a duct of doubly infinite extent ($-\infty < x < \infty$). Since ϕ contains only outgoing waves this field can be modelled in terms of an even source (or delta function) situated at the membrane edge, $y = a$, $x = 0$ (see Leppington, 1978). Thus, the membrane equation becomes

$$\left\{\frac{\partial^2}{\partial x^2} + \mu^2\right\}\frac{\partial \phi}{\partial y} + \alpha\phi = 2D\delta(x), \quad y = a, \quad -\infty < x < \infty, \tag{3.11}$$

where the constant D is determined from the edge condition. The boundary value problem is now amenable to solution by Fourier transform techniques and omitting all analysis for the sake of brevity, the solution expressed in integral form is

$$\phi(x,y) = -\frac{D}{\pi}\int_{-\infty}^{\infty} \frac{\cosh\{\gamma(s)y\}e^{-isx}}{K(s)}\, ds \tag{3.12}$$

where

$$K(s) = (s^2 - \mu^2)\gamma\sinh(\gamma a) - \alpha\cosh(\gamma a), \quad \gamma = (s^2-1)^{1/2} \tag{3.13}$$

and the required edge behaviour (3.10) is ensured by setting

$$D = 2\pi A\gamma_0 \sinh(\gamma_0 a)\left\{\int_{-\infty}^{\infty} \frac{\gamma\sinh(\gamma a)}{K(s)}\, ds\right\}^{-1}. \tag{3.14}$$

The integration path is indented above any singularities on the negative real axis and below any on the positive real axis. That (3.12) is the exact solution of the boundary value problem is easily verified by direct substitution into the relevant equations. The function $K(s)$ is the characteristic function for the problem and in fact is the dispersion function cited in (2.10) specialized to the case of boundary conditions (3.6), (3.8). The roots of the dispersion relation $K(s) = 0$, as discussed in the previous section for this class of boundary value problems, are discrete and occur in pairs; hence $s = \pm\nu_n$, $n = 0, 1, 2, 3, \ldots$ and these represent the wavenumbers, in the x direction, of travelling or evanescent waves that comprise the scattered field. Associated with the ν_n are

$$\gamma_n = (\nu_n^2 - 1)^{1/2}, \qquad n = 0, 1, 2, 3, \ldots \tag{3.15}$$

of which γ_0 is positive real and γ_n, $n > 0$ are positive imaginary for all ranges of the parameters a, μ, α. This can be shown to be the case simply by examining the dispersion relation $K(s) = 0$ in the form

$$\gamma\tanh(\gamma a) = \alpha/(\gamma^2 + 1 - \mu^2), \tag{3.16}$$

and plotting both sides of this equation for real and purely imaginary values of γ. The intersections of these curves reveal the location of the ν_n; note that a finite, and increasing as a increases, number of the roots lies on the real axis between 0 and 1. In fact it can be shown that all the $+\nu_n$ roots lie on the semi-infinite lines $\Re(\nu_n) \geq 0$ or $\Im(\nu_n) \geq 0$ and $\nu_n \sim \gamma_n \sim in\pi/a$, $n >> 1$. Again, the precise detail of the location of the roots is not germane to the main thrust of this article but has been very well studied elsewhere (cf. the work by Cannell on elastic plate scattering (Cannell, 1975)). The root ν_0 corresponds to the subsonic fluid-coupled structural wave and thus appears in the forcing term, (3.3). Due to the nature of the sound field in a duct, which comprises only discrete modes (i.e. the dispersion function is free of branch-cuts), the integral (3.12) can be expressed as an infinite sum of modes. This is achieved by deforming the contour of integration in (3.12) into the lower half-plane and picking up the residue contributions at each pole $s = -\nu_n$. Omitting details it is found that

$$\phi = -iD \sum_{n=0}^{\infty} \frac{\gamma_n \sinh(\gamma_n a)}{\nu_n C_n} \cosh(\gamma_n y) e^{i\nu_n x} \tag{3.17}$$

where D can be written as

$$D = -2iA\gamma_0 \sinh(\gamma_0 a) \left[\sum_{n=0}^{\infty} \frac{\gamma_n^2 \sinh^2(\gamma_n a)}{\nu_n C_n} \right]^{-1} \tag{3.18}$$

in which

$$C_n = \frac{1}{2}\{\alpha a + (3\gamma_n^2 + 1 - \mu^2)\sinh^2(\gamma_n a)\}. \tag{3.19}$$

