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# IDEALS OF $A(G)$ AND BIMODULES OVER MAXIMAL ABELIAN SELFADJOINT ALGEBRAS

M. ANOUSSIS, A. KATAVOLOS AND I. G. TODOROV

*To the memory of Bill Arveson, with gratitude*

ABSTRACT. This paper is concerned with weak\* closed masa-bimodules generated by  $A(G)$ -invariant subspaces of  $\text{VN}(G)$ . An annihilator formula is established, which is used to characterise the weak\* closed subspaces of  $\mathcal{B}(L^2(G))$  which are invariant under both Schur multipliers and a canonical action of  $M(G)$  on  $\mathcal{B}(L^2(G))$  via completely bounded maps. We study the special cases of extremal ideals with a given null set and, for a large class of groups, we establish a link between relative spectral synthesis and relative operator synthesis.

## 1. INTRODUCTION

Let  $G$  be a locally compact group. The algebra  $M^{\text{cb}}A(G)$  of completely bounded multipliers of the Fourier algebra  $A(G)$ , introduced in [4], has played a pivotal role in both Harmonic Analysis and Operator Algebra Theory. It was shown by J. E. Gilbert and by M. Bożejko and G. Fendler in [3] (see also [16] and [30]) that the map  $N$  which sends a function  $f : G \rightarrow \mathbb{C}$  to the function  $Nf : G \times G \rightarrow \mathbb{C}$  given by  $Nf(s, t) = f(ts^{-1})$ , carries  $M^{\text{cb}}A(G)$  isometrically into the algebra of Schur multipliers  $\mathfrak{S}(G)$  on  $G \times G$ . This result has led to fruitful interaction between the two areas, see *e.g.* [25], [22] and [30].

The weak\* closed subspaces of the von Neumann algebra  $\text{VN}(G)$  that are invariant under  $A(G)$  are precisely the annihilators of (closed) ideals  $J \subseteq A(G)$ . On the other hand, the weak\* closed subspaces of the space  $\mathcal{B}(L^2(G))$  of bounded operators on  $L^2(G)$  which are invariant under all Schur multipliers are precisely the (weak\* closed) masa-bimodules in  $\mathcal{B}(L^2(G))$ , that is, invariant under the map  $T \rightarrow M_f T M_g$  where  $f, g \in L^\infty(G)$  (or, under left and right composition with multiplication operators from  $L^\infty(G)$ ).

Thus, given a closed ideal  $J \subseteq A(G)$ , there are two natural ways to construct a weak\* closed masa-bimodule in  $\mathcal{B}(L^2(G))$ : (a) one may first consider the norm closed masa-bimodule  $\text{Sat}(J)$  of  $T(G)$  suitably generated by  $N(J)$  and then take the annihilator of  $\text{Sat}(J)$  in  $\mathcal{B}(L^2(G))$ , or (b) one may first take the annihilator  $J^\perp$  of  $J$  in  $\text{VN}(G)$  and then generate a weak\* closed masa-bimodule  $\text{Bim}(J^\perp)$ . One of our main results, Theorem 3.2,

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*Key words and phrases.* Fourier algebra, masa-bimodule, invariant subspaces.

is that these two operations have the same outcome; in other words, the diagram

$$\begin{array}{ccc} J & \xrightarrow{\perp} & J^\perp \\ \downarrow & & \downarrow \\ \text{Sat}(J) & \xrightarrow{\perp} & \text{Bim}(J^\perp) \end{array}$$

is commutative. The proof uses the techniques developed by J. Ludwig, N. Spronk and L. Turowska in [31] and [20]. Some of the results in Section 3 also appear in the aforementioned papers; we have chosen to present complete arguments in order to clarify some details.

Using this result, we present a unified approach to some problems of Harmonic Analysis on  $G$ . In particular, in Section 5, we look at the special cases where  $J$  is the minimal, or the maximal, ideal of  $A(G)$  with a given null set  $E \subseteq G$ . The main result here is Theorem 5.3; as a corollary, we obtain the result established in [20] that if  $A(G)$  possesses an approximate identity then a closed set  $E \subseteq G$  satisfies spectral synthesis if and only if the set  $E^* = \{(s, t) : ts^{-1} \in E\}$  satisfies operator synthesis.

The connection between spectral synthesis and operator synthesis was discovered by W. B. Arveson in [1]. The above result is due to J. Froelich [11] for  $G$  abelian and to N. Spronk and L. Turowska [31] for  $G$  compact. J. Ludwig and L. Turowska [20] show that a closed subset  $E$  of a locally compact group  $G$  satisfies *local* spectral synthesis if and only if  $E^*$  satisfies operator synthesis; local spectral synthesis coincides with spectral synthesis when  $A(G)$  has an approximate identity.

Spectral synthesis relative to a fixed  $A(G)$ -invariant subspace of  $\text{VN}(G)$  was introduced for locally compact groups by E. Kaniuth and A.T. Lau in [17]. In [26], the authors define relative operator synthesis for subsets of  $G \times G$ , where  $G$  is compact, and link it to relative spectral synthesis. In Section 6, using our results, and assuming that the  $A(G)$ -invariant subspace of  $\text{VN}(G)$  is weak\* closed we prove an analogous relation for locally compact groups for which  $A(G)$  possesses an approximate identity. We note that this class contains, but is larger than, the class of amenable groups.

As another application, we are able to identify the weak\* closed subspaces of  $\mathcal{B}(L^2(G))$  that are invariant under both Schur multiplication and an action of the measure algebra  $M(G)$ . More precisely, let  $\Gamma : M(G) \rightarrow \mathcal{B}(\mathcal{B}(L^2(G)))$  be the representation of  $M(G)$  given by

$$\Gamma(\mu)(T) = \int_G \rho_r T \rho_r^* d\mu(r), \quad T \in \mathcal{B}(L^2(G)).$$

This action was studied by F. Ghahramani, M. Neufang, Zh.-J. Ruan, R. Smith, N. Spronk and E. Størmer in [12], [21], [22], [29], [32], among others.

The maps  $\Gamma(\mu)$  are precisely those weak\* continuous completely bounded maps on  $\mathcal{B}(L^2(G))$  that are  $\text{VN}(G)$ -bimodule maps and leave the multiplication masa invariant [21], [22]. In Section 4, we show that the set  $\mathcal{L}$  of all weak\* closed subspaces of  $\mathcal{B}(L^2(G))$  that are invariant under both  $\mathfrak{S}(G)$

and  $\Gamma(M(G))$  consists precisely of the masa-bimodules of the form  $\text{Bim}(J^\perp)$ , where  $J \subseteq A(G)$  is a closed ideal; we also determine the lattice structure of  $\mathcal{L}$ .

In the presence of an approximate identity in  $A(G)$ , we show that the generating invariant subspace of a bimodule of the form  $\text{Bim}(\mathcal{X})$  can be recovered by taking the intersection with  $\text{VN}(G)$ . Thus, the map  $\mathcal{X} \rightarrow \text{Bim}(\mathcal{X})$  from the class of weak\* closed invariant subspaces of  $\text{VN}(G)$  to the class of weak\* closed masa bimodules of  $B(L^2(G))$  is in this case one-to-one.

## 2. PRELIMINARIES

If  $(X, m)$  is a  $\sigma$ -finite measure space, we write  $L^p(X)$  for  $L^p(X, m)$ . For  $\phi \in L^\infty(X)$ , let  $M_\phi$  be the operator on  $L^2(X)$  of multiplication by  $\phi$ . The collection  $\mathcal{D}_X = \{M_\phi : \phi \in L^\infty(X)\}$  is a maximal abelian selfadjoint algebra (masa, for short).

Let  $X$  and  $Y$  be standard Borel spaces (that is, Borel isomorphic to Borel subsets of complete separable metric spaces), equipped with  $\sigma$ -finite measures  $m$  and  $n$ . A subset  $E \subseteq X \times Y$  is called *marginally null* if  $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$ , where  $m(X_0) = n(Y_0) = 0$ ; we write  $E \simeq \emptyset$ . Two functions  $h_1, h_2 : X \times Y \rightarrow \mathbb{C}$  are said to be equal *marginally almost everywhere* (*m.a.e.*) or *marginally equivalent* if the set  $\{(x, y) : h_1(x, y) \neq h_2(x, y)\}$  is marginally null.

Let  $T(X, Y)$  be the projective tensor product  $L^2(X) \hat{\otimes} L^2(Y)$ . Every element  $h \in T(X, Y)$  is an absolutely convergent series

$$h = \sum_{i=1}^{\infty} f_i \otimes g_i, \quad f_i \in L^2(X), g_i \in L^2(Y), i \in \mathbb{N},$$

where  $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$  and  $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$ . Such an element  $h$  may be considered either as a function  $h : X \times Y \rightarrow \mathbb{C}$ , defined marginally almost everywhere and given by

$$h(x, y) = \sum_{i=1}^{\infty} f_i(x)g_i(y),$$

or as an element of the predual of the space  $\mathcal{B}(L^2(X), L^2(Y))$  of all bounded linear operators from  $L^2(X)$  into  $L^2(Y)$  via the pairing

$$\langle T, h \rangle_t := \sum_{i=1}^{\infty} (Tf_i, \bar{g}_i).$$

We denote by  $\|h\|_t$  the norm of  $h \in T(X, Y)$  and note that if  $\phi \in L^\infty(X)$  and  $\psi \in L^\infty(Y)$ , then the function  $(\phi \otimes \psi)h$  belongs to  $T(X, Y)$ ; thus,  $T(G)$  has a natural  $(L^\infty(X), L^\infty(Y))$ -module structure.

Let  $\mathfrak{S}(X, Y)$  be the multiplier algebra of  $T(X, Y)$ ; by definition, a measurable function  $w : X \times Y \rightarrow \mathbb{C}$  belongs to  $\mathfrak{S}(X, Y)$  if the map  $m_w : h \rightarrow wh$  leaves  $T(X, Y)$  invariant, that is, if  $wh$  is marginally equivalent to a function from  $T(X, Y)$ , for every  $h \in T(X, Y)$ . The elements of  $\mathfrak{S}(X, Y)$  are called

(measurable) Schur multipliers. The closed graph theorem can be used to show that  $m_w$  is automatically a bounded operator; hence it has a dual

$$S_w : \mathcal{B}(L^2(X), L^2(Y)) \rightarrow \mathcal{B}(L^2(X), L^2(Y)),$$

given by

$$\langle S_w(T), h \rangle_t = \langle T, wh \rangle_t, \quad h \in T(X, Y), \quad T \in \mathcal{B}(L^2(X), L^2(Y)).$$

If  $\phi \in L^\infty(X)$  and  $\psi \in L^\infty(Y)$ , one finds that  $S_{\phi \otimes \psi}(T) = M_\psi T M_\phi$ ,  $T \in \mathcal{B}(L^2(X), L^2(Y))$ . It follows that if  $k \in L^2(Y \times X)$ ,  $T_k$  is the Hilbert-Schmidt operator from  $L^2(X)$  into  $L^2(Y)$  given by  $(T_k f)(y) = \int_X k(y, x) f(x) dm(x)$  ( $f \in L^2(X)$ ) and  $w \in \mathfrak{S}(X, Y)$ , then  $S_w(T_k) = T_{w^\flat k}$  where  $w^\flat : Y \times X \rightarrow \mathbb{C}$  is the function  $w^\flat(x, y) = w(y, x)$ .

It can be shown ([23], see also [13, 18] and [30]) that  $w \in \mathfrak{S}(X, Y)$  if and only if  $w$  can be represented in the form

$$w(x, y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{for almost all } (x, y) \in X \times Y,$$

where  $(a_k)_{k \in \mathbb{N}} \subseteq L^\infty(X)$  and  $(b_k)_{k \in \mathbb{N}} \subseteq L^\infty(Y)$  are sequences of functions with  $\text{esssup}_{x \in X} \sum_{k=1}^{\infty} |a_k(x)|^2 < \infty$  and  $\text{esssup}_{y \in Y} \sum_{k=1}^{\infty} |b_k(y)|^2 < \infty$ . In this case,

$$S_w(T) = \sum_{k=1}^{\infty} M_{b_k} T M_{a_k}, \quad T \in \mathcal{B}(L^2(X), L^2(Y)),$$

where, for every  $T \in \mathcal{B}(L^2(X), L^2(Y))$ , the series converges in the weak\* topology. We write  $w = \sum_{k=1}^{\infty} a_k \otimes b_k$  as a formal series. Moreover, the norm of  $S_w$  as an operator on  $\mathcal{B}(L^2(X), L^2(Y))$  (which of course also equals the norm of the operator  $m_w$  on  $T(X, Y)$ ) is given by

$$\|S_w\| = \inf \left\{ \left\| \sum_{k=1}^{\infty} |a_k|^2 \right\|_{\infty}^{1/2} \left\| \sum_{k=1}^{\infty} |b_k|^2 \right\|_{\infty}^{1/2} : \text{all rep's } w = \sum_{k=1}^{\infty} a_k \otimes b_k \right\}$$

We denote this quantity by  $\|w\|_{\mathfrak{S}}$ . Note that if  $w = \sum_{k=1}^{\infty} a_k \otimes b_k \in \mathfrak{S}(X, Y)$  then  $w^\flat = \sum_{k=1}^{\infty} b_k \otimes a_k \in \mathfrak{S}(Y, X)$  and  $\|w\|_{\mathfrak{S}} = \|w^\flat\|_{\mathfrak{S}}$ .

