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A Belief Revision Framework for Revising Epistemic States with Partial Epistemic States[*]

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Abstract

Belief revision performs belief change on an agent's beliefs when new evidence (either of the form of a propositional formula or of the form of a total pre-order on a set of interpretations) is received. Jeffrey's rule is commonly used for revising probabilistic epistemic states when new information is probabilistically uncertain. In this paper, we propose a general epistemic revision framework where new evidence is of the form of a partial epistemic state. Our framework extends Jeffrey's rule with uncertain inputs and covers well-known existing frameworks such as ordinal conditional function (OCF) or possibility theory. We then define a set of postulates that such revision operators shall satisfy and establish representation theorems to characterize those postulates. We show that these postulates reveal common characteristics of various existing revision strategies and are satisfied by OCF conditionalization, Jeffrey's rule of conditioning and possibility conditionalization. Furthermore, when reducing to the belief revision situation, our postulates can induce Darwiche and Pearl's postulates C1 and C2.

Keywords Epistemic state; Epistemic revision; Belief revision; Probability updating; Iterated revision; Ordinal conditional function; Possibility theory; Jeffrey's Rule

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1 Introduction

Information used in real applications is often uncertain, which reflects a kind of reliability of sources and sensors. In addition, knowledge bases are not static. There are always new information that should be taken into account. In probabilistic frameworks, these two aspects (i.e., uncertain and dynamic) are handled in homogeneous way by representing uncertainty associated with information in the form of probability distributions, and using different forms of conditioning for updating.

In Artificial Intelligent (AI) community, since 1985, the process of changing beliefs with new information is known as belief revision. Belief revision [1, 28, 34] performs belief change on an agent's beliefs when new evidence is received. It has been observed that a pure logic-based revision framework, e.g., AGM postulates based framework, may permit some counterintuitive results in iterated revision[1]. As a result, revision on epistemic states should be introduced accordingly [14, 4, 51, 5, 33, 42], etc.

However, in most of these research efforts, new evidence is still represented as a *propositional* formula, not an epistemic state (even if initial epistemic state may be a propositional formula or a totally pre-ordered relation on a set of possible worlds). Therefore, these methods do not fully implement a revision that reflects the effect of epistemic states, e.g., new information could be uncertain [14, 15]. Although an effort has been made to address this problem in a couple of papers (e.g., [4]), in which new evidence is represented as a full epistemic state. The revision methods proposed still cannot manage the strengths over partitions on a set of interpretations, which, in probability or possibility settings, is already accomplished by Jeffrey's rule [3]. That is, we need to develop a revision framework which can deal with new information with strengths that could be modeled by partial epistemic states similar to the probability counterparts of Jeffrey's rule. Here we should note that new information not only comes from observations of the agent on the environment but can also be conveyed by other agents where *epistemic* makes sense. So here we use partial epistemic states as inputs which we aim to cover both situations, even if there might be some abuse of concept.

Jeffrey's rule is widely applied when an agent's current belief and new evidence are both modeled in probability measures. More precisely, in Jeffrey's rule, the prior state is a probability distribution representing an agent's current beliefs or generic knowledge whilst new evidence is a partial probability measure solely on a partitioned subsets of the world. Similar strategies were also proposed for ordinal conditional functions (OCFs) [58, 59], for possibility measures [19, 3], etc. However, despite of the need to handle new, input information with strengths that may be present in different forms, to the best of our knowledge, there does not exist a common revision strategy (and its corresponding postulates) to address this issue. In another words, can we develop a general revision framework that subsumes these individual revision strategies (in different frameworks) with a set of common postulates? A significant advantage of this, if achievable, is to facilitate further understanding of the nature of revision, regardless of which formalism may be deployed to represent an agent's beliefs and new uncertain

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[1]Note that AGM postulates are not designed for iterated revision. So it is not surprising they permit counterintuitive results in iterated revision.

evidence.

To answer this question, we first propose a framework to represent an agent's epistemic beliefs, which generalizes various definitions of epistemic states in the literature (e.g., a weighted formula [33, 48], a total pre-order [4], an OCF-based epistemic state [49, 58, 59], a probability measure [30], a partial pre-order [45, 41], a mass function [46, 47], etc). This framework takes inspirations from Jeffrey's rule of conditioning under uncertain inputs. We then investigate how a set of rational postulates should be derived to regulate revision operators defined from this framework and provide representation theorems for these postulates. We prove that these postulates are satisfied by OCF conditionalization, possibility conditionalization, and most significantly Jeffrey'rule of conditioning.

Our main objective of defining a general iterated revision framework is to implement the revision of an agent's current beliefs (represented as a full epistemic state) with new, uncertain evidence (represented as a partial epistemic state). In standard AGM framework, there are no explicit representations of strengths associated with initial beliefs and inputs (even if any revision operator that satisfies AGM postulates is implicitly based on some total pre-order on interpretations). Our framework, however, supports the explicit representation of strengths which will help in determining the result of revision.

Furthermore, we investigate the relationships between this general framework with logic-based belief/epistemic revision, especially with Darwiche and Pearl's (DP's) belief revision framework [14]. We prove that when reducing to the belief revision situation, our postulates can induce DP's postulates C1 and C2.

To summarize, the main contributions of the paper are:

- Our definition of epistemic states subsumes many existing definitions of epistemic states.

- We provide a generalized revision strategy and corresponding postulates.

- We prove two presentations theorems which show clear and succinct *kinematic* semantics in revision.

- We prove that our framework can recover many existing numerical revision operators, e.g., Jeffrey's rule, OCF conditionalization, possibilistic revision, etc.

- When an input is a formula, our postulates can induce all the AGM-KM postulates and Darwiche and Pearl's C1, C2 postulates.

In other words, our framework provides an important one step forward of extending the various existing revision strategies and revision operators.

The rest of the paper is organized as follows. We provide the preliminaries and Jeffrey's rule in Section 2 and 3 respectively. In Section 4, formal definitions of epistemic space and epistemic state are introduced. In Section 5, we propose a set of postulates for epistemic revision and their corresponding representation theorems. In Section 6 and Section 7, we discuss how our framework subsumes existing revision strategies. Finally, we conclude the paper in Section 8.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ defined on a finite set $\mathcal{A}$ of propositional atoms, which are denoted by p, q, r etc. A proposition ϕ is constructed by propositional atoms with logic connectives $\neg, \wedge, \vee, \rightarrow$ in the standard way. An interpretation ω (or possible world) is a function that maps $\mathcal{A}$ onto the set $\{0, 1\}$. The set of all possible interpretations on $\mathcal{A}$ is denoted as W. Function ω can be extended to any proposition in $\mathcal{L}$ in the usual way. An interpretation ω is a model of (or satisfies) ϕ iff $\omega(\phi) = 1$, denoted as $\omega \models \phi$. We use $Mod(\phi)$ to denote the set of models for ϕ. We write $\phi \vdash \psi$ if $Mod(\phi) \subseteq Mod(\psi)$ and $\phi \equiv \psi$ if $Mod(\phi) = Mod(\psi)$. Furthermore, we also take the view that a proposition ϕ can be equivalently represented by a subset of the set of possible worlds W. That is, for any proposition ϕ, there is a subset A of W such that $Mod(\phi) = A$. Let $|A|$ denote the cardinality of A.

$\{A_1, \ldots, A_n\}$ is a *partition* of set W iff $\forall i, A_i \neq \emptyset, \bigcup_{i=1}^n A_i = W$ and for $i \neq j$, $A_i \cap A_j = \emptyset$. For convenience, we also call $\{\mu_1, \ldots, \mu_n\}$ a *partition* of set W when $\{A_1, \ldots, A_n\}$ is a partition and for any A_i, $Mod(\mu_i) = A_i$, and hence any μ_i is consistent.

Definition 1 *A partition $\{B_1, \ldots, B_k\}$ (resp., $\{\phi_1, \ldots, \phi_k\}$) is called a* **refinement** *of partition $\{A_1, \ldots, A_n\}$ (resp., $\{\mu_1, \ldots, \mu_n\}$) if $\forall i, 1 \leq i \leq k, \exists j, 1 \leq j \leq n$, s.t. $B_i \subseteq A_j$ (resp., $\phi_i \vdash \mu_j$).*

3 Jeffrey's Rule

In probability theory framework, a well-known revision method is Jeffrey's rule [32].

Definition 2 *(Jeffrey's rule) Let P be the prior probability distribution on W and $\mathscr{F} = \{\mu_1, \ldots, \mu_n\}$ be a partition of W. Assume that a new piece of evidence gives a probability measure $(W, \mathscr{F}, P^{\mathscr{F}})$ such that $P^{\mathscr{F}}(\mu_i) = \alpha_i \geq 0, 1 \leq i \leq n$ with $\sum_{1 \leq i \leq n} \alpha_i = 1$. Then Jeffrey's Rule revises P with $P^{\mathscr{F}}$ with operator $\circ_p$ and obtains*

$$(P \circ_p P^{\mathscr{F}})(w) = \alpha_i P(w)/P(\mu_i) \quad for \ w \models \mu_i \tag{1}$$

A conventional extension for Equation (1) is that if $P(\mu_i) = 0$, then $P^{\mathscr{F}}(\mu_i) = \alpha_i$ must be 0, and consequently $\alpha_i P(w)/P(\mu_i)$ is defined as 0 (otherwise it is not defined). This setting is also proposed in [3] for possibility measures that an impossible event should be always impossible. That is, if an event is impossible in terms of the initial epistemic state, then it should also be impossible in the input.

Jeffrey's rule revises the prior probability distribution P to P' given an uncertain input with probabilities bearing on a partition of W. It produces a unique distribution that satisfies the following two equations [12]:

$$P'(\mu_i) = P^{\mathscr{F}}(\mu_i) = \alpha_i \tag{2}$$

which shows that the new information is preserved and

$$\forall \mu_i, \forall \phi \vdash \mu_i, P(\phi|\mu_i) = P'(\phi|\mu_i) \tag{3}$$

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which states that the revised (new) probability distribution P' must retain the degree of conditional probability of any event ϕ that implies μ_i.

Note that Equation (3) can be equivalently written as:

$$\forall \mu_i, \forall \phi, \phi' \vdash \mu_i, \frac{P(\phi)}{P'(\phi)} = \frac{P(\phi')}{P'(\phi')} \tag{4}$$

This is often called *probability kinematics* [32, 12].

The following example shows how Jeffrey's rule is applied.

Example 1 *(Adapted from [3]) Let us consider the following example where we are interested in knowing if a given researcher, named JM who works in a laboratory in computer science, is attending a given conference. We are also interested in knowing whether JM is lodging in the hotel recommended by the conference. Lastly, we would like to know whether JM has a biometric passport. For simplicity, we only use the following two variables to encode available information:*

H: *to express that JM booked a room in conference hotel,*

B: *to express that JM has a biometric passport.*

The following probability distribution provides an encoding of our initial beliefs.

$$P(H \wedge B) = 0.4, P(H \wedge \neg B) = 0.2, P(\neg H \wedge B) = 0.3, P(\neg H \wedge \neg B) = 0.1$$

And assume that we have a new piece of information, where the director of laboratory states that now booking the conference hotel is less plausible than booking a non-conference hotel. This new information is represented by the following uncertain input.

$$P'(H) = 0.3, P'(\neg H) = 0.7$$

Then using Jeffrey's rule, we get the revised probability distribution as follows.

$$(P \circ_p P')(H \wedge B) = 0.2, (P \circ_p P')(H \wedge \neg B) = 0.1,$$

$$(P \circ_p P')(\neg H \wedge B) = 0.525, (P \circ_p P')(\neg H \wedge \neg B) = 0.175$$

For the resulting epistemic state $(P \circ_p P')$, we have

$$(P \circ_p P')(H) = P'(H) = 0.3,$$

$$(P \circ_p P')(\neg H) = P'(\neg H) = 0.7.$$

In the context of H (resp. $\neg H$), the plausibility ordering between B and $\neg B$ in the initial epistemic state is the same as that in the resulting epistemic state. Hence, Jeffrey's rule admits input while respecting the minimal change principle.

4 Epistemic Space and Epistemic State

In order to define a general revision framework with uncertain input, we first provide formal definitions of epistemic space and epistemic state. Let D denote an infinite[2] set of values with two special elements $\bot, \top$ in D, and there is a total pre-order $\preceq_D$ on D (corresponding notations like $\prec_D, \succ_D$ are defined as usual) such that $\forall x \in D$, $\bot \preceq_D x \preceq_D \top$. For example, if $D = [0,1]$, then we have $\bot = 0$ and $\top = 1$. $x \preceq_D y$ can be seen as x is at most as preferred (or plausible, important, etc) as y.

4.1 Extension functions

Since a revision process usually involves some kind of set operation, so when strengths are attached to formulae/sets (as mentioned before, a formula corresponds to a set), we need some function being introduced to handle the operations on strengths of the sets, in particular, a function that associates the strength of a set to the strengths of its subsets. To this end, before defining epistemic states, we first define the notion of **extension function**. A function f associating a value in D to a multi-set of values in D is called an extension function if it satisfies

Identity $f(\{x\}) = x$
Minimality $f(\{x_1, \ldots, x_k\}) = \bot$ iff $x_1 = \ldots = x_k = \bot$
Monotonicity $f(\{x\}) \preceq_D f(\{x, y\})$

We do not define $f : 2^D \to D$ since this function is not simply a mapping from 2^D to D, rather it is a mapping from any tuple of values from D to a value in D with the tuple size varies. For instance, we can have $f(\{x, x, x\}) = x'$, but $(\{x, x, x\})$ is not an element of 2^D. This also follows the definition of aggregation functions in [36].

For simplicity, if there is no confusion from the context, for $f(\{x_1, \ldots, x_k\})$ hereafter we will omit the multi-set sign and write it as $f(x_1, \ldots, x_k)$.

In [36], an aggregation function f is defined as a function associating a single non-negative integer to a set of non-negative integers and satisfies the following three properties:

Identity $f(x) = x$,
Minimality $f(x_1, \cdots, x_k) = 0$ iff $x_1 = \cdots = x_k = 0$,
Non-decreasingness If $x \preceq_D y$, then $f(x_1, \cdots, x, \cdots, x_k) \preceq_D f(x_1, \cdots, y, \cdots, x_k)$.

An extension function is similar to an aggregation function in the sense that both of them attempt to associate a set of values to a single value within a given domain. The differences between them are (i) an extension function is defined on D instead of a set of integers; and (ii) it satisfies the Monotonicity property above instead of the *Non-decreasingness* property.

Note that *Monotonicity* property and *Non-decreasingness* property define two different classes of extension functions. For example, functions like *median*, *mode* satisfy

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[2]Here we should note that the finiteness of W and the infiniteness of D are not inconsistent. For instance, let $W = \{w_1, w_2\}$, then W is finite, but as we can define infinite numbers of probability functions on W, each value in $D = [0,1]$ can be obtained. So they are not inconsistent.