It is easy to show that C_n is related to the dispersion function via

$$C_n = \frac{1}{2}\sinh(\gamma_n a)\left.\frac{dK(s)}{d\gamma}\right|_{s=\nu_n}. \tag{3.20}$$

Finally, from (3.17) one can determine the reflected fluid-coupled membrane wave term. In total (recall the part subtracted from ϕ_{tot} in (3.2), (3.3)) it is found to be

$$\Re\left[\left(A - iD\frac{\gamma_0 \sinh(\gamma_0 a)}{\nu_0 C_0}\right)\cosh(\gamma_0 y) e^{i\nu_0 x}\right]. \tag{3.21}$$

The exact solution determined above by the Fourier transformation method can now be used as a convenient check on the direct approach using eigenfunction expansions and the orthogonality relation established in the previous section. The scattered field can easily be shown, by separation of variables, to be representable in the form

$$\phi = \sum_{n=0}^{\infty} B_n \cosh(\gamma_n y) e^{i\nu_n x} \tag{3.22}$$

(cf. (3.17)), which satisfies Helmholtz' equation and boundary conditions (3.6) and (3.8). The eigenvalues ν_n are just those discussed above. The coefficients B_n, $n = 0, 1, 2, \ldots$ are to be determined by applying both an appropriate condition along the vertical interface $x = 0$, $0 \le y \le a$, and the edge condition. For the boundary value problem described above, the most convenient form of the orthogonality relation is that given in (2.14). From boundary conditions (3.6) and (3.8) it is easily shown that

$$\frac{B_{nm}}{A_{mn}} = \frac{B_{mn}}{A_{mn}} = \frac{-\alpha}{\nu_m^2 - \nu_n^2}, \tag{3.23}$$

so that we obtain

$$(Y_m, Y_n) = \alpha \int_0^a Y_m(y) Y_n(y)\, dy + Y_m'(a) Y_n'(a) = \begin{cases} 0, & m \ne n, \\ C_n, & n = m, \end{cases} \tag{3.24}$$

where $Y_n(y) = \cosh(\gamma_n y)$. The constant C_n is easily shown to take the value given by (3.19). The inner-product relation (2.29) can now be employed on $\phi_x(x, y)$, which yields

$$(\phi_x, Y_n) = \alpha \int_0^a \phi_x(x, y) Y_n(y)\, dy + \phi_{xy}(x, a) Y_n'(a) = i\nu_n B_n C_n e^{i\nu_n x}. \tag{3.25}$$

Thus, taking the limit $x \downarrow 0$ and employing the boundary condition (3.7) gives the value of the coefficients:

$$B_n = -i\frac{\gamma_n \sinh(\gamma_n a)}{\nu_n C_n}\phi_{xy}(0+,a) \tag{3.26}$$

which are given in terms of the constant

$$E = \phi_{xy}(0+,a) \equiv \lim_{x\downarrow 0}\phi_{xy}(x,a) \tag{3.27}$$

and this is just the value of the membrane slope at its edge. Hence, the solution to the boundary value problem is

$$\phi(x,y) = -iE\sum_{n=0}^{\infty}\frac{\gamma_n \sinh(\gamma_n a)}{\nu_n C_n}\cosh(\gamma_n y)e^{i\nu_n x} \tag{3.28}$$

and, applying the edge condition (3.10), we also have

$$\sum_{n=0}^{\infty}\gamma_n B_n \sinh(\gamma_n a) = -2A\gamma_0 \sinh(\gamma_0 a) \tag{3.29}$$

from which we finally deduce the value of the constant as

$$E = -2iA\gamma_0 \sinh(\gamma_0 a)\left[\sum_{n=0}^{\infty}\frac{\gamma_n^2 \sinh^2(\gamma_n a)}{\nu_n C_n}\right]^{-1}. \tag{3.30}$$

By inspection of (3.30) and (3.18), $D \equiv E$ and so the two solutions, (3.12) and (3.28), are identical. In particular, this verifies that the set of eigenfunctions (3.22) is complete, and that the orthogonality relation (2.12) or (2.14) can be used constructively for dealing with problems with corner or edge discontinuities. As a final note, we highlight the corner discontinuity in this example:

$$\lim_{x\downarrow 0}\phi_{xy}(x,a) = \lim_{x\downarrow 0}\phi_{yx}(x,a) \neq \lim_{y\uparrow a}\phi_{xy}(0,y) = \lim_{y\uparrow a}\phi_{yx}(0,y). \tag{3.31}$$

It is easy to show from (3.28) and the boundary condition (3.7) that the expressions on the right-hand side of the inequality are zero, whilst the left-hand side takes the value E.