If  $w \in \mathfrak{S}(X, Y)$ , the operator  $S_w$  is a  $(\mathcal{D}_Y, \mathcal{D}_X)$ -module map, while the operator  $m_w$  is a  $(L^\infty(X), L^\infty(Y))$ -module map. In fact, a weak\* closed subspace  $\mathcal{U} \subseteq \mathcal{B}(L^2(X), L^2(Y))$  is a *masa-bimodule* in the sense that  $BTA \in \mathcal{U}$  for all  $A \in \mathcal{D}_X$ ,  $B \in \mathcal{D}_Y$  and  $T \in \mathcal{U}$ , if and only if  $\mathcal{U}$  is invariant under the mappings  $S_w$ ,  $w \in \mathfrak{S}(X, Y)$  [6, Proposition 3.2]. It follows by duality that a norm closed subspace  $V \subseteq T(X, Y)$  is an  $(L^\infty(X), L^\infty(Y))$ -module if and only if it is invariant under the mappings  $m_w$ ,  $w \in \mathfrak{S}(X, Y)$ .

Throughout,  $G$  will denote a second countable locally compact group.

We now summarise some results from non-commutative harmonic analysis. All spaces  $L^p(G)$  are with respect to left Haar measure  $m$ ;  $dm(x)$  is shortened to  $dx$  and the modular function is denoted by  $\Delta$ . If  $A, B \subseteq G$  we write  $A^{-1} = \{x^{-1} : x \in A\}$  and  $AB = \{xy : x \in A, y \in B\}$ . Denote by  $\lambda : G \rightarrow \mathcal{B}(L^2(G))$ ,  $s \rightarrow \lambda_s$ , the left regular representation and write  $(f, g)$  for the inner product of the elements  $f, g \in L^2(G)$ . We set  $T(G) = T(G, G)$ ,  $\mathcal{B}(L^2(G)) = \mathcal{B}(L^2(G), L^2(G))$  and  $\mathfrak{S}(G) = \mathfrak{S}(G, G)$ . The *group von Neumann algebra* of  $G$  is the algebra

$$\text{VN}(G) = \text{span}\{\lambda_x : x \in G\}^{-w*},$$

acting on  $L^2(G)$ , while the *Fourier algebra*  $A(G)$  of  $G$  [9] is the (commutative, regular, semi-simple) Banach algebra consisting of all complex functions  $u$  on  $G$  of the form

$$(1) \quad u(x) = (\lambda_x \xi, \eta), \quad x \in G, \text{ where } \xi, \eta \in L^2(G).$$

Multiplication in  $A(G)$  is pointwise, while the norm  $\|u\|$  of an element  $u \in A(G)$  is the infimum of the products  $\|\xi\|_2 \|\eta\|_2$  over all representations (1) of  $u$ . The spectrum of  $A(G)$  is identified with  $G$  via point evaluations.

Every element  $\tau$  of the dual  $A(G)^*$  defines a bounded operator  $T_\tau$  on  $L^2(G)$  by the formula

$$\langle \tau, u \rangle_a := (T_\tau \xi, \eta)$$

(the symbol  $\langle \cdot, \cdot \rangle_a$  is used to denote the duality between  $A(G)$  and  $A(G)^*$ ), where  $u \in A(G)$  is given by (1). The map

$$\tau \rightarrow T_\tau : A(G)^* \rightarrow \mathcal{B}(L^2(G))$$

sends  $A(G)^*$  isometrically and weak\* homeomorphically onto  $\text{VN}(G)$ .

Note that the spaces  $A(G)^*$  and  $\text{VN}(G)$  are usually identified in the literature and the map  $\tau \rightarrow T_\tau$  is suppressed; we have chosen to retain it in order to emphasize the different dualities used in this paper. The algebra  $\text{VN}(G)$  is a Banach  $A(G)$ -module under the operation

$$(u, T_\tau) \rightarrow uT_\tau = T_{\tau'},$$

where  $\tau'$  is defined by the relation

$$\langle \tau', v \rangle_a = \langle \tau, uv \rangle_a, \quad v \in A(G).$$

The predual  $P : T(G) \rightarrow A(G)$  of the map  $\tau \rightarrow T_\tau : A(G)^* \rightarrow \mathcal{B}(L^2(G))$  is the contraction given by

$$(2) \quad \langle \tau, P(h) \rangle_a = \langle T_\tau, h \rangle_t, \quad \tau \in A(G)^*, \quad h \in T(G).$$

To obtain an explicit formula for  $P$ , take  $\tau$  such that  $T_\tau = \lambda_s$  and recall that  $\langle \tau, u \rangle_a = u(s)$ ,  $s \in G$ . If  $h \in T(G)$  is of the form  $h = \sum_{i=1}^{\infty} f_i \otimes g_i$ ,

where  $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$  and  $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$ , then

$$\begin{aligned} P(h)(s) &= \langle \tau, P(h) \rangle_a = \langle \lambda_s, h \rangle_t = \sum_{i=1}^{\infty} (\lambda_s f_i, \bar{g}_i) \\ &= \sum_{i=1}^{\infty} \int_G f_i(s^{-1}t) g_i(t) dt = \int_G \sum_{i=1}^{\infty} f_i(s^{-1}t) g_i(t) dt \end{aligned}$$

by Proposition 3.3 below. Thus

$$(3) \quad P(h)(s) = \int_G h(s^{-1}t, t) dt, \quad s \in G.$$

The space  $A(G)$  has a canonical operator space structure arising from its identification with the predual of  $VN(G)$  (the reader is referred to [7], [24], [25] for the basic notions of operator space theory). We write

$$MA(G) = \{v : G \rightarrow \mathbb{C} : vu \in A(G) \text{ for all } u \in A(G)\}$$

for the multiplier algebra of  $A(G)$ ; the set of all  $v \in MA(G)$  for which the map  $u \rightarrow vu$  on  $A(G)$  is completely bounded will be denoted  $M^{\text{cb}}A(G)$  and equipped with the completely bounded norm.

Define

$$N : L^\infty(G) \rightarrow L^\infty(G \times G) \quad \text{by} \quad N(f)(x, y) = f(yx^{-1}).$$

We warn the reader that our definition of the map  $N$  differs from the one used in [31, 20], where the expression  $f(xy^{-1})$  is used instead of  $f(yx^{-1})$ .

The following result [3], [30] (see also [16]) will be used in the sequel.

**Theorem 2.1.** *The map  $u \rightarrow N(u)$  is an isometry from  $M^{\text{cb}}A(G)$  into  $\mathfrak{S}(G)$ . Moreover,  $N(M^{\text{cb}}A(G))$  equals the space of those  $w \in \mathfrak{S}(G)$  for which  $w(sr, tr) = w(s, t)$  for every  $r \in G$  and marginally almost all  $s, t$ .*

If  $G$  is compact then  $T(G)$  contains the constant functions, and Theorem 2.1 implies that  $N$  takes values in  $T(G)$ .

If  $u \in A(G)$ ,  $h \in T(G)$  and  $t \in G$  then, using (3), we have

$$\begin{aligned} P(N(u)h)(t) &= \int N(u)(t^{-1}s, s) h(t^{-1}s, s) ds \\ &= \int u(s(t^{-1}s)^{-1}) h(t^{-1}s, s) ds = u(t)P(h)(t) \end{aligned}$$

$$(4) \quad \text{so} \quad P(N(u)h) = uP(h).$$

### 3. IDEALS AND BIMODULES

The main result of this section is the annihilator formula of Theorem 3.2. We start by explaining its main ingredients. Given a closed ideal  $J$  of  $A(G)$ , we will abuse notation and identify its annihilator  $J^\perp$  with a (weak\* closed) subspace of  $VN(G)$ . The space  $J^\perp$  is *invariant*, that is, it is an  $A(G)$ -submodule of  $VN(G)$ ; it is easy to see that every weak\* closed invariant subspace of  $VN(G)$  arises in this way. Similarly, there is a bijective

correspondence between the class of all norm closed  $L^\infty(G)$ -bimodules in  $T(G)$  and the class of all weak\* closed masa-bimodules in  $\mathcal{B}(L^2(G))$ , given by taking annihilators and pre-annihilators.

Given any weak\* closed invariant subspace  $\mathcal{X}$  of  $\text{VN}(G)$ , we let  $\text{Bim}(\mathcal{X}) \subseteq \mathcal{B}(L^2(G))$  be the weak\* closed masa-bimodule generated by  $\mathcal{X}$ . It is not hard to see that

$$\text{Bim}(\mathcal{X}) = \overline{[\mathfrak{S}(G)\mathcal{X}]^{w*}}.$$

Note that if  $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$  is a weak\* closed masa-bimodule then  $\mathcal{U} \cap \text{VN}(G)$  is a weak\* closed invariant subspace of  $\text{VN}(G)$ ; indeed, if  $u \in A(G)$  and  $T \in \mathcal{U} \cap \text{VN}(G)$  then  $uT = S_{N(u)}(T)$ .

Given a closed ideal  $J \subseteq A(G)$ , we wish to define, similarly, a norm closed  $L^\infty(G)$ -bimodule in  $T(G)$  “generated by”  $J$ . To this end, suppose first that  $G$  is compact. Then, as pointed out in Section 2,  $N(J) \subseteq T(G)$ . Hence, one may consider the norm closed  $L^\infty(G)$ -bimodule of  $T(G)$  generated by  $N(J)$ , that is, the space  $\overline{[\mathfrak{S}(G)N(J)]^{\|\cdot\|_t}}$ . If  $G$  is not compact, the map  $N$  does not take values in  $T(G)$  but in  $\mathfrak{S}(G)$ . However, if  $u \in A(G)$  then  $N(u)$  belongs to  $T(G)$  ‘locally’ in the sense that  $N(u)\chi_{L \times L} \in T(G)$  for every compact subset  $L \subseteq G$  (indeed, for such  $L$ , the function  $\chi_{L \times L}$  is in  $T(G)$  and since  $N(u)$  is a multiplier of  $T(G)$ , the claim follows). Hence we may consider the closed  $L^\infty(G)$ -bimodule of  $T(G)$  generated by the set

$$\{N(u)\chi_{L \times L} : u \in J, L \text{ compact, } L \subseteq G\}.$$

We will denote this bimodule by  $\text{Sat}(J)$ . This bimodule may also be written as follows

**Proposition 3.1.** *If  $J \subseteq A(G)$  is a closed ideal, then*

$$\text{Sat}(J) = \overline{[N(J)T(G)]^{\|\cdot\|_t}}.$$

*Proof.* It is clear that  $\text{Sat}(J) \subseteq \overline{[N(J)T(G)]^{\|\cdot\|_t}}$ . For the converse, consider  $u \in J$  and  $h \in T(G)$ . It follows from Lemma 3.13 below that there are compact subsets  $K_n$  and  $L_n$  of  $G$  such that  $h\chi_{K_n \times L_n} \in \mathfrak{S}(G)$  and  $h$  is the  $\|\cdot\|_t$ -limit of the sequence  $(h\chi_{K_n \times L_n})_n$ . Set  $M_n = K_n \cup L_n$ . Then  $h\chi_{K_n \times L_n} = h\chi_{K_n \times L_n}\chi_{M_n \times M_n}$  and  $N(u)h = \lim_n N(u)h\chi_{K_n \times L_n}\chi_{M_n \times M_n}$ . But  $N(u)\chi_{M_n \times M_n} \in \text{Sat}(J)$  and  $h\chi_{K_n \times L_n} \in \mathfrak{S}(G)$ . Thus

$$N(u)h = \lim_{n \rightarrow \infty} N(u)h\chi_{K_n \times L_n}\chi_{M_n \times M_n} \in \text{Sat}(J).$$

□

The rest of the section is devoted to the proof of the following theorem.