Non-decreasingness but not *Monotonicity* and the function f in the following example satisfy *Monotonicity* but not *Non-decreasingness*. In some sense, Monotonicity imposes more constraints on the set-structure of the functions than Non-decreasingness, e.g., it excludes the mode function. In this paper, to prove the representation theorems, we require the *Monotonicity* property instead of the *Non-decreasingness* property.

Example 2 *Let* $D = \{2^a 3^b | a \in N, b \in N \cup \{\infty\}\}$ *such that* $\preceq_D$ *is defined as arithmetic* $\geq$, $\top = 1$, *and* $\perp = \infty$. *Let* f *be defined as* $f(2^{a_1} 3^{b_1}, \cdots, 2^{a_n} 3^{b_n}) = 2^{min(a_1, \cdots, a_n)} 3^{min(b_1, \cdots, b_n)}$. *Obviously,* f *satisfies Identity, Minimality and Monotonicity, but* f *does not satisfy the Non-decreasingness property. For example,* $4 \prec_D 3$, *but* $f(4, 9) = f(2^2 3^0, 2^0 3^2) = 1 \succ_D 3 = f(3, 9) = f(2^0 3^1, 2^0 3^2)$.

4.2 Partial and full epistemic states

Now we define epistemic states and epistemic spaces, which are similar to the definition of probability spaces and probability measures.

Definition 3 *Let* $\mathscr{F}$ *be a partition of* W, *a* **partial epistemic state** Φ *on* $\mathscr{F}$ *is a mapping associating a value in* D *to each element of* $\mathscr{F}$.

If there is no confusion from the context, we simply call Φ a partial epistemic state.

Definition 4 *A* **partial epistemic space** *is a tuple* $(W, \mathscr{F}, \Phi, D, f)$, *where:*

- $\mathscr{F}$ *is a partition of* W,

- Φ *is a partial epistemic state, and*

- f *is an extension function.*

Φ *can be extended from* $\mathscr{F}$ *to* $2^{\mathscr{F}}$ *by* f[3] *such that for* $A_1, \cdots, A_k \in \mathscr{F}$, $\Phi(\bigcup_{i=1}^{k} A_i) = f(\Phi(A_1), \cdots, \Phi(A_k))$.

Note that it would be more accurate to use $\{A_1, \cdots, A_k\}$ instead of $\bigcup_{i=1}^{k} A_i$, but as $A_i \cap A_j = \emptyset$ when $i \neq j$, we simply use $\bigcup_{i=1}^{k} A_i$.

Example 3 *Let* $W = \{w_1, w_2, w_3\}$ *and a partition* $\mathscr{F}$ *on* W *be* $\{\{w_1, w_2\}, \{w_3\}\}$. *Also, let* $D = \{Good, \cdots, Neutral, \cdots, Bad\}$ *be the (infinite) set of values with* $\perp = Bad \prec_D \cdots \prec_D Neutral \prec_D \cdots \prec_D Good = \top$ *and* $f = max$ *(f satisfies Identity, Minimality and Monotonicity). Let* Φ *define the following mapping:* $\Phi(\{w_1, w_2\}) = Good$ *and* $\Phi(\{w_3\}) = Bad$, *then* $(W, \mathscr{F}, \Phi, D, f)$ *is a partial epistemic space.*

An epistemic space $(W, \{\{w_1\}, ... \{w_n\}\}, \Phi, D, f)$ is a special case of partial epistemic space $(W, \mathscr{F}, \Phi, D, f)$ where the partition of W is the set of all singleton sets. To differentiate the former from the latter, we call $(W, \{\{w_1\}, ... \{w_n\}\}, \Phi, D, f)$ a **full epistemic space** and its corresponding Φ a **full epistemic state.**

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[3]Note that f is hence defined on $2^{\mathscr{F}}$ instead of on W. This is not a limitation since in our framework, we only need the f values over $2^{\mathscr{F}}$. In addition, a function over W always induces a unique function over $2^{\mathscr{F}}$ while the converse is not.

Note that a partial epistemic state Φ such that $\Phi(A) = \alpha$, $\Phi(\overline{A}) = \beta$ is not equal to a full epistemic state Ψ such that $\forall w \in A, \Psi(w) = a$ (e.g., in the probabilistic case, $a = \alpha/|A|$, and in the OCF case, $a = \alpha$), $\forall w \notin A, \Psi(w) = b$ and $\Psi(A) = \alpha$, $\Psi(\overline{A}) = \beta$ by f_Ψ. For example, in probability theory, a probability measure P with $P(\{man, woman\}) = 0.8$ does not mean $P(\{man\}) = P(\{woman\}) = 0.4$. Instead, from $P(\{man, woman\}) = 0.8$ we can obtain a family of possible probability distributions over $W = \{man, woman\}$. So a partial epistemic state can not be encoded by a full epistemic state.

Obviously, if Φ is a probability measure ($D = [0, 1]$, f is $+$), then the above definition degenerates to the definition of probability space.

In this paper we use Φ, Ψ, Θ etc (possibly with a subscript) to denote an epistemic state.

Literally, although there have been many papers focusing on epistemic revision and merging, there does not exist a commonly accepted definition of epistemic state. In some papers (e.g. [14]), no formal definitions of epistemic state are given, though the concept is used. In papers like [4], the definition of epistemic state is always associated with an epistemic space which contains the observable parts of the epistemic state and the projection function to obtain the observable parts. In papers for knowledge base merging or arbitration (e.g., [39, 21]), knowledge bases play the role of epistemic states. In some other papers, definitions for epistemic states are mainly based on plausibility orderings on possible worlds [49, 58, 59, 33], etc. In addition, concrete representations like probability measures, OCFs, possibility measures could also be considered as epistemic states.

It is easy to see that an epistemic state as a plausibility ordering can be induced from a *full epistemic state*. That is,

Definition 5 *For any full epistemic state Φ, an ordering $\leq_\Phi$ between interpretations is defined as $\forall w, w'$, $w \leq_\Phi w'$ iff $\Phi(w') \preceq_D \Phi(w)$.*

Furthermore, Def. 4 not only generalizes the notion of probability space, it also takes definitions of OCFs (when D is set a set of ordinals with $\top = 0$, $\bot = +\infty$ and $\preceq_D = \geq$, and $f = min$) and possibility measures (when D is $[0, 1]$ and $f = max$) as special cases. Therefore, Def. 4 indeed provides a general framework to model epistemic states defined in different formalisms.

Value $\Phi(A)$ can be interpreted as an agent's *epistemic firmness* on A. Note that $\Phi(A)$ encodes *all* the information an agent provides on A, in particular, if the agent changes the value $\Phi(\overline{A})$ while maintaining $\Phi(A)$ unchanged, we should consider that the agent maintains its belief on A, despite that the agent has changed its belief on $\overline{A}$. Usually, constraints are placed on $\Phi(A)$ and $\Phi(\overline{A})$, e.g., if Φ is a probability measure, then $\Phi(A) + \Phi(\overline{A}) = 1$, or if Φ is an OCF , then $min(\Phi(A), \Phi(\overline{A})) = 0$. Here for generality, we do not assume any constraints on $\Phi(A)$ and $\Phi(\overline{A})$.

Intuitively, the *Minimality* property of f, when considered in Def. 4, ensures that if an agent thinks the true world is definitely not in a particular set, then the true world should not be in any of its subsets, and vice versa, i.e., $\Phi(\bigcup_{i=1}^k A_i) = \bot$ iff $\Phi(A_1) = \cdots = \Phi(A_k) = \bot$. Similar properties are also introduced in [13] as the *impossibility* property and in [52] as the *zero-preservation* property. The *Monotonicity* property

indicates that if $A \subseteq B$, then $\Phi(A) \preceq_D \Phi(B)$, especially when $\Phi(A)$ is interpreted as a kind of plausibility value (epistemic firmness) of A. This property is very similar to Axiom A1: if $A \subseteq B$ then $Pl(A) \leq Pl(B)$ for a plausibility measure Pl [26] which was also mentioned in [30].

Parallel to probability theory, probability distributions are applied (and discussed) more frequently than their corresponding probability spaces. In the following, most of the time we will only mention epistemic states without explicitly discussing their corresponding epistemic spaces too.

To ensure that a domain D contains sufficient but non-redundant elements, we impose two requirements on D.

Definition 6 D *is* **strict** *iff it satisfies the following*

- *for any W, $\forall x \in D$, $\exists \Phi$, $\exists A \subseteq W$ s.t., $\Phi(A) = x$.*

- *for any W, $\forall x, y \in D$ and $x \preceq_D y$, $\exists \Phi$, $\exists A, B$ s.t. $A \subseteq B \subseteq W$, $\Phi(A) = x$ and $\Phi(B) = y$.*

The intuition of the first condition is that D does not contain redundant elements. The second condition corresponds to the *Monotonicity* requirement of f. Both conditions are used to relate D to epistemic states. Observe that requiring D *strict* is not a major issue because Φ is selected freely in the above definition (not restricted by D). That is, D is independent of Φ. In addition, D is independent of W. In the following, we always assume D is strict. This assumption is well fitted in many uncertainty representation formalisms. For example, as long as $|W| > 1$, $D = [0, 1]$ is strict in probabilistic and possibilistic settings.

4.3 Entailment of epistemic states

As W, D and f are assumed to be clear and unchanged throughout, an epistemic state Φ defined on $\mathscr{F}$ is denoted as $\Phi^{\mathscr{F}}$.

By abuse of notations, we also write $\Phi^{\mathscr{F}}(\mu) = \alpha$ when $\Phi^{\mathscr{F}}(A) = \alpha$ and $Mod(\mu) = A$, i.e., a proposition is assigned a plausibility value which is the value assigned to the set of its models. In the following, we will use propositions rather than their corresponding sets of models.

We define the entailment of epistemic states as follows.

Definition 7 *Let $\Phi^{\mathscr{F}_1}$ and $\Psi^{\mathscr{F}_2}$ be two epistemic states, then $\Phi^{\mathscr{F}_1}$ entails $\Psi^{\mathscr{F}_2}$, denoted as $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$, iff $\mathscr{F}_1$ is a refinement of $\mathscr{F}_2$ and $\forall \mu \in \mathscr{F}_2$, $\Phi^{\mathscr{F}_1}(\mu) = \Psi^{\mathscr{F}_2}(\mu)$.*

From this definition, we can see that if $\Phi^{\mathscr{F}_1}$ entails $\Psi^{\mathscr{F}_2}$, then $\Phi^{\mathscr{F}_1}$ contains more specific information than $\Psi^{\mathscr{F}_2}$.

Note that each element μ of $\mathscr{F}_2$ is necessarily the union of several elements of $\mathscr{F}_1$. Hence $\Psi_1^{\mathscr{F}}(\mu)$ can always be obtained using extension functions.

Example 4 *Let $W = \{w_1, w_2\}$ and $f = +$, $\Phi^{\mathscr{F}_1}$ be such that $\mathscr{F}_1 = \{\{w_1\}, \{w_2\}\}$, $\Phi^{\mathscr{F}_1}(\{w_1\}) = 1$, $\Phi^{\mathscr{F}_1}(\{w_2\}) = 2$ (hence $\Phi^{\mathscr{F}_1}(\{w_1, w_2\}) = 3$), $\Psi^{\mathscr{F}_2}$ be such that $\mathscr{F}_2 = \{\{w_1, w_2\}\}$, $\Psi^{\mathscr{F}_2}(\{w_1, w_2\}) = 3$, then we have $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$.*

In the rest of the paper, to differentiate, a full epistemic state will be represented without a superscript describing a partition (e.g., Φ) and a partial epistemic state always with a superscript describing a partition (e.g., $\Psi^{\mathscr{F}}$).

5 Revising Epistemic States by Partial Epistemic States

5.1 Epistemic Revision

5.1.1 Postulates

Motivated by the principle of Jeffrey's rule on conditioning on probability spaces and the ideal requirement that only the strengths of prior beliefs and evidence should determine the outcome of belief revision [14], we propose the following constraints on revision in our epistemic space framework.

- Revision should be focused on a full epistemic state (representing prior beliefs or generic knowledge) revised by a partial epistemic state (representing a new, uncertain input). This is the spirit of Jeffrey's rule (revising a probability distribution with an uncertain input) and existing revision frameworks (e.g., prior beliefs are total pre-orders whilst an input is a propositional formula). Hence we use full epistemic states to encode current beliefs and partial epistemic states to encode new, uncertain inputs.

- Only the strengths of beliefs and new evidence determine the outcome of revision. This is the main argument in [14]. This postulate is intuitively in agreement with the *Neutrality with respect to the intensity scale* condition proposed in [17] which says in a social choice scenario, an aggregation function should not depend on the semantic meanings of a set of social choice functions, but only focus on their intensities of choices.

- New, most recent evidence has the priority. For one-shot revision, this is explained as that new evidence is preserved.

In addition, we also have a default assumption that an impossible event in the initial epistemic state should also always be impossible in the revision. That is,

Impossibility Preservation for any $\mu \in \mathscr{F}$, if $\Phi(\mu) = \bot$, then $\Psi^{\mathscr{F}}(\mu) = \bot$.

This is a requirement of consistency between the initial state and the input, just similar to the settings for Jeffrey's rule and revision of possibility measures in [3].

Based on these constraints, we propose the following three postulates. Let $\circ$ be a revision operator.

ER1 If Φ is a full epistemic state and $\Psi^{\mathscr{F}}$ is a partial epistemic state, then $\Phi \circ \Psi^{\mathscr{F}}$ is a full epistemic state.
Explanation: This postulate suggests that the revision operator $\circ$ is a mapping from a full epistemic state and a partial epistemic state to a full epistemic state. Again, it is worth pointing out that Φ and $\Psi^{\mathscr{F}}$ share the same D, W and f. This postulate can be called **Preservation**.

ER2 $\Phi \circ \Psi^{\mathscr{F}} \models \Psi^{\mathscr{F}}$.

Explanation: New evidence is preserved. More precisely, $\Psi^{\mathscr{F}}$ can be recovered from $\Phi \circ \Psi^{\mathscr{F}}$ with the extension function f. By convention, this postulate can be called **Success**.

ER3 For any $\mu \in \mathscr{F}$, and $\mu' \in \mathscr{F}'$, if $\Phi(\mu) = \Phi'(\mu')$ and $\Psi^{\mathscr{F}}(\mu) = \Psi'^{\mathscr{F}'}(\mu')$, then for $\psi \vdash \mu$ and $\psi' \vdash \mu'$, $(\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi')$ iff $\Phi(\psi) = \Phi'(\psi')$.

To get a better understanding, ER3 can be stated with two separate steps as follows:

For any $\mu \in \mathscr{F}$, and $\mu' \in \mathscr{F}'$, if

- $\Phi(\mu) = \Phi'(\mu')$ and
- $\Psi^{\mathscr{F}}(\mu) = \Psi'^{\mathscr{F}'}(\mu')$

Then $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\mu')$.