4 Application to the scattering of sound by an abrupt change in the boundary conditions of a waveguide

The problem under consideration in this section is the sound field generated when a plane acoustic wave is scattered by a material discontinuity in a two-dimensional waveguide. The waveguide is formed by an infinite rigid surface lying along $y = 0$ together with a rigid surface lying along the line $y = a$, $x < 0$ and a membrane, with parameters α and μ as in (3.8), occupying $y = a$, $x > 0$. The interior of the waveguide contains a compressible fluid of sound speed c and density ρ whilst the region exterior to the duct is *in vacuo*; see figure 3. A plane acoustic wave of unit amplitude and harmonic time dependence is incident in the positive x-direction along the waveguide towards $x = 0$. Once again the non-dimensionalization introduced in section 3 is employed.

This is a standard Wiener-Hopf problem; however, the aim of this section is to demonstrate how the eigenfunction technique yields a solution in a form that is very convenient for numerical evaluation. As mentioned in section 3, the membrane boundary condition is of sufficiently high-order to require an edge condition to be imposed at $y = a$, $x = 0$. Although the technique of section 3 could be applied here, an alternative simple approach is used to enforce the edge condition for this problem. The method used here, whilst elegant, is not quite as general as

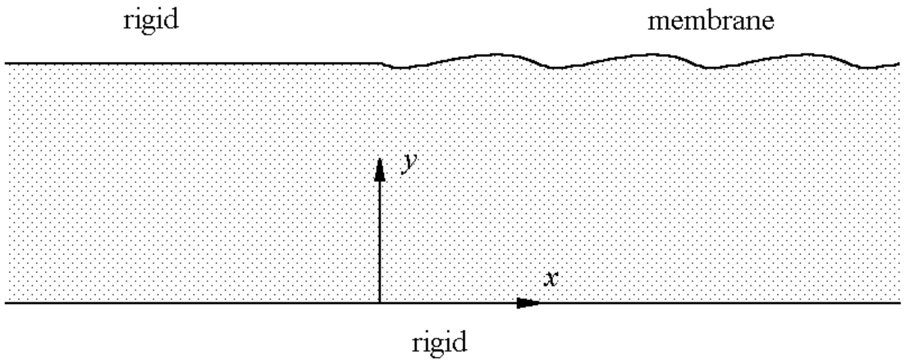

FIG. 3. A duct containing a compressible fluid. On $y = 0$, $-\infty < x < \infty$ and on $y = a$, $x < 0$ the walls are hard; on $y = a$, $x > 0$ the duct boundary is a thin membrane with condition (3.8).

that of section 3 being applicable only for jumps in boundary conditions and not usually for discontinuities in geometry.

It is convenient to express the total non-dimensional fluid velocity potential, $\phi_{tot}(x, y)$, in terms of two separate scattered fields for $x < 0$ and $x > 0$, both of which are assumed to have harmonic time dependence. Thus

$$\phi_{tot}(x,y,t) = \begin{cases} \Re\left[\{\phi_1(x,y) + e^{ix} + e^{-ix}\}e^{-it}\right], & x < 0, \\ \Re\left[\{\phi_2(x,y)\}e^{-it}\right], & x > 0, \end{cases} \quad (4.1)$$

where the incident field again includes a reflected wave of equal amplitude for convenience. The boundary value problem may be formulated in terms of the scattered potentials. These quantities satisfy Helmholtz' equation, that is

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 1\right\}\phi_j(x,y) = 0, \quad j = 1, 2. \quad (4.2)$$

On the two rigid duct surfaces the normal component of fluid velocity vanishes, that is

$$\frac{\partial \phi_j}{\partial y} = 0, \quad j = 1, 2, \quad y = 0, \quad -\infty < x < \infty, \quad (4.3)$$

$$\frac{\partial \phi_1}{\partial y} = 0, \quad y = a, \quad x < 0. \quad (4.4)$$