**Theorem 3.2.** *Let  $J \subseteq A(G)$  be a closed ideal. Then  $\text{Sat}(J)^\perp = \text{Bim}(J^\perp)$ .*

We need several preliminary results.

**Proposition 3.3.** *Let  $h = \sum_{i=1}^\infty f_i \otimes g_i \in T(G)$  and  $s, t \in G$ . Then the function  $h_{s,t} : G \rightarrow \mathbb{C}$  given by  $h_{s,t}(r) = h(sr, tr)$ ,  $r \in G$ , belongs to  $L^1(G)$  and  $\|h_{s,t}\|_1 \leq \|h\|_t$ .*



In particular, the sequence  $(u_n)_{n \in \mathbb{N}}$  where  $u_n(r) = \sum_{i=1}^n f_i(sr)g_i(tr)$ , converges in the norm of  $L^1(G)$  and hence, for all  $f \in L^\infty(G)$ ,

$$\int_G f(r)h(sr, tr)dr = \sum_{i=1}^{\infty} \int f(r)f_i(sr)g_i(tr)dr.$$

*Proof.* The argument below is due to Ludwig - Turowska [20, Proof of Theorem 4.11]. We reproduce it for completeness: For each  $s, t \in G$ , applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \int_G |h(sr, tr)|dr &\leq \int_G \sum_{i=1}^{\infty} |f_i(sr)g_i(tr)|dr \\ &\leq \int_G \left( \sum_{i=1}^{\infty} |f_i(sr)|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |g_i(tr)|^2 \right)^{1/2} dr \\ &\leq \left( \sum_{i=1}^{\infty} \int_G |f_i(sr)|^2 dr \right)^{1/2} \left( \sum_{i=1}^{\infty} \int_G |g_i(tr)|^2 dr \right)^{1/2} \\ &= \left( \sum_{i=1}^{\infty} \|f_i\|_2^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} \|g_i\|_2^2 \right)^{1/2} < \infty. \end{aligned}$$

Taking the infimum over all representations of  $h$ , we obtain

$$(5) \quad \|h_{s,t}\|_1 \leq \|h\|_t.$$

The remaining assertions are clear from this inequality after an application of the Lebesgue Dominated Convergence Theorem.  $\square$

Denote by  $\widehat{G}$  the set of (equivalence classes of) unitary irreducible representations of  $G$ . For  $\pi \in \widehat{G}$ , write  $H_\pi$  for the Hilbert space where the representation  $\pi$  acts. Fixing an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}_\pi}$  of  $H_\pi$  (where  $\mathbb{N}_\pi$  is either finite or equals  $\mathbb{N}$ ), we write  $u_{i,j}^\pi(r) = (\pi(r)e_j, e_i)$  for the coefficients of  $\pi$ .

Let  $\pi \in \widehat{G}$  and  $h \in T(G)$ . Define

$$h_r(s, t) = h(sr, tr), \quad r, s, t \in G;$$

$$h^\pi(s, t) = \int_G h_r(s, t)\pi(r)dr \in \mathcal{B}(H_\pi);$$

$$\tilde{h}^\pi(s, t) = \pi(s)h^\pi(s, t) = \int_G h_r(s, t)\pi(sr)dr \in \mathcal{B}(H_\pi),$$

where the integrals are understood in the weak sense. We also let

$$(6) \quad \begin{aligned} h_{i,j}^\pi(s, t) &= (h^\pi(s, t)e_j, e_i) = \int_G h_r(s, t)u_{i,j}^\pi(r)dr; \\ \tilde{h}_{i,j}^\pi(s, t) &= (\tilde{h}^\pi(s, t)e_j, e_i) = \int_G h_r(s, t)u_{i,j}^\pi(sr)dr. \end{aligned}$$

If  $\phi$  is a function on  $G$ , we denote by  $\check{\phi}$  the function given by  $\check{\phi}(s) = \phi(s^{-1})$ ,  $s \in G$ .

**Lemma 3.4.** *Let  $h \in T(G)$ . Then  $\tilde{h}_{i,j}^\pi \in \mathfrak{S}(G)$  and  $\|\tilde{h}_{i,j}^\pi\|_{\mathfrak{S}} \leq \|h\|_t$ .*

*Proof.* When  $h = f \otimes g$  is an elementary tensor, (6) gives

$$\tilde{h}_{i,j}^\pi(s, t) = \int_G f(x) u_{i,j}^\pi(x) g(ts^{-1}x) dx = \phi(st^{-1}),$$

where  $\phi = (f u_{i,j}^\pi) * \check{g} \in A(G)$ . Thus,

$$\tilde{h}_{i,j}^\pi(s, t) = \check{\phi}(ts^{-1}) = (N\check{\phi})(s, t)$$

and so, since  $\check{\phi} \in A(G) \subseteq M^{\text{cb}}A(G)$ , Theorem 2.1 shows that  $\tilde{h}_{i,j}^\pi$  is in  $\mathfrak{S}(G)$  and that  $\|\tilde{h}_{i,j}^\pi\|_{\mathfrak{S}} = \|\check{\phi}\|_{M^{\text{cb}}A(G)}$ . Now  $A(G)$  embeds contractively in  $M^{\text{cb}}A(G)$  [4] and so

$$\|\check{\phi}\|_{M^{\text{cb}}A(G)} \leq \|\check{\phi}\|_{A(G)} = \|\phi\|_{A(G)} \leq \|f u_{i,j}^\pi\|_2 \|g\|_2.$$

Thus,

$$\|\tilde{h}_{i,j}^\pi\|_{\mathfrak{S}} \leq \|f u_{i,j}^\pi\|_2 \|g\|_2 \leq \|f\|_2 \|g\|_2 = \|h\|_t.$$

The same inequality holds for linear combinations  $\sum_{n=1}^N f_n \otimes g_n$ , and hence the linear operator  $\Phi$  given by  $\Phi(h) = \tilde{h}_{i,j}^\pi$ , defined on the algebraic tensor product  $L^2(G) \otimes L^2(G)$  extends to a bounded operator  $\Phi : T(G) \rightarrow \mathfrak{S}(G)$ ; clearly,  $\|\Phi(h)\|_{\mathfrak{S}} \leq \|h\|_t$ .

Now let  $h = \sum_{i=1}^{\infty} f_i \otimes g_i$  be an arbitrary element of  $T(G)$ , and set  $h_n = \sum_{i=1}^n f_i \otimes g_i$ ,  $n \in \mathbb{N}$ ; we show that  $\Phi(h) = \tilde{h}_{i,j}^\pi$ . Since  $h_n \rightarrow_{n \rightarrow \infty} h$  in  $T(G)$ , we have that  $\Phi(h_n) \rightarrow_{n \rightarrow \infty} \Phi(h)$  in  $\mathfrak{S}(G)$  and hence  $\Phi(h_n) \chi_{L \times L} \rightarrow \Phi(h) \chi_{L \times L}$  in  $T(G)$  for every compact set  $L \subseteq G$ . By [28, Lemma 2.1], a subsequence of  $(\Phi(h_n) \chi_{L \times L})_{n \in \mathbb{N}}$  converges m.a.e. to  $\Phi(h) \chi_{L \times L}$ .

On the other hand, Proposition 3.3 shows that  $\Phi(h_n) \rightarrow_{n \rightarrow \infty} \tilde{h}_{i,j}^\pi$  pointwise. It follows that  $\Phi(h) = \tilde{h}_{i,j}^\pi$ .  $\square$

The proof of the following lemma, which follows readily from the definitions, is left to the reader.

**Lemma 3.5.** (i) *If  $h \in \mathfrak{S}(G)$  then  $h_r \in \mathfrak{S}(G)$  and  $\|h_r\|_{\mathfrak{S}} = \|h\|_{\mathfrak{S}}$ .*  
 (ii) *If  $h \in T(G)$  then  $h_r \in T(G)$  and  $\|h_r\|_t \leq \Delta(r)^{-1} \|h\|_t$ .*

We do not know if  $h_{i,j}^\pi$  always defines a Schur multiplier; however, it suffices for our purposes to show that its restriction to a compact set does define a Schur multiplier; this is done in Lemma 3.8. In Lemma 3.12 we express this restriction in terms of  $\tilde{h}_{k,j}^\pi$ .

We thank the referee for the following remark.

**Remark 3.6.** *If  $h \in \mathfrak{S}(G)$  is compactly supported, say  $\text{supp } h \subset K \times K$  where  $K \subseteq G$  is compact, then  $h \in T(G)$ .*

*Proof.* Indeed, given  $\varepsilon > 0$ , writing  $h = \sum_n \varphi_n \otimes \psi_n$  with

$$\left\| \sum_n |\varphi_n|^2 \right\|_\infty \left\| \sum_n |\psi_n|^2 \right\|_\infty < (\|S_h\| + \varepsilon)^2,$$

we have

$$\begin{aligned} \|h\|_t^2 &\leq \left( \sum_n \|\varphi_n\|_2 \|\psi_n\|_2 \right)^2 \leq \left( \sum_n \int_K |\varphi_n|^2 \right) \left( \sum_n \int_K |\psi_n|^2 \right) \\ &= \left( \int_K \sum_n |\varphi_n|^2 \right) \left( \int_K \sum_n |\psi_n|^2 \right) \\ &\leq m(K)^2 \left\| \sum_n |\varphi_n|^2 \right\|_\infty \left\| \sum_n |\psi_n|^2 \right\|_\infty = m(K)^2 (\|S_h\| + \varepsilon)^2. \end{aligned}$$

□

For the next lemma, recall (see for example [2] or [30, Section 3]) that  $\mathfrak{S}(G)$  can be identified with the weak\* Haagerup tensor product  $L^\infty(G) \otimes^{w^*h} L^\infty(G)$  which coincides with the dual of the Haagerup tensor product  $L^1(G) \otimes^h L^1(G)$ , the duality being given by

$$(7) \quad \langle w, f \otimes g \rangle = \iint w(s, t) f(s) g(t) ds dt, \quad w \in \mathfrak{S}(G), \quad f, g \in L^1(G).$$

**Lemma 3.7.** *Let  $L \subseteq G$  be a compact set. If  $h \in \mathfrak{S}(G)$  is supported in a compact set  $K \times K$ , then the function  $r \rightarrow \langle (\chi_{L \times L} h_r), \omega \rangle$  is continuous for every  $\omega \in L^1(G) \otimes^h L^1(G)$ .*

*Proof.* For  $\omega = f \otimes g$  where  $f, g \in L^1(G)$ , we have

$$\begin{aligned} \langle (\chi_{L \times L} h_r), \omega \rangle &= \iint \chi_L(s) \chi_L(t) h(sr, tr) f(s) g(t) ds dt \\ &= \iint \chi_L(sr^{-1}) \chi_L(tr^{-1}) h(s, t) \Delta(r)^{-1} f(sr^{-1}) \Delta(r)^{-1} g(tr^{-1}) ds dt \end{aligned}$$

But, as  $r \rightarrow e$ , the function  $s \rightarrow \chi_L(sr^{-1}) f(sr^{-1}) \Delta(r)^{-1}$  tends to  $\chi_L f$  in the norm of  $L^1(G)$ ; similarly for  $\chi_L g$ . Therefore, since  $h$  is bounded,

$$\langle (\chi_{L \times L} h_r), \omega \rangle \xrightarrow{r \rightarrow e} \iint \chi_L(s) h(s, t) \chi_L(t) f(s) g(t) ds dt.$$

It follows that  $r \rightarrow \langle (\chi_{L \times L} h_r), \omega \rangle$  is continuous for every finite sum  $\omega = \sum f_n \otimes g_n$ . Since such elements  $\omega$  form a dense subset of  $L^1(G) \otimes^h L^1(G)$ , and  $\|h\|_{\mathfrak{S}} = \|h_r\|_{\mathfrak{S}}$  for all  $r \in G$  (Lemma 3.5 (i)), the conclusion follows. □

Suppose  $h \in \mathfrak{S}(G)$  is supported in a compact set  $K \times K$  and let  $u : G \rightarrow \mathbb{C}$  be bounded and continuous. For a compact set  $L \subseteq G$ , let

$$(8) \quad \begin{aligned} w_{h,L}(s,t) &= \chi_{L \times L}(s,t) \int_G h(sr, tr) u(r) dr \\ &= \chi_{L \times L}(s,t) \int_G h(sr, tr) u(r) \chi_{L^{-1}K}(r) dr \end{aligned}$$

(the second equality follows from the fact that  $sr \in K$  forces  $r \in s^{-1}K \subseteq L^{-1}K$  if  $s \in L$ ). In the next lemma, we show that  $w_{h,L} \in \mathfrak{S}(G)$ . First note that, since  $h \in \mathfrak{S}(G)$ , the function  $r \rightarrow \langle (\chi_{L \times L} h_r), \omega \rangle$  is bounded and hence the integral  $\int \langle (\chi_{L \times L} h_r), \omega \rangle u(r) \chi_{L^{-1}K}(r) dr$  exists.

