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Break Date Estimation for Models with Deterministic Structural Change

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Abstract

In this paper we consider estimating the timing of a break in level and/or trend when the order of integration and autocorrelation properties of the data are unknown. For stationary innovations, break point estimation is commonly performed by minimizing the sum of squared residuals across all candidate break points, using a regression of the levels of the series on the assumed deterministic components. For unit root processes, the obvious modification is to use a first differenced version of the regression, while a further alternative in a stationary autoregressive setting is to consider a GLS-type quasi-differenced regression. Given uncertainty over which of these approaches to adopt in practice, we develop a hybrid break fraction estimator that selects from the levels-based estimator, the first-difference-based estimator, and a range of quasi-difference-based estimators, according to which achieves the global minimum sum of squared residuals. We establish the asymptotic properties of the estimators considered, and compare their performance in practically relevant sample sizes using simulation. We find that the new hybrid estimator has desirable asymptotic properties and performs very well in finite samples, providing a reliable approach to break date estimation without requiring decisions to be made regarding the autocorrelation properties of the data.

Keywords: Break point estimation; Break in level; Break in trend; Local-to-zero breaks.

JEL Classification: C22.

1 Introduction

The recent literature is replete with analysis focusing on structural change in the trend function of a time series, motivated by the apparent prevalence of breaks in level and/or trend in macroeconomic time series; see, for example, Stock and Watson (1996, 1999, 2005) and Perron and Zhu (2005). Correct specification of a break in the deterministic trend path of a series is vital for modelling, estimation and forecasting efforts, and is crucial for achieving reliable unit root test inference (see, *inter alia*, Perron (1989)). Given that in most macroeconomic series, uncertainty also exists as to whether the underlying stochastic component is best modelled by a stationary (I(0)) or unit root (I(1)) process, much recent work (e.g. Harvey *et al.* (2009, 2010), Perron and Yabu (2009), Sayginsoy and Vogelsang (2011) and Vogelsang (1998)) has been directed at testing for the presence of structural break(s) when the true order of integration of the series is assumed unknown.

Of equal importance to the *presence* of a break in level and/or trend in a series is the related issue of the *timing* of the change, and it is the estimation of such break-points that this paper addresses. While a number of methods of break date estimation have been proposed in the literature, selection of an efficient break fraction estimator is complicated by the aforementioned fact that the order of integration is typically not known with certainty. In the context of stationary innovations, Bai (1994) and Bai and Perron (1998), *inter alia*, consider choosing the break date which corresponds to minimizing the sum of squared residuals, across all candidate break points, from a regression of the level of the series on the appropriate deterministic regressors. In a unit root setting, a more efficient approach is obtained by minimizing the sum of squared residuals from a first-differenced version of the relevant regression; see Harris *et al.* (2009). A further alternative, adopted by Carrion-i-Silvestre *et al.* (2009) in an assumed local-to-unity setting, is to again date the break according to the minimum residual sum squares, but using a quasi-differenced regression. Practitioners are then faced with choosing between a number of candidate break fraction estimators, inevitably without knowledge of the underlying integration properties of the series.

In this paper we focus on developing a minimum sum of squared residuals-based break fraction estimator that performs well across unit root and stationary processes, and in the stationary case, across a range of serial correlation structures. In common with recent literature on this topic, e.g. Perron and Zhu (2005) and Yang (2012), we view our analysis as complementary to methods of break detection, for two reasons. First, many testing procedures explicitly require an estimated break date, and the power of such break detection tests is inherently limited by the accuracy of the dating procedure. An accurate dating procedure is what this paper provides, hence our proposed estimator could feed into a number of break detection methods. Second, even for break detection procedures that do not require an *a priori* break date estimator (e.g. the exp-Wald statistic proposed by Perron and Yabu, 2009), interest still lies in the timing of the break should one be detected, therefore our proposed procedure is equally relevant there. The relevance also extends to unit root testing in the presence

of a break at an unknown time.

The outline of the paper is as follows. In section 2, we begin by establishing, using a local-to-zero break magnitude assumption, the asymptotic properties of break fraction estimators based on both quasi-differenced (which includes levels as a special case) and first-differenced regressions, and confirm that the former is to be preferred for a stationary series, and in general the latter for a unit root process. In section 3, we then develop a hybrid estimator which selects between the first-differenced-based estimator and a range of quasi-differenced-based estimators according to which achieves the global minimum sum of squared residuals. The large sample behaviour of the new estimator is also established. Section 4 demonstrates through a finite sample Monte Carlo analysis that the hybrid estimator performs extremely well across a wide range of possible DGPs, outperforming established break fraction estimators (which perform badly outside of their respective assumptions regarding the integration order of the data). The hybrid estimator is simple to compute, and is found to comprise a reliable approach to break date estimation without requiring *a priori* decisions to be made regarding the autocorrelation properties of the data. Section 5 concludes. Proofs of the main results are provided in the online Appendix to this paper.

In the following, ‘ $\lfloor \cdot \rfloor$ ’ denotes the integer part of its argument, ‘ \Rightarrow ’ denote weak convergence, and $1(\cdot)$ denotes the indicator function.

2 The model and standard break date estimators

We consider the following model allowing for a break in level and trend

$$y_t = \alpha_1 + \alpha_2 t + \gamma_1 DU_t(\tau^*) + \gamma_2 DT_t(\tau^*) + u_t, \quad t = 1, \dots, T, \quad (1)$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad t = 2, \dots, T \quad (2)$$

with $u_1 = \varepsilon_1$, where $DU_t(\tau^*) = 1(t > \lfloor \tau^* T \rfloor)$ and $DT_t(\tau^*) = 1(t > \lfloor \tau^* T \rfloor)(t - \lfloor \tau^* T \rfloor)$ with $\lfloor \tau^* T \rfloor$ the break point with associated break fraction τ^* and level and trend break magnitudes γ_1 and γ_2 , respectively. Here τ^* is unknown but satisfies $\tau^* \in \Lambda$, where $\Lambda = [\tau_L, \tau_U]$ with $0 < \tau_L < \tau_U < 1$. To make our theoretical developments as transparent as possible, we assume that the innovation process $\{\varepsilon_t\}$ of (2) is an *IID* sequence with variance ω_ε^2 and finite fourth moment. The partial sum process of $\{\varepsilon_t\}$ then satisfies a functional central limit theorem [FCLT],

$$T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} \varepsilon_t \Rightarrow \omega_\varepsilon W(r)$$

where $W(r)$ is a standard Brownian motion process on $[0, 1]$.

We consider two cases for the order of integration of the autoregressive process, u_t . The $I(0)$ case for u_t is represented by setting $|\rho| < 1$ in (2). In this situation the long run variance of u_t is given by $\omega_u^2 = \omega_\varepsilon^2 / (1 - \rho)^2$. Here we will also assume that $\gamma_1 = \gamma_{1,T} = \kappa_1 T^{-1/2}$ and $\gamma_2 = \gamma_{2,T} = \kappa_2 T^{-3/2}$. The $T^{-1/2}$ and $T^{-3/2}$ scalings provide

the appropriate Pitman drifts for the asymptotic analysis of break date estimators in this case. The I(1) case for u_t is represented by setting $\rho = 1$ in (2). Here we assume $\gamma_1 = \kappa_1$ and $\gamma_2 = \gamma_{2,T} = \kappa_2 T^{-1/2}$ which are now the appropriate scalings.¹ For future brevity, the two cases are summarized as follows:

Assumption I(0): $|\rho| < 1$ with $\gamma_1 = \gamma_{1,T} = \kappa_1 T^{-1/2}$ and $\gamma_2 = \gamma_{2,T} = \kappa_2 T^{-3/2}$.

Assumption I(1): $\rho = 1$ with $\gamma_1 = \kappa_1$ and $\gamma_2 = \gamma_{2,T} = \kappa_2 T^{-1/2}$.