The remaining duct surface, formed by a semi-infinite membrane, is governed by the non-dimensional equation (3.11),

$$\left\{\frac{\partial^2}{\partial x^2} + \mu^2\right\}\frac{\partial \phi_2}{\partial y} + \alpha\phi_2 = 0, \quad y = a, \quad x > 0, \quad (4.5)$$

where the parameters μ and α are defined in (3.9). The two scattered potentials must be matched across the interface at the boundary discontinuity and the appropriate conditions comprise continuity of pressure and velocity across $x = 0$. That is,

$$\phi_1(0,y) + 2 = \phi_2(0,y), \quad 0 \le y \le a, \quad (4.6)$$

$$\frac{\partial \phi_1}{\partial x}(0,y) = \frac{\partial \phi_2}{\partial x}(0,y), \quad 0 \le y \le a, \quad (4.7)$$

respectively. An edge condition must be applied at the membrane junction and the most general form for this is

$$\bar{b}\frac{\partial^2\phi_2}{\partial x\partial y}(0+,a)+\bar{a}\frac{\partial\phi_2}{\partial y}(0+,a)=0 \tag{4.8}$$

where as before $f(0+,a)$ stands for $\lim_{x\downarrow 0} f(x,a)$, and $\bar{a}$ and $\bar{b}$ are given constants. In this example, the usual membrane edge condition is that with zero displacement, $\bar{b}=0$. However, it is possible to have other conditions, such as zero membrane slope ($\bar{a}=0$) where the membrane slides without friction over a perpendicular boundary. The full condition (4.8), with $\bar{a}\neq 0$, $\bar{b}\neq 0$, is perhaps artificial when applied to a membrane, but is illustrative of the more complicated edge conditions which may arise in problems involving elastic plates, see for example Lawrie & Abrahams (1997). Finally, as usual, the scattered fields comprise outward going waves only, that is, all scattered waves travel away from the discontinuity.

It is a simple matter to show, by means of separation of variables, that the following forms for ϕ_1 and ϕ_2 can be obtained:

$$\phi_1(x,y) = \sum_{n=0}^{\infty} A_n\cos(n\pi y/a)e^{-i\lambda_n x}, \tag{4.9}$$

$$\phi_2(x,y) = \sum_{n=0}^{\infty} B_n\cosh(\gamma_n y)e^{i\nu_n x}. \tag{4.10}$$

Here $\lambda_n=(1-n^2\pi^2/a^2)^{1/2}$, $n=0,1,2,\ldots$, are positive real for $n<a/\pi$ or positive imaginary, and $\gamma_n=(\nu_n^2-1)^{1/2}$, $n=0,1,2,\ldots$, with ν_n defined, as before, to be the roots of $K(s)=0$, see (3.13), with $\Re(\nu_n)>0$ and/or $\Im(\nu_n)>0$. Then, (4.6) and (4.7) give

$$\sum_{n=0}^{\infty} B_n\cosh(\gamma_n y)=2+\sum_{n=0}^{\infty} A_n\cos(n\pi y/a), \quad 0\le y\le a \tag{4.11}$$

and

$$\sum_{n=0}^{\infty}\nu_n B_n\cosh(\gamma_n y)=-\sum_{n=0}^{\infty}\lambda_n A_n\cos(n\pi y/a), \quad 0\le y\le a. \tag{4.12}$$

The eigenfunction expansions for $x<0$ and $x>0$ satisfy two different orthogonality relations. For $x<0$ the standard Fourier cosine series orthogonality relation holds (Churchill & Brown, 1987), whereas for $x>0$ equation (3.24) holds. Whilst both of these relations must be used, there is an element of choice as to which is applied to (4.11) and which to (4.12). It is now demonstrated that the choice of application of orthogonality relations is commensurate with the edge-condition in force. The cases $\bar{a}=0$ and $\bar{b}=0$ in (4.8) are examined in turn.

4.1 Case A: zero slope membrane edge condition

In this example the membrane is chosen to have zero slope at $x=0$, $y=a$, that is when $\phi_{2xy}(0+,a)=0$ or $\bar{a}=0$. It will now be shown to be expedient to apply the Fourier cosine series orthogonality relation to (4.11). This yields

$$A_n=-2\delta_{n0}+\frac{\epsilon_n}{a}\sum_{m=0}^{\infty}B_m R_{mn} \tag{4.13}$$

where δ_{nm} is the Kronecker delta and ϵ_n, R_{mn} are defined by

$$\epsilon_n=\begin{cases}1, & n=0\\ 2, & n>0\end{cases} \tag{4.14}$$

and

$$R_{mn} = \int_0^a \cosh(\gamma_m y)\cos(n\pi y/a)\, dy = (-1)^n \frac{\sinh(\gamma_m a)}{\gamma_m\{1 + (\frac{n\pi}{a\gamma_m})^2\}}. \tag{4.15}$$