**Lemma 3.8.** *If  $h \in \mathfrak{S}(G)$  is compactly supported, then for every compact  $L \subseteq G$  and every bounded continuous function  $u : G \rightarrow \mathbb{C}$ , the function  $w_{h,L}$  defined in (8) is a Schur multiplier. In particular,  $\chi_{L \times L} h_{i,j}^T$  is a Schur multiplier.*

*Proof.* Suppose  $h$  is supported in a compact set  $K \times K$ . By Lemmas 3.5(i) and 3.7, the linear mapping

$$L^1(G) \otimes^h L^1(G) \rightarrow \mathbb{C} : \omega \rightarrow \int \langle (\chi_{L \times L} h_r), \omega \rangle u(r) \chi_{L^{-1}K}(r) dr$$

is bounded with norm not exceeding  $\|h\|_{\mathfrak{S}} \|u\|_{\infty} m(L^{-1}K)$ ; hence it defines an element  $v_{h,L} \in (L^1(G) \otimes^h L^1(G))^* = \mathfrak{S}(G)$ , that is

$$(9) \quad \langle v_{h,L}, \omega \rangle = \int_G \langle (\chi_{L \times L} h_r), \omega \rangle u(r) \chi_{L^{-1}K}(r) dr,$$

for all  $\omega \in L^1(G) \otimes^h L^1(G)$ .

We will show that  $v_{h,L} = w_{h,L}$  almost everywhere.

By (7) and (9), if  $\omega = f \otimes g$  with  $f, g \in L^1(G)$  then, applying Fubini's theorem (note that the integration with respect to  $r$  is over a compact set), we obtain

$$\begin{aligned} \iint v_{h,L}(s,t) f(s) g(t) ds dt &= \langle v_L, \omega \rangle = \int \langle (\chi_{L \times L} h_r), \omega \rangle u(r) \chi_{L^{-1}K}(r) dr \\ &= \int \left( \iint (\chi_{L \times L} h_r)(s,t) f(s) g(t) ds dt \right) u(r) \chi_{L^{-1}K}(r) dr \\ &= \iint \left( \int (\chi_{L \times L} h_r)(s,t) u(r) \chi_{L^{-1}K}(r) dr \right) f(s) g(t) ds dt \\ &= \iint w_{h,L}(s,t) f(s) g(t) ds dt. \end{aligned}$$

This shows that the function  $v_{h,L} - w_{h,L}$  annihilates the algebraic tensor product  $L^1(G) \otimes L^1(G)$ , hence all of  $L^1(G \times G)$ . It follows that it is zero as an element of  $L^\infty(G \times G)$ .  $\square$

**Corollary 3.9.** *Suppose  $h \in \mathfrak{S}(G)$  is compactly supported. For every continuous bounded function  $u : G \rightarrow \mathbb{C}$ , every  $T \in \mathcal{B}(L^2(G))$  and every  $f, g \in L^2(G)$  supported in a compact set  $L \subseteq G$  we have*

$$(S_{w_{h,L}}(T)f, g) = \int_G u(r)(S_{h_r}(T)f, g)dr,$$

where  $w_{h,L}$  is as in (8). In particular,

$$(S_{h_{i,j}^\pi \chi_{L \times L}}(T)f, g) = \int_G u_{i,j}^\pi(r)(S_{h_r}(T)f, g)dr$$

for all  $\pi \in \widehat{G}$  and all  $i, j \in \mathbb{N}_\pi$ .

*Proof.* Let  $K \subseteq G$  be a compact set such that  $\text{supp } h \subseteq K \times K$ . Suppose first that  $T = T_k$  is a Hilbert-Schmidt operator (here  $k \in L^2(G \times G)$ ). Using Fubini's theorem, we have

$$\begin{aligned} (S_{w_{h,L}}(T_k)f, g) &= \int_{G \times G} w_{h,L}(s, t)k(t, s)f(s)\overline{g(t)}dsdt \\ &= \int_{G \times G} \left( \int_{L^{-1}K} h(sr, tr)u(r)dr \right) k(t, s)f(s)\overline{g(t)}dsdt \\ &= \int_{L^{-1}K} u(r) \left( \int_{G \times G} h(sr, tr)k(t, s)f(s)\overline{g(t)}dsdt \right) dr \\ &= \int_{L^{-1}K} u(r)(S_{h_r}(T_k)f, g)dr. \end{aligned}$$

If  $T$  is arbitrary, let  $(T_n)$  be a sequence of Hilbert-Schmidt operators (with operator norms uniformly bounded by  $\|T\|$ ) such that  $T_n \rightarrow T$  in the weak\* topology. Then

$$(S_{w_{h,L}}(T_n)f, g) \rightarrow (S_{w_{h,L}}(T)f, g)$$

by the weak\* continuity of  $S_{w_{h,L}}$ . On the other hand, since

$$|(S_{h_r}(T_n)f, g)| \leq \|T_n\| \|f\|_2 \|g\|_2 \|h_r\|_{\mathfrak{S}} \leq \|T\| \|f\|_2 \|g\|_2 \|h\|_{\mathfrak{S}}$$

and  $L^{-1}K$  has finite Haar measure, the Lebesgue Dominated Convergence Theorem implies that

$$\int_G u(r)(S_{h_r}(T_n)f, g)dr \rightarrow \int_G u(r)(S_{h_r}(T)f, g)dr.$$

The conclusion follows.  $\square$

**Lemma 3.10.** *If  $h \in T(G)$  then*

$$(10) \quad \sum_{k \in \mathbb{N}_\pi} \left\| \tilde{h}_{k,j}^\pi \right\|_{\mathfrak{S}}^2 \leq \|h\|_t^2.$$

*Proof.* Suppose first that  $h$  is an elementary tensor, say  $h = \phi \otimes \psi$ , and recall that in this case  $\|\tilde{h}_{k,j}^\pi\|_{\mathfrak{S}} \leq \|u_{k,j}^\pi \phi\|_2 \|\psi\|_2$  (see the proof of Lemma 3.4). Now

$$\begin{aligned} \sum_k \|u_{i,k}^\pi \phi\|^2 &= \sum_k \int_G |u_{i,k}^\pi(s) \phi(s)|^2 ds = \int_G \sum_k |(\pi(s)e_k, e_i)|^2 |\phi(s)|^2 ds \\ &= \int_G \sum_k |(e_k, \pi(s^{-1})e_i)|^2 |\phi(s)|^2 ds \\ &= \int_G \|\pi(s^{-1})e_i\|^2 |\phi(s)|^2 ds = \int_G |\phi(s)|^2 ds = \|\phi\|_2^2 \end{aligned}$$

and so  $\sum_k \|\tilde{h}_{k,j}^\pi\|_{\mathfrak{S}}^2 \leq \|h\|_t^2$ .

The same estimate persists when  $h$  is a finite sum  $h = \sum_l \phi_l \otimes \psi_l$ :

$$\begin{aligned} \sum_k \|\tilde{h}_{k,j}^\pi\|_{\mathfrak{S}}^2 &\leq \sum_k \left( \sum_l \|u_{k,j}^\pi \phi_l\|_2 \|\psi_l\|_2 \right)^2 \leq \sum_l \sum_k \|u_{k,j}^\pi \phi_l\|_2^2 \sum_l \|\psi_l\|_2^2 \\ &= \sum_l \|\phi_l\|_2^2 \sum_l \|\psi_l\|_2^2. \end{aligned}$$

In particular,  $\sum_{k=1}^N \|\tilde{h}_{k,j}^\pi\|_{\mathfrak{S}}^2 \leq \|h\|_t^2$  for all  $N \in \mathbb{N}_\pi$ . Since the map  $h \rightarrow \tilde{h}_{k,j}^\pi : T(G) \rightarrow \mathfrak{S}(G)$  is contractive (see Lemma 3.4), the last inequality holds for all  $h \in T(G)$ . But  $N$  is arbitrary, and so (10) is proved for an arbitrary  $h$ .  $\square$

We thank V. S. Shulman and L. Turowska for letting us include a proof of the following lemma from an earlier version of [27].

**Lemma 3.11.** *For each  $k$  and  $\pi \in \hat{G}$ , the functions  $\sum_{l=m}^\infty |u_{k,l}^\pi|^2$  converge to zero uniformly on compact sets, as  $m \rightarrow \infty$ .*

*Proof.* It suffices to consider the case where  $H_\pi$  is infinite dimensional. Fix  $k \in \mathbb{N}$  and let

$$f_m(r) = \sum_{l=m}^\infty |u_{k,l}^\pi(r)|^2 = \sum_{l=m}^\infty |(\pi(r)e_k, e_l)|^2 = \|P_m \pi(r)e_k\|^2,$$

where  $P_m$  is the projection on the closed subspace generated by  $\{e_l : l \geq m\}$ .

Since the function  $r \rightarrow \pi(r)e_k : G \rightarrow H_\pi$  is continuous, so is the function  $r \rightarrow P_m \pi(r)e_k : G \rightarrow H_\pi$ ; thus each  $f_m$  is a continuous function.

Since  $(P_m)_{m \in \mathbb{N}}$  decreases to 0, the sequence  $(f_m(r))_{m \in \mathbb{N}}$  decreases to 0 for each  $r \in G$ . By Dini's Theorem, the convergence is uniform on compact subsets of  $G$ .  $\square$

**Lemma 3.12.** *Assume  $h \in T(G)$ . Let  $\pi \in \widehat{G}$  and  $i, j \in \mathbb{N}_\pi$ . For each compact set  $L \subseteq G$ , and all  $f, g \in L^2(G)$ , we have that*

$$h_{i,j}^\pi(\chi_L f \otimes \chi_L g) = \sum_k (\check{u}_{i,k}^\pi \otimes \mathbf{1}) \tilde{h}_{k,j}^\pi(\chi_L f \otimes \chi_L g)$$

and

$$\tilde{h}_{i,j}^\pi(\chi_L f \otimes \chi_L g) = \sum_k (u_{i,k}^\pi \otimes \mathbf{1}) h_{k,j}^\pi(\chi_L f \otimes \chi_L g)$$

in the norm of  $T(G)$ .

*Proof.* We prove the first formula; the second follows similarly. We show first that the series

$$(11) \quad \sum_k (\check{u}_{i,k}^\pi \otimes \mathbf{1}) \tilde{h}_{k,j}^\pi(\chi_L f \otimes \chi_L g)$$

converges in the norm of  $T(G)$ . Fix  $\epsilon > 0$  and let  $T \in \mathcal{B}(L^2(G))$  be a contraction. For all  $n < m$ ,

$$\begin{aligned} \sum_{k=n}^m \|(\check{u}_{i,k}^\pi \chi_L f)\|^2 &= \sum_{k=n}^m \int_L |\check{u}_{i,k}^\pi(r)|^2 |f(r)|^2 dr = \int_L \sum_{k=n}^m |\check{u}_{i,k}^\pi(r)|^2 |f(r)|^2 dr \\ &\leq \|f\|_2^2 \sup_{r \in L^{-1}} \sum_{k=n}^m |u_{i,k}^\pi(r)|^2. \end{aligned}$$

Using Lemma 3.11, we can choose  $n < m$  so that

$$(12) \quad \sum_{k=n}^m \|(\check{u}_{i,k}^\pi \chi_L f)\|^2 < \epsilon^2 \|h\|_t^{-2} \|\chi_L \bar{g}\|^{-2}.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left| \left\langle T, \sum_{k=n}^m (\check{u}_{i,k}^\pi \otimes \mathbf{1}) \tilde{h}_{k,j}^\pi(\chi_L f \otimes \chi_L g) \right\rangle \right|^2 \\ &\leq \left( \sum_{k=n}^m \left| (S_{\tilde{h}_{k,j}^\pi}^\pi(T)(\check{u}_{i,k}^\pi \chi_L f), \chi_L \bar{g}) \right| \right)^2 \leq \left( \sum_{k=n}^m \left\| S_{\tilde{h}_{k,j}^\pi}^\pi(T)(\check{u}_{i,k}^\pi \chi_L f) \right\| \|\chi_L \bar{g}\| \right)^2 \\ &\leq \sum_{k=n}^m \left\| S_{\tilde{h}_{k,j}^\pi}^\pi(T) \right\|^2 \sum_{k=n}^m \|(\check{u}_{i,k}^\pi \chi_L f)\|^2 \|\chi_L \bar{g}\|^2 < \epsilon, \end{aligned}$$

where for the last inequality we have used (12) and Lemma 3.10. It follows that the series (11) converges in the norm of  $T(G)$ ; let  $\Lambda$  be its sum. By [28, Lemma 2.1], there exists a sequence of partial sums of (11) that converges marginally almost everywhere to  $\Lambda$ .