We consider estimating τ^* by minimizing the residual sum of squares from a quasi-differenced version of (1), i.e.

$$\hat{\tau}_{\bar{\rho}} = \arg \min_{\tau \in \Lambda} S(\bar{\rho}, \tau)$$

where $S(\bar{\rho}, \tau)$ denotes the residual sum of squares from an OLS regression of $\mathbf{y}_{\bar{\rho}}$ on $\mathbf{Z}_{\bar{\rho}, \tau}$ with

$$\begin{aligned} \mathbf{y}_{\bar{\rho}} &= [y_1, y_2 - \bar{\rho}y_1, \dots, y_T - \bar{\rho}y_{T-1}]', \\ \mathbf{Z}_{\bar{\rho}, \tau} &= [\mathbf{x}_1, \mathbf{x}_2 - \bar{\rho}\mathbf{x}_1, \dots, \mathbf{x}_T - \bar{\rho}\mathbf{x}_{T-1}]' \text{ with } \mathbf{x}_t = [1, t, DU_t(\tau), DT_t(\tau)]'. \end{aligned}$$

If ρ were known, standard GLS-based efficiency considerations would lead us to set $\bar{\rho} = \rho$. For example, if $\rho = 0$, we would obtain $\hat{\tau}_0$ from the levels of y_t regressions as

$$\begin{aligned} \hat{\tau}_0 &= \arg \min_{\tau \in \Lambda} S(0, \tau) \\ &= \arg \min_{\tau \in \Lambda} \sum_{t=1}^T \{y_t - \hat{\alpha}_1 - \hat{\alpha}_2 t - \hat{\gamma}_1 DU_t(\tau) - \hat{\gamma}_2 DT_t(\tau)\}^2 \end{aligned}$$

while if $\rho = 1$, we would obtain $\hat{\tau}_1$ from the first differences of y_t regressions as

$$\begin{aligned} \hat{\tau}_1 &= \arg \min_{\tau \in \Lambda} S(1, \tau) \\ &= \arg \min_{\tau \in \Lambda} \sum_{t=2}^T \{\Delta y_t - \hat{\alpha}_2 - \hat{\gamma}_1 D_t(\tau) - \hat{\gamma}_2 DU_t(\tau)\}^2 \end{aligned}$$

with $D_t(\tau) = 1(t = \lfloor \tau T \rfloor + 1)$.²

In practice of course, the value of ρ is typically unknown, so we begin by establishing the asymptotic behaviour of different estimators under both I(0) and I(1) specifications. To this end, the next two theorems detail the large sample properties of $\hat{\tau}_{\bar{\rho}}$ for an arbitrary $\bar{\rho}$ where $-1 < \bar{\rho} \leq 1$.

¹Note that the appropriate ‘‘scaling’’ on the level break in the I(1) case is $O(1)$, and differs from the $O(T^{1/2})$ Pitman drift appropriate when testing for the presence of a level break in an I(1) process.

²Here the OLS estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are used in a generic sense. They are also functions of τ but we suppress this dependence for notational brevity.

Theorem 1 Under Assumption I(0),

(i) for $|\bar{\rho}| < 1$

$$\begin{aligned}\hat{\tau}_{\bar{\rho}} &\Rightarrow \arg \sup_{\tau \in \Lambda} [\mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*)]' \mathbf{B}_2(\tau)^{-1} [\mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*)] \\ &\equiv \mathcal{L}_0(\kappa'_1, \kappa'_2, \tau^*)\end{aligned}\quad (3)$$

where

$$\begin{aligned}\mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) &= \begin{bmatrix} \kappa'_1 G_{11}(\tau^*, \tau) + \kappa'_2 G_{21}(\tau^*, \tau) + W(1) - W(\tau) \\ \kappa'_1 G_{12}(\tau^*, \tau) + \kappa'_2 G_{22}(\tau^*, \tau) + \int_{\tau}^1 (r - \tau) dW(r) \end{bmatrix} \\ \mathbf{B}_1(\tau) &= \begin{bmatrix} (1 - \tau)(1 - 3\tau) & 6\tau(1 - \tau) \\ -\tau(1 - \tau)^2 & (1 - \tau)^2(1 + 2\tau) \end{bmatrix} \\ \mathbf{B}_2(\tau) &= \begin{bmatrix} \tau(1 - \tau)(1 - 3\tau + 3\tau^2) & \tau^2(2\tau - 1)(1 - \tau)^2/2 \\ \tau^2(2\tau - 1)(1 - \tau)^2/2 & \tau^3(1 - \tau)^3/3 \end{bmatrix} \\ \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*) &= \begin{bmatrix} \kappa'_1(1 - \tau^*) + \kappa'_2(1 - \tau^*)^2/2 + W(1) \\ \kappa'_1(1 - \tau^{*2})/2 + \kappa'_2(1 - \tau^*)^2(2 + \tau^*)/6 + \int_0^1 r dW(r) \end{bmatrix}\end{aligned}$$

with $\kappa'_i = \kappa_i/\omega_u$ and

$$\begin{aligned}G_{11}(\tau^*, \tau) &= \begin{cases} 1 - \tau & \tau \geq \tau^* \\ 1 - \tau^* & \tau \leq \tau^* \end{cases} \\ G_{21}(\tau^*, \tau) &= \begin{cases} (1 - \tau)(1 + \tau - 2\tau^*)/2 & \tau \geq \tau^* \\ (1 - \tau^*)^2/2 & \tau \leq \tau^* \end{cases} \\ G_{12}(\tau^*, \tau) &= \begin{cases} (1 - \tau)^2/2 & \tau \geq \tau^* \\ (1 - \tau^*)(1 + \tau^* - 2\tau)/2 & \tau \leq \tau^* \end{cases} \\ G_{22}(\tau^*, \tau) &= \begin{cases} (1 - \tau)^2(2 + \tau - 3\tau^*)/6 & \tau \geq \tau^* \\ (1 - \tau^*)^2(2 + \tau^* - 3\tau)/6 & \tau \leq \tau^* \end{cases},\end{aligned}$$

(ii) for $\bar{\rho} = 1$

$$\hat{\tau}_1 \Rightarrow \arg \sup_{\tau \in \Lambda} \lim_{T \rightarrow \infty} \Delta u_{\tau T+1}^2. \quad (4)$$

Remark 1 It is shown by (3) that $\hat{\tau}_{\bar{\rho}}$ has a limit distribution which is invariant to all $|\bar{\rho}| < 1$; this follows as a consequence of the asymptotic equivalence between levels and quasi-differenced regression estimates in this context (see Grenander and Rosenblatt, 1957). However, the distribution does depend on ρ via ω_u in κ'_i .

Remark 2 The corresponding limit (4) for $\hat{\tau}_{\bar{\rho}}$ with $\bar{\rho} = 1$ can only be written in an implicit form, because it only depends on the two disturbances $u_{\tau T+1}$ and $u_{\tau T}$ hence no FCLT is applicable. The more pertinent feature here, however, is that (4) does *not* involve τ^* . Hence, $\hat{\tau}_1$ can never be considered an effective estimator of τ^* under Assumption I(0).

Figure 1 provides histograms of the limit distribution $\mathcal{L}_0(\kappa'_1, \kappa'_2, \tau^*)$ for various combinations of non-zero values of κ_1 and κ_2 for the case of $\tau^* = 0.5$. Here we set $\rho = 0$ and $\omega_\varepsilon = 1$, such that $\omega_u = 1$ and hence $\kappa'_i = \kappa_i$. We approximate the limit functionals by normalized sums of 1,000 steps using normal $IID(0, 1)$ random variates. In the simulations here and in the remainder of the paper we set $\Lambda = [0.15, 0.85]$ and employ 10,000 Monte Carlo replications. All simulations were programmed in Gauss 9.0. The results are largely as we would expect; accuracy of $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$ as an estimator of τ^* , measured subjectively, improves with increasing κ_1 and/or κ_2 . When κ_1 is zero, $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$ has a bimodal and (near) symmetric distribution around $\tau^* = 0.5$, and when neither κ_1 or κ_2 are zero the distribution is bimodal but not symmetric; both these properties are consistent with the results documented by Perron and Zhu (2005) under an assumption of fixed magnitude (as opposed to local-to-zero) breaks.³

Theorem 2 *Under Assumption I(1),*

(i) for $|\bar{\rho}| < 1$

$$\hat{\tau}_{\bar{\rho}} \Rightarrow \arg \sup_{\tau \in \Lambda} [\mathbf{J}(\kappa''_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{K}(\kappa''_2, \tau^*)]' \mathbf{B}_2(\tau)^{-1} [\mathbf{J}(\kappa''_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{K}(\kappa''_2, \tau^*)] \quad (5)$$

where $\kappa''_2 = \kappa_2/\omega_\varepsilon$ and

$$\mathbf{J}(\kappa''_2, \tau^*, \tau) = \begin{bmatrix} \kappa''_2 G_{21}(\tau^*, \tau) + \int_\tau^1 W(r) dr \\ \kappa''_2 G_{22}(\tau^*, \tau) + \int_\tau^1 (r - \tau) W(r) dr \end{bmatrix}$$

$$\mathbf{K}(\kappa''_2, \tau^*) = \begin{bmatrix} \kappa''_2 (1 - \tau^*)^2 / 2 + \int_0^1 W(r) dr \\ \kappa''_2 (1 - \tau^*)^2 (2 + \tau^*) / 6 + \int_0^1 r W(r) dr \end{bmatrix}$$

with $\mathbf{B}_1(\tau)$, $\mathbf{B}_2(\tau)$, $G_{21}(\tau^*, \tau)$ and $G_{22}(\tau^*, \tau)$ as defined in Theorem 1,