More generally, we require: for $\psi \vdash \mu$ and $\psi' \vdash \mu'$,

$$(\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi') \text{ iff } \Phi(\psi) = \Phi'(\psi').$$

Explanation: This postulate implements the constraint that the strengths of beliefs and evidence determine the outcome of revision. More specifically, as evidence $\Psi^{\mathscr{F}}$ (resp. $\Psi'^{\mathscr{F}'}$) provides no information on ψ (resp. ψ') directly, the only information related to ψ (resp. ψ') is μ (resp. μ') as $\psi \vdash \mu$ (resp. $\psi' \vdash \mu'$), so the strength of ψ (resp. ψ') after revision should only rely on its own strength before revision and the strengths of μ (resp. μ') before and after revision.

ER3 stems from Equation (4) of Jeffrey's rule. If we let $\Phi = \Phi'$, $\Psi^{\mathscr{F}} = \Psi'^{\mathscr{F}'}$, $\mu = \mu'$, then ER3 is reduced to:

$$\forall \mu \text{ and } \forall \phi, \phi' \vdash \mu, (\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi \circ \Psi^{\mathscr{F}})(\psi') \text{ iff } \Phi(\psi) = \Phi(\psi').$$

Comparing to Equation (4), $\Phi \circ \Psi^{\mathscr{F}}$ can be viewed as the counterpart of the revised probability distribution P' in Equation (4) and Φ as P. From this point of view, ER3 can be seen as an extension of Equation (4) to the epistemic state case. In addition, this postulate is intuitively in agreement with the *Neutrality with respect to the intensity scale* condition proposed in [17] which says in a social choice scenario, an aggregation function should not depend on the semantic meanings of a set of social choice functions, but only focus on their intensities (numerical values in $[0, 1]$) of choices.

A simple proposition induced from **ER1** and **ER2** is stated as follows.

Proposition 1 *Let Φ be a full epistemic state, then $\Phi \circ \Phi = \Phi$.*

5.1.2 Relationship with the AGM-KM postulates

In this subsection, we will make some remarks about the relationship between our postulates and the AGM-KM postulates [1, 34].

In [1], a set of belief revision postulates are proposed and in [34], this set of postulates are reformulated. In [14], Darwiche and Pearl presented a series of beautiful examples to illustrate the weakness of AGM-KM postulates on iterated belief revision.

But we should point out that the comparison with AGM-KM postulates and Darwiche and Pearl's postulates [14] have been done in two parts in this paper. Here in Section 5 we do a simple comparison aiming to show what is the counterpart of each the standard AGM-KM postulates and DP's postulates in our framework, and whether they can be defined in our framework. This is also a way to justify our postulates. Later in Section 7, we will provide a detailed comparison between the AGM-KM (and DP) framework and our framework which shows that when reducing to the belief revision situation, our postulates induce the same belief set as done by the AGM-KM (and DP) postulates.

Darwiche and Pearl recommended that to ensure the rational preservation of conditional beliefs[4]during (iterated) belief revision, a revision process shall perform operations on epistemic states not just on their belief sets.

With this intention, they modified the AGM-KM postulates to obtain a set of revised postulates in which an agent's original belief is in the form of epistemic states and new evidence is a propositional formula. The revised postulates for epistemic revision are

R1 $\Phi \circ \mu$ implies μ.
R2 If $\Phi \wedge \mu$ is satisfiable, then $\Phi \circ \mu \equiv \Phi \wedge \mu$.
R3 If μ is satisfiable, then $\Phi \circ \mu$ is also satisfiable.
R4 If $\Phi_1 = \Phi_2$ and $\mu_1 \equiv \mu_2$, then $\Phi_1 \circ \mu_1 \equiv \Phi_2 \circ \mu_2$.
R5 $(\Phi \circ \mu) \wedge \phi \vdash \Phi \circ (\mu \wedge \phi)$.
R6 If $(\Phi \circ \mu) \wedge \phi$ is satisfiable, then $\Phi \circ (\mu \wedge \phi) \vdash (\Phi \circ \mu) \wedge \phi$.

Here, Φ (possibly with a subscript) stands for an epistemic state[5] and μ and ϕ are propositional formulae. $\Phi \circ \mu$ is an epistemic state after revising Φ with revision operator $\circ$ by μ. When Φ is embedded in a propositional formula, it is used to stand for $Bel(\Phi)$ (its belief set which is a formula) not an epistemic state for simplification purposes, for example, $\Phi \wedge \phi$ means $Bel(\Phi) \wedge \phi$. These

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[4]Let ψ and α be two propositional formulae and let $\circ$ be a belief revision operator, the revision of ψ by α is a new propositional formula and is denoted as $\psi \circ \alpha$. $\beta|\alpha$ is a conditional belief of ψ if $\psi \circ \alpha \vdash \beta$. Please refer to [55, 27, 38, 6, 8, 2] for detailed studies of conditionals in belief revision.

[5]It did not explicitly define what an epistemic state is in [14], but it can be considered as an agent's current beliefs together with the relative plausibility orderings of possible worlds (represented by a total pre-order on W) which are inconsistent with the current beliefs.

postulates are natural extensions of the AGM-KM postulates to epistemic revision except that postulate (R4) is a weaker version of the original KM postulate[6] which states that if two equivalent formulae are revised by two other equivalent formulae respectively, then the revised results should be equivalent. Since different epistemic states can have the same belief set, (R4) requires that not just the two initial belief sets, but the two epistemic states must be identical:[7] $\Phi_1 = \Phi_2$. It was argued in [33] that since there was no formal definition on epistemic states in [14], it was not possible to define the equivalence of two such states. So (R4) is changed into

R4$'$ If $\mu_1 \equiv \mu_2$, then $\Phi \circ \mu_1 \equiv \Phi \circ \mu_2$.

Obviously, ER2 is a straightforward generalization of R1, whilst ER1 extends R3 in the epistemic revision situation where new evidence is also an epistemic state. ER3, however, not only generalizes R4, but also is a key characteristic postulate of revision considering with strengths of beliefs and evidence. Actually this postulate shows that we do not need to care about the semantics of propositional formulae, but only their syntactical relations and their strengths determine the revision result.

There are no obvious generalizations for R5 and R6 in our postulates, because the conjunction of two formulae (for two belief sets) used in DP postulates is hardly generalizable on epistemic revision in our framework. In another words, the conjunction of two epistemic states are undefinable [8]. For instance, let $W = \{w_1, w_2, w_3\}$, and let $\Phi^{\mathscr{F}}(\{w_1, w_2\}) = \Phi^{\mathscr{F}}(\{w_3\}) = \alpha$ and $\Psi^{\mathscr{F}'}(\{w_1\}) = \Psi^{\mathscr{F}'}(\{w_2, w_3\}) = \beta$ be two epistemic states, then it is not obvious how to define the conjunction of $\Phi^{\mathscr{F}}$ and $\Psi^{\mathscr{F}'}$.

As for postulate R2, the following proposition shows why we do not need to provide a separate postulate as its generalization.

Proposition 2 *Let Φ be a full epistemic state, $\Psi^{\mathscr{F}}$ be a partial epistemic state and $\circ$ be an epistemic revision operator satisfying* **ER1-3**. *For any $\mu \in \mathscr{F}$, if $\Phi(\mu) = \Psi^{\mathscr{F}}(\mu)$, then $\forall \phi \vdash \mu$, $\Phi(\phi) = (\Phi \circ \Psi^{\mathscr{F}})(\phi)$.*

This proposition shows that when **ER1-3** hold, then if new evidence is partially consistent with the prior state, then the consistent part is not changed, which can be seen as an extension of R2.

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[6]The postulate can be written as: If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.

[7]In [29], an example nicely illustrating the difference between the original postulate and R4 is as follows. *Two jurors in a murder trial possess different biases; juror l believes "A is the murderer, B is a remote but unbelievable possibility while C is definitely innocent". Juror 2 believes "A is the murderer, C is a remote but unbelievable possibility while B is definitely innocent". The two jurors share the same belief set represented by the consequences of "A is the only murderer". A surprising new piece of evidence now states: "A is not the murderer" (A has produced a reliable alibi). Clearly, any rational account of belief revision should allow juror 1 to uphold a different belief set from juror 2.* Here the two jurors have the same belief set but different epistemic states.

[8]In a similar sense, in [18], it is argued that the case of expansion never occurs for probability functions.

5.2 Iterated Epistemic Revision

5.2.1 Postulates

In this subsection, we propose postulates for iterated epistemic revision. Here the constraint that new, most recent evidence has the priority can also be explained under the context that when two pieces of new information happen to have the same partition $\mathscr{F}$ (which implies that both pieces of evidence refer to the same sets of hypotheses) but with different strengths of belief on them, then the most recent evidence overrules the previous one (as the latter (evidence) is assumed to represent the most recently received (and acceptable) information about a situation).

We have

ER4 $\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}} = \Phi \circ \Theta^{\mathscr{F}}$.

    Explanation: W.r.t. the same hypotheses with different strengths, the latest evidence overrules previous ones.

    Note that in ER4, by the default assumption, for any $\mu \in \mathscr{F}$, if $\Phi(\mu) = \bot$, then we must have $\Psi^{\mathscr{F}}(\mu) = \bot$ and $\Theta^{\mathscr{F}}(\mu) = \bot$. Again this is for the consistency between the initial epistemic states and the inputs.

ER4* $\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'} = \Phi \circ \Theta^{\mathscr{F}'}$ where partition $\mathscr{F}'$ is a refinement of partition $\mathscr{F}$.

    Explanation: The latest fine-grained evidence overrules old ones.

    By the default assumption mentioned in Section 5.1 (Impossibility Preservation), in ER4*, for any $\mu \in \mathscr{F}$, if $\Phi(\mu) = \bot$, then we must have $\Psi^{\mathscr{F}}(\mu) = \bot$. In addition, from $\Phi(\mu) = \bot$, for any $\phi \vdash \mu$, we must have $\Phi(\phi) = \bot$ (Minimality property in Section 4.1), then if $\phi \in \mathscr{F}'$, it should be $\Theta^{\mathscr{F}'}(\phi) = \bot$ (Impossibility Preservation again). Therefore, Impossibility Preservation is also respected by ER4*. These two postulates can be called **Iteration** postulates.

Example 5 *(Example 1 Cont') Recall the initial probability distribution for whether JM is lodging in a conference hotel and whether he has a biometric passport is:*

$$P(H \wedge B) = 0.4, P(H \wedge \neg B) = 0.2, P(\neg H \wedge B) = 0.3, P(\neg H \wedge \neg B) = 0.1$$

and the new evidence gives:

$$P'(H) = 0.3, P'(\neg H) = 0.7$$

Now assume that the secretary of the director tells us that she remembers JM had booked a conference hotel. This information is represented by the following input:

$$P''(H) = 0.9, P''(\neg H) = 0.1$$

Then from Jeffrey's rule, we can easily have the follows:

$$(P \circ_p P' \circ_p P'')(H \wedge B) = 0.6, (P \circ_p P' \circ_p P'')(H \wedge \neg B) = 0.3,$$

$$(P \circ_p P' \circ_p P'')(\neg H \wedge B) = 0.075, (P \circ_p P' \circ_p P'')(\neg H \wedge \neg B) = 0.025$$

$$(P \circ_p P' \circ_p P'')(H \wedge B) = 0.6, (P \circ_p P'')(H \wedge \neg B) = 0.3,$$

$$(P \circ_p P'')(\neg H \wedge B) = 0.075, (P \circ_p P'')(\neg H \wedge \neg B) = 0.025$$

That is, $P \circ_p P' \circ_p P'' = P \circ_p P''$.

Probabilistic revision by Jeffrey's rule is an example that follows all the above ER postulates.

5.2.2 Relationship with DP's iterated postulates

To regulate iterated epistemic revision to preserve conditional beliefs, Darwiche and Pearl gave the following four additional postulates which are for four disjoint types of conditional beliefs.

C1 If $\alpha \vdash \mu$, then $(\Phi \circ \mu) \circ \alpha \equiv \Phi \circ \alpha$.
C2 If $\alpha \vdash \neg\mu$, then $(\Phi \circ \mu) \circ \alpha \equiv \Phi \circ \alpha$.
C3 If $\Phi \circ \alpha \vdash \mu$, then $(\Phi \circ \mu) \circ \alpha \vdash \mu$.
C4 If $\Phi \circ \alpha \nvdash \neg\mu$, then $(\Phi \circ \mu) \circ \alpha \nvdash \neg\mu$.

$\Phi \circ \alpha \vdash \beta$ here stands for $Bel(\Phi \circ \alpha) \vdash \beta$.

As C1-C4 play an important role in the rest of the paper, here we repeat their explanations given in [14]. C1 states that when two pieces of evidence arrive, the second being more specific than the first, the first is redundant; that is, the second evidence alone would yield the same belief set. C2 says when two contradictory pieces of evidence arrive, the last one prevails; that is, the second evidence alone would yield the same belief set. C3 describes an evidence μ should be retained after accommodating more recent evidence α that implies μ given current beliefs. C4 gives that no evidence can contribute to its own demise. If μ is not contradicted after seeing α, then it should remain un-contradicted when α is preceded by μ itself.

In our framework, ER4 and ER4* are closely related to C1 and C2, but in general ER4 does not imply C1 and C2 while ER4* does.

5.3 Representation Theorems

In this subsection, we present our two representation theorems which provide semantic interpretations for postulates. For convenience, we recall Darwiche and Pearl's representation theorems [14].

In [14], two representation theorems are given to characterize the two sets of postulates, i.e., R1-R6 and C1-C4. First we introduce the definition of faithful assignment.

Definition 8 *([34, 14]) Let W be the set of all worlds (interpretations) of a propositional language $\mathcal{L}$ and suppose that the belief set of any epistemic state belongs to $\mathcal{L}$. A function that maps each epistemic state Φ to a total pre-order $\leq_\Phi$ on worlds W is said to be a* faithful assignment *if and only if:*

1. $w_1, w_2 \models \Phi$ only if $w_1 =_\Phi w_2$.
2. $w_1 \models \Phi$ and $w_2 \not\models \Phi$ only if $w_1 <_\Phi w_2$.
3. $\Phi \equiv \Psi$ only if $\leq_\Phi = \leq_\Psi$ where Ψ is also an epistemic state.

Here $w_1 <_\Phi w_2$ iff $w_1 \leq_\Phi w_2$ and $w_2 \not\leq_\Phi w_1$, $w_1 =_\Phi w_2$ iff $w_1 \leq_\Phi w_2$ and $w_2 \leq_\Phi w_1$.

Theorem 1 *([14]) A revision operator $\circ$ satisfies postulates **R1-R6** if and only if there exists a faithful assignment that maps each epistemic state Φ to a total pre-order $\leq_\Phi$ such that:*
$$Mod(\Phi \circ \mu) = min(Mod(\mu), \leq_\Phi).$$

Theorem 2 *([14]) Suppose that a revision operator $\circ$ satisfies postulates **R1-R6**. The operator satisfies postulates **C1**-**C4** iff the operator and its corresponding faithful assignment satisfy:*

CR1 *If $w_1 \models \mu$ and $w_2 \models \mu$, then $w_1 \leq_\Phi w_2$ iff $w_1 \leq_{\Phi \circ \mu} w_2$.*
CR2 *If $w_1 \models \neg\mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_\Phi w_2$ iff $w_1 \leq_{\Phi \circ \mu} w_2$.*
CR3 *If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 <_\Phi w_2$ only if $w_1 <_{\Phi \circ \mu} w_2$.*
CR4 *If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_\Phi w_2$ only if $w_1 \leq_{\Phi \circ \mu} w_2$.*

In [14], the proof of this representation theorem shows that an epistemic revision operator $\circ$ satisfies postulate Ci iff condition CRi is satisfied, $1 \leq i \leq 4$.