On the other hand, for every  $s, t \in G$ , we have

$$\begin{aligned} h_{i,j}^\pi(s, t) &= (h^\pi(s, t)e_j, e_i) = (\tilde{h}^\pi(s, t)e_j, \pi(s)e_i) \\ &= \sum_k (\tilde{h}^\pi(s, t)e_j, e_k)(e_k, \pi(s)e_i) = \sum_k u_{i,k}^\pi(s^{-1}) \tilde{h}_{k,j}^\pi(s, t), \end{aligned}$$

and hence the series (11) converges pointwise to the function  $h_{i,j}^\pi(\chi_L f \otimes \chi_L g)$ . It follows that  $\Lambda = h_{i,j}^\pi(\chi_L f \otimes \chi_L g)$  and the proof is complete.  $\square$

**Lemma 3.13.** *Let  $\mathcal{S} \subseteq T(G)$  be a norm closed  $L^\infty(G)$ -bimodule. Each  $h \in \mathcal{S}$  is the norm limit of a sequence  $(h_n)$  with  $h_n = h\chi_{K_n \times L_n} \in \mathcal{S} \cap \mathfrak{S}(G)$  where  $K_n$  and  $L_n$  are compact sets.*

*Proof.* Let  $h = \sum_{i=1}^\infty f_i \otimes g_i$ , where  $\sum_{i=1}^\infty \|f_i\|_2^2$  and  $\sum_{i=1}^\infty \|g_i\|_2^2$  are finite. Given  $n \in \mathbb{N}$ , let

$$A_n = \left\{ s \in G : \sum_{i=1}^\infty |f_i(s)|^2 \leq n \right\}, \quad B_n = \left\{ t \in G : \sum_{i=1}^\infty |g_i(t)|^2 \leq n \right\}$$

and choose compact sets  $K_n \subseteq A_n$  and  $L_n \subseteq B_n$  such that  $m(A_n \setminus K_n) < 2^{-n}$  and  $m(B_n \setminus L_n) < 2^{-n}$ . Setting  $h_n = h\chi_{K_n \times L_n}$ , we see that

(a)  $h_n \in \mathcal{S}$  because  $\mathcal{S}$  is an  $L^\infty(G)$ -bimodule, and  
(b)  $h_n \in \mathfrak{S}(G)$  because  $h_n(s, t) = \sum_{i=1}^\infty (\chi_{K_n} f_i)(s) (\chi_{L_n} g_i)(t)$  ( $s, t \in G$ ), where  $\sum_{i=1}^\infty |(\chi_{K_n} f_i)(s)|^2 \leq n$  and  $\sum_{i=1}^\infty |(\chi_{L_n} g_i)(t)|^2 \leq n$  a.e..

It is straightforward to see that  $\|h - h_n\|_t \rightarrow 0$ .  $\square$

**Lemma 3.14.** *Let  $h \in T(G)$  be supported in a compact set  $K \times K$ . Then  $h$  belongs to the  $T(G)$ -closed linear span of*

$$\{\chi_{K \times K} h_{i,j}^\pi : \pi \in \widehat{G}, i, j \in \mathbb{N}_\pi\}.$$

*Proof.* Suppose  $T \in B(L^2(G))$  satisfies

$$\langle T, \chi_{K \times K} h_{i,j}^\pi \rangle_t = 0 \quad \text{for all } \pi \in \widehat{G}, i, j \in \mathbb{N}_\pi.$$

We will show that  $\langle T, h \rangle_t = 0$ .

Recall that  $h_{i,j}^\pi(s, t) = \int_G h(sr, tr) u_{i,j}^\pi(r) dr$ . We may write  $\chi_{K \times K} h_{i,j}^\pi$  in the form

$$\chi_{K \times K} h_{i,j}^\pi = \chi_{K \times K} (\chi_{K^{-1}K} u_{i,j}^\pi \Delta^{-1}) \star h$$

where  $\star$  is the action of  $L^1(G)$  on  $T(G)$  given by

$$(g \star h)(s, t) := \int_G h(sr, tr) g(r) \Delta(r) dr$$

which satisfies  $\|g \star h\|_{T(G)} \leq \|g\|_{L^1(G)} \|h\|_{T(G)}$ . Thus the hypothesis gives

$$\langle S_{\chi_{K \times K}}(T), (\chi_{K^{-1}K} u_{i,j}^\pi \Delta^{-1}) \star h \rangle_t = \langle T, \chi_{K \times K} (\chi_{K^{-1}K} u_{i,j}^\pi \Delta^{-1}) \star h \rangle_t = 0.$$

Now let  $f$  be a continuous function supported in the compact set  $K^{-1}K$ . Then  $f\Delta$  is continuous and vanishes outside  $K^{-1}K$ ; hence it is the limit, uniformly in  $K^{-1}K$ , of a sequence  $(g_n)$  of linear combinations of coefficients  $u_{i,j}^\pi$  of irreducible representations  $\pi$  of  $G$  (see [10, Theorem 3.27, 3.31 and Proposition 3.33] or [5, 13.6.5 and 13.6.4]). Hence  $g_n \Delta^{-1} \rightarrow f$  uniformly in  $K^{-1}K$  (observe that  $\Delta^{-1}$  is continuous, hence bounded, on compact sets). Each  $g_n$  is a linear combination of coefficients  $u_{i,j}^\pi$  and therefore  $\langle S_{\chi_{K \times K}}(T), (\chi_{K^{-1}K} g_n \Delta^{-1}) \star h \rangle_t = 0$  for each  $n$ . Since  $f = f\chi_{K^{-1}K}$ , it follows that  $\langle S_{\chi_{K \times K}}(T), f \star h \rangle_t = 0$ . Now let  $\{f_\alpha\}$  be an approximate identity



for  $L^1(G)$  consisting of non-negative continuous functions with  $\|f_\alpha\|_1 = 1$ , all supported in  $K^{-1}K$ . Then a standard argument shows that  $\|f_\alpha \star h - h\|_t \rightarrow 0$  and we obtain

$$\langle T, h \rangle_t = \langle T, \chi_{K \times K} h \rangle_t = \langle S_{\chi_{K \times K}}(T), h \rangle_t = \lim_{\alpha} \langle S_{\chi_{K \times K}}(T), f_\alpha \star h \rangle_t = 0$$

which proves the lemma.  $\square$

We now proceed to the proof of Theorem 3.2. We first show that

$$(13) \quad \text{Sat}(J) \subseteq (\text{Bim}(J^\perp))_\perp.$$

Let  $u \in J$ ,  $h \in T(G)$ ,  $w \in \mathfrak{S}(G)$  and  $T \in J^\perp \subseteq \text{VN}(G)$ . Then, if  $\tau \in A(G)^*$  satisfies  $P^*(\tau) = T$ , using (4), we have

$$\begin{aligned} \langle S_w(T), N(u)h \rangle_t &= \langle T, N(u)wh \rangle_t = \langle P^*(\tau), N(u)wh \rangle_t \\ &= \langle \tau, P(N(u)wh) \rangle_a = \langle \tau, uP(wh) \rangle_a = 0 \end{aligned}$$

since  $u \in J$  and  $P(wh) \in A(G)$ , hence  $uP(wh) \in J$ . Thus,  $S_w(T)$  annihilates  $\text{Sat}(J)$  by Proposition 3.1. Since  $\{S_w(T) : T \in J^\perp, w \in \mathfrak{S}(G)\}$  generates  $\text{Bim}(J^\perp)$ , (13) is established.

For the reverse inclusion, suppose that  $h \in (\text{Bim}(J^\perp))_\perp$ . By Lemma 3.13, we may assume that there exists a compact set  $K \subseteq G$  such that  $\text{supp } h \subseteq K \times K$  and  $h \in \mathfrak{S}(G) \cap (\text{Bim}(J^\perp))_\perp$ .

The steps of the argument are the following:

**Step 1.** *If  $T \in J^\perp$  then  $S_{h_r}(T) = 0$  for every  $r \in G$ .*

*Proof.* A direct verification using relation (3) shows that if  $r \in G$  then  $\Delta(r)^{-1}P(h) = P(h_r)$ . By Lemma 3.5 (i),  $h_r \in \mathfrak{S}(G)$ . It suffices to prove that  $(S_{h_r}(T)\xi, \eta) = 0$  whenever  $\xi$  and  $\eta$  are in  $L^\infty(G) \cap L^2(G)$ . In this case,  $w := \xi \otimes \bar{\eta}$  is in  $T(G) \cap \mathfrak{S}(G)$  and, if  $\tau \in A(G)^*$  is such that  $T_\tau = T$ , then

$$\begin{aligned} (S_{h_r}(T)\xi, \eta) &= \langle S_{h_r}(T), w \rangle_t = \langle T, h_r w \rangle_t = \langle \tau, P(h_r w) \rangle_a \\ &= \Delta(r)^{-1} \langle \tau, P(h w_{r^{-1}}) \rangle_a = \Delta(r)^{-1} \langle T, h w_{r^{-1}} \rangle_t \\ &= \Delta(r)^{-1} \langle S_{w_{r^{-1}}}(T), h \rangle_t = 0, \end{aligned}$$

since  $h$  annihilates  $\text{Bim}(J^\perp)$ . Hence,  $S_{h_r}(T) = 0$  for every  $r \in G$ .

**Step 2.** *If  $T \in J^\perp$  then  $S_{\chi_{L \times L} h_{i,j}^\pi}(T) = 0$  for all  $\pi \in \widehat{G}$ , all  $i, j$ , and all compact sets  $L \subseteq G$ .*

This follows from Step 1 and Corollary 3.9.

**Step 3.** *If  $T \in J^\perp$  then  $S_{h_{i,j}^\pi}(T) = 0$ .*

*Proof.* Step 2 and Lemma 3.12 imply that, for every  $f, g \in L^2(G)$  and every compact set  $L \subseteq G$ , we have

$$(S_{h_{i,j}^\pi}(T)\chi_L f, \chi_L g) = \sum_{k=1}^{\infty} (S_{\chi_{L \times L} h_{k,j}^\pi}(T) u_{i,k}^\pi \chi_L f, \chi_L g) = 0.$$

This implies that  $S_{\tilde{h}_{i,j}^\pi}(T) = 0$ .

**Step 4.** *If  $\pi \in \widehat{G}$  and  $L$  is a compact subset of  $G$ , then  $\tilde{h}_{i,j}^\pi \chi_{L \times L} \in \text{Sat}(J)$  for all  $i, j$ .*

*Proof.* Since  $\tilde{h}_{i,j}^\pi \in \mathfrak{S}(G)$  (Lemma 3.4) and  $\tilde{h}_{i,j}^\pi(sr, tr) = \tilde{h}_{i,j}^\pi(s, t)$ , Theorem 2.1 implies that  $\tilde{h}_{i,j}^\pi = N(v)$ , for some  $v \in M^{\text{cb}}A(G)$ . We claim that  $vA(G) \subseteq J$ . Indeed, for every  $\tau \in J^\perp$  and every  $w \in T(G)$ , we have, by Step 3,

$$\begin{aligned} 0 &= \langle S_{\tilde{h}_{i,j}^\pi}(T_\tau), w \rangle_t = \langle S_{N(v)}(T_\tau), w \rangle_t = \langle T_\tau, N(v)w \rangle_t \\ &= \langle \tau, P(N(v)w) \rangle_a = \langle \tau, vP(w) \rangle_a \end{aligned}$$

where we have used relation (4). Note that  $vP(w) \in A(G)$  since  $P(w) \in A(G)$  and  $v \in M^{\text{cb}}A(G)$ . This equality shows, by the Hahn-Banach theorem, that  $vP(w) \in J$ . Thus, since  $P$  is surjective,  $vA(G) \subseteq J$ . We may choose  $w \in T(G)$  such that  $u := P(w)$  satisfies  $u|_{LL^{-1}} = 1$ . Then  $N(u)\chi_{L \times L} = \chi_{L \times L}$  and so

$$\tilde{h}_{i,j}^\pi \chi_{L \times L} = N(v)\chi_{L \times L} = N(v)N(u)\chi_{L \times L} = N(vu)\chi_{L \times L}.$$

But  $vu = vP(w) \in J$ . Thus,  $\tilde{h}_{i,j}^\pi \chi_{L \times L} \in \text{Sat}(J)$ .