(ii) for $\bar{\rho} = 1$

$$\hat{\tau}_1 \Rightarrow \arg \sup_{\tau \in \Lambda} \begin{bmatrix} \kappa''_1 H_1(\tau^*, \tau) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \kappa''_2 H_2(\tau^*, \tau) + \tau W(1) - W(\tau) \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & \tau(1 - \tau) \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} \kappa''_1 H_1(\tau^*, \tau) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \kappa''_2 H_2(\tau^*, \tau) + \tau W(1) - W(\tau) \end{bmatrix}$$

$$\equiv \mathcal{L}_1(\kappa''_1, \kappa''_2, \tau^*) \quad (6)$$

where $\kappa''_i = \kappa_i/\omega_\varepsilon$ and

$$H_1(\tau^*, \tau) = \begin{cases} 0 & \tau \neq \tau^* \\ 1 & \tau = \tau^* \end{cases}$$

$$H_2(\tau^*, \tau) = \begin{cases} \tau^*(1 - \tau) & \tau \geq \tau^* \\ \tau(1 - \tau^*) & \tau \leq \tau^* \end{cases} .$$

³We do not explore the consequences of different ρ or ω_ε here since these only effect the values of the κ'_i . Therefore, the same effect could always be obtained by maintaining $\rho = 0$ and $\omega_\varepsilon = 1$ and simply altering the κ_i appropriately.

Remark 3 It is shown by (5) that $\hat{\tau}_{\bar{\rho}}$ has a limit distribution which is invariant to all $|\bar{\rho}| < 1$. The more pertinent feature here, however, is that (5) does *not* involve κ_1 . Hence, $\hat{\tau}_{\bar{\rho}}$ with $|\bar{\rho}| < 1$ cannot be used to detect level breaks under Assumption I(1). Also, the limit expression (5) is very similar in structure to that of (3) - it is obtained from (3) by replacing $dW(r)$ with $W(r)$ and setting $\kappa'_1 = 0$, $\kappa'_2 = \kappa''_2$. In contrast, as would be expected given its appropriateness under Assumption I(1), the limit of $\hat{\tau}_1$ involves both κ_1 and κ_2 .

Figure 2 provides histograms of the limit distribution $\mathcal{L}_1(\kappa''_1, \kappa''_2, \tau^*)$ for various non-zero values of κ_1 and κ_2 when $\tau^* = 0.5$. Again $\omega_\varepsilon = 1$, so that $\kappa''_i = \kappa_i$. Once more we observe the accuracy of $\hat{\tau}_1$ improving with increasing κ_1 and/or κ_2 . Figure 3 gives the corresponding histograms for $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$; note that Figures 3(a) and 3(b) are identical, since $\kappa_2 = 0$ here and the limit does not depend on κ_1 (see Remark 3). For non-zero trend break magnitudes, $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$ detects the break with increasing accuracy as κ_2 rises, but a comparison with the corresponding histograms for $\hat{\tau}_1$ in Figure 2 shows that while it is competitive for $\kappa_2 = 5$ (although neither estimator could be considered anyway decent here), it is clearly inferior to $\hat{\tau}_1$ for $\kappa_2 = 15$. In related work, Yang (2012) considers the relative performance of levels- and first-differenced-based estimators for a model with unit root errors and a local break in trend only, showing that the levels estimator can outperform the first-differenced estimator for very small breaks. However, for such small break magnitudes in this region, both break point estimators display very poor accuracy (cf. Figures 2(c) and 3(c)), so the relative differences here are of limited practical importance.

3 A hybrid break date estimator

The above asymptotic results suggest, fairly unambiguously, that in constructing $\hat{\tau}_{\bar{\rho}}$, we should choose some $|\bar{\rho}| < 1$ if $|\rho| < 1$, and choose $\bar{\rho} = 1$ if $\rho = 1$. This follows since $\hat{\tau}_1$ cannot be considered a suitable estimator of the break fraction under Assumption I(0), and $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$ is effectively outperformed by $\hat{\tau}_1$ under Assumption I(1). However, given that we consider the true value of ρ to be unknown in practice, we now consider developing a hybrid break fraction estimator that selects between the $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$ and $\hat{\tau}_1$ possibilities depending on the sample's properties. To begin, if we consider just two possible values for $\bar{\rho}$: $\bar{\rho} = \rho'$ where $|\rho'| < 1$, and $\bar{\rho} = 1$, i.e. $\hat{\tau}_{\rho'}$ and $\hat{\tau}_1$ are the only possible estimators of τ^* , then we might consider selecting between $\hat{\tau}_{\rho'}$ or $\hat{\tau}_1$ according to which corresponds to the lowest residual sum of squares, i.e. choose

$$\begin{aligned} \hat{\tau}_{\rho'} & \text{ if } \min_{\tau \in \Lambda} S(\rho', \tau) < \min_{\tau \in \Lambda} S(1, \tau) \\ \hat{\tau}_1 & \text{ if } \min_{\tau \in \Lambda} S(\rho', \tau) > \min_{\tau \in \Lambda} S(1, \tau) \end{aligned} .$$

Another way of writing this is to define the hybrid estimator

$$\hat{\tau}_{D_2} = \arg \min_{\tau \in \Lambda, \bar{\rho} \in D_2} S(\bar{\rho}, \tau)$$

where $D_2 = \{\rho', 1\}$.⁴

To examine the asymptotic behaviour of this hybrid estimator $\hat{\tau}_{D_2}$, we first establish the limiting properties of $S(\bar{\rho}, \tau)$ under Assumption I(0) and Assumption I(1).

Theorem 3 *Under Assumption I(0), for $|\bar{\rho}| < 1$ and $\bar{\rho} = 1$*

$$T^{-1}S(\bar{\rho}, \tau) \Rightarrow \frac{\omega_\varepsilon^2}{1 - \rho^2}(1 + \bar{\rho}^2 - 2\bar{\rho}\rho)$$

for all τ .

Theorem 4 *Under Assumption I(1),*

(i) for $|\bar{\rho}| < 1$

$$T^{-2}S(\bar{\rho}, \tau) \Rightarrow Q(\bar{\rho}, \omega_\varepsilon^2, \kappa_2'', \tau, \tau^*)$$

where $Q(\cdot)$ is some non-degenerate distribution for all τ ,⁵

(ii) for $\bar{\rho} = 1$

$$T^{-1}S(1, \tau) \Rightarrow \omega_\varepsilon^2$$

for all τ .

If we first consider behaviour under Assumption I(1), where $\rho = 1$, it follows from Theorem 4 that $T^{-2}S(\rho', \tau)$ converges to a distribution while $T^{-1}S(1, \tau)$ converges to ω_ε^2 for all τ . Asymptotically then, $\min_{\tau \in \Lambda} S(\rho', \tau) > \min_{\tau \in \Lambda} S(1, \tau) > 0$. Next, under Assumption I(0), where $|\rho| < 1$, Theorem 3 implies that

$$\begin{aligned} T^{-1}S(\rho', \tau) - T^{-1}S(1, \tau) &\Rightarrow \frac{\omega_\varepsilon^2}{1 - \rho^2} \{(1 + \rho'^2 - 2\rho'\rho) - 2(1 - \rho)\} \\ &= \frac{\omega_\varepsilon^2}{1 - \rho^2} (1 - \rho') (2\rho - \rho' - 1) \end{aligned}$$

for all τ . Since $1 - \rho' > 0$ here, it follows that, asymptotically,

$$\begin{aligned} \min_{\tau \in \Lambda} S(\rho', \tau) &< \min_{\tau \in \Lambda} S(1, \tau) && \text{if } \rho < (\rho' + 1)/2 \\ \min_{\tau \in \Lambda} S(\rho', \tau) &> \min_{\tau \in \Lambda} S(1, \tau) && \text{if } \rho > (\rho' + 1)/2 \end{aligned} \quad (7)$$

Consequently, we find that $\hat{\tau}_{D_2} = \hat{\tau}_1$ asymptotically if $\rho = 1$ which is as we would desire. For $|\rho| < 1$, (7) shows us that $\hat{\tau}_{D_2} = \hat{\tau}_{\rho'}$, the desired outcome, unless $\rho > \rho'$ and ρ is closer to 1 than it is to ρ' , in which case $\hat{\tau}_{D_2} = \hat{\tau}_1$ which is the ineffective estimator in the I(0) case. By way of an example, suppose $|\rho| < 1$ and we set $\rho' = 0$ (so that $\hat{\tau}_{D_2}$ selects between the levels- and the first differences-based estimators $\hat{\tau}_0$ and $\hat{\tau}_1$); we find that $\hat{\tau}_{D_2} = \hat{\tau}_1$ (the ineffective estimator) in the region $\rho > 0.5$. If on the other hand

⁴That is, D_2 is a discrete set consisting of the two elements ρ' and 1.