Now, the inner-product relation (2.29) employed on $\partial\phi_2/\partial x$, can be simplified using (4.10) and the definition of the inner product (3.24), and this yields

$$\alpha \sum_{n=0}^{\infty} \nu_n B_n e^{i\nu_n x} \int_0^a Y_m Y_n\, dy + \sum_{n=0}^{\infty} \nu_n B_n Y_n'(a) e^{i\nu_n x} Y_m'(a) = B_m \nu_m C_m e^{i\nu_m x} \tag{4.16}$$

for all m, where as before $Y_n(y) = \cosh(\gamma_n y)$. Taking $x \downarrow 0$, equation (4.16) may be recast into the form

$$\alpha \sum_{n=0}^{\infty} \nu_n B_n \int_0^a Y_m Y_n\, dy - iY_m'(a) \frac{\partial^2 \phi_2}{\partial y \partial x}(0+, a) = B_m \nu_m C_m, \tag{4.17}$$

so that, for the edge condition $\phi_{2xy}(0+, a) = 0$, the orthogonality relation reduces simply to

$$\alpha \sum_{n=0}^{\infty} \nu_n B_n \int_0^a Y_m Y_n\, dy = B_m \nu_m C_m, \quad m = 0, 1, 2, 3, \ldots. \tag{4.18}$$

Note that this now resembles the usual form found in Sturm-Liouville problems. Hence, multiplying (4.12) by $Y_n(y)$, integrating term by term over the duct $0 \leq y \leq a$ and using (4.18) gives

$$\alpha \sum_{m=0}^{\infty} \nu_m B_m \int_0^a Y_n Y_m\, dy = B_n \nu_n C_n = -\alpha \sum_{\ell=0}^{\infty} \lambda_\ell A_\ell R_{n\ell} \tag{4.19}$$

for all integers $n \geq 0$ and hence,

$$B_n = -\frac{\alpha}{\nu_n C_n} \sum_{\ell=0}^{\infty} \lambda_\ell A_\ell R_{n\ell}. \tag{4.20}$$

Finally, on eliminating A_ℓ from (4.20) using (4.13), it is found that B_n satisfies the algebraic system

$$B_n = \frac{\alpha}{\nu_n C_n} \left\{ 2R_{n0} - \frac{1}{a} \sum_{m=0}^{\infty} B_m \sum_{\ell=0}^{\infty} \epsilon_l \lambda_\ell R_{m\ell} R_{n\ell} \right\}, \tag{4.21}$$

which is consistent with the equivalent expression derived by Warren & Lawrie (1996). Once solved, the B_n are substituted into (4.13) to determine the A_n.

4.2 Case B: zero displacement membrane edge condition

To obtain the coefficient relations when the membrane is pinned at the point $x = 0$, $y = a$, i.e. when $\phi_{2y}(0+, a) = 0$ or $\bar{b} = 0$, the alternative approach is employed. The Fourier cosine series orthogonality relation is applied to (4.12) instead of (4.11). In this case it is found that

$$A_n = -\frac{\epsilon_n}{a\lambda_n} \sum_{m=0}^{\infty} \nu_m B_m R_{mn}. \tag{4.22}$$

Similarly, the inner-product relation (2.29), (3.24) is employed on $\phi_2(x, y)$ from (4.10), which yields as $x \downarrow 0$

$$\alpha \sum_{n=0}^{\infty} B_n \int_0^a Y_m Y_n\, dy + \sum_{n=0}^{\infty} B_n Y_n'(a) Y_m'(a) = B_m C_m, \quad m = 0, 1, 2, 3, \ldots. \tag{4.23}$$

It follows that the orthogonality relation may be recast as

$$\alpha \sum_{n=0}^{\infty} B_n \int_0^a Y_m Y_n \, dy + Y_m'(a) \frac{\partial \phi_2}{\partial y}(0+, a) = B_m C_m, \tag{4.24}$$

so that, for the edge condition $\phi_{2y}(0+, a) = 0$, it becomes

$$\alpha \sum_{n=0}^{\infty} B_n \int_0^a Y_m Y_n \, dy = B_m C_m, \quad m = 0, 1, 2, 3, \ldots. \tag{4.25}$$

Equation (4.25) can now be employed in (4.11) to obtain

$$\alpha \sum_{m=0}^{\infty} B_m \int_0^a Y_n Y_m \, dy = B_n C_n = 2\alpha R_{n0} + \alpha \sum_{\ell=0}^{\infty} A_\ell R_{n\ell}. \tag{4.26}$$

It follows that

$$B_n = \frac{2\alpha}{C_n} R_{n0} + \frac{\alpha}{C_n} \sum_{\ell=0}^{\infty} A_\ell R_{n\ell}. \tag{4.27}$$

Finally, on eliminating A_ℓ from (4.27), it is found that

$$B_n = \frac{2\alpha}{C_n} R_{n0} - \frac{\alpha}{a C_n} \sum_{m=0}^{\infty} \nu_m B_m \sum_{\ell=0}^{\infty} \frac{\epsilon_\ell}{\lambda_\ell} R_{m\ell} R_{n\ell}. \tag{4.28}$$

As in the previous case, once this is solved for the B_n the other coefficients, A_n, are found from (4.22).