Now we present our representation theorems. In order to establish them, we need to define the **retentive** and **conductive** operators on D.

Definition 9 *An operator $\ominus$ defined on D is called **retentive** if for any $a_1, a_2, b_1, b_2 \in D$ s.t. $a_1 \preceq_D a_2$, $b_1 \preceq_D b_2$, the following statement holds:*

If	a_1	$\ominus$	$a_2 = b_1$	$\ominus$	b_2
and			$a_2 =$		$b_2,$
then	a_1		$= b_1.$		

The word *retentive* here intuitively means that when eliminating the equivalent second operands, the equivalence is still retained for the first operands.

Definition 10 *An operator $\ominus$ defined on D is called **conductive** if for any $a_1, a_2, a_3, b_1, b_2, b_3 \in D$ s.t. $a_1 \preceq_D a_2 \preceq_D a_3$, $b_1 \preceq_D b_2 \preceq_D b_3$, the following statement holds:*

If	a_1	$\ominus a_3 = b_1$	$\ominus b_3$
and		$a_2 \ominus a_3 =$	$b_2 \ominus b_3,$
then	$a_1 \ominus a_2$	$= b_1 \ominus b_2.$	

The word *conductive* here intuitively means that when eliminating same items from two equations (the second operands in both the first and second equations), the remaining operands can be combined naturally to form a new equation.

The retentive requirement specifies how to remove equivalent item in one equation and the conductive requirement tells how to remove same items from multiple equations to form a new equation.

There are many concrete retentive and conductive operators, for example, if $\ominus$ is the subtraction ('-') or division ('/') operator in mathematics, then it is retentive and conductive. Also, if $D = \{0, 1, \cdots, n - 1\}$, where n is a natural number, then $\ominus$ defined as $x \ominus y = (x - y) \bmod n$ is also retentive and conductive.

Theorem 3 *A revision operator $\circ$ satisfies postulates* **ER1-ER4** *iff there exists a* **retentive** *operator $\ominus$ defined on D such that for any full epistemic state Φ and any epistemic state $\Psi^{\mathscr{F}}$, $\forall \mu \in \mathscr{F}$ and $\forall \phi \vdash \mu$,*

$$(\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$$

and

$$(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu).$$

This theorem is a direct extension of Jeffrey's rule, as can be seen from the similarity between the two equations in the theorem and Equation (2) and Equation (3) of Jeffrey's rule (where $\ominus$ is division). If the epistemic states in this theorem are probability measures, then we immediately obtain the two requirements used for defining Jeffrey's rule.

To some extent, this theorem shows *how minimal change* happens during epistemic revision in the sense that it preserves a kind of *distance* based on $\ominus$ (although $\ominus$ is not constructively given). Furthermore, evidently $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ and $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$ are counterparts of Equation (2) and (4), respectively. As mentioned before, Equation (2) and (4) are necessary and sufficient conditions for Jeffrey's rule to yield a unique distribution. Therefore, this theorem presents a generalization of Jeffrey's rule.

With postulates **ER1-ER3** and **ER4***, we get the following representation theorem.

Theorem 4 *A revision operator $\circ$ satisfies postulates* **ER1-ER3** *and* **ER4*** *iff there exists a* **retentive and conductive** *operator $\ominus$ defined on D such that for any full epistemic state Φ and any epistemic state $\Psi^{\mathscr{F}}$, $\forall \mu \in \mathscr{F}$ and $\forall \phi \vdash \mu$, $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ and $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$.*

Example 6 *(An instance of revision operator) Let $W = \{(a, b) | a \in N, b \in N \cup \{\infty\}\}$, $D = \{2^a 3^b | a \in N, b \in N \cup \{\infty\}\}$ such that $\preceq_D$ is defined as the arithmetic $\geq$, $\top = 1$, and $\bot = \infty$. Let f be defined as $f(2^{a_1} 3^{b_1}, \cdots, 2^{a_n} 3^{b_n}) = 2^{min(a_1, \cdots, a_n)} 3^{min(b_1, \cdots, b_n)}$ and Φ be such that $\Phi((a, b)) = 2^a 3^b$. Let a new piece of evidence taken on partition $\mathscr{F} = \{\mu_1, \cdots, \mu_n\}$ be such that $\Psi^{\mathscr{F}}(\mu_i) = \alpha_i$, $1 \leq i \leq n$, then we can define a revision operator $\circ_n$ as*

$$(\Phi \circ_n \Psi^{\mathscr{F}})(w) = \alpha_i \Phi(w) / \Phi(\mu_i) \text{ for } w \models \mu_i$$

$\circ_n$ also satisfies postulates **ER1-ER4** *and* **ER4*** *, and f satisfies the* Monotonicity *Property but not the* Non-decreasingness *property.*

Example 7 *(An instance of an improvement operator) Improvement operators are introduced in [37] with a* Weak Primacy of Update *intuition such that the plausibility of the new information must be increased after the improvement, instead of having to be accepted as in the AGM framework. Let $D = \{1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^i}, \cdots, 0\}$, $f = max$ and $\ominus = /$ (division) in theorem 4, for any epistemic state Φ and any formula μ, let a partial epistemic state $(\Psi^{\mathscr{F}})$ such that $(\Psi^{\mathscr{F}})(\mu) = \Phi(\mu)$ and $(\Psi^{\mathscr{F}})(\neg\mu) = \frac{\Phi(\neg\mu)}{2}$ represents μ, then it is easy to check that $\circ$ reduces to an improvement operator for Φ and μ.*

6 Comparison with numeric revision strategies

6.1 Jeffrey's rule

Since our framework aims to extend Jeffrey's rule, in this section, we look into how the postulates we proposed on iterated epistemic revision would coincide with Jeffrey's rule of conditioning.

Formally, let P be a prior epistemic state which is a probability distribution on W, and a new piece of evidence shows that a partition $\mathscr{F} = \{\mu_1, \cdots, \mu_n\}$ of W should take new probabilities as $P^{\mathscr{F}}(\mu_i) = \alpha_i$, $1 \leq i \leq n$ with $\sum_{1 \leq i \leq n} \alpha_i = 1$.

Let P be a probabilistic epistemic state (i.e., a probability distribution extended to 2^W by $f = +$), and $P^{\mathscr{F}}$ be a probabilistic partial epistemic state, an epistemic state revision operator $\bullet$ constructed from Jeffrey's Rule (i.e. Definition 2) is defined as follows

$$(P \bullet P^{\mathscr{F}})(w) \stackrel{def}{=} P^{\mathscr{F}}(\mu_i) \frac{P(w)}{P(\mu_i)} \text{ for } w \models \mu_i. \tag{5}$$

We use the table below to compare our framework with Jeffrey's rule.

Our Framework	Jeffrey's Rule
W	W
D	[0,1]
f	+
Φ (Epistemic state)	P
$\Psi^{\mathscr{F}}$ (Evidence)	$P^{\mathscr{F}}$
$\Psi^{\mathscr{F}}(\mu_i) = \alpha_i, 1 \leq i \leq n$	$P^{\mathscr{F}}(\mu_i) = \alpha_i, 1 \leq i \leq n$
$(\Phi \circ \Psi^{\mathscr{F}})(form(w)) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu_i) = \Phi(form(w)) \ominus \Phi(\mu_i)$	$\frac{(P \circ_p P^{\mathscr{F}})(w)}{(P \circ_p P^{\mathscr{F}})(\mu_i)} = \frac{P(w)}{P(\mu_i)}$

Table 1: Comparison between our framework and Jeffrey's rule

Table 1 clearly shows that how our framework is reduced to the probability case and recovers Jeffrey's framework.

The following theorem shows that the above revision operator for probability updating satisfies our postulates.

Theorem 5 *The revision operator $\bullet$ defined in Equation (5) satisfies postulates* **ER1-ER4** *and* **ER4***.

In fact, Equation (2) of Jeffrey's rule can be directly recovered by postulate **ER2**, whilst Equation (3) of Jeffrey's rule, i.e., the probability kinematics, is implied in the ER postulates.

6.2 OCF Conditionalization and Possibility Revision

OCF conditionalization: An OCF [58] κ is a function from a set of possible worlds W to the set of ordinals with $\kappa^{-1}(0) \neq \emptyset$. It can be extended to a set of propositions as $\kappa(\mu) = min_{w \models \mu}\kappa(w)$. Given κ as the prior OCF, $\mathscr{F} = \{\mu_1, \cdots, \mu_n\}$ as a partition and a new piece of evidence as $\lambda^{\mathscr{F}}(\mu_i) = \alpha_i$, $1 \leq i \leq n$ s.t. $min_{1 \leq i \leq n}(\alpha_i) = 0$, then the conditionalization of κ w.r.t $\lambda^{\mathscr{F}}$ is

$$(\kappa \circ_c \lambda^{\mathscr{F}})(w) = \alpha_i + \kappa(w) - \kappa(\mu_i) \text{ for } w \models \mu_i \tag{6}$$

If we consider the conditionalization operator $\circ_c$ as a revision operator, then Equation (6) can be seen as a revision strategy, and we have

Theorem 6 *The revision operator $\circ_c$ defined in Equation (6) satisfies postulates* **ER1-ER4** *and* **ER4***.

Possibility revision: A possibility distribution [20] π is a mapping from W to $[0, 1]$. It induces a possibility measure Π and a necessity measure N as follows:

$$\Pi(\mu) = max_{w \models \mu}\pi(w) \text{ and } N(\mu) = 1 - \Pi(\neg\mu).$$

$\Pi(\mu)$ estimates to what extent an agent believes μ can be true while $N(\mu)$ estimates the degree the agent believes that μ is necessarily true.

One of the possibility conditioning methods is defined as

$$\Pi(\phi|\mu) \stackrel{def}{=} \frac{\Pi(\phi \wedge \mu)}{\Pi(\mu)} \tag{7}$$

A counterpart of Spohn's (μ, α)-conditionalization was suggested in [20] in possibility theory such that if new evidence suggests that $\Pi'(\mu) = 1$ and $\Pi'(\neg\mu) = 1 - \alpha$ (which implies $N'(\mu) = \alpha$), then the belief change can take the following form

$$\Pi'(w) = \begin{cases} \Pi(w|\mu) & \text{for } w \models \mu \\ (1-\alpha)\Pi(w|\neg\mu) & \text{for } w \not\models \mu \end{cases} \tag{8}$$

This simple revision operator can be easily generalized to a more complex situation which is a counterpart of Jeffrey's Rule and Spohn's general conditionalization. Formally, let Π be a prior epistemic state which is a possibility measure (Π can be seen as defined on W and extended to 2^W by $f = max$), and let $\mathscr{F} = \{\mu_1, \cdots, \mu_n\}$ be a partition of W, the partial epistemic state for new evidence is defined as $\Pi'^{\mathscr{F}}(\mu_i) = \alpha_i$, $1 \leq i \leq n$ with $max_{1 \leq i \leq n}\alpha_i = 1$.

Hence an epistemic revision operator $\circ'_p$ in possibility theory can be constructed by a generalization of Equation (8) as follows

$$(\Pi \circ'_p \Pi'^{\mathscr{F}})(w) \stackrel{def}{=} \alpha_i \frac{\Pi(w)}{\Pi(\mu_i)} \text{ for } w \models \mu_i. \tag{9}$$

The following theorem shows that the above revision operator for possibility revision satisfies our postulates.

Theorem 7 *The revision operator $\circ'_p$ defined in Equation (9) satisfies postulates* **ER1-ER4** *and* **ER4***.

7   Comparison with Logic-based Iterated Belief/Epistemic Revision

In this section, we compare our framework with other logic-based (iterated) belief revision frameworks. To do so, we need to ensure that each epistemic state has a non-empty belief set, hence we exclude epistemic states with empty belief sets.

Definition 11 *Let $\Phi^{\mathscr{F}}$ be a partial epistemic state and μ be any propositional formula that is a disjunction of some formulae in $\mathscr{F}$. $\Phi^{\mathscr{F}}$ is said to satisfy the* **Maximality** *property iff $\Phi^{\mathscr{F}}$ satisfies $\Phi^{\mathscr{F}}(W) = \top$, and $\Phi^{\mathscr{F}}(\mu) = \top$ iff $\exists \phi \in \mathscr{F}, \phi \vdash \mu, \Phi^{\mathscr{F}}(\phi) = \top$.*

Particularly, if Ψ is a full epistemic state and μ is any propositional formula, Ψ satisfies the Maximality property iff it satisfies $\Psi(W) = \top$, and $\Psi(\mu) = \top$ iff $\exists w \models \mu$, $\Psi(w) = \top$. For instance, OCF and possibility measures satisfy the Maximality property.

In this section, if there is no other specification, we always take the Maximality property as a default assumption.

Now we can define the belief set of an epistemic state as follows.

Definition 12 *Let $\Phi^{\mathscr{F}}$ be a partial epistemic state which satisfies property Maximality, then its belief set $Bel(\Phi^{\mathscr{F}})$ is defined as*

$$Bel(\Phi^{\mathscr{F}}) = \{\mu : \Phi^{\mathscr{F}}(\mu) = \top\} \tag{10}$$

In other words, the belief set of an epistemic state (with property Maximality) is the set of propositions with a plausibility value $\top$. An alternative but not equivalent, weaker definition of belief set is $Bel(\Phi^{\mathscr{F}}) = \{\mu : \Phi^{\mathscr{F}}(\mu) > \Phi^{\mathscr{F}}(\neg\mu)\}$. In the following, we only concentrate on epistemic states with non-empty belief sets. We can prove that the definition of entailment on epistemic states generalizes the classical definition of entailment on beliefs of epistemic states.

Proposition 3 *Let $\Phi^{\mathscr{F}_1}$ and $\Psi^{\mathscr{F}_2}$ be two epistemic states, if $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$, then $\bigvee Bel(\Phi^{\mathscr{F}_1}) \vdash \bigvee Bel(\Psi^{\mathscr{F}_2})$.*

This proposition shows that our postulate **ER2** is truly an extension of the success postulate of KM postulates [34].

7.1 DP's Iterated Belief Revision

In this subsection, we aim to compare our framework with Darwiche and Pearl's iterated belief revision framework [14]. We show that only half of the DP postulates (C1 and C2) are covered by our framework.