⁵The precise form of $Q(\cdot)$ can easily be established from the results in the Appendix; we omit the details here since it is only the order of $S(\bar{\rho}, \tau)$ that proves important for our purposes.

we choose $\rho' = 0.9$, we find that $\hat{\tau}_{D_2} = \hat{\tau}_1$ now only in the region $\rho > 0.95$. Purely asymptotic considerations would therefore indicate that the problem region associated with $|\rho| < 1$ can be made arbitrarily small by setting $\rho' = 1 - \epsilon$, where $\epsilon > 0$ is made arbitrarily close to 0, thereby reducing the problem region to $\rho > 1 - \epsilon/2$.

Notwithstanding our asymptotic results under Assumption I(0), in finite samples the choice of ρ' will have a significant influence of the behaviour of $\hat{\tau}_{D_2}$ even when the condition $\rho < (\rho' + 1)/2$ is satisfied. Therefore, from a finite sample (i.e. empirical) perspective the idea of setting $\rho' = 1 - \epsilon$ is unlikely to prove an attractive proposition unless ρ is actually very close to 1, despite its asymptotic appeal.

As noted above, efficiency considerations suggest we should (infeasibly) always set $\bar{\rho} = \rho$. As a step in this direction we consider generalizing the hybrid estimator by at least allowing $\bar{\rho}$ to cover a subset of possible values for ρ , replacing the 2 element set D_2 with the m element set $D_m = \{\rho'_1, \rho'_2, \dots, \rho'_{m-1}, 1\}$ where $|\rho'_i| < 1$ for all i and, without loss of generality, $-1 < \rho'_1 < \rho'_2 < \dots < \rho'_{m-1} < 1$. Therefore, we now consider the following hybrid (pseudo GLS) estimator

$$\hat{\tau}_{D_m} = \arg \min_{\tau \in \Lambda, \bar{\rho} \in D_m} S(\bar{\rho}, \tau) \quad (8)$$

Note that $\hat{\tau}_{D_m}$ could equivalently be defined as choosing one of $\hat{\tau}_{\rho'_1}, \hat{\tau}_{\rho'_2}, \dots, \hat{\tau}_{\rho'_{m-1}}, \hat{\tau}_1$ according to which of these estimators achieves the smallest residual sum of squares.

The next Corollary is a useful initial step in explaining the limit behaviour of $\hat{\tau}_{D_m}$.

Corollary 5

(i) Under Assumption I(0),

$$\arg \min_{\bar{\rho} \in D_m} S(\bar{\rho}, \tau) \Rightarrow \begin{cases} \rho'_1 & \text{if } \rho < \rho'_1 \\ \arg \min_{\bar{\rho} \in D'_m} |\bar{\rho} - \rho| & \text{if } \rho'_1 \leq \rho \leq \rho'_{m-1} \\ \rho'_{m-1} & \text{if } \rho'_{m-1} < \rho < (\rho'_{m-1} + 1)/2 \\ 1 & \text{if } \rho > (\rho'_{m-1} + 1)/2 \end{cases}$$

for all τ .

(ii) Under Assumption I(1),

$$\arg \min_{\bar{\rho} \in D_m} S(\bar{\rho}, \tau) \Rightarrow 1$$

for all τ .

Part (ii) follows directly from Theorem 4, since $T^{-1}S(\bar{\rho}, \tau)$ diverges to $+\infty$ in T unless $\bar{\rho} = 1$. Part (i) follows from Theorem 3 since we have

$$\begin{aligned} \arg \min_{\bar{\rho} \in D_m} S(\bar{\rho}, \tau) &= \arg \min_{\bar{\rho} \in D_m} T^{-1}S(\bar{\rho}, \tau) \\ &\Rightarrow \arg \min_{\bar{\rho} \in D_m} \frac{\omega_\epsilon^2}{1 - \rho^2} (1 + \bar{\rho}^2 - 2\bar{\rho}\rho) \\ &= \arg \min_{\bar{\rho} \in D_m} \bar{\rho}^2 - 2\bar{\rho}\rho. \end{aligned}$$

Now

$$\frac{d(\bar{\rho}^2 - 2\bar{\rho}\rho)}{d\bar{\rho}} = 2(\bar{\rho} - \rho)$$

and so $\arg \min_{\bar{\rho} \in D_m} S(\bar{\rho}, \tau) \Rightarrow \arg \min_{\bar{\rho} \in D_m} |\bar{\rho} - \rho|$. When $\rho < \rho'_1$, clearly $\arg \min_{\bar{\rho} \in D_m} |\bar{\rho} - \rho| = \rho'_1$. Also, when $\rho'_1 \leq \rho \leq \rho'_{m-1}$, $\arg \min_{\bar{\rho} \in D_m} |\bar{\rho} - \rho| = \arg \min_{\bar{\rho} \in D'_m} |\bar{\rho} - \rho|$.⁶ For $\rho > \rho'_{m-1}$, $\arg \min_{\bar{\rho} \in D_m} |\bar{\rho} - \rho| = \rho'_{m-1}$ unless $\rho > (\rho'_{m-1} + 1)/2$ in which case $\arg \min_{\bar{\rho} \in D_m} |\bar{\rho} - \rho| = 1$. Notice that this condition coincides with that in the second part of (7) (on replacing ρ' with ρ'_{m-1}). Intuitively, in the limit, $\arg \min_{\bar{\rho} \in D_m} S(\bar{\rho}, \tau)$ is always the element of D_m closest to the true value of ρ .

The asymptotic behaviour of $\hat{\tau}_{D_m}$ can now be established under both Assumption I(0) and Assumption I(1).

Corollary 6

(i) Under Assumption I(0),

$$\hat{\tau}_{D_m} \Rightarrow \begin{cases} \arg \sup_{\tau \in \Lambda} \lim_{T \rightarrow \infty} \Delta u_{\tau T+1}^2 & \text{if } \rho > (\rho'_{m-1} + 1)/2 \\ \mathcal{L}_0(\kappa'_1, \kappa'_2, \tau^*) & \text{otherwise} \end{cases}.$$

(ii) Under Assumption I(1),

$$\hat{\tau}_{D_m} \Rightarrow \mathcal{L}_1(\kappa''_1, \kappa''_2, \tau^*).$$

For part (ii), Corollary 5 (ii) implies that, asymptotically, $\arg \min_{\tau \in \Lambda, \bar{\rho} \in D_m} S(\bar{\rho}, \tau) = \arg \min_{\tau \in \Lambda} S(1, \tau)$ which has the limit distribution $\mathcal{L}_1(\kappa''_1, \kappa''_2, \tau^*)$ of Theorem 2 (ii). Part (i) follows from Corollary 5 (i). Using the final limit of Corollary 5 (i) we find that when $\rho > (\rho'_{m-1} + 1)/2$ then, asymptotically, $\arg \min_{\tau \in \Lambda, \bar{\rho} \in D_m} S(\bar{\rho}, \tau) = \arg \min_{\tau \in \Lambda} S(1, \tau)$ which has the limit given in (4). Otherwise, using the first three limits of Corollary 5 (i) we find that, asymptotically, $\arg \min_{\tau \in \Lambda, \bar{\rho} \in D_m} S(\bar{\rho}, \tau) = \arg \min_{\tau \in \Lambda} S(h, \tau)$ where h is some element of $\{\rho'_1, \rho'_2, \dots, \rho'_{m-1}\}$. Since $|h| < 1$ the limit distribution is then $\mathcal{L}_0(\kappa'_1, \kappa'_2, \tau^*)$ of Theorem 1 (i).

We find, therefore, that under Assumption I(1), the hybrid estimator $\hat{\tau}_{D_m}$ has the same limit behaviour as $\hat{\tau}_1$, as we would wish, while under Assumption I(0), it has the same asymptotic properties as $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$, again as desired, unless ρ happens to lie in the problem region $\rho > (\rho'_{m-1} + 1)/2$. The hybrid estimator therefore has a clear asymptotic appeal, particularly if one makes the judicious choice of setting ρ'_{m-1} very close to 1. In the next section, we examine the behaviour of $\hat{\tau}_{D_m}$ in samples of practically relevant size.