**Step 5.** *If  $\pi \in \widehat{G}$  and  $L$  is a compact subset of  $G$ , then  $h_{i,j}^\pi \chi_{L \times L} \in \text{Sat}(J)$  for all  $i, j$ .*

*Proof.* This is a direct consequence of Lemma 3.12.

**Step 6.**  $h \in \text{Sat}(J)$ .

*Proof.* By Lemma 3.14,  $h$  is in the  $T(G)$ -norm closed linear span of elements of the form  $h_{i,j}^\pi \chi_{L \times L}$ , so  $h \in \text{Sat}(J)$ .

The proof of Theorem 3.2 is complete.

#### 4. JOINTLY INVARIANT SUBSPACES

In this section, we characterise the common weak\* closed invariant subspaces of Schur multipliers and a class of completely bounded maps arising from a canonical representation of the measure algebra of  $G$ .

Let  $\rho : G \rightarrow \mathcal{B}(L^2(G))$ ,  $r \rightarrow \rho_r$ , be the right regular representation of  $G$  on  $L^2(G)$ , that is, the representation given by  $(\rho_r f)(s) = \Delta(r)^{1/2} f(sr)$ ,  $s, r \in G$ ,  $f \in L^2(G)$ .

Let  $M(G)$  be the Banach algebra of all bounded complex Borel measures on  $G$ . Following [22] (see also [29]), we define a representation  $\Gamma$  of  $M(G)$  on  $\mathcal{B}(L^2(G))$  by letting

$$\Gamma(\mu)(T) = \int_G \rho_r T \rho_r^* d\mu(r), \quad T \in \mathcal{B}(L^2(G)),$$

the integral being understood in the weak sense (that is, for every  $h \in T(G)$  and every  $T \in \mathcal{B}(L^2(G))$  the formula  $\langle \Gamma(\mu)(T), h \rangle_t = \int_G \langle \rho_r T \rho_r^*, h \rangle_t d\mu(r)$

holds). This representation was studied by E. Størmer [32], F. Ghahramani [12], M. Neufang [21] and M. Neufang, Zh.-J. Ruan and N. Spronk [22], among others.

Denote, as is customary, by  $\text{Ad}\rho_r$  the map on  $\mathcal{B}(L^2(G))$  given by  $\text{Ad}\rho_r(T) = \rho_r T \rho_r^*$ ,  $T \in \mathcal{B}(L^2(G))$ ; since  $\text{Ad}\rho_r$  is a (bounded) weak\* continuous map, it has a (bounded) predual  $\theta_r : T(G) \rightarrow T(G)$ .

**Lemma 4.1.** *Let  $r \in G$ . Then  $\theta_r(h) = \Delta(r^{-1})h_{r^{-1}}$ , for every  $h \in T(G)$ .*

*Proof.* By linearity and continuity (see Lemma 3.5 (ii)), it suffices to check the formula when  $h$  is an elementary tensor, say  $h = f \otimes \bar{g}$  for some  $f, g \in L^2(G)$ . For  $T \in \mathcal{B}(L^2(G))$ , we have

$$\begin{aligned} \langle T, \theta_r(h) \rangle_t &= \langle \text{Ad}\rho_r(T), h \rangle_t = \langle \rho_r T \rho_{r^{-1}}, f \otimes \bar{g} \rangle_t = \langle T(\rho_{r^{-1}}f), \rho_{r^{-1}}g \rangle_t \\ &= \langle T, (\rho_{r^{-1}}f) \otimes \overline{\rho_{r^{-1}}g} \rangle_t. \end{aligned}$$

However, if  $s, t \in G$  then

$$\begin{aligned} (\rho_{r^{-1}}f \otimes \overline{\rho_{r^{-1}}g})(s, t) &= (\rho_{r^{-1}}f)(s)(\overline{\rho_{r^{-1}}g})(t) = \Delta(r^{-1})f(sr^{-1})\bar{g}(tr^{-1}) \\ &= \Delta(r^{-1})h_{r^{-1}}(s, t). \end{aligned}$$

The proof is complete.  $\square$

**Lemma 4.2.** *Let  $V \subseteq T(G)$  be a norm closed  $L^\infty(G)$ -bimodule such that  $\theta_r(V) \subseteq V$  for each  $r \in G$ . Then there exists a closed ideal  $J \subseteq A(G)$  such that  $V = \text{Sat}(J)$ .*

*Proof.* Let

$$J = \{u \in A(G) : N(u)\chi_{L \times L} \in V \text{ for every compact set } L \subseteq G\}.$$

Since  $A(G)$  embeds contractively into  $M^{cb}A(G)$  and the map  $N$  is continuous, it is clear that  $J$  is a closed subspace of  $A(G)$ . We check that  $J$  is an ideal: if  $u \in J$ ,  $v \in A(G)$  and  $L \subseteq G$  is a compact subset, then

$$N(uv)\chi_{L \times L} = (Nu)(Nv)\chi_{L \times L} \in \mathfrak{S}(G)V \subseteq V$$

since  $N(v) \in \mathfrak{S}(G)$  by Theorem 2.1 and  $V$ , being a closed  $L^\infty(G)$ -bimodule, is invariant under  $\mathfrak{S}(G)$ .

Clearly  $\text{Sat}(J) \subseteq V$ . To show that  $V \subseteq \text{Sat}(J)$ , let  $h \in V$ . By Lemma 3.13, we may assume that  $\text{supp } h \subseteq K \times K$  and  $h \in \mathfrak{S}(G) \cap V$  for some compact set  $K \subseteq G$ . In order to conclude that  $h \in \text{Sat}(J)$  it suffices, by Lemma 3.14, to prove that  $h_{i,j}^\pi \chi_{L \times L} \in \text{Sat}(J)$  for every irreducible representation  $\pi$  of  $G$ , every  $i, j \in \mathbb{N}_\pi$  and every compact set  $L \subseteq G$ .

The function  $r \rightarrow u_{i,j}^\pi(r)h_r$ ,  $G \rightarrow T(G)$ , is continuous (Lemma 3.5(ii)) and hence the integral

$$\omega := \int_{L^{-1}K} u_{i,j}^\pi(r)h_r dr = \int_{L^{-1}K} u_{i,j}^\pi(r)\Delta(r)\theta_r(h)dr$$

exists as a Bochner integral and defines an element of  $T(G)$ . The second equality shows that  $\omega$  is in the closed linear span of  $\{\theta_r(h) : r \in G\}$ . But  $V$  is invariant under  $\theta_r$ , and hence  $\omega \in V$ .

We claim that  $\chi_{L \times L} \omega = \chi_{L \times L} h_{i,j}^\pi$ . To see this, let  $T = T_k$  be a Hilbert-Schmidt operator of the form  $k = f \otimes g$  with  $f, g \in L^2(G)$ . Then

$$\begin{aligned} \langle \chi_{L \times L} \omega, T \rangle_t &= \int_{L^{-1}K} u_{i,j}^\pi(r) \langle \chi_{L \times L} h_r, T \rangle_t dr \\ &= \int_{L^{-1}K} \iint_{G \times G} u_{i,j}^\pi(r) \chi_{L \times L}(s, t) h_r(s, t) f(t) g(s) ds dt dr \\ &= \langle \chi_{L \times L} h_{i,j}^\pi, T \rangle_t \end{aligned}$$

(the last equality follows as in the proof of Corollary 3.9). This proves the claim.

Thus  $\chi_{L \times L} h_{i,j}^\pi$  is in  $V$ . Since  $V$  is a norm closed  $L^\infty(G)$  bimodule, using Lemma 3.12 we conclude that  $\chi_{L \times L} \tilde{h}_{i,j}^\pi \in V$ . By Theorem 2.1, since  $\tilde{h}_{i,j}^\pi \in \mathfrak{S}(G)$ , there exists  $v \in M^{\text{cb}}A(G)$  such that  $\tilde{h}_{i,j}^\pi = N(v)$ .

We claim that  $vA(G) \subseteq J$ . Indeed, for every  $u \in A(G)$ , if  $L \subseteq G$  is a compact set,

$$\chi_{L \times L} N(vu) = (\chi_{L \times L} \tilde{h}_{i,j}^\pi) N(u) \in V \mathfrak{S}(G) \subseteq V$$

and thus  $vu \in J$  by the definition of  $J$ .

Since  $P$  is surjective, we may choose  $w \in T(G)$  such that  $u := P(w)$  satisfies  $u|_{LL^{-1}} = 1$ . Then  $N(u)\chi_{L \times L} = \chi_{L \times L}$  and so

$$\tilde{h}_{i,j}^\pi \chi_{L \times L} = N(v) \chi_{L \times L} = N(v) N(u) \chi_{L \times L} = N(vu) \chi_{L \times L}.$$

Since  $vA(G) \subseteq J$ , we obtain  $vu = vP(w) \in J$ . Thus,  $\tilde{h}_{i,j}^\pi \chi_{L \times L} \in \text{Sat}(J)$ .

Using Lemma 3.12 again, we obtain  $\chi_{L \times L} h_{i,j}^\pi \in \text{Sat}(J)$  and the proof is complete.  $\square$

**Theorem 4.3.** *Let  $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$  be a weak\* closed subspace. The following are equivalent:*

- (i) *the space  $\mathcal{U}$  is invariant under the mappings  $S_w$  and  $\Gamma(\mu)$ , for all  $w \in \mathfrak{S}(G)$  and all  $\mu \in M(G)$ ;*
- (ii) *the space  $\mathcal{U}$  is invariant under the mappings  $S_w$  and  $\text{Ad}\rho_r$ , for all  $w \in \mathfrak{S}(G)$  and all  $r \in G$ ;*
- (iii) *there exists a closed ideal  $J \subseteq A(G)$  such that  $\mathcal{U} = \text{Bim}(J^\perp)$ .*

*Proof.* (i) $\Rightarrow$ (ii) This follows by choosing  $\mu$  to be the point mass at  $r \in G$ .

(ii) $\Rightarrow$ (iii) Let  $V = \mathcal{U}_\perp$ ; then  $V$  is a norm closed subspace of  $T(G)$ , invariant under the maps of the form  $m_w$  ( $w \in \mathfrak{S}(G)$ ) and  $\theta_r$  ( $r \in G$ ). By Lemma 4.2, there exists a closed ideal  $J \subseteq A(G)$  such that  $V = \text{Sat}(J)$ . Hence  $\mathcal{U} = \text{Bim}(J^\perp)$  by Theorem 3.2.

(iii) $\Rightarrow$ (i) Let  $T \in \mathcal{U}$  and  $\mu \in M(G)$ . To show that  $\Gamma(\mu)(T) \in \mathcal{U}$  it suffices, by Theorem 3.2, to show that  $\langle \Gamma(\mu)(T), w \rangle_t = 0$  for every  $w \in \text{Sat}(J)$ . But,

if  $w \in \text{Sat}(J)$  then, for all  $r \in G$ ,

$$\langle \rho_r T \rho_r^*, w \rangle_t = \langle T, \theta_r(w) \rangle_t = 0$$

since  $\text{Sat}(J)$  is clearly  $\theta_r$ -invariant. It follows that

$$\langle \Gamma(\mu)(T), w \rangle_t = \int_G \langle \rho_r T \rho_r^*, w \rangle_t d\mu(r) = 0.$$

Since  $\mathcal{U} = \text{Bim}(J^\perp)$  is invariant under all Schur multipliers, the proof is complete.  $\square$

**Corollary 4.4.** *Let  $\mathcal{L}$  be the set of all weak\* closed subspaces of  $\mathcal{B}(L^2(G))$  which are invariant under both  $\mathfrak{S}(G)$  and  $\Gamma(M(G))$ . Then*

$$\mathcal{L} = \{\text{Bim}(J^\perp) : J \subseteq A(G) \text{ a closed ideal}\}$$

and  $\mathcal{L}$  is a lattice under the operations of intersection and closed linear span  $\vee$ . Moreover,

$$\begin{aligned} \text{Bim}(J_1^\perp) \cap \text{Bim}(J_2^\perp) &= \text{Bim}((J_1 + J_2)^\perp), \\ \text{Bim}(J_1^\perp) \vee \text{Bim}(J_2^\perp) &= \text{Bim}((J_1 \cap J_2)^\perp). \end{aligned}$$

*Proof.* The description of  $\mathcal{L}$  is contained in Theorem 4.3. The first identity will be proved in Proposition 6.1. The inclusion  $\text{Bim}(J_1^\perp) \vee \text{Bim}(J_2^\perp) \subseteq \text{Bim}((J_1 \cap J_2)^\perp)$  is trivial. The reverse inclusion follows directly from the definition of  $\text{Bim}$  and the fact that  $(J_1 \cap J_2)^\perp = \overline{J_1^\perp + J_2^\perp}^{w*}$ .  $\square$