For convenience, we use $\Delta^{\mathscr{F}_\mu}$ to denote a partial epistemic state such that its corresponding partition $\mathscr{F}_\mu$ is $\mathscr{F}_\mu = \{\mu, \neg\mu\}$, and the values are $\Delta^{\mathscr{F}_\mu}(\mu) = \top$, $\Delta^{\mathscr{F}_\mu}(\neg\mu) \prec_D \top$ ($\Delta^{\mathscr{F}_\mu}(\neg\mu)$ can be any value in D other than $\top$). Hence we have $Bel(\Delta^{\mathscr{F}_\mu}) = \{\mu\}$. In the following, we use $\Delta^{\mathscr{F}_\mu}$ to encode new evidence where in logic-based revision frameworks, e.g., [14], etc, new evidence is simply represented as a single formula μ.

In addition, we use notation Γ^μ to denote a partial epistemic state such that its partition $\mathscr{F}_{\Gamma^\mu} = \{\mu, \neg\mu\}$ and $\Gamma^\mu(\mu) = \top$, $\Gamma^\mu(\neg\mu) = \bot$. Note that Γ is different from the notation Δ. Based on Γ, we have the following definition.

Definition 13 *Let w_1, w_2 be two possible worlds of W, $\circ$ be an epistemic revision operator satisfying all the ER postulates, and Φ be a full epistemic state. We define $w_1 \leq_\Phi w_2$ if $(\Phi \circ \Gamma^{form(w_1,w_2)})(w_1) = \top$[9]. Also, $w_1 <_\Phi w_2$ iff $w_1 \leq_\Phi w_2$ and $w_2 \not\leq_\Phi w_1$, and $w_1 =_\Phi w_2$ iff $w_1 \leq_\Phi w_2$ and $w_2 \leq_\Phi w_1$.*

Conventionally, we just write $\leq_\Phi$ instead of writing $\leq_{\circ,\Phi}$.

$w_1 \leq_\Phi w_2$ shows that w_1 is *more plausible* than w_2 in epistemic state Φ. Intuitively, if some w_is are the "most likely", i.e., $\Phi(w_i) = \top$, then we should expect that for any w_j, $w_i \leq_\Phi w_j$. This is verified by the following proposition.

Proposition 4 *Let Φ be a full epistemic state, if $\Phi(w_1) = \top$, then $w_1 \leq_\Phi w_2$ for any $w_2 \in W$.*

Proposition 5 *Let Φ be a full epistemic state, then $\leq_\Phi$ defined based on Definition 13 is a total pre-order.*

The following definition of *faithful* assignment is a counterpart of that in [34].

Definition 14 *Let W be the set of all possible worlds and $\mathcal{E}$ be the set of all full epistemic states. A function that maps from each full epistemic state Φ in $\mathcal{E}$ to a total pre-order $\leq_\Phi$ on W is called a **faithful assignment** if and only if:*

1. If $\Phi(w_1) = \Phi(w_2) = \top$, then $w_1 =_\Phi w_2$.

2. If $\Phi(w_1) = \top$ and $\Phi(w_2) \neq \top$, then $w_1 <_\Phi w_2$.

3. If $\Phi \simeq \Psi$, then $\leq_\Phi = \leq_\Psi$ where $\Psi \in \mathcal{E}$ is also a full epistemic state and $\Phi \simeq \Psi$ means that $\forall w \in W$, $\Phi(w) = \Psi(w)$.

The following proposition shows that our Definition 13 from Φ to $\leq_\Phi$ is a faithful assignment.

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[9]This definition is more general than the definition that $w_1 \leq_\Phi w_2$ iff $\Phi(w_2) \preceq_D \Phi(w_1)$. And this definition is sufficient for our purpose to compare with DP's postulates. Furthermore, this definition is still valid when $\preceq_D$ is not a total pre-order. Also note that this definition is intuitively similar to the one defined in [34].

Proposition 6 *The mapping from each full epistemic state Φ to $\leq_\Phi$ based on Definition 13 is faithful.*

Theorem 8 *Let Φ be a full epistemic state, μ be a propositional formula and $\circ$ be an epistemic revision operator, if $\circ$ satisfies postulates* **ER1-ER3** *and* **ER4***, *then we have*

$$Mod(Bel(\Phi \circ \Delta^{\mathscr{F}_\mu})) = min(Mod(\mu), \leq_\Phi), and$$

C1* *If $\alpha \vdash \mu$, then $Bel(\Phi \circ \Delta^{\mathscr{F}_\mu} \circ \Delta^{\mathscr{F}_\alpha}) = Bel(\Phi \circ \Delta^{\mathscr{F}_\alpha})$.*

C2* *If $\alpha \vdash \neg\mu$, then $Bel(\Phi \circ \Delta^{\mathscr{F}_\mu} \circ \Delta^{\mathscr{F}_\alpha}) = Bel(\Phi \circ \Delta^{\mathscr{F}_\alpha})$.*

This theorem shows that the belief set from epistemic revision on an epistemic state Φ with $\Delta^{\mathscr{F}_\mu}$ is equal to the belief set from belief revision on Φ with formula μ. It also reveals that our revision postulates imply DP's iterated belief revision postulates C1 and C2. Furthermore, this theorem, together with Proposition 5 and Proposition 6, also conclude that our postulates (**ER1-ER3, ER4***) indeed imply **R1-R6** when epistemic states have belief sets.

In general our postulates do not induce C3 and C4. In the epistemic states settings, C3 and C4 can be re-written as follows.

C3* $(\Phi \circ \Delta^\alpha)(\mu) = \top$, then $(\Phi \circ \Delta^\mu \circ \Delta^\alpha)(\mu) = \top$.

C4* $(\Phi \circ \Delta^\alpha)(\neg\mu) \neq \top$, then $(\Phi \circ \Delta^\mu \circ \Delta^\alpha)(\neg\mu) \neq \top$.

Example 8 *A father believes that his child X is clever(c) (i.e. as X usually gets high marks in exams) and very honest(h) (so he thinks X gets high marks not by cheating). His epistemic state at that time can be described by an ordinal conditional function κ such that $\kappa(c, h) = 0$, $\kappa(\neg c, h) = 3$, $\kappa(c, \neg h) = 10$ and $\kappa(\neg c, \neg h) = 15$. Now some event made the father realized that the child told a little lie, still he believes that his child is largely honest but his belief in the child's dishonesty increased. This leads to a new kappa function $\kappa'(h) = 0$ and $\kappa'(\neg h) = 2$. Then he revised his original epistemic state to $\kappa'(c, h) = 0$, $\kappa'(\neg c, h) = 3$, $\kappa'(c, \neg h) = 2$ and $\kappa'(\neg c, \neg h) = 7$.*

Let $\mu = h$, Δ^μ be such that $\Delta^\mu(\mu) = 0$ and $\Delta^\mu(\neg\mu) = 2$, $w_1 = \{\neg c, h\}$, $w_2 = \{c, \neg h\}$, $\alpha = form(w_1, w_2)$ and Δ^α be such that $\Delta^\alpha(\alpha) = 0$ and $\Delta^\alpha(\neg\alpha) = 2$. Then we can easily check that $(\kappa \circ \Delta^\alpha)(\mu) = \top$ and $(\kappa \circ \Delta^\alpha)(\neg\mu) \neq \top$, but we have $(\Phi \circ \Delta^\mu \circ \Delta^\alpha)(\mu) \neq \top$ and $(\Phi \circ \Delta^\mu \circ \Delta^\alpha)(\neg\mu) = \top$ which contradict to C3* and C4*, respectively.

The revision in the above example is in fact an extension of classical revision in the sense that the propositional part of new information is consistent with the prior state[10] but the strengths on some propositions are different. This is a natural extension, at least not surprising, as a revision is to revise the prior information with new information in the sense that new information is inconsistent with the prior information. In a belief revision setting, information is only represented as a formula, hence

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[10]This situation is also considered in the non-prioritized revision in [22] which gives the following postulate. Let A and K both be a set of sentences, if $A \subseteq K$, and $K \not\models \bot$, then $K \circ A = K$. In [59], this situation is named "reconstructing" in an OCF setting.

the inconsistency appears in the form of logical inconsistency of formulae. While in an epistemic revision setting, information is a formula plus its epistemic firmness, hence the inconsistency can also appear as that the same formula having different epistemic firmness, and therefore, needs a revision. This is also suggested in [9] that *one might revise one's epistemic commitments without thereby revising one's beliefs.* Our framework can encode this kind of revision but DP framework does not accept this. In this kind of revision, postulates C3 and C4 do not hold.

7.2 Three Kinds of Iterated Revision Strategies

In [24], a review of belief change literature is presented, among which three well-known revision strategies are mentioned, i.e., the conservative revision, moderate revision and radical revision.

In [8], natural revision, or conservative revision is proposed that after revision, only the rank of the most plausible worlds of the evidence is changed to *most plausible* in the revised beliefs. That is, let $\circ_N$ be the natural revision operator, Φ be the initial state, μ be the evidence, w_1, w_2 be any of the most plausible world of μ, i.e., $w_1, w_2 \in min(Mod(mu), \leq_\Phi)$, and w'_1, w'_2 be any other worlds, then natural revision can be described by the following revised plausibility orderings:

$$w_1 =_{\Phi \circ_N \mu} w_2, w_1 <_{\Phi \circ_N \mu} w'_1, \ and \ w'_1 \leq_{\Phi \circ_N \mu} w'_2 \ \text{iff} \ w'_1 \leq_\Phi w'_2.$$

It is not difficult to see from Example 8 that our framework does not extend natural revision.

In [50, 51], revision of epistemic entrenchment is proposed which in fact revises a full epistemic state by a full epistemic state in a lexicographic way. Epistemic entrenchment considers a kind of partial preorder based strength, but it still cannot express strengths in a general manner. Lexicographic revision is also called moderate revision. In moderate revision, we usually consider the Recalcitrance (Rec) postulate [51] and Independence (Ind) postulate [5, 33] as follows:

Rec If $\alpha \not\vdash \neg\mu$, then $(\Phi \circ \mu) \circ \alpha \vdash \mu$.

Ind If $\Phi \circ \neg\alpha \not\vdash \neg\mu$, then $(\Phi \circ \mu) \circ \neg\alpha \vdash \mu$.

We can also prove that these two postulates do not hold when epistemic revision is reduced to belief revision. Note that from the semantics, it is easy to see that Rec implies Ind which implies C3 and C4. Hence it is natural that in Example 8, the same settings of Φ, α and μ also show that postulates Rec and Ind do not hold.

It is interesting to investigate further why postulates C3, C4, Rec, and Ind do not hold. We believe that the problem is rooted from postulate R*2. Postulate R*2 says that if the new information μ is consistent with the prior belief $Bel(\Phi)$ of the prior epistemic state Φ, then the belief set of revised epistemic state is the conjunction of the belief set of prior state and the new information (i.e., $\mu \wedge Bel(\Phi)$, the consistent part). Note that R*2 simply ignores the inconsistent part (i.e., $\mu \wedge \neg Bel(\Phi)$ and $\neg\mu \wedge Bel(\Phi)$). In a belief revision situation, this ignorance does not affect the revision result as the result only contains the consistent part. However, in our epistemic revision

situation, as we want to get a full epistemic state after revision (hence the revision result should consider both the consistent and the inconsistent part), this ignorance should be properly handled. Example 8 shows that the belief set is not changed, but the firmness of other formulae are weakened. R*2 cannot distinguish these two situations, so C3, C4, Rec, and Ind are all failed here.

However, it is worth pointing out that the above investigation does not against iterated revision axioms in belief revision, since in belief revision, the belief set is the main set that axioms are designed for. But in epistemic revision, not only the belief set but also the revision strategy need to be considered too. Therefore, it is expected that some axioms for belief revision will not be suitable for epistemic revision.

Let $\circ_L$ be the lexicographic revision operator, it is not difficult to show the following results.

Proposition 7 *Let Φ be the initial state, μ be the evidence, then we have $\Phi \circ_N \mu \equiv \Phi \circ_L min(Mod(\mu), \leq_\Phi)$.*

The last one, radical revision, or irrevocable revision, is in fact conditioning. A radical revision on evidence μ can be translated in our epistemic revision framework as revision on evidence Γ^μ which is defined in Section 7.1, before Definition 13.

7.3 Revising Full Epistemic States by Full Epistemic States

A set of axioms (i.e., REE*1-REE*4, REE*It) for characterizing iterated revision of full epistemic states (total pre-orders) by full epistemic states was presented in [4] as follows.

> **REE*1** $\Phi \circ \Psi \models \Psi$
> **REE*2** If $\Phi \wedge \Psi$ is consistent, then $\Phi \circ \Psi \equiv \Phi \wedge \Psi$
> **REE*3** If Ψ is consistent, then $\Phi \circ \Psi$ is consistent
> **REE*4** If $\Psi_1 \equiv \Psi_2$, then $\Phi \circ \Psi_1 \equiv \Phi \circ \Psi_2$
> **REE*It** $(\Phi \circ \Theta) \circ \Gamma \equiv \Phi \circ (\Theta \circ \Gamma)$

Similarly, here an epistemic state Φ embedded in a formula stands for $Bel(\Phi)$.

The following result presents the relationship between the REE Axioms and our postulates.

Proposition 8 *A revision operator $\circ$ satisfying **REE*1-4** and **REE*It** also satisfies **ER1-2** and **ER4***.*

ER3 is not mentioned in the above proposition since it is not interpretable (e.g., $\Phi(\mu)$, etc) in the settings of [4]. In addition, the converse is false. This is not surprising since the framework of [4] leads to a unique solution.

7.4 Iterated Conditional Revision

In [35], conditional preservation for belief revision is studied through the link of Ramsey Test [54]. This paper introduces an algebraic structure $\mathscr{A} = (\mathscr{A}, \leq_{\mathscr{A}}, \oplus, \odot, 0^{\mathscr{A}}, 1^{\mathscr{A}})$

(readers can refer to [35] for details) which is a nice morphism to probabilistic algebra. They also introduced a notion *conditional valuation function* enforcing algebraic properties on $\mathscr{A}$ which makes $\mathscr{A}$ almost the full morphism to probabilistic algebra. Based on this algebraic structure, in [35], epistemic states are defined as logic formulae equipped with conditional valuation functions, and then iterated conditional belief revision is axiomatized and investigated, showing that, with proper definitions, all the DP postulates (adapted in a conditional form) can be recovered. It is not so surprising since the algebraic structure defined in [35] is more specific than the definition of epistemic state in our paper. However, this also suggests that the framework in [35] may be too strict as recovering all the DP postulates implies rejecting Example 8 in Section 7.1 where our framework allows. Nevertheless, the algebraic structure is a good hint on developing future extensions of our framework to yield a unique solution for belief revision.

7.5 Two-dimensional Belief Revision

In [10, 25, 57, 56], two kinds of two-dimensional belief revision, revision by comparison and bounded revision, are introduced. Unlike our epistemic states approach which is largely quantitatively, the two-dimensional revision *lies between quantitative and qualitative approaches in that they do not use numbers and still able to specify the extent or degree to which a new piece of information is to be accepted* [57]. But since two-dimensional revision only imposes a constraint on the acceptance degree of new information and another given formula, it does not have the same expressive power as our epistemic states revision does, although it is better than that of traditional belief revision.