4.3 Case C

Having demonstrated the efficient application of the reduced orthogonality relations for the cases $\bar{a} = 0$, $\bar{b} = 0$ in the membrane edge condition (4.8), it is now straightforward to see how to apply it in general. That is, the relations (4.17) and (4.24) can be combined as

$$\sum_{n=0}^{\infty} \alpha B_n (i\bar{b}\nu_n + \bar{a}) \int_0^a Y_n Y_m \, dy + Y_m'(a) \left(\bar{b}\phi_{xy}(0+, a) + \bar{a}\phi_y(0+, a)\right) = B_m C_m (i\bar{b}\nu_m + \bar{a}) \tag{4.29}$$

which is deliberately chosen as $\bar{a}$ times (4.11) plus $i\bar{b}$ times (4.12) in order that the second term on the left-hand side vanishes owing to (4.8). Therefore,

$$(i\bar{b}\nu_n + \bar{a}) B_n C_n = 2\bar{a}\alpha R_{n0} + \alpha \sum_{\ell=0}^{\infty} A_\ell R_{n\ell} (\bar{a} - i\bar{b}\lambda_\ell). \tag{4.30}$$

Similarly, the orthogonality relation for Fourier series can be applied to any other linear combination of (4.11), (4.12). For example, it is possible to take the sum of $i\bar{b}$ times (4.11) and $\bar{a}$ times (4.12), which yields

$$(i\bar{b} - \bar{a}\lambda_n) A_n = -2i\bar{b}\delta_{n0} + \frac{\epsilon_n}{a} \sum_{m=0}^{\infty} B_m R_{mn} (i\bar{b} + \bar{a}\nu_m). \tag{4.31}$$

Note that when this combination is linearly independent of (4.30), i.e. for $\bar{a} \neq \pm i\bar{b}$, equations (4.30) and (4.31) together constitute an infinite system of linear equations for the unknowns A_n, B_n. Notice that they reduce to the coupled pairs (4.13), (4.20) or (4.22), (4.27) when $\bar{a} = 0$ or $\bar{b} = 0$ respectively.

4.4 The behaviour of the coefficients

It is easy to verify that, for the three cases above, the equations satisfied by the B_n coefficients lead to different solutions. In particular, it is sufficient to prove this by showing that the decay in the value of B_n as $n \to \infty$ is different in the cases A and B. Furthermore, by examining the convergence of series, it is possible to *prove* that the infinite systems are solvable. This will be addressed in a later article.

The matrix R_{mn} (equation (4.15)) appearing in (4.13), (4.20), (4.22) and (4.27) can be rewritten via the dispersion relation as

$$R_{mn} = \frac{\alpha(-1)^n \cosh(\gamma_m a)}{(\gamma_m^2 + (n\pi/a)^2)(\gamma_m^2 + 1 - \mu^2)} \tag{4.32}$$

and so by inspection,

$$R_{mn} \sim \alpha(-1)^{n+m}(m\pi/a)^{-4}, \quad m \to \infty, \quad n \text{ fixed}, \tag{4.33}$$

$$R_{mn} \sim (-1)^n \gamma_m \sinh(\gamma_m a)(n\pi/a)^{-2}, \quad n \to \infty, \quad m \text{ fixed}. \tag{4.34}$$

However, γ_m approaches $im\pi/a$ as $m \to \infty$ and so for the diagonal elements, the denominator contains a term which can be shown to behave as

$$\gamma_m^2 + (m\pi/a)^2 \sim \frac{-2\alpha}{a}\left(\frac{a}{m\pi}\right)^2, \quad m \to \infty. \tag{4.35}$$