**Lemma 4.5.** *Assume that  $A(G)$  has a (possibly unbounded) approximate identity. Then  $J^\perp = \text{Bim}(J^\perp) \cap \text{VN}(G)$  for every ideal  $J$  of  $A(G)$ .*

*Proof.* Let  $T \in \text{Bim}(J^\perp) \cap \text{VN}(G)$ . Since  $T$  is in  $\text{VN}(G)$ , it is of the form  $T = P^*(\tau)$  for some  $\tau \in A(G)^*$  (see relation (2)). By Theorem 3.2,  $T \in (\text{Sat}(J))^\perp$ . By Proposition 3.1, for all  $v \in J$  and all  $h \in T(G)$ ,

$$0 = \langle T, (N(v)h) \rangle_t = \langle P^*(\tau), N(v)h \rangle_t = \langle \tau, P(N(v)h) \rangle_a = \langle \tau, vP(h) \rangle_a$$

(using relation (4)). Since  $A(G)$  has an approximate identity and the map  $P : T(G) \rightarrow A(G)$  is surjective, there is a net  $(h_i)$  in  $T(G)$  such that  $vP(h_i) \rightarrow v$  in  $A(G)$ . It follows that  $\langle \tau, v \rangle_a = 0$  for all  $v \in J$  and therefore  $T \in J^\perp$  as claimed.  $\square$

**Remark 4.6.** It follows from Theorem 4.3 that  $\text{Bim}$  maps the set of all weak\* closed  $A(G)$ -invariant subspaces in  $\text{VN}(G)$  onto the set of all weak\* closed masa-bimodules in  $\mathcal{B}(L^2(G))$  which are invariant under conjugation by  $\rho_r$ ,  $r \in G$ .

Using Theorem 3.2 we see that  $\text{Sat}$  maps the set of all closed ideals of  $A(G)$  onto the set of all closed  $L^\infty(G)$ -bimodules in  $T(G)$  which are invariant under  $h \rightarrow h_r$ .

Combining this with Lemma 4.5 gives the following:

**Corollary 4.7.** *If  $A(G)$  has an approximate identity, the maps  $\text{Bim}$  and  $\text{Sat}$  are bijective.*

Note that the class of groups for which  $A(G)$  possesses an approximate identity contains, but is strictly larger than, the class of all amenable groups (see [20, Remark 4.5] for a relevant discussion). It is unknown whether this class contains all locally compact groups; it does contain those groups having the ‘approximation property’ of Haagerup and Kraus [14]. It is now known that there are groups failing the approximation property [19, 15].

**Question 4.8.** *Does the conclusion of Lemma 4.5 hold for an arbitrary second countable locally compact group?*

**Remark** Theorem 4.3 describes the class of all weak\* closed subspaces of  $\mathcal{B}(L^2(G))$  which are invariant under  $\Gamma(M(G))$  and under all Schur multipliers. If instead we consider only invariant Schur multipliers, namely  $N(M^{\text{cb}}A(G))$ , we obtain a strictly larger class; consider, for example,  $\text{VN}(G)$ .

## 5. THE EXTREMAL BIMODULES

In this section, we relate Theorem 3.2 to the extremal masa-bimodules associated with a subset of  $G \times G$  “of Toeplitz type”. We start by recalling some notions and results from [1] and [8] in the special case that we will use.

A subset  $E$  of  $G \times G$  is called  $\omega$ -open if it is marginally equivalent to the union of a countable set of Borel rectangles. The complements of  $\omega$ -open sets are called  $\omega$ -closed. If  $F \subseteq G \times G$  is an  $\omega$ -closed set, an operator  $T \in \mathcal{B}(L^2(G))$  is said to be *supported by*  $F$  if

$$(A \times B) \cap F \simeq \emptyset \implies P(B)TP(A) = 0,$$

for all measurable rectangles  $A \times B \subseteq G \times G$ , where  $P(A)$  denotes the orthogonal projection from  $L^2(G)$  onto  $L^2(A)$ . Given a masa-bimodule  $\mathcal{U}$ , there exists a smallest, up to marginal equivalence,  $\omega$ -closed subset  $F \subseteq G \times G$  such that every operator in  $\mathcal{U}$  is supported by  $F$ ; we call  $F$  the *support* of  $\mathcal{U}$ . Given an  $\omega$ -closed set  $F \subseteq G \times G$ , there exists [1] a largest weak\* closed masa-bimodule  $\mathfrak{M}_{\max}(F)$  and a smallest weak\* closed masa-bimodule  $\mathfrak{M}_{\min}(F)$  with support  $F$ . The masa-bimodule  $\mathfrak{M}_{\max}(F)$  is the space of all  $T \in \mathcal{B}(L^2(G))$  supported on  $F$ . We say that an  $\omega$ -closed set  $F \subseteq X \times Y$  satisfies *operator synthesis* if  $\mathfrak{M}_{\max}(F) = \mathfrak{M}_{\min}(F)$ .

Let  $F$  be an  $\omega$ -closed subset of  $G \times G$ ; define

$$\Phi(F) = \{\psi \in T(G) : \psi \chi_F = 0 \text{ m.a.e.}\}$$

and

$$\Psi(F) = \overline{\{\psi \in T(G) : \psi \text{ vanishes on an } \omega\text{-open neighbourhood of } F\}}^{\|\cdot\|_t}.$$

It was shown in [28] that  $\Phi(F)^\perp = \mathfrak{M}_{\min}(F)$  and  $\Psi(F)^\perp = \mathfrak{M}_{\max}(F)$ . For an  $L^\infty(G)$ -bimodule  $V$  in  $T(G)$ , we let  $\text{null}(V)$  be the largest, up to marginal equivalence,  $\omega$ -closed subset  $F$  of  $G \times G$  such that  $h|_F = 0$  for all  $h \in V$  [28]. Then a closed  $L^\infty(G)$ -bimodule  $V \subseteq T(G)$  satisfies  $\Psi(F) \subseteq V \subseteq \Phi(F)$  if and only if  $\text{null}(V) = F$ .

For a closed set  $E \subseteq G$ , let

$$I(E) = \{f \in A(G) : f(x) = 0, x \in E\} \quad \text{and} \quad J(E) = \overline{J_0(E)},$$

where  $J_0(E) = \{f \in A(G) : f \text{ vanishes on a neighbourhood of } E\}$ .

If  $J \subseteq A(G)$  is a closed ideal, denote by  $Z(J)$  the set of common zeroes of functions in  $J$ :

$$Z(J) = \{s \in G : f(s) = 0 \text{ for all } f \in J\}.$$

Then  $J(E) \subseteq J \subseteq I(E)$  if and only if  $Z(J) = E$ . If  $J(E) = I(E)$  then one says that  $E$  satisfies *spectral synthesis*.

For a subset  $E \subseteq G$ , we set

$$E^* = \{(s, t) : ts^{-1} \in E\}.$$

The relation between the notions of a null set and a zero set is described in the next proposition.

**Proposition 5.1.** *Let  $J \subseteq A(G)$  be a closed ideal. Then  $\text{null}(\text{Sat}(J)) = (Z(J))^*$ .*

*Proof.* Let  $E = Z(J)$ . By the definition of  $\text{Sat}(J)$ , using [28, Lemma 2.1] every element  $h$  of  $\text{Sat}(J)$  is a m.a.e. limit of a sequence of finite sums of elements of the form  $\phi_i N(u_i) \chi_{L_i \times L_i}$ , where  $\phi_i \in \mathfrak{S}(G)$ ,  $u_i \in J$  and  $L_i$  is a compact subset of  $G$ . But  $N(u) \chi_{L \times L}$  vanishes on  $E^*$  for every  $u \in J$  and every compact subset  $L$  of  $G$ . Thus  $h$  vanishes m.a.e. on  $E^*$  and it follows that  $E^* \subseteq \text{null Sat}(J)$ .

Conversely, suppose that  $(s, t) \notin E^*$ , that is,  $ts^{-1} \notin E$ . Let  $U$  be a compact neighbourhood of  $ts^{-1}$  disjoint from  $E$  and let  $v \in A(G)$  be a function vanishing on an open neighbourhood of  $E$  such that  $v|_U = 1$  [9, Lemma 3.2]. Then  $v \in J$  and, if  $K, L \subseteq G$  are compact neighbourhoods of  $s$  and  $t$  such that  $LK^{-1} \subseteq U$ , then  $(Nv) \chi_{U \times U}$  takes the value 1 on  $K \times L$ . Thus every  $(s, t) \notin E^*$  has a relatively compact open neighbourhood  $W_{(s,t)} \subseteq K \times L$  disjoint from  $\text{null Sat}(J)$  up to a marginally null set. Taking a countable subcover of the cover  $\{W_{(s,t)}\}$  of  $(E^*)^c$  we obtain an  $\omega$ -open neighbourhood of  $(E^*)^c$  disjoint from  $\text{null Sat}(J)$  up to a marginally null set. It follows that  $\text{null Sat}(J) \subseteq E^*$  up to a marginally null set and the proof is complete.  $\square$

**Corollary 5.2.** *Let  $V \subseteq T(G)$  be a norm closed  $L^\infty(G)$ -bimodule such that if  $h \in V$  then  $h_r \in V$ , for every  $r \in G$ . Then there exists a closed subset  $E \subseteq G$  such that  $\text{null } V = E^*$ .*

*Proof.* By Lemma 4.2, there exists an ideal  $J \subseteq A(G)$  such that  $V = \text{Sat}(J)$ . Let  $E = Z(J)$ ; by Proposition 5.1,  $\text{null } V = E^*$ .  $\square$

**Theorem 5.3.** *Let  $E \subseteq G$  be a closed set. The following hold:*

- (i)  $\text{Bim}(I(E)^\perp) = \mathfrak{M}_{\min}(E^*)$ ;
- (ii)  $\text{Bim}(J(E)^\perp) = \mathfrak{M}_{\max}(E^*)$ .

*Proof.* (i) By Theorem 3.2,  $\text{Bim}(I(E)^\perp)_\perp = \text{Sat}(I(E))$ . By Proposition 5.1,  $\text{null Sat}(I(E)) = E^*$  and hence  $\mathfrak{M}_{\min}(E^*) \subseteq \text{Bim}(I(E)^\perp)$  by the minimality of  $\mathfrak{M}_{\min}(E^*)$ .

To prove the reverse inclusion, let  $T = T_\tau \in I(E)^\perp$  and  $w \in \mathfrak{S}(G)$ . If  $h \in T(G)$  vanishes on  $E^*$  then  $wh$  vanishes on  $E^*$ . Relation (3) now shows that  $P(wh) \in I(E)$  and so

$$\langle S_w(T), h \rangle_t = \langle T, wh \rangle_t = \langle \tau, P(wh) \rangle_a = 0,$$

showing that  $S_w(T) \in \Phi(E^*)^\perp = \mathfrak{M}_{\min}(E^*)$ . Thus,  $\text{Bim}(I(E)^\perp) \subseteq \mathfrak{M}_{\min}(E^*)$  and the proof is complete.

(ii) Observe that for each  $v \in J_0(E)$  and each compact set  $L \subseteq G$ , the element  $N(v)\chi_{L \times L}$  is in  $\Psi(E^*)$ . By continuity of the map  $N$ , the same holds for  $v \in J(E)$ . It follows from the definition of  $\text{Sat}(J(E))$  that  $\text{Sat}(J(E)) \subseteq \Psi(E^*)$ .

On the other hand, by Proposition 5.1,  $\text{null Sat}(J(E)) = E^*$  and, since  $\text{Sat}(J(E))$  is a closed  $L^\infty(G)$ -bimodule, the minimality property of  $\Psi(E^*)$  shows that  $\Psi(E^*) \subseteq \text{Sat}(J(E))$ .

Hence  $\text{Sat}(J(E)) = \Psi(E^*)$ . By [28] and Theorem 3.2,  $\text{Bim}(J(E)^\perp) = \mathfrak{M}_{\max}(E^*)$ .  $\square$

It is worthwhile to isolate the following characterisation of reflexive jointly invariant subspaces, which is an immediate consequence of Theorem 4.3.