7.6 Revision on possibility functions

[3] studies revision on possibility functions and proposes five axioms for the revision process. Let Π be the prior possibility function, $\mathscr{F} = \{\mu_1, \cdots, \mu_n\}$ be a partition of W, Π_I be the possibility function defined over $2^{\mathscr{F}}$ such that $\Pi_I(\mu_i) = \lambda_i$ with $max_i \lambda_i = 1$, $\circ_P$ be any revision operator on possibility functions and $\Pi' = \Pi \circ_P \Pi_I$ be the possibility function after revision, then the five axioms are listed as follows:

A1: Consistency Π' is a possibility function
A2: Priority to Input $\forall \mu_i, \Pi'(\mu_i) = \lambda_i$
A3: Faithfulness $\forall w, w' \models \mu_i$, if $\Pi(w) \geq \Pi(w')$, then $\Pi'(w) \geq \Pi'(w')$
A4: Inertia $\forall \mu_i$, if $\Pi(\mu_i) = \lambda_i$, then $\forall w \models \mu_i, \Pi'(w) = \Pi(w)$
A5: Impossibility Preservation If $\Pi(w) = 0$, then $\Pi'(w) = 0$

According to [3], A1 means that the revised state is consistent, A2 is the success postulate, A3 says the new function keeps the previous relative order between models of each μ_i, A4 illustrates that means that when the partial epistemic state is in agreement with prior possibility levels of μ_i, then revision does not affect models of μ_i, and A5 restricts that impossible worlds remains impossible after revision. These axioms are

basic principles a revision strategy on possibility functions should satisfy. However, they do not lead to a unique solution for revision.

We can see that the revision operator defined in Section 6.2 satisfies all the axioms.

7.7 Probabilistic Revision using Counterfactual Probabilistic Functions

In [40], belief revision using counterfactual probabilistic functions, or Popper functions, is proposed. In this approach, each probability function is attached with a unique Popper function which determines the revised probability function. In [7], this approach is extended to revision by probabilistic ordinal conditional functions, which is also associated with corresponding Popper functions. However, in [7], it shows that this Popper function approach has an unattractive feature that the relationship between a Popper function and its revision can be arbitrary. In addition, in both papers, it is the ranks of possible worlds (determined by a simple plausibility ordering, or an pre-given OCF), rather than the strengths (i.e., the probability values), that play a role in revision.

7.8 Other research works on belief revision

Apart from the research works addressed above, there are still a lot of other papers proposing alternative belief revision models that are relevant or partially relevant to our revision framework. In [11], it proposes a way to deal with contradicting information by considering a support ordering on that information. In [23], a framework of belief revision is proposed where the epistemic input is a pair $< A, i >$ where A is a set of sentences and i represents the layer of the stratified belief base to be revised. In some sense, the layer of the stratified belief base to be revised can be seen as the set of beliefs (with the same strength) to be affected by the revision process. In [16], revision postulates are verified in situations that new evidence (a formula) is accepted only when its certainty degree exceeds some confidence level c, and it shows that in this case, many revision postulates do not hold. In [43], a revision framework is investigated where the revision result is determined by the strength of information instead of the arrival order, and shows that it actually leads to a merging operator.

8 Conclusion

Jeffrey's rule is a well-known and important rule that guides the probability kinematics. It can also be seen as a revision strategy for probability measures. Due to its importance in revision and practical usage, OCF and possibility measures, etc, have introduced its counterparts to serve as their revision strategies. But since these counterparts are all for particular kinds of epistemic states, an investigation of extending Jeffrey's rule to generalized epistemic states is desirable.

In this paper, we have proposed a general definition of epistemic states and studied its revision strategy where new, uncertain evidence is represented as a partial epistemic state. We introduced a formal definition of epistemic state which can cover a variety

of definitions of epistemic states in the literature. A set of epistemic revision postulates and their corresponding representation theorems were then provided from which we can recover several well-known revision strategies including Jeffrey's probabilistic kinematics and the revision of full epistemic states by full epistemic states. A comparison with logic-based belief revision frameworks was also presented.

When reducing to the belief revision situation by Darwiche and Pearl (where new evidence is a propositional formula and each epistemic state has a belief set), our postulates subsume two of the DP's postulates while other remaining postulates are not suitable for iterated epistemic revision.

The underlying assumption of belief revision is that the most recent evidence has the highest priority. This assumption has its drawbacks. Darwiche and Pearl realized this issue and concluded that a natural way to resolve this is to allow *the outcome of belief change depends on the strength of evidence triggering the change*. In the Future work section [14], they briefly discussed an idea of using multiple revision operators $\circ_m$ instead of a single revision operator, where $\circ_m$ means revising an epistemic state with evidence having strength m. However, how to design such operators and how to manage a sequence of revision operators $\circ_{m_1}, \circ_{m_2}, ..., \circ_{m_n}$ remain to be investigated.

The assumption of giving priority to the most recent evidence is also questioned in [15]. To get around this assumption, iterated revision is taken as a prioritized merging where a set of evidence is prioritized according to their reliabilities rather than the *time points* these pieces of evidence arrive. The revised (or the merged) result is a consistent belief set such that when a more reliable piece of evidence is inconsistent with a less reliable piece of evidence, then the reliable evidence should be preserved in the belief set.

A minor drawback with the method in [15] is that it should preserve all the previous pieces of evidence to form a prioritized observable base before merging. But usually (or from a limited memory perspective) revision is only based on current state and recent evidence. In [42], a belief change framework is proposed to deal with the belief revision on epistemic states that the uncertain input is not surely accepted which does not need to record all the previous pieces of evidence.

In the literature, the success postulate, i.e., giving new information primacy, is also violated by non-prioritized belief revision operators[11] (cf. [31, 24] for an overview). However, usually strength does not play a role in non-prioritized belief revision. Also, non-prioritized revision is more closely to fusion than revision. Let K be the initial state and μ be the evidence, a simple example of non-prioritized revision gives $K \circ \mu \vee \mu \circ K$ where $\circ$ is an ordinary revision operator. The difference between fusion and non-prioritized revision is largely the starting point.

For our future work, we will investigate belief expansion and contraction in our epistemic framework. Furthermore, study on revising a partial epistemic state with a partial epistemic state with some additional constraints (e.g., maximum-entropy) is also an interesting topic. Finally, postulational approaches to belief revision are criticized in [53] that belief revision operators defined in these approaches are in fact ill-defined, and belief revision hence should be better studied with an independently motivated epistemological theory. We are interested in whether this conclusion still holds on our

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[11]In fact, it is the definition of non-prioritized revision [24].

epistemic revision.

References

[1] C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet functions for contraction and revision. *J. Sym. Log.*, 50:510–530, 1985.

[2] H. Arló-Costa. Belief revision conditionals: basic iterated systems. *Annals of Pure and Applied Logic*, 96:3–28, 1999.

[3] S. Benferhat, D. Dubois, H. Prade, and M. A. Williams. A framework for iterated belief revision using possibilistic counterparts to Jeffrey's rule. *Funda. Inform.*, 99(2):147–168, 2010.

[4] S. Benferhat, S. Konieczny, O. Papini, and R. P. Pérez. Iterated revision by epistemic states: Axioms, semantics and syntax. In *ECAI'00*, pages 13–17, 2000.

[5] R. Booth and T. Meyer. Admissible and restrained revision. *JAIR*, 26:127–151, 2006.

[6] C. Boutilier. Revision sequences and nested conditionals. In *Procs. of IJCAI*, pages 519–525, 1993.

[7] C. Boutilier. On the revision of probabilistic belief states. *Notre Dame J. Formal Logic*, 36(1):158–183, 1995.

[8] C. Boutilier. Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic*, 25(3):262–305, 1996.

[9] H. Brent. *Belief revision and epistemic value.* https://dspace.ucalgary.ca/handle/1880/26655, Department of Philosophy, University of Calgary, 1997.

[10] J. Cantwell. On the logic of small changes in hypertheories. *Theoria*, 63:54–89, 1997.

[11] J. Cantwell. Resolving conflicting information. *Journal of Logic, Language and Information*, 7(2):191–220, 1998.

[12] H. Chan and A. Darwiche. On the revision of probabilistic beliefs using uncertain evidence. *Artif. Intel.*, 163:67–90, 2005.

[13] R. M. Cooke. *Experts in Uncertainty*. Department of Mathematics, Delft University of Technology, Delft, 1990.

[14] A. Darwiche and J. Pearl. On the logic of iterated belief revision. *Artif. Intel.*, 89:1–29, 1997.

[15] J. Delgrande, D. Dubois, and J. Lang. Iterated revision as prioritized merging. In *KR'06*, pages 210–220, 2006.

[16] J. P. Delgrande. Revising beliefs on the basis of evidence. *International Journal of Approximate Reasoning*, 53(3):396–412, 2012.

[17] D. Dubois and J. L. Koning. Social choice axioms for fuzzy set aggregation. *Fuz. sets and Sys.*, 43:257–274, 1991.

[18] D. Dubois, S. Moral, and H. Prade. Belief change rules in ordinal and numerical uncertainty theories. In *D. Gabbay and P. Smets, editors, Handbook of Defeasible Reasoning and Uncertainty Management Systems, volume 3*, pages 311–392. Kluwer Academic Pub., 1998.

[19] D. Dubois and H. Prade. Representation and combination of uncertainty with belief functions and possibility measures. *Computational Intelligence*, 4:244–264, 1988.

[20] D. Dubois and H. Prade. Belief revision and updates in numerical formalisms: An overview, with new results for the possibilistic framework. In *IJCAI*, pages 620–625, 1993.

[21] P. Everaere, S. Konieczny, and P. Marquis. Disjunctive merging: Quota and gmin merging operators. *Artificial Intelligence*, 174(12-13):824–849, 2010.

[22] M. Falappa, G. Kern-Isberner, and G. Simari. Explanations, belief revision and defeasible reasoning. *Artificial Intelligence*, 141(1):1–28, 2002.

[23] M. A. Falappa, A. J. Garcia, G. Kern-Isberner, and G. R. Simari. Stratified belief bases revision with argumentative inference. *J. Philosophical Logic*, 42(1):161–193, 2013.

[24] E. Fermé and S. O. Hansson. AGM 25 years twenty-five years of research in belief change. *Journal of Philosophical Logic*, 40(2):295–331, 2011.

[25] E Fermé and H Rott. Revision by comparison. *Artif. Intel.*, 157(1-2):5–47, 2004.

[26] N. Friedman and J. Halpern. Plausibility measures: A user's guide. In *Proceedings of Eleventh Conference on Uncertainty in Artificial Intelligence (UAI-95)*, pages 175–18, San Francisco, CA, 1995. Morgan Kaufmann.

[27] P. Gärdenfors. Belief revision and the Ramsey test for conditionals. *Philosophical Review*, 95:81–93, 1986.

[28] P. Gärdenfors. Knowledge in flux: modeling the dynamics of epistemic states. 1988.

[29] M. Goldszmidt and J. Pearl. Qualitative probabilites for default reasoning, belief revision, and causal modeling. *Artificial Intelligence*, 84:57–112, 1996.

[30] J. Halpern. *Reasoning about Uncertainty*. MIT Press, 2003.

[31] S. O. Hansson. A survey of non-prioritized belief revision. *Erkenntnis*, 50(2-3):413–427, 1999.

[32] J. Jeffrey. *The logic of decision*. McGraw-Hill Press, 1965.

[33] Y. Jin and M. Thielscher. Iterated belief revision, revised. *Artif. Intel.*, 171:1–18, 2007.

[34] H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. *Artif. Intel.*, 52:263–294, 1991.

[35] G. Kern-Isberner. A thorough axiomatization of a principle of conditional preservation in belief revision. *Ann. Math. Artif. Intel.*, 40(1-2):127–164, 2004.

[36] S. Konieczny, J. Lang, and P. Marquis. DA^2 merging operators. *Artif. Intel.*, 157:49–79, 2004.

[37] S. Konieczny and R. P. Pérez. Improvement operators. In *Procs. of KR*, pages 177–186, 2008.

[38] I. Levi. Iteration of conditionals and the Ramsey test. *Synthese*, 76:49–81, 1988.

[39] P. Liberatore and M. Schaerf. Arbitration (or how to merge knowledge bases). *IEEE Trans. Knowledge Data Engineering*, 10(1):76–90, 1998.

[40] S. Lindström and W. Rabinowicz. On the probabilistic representation of non-probabilistic belief revision. *Journal of Philosophical Logic*, 18:69–101, 1989.

[41] J. Ma, S. Benferhat, and W. Liu. Revising partial pre-orders with partial pre-orders: A unit-based revision framework. In *Proceedings of 13th International Conference on Principles of Knowledge Representation and Reasoning (KR2012)*, pages 633–637, 2012.

[42] J. Ma and W. Liu. Modeling belief change on epistemic states. In *Procs. of FLAIRS*, pages 553–558, 2009.

[43] J. Ma and W. Liu. A framework for managing uncertain inputs: An axiomization of rewarding. *International Journal of Approximate Reasoning*, 52(7):917–934, 2011.

[44] J. Ma, W. Liu, and S. Benferhat. A belief revision framework for revising epistemic states with partial epistemic states. In *Procs. of AAAI*, pages 333–338, 2010.

[45] J. Ma, W. Liu, and S. Benferhat. Revision over partial pre-orders: a postulational study. In *Proceedings of international conference on Scalable Uncertainty Management (SUM)*, pages 219–232, 2012.

[46] J. Ma, W. Liu, S. Dubois, and H. Prade. Revision rules in the theory of evidence. In *Proceedings of the 22nd International Conference on Tools of Artificial Intelligence (ICTAI'10)*, pages 295–302, 2010.

[47] J. Ma, W. Liu, S. Dubois, and H. Prade. Bridging Jeffery's rule, agm revision, and dempster conditioning in the theory of evidence. *International Journal of Artificial Intelligence Tools*, 20(4):691–720, 2011.

[48] J. Ma, W. Liu, and P. Miller. Belief change with noisy sensing in the situation calculus. In *Proceedings of the 27th Conference on Uncertainty in Artificial Intelligence (UAI'2011)*, pages 471–478, 2011.

[49] T. Meyer. Merging epistemic states. In *PRICAI'00*, pages 286–296, 2000.

[50] A. Nayak. Iterated belief change based on epistemic entrenchment. *Erkenntnis*, 41:353–390, 1994.

[51] A. C. Nayak, M. Pagnucco, and P. Peppas. Dynamic belief revision operators. *Artif. Intel.*, 146:193–228, 2003.

[52] M. Oussalah. Data fusion: a study of adaptative combinations with incomplete certainty qualification part ii: local versus global certainty qualification. *Kybernetes*, 35(10):1607–1627, 2006.

[53] J. Pollock and A. Gillies. Belief revision and epistemology. *Synthese*, 122:69–92, 2000.

[54] F. Ramsey. *Truth and probabilities*. Harcourt Brace, New York, 1931.

[55] F. P. Ramsey. *Foundations of Mathematics and Other Logical Essays, 2nd ed.* Routledge and Kegan Paul, London, 1950.