Finally, our asymptotic results imply that $\hat{\tau}_{D_m}$ has the same asymptotic properties as the following sequential break fraction estimator: in the first step, minimize $S(\bar{\rho}, \tau)$

⁶Notice that if it happens to be the case that $\rho \in D_m$ then $\arg \min_{\bar{\rho} \in D'_m} |\bar{\rho} - \rho| = \rho$ (at which point the limit of $T^{-1}S(\bar{\rho}, \tau)$ is ω_ε^2).

across D_m for any single value of $\tau \in \Lambda$, in the second step, minimize $S(\bar{\rho}, \tau)$ across τ imposing the value of $\bar{\rho}$ obtained from the first step. Such a sequential approach, while asymptotically valid, is likely to perform rather poorly in finite samples, since it is entirely possible that two different choices for τ in the initial minimization will lead to two different $\bar{\rho}$ and, consequently, two different break fraction estimates. We therefore do not advocate use of this two step approach, instead recommending $\hat{\tau}_{D_m}$ as defined in (8).

4 Finite sample performance

In this section we compare the finite sample performance of the hybrid estimator $\hat{\tau}_{D_m}$ with the levels- and first-difference-based estimators $\hat{\tau}_0$ and $\hat{\tau}_1$. The simulation DGP is (1) and (2), with $u_1 = \varepsilon_1$, $\varepsilon_t \sim NIID(0, 1)$ and $\alpha_1 = \alpha_2 = 0$ (without loss of generality). As in the asymptotic simulations, we use $\tau^* = 0.5$. For cases where u_t is I(0), we set $\gamma_1 = \kappa_1 T^{-1/2}$, $\gamma_2 = \kappa_2 T^{-3/2}$ with $\kappa_1 = \{0, 10, 20, 30\}$ and $\kappa_2 = \{0, 250, 500, 750\}$; where u_t is I(1), we set $\gamma_1 = \kappa_1$, $\gamma_2 = \kappa_2 T^{-1/2}$ with $\kappa_1 = \{0, 1, 2, 3\}$ and $\kappa_2 = \{0, 5, 10, 15\}$. The combination $\kappa_1 = \kappa_2 = 0$ is provided as a no-break point of reference. We consider the sample sizes $T = 150$ and $T = 300$ and a range of values of ρ .

A choice must be made regarding D_m . Here we set $D_m = \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.975, 1\}$. This choice is motivated by two empirical observations regarding economic time series. The first is that serial correlation is not usually found to be negative, so that we exclude negative values of $\bar{\rho}$. The second is that the serial correlation is often found to be very strongly positive (as exemplified by the ongoing I(0)/I(1) debate), so we include some large values of $\bar{\rho} < 1$ as well as $\bar{\rho} = 1$; moreover, inclusion of the value 0.975 confines the problem region discussed above to the small interval region $0.9875 < \rho < 1$.

For further comparison, we also examine an AR(1)-based estimator of τ^* . This is calculated from minimizing the residual sum of squares from the fitted OLS regression

$$y_t = \beta_1 + \beta_2 t + \phi y_{t-1} + \pi_1 D_t(\tau) + \pi_2 DU_t(\tau) + \pi_3 DT_t(\tau) + e_t, \quad t = 2, \dots, T,$$

across τ . Here the one-time dummy variable $D_t(\tau)$ is included to identify a level break in the I(1) case (corresponding here to $\phi = 1$). We denote this estimator as $\hat{\tau}_{AR}$.

With four different break fraction estimators, sixteen combinations of break magnitude and two sample sizes, it is not practical to show full histograms across different values of ρ . Instead, in Tables 1-5, we simply provide the empirical probability that each estimator lies in the range $\tau^* \pm 0.010$, $\tau^* \pm 0.025$ and $\tau^* \pm 0.050$. Other things equal, the larger these probabilities, then the better the estimator.

Table 1 concerns the case of I(0), white noise errors, a situation where $\hat{\tau}_0$ represents the optimal estimator. What is immediately apparent is that $\hat{\tau}_1$ is by some considerable margin the poorest performing estimator across all κ_1 and κ_2 . When $T = 150$ it does have some ability to detect the larger breaks, but even then remains much inferior to the other three; for $T = 300$ its performance levels for non-zero κ_1 and κ_2 are similar

to the no-break reference case, in line with the result of Theorem 1 (ii) which showed that $\hat{\tau}_1$ is asymptotically ineffective in this setting. We also see that, on balance, $\hat{\tau}_{AR}$ does not perform as well as either $\hat{\tau}_0$ or $\hat{\tau}_{D_m}$. It is competitive when there is a pure trend break, but loses out everywhere else. The estimators $\hat{\tau}_0$ and $\hat{\tau}_{D_m}$ show almost identical behaviour everywhere, highlighting the attractive performance of the hybrid estimator in this white noise case.

In Table 2 the errors are I(0), AR(1) with $\rho = 0.5$. The comparative behaviour of the estimators remains pretty much the same as seen in Table 1, only with the differences between worst and best being slightly less emphasized. Table 3 considers the case of I(0), AR(1) errors with $\rho = 0.925$. Here, the stronger autoregressive component begins to diminish the ability of the estimators to identify the true break point, other than $\hat{\tau}_1$, which improves relative to the $\rho = 0.5$ case. However, rather encouragingly, it is the hybrid estimator $\hat{\tau}_{D_m}$ that shows the best performance overall.

Table 4 represents a problem case of I(0) but near I(1) errors, with $\rho = 0.994$, this value of ρ being chosen to lie in the middle of the asymptotic problem region for $\hat{\tau}_{D_m}$. For $T = 150$, perhaps not surprisingly, given the strength of the autoregressive component, $\hat{\tau}_1$ is generally the best performing estimator here, although again $\hat{\tau}_{D_m}$ performs very well and is a close competitor to $\hat{\tau}_1$, comfortably out-performing $\hat{\tau}_{AR}$ and $\hat{\tau}_0$. When $T = 300$, although the probabilities associated with all the procedures are lower, it could be argued that $\hat{\tau}_1$ is still the best performing estimator (despite its asymptotic shortcomings); once again, $\hat{\tau}_{D_m}$ behaves very similarly to $\hat{\tau}_1$ here, so it appears that the asymptotic problem region associated with $\hat{\tau}_{D_m}$ is unlikely to be of much concern in any practical setting.

Table 5 shows the results for I(1), random walk errors, where $\hat{\tau}_1$ now assumes the role of the optimal estimator. The worst performing estimator is now $\hat{\tau}_0$, except for the pure trend break case, where $\hat{\tau}_{AR}$ performs most poorly. Both $\hat{\tau}_1$ and $\hat{\tau}_{D_m}$ are very clearly better performing estimators and behave very similarly everywhere. The results for $T = 300$ also show that the probabilities associated with $\hat{\tau}_0$ are largely insensitive to κ_1 , and close to the no-break reference case when $\kappa_2 = 0$, which accords with our asymptotic results in Theorem 2 (i).

Results for additional values of $0 < \rho < 1$ are reported in Harvey and Leybourne (2013). These further highlight the attractive properties of $\hat{\tau}_{D_m}$ discussed above; in particular, we find that as the value of ρ increases further towards unity, results for $\hat{\tau}_{D_m}$ become increasingly similar to those for $\hat{\tau}_1$, with both these estimators outperforming $\hat{\tau}_0$ and $\hat{\tau}_{AR}$. This discussion paper also presents results for where ε_t follows an MA(1) process. In summary, when the errors are ARMA(0,1), the behaviour of the estimators is actually qualitatively similar to that in Table 1, hence similar comments made above for this case apply here also. When the errors are ARIMA(0,1,1), the rankings of $\hat{\tau}_0$, $\hat{\tau}_1$ and $\hat{\tau}_{AR}$ appear quite highly dependent on the particular κ_1 , κ_2 settings; what is clear throughout, however, is that $\hat{\tau}_{D_m}$ either performs the best, or if not, then virtually always as well as the highest ranking of the other three estimators.

Finally, we also experimented with a finer grid of values for D_m , re-running all the finite sample simulations with $D_m = \{0, 0.01, 0.02, \dots, 0.99, 1\}$. We found the re-

sults to be almost identical to those obtained using our recommended grid, with the probabilities being within ± 0.01 of the values reported in the tables.