Therefore,

$$R_{mm} \sim \frac{a}{2}, \quad m \to \infty \tag{4.36}$$

and it is easy to show that

$$C_m \sim \frac{\alpha a}{2} \tag{4.37}$$

in the same limit. The $O(1)$ diagonal elements can be removed from (4.13), (4.20) to give the system

$$A_n - \frac{\epsilon_n}{2}B_n = -2\delta_{n0} + \frac{\epsilon_n}{a}\sum_{m=0}^{\infty} B_m S_{mn}, \tag{4.38}$$

$$\frac{\alpha a \lambda_n}{2\nu_n C_n}A_n + B_n = -\frac{\alpha}{\nu_n C_n}\sum_{m=0}^{\infty} \lambda_m A_m S_{nm}, \tag{4.39}$$

where

$$S_{mn} = R_{mn} - \frac{a}{2}\delta_{mn}, \tag{4.40}$$

which decays rapidly with m and n. The pair of equations is solvable by truncation or by iteration; the latter yields analytically that

$$A_m \sim -B_m = (-1)^m m^{-2} k_1, \tag{4.41}$$

for large m, where

$$k_1 = \frac{a}{\pi^2}\sum_{n=0}^{\infty} \gamma_n B_n \sinh(\gamma_n a) = \frac{a}{\pi^2}\phi_{2y}(0+, a), \tag{4.42}$$

for the particular edge condition $\phi_{xy}(0+, a) = 0$. The coupled pair of equations, and the asymptotic result (4.41), can be confirmed from the Wiener-Hopf solution and indeed an exact solution is obtainable by this method. In general it will be unlikely that there is a better alternative approach to the method employed here as the system in the form (4.38), (4.39) constitutes a particularly efficient route for determining the unknown coefficients.

For the alternative edge condition, $\phi_y(0+,a)=0$, equations (4.22), (4.27) may be recast as

$$A_n + \frac{\epsilon_n \nu_n}{2\lambda_n} B_n = -\frac{\epsilon_n}{a\lambda_n}\sum_{m=0}^{\infty}\nu_m B_m S_{mn}, \tag{4.43}$$

$$-\frac{\alpha a}{2C_n}A_n + B_n = \frac{2\alpha}{C_n}R_{n0} + \frac{\alpha}{C_n}\sum_{m=0}^{\infty}A_m S_{nm}, \tag{4.44}$$

which result can again be proved by Wiener-Hopf methods. Iteration reveals that

$$A_m \sim B_m \sim (-1)^m m^{-3} k_2, \quad m \to \infty, \tag{4.45}$$

where

$$k_2 = \frac{ia^2}{\pi^3}\sum_{n=0}^{\infty}\nu_n\gamma_n B_n \sinh(\gamma_n a) = \frac{a^2}{\pi^3}\phi_{2xy}(0+,a). \tag{4.46}$$

Clearly, (4.41) and (4.45) indicate that cases A and B have quite different solutions, as indeed does case C.

As a final point to note, if one employs the orthogonality relation (4.16) for the edge condition $\phi_y(0+,a)=0$ rather than $\phi_{xy}(0+,a)=0$, then the second term in (4.17) does not vanish. Thus, an extra constraint to force the requisite edge condition is needed, and this can be incorporated in the fashion discussed in the previous section.

5 Concluding remarks

The two examples in sections 3 and 4 have demonstrated that duct model problems with high-order boundary conditions can be reduced, by eigenfunction expansions and mode-matching, to systems of coupled equations, for example (4.11) and (4.12). Via the application of an orthogonality relation detailed in section 2, together with explicit procedures for 'building-in' the edge conditions, the coupled systems were re-expressed as algebraic equations which are straightforward to solve by truncation. In practise, such systems demonstrate rapid numerical convergence; this, together with other analytical results (cf. system investigated in Abrahams & Wickham, 1991) suggest that these systems are in fact ℓ^2. An alternative approach to verifying this convergence result is to obtain the exact expressions for the eigenfunction expansion coefficients (A_n, B_n) via the Wiener-Hopf technique. The two examples were illustrative, in that they demonstrated the new approach taken by the authors, yet are simple enough to be amenable to exact solution; this provided a direct check on the efficacy of the proposed method. It is important to note that, as in section 4, the nature of the solution usually changes significantly for different edge conditions. This was demonstrated by the different behaviour in n of the modal coefficients A_n, B_n in (4.41), (4.45). Without specification of the edge conditions, the lack of uniqueness in the boundary value problems is manifested in the occurrence of extra (non-integral) terms in the orthogonality relation (2.14). The non-uniqueness is, in general, resolved by applying the edge conditions as shown in section 3. Sometimes, a judicious application of the orthogonality relation to an appropriate choice of mode-matching equation (see section 4), ensures automatic satisfaction of the required edge-behaviour and leads to a unique solution.