**Theorem 5.4.** *Let  $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$  be a reflexive subspace. Then  $\mathcal{U}$  is invariant under all mappings  $S_w, w \in \mathfrak{S}(G)$  and  $\text{Ad}\rho_r, r \in G$ , if and only if there exists a closed set  $E \subseteq G$  such that  $\mathcal{U} = \mathfrak{M}_{\max}(E^*)$ .*

As a corollary to Theorem 5.3, we obtain the following result of [20]:

**Theorem 5.5** ([20]). *Assume that  $A(G)$  has an approximate identity. Then a closed set  $E \subseteq G$  satisfies spectral synthesis if and only if the set  $E^* \subseteq G \times G$  satisfies operator synthesis.*

*Proof.* Assume  $E$  satisfies spectral synthesis. Then  $I(E) = J(E)$  and it follows from Theorem 5.3 that  $\mathfrak{M}_{\min}(E^*) = \mathfrak{M}_{\max}(E^*)$ . Conversely, if  $\mathfrak{M}_{\min}(E^*) = \mathfrak{M}_{\max}(E^*)$  then, by Theorem 5.3,  $\text{Bim}(I(E)^\perp) = \text{Bim}(J(E)^\perp)$  and now, by Lemma 4.5,  $I(E) = J(E)$ .  $\square$

**Question 5.6.** *Let  $E \subseteq G$  be a closed subset. Is every weak\* closed masa-bimodule  $\mathcal{U}$  with  $\mathfrak{M}_{\min}(E^*) \subseteq \mathcal{U} \subseteq \mathfrak{M}_{\max}(E^*)$  of the form  $\mathcal{U} = \text{Bim}(J^\perp)$  for some closed ideal  $J \subseteq A(G)$ ?*

In view of Theorem 4.3, the above question asks, in other words, whether there exist closed sets  $E \subseteq G$  such that  $E^*$  supports a weak\* closed masa-bimodule not invariant under conjugation by the unitaries  $\rho_s, s \in G$  (see Section 4). If such a set exists, it will necessarily be non-synthetic; for if  $E$  is synthetic, then Theorem 5.5 gives  $\mathfrak{M}_{\min}(E^*) = \mathfrak{M}_{\max}(E^*)$  and consequently no such bimodule exists.



## 6. RELATIVE SYNTHESIS

In this section, we obtain an extension of Theorem 5.5 which links relative spectral synthesis to relative operator synthesis. Theorem 6.2 was proved by K. Parthasarathy and R. Prakash in [26, Theorem 4.6] under the assumption that  $\mathcal{X}$  is an  $A(G)$ -invariant subspace of  $\text{VN}(G)$  and  $G$  is compact. In our result we assume that  $\mathcal{X}$  is weak\* closed and  $A(G)$  possesses an approximate identity.

We recall the relevant definitions. Let  $\mathcal{X} \subseteq \text{VN}(G)$  be an  $A(G)$ -invariant subspace. A closed subset  $E \subseteq G$  is called  $\mathcal{X}$ -spectral [17] if  $\mathcal{X} \cap J(E)^\perp = \mathcal{X} \cap I(E)^\perp$ . This notion has a natural operator theoretic version: if  $\mathcal{U}$  is a weak\* closed masa-bimodule and  $F$  is an  $\omega$ -closed set, we say that  $F$  is  $\mathcal{U}$ -operator synthetic if  $\mathcal{U} \cap \mathfrak{M}_{\min}(F) = \mathcal{U} \cap \mathfrak{M}_{\max}(F)$ . The latter notion was defined in [26] for subsets of  $G \times G$ , where  $G$  is a compact group.

**Proposition 6.1.** (i) Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be weak\* closed invariant subspaces of  $\text{VN}(G)$ . Then  $\text{Bim}(\mathcal{X}_1 \cap \mathcal{X}_2) = \text{Bim}(\mathcal{X}_1) \cap \text{Bim}(\mathcal{X}_2)$ .

(ii) Let  $J_1$  and  $J_2$  be closed ideals of  $A(G)$ . Then  $\text{Sat}(J_1) \cap \text{Sat}(J_2) = \text{Sat}(J_1 \cap J_2)$ .

*Proof.* (i) Let  $J_i \subseteq A(G)$  be a closed ideal with  $J_i^\perp = \mathcal{X}_i$ ,  $i = 1, 2$ . Let  $J = \overline{J_1 + J_2}$ ; then  $J$  is the smallest closed ideal of  $A(G)$  containing both  $J_1$  and  $J_2$ . Note that

$$(14) \quad \text{Sat}(J_1)^\perp \cap \text{Sat}(J_2)^\perp \subseteq \text{Sat}(J)^\perp.$$

Indeed, if  $T \in \mathcal{B}(L^2(G))$  annihilates both  $\text{Sat}(J_1)$  and  $\text{Sat}(J_2)$  then, by Proposition 3.1,  $T$  annihilates  $N(J_1)T(G)$  and  $N(J_2)T(G)$ , hence their sum; continuity of the map  $N$  shows that  $T$  must annihilate  $N(J)T(G)$  and hence  $\text{Sat}(J)$ .

It is obvious that  $\text{Bim}(\mathcal{X}_1 \cap \mathcal{X}_2) \subseteq \text{Bim}(\mathcal{X}_1) \cap \text{Bim}(\mathcal{X}_2)$ . Suppose that  $T \in \text{Bim}(\mathcal{X}_1) \cap \text{Bim}(\mathcal{X}_2)$ . By Theorem 3.2 and (14),  $T \in \text{Bim}(J^\perp)$ . But  $\text{Bim}(J^\perp) \subseteq \text{Bim}(J_1^\perp \cap J_2^\perp) = \text{Bim}(\mathcal{X}_1 \cap \mathcal{X}_2)$ , since  $J_i \subseteq J$ ,  $i = 1, 2$ . Thus,  $T \in \text{Bim}(\mathcal{X}_1 \cap \mathcal{X}_2)$  and the proof is complete.

(ii) It follows from Corollary 4.4 and Theorem 3.2 that

$$\overline{\text{Sat}(J_1)^\perp + \text{Sat}(J_2)^\perp}^{w*} = \text{Sat}(J_1 \cap J_2)^\perp.$$

Taking pre-annihilators, the result follows.  $\square$

**Theorem 6.2.** Assume that  $A(G)$  possesses an approximate identity. Let  $E \subseteq G$  be a closed set,  $\mathcal{X} \subseteq \text{VN}(G)$  a weak\* closed invariant subspace and  $\mathcal{U} = \text{Bim}(\mathcal{X})$ . The following are equivalent:

- (i)  $E$  is  $\mathcal{X}$ -spectral;
- (ii)  $E^*$  is  $\mathcal{U}$ -operator synthetic.

*Proof.* Suppose  $E$  is  $\mathcal{X}$ -spectral. Then  $\mathcal{X} \cap J(E)^\perp = \mathcal{X} \cap I(E)^\perp$ . By Proposition 6.1 and Theorem 5.3,  $\mathcal{U} \cap \mathfrak{M}_{\min}(E^*) = \mathcal{U} \cap \mathfrak{M}_{\max}(E^*)$ . Thus,  $E^*$  is  $\mathcal{U}$ -operator synthetic.

Conversely, suppose  $E^*$  is  $\mathcal{U}$ -operator synthetic. Then

$$\mathcal{U} \cap \mathfrak{M}_{\min}(E^*) \cap \text{VN}(G) = \mathcal{U} \cap \mathfrak{M}_{\max}(E^*) \cap \text{VN}(G).$$

By Lemma 4.5 and Theorem 5.3,  $\mathcal{X} \cap J(E) = \mathcal{X} \cap I(E)$ , that is,  $E$  is  $\mathcal{X}$ -spectral.  $\square$

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#### REFERENCES

- [1] W.B. ARVESON, *Operator algebras and invariant subspaces*, Ann. Math. (2) 100 (1974), 433-532.
- [2] D. BLECHER AND R. R. SMITH, *The dual of the Haagerup tensor product*, J. Lond. Math. Soc. 45 (1992), 126-144.
- [3] M. BOŹEJKO AND G. FENDLER, *Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group*, Boll. Un. Mat. Ital. A (6) 2 (1984), no. 2, 297-302.
- [4] J. DE CANNIERE AND U. HAAGERUP, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. 107 (1985), no. 2, 455-500.
- [5] J. DIXMIER, *Les  $C^*$ -algèbres et leurs représentations*, Gauthier-Villars, 1996.
- [6] G. K. ELEFThERAKIS AND I. G. TODOROV, *Ranges of bimodule projections and reflexivity*, J. Funct. Anal. 262 (2012), no. 11, 4891-4915.
- [7] E. G. EFFROS AND Z-J. RUAN, *Operator spaces*, Oxford University Press, New York, 2000.
- [8] J.A. ERDOS, A. KATAVOLOS AND V.S. SHULMAN, *Rank one subspaces of bimodules over maximal abelian selfadjoint algebras*, J. Funct. Anal. 157 No.2 (1998), 554-587.
- [9] P. EYMARD, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181-236.
- [10] G. FOLLAND, *A course in abstract harmonic analysis*, CRC press, London, 1995.
- [11] J. FROELICH, *Compact operators, invariant subspaces and spectral synthesis*, J. Funct. Analysis 81 (1988), 1-37.
- [12] F. GHAMRANI, *Isometric representation of  $M(G)$  on  $B(H)$* , Glasgow Math. J. 23 (1982) no. 2, 119-122.
- [13] U. HAAGERUP, *Decomposition of completely bounded maps on operator algebras*, Odense University, Denmark (1980).
- [14] U. HAAGERUP AND J. KRAUS, *Approximation properties for group  $C^*$ -algebras and group von Neumann algebras*, Trans. Amer. Math. Soc. 344 (1994), no. 2, 667-699.
- [15] U. HAAGERUP AND T. DE LAAT, *Simple Lie groups without the approximation property*, Duke Math. J. 162 (2013), no. 5, 925-964.
- [16] P. JOLISSAINT, *A characterisation of completely bounded multipliers of Fourier algebras*, Colloquium Math. 63 (1992) 311-313.
- [17] E. KANIUTH AND A.T. LAU, *Spectral synthesis for  $A(G)$  and subspaces of  $\text{VN}(G)$* , Proc. Amer. Math. Soc. 129 (2001), no. 11, 3253-3263.
- [18] A. KATAVOLOS AND V. PAULSEN, *On the ranges of bimodule projections*, Canad. Math. Bull. 48 (2005), no. 1, 97-111.
- [19] V. LAFFORGUE AND M. DE LA SALLE, *Noncommutative  $L_p$ -spaces without the completely bounded approximation property*, Duke Math. J. 160 (2011), no. 1, 71-116.
- [20] J. LUDWIG AND L. TUROWSKA, *On the connection between sets of operator synthesis and sets of spectral synthesis for locally compact groups*, J. Funct. Anal. 233 (2006), 206-227.

- [21] M. NEUFANG, *Abstrakte harmonische Analyse und Modulhomomorphismen über von Neumann-Algebren*, PhD thesis, University of Saarland, Saarbrücken, 2000.
- [22] M. NEUFANG, ZH.-J. RUAN AND N. SPRONK, *Completely isometric representations of  $M_{cb}A(G)$  and  $UCB(\hat{G})^*$* , Trans. Amer. Math. Soc. 360 (2008), no. 3, 1133-1161.
- [23] V.V. PELLER, *Hankel operators in the perturbation theory of unitary and selfadjoint operators*, Funktsional. Anal. i Prilozhen. 19 (1985), no. 2, 37-51, 96.
- [24] V. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge University Press, Cambridge, 2002.
- [25] G. PISIER, *An introduction to the theory of operator spaces*, Cambridge University Press, Cambridge, 2003.
- [26] K. PARTHASARATHY AND R. PRAKASH, *Spectral synthesis and operator synthesis*, Studia Math. 177 (2006), no. 2, 173-181.
- [27] V. S. SHULMAN, I. G. TODOROV AND L. TUROWSKA, *Sets of multiplicity and closable multipliers of group algebras*, preprint, arXiv:1401.2620.
- [28] V. S. SHULMAN AND L. TUROWSKA, *Operator synthesis I: synthetic sets, bilattices and tensor algebras*, J. Funct. Anal. 209 (2004), 293-331.
- [29] R. R. SMITH AND N. SPRONK, *Representations of group algebras in spaces of completely bounded maps*, Indiana Univ. Math. J. 54 (2005), no. 3, 873-896.
- [30] N. SPRONK, *Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras*, Proc. London Math. Soc. (3) 89 (2004), 161-192.
- [31] N. SPRONK AND L. TUROWSKA, *Spectral synthesis and operator synthesis for compact groups*, J. London Math. Soc. (2) 66 (2002), 361-376.
- [32] E. STØRMER, *Regular abelian Banach algebras of linear maps of operator algebras*, J. Funct. Anal. 37 (1980), 331-373.

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