5 Conclusion

In summary, we have considered the asymptotic and finite sample performance of a number of minimum sum of squared residuals-based break fraction estimators. We first considered the asymptotic performance of estimators based on a levels or quasi-differenced regression of y_t on the relevant deterministic components, and also an estimator based on a first differenced version of the regression. It was found that the levels/quasi-differenced approach performed well under an assumption of $I(0)$ errors, while the first differenced-based estimator was inappropriate in this context. Essentially the reverse was observed under $I(1)$ errors, with the first differenced approach now preferred. Given this inherent lack of robustness in the performance of the estimators across $I(0)$ and $I(1)$ environments, we proposed a hybrid estimator, $\hat{\tau}_{D_m}$, which selects between the first differenced estimator and a number of quasi-differenced alternatives according to which achieves the smallest minimum sum of squared residuals.

This new procedure was found to achieve most of the desirable properties of the appropriate estimators for the stationary and unit root worlds, without the inherent downsides involved in selecting purely one approach. This finding was also shown to carry over to sample sizes of practical relevance, with the hybrid estimator always competitive with the better of the levels- and first differenced-based estimators across a range of $I(0)$ and $I(1)$ data generating processes. Indeed, the qualitative behaviour of $\hat{\tau}_{D_m}$ would extend to more general dynamic processes, since the autoregressive filtering inherent in $\hat{\tau}_{D_m}$ is only ever intended to remove the dominant autoregressive behaviour present in a series; there is no need to whiten the series entirely.

An alternative approach to constructing $\hat{\tau}_{D_m}$ using a discrete number of quasi-difference parameters in D_m would be to use all values in the continuous set $(-1, 1]$, i.e. modify the pseudo GLS hybrid estimator (8) to instead minimize $S(\bar{\rho}, \tau)$ over $\tau \in \Lambda$ and $\bar{\rho} \in (-1, 1]$. Such an approach would represent a considerable increase in the computational burden of the procedure, due to the requirement for numerical Newton-type minimization methods. Moreover, marginal changes in $\bar{\rho}$ have an almost negligible effect on the resulting quasi-differenced break fraction estimator, and so implementing $\hat{\tau}_{D_m}$ using a reasonably fine set of discrete values for D_m (as in our simulations) is entirely sufficient.

In practical applications, where knowledge of the integration order of the stochastic component of a series cannot typically be taken as known, it is desirable to have available a break fraction estimator that works well without having to take a potentially incorrect and therefore costly stand on the data's order of integration. We consider that the hybrid estimator $\hat{\tau}_{D_m}$ proposed in this paper goes a long way in fulfilling this role, and should therefore have practical appeal; moreover, extension of the hybrid procedure to the case of estimating the timing of multiple breaks is entirely straightforward.

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TABLE 1

Probability of break fraction estimators lying in the range $\tau^ \pm \xi$; $\rho = 0$*

Panel A. $T = 150$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.02	0.03	0.03	0.02	0.05	0.06	0.06	0.05	0.11	0.13	0.13	0.11
	250	0.09	0.03	0.15	0.09	0.26	0.07	0.35	0.25	0.63	0.15	0.69	0.63
	500	0.18	0.04	0.25	0.18	0.50	0.09	0.56	0.50	0.93	0.19	0.92	0.93
	750	0.28	0.06	0.33	0.27	0.70	0.13	0.68	0.70	0.99	0.26	0.98	0.99
10	0	0.29	0.04	0.18	0.29	0.44	0.08	0.32	0.44	0.59	0.15	0.46	0.59
	250	0.37	0.04	0.25	0.37	0.55	0.09	0.44	0.55	0.70	0.16	0.66	0.69
	500	0.44	0.06	0.32	0.43	0.61	0.11	0.55	0.61	0.86	0.20	0.83	0.86
	750	0.49	0.09	0.37	0.49	0.69	0.16	0.64	0.68	0.96	0.28	0.94	0.96
20	0	0.73	0.09	0.61	0.73	0.89	0.12	0.82	0.89	0.96	0.19	0.93	0.96
	250	0.74	0.10	0.59	0.74	0.87	0.14	0.77	0.87	0.93	0.20	0.86	0.93
	500	0.75	0.12	0.59	0.74	0.86	0.17	0.76	0.86	0.90	0.25	0.86	0.90
	750	0.76	0.16	0.60	0.75	0.85	0.23	0.76	0.85	0.94	0.34	0.91	0.94
30	0	0.92	0.18	0.87	0.92	0.99	0.22	0.98	0.99	1.00	0.27	1.00	1.00
	250	0.93	0.20	0.87	0.93	0.98	0.24	0.96	0.98	0.99	0.30	0.98	0.99
	500	0.92	0.24	0.86	0.92	0.97	0.29	0.94	0.97	0.98	0.36	0.97	0.98
	750	0.92	0.30	0.85	0.91	0.96	0.36	0.92	0.96	0.98	0.45	0.96	0.98
Panel B. $T = 300$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.02	0.02	0.02	0.02	0.06	0.07	0.07	0.06	0.11	0.14	0.13	0.11
	250	0.07	0.03	0.13	0.07	0.27	0.07	0.38	0.27	0.62	0.15	0.70	0.62
	500	0.12	0.03	0.22	0.12	0.52	0.08	0.61	0.52	0.93	0.16	0.93	0.93
	750	0.18	0.03	0.29	0.18	0.71	0.09	0.76	0.71	0.99	0.17	0.99	0.99
10	0	0.26	0.03	0.16	0.26	0.46	0.08	0.33	0.46	0.59	0.15	0.46	0.59
	250	0.32	0.03	0.22	0.32	0.54	0.08	0.46	0.54	0.69	0.15	0.67	0.69
	500	0.36	0.03	0.28	0.36	0.61	0.08	0.59	0.61	0.87	0.16	0.88	0.87
	750	0.40	0.04	0.33	0.40	0.68	0.09	0.69	0.68	0.97	0.17	0.96	0.97
20	0	0.68	0.04	0.56	0.68	0.90	0.09	0.84	0.90	0.96	0.16	0.93	0.96
	250	0.68	0.04	0.53	0.68	0.87	0.09	0.77	0.87	0.92	0.16	0.85	0.92
	500	0.68	0.05	0.52	0.68	0.86	0.10	0.74	0.86	0.90	0.17	0.87	0.90
	750	0.70	0.05	0.52	0.69	0.84	0.11	0.75	0.84	0.95	0.19	0.94	0.95
30	0	0.89	0.07	0.83	0.89	0.99	0.12	0.98	0.99	1.00	0.18	1.00	1.00
	250	0.89	0.07	0.82	0.89	0.98	0.12	0.96	0.98	0.99	0.19	0.98	0.99
	500	0.88	0.08	0.80	0.88	0.97	0.13	0.93	0.97	0.98	0.20	0.96	0.98
	750	0.88	0.08	0.78	0.88	0.96	0.14	0.91	0.96	0.98	0.22	0.96	0.98

TABLE 2

Probability of break fraction estimators lying in the range $\tau^ \pm \xi$; $\rho = 0.5$*

Panel A. $T = 150$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.02	0.03	0.03	0.02	0.06	0.07	0.06	0.05	0.12	0.14	0.13	0.12
	250	0.07	0.03	0.09	0.06	0.17	0.07	0.20	0.16	0.41	0.15	0.42	0.37
	500	0.14	0.05	0.15	0.12	0.34	0.11	0.35	0.31	0.73	0.22	0.68	0.69
	750	0.21	0.08	0.21	0.19	0.52	0.18	0.46	0.48	0.90	0.34	0.82	0.88
10	0	0.11	0.05	0.07	0.11	0.18	0.08	0.14	0.17	0.28	0.15	0.23	0.27
	250	0.18	0.06	0.15	0.18	0.30	0.10	0.27	0.29	0.49	0.17	0.46	0.47
	500	0.25	0.08	0.22	0.25	0.42	0.14	0.39	0.41	0.71	0.24	0.64	0.68
	750	0.31	0.13	0.27	0.32	0.54	0.22	0.47	0.53	0.86	0.37	0.76	0.85
20	0	0.37	0.12	0.27	0.38	0.50	0.15	0.40	0.49	0.63	0.22	0.53	0.62
	250	0.42	0.14	0.32	0.43	0.56	0.18	0.47	0.56	0.69	0.25	0.61	0.68
	500	0.48	0.19	0.37	0.50	0.62	0.24	0.53	0.63	0.75	0.33	0.69	0.75
	750	0.52	0.26	0.42	0.55	0.66	0.34	0.59	0.68	0.84	0.47	0.77	0.84
30	0	0.67	0.28	0.58	0.70	0.80	0.31	0.73	0.80	0.89	0.36	0.83	0.89
	250	0.68	0.32	0.59	0.71	0.79	0.35	0.72	0.80	0.87	0.40	0.81	0.87
	500	0.70	0.38	0.62	0.74	0.80	0.42	0.74	0.82	0.86	0.49	0.82	0.88
	750	0.72	0.47	0.64	0.77	0.81	0.52	0.76	0.84	0.87	0.62	0.85	0.89