The generalization of the present approach to complex boundary value problems of the type illustrated in figure 1 is straightforward. Eigenfunction expansions are generated in each region of continuous material and geometrical properties, and these are matched across the interface at the points of discontinuous boundary. The orthogonality relation (2.12) is employed at each interface with coefficients chosen for the given boundary equation, and any edge conditions employed at this stage. The coupled second-kind Fredholm systems of equations resulting from each boundary discontinuity are combined to generate a large, but easily solved, linear algebraic matrix equation. Note that the duct may be of infinite or finite extent in x, and the procedure is easily generalizable to three-dimensional models. That is, wave propagation problems involving pipes or tubes of arbitrary cross-section, with flexible walls, will have an analogous orthogonality relation to that in (2.12).

For models of wave propagation in non-ducted regions the wavenumber spectrum is continuous. Thus, the present approach is not applicable and instead transform methods, including the Wiener-Hopf technique (Cannell, 1975) and its extensions, may be appropriate. However, for pipes and duct models, where both mode-matching and transform methods are valid, the former is usually simpler and easier to employ. Therefore, the present approach for dealing with complicated high-order boundary conditions, which yields tractable and versatile orthogonality relations, appears to offer an attractive solution method (Warren & Lawrie, 1996).

acknowledgement

The authors are extremely grateful to Professor J. Boersma for his painstaking review of this article. He pointed out many ways in which the work could, and has, been improved. In particular, by indicating that the most useful form of the orthogonality relation for problems of the type shown in section 3 is (2.14), it allows herein a much more direct approach to be implemented than was employed originally.

References

[1] I. D. Abrahams, *Q. Jl. Mech. Appl. Math.* **35** (1982) 91–101.

[2] I. D. Abrahams and G. R. Wickham, *Wave Motion* **14** (1991) 145–168.

[3] P. A. Cannell, *Proc. R. Soc. Lond. A* **347** (1975) 213–238.

[4] R. V. Churchill and J. W. Brown, *Fourier series and boundary value problems*, McGraw-Hill, New York 1987.

[5] R. A. Dalrymple and P. A. Martin, *Appl. Ocean Res.* **18** (1996) 1–11.

[6] P. G. Drazin and W. H. Reid. *Hydrodynamic Stability*, Cambridge University Press, Cambridge 1981.

[7] D. V. Evans and R. Porter, *Appl. Ocean Res.* **17** (1995) 155–164.

[8] D. V. Evans and C. M. Linton, *J. Fluid Mech.* **225** (1991) 153–175.

[9] M. E. Fama, *Quart. J. Mech. Appl. Math.* **25** (1972) 479–495.

[10] R. T. Folk and A. Herczynski, *J. Acoust. Soc. Am.* **80** (1986) 1103–1110.

[11] A. Herczynski and R. T. Folk, *J. Mech. Appl. Math.* **42** (1989) 523–536.

[12] M. C. Junger and D. Feit. *Sound, structures and their interaction*, 2nd ed., MIT Press, Cambridge, USA 1986.

[13] J. B. Lawrie and I. D. Abrahams. *IMA J. Appl. Math.* **59**, (1997) 1–23.

[14] N. N. Lebedev, I. P. Skalskaya and Y. S. Uflyand. *Worked problems in applied mathematics*, Dover, New York 1979.

[15] F. G. Leppington, *Q. Jl. Mech. Appl. Math.* **29** (1976) 527.

[16] F. G. Leppington, *J. Sound Vib.* **58** (1978) 319–332.

[17] J. E. Murphy, L. Gongqin and S. A. Chin-Bing, *J. Acoust. Soc. Am.* **96** (1994) 2313–2317.

[18] I. Papanikolaou, *PhD thesis*, University of London, 1997.

[19] K. S. Peat, *J. Sound Vib.* **150** (1991) 101–110.

[20] B. S. R. Rao and H. S. Rao, *Mech. Struct. Mach.* **16** (1988) 167–186.

[21] C. L. Scandrett and C. L. Frenzen, *J. Acoust. Soc. Am.* **98** (1995) 1199–1203.

[22] J. Seligson, *J. Lightwave Tech.* **6** (1988) 1260–1264.

[23] I. Y. Shen and C. D. Mote, Jr., *J. Acoust. Soc. Am.* **91** (1992) 2703–2707.

[24] D. P. Warren and J. B. Lawrie. *Technical Report, TR/34/96*, Brunel University, Mathematics & Statistics Department, U.K. 1996.

[25] C. Wu, E. Watanabe and T. Utsunomiya, *Appl. Ocean Res.* **17** (1995) 301–310.

[26] M. Zheng-Dong and I. Hagiwara, *AIAA J.* **29** (1991) 1720–1726.