[56] H. Rott. Bounded revision: Two-dimensional belief change between conservative and moderate revision. In T. Rønnow-Rasmussen, B. Petersson, J. Josefsson, and D. Egonsson, editors, *Hommage à Wlodek. Philosophical Papers Dedicated to Wlodek Rabinowicz.* http://www.fil.lu.se/hommageawlodek/site/papper/RottHans.pdf, 2007.

[57] H. Rott. Two-dimensional belief change, an advertisement. In *drops.dagstuhl.de/opus/volltexte/2007/1240/pdf/07351.RottHans.Paper.1240.pdf*, 2007.

[58] W. Spohn. Ordinal conditional functions: A dynamic theory of epistemic states. *Causation in Decision, Belief Change, and Statistics*, 2:105–134, 1988.

[59] M. A. Williams. Transmutations of knowledge systems. In *KR'94*, pages 619–629, 1994.

Appendix

Proof of Proposition 1: From **ER2**, we have $\Phi \circ \Phi \models \Phi$. From **ER1**, we have $\Phi \circ \Phi$ is a full epistemic state. From Definition 7, it is easy to see that the only full epistemic state that entails Φ is Φ itself. Hence we have $\Phi \circ \Phi = \Phi$.

Proof of Proposition 2: By Proposition 1, we know $\Phi \circ \Phi = \Phi$. Therefore, by setting Φ and Φ' to Φ, $\Psi^{\mathscr{F}}$ to $\Psi^{\mathscr{F}}$, $\Psi'^{\mathscr{F}'}$ to Φ, μ and μ' to μ, and ψ and ψ' to ϕ in **ER3**, we immediately get $(\Phi \circ \Psi^{\mathscr{F}})(\phi) = (\Phi \circ \Phi)(\phi) = \Phi(\phi)$.

Proof of Theorem 3: The proof is similar to the one in [34] and [14]. ($\Rightarrow$) Suppose there is a revision operator $\circ$ satisfying postulates **ER1-ER4**. $\forall \mu \in \mathscr{F}, (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ is ensured by **ER1**.

For any $x, y \in D$ s.t., $x \preceq_D y$, as D is strict, we have $\exists \Phi, A, B$ s.t., $A \subseteq B$, $\Phi(A) = x$ and $\Phi(B) = y$. Let $Mod(A) = \phi_1$ and $Mod(B) = \phi_2$ (hence $\phi_1 \vdash \phi_2$), we rewrite $\Phi(\phi_1) = x$ and $\Phi(\phi_2) = y$, then we define $x \ominus y \overset{def}{=} (\Phi \circ \Gamma^{\phi_2})(\phi_1)$ (see the definition of Γ in Section 7.1). Later we will show that this definition is independent of Φ, ϕ_1, ϕ_2.

As Γ^{ϕ_2} is a partial epistemic state (since ϕ_2 is satisfiable), from **ER2**, $\Phi \circ \Gamma^{\phi_2}$ is a full epistemic state, so our definition holds.

Now first we prove that for any $\Psi^{\mathscr{F}}$ s.t. $\phi_2 \in \mathscr{F}$, we have $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Phi \circ \Psi^{\mathscr{F}} \circ \Gamma^{\phi_2})(\phi_1)$.

From the *Minimality* property, we get that Γ^{ϕ_2} is equal to an epistemic state $\Theta^{\mathscr{F}}$ such that $\Theta^{\mathscr{F}}(\phi_2) = \top$ and $\forall \psi \in \mathscr{F}, \psi \neq \phi_2, \Theta^{\mathscr{F}}(\psi) = \bot$. Hence by **ER4**, we immediately know that $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Phi \circ \Psi^{\mathscr{F}} \circ \Gamma^{\phi_2})(\phi_1)$ does hold.

Then we prove that $\ominus$ is independent of Φ, ϕ_1 and ϕ_2. It suffices to prove that $\forall \Phi, \Psi, \forall \phi_1 \vdash \phi_2, \psi_1 \vdash \psi_2$, if $\Phi(\phi_1) = \Psi(\psi_1)$ and $\Phi(\phi_2) = \Psi(\psi_2)$, then $\Phi(\phi_1) \ominus \Phi(\phi_2) = \Psi(\psi_1) \ominus \Psi(\psi_2)$, or $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Psi \circ \Gamma^{\psi_2})(\psi_1)$.

From **ER1**, $(\Phi \circ \Gamma^{\phi_2})(\phi_2) = \top = (\Psi \circ \Gamma^{\psi_2})(\psi_2)$, then by setting Φ to Φ, Φ' to Ψ, $\Psi^{\mathscr{F}}$ to Γ^{ϕ_2}, $\Psi'^{\mathscr{F}'}$ to Γ^{ψ_2}, μ to ϕ_2, μ' to ψ_2, ψ to ϕ_1 and ψ' to ψ_1 in **ER3**, we immediately know that $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Psi \circ \Gamma^{\psi_2})(\psi_1)$. Thus $\Phi(\phi_1) \ominus \Phi(\phi_2) = \Psi(\psi_1) \ominus \Psi(\psi_2)$ which implies $\ominus$ is independent of Φ, A, B.

Now we prove that $\ominus$ is retentive. Suppose Φ and Ψ are two epistemic states, and $\phi_1, \phi_2, \psi_1, \psi_2$ are propositions s.t. $\phi_1 \vdash \phi_2, \psi_1 \vdash \psi_2$. Then we need to show if $\Phi(\phi_2) = \Psi(\psi_2)$ and $\Phi(\phi_1) \ominus \Phi(\phi_2) = \Psi(\psi_1) \ominus \Psi(\psi_2)$ (i.e., $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Psi \circ \Gamma^{\psi_2})(\psi_1)$), it should be $\Phi(\phi_1) = \Psi(\psi_1)$.

As $(\Phi \circ \Gamma^{\phi_2})(\phi_2) = (\Psi \circ \Gamma^{\psi_2})(\psi_2) = \top$, by setting Φ to Φ, Φ' to Ψ, $\Psi^{\mathscr{F}}$ to Γ^{ϕ_2}, $\Psi'^{\mathscr{F}'}$ to Γ^{ψ_2}, μ to ϕ_2, μ' to ψ_2, ψ to ϕ_1 and ψ' to ψ_1 in **ER3**, we really get $\Phi(\phi_1) = \Psi(\psi_1)$.

Finally we show that $\forall \phi \vdash \mu, (\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$.

In fact, we have $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = (\Phi \circ \Psi^{\mathscr{F}} \circ \Gamma^{\mu})(\phi)$ and $\Phi(\phi) \ominus \Phi(\mu) = (\Phi \circ \Gamma^{\mu})(\phi)$, and notice that we have already proved $\Phi \circ \Psi^{\mathscr{F}} \circ \Gamma^{\mu} = \Phi \circ \Gamma^{\mu}$ in the above.

($\Leftarrow$) Suppose there is a retentive operator $\ominus$ defined for any $x \preceq_D y$ on D which satisfies $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$ for $\phi \vdash \mu \in \mathscr{F}$.

(ER1) $\Phi \circ \Psi^{\mathscr{F}} \models \Psi^{\mathscr{F}}$.

Follows immediately from $\forall \mu \in \mathscr{F}, (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Psi^{\mathscr{F}}(\mu)$ and the ex-

tension function f.

(ER2) If Φ is a full epistemic state and $\Psi^{\mathscr{F}}$ is a partial epistemic state, then $\Phi \circ \Psi^{\mathscr{F}}$ is a full epistemic state.

Let $\phi_j \in \mu$ be a formula such that $Mod(\phi_j) = \{w_j\}$, then from $(\Phi \circ \Psi^{\mathscr{F}})(\phi_j) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi_j) \ominus \Phi(\mu)$ and $\ominus$ is retentive, we have $(\Phi \circ \Psi^{\mathscr{F}})(\phi_j)$ is well defined. Thus $\Phi \circ \Psi^{\mathscr{F}}$ is a full epistemic state.

(ER3) For any $\mu \in \mathscr{F}$, and $\mu' \in \mathscr{F}'$, if $\Phi(\mu) = \Phi'(\mu')$ and $\Psi^{\mathscr{F}}(\mu) = \Psi'^{\mathscr{F}'}(\mu')$, then for $\psi \vdash \mu$ and $\psi' \vdash \mu'$, $(\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi')$ iff $\Phi(\psi) = \Phi'(\psi')$.

From **ER1** and $\Psi^{\mathscr{F}}(\mu) = \Psi'^{\mathscr{F}'}(\mu')$, we know that $(\Phi \circ \Psi^{\mathscr{F}})(\mu) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\mu')$. Then we have:

1. If $(\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi')$, then we get

$$
\begin{aligned}
\Phi(\psi) \ominus \Phi(\mu) &= (\Phi \circ \Psi^{\mathscr{F}})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) \\
&= (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi') \ominus (\Phi' \circ \Psi'^{\mathscr{F}'})(\mu') \\
&= \Phi'(\psi') \ominus \Phi'(\mu').
\end{aligned}
$$

Now as $\ominus$ is retentive and $\Phi(\mu) = \Phi'(\mu')$, we obtain $\Phi(\psi) = \Phi'(\psi')$.

2. If $\Phi(\phi) = \Phi'(\phi')$, then similarly, it is easy to show that $(\Phi \circ \Psi^{\mathscr{F}})(\psi) = (\Phi' \circ \Psi'^{\mathscr{F}'})(\psi')$.

(ER4) $\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}} = \Phi \circ \Theta^{\mathscr{F}}$.

First we prove that $\forall \phi \vdash \mu \in \mathscr{F}$, $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\phi) = (\Phi \circ \Theta^{\mathscr{F}})(\phi)$. From **ER1**, we have $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\mu) = \Theta^{\mathscr{F}}(\mu) = (\Phi \circ \Theta^{\mathscr{F}})(\mu)$, hence as $\ominus$ is retentive, it is equal to prove that $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\mu) = (\Phi \circ \Theta^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Theta^{\mathscr{F}})(\mu)$.

As $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\mu) = (\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$ and $(\Phi \circ \Theta^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Theta^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$, we easily get that they are equivalent.

For an arbitrary ϕ, without loss of generality, suppose $\mathscr{F} = \{\mu_1, \cdots, \mu_n\}$, we have $\phi = \phi \wedge (\mu_1 \vee \cdots \vee \mu_n) = (\phi \wedge \mu_1) \vee \cdots (\phi \wedge \mu_n)$, as $\forall i, 1 \leq i \leq n$, $\phi \wedge \mu_i \vdash \mu_i$, we know that $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\phi \wedge \mu_i) = (\Phi \circ \Theta^{\mathscr{F}})(\phi \wedge \mu_i)$, thus according to the extension function f, we get $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}})(\phi) = (\Phi \circ \Theta^{\mathscr{F}})(\phi)$.

Proof of Theorem 4: $(\Rightarrow)$ As **ER4*** is a stronger version of **ER4**, from Theorem 3, we immediately get a retentive operator $\ominus$ as defined in the proof of Theorem 3. The remain problem is to show that $\ominus$ is conductive.

Suppose Φ and Ψ are two epistemic states, and $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3$ are propositions s.t. $\phi_1 \vdash \phi_2 \vdash \phi_3$, $\psi_1 \vdash \psi_2 \vdash \psi_3$. We need to show if $\Phi(\phi_1) \ominus \Phi(\phi_3) = \Psi(\psi_1) \ominus \Psi(\psi_3)$, and $\Phi(\phi_2) \ominus \Phi(\phi_3) = \Psi(\psi_2) \ominus \Psi(\psi_3)$, then $\Phi(\phi_1) \ominus \Phi(\phi_2) = \Psi(\psi_1) \ominus \Psi(\psi_2)$.

From the definition of $\ominus$ in the proof of Theorem 3, we just need to prove that if $(\Phi \circ \Gamma^{\phi_3})(\phi_1) = (\Psi \circ \Gamma^{\psi_3})(\psi_1)$, and $(\Phi \circ \Gamma^{\phi_3})(\phi_2) = (\Psi \circ \Gamma^{\psi_3})(\psi_2)$, then $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Psi \circ \Gamma^{\psi_2})(\psi_1)$.

From the *Minimality* property, we get $\Gamma^{\phi_2} = \Theta^{\mathscr{F}'}$ such that $\mathscr{F}' = \{\phi_2, \phi_3 \wedge \neg\phi_2, \neg\phi_3\}$ and $\Theta^{\mathscr{F}'}(\phi_2) = \top$, $\Theta^{\mathscr{F}'}(\phi_3 \wedge \neg\phi_2) = \bot$ and $\Theta^{\mathscr{F}'}(\neg\phi_3) = \bot$. While for Γ^{ϕ_3}, the corresponding $\mathscr{F}_{\Gamma^{\phi_3}} = \{\phi_3, \neg\phi_3\}$, hence it is easy to check that $\mathscr{F}'$ is a

refinement of $\mathscr{F}_{\phi_3}$, therefore, from **ER4***, we have $(\Phi \circ \Gamma^{\phi_3} \circ \Gamma^{\phi_2}) = (\Phi \circ \Gamma^{\phi_2})$. And similarly $(\Psi \circ \Gamma^{\psi_3} \circ \Gamma^{\psi_2}) = (\Psi \circ \Gamma^{\psi_2})$.

Thus as $(\Phi \circ \Gamma^{\phi_3})(\phi_1) = (\Psi \circ \Gamma^{\psi_3})(\psi_1)$, and $(\Phi \circ \Gamma^{\phi_3})(\phi_2) = (\Psi \circ \Gamma^{\psi_3})(\psi_2)$, and $(\Phi \circ \Gamma^{\phi_3} \circ \Gamma^{\phi_2})(\phi_2) = \top = (\Psi \circ \Gamma^{\psi_3} \circ \Gamma^{\psi_2})(\psi_2)$, by setting Φ to $\Phi \circ \Gamma^{\phi_3}$, Φ' to $\Psi \circ \Gamma^{\psi_3}$, $\Psi^{\mathscr{F}}$ to Γ^{ϕ_2}, $\Psi'^{\mathscr{F}'}$ to Γ^{ψ_2}, μ to ϕ_2, μ' to ψ_2, ψ to ϕ_1 and ψ' to ψ_1 in **ER3**, we get $(\Phi \circ \Gamma^{\phi_3} \circ \Gamma^{\phi_2})(\phi_1) = (\Psi \circ \Gamma^{\psi_3} \circ \Gamma^{\psi_2})(\psi_1)$. Therefore from $(\Phi \circ \Gamma^{\phi_3} \circ \Gamma^{\phi_2}) = (\Phi \circ \Gamma^{\phi_2})$ and $(\Psi \circ \Gamma^{\psi_3} \circ \Gamma^{\psi_2}) = (\Psi \circ \Gamma^{\psi_2})$, we have $(\Phi \circ \Gamma^{\phi_2})(\phi_1) = (\Psi \circ \Gamma^{\psi_2})(\psi_1)$.

($\Leftarrow$) The proofs of **ER1-ER3** are already done in the proof of Theorem 3. We just need to show **ER4*** holds.