Panel B. $T = 300$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.02	0.02	0.02	0.02	0.06	0.07	0.07	0.06	0.12	0.14	0.13	0.11
	250	0.05	0.02	0.08	0.05	0.17	0.07	0.22	0.15	0.37	0.14	0.43	0.33
	500	0.09	0.03	0.13	0.07	0.32	0.08	0.38	0.27	0.68	0.16	0.70	0.63
	750	0.13	0.03	0.18	0.10	0.48	0.09	0.50	0.42	0.87	0.18	0.85	0.84
10	0	0.09	0.03	0.06	0.08	0.18	0.07	0.13	0.17	0.27	0.14	0.22	0.26
	250	0.13	0.03	0.11	0.12	0.28	0.08	0.27	0.26	0.45	0.15	0.45	0.41
	500	0.18	0.03	0.16	0.16	0.39	0.09	0.39	0.36	0.68	0.16	0.67	0.64
	750	0.22	0.04	0.21	0.20	0.51	0.10	0.50	0.46	0.85	0.18	0.81	0.82
20	0	0.30	0.05	0.20	0.29	0.48	0.10	0.36	0.47	0.61	0.16	0.48	0.59
	250	0.33	0.05	0.22	0.32	0.53	0.10	0.41	0.51	0.65	0.17	0.56	0.63
	500	0.36	0.06	0.26	0.36	0.58	0.11	0.48	0.56	0.72	0.19	0.68	0.70
	750	0.40	0.07	0.30	0.39	0.62	0.13	0.55	0.60	0.82	0.21	0.78	0.80
30	0	0.56	0.10	0.45	0.56	0.77	0.14	0.67	0.76	0.87	0.21	0.79	0.86
	250	0.56	0.11	0.44	0.56	0.76	0.15	0.65	0.75	0.84	0.22	0.75	0.84
	500	0.58	0.12	0.45	0.58	0.76	0.16	0.66	0.77	0.84	0.24	0.77	0.84
	750	0.59	0.13	0.47	0.60	0.77	0.18	0.67	0.77	0.85	0.26	0.81	0.84

TABLE 3

Probability of break fraction estimators lying in the range $\tau^ \pm \xi$; $\rho = 0.925$*

Panel A. $T = 150$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.03	0.03	0.03	0.03	0.08	0.06	0.07	0.06	0.17	0.13	0.14	0.13
	250	0.04	0.03	0.04	0.04	0.09	0.08	0.08	0.10	0.19	0.16	0.17	0.20
	500	0.06	0.05	0.05	0.08	0.14	0.12	0.12	0.17	0.29	0.23	0.24	0.33
	750	0.09	0.09	0.08	0.13	0.22	0.19	0.17	0.27	0.44	0.36	0.33	0.50
10	0	0.05	0.05	0.05	0.05	0.09	0.09	0.09	0.09	0.18	0.15	0.16	0.16
	250	0.05	0.06	0.06	0.07	0.11	0.11	0.11	0.13	0.21	0.18	0.20	0.22
	500	0.07	0.10	0.09	0.12	0.16	0.16	0.16	0.21	0.31	0.26	0.27	0.37
	750	0.11	0.15	0.13	0.19	0.23	0.25	0.22	0.32	0.45	0.40	0.36	0.53
20	0	0.08	0.15	0.12	0.14	0.13	0.18	0.16	0.18	0.22	0.24	0.23	0.24
	250	0.09	0.18	0.15	0.18	0.15	0.22	0.20	0.22	0.25	0.29	0.28	0.31
	500	0.12	0.24	0.20	0.26	0.20	0.29	0.26	0.33	0.34	0.38	0.36	0.45
	750	0.16	0.31	0.26	0.35	0.27	0.39	0.34	0.44	0.48	0.51	0.45	0.61
30	0	0.14	0.37	0.29	0.35	0.20	0.39	0.33	0.37	0.28	0.44	0.39	0.42
	250	0.15	0.41	0.33	0.41	0.21	0.44	0.37	0.43	0.31	0.48	0.44	0.49
	500	0.19	0.48	0.40	0.50	0.26	0.51	0.45	0.54	0.40	0.57	0.52	0.61
	750	0.24	0.55	0.47	0.58	0.34	0.60	0.52	0.64	0.52	0.68	0.60	0.73
Panel B. $T = 300$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.03	0.02	0.02	0.03	0.07	0.07	0.07	0.07	0.14	0.14	0.14	0.13
	250	0.03	0.02	0.03	0.03	0.08	0.07	0.08	0.08	0.16	0.14	0.15	0.16
	500	0.03	0.03	0.03	0.05	0.11	0.08	0.10	0.13	0.21	0.16	0.19	0.24
	750	0.04	0.03	0.05	0.07	0.14	0.10	0.14	0.19	0.29	0.19	0.26	0.36
10	0	0.03	0.03	0.03	0.03	0.08	0.08	0.08	0.08	0.15	0.14	0.14	0.14
	250	0.03	0.03	0.03	0.04	0.09	0.08	0.09	0.09	0.17	0.15	0.15	0.17
	500	0.04	0.04	0.04	0.06	0.11	0.09	0.11	0.14	0.22	0.17	0.20	0.25
	750	0.05	0.05	0.06	0.09	0.16	0.11	0.15	0.21	0.30	0.20	0.27	0.37
20	0	0.04	0.06	0.05	0.06	0.10	0.10	0.10	0.11	0.17	0.17	0.17	0.17
	250	0.05	0.06	0.06	0.07	0.11	0.11	0.12	0.12	0.19	0.18	0.18	0.20
	500	0.06	0.07	0.08	0.10	0.14	0.12	0.15	0.18	0.24	0.20	0.23	0.29
	750	0.07	0.09	0.10	0.14	0.18	0.15	0.19	0.26	0.32	0.24	0.30	0.41
30	0	0.07	0.13	0.11	0.14	0.13	0.17	0.16	0.18	0.21	0.23	0.23	0.24
	250	0.07	0.14	0.12	0.16	0.14	0.18	0.18	0.20	0.22	0.24	0.24	0.27
	500	0.08	0.15	0.15	0.20	0.17	0.20	0.22	0.27	0.27	0.27	0.30	0.36
	750	0.10	0.18	0.18	0.25	0.21	0.23	0.27	0.35	0.34	0.31	0.36	0.48

TABLE 4

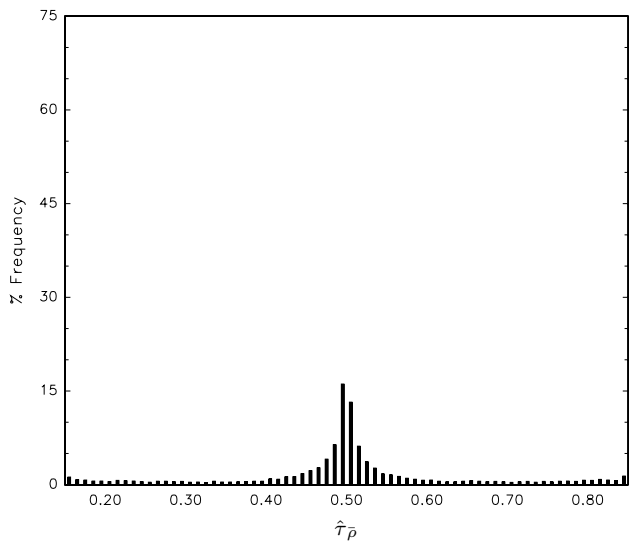
Probability of break fraction estimators lying in the range $\tau^ \pm \xi$; $\rho = 0.994$*