(ER4*) $\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'} = \Phi \circ \Theta^{\mathscr{F}'}$ where $\mathscr{F}'$ is a refinement of $\mathscr{F}$.

We still only need to show that $\forall \phi \in \mathscr{F}'$ and any $\psi \vdash \phi$, $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'})(\psi) = (\Phi \circ \Theta^{\mathscr{F}'})(\psi)$, for arbitrary ψ, the same proof can be found in the proof of Theorem 3 for **ER4** using equivalent instantiation.

From **ER1**, for any $\phi \in \mathscr{F}'$, $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'})(\phi) = \Theta^{\mathscr{F}'}(\phi) = (\Phi \circ \Theta^{\mathscr{F}'})(\phi)$, then as $\ominus$ is retentive, it suffices to show $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'})(\phi) = (\Phi \circ \Theta^{\mathscr{F}'})(\psi) \ominus (\Phi \circ \Theta^{\mathscr{F}'})(\phi)$. As $(\Phi \circ \Theta^{\mathscr{F}'})(\psi) \ominus (\Phi \circ \Theta^{\mathscr{F}'})(\phi) = \Phi(\psi) \ominus \Phi(\phi)$ and $(\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}} \circ \Theta^{\mathscr{F}'})(\phi) = (\Phi \circ \Psi^{\mathscr{F}})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\phi)$. It is equal to show $(\Phi \circ \Psi^{\mathscr{F}})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\phi) = \Phi(\psi) \ominus \Phi(\phi)$. From the definition of refinement in Def. 1, let $\phi \vdash \mu \in \mathscr{F}$, then we have $\psi \vdash \mu$ and it leads to $(\Phi \circ \Psi^{\mathscr{F}})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\psi) \ominus \Phi(\mu)$. And $\phi \vdash \mu$ leads to $(\Phi \circ \Psi^{\mathscr{F}})(\phi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\mu) = \Phi(\phi) \ominus \Phi(\mu)$. Therefore as $\ominus$ is conductive, we have $(\Phi \circ \Psi^{\mathscr{F}})(\psi) \ominus (\Phi \circ \Psi^{\mathscr{F}})(\phi) = \Phi(\psi) \ominus \Phi(\phi)$ which completes the proof.

Proof of Theorem 5: According to Theorem 3 and Theorem 4, we only need to prove that there exists a retentive and conductive operator $\ominus$ such that $P(w) \ominus P(\mu_i) = (P \bullet P^{\mathscr{F}})(w) \ominus (P \bullet P^{\mathscr{F}})(\mu_i)$ if $w \models \mu_i$. Let $\ominus = /$, the proof is straightforward.

Proof of Theorem 6: Similar to the proof of Theorem 5, except that $\ominus = -$.

Proof of Theorem 7: Similar to the proof of Theorem 5 and omitted.

Proof of Proposition 3: As $\Phi^{\mathscr{F}_1} \models \Psi^{\mathscr{F}_2}$, we have for any $\mu \in \mathscr{F}_2$, s.t., $\Psi^{\mathscr{F}_2}(\mu) = \top$, it should be $\Phi^{\mathscr{F}_1}(\mu) = \top$, hence from the Maximality property, we get $\exists \phi \in \mathscr{F}_1, \phi \vdash \mu$, s.t., $\Phi^{\mathscr{F}_1}(\phi) = \top$. Conversely, for any $\phi \in \mathscr{F}_1$, $\Phi^{\mathscr{F}_1}(\phi) = \top$, as $\mathscr{F}_1$ is a refinement of $\mathscr{F}_2$, we have $\exists \mu \in \mathscr{F}_2$, s.t., $\phi \vdash \mu$ (hence $\Phi^{\mathscr{F}_1}(\mu) = \top$) and $\Psi^{\mathscr{F}_2}(\mu) = \Phi^{\mathscr{F}_1}(\mu) = \top$. Therefore, let $Bel(\Psi^{\mathscr{F}_2}) = \bigvee_{i=1}^{n} \mu_i$, $Bel(\Phi^{\mathscr{F}_1})$ should be in the form of $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k_i} \phi_{ij}$ such that $\phi_{ij} \vdash \mu_i$, hence we have $Bel(\Phi^{\mathscr{F}_1}) \vdash Bel(\Psi^{\mathscr{F}_2})$.

Proof of Proposition 4: As $\Phi(w_1) = \top$, we get $\Phi(form(w_1, w_2)) = \top$ by the *Maximality* property. Hence by Proposition 2, we get $(\Phi \circ \Gamma^{form(w_1,w_2)})(w_1) = \top$, therefore $w_1 \leq_\Phi w_2$.

Proof of Proposition 5: We prove that $\leq_\Phi$ is total, reflexive and transitive. This proof is to some extent similar to the one in [34] and [14].

1. *total*: $\forall w_1, w_2$, as $\Gamma^{form(w_1,w_2)}$ is a partial epistemic state, from **ER2**, $\Phi \circ \Gamma^{form(w_1,w_2)}$ is a full epistemic state. Then from **ER1**, $(\Phi \circ \Gamma^{form(w_1,w_2)})(form(w_1, w_2)) = \top$. So from the *Maximality* property, it should be $(\Phi \circ \Gamma^{form(w_1,w_2)})(w_1) = \top$ or $(\Phi \circ \Gamma^{form(w_1,w_2)})(w_2) = \top$. Therefore, from Proposition 4, we have either $w_1 \leq_\Phi w_2$ or $w_2 \leq_\Phi w_1$ which means $\leq_\Phi$ is total.

2. *Reflexive*: $\forall w$, by **ER1** and **ER2**, it is easy to get $(\Phi \circ \Gamma^{form(w)})(w) = \top$. Therefore, from Proposition 4, we have $w \leq_\Phi w$ and $\leq_\Phi$ is reflexive.

3. *Transitive*: Suppose $w_1 \leq_\Phi w_2$ and $w_2 \leq_\Phi w_3$. We need to show $w_1 \leq_\Phi w_3$.

   We consider the following two cases.

   (a) $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_3) = \top$.
   As $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_3) = \top$, from the *Maximality* property, we get
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(form(w_2,w_3)) = \top = \Gamma^{form(w_2,w_3)}(form(w_2,w_3))$,
   then from Proposition 2, we get
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_2,w_3)})(w_3)$
   $= (\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_3) = \top$.
   From the *Minimality* property, the partition for $\Gamma^{form(w_2,w_3)}$ can be seen as a refinement of the one for $\Gamma^{form(w_1,w_2,w_3)}$, thus from **ER4***, we know that $\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_2,w_3)} = \Phi \circ \Gamma^{form(w_2,w_3)}$. Therefore, we get
   $(\Phi \circ \Gamma^{form(w_2,w_3)})(w_3) = \top$.
   As $w_2 \leq_\Phi w_3$, we have $(\Phi \circ \Gamma^{form(w_2,w_3)})(w_2) = \top$. From **ER4***, we get
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_2,w_3)})(w_2)$
   $= (\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_2) = \top$.
   By setting Φ and Φ' to Φ, μ and μ' to $form(w_1,w_2)$, $\Psi^{\mathscr{F}}$ and $\Psi'^{\mathscr{F}'}$ to $\Gamma^{form(w_1,w_2,w_3)}$ and ψ to w_2, ψ' to w_3 in **ER3**, we get
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_2) = (\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_3) = \top$.
   A similar process on w_1 and w_2 also induces that
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_1) = \top$.
   By setting Φ and Φ' to $\Phi \circ \Gamma^{form(w_1,w_2,w_3)}$, μ and μ' to $form(w_1,w_3)$, $\Psi^{\mathscr{F}}$ and $\Psi'^{\mathscr{F}'}$ to $\Gamma^{form(w_1,w_3)}$ and ψ to w_1, ψ' to w_3 in **ER3**, we get
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_1,w_3)})(w_1)$
   $= (\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_1,w_2,w_3)})(w_3)$.
   From **ER4***, we hence obtain
   $(\Phi \circ \Gamma^{form(w_1,w_3)})(w_1) = (\Phi \circ \Gamma^{form(w_1,w_3)})(w_3)$.
   However, from **ER1**, it should be
   $(\Phi \circ \Gamma^{form(w_1,w_3)})(form(w_1,w_3)) = \top$.
   Therefore from the *Maximality* property, we should have
   $(\Phi \circ \Gamma^{form(w_1,w_3)})(w_1) = (\Phi \circ \Gamma^{form(w_1,w_3)})(w_3) = \top$.
   Therefore $w_1 \leq_\Phi w_3$ does hold.

   (b) $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_3) \neq \top$.
   From $w_1 \leq_\Phi w_2$, we have $(\Phi \circ \Gamma^{form(w_1,w_2)})(w_1) = \top$. Since
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(form(w_1,w_2,w_3)) = \top$
   and
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_3) \neq \top$,
   from the *Maximality* property, we can easily get
   $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(form(w_1,w_2)) = \top$, then from Proposition 2, we

get $(\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_1,w_2)})(w_1) = (\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_1)$.
As **ER4*** gives $\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_1,w_2)} = \Phi \circ \Gamma^{form(w_1,w_2)}$,
we get

$(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_1) = (\Phi \circ \Gamma^{form(w_1,w_2)})(w_1) = \top.$

Then from the *Maximality* property, we have

$(\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(form(w_1,w_3)) = \top.$

From Proposition 2, we have

$(\Phi \circ \Gamma^{form(w_1,w_2,w_3)} \circ \Gamma^{form(w_1,w_3)})(w_1)$
$= (\Phi \circ \Gamma^{form(w_1,w_2,w_3)})(w_1) = \top.$

Therefore from **ER4***, we get $(\Phi \circ \Gamma^{form(w_1,w_3)})(w_1) = \top$. It shows that $w_1 \leq_\Phi w_3$ which completes the proof.

Proof of Proposition 6:

1. $\Phi(w_1) = \Phi(w_2) = \top$ only if $w_1 =_\Phi w_2$.
   Follows immediately from Proposition 4.

2. $\Phi(w_1) = \top$ and $\Phi(w_2) \neq \top$ only if $w_1 <_\Phi w_2$.
   From Proposition 4 and $\Phi(w_1) = \top$, we know $w_1 \leq_\Phi w_2$.

   From the *Maximality* property, we get $\Phi(form(w_1,w_2)) = \top$. Then from Proposition 2, we obtain $(\Phi \circ \Gamma^{form(w_1,w_2)})(w_2) = \Phi(w_2) \neq \top$. So $w_2 \nleq_\Phi w_1$. Therefore $w_1 <_\Phi w_2$.

3. $\Phi \simeq \Psi$ only if $\leq_\Phi = \leq_\Psi$.
   Follows immediately from the definition of $\leq_\Phi$ and $\leq_\Psi$ and **ER3**.

Proof of Theorem 8:

1. $Bel(\Phi \circ \Delta^\mu) \subseteq min(Mods(\mu), \leq_\Phi)$.
   Suppose $(\Phi \circ \Delta^\mu)(w) = \top$, from **ER1**, we know that $(\Phi \circ \Delta^\mu)(\mu) = \top$ and $(\Phi \circ \Delta^\mu)(\neg\mu) \prec_D \top$, thus from the *Maximality* property, it should be $w \models \mu$. For $\forall w' \models \mu$, from the *Maximality* property, we get $(\Phi \circ \Delta^\mu)(form(w,w')) = \top = \Gamma^{form(w,w')}(form(w,w'))$. Hence from Proposition 2, we obtain $(\Phi \circ \Delta^\mu \circ \Gamma^{form(w,w')})(w) = (\Phi \circ \Delta^\mu)(w) = \top$. But from the *Minimality* property and $form(w,w') \vdash \mu$, we know that the partition for $\Gamma^{form(w,w')}$ can be seen as a refinement of the one for Δ^μ, thus from **ER4***, we get $\Phi \circ \Delta^\mu \circ \Gamma^{form(w,w')} = \Phi \circ \Gamma^{form(w,w')}$. So finally we get $(\Phi \circ \Gamma^{form(w,w')})(w) = \top$, thus from the definition of $\leq_\Phi$, we get $w \leq_\Phi w'$. Therefore $w \in min(Mods(\mu), \leq_\Phi)$.

2. $min(Mods(\mu), \leq_\Phi) \subseteq Bel(\Phi \circ \Delta^\mu)$.
   Suppose $w \in min(Mods(\mu), \leq_\Phi)$, and $(\Phi \circ \Delta^\mu)(w) \prec_D \top$. We will prove a contradiction. From **ER1**, $(\Phi \circ \Delta^\mu)(\mu) = \top$, then from the *Maximality* property, there exists a w' such that $w' \models \mu$ and $(\Phi \circ \Delta^\mu)(w') = \top$. From the *Maximality* property, we get $(\Phi \circ \Delta^\mu)(form(w,w')) = \top$. Then from **ER4*** and Proposition 2, we obtain $(\Phi \circ \Gamma^{form(w,w')})(w') = (\Phi \circ \Delta^\mu \circ \Gamma^{form(w,w')})(w') = (\Phi \circ \Delta^\mu)(w') = \top$, and $(\Phi \circ \Gamma^{form(w,w')})(w) = (\Phi \circ \Delta^\mu \circ \Gamma^{form(w,w')})(w)$
   $= (\Phi \circ \Delta^\mu)(w) \prec_D \top,$

   thus $w' <_\Phi w$ which contradicts to $w \in min(Mods(\mu), \leq_\Phi)$.

C1* If $\alpha \vdash \mu$, then $Bel(\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}) = Bel(\Phi \circ \Delta^{\alpha})$.

We need to show that for any w, if $\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}(w) = \top$, then $\Phi \circ \Delta^{\alpha}(w) = \top$, and vice versa.

First, assume $\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}(w) = \top$, as $\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha} \models \Delta^{\alpha}$, we get $\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}(\alpha) = \top$ and $\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}(\neg\alpha) \neq \top$, then we have $w \models \alpha \vdash \mu$. Hence for any $w' \models \alpha \vdash \mu$, we have $w \leq_{\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}} w'$. From CR1*, we have $w \leq_{\Phi \circ \Delta^{\mu}} w'$, and again from CR1*, we have $w \leq_{\Phi} w'$. Finally, from CR1* again, we get $w \leq_{\Phi \circ \Delta^{\alpha}} w'$, as $\Phi \circ \Delta^{\alpha}(\alpha) = \top$, we should have $\Phi \circ \Delta^{\alpha}(w) = \top$. Conversely, if $\Phi \circ \Delta^{\alpha}(w) = \top$, we can similarly prove that $\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}(w) = \top$.

C2* If $\alpha \vdash \neg\mu$, then $Bel(\Phi \circ \Delta^{\mu} \circ \Delta^{\alpha}) = Bel(\Phi \circ \Delta^{\alpha})$.

The proof is similar to the above except that we should refer to CR2*.

Proof of Proposition 7: It is straightforward and omitted.

Proof of Proposition 8: From REE*1, REE*3 and REE*It, it is straightforward to see that $\circ$ must follow ER1, ER2 and ER4*, respectively.