Panel A. $T = 150$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.04	0.03	0.03	0.03	0.09	0.06	0.08	0.07	0.19	0.13	0.16	0.14
	250	0.04	0.03	0.03	0.03	0.10	0.07	0.08	0.08	0.21	0.15	0.17	0.17
	500	0.05	0.05	0.05	0.06	0.13	0.11	0.11	0.13	0.27	0.22	0.21	0.25
	750	0.07	0.08	0.06	0.09	0.17	0.17	0.14	0.20	0.36	0.32	0.27	0.37
10	0	0.05	0.05	0.05	0.05	0.10	0.08	0.09	0.09	0.19	0.15	0.18	0.16
	250	0.05	0.06	0.06	0.06	0.11	0.10	0.11	0.11	0.22	0.18	0.19	0.19
	500	0.07	0.09	0.08	0.09	0.14	0.15	0.14	0.16	0.27	0.25	0.24	0.27
	750	0.09	0.14	0.11	0.14	0.19	0.22	0.18	0.24	0.36	0.36	0.30	0.39
20	0	0.08	0.15	0.12	0.14	0.12	0.18	0.16	0.17	0.22	0.24	0.24	0.23
	250	0.08	0.18	0.14	0.16	0.13	0.22	0.18	0.20	0.24	0.28	0.26	0.27
	500	0.10	0.22	0.18	0.21	0.16	0.27	0.23	0.27	0.29	0.36	0.32	0.36
	750	0.12	0.29	0.23	0.28	0.21	0.36	0.29	0.36	0.38	0.47	0.39	0.48
30	0	0.12	0.37	0.28	0.33	0.16	0.39	0.32	0.35	0.25	0.44	0.38	0.40
	250	0.13	0.41	0.32	0.37	0.17	0.43	0.36	0.39	0.27	0.48	0.41	0.44
	500	0.14	0.47	0.37	0.43	0.20	0.50	0.41	0.47	0.32	0.56	0.47	0.53
	750	0.17	0.55	0.43	0.51	0.25	0.58	0.48	0.55	0.40	0.65	0.55	0.63
Panel B. $T = 300$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.03	0.02	0.03	0.02	0.09	0.07	0.08	0.07	0.19	0.13	0.16	0.14
	250	0.03	0.03	0.03	0.03	0.10	0.07	0.08	0.07	0.20	0.14	0.16	0.14
	500	0.03	0.03	0.03	0.03	0.10	0.08	0.09	0.08	0.21	0.15	0.18	0.16
	750	0.04	0.04	0.04	0.04	0.11	0.10	0.10	0.11	0.23	0.18	0.20	0.20
10	0	0.03	0.03	0.03	0.03	0.10	0.08	0.09	0.07	0.19	0.14	0.17	0.14
	250	0.03	0.03	0.04	0.03	0.10	0.08	0.09	0.08	0.19	0.14	0.17	0.15
	500	0.04	0.04	0.04	0.04	0.11	0.09	0.10	0.09	0.21	0.16	0.18	0.17
	750	0.04	0.05	0.05	0.05	0.12	0.11	0.11	0.11	0.23	0.19	0.20	0.20
20	0	0.04	0.06	0.05	0.06	0.11	0.10	0.11	0.10	0.20	0.16	0.19	0.16
	250	0.04	0.06	0.06	0.06	0.11	0.11	0.11	0.10	0.20	0.17	0.19	0.17
	500	0.04	0.07	0.06	0.07	0.11	0.12	0.12	0.12	0.21	0.19	0.20	0.19
	750	0.05	0.08	0.07	0.09	0.13	0.14	0.14	0.15	0.24	0.22	0.22	0.23
30	0	0.05	0.13	0.11	0.12	0.12	0.17	0.16	0.16	0.21	0.23	0.23	0.21
	250	0.05	0.14	0.11	0.13	0.12	0.18	0.16	0.17	0.21	0.24	0.24	0.23
	500	0.06	0.16	0.12	0.14	0.12	0.20	0.18	0.19	0.22	0.26	0.25	0.25
	750	0.06	0.17	0.14	0.17	0.14	0.22	0.20	0.22	0.25	0.29	0.28	0.29

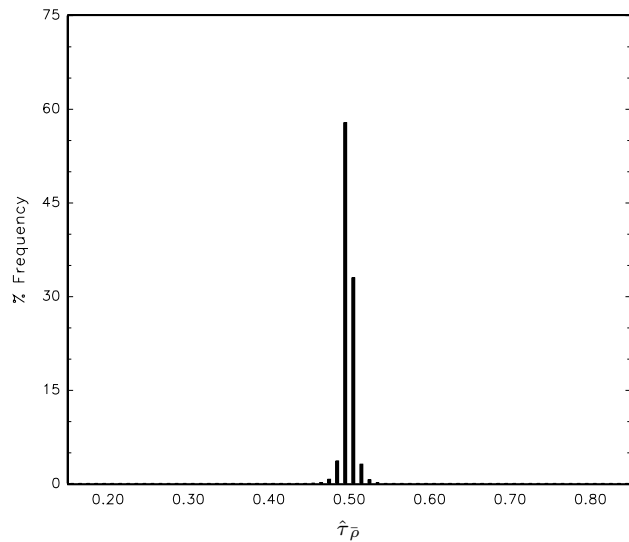
TABLE 5

Probability of break fraction estimators lying in the range $\tau^ \pm \xi$; $\rho = 1$*

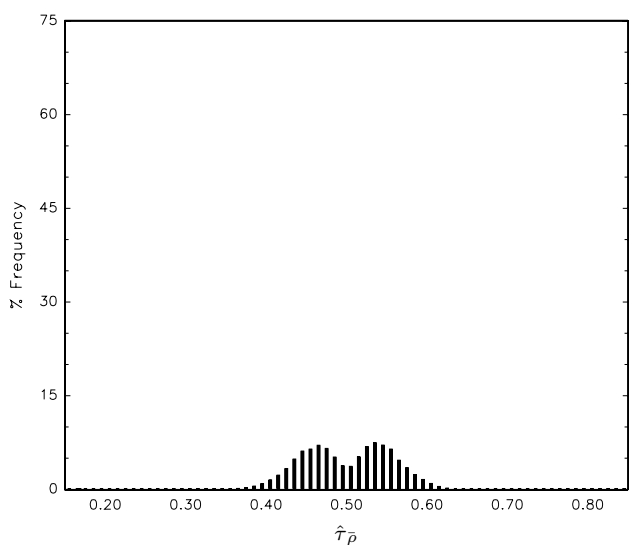
Panel A. $T = 150$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.04	0.03	0.03	0.03	0.09	0.06	0.08	0.07	0.19	0.13	0.16	0.14
	5	0.07	0.08	0.07	0.09	0.17	0.17	0.14	0.20	0.36	0.32	0.27	0.36
	10	0.15	0.26	0.16	0.26	0.36	0.47	0.29	0.48	0.67	0.70	0.47	0.72
	15	0.24	0.46	0.29	0.46	0.54	0.72	0.46	0.72	0.88	0.91	0.62	0.91
1	0	0.05	0.07	0.06	0.06	0.10	0.10	0.10	0.10	0.20	0.17	0.19	0.17
	5	0.09	0.16	0.13	0.17	0.19	0.25	0.21	0.26	0.36	0.37	0.32	0.40
	10	0.18	0.39	0.27	0.38	0.37	0.56	0.38	0.56	0.66	0.74	0.52	0.76
	15	0.27	0.60	0.43	0.58	0.54	0.79	0.56	0.78	0.87	0.93	0.68	0.93
2	0	0.09	0.23	0.18	0.21	0.13	0.26	0.22	0.24	0.23	0.32	0.29	0.29
	5	0.14	0.41	0.31	0.38	0.23	0.46	0.37	0.44	0.39	0.55	0.46	0.54
	10	0.24	0.64	0.50	0.61	0.40	0.73	0.57	0.71	0.67	0.83	0.66	0.82
	15	0.32	0.78	0.65	0.76	0.55	0.88	0.72	0.87	0.87	0.96	0.79	0.95
3	0	0.15	0.57	0.45	0.51	0.20	0.58	0.47	0.52	0.28	0.61	0.52	0.56
	5	0.21	0.71	0.60	0.67	0.28	0.74	0.63	0.70	0.42	0.77	0.68	0.74
	10	0.30	0.86	0.75	0.83	0.44	0.89	0.79	0.87	0.68	0.93	0.82	0.91
	15	0.38	0.93	0.84	0.91	0.57	0.96	0.88	0.94	0.86	0.98	0.90	0.97
Panel B. $T = 300$													
κ_1	κ_2	$\xi = 0.010$				$\xi = 0.025$				$\xi = 0.050$			
		$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$	$\hat{\tau}_0$	$\hat{\tau}_1$	$\hat{\tau}_{AR}$	$\hat{\tau}_{D_m}$
0	0	0.03	0.02	0.03	0.02	0.10	0.07	0.08	0.07	0.19	0.14	0.17	0.13
	5	0.06	0.07	0.06	0.08	0.18	0.18	0.16	0.21	0.36	0.32	0.28	0.36
	10	0.12	0.23	0.15	0.24	0.38	0.49	0.31	0.51	0.68	0.70	0.47	0.73
	15	0.20	0.43	0.27	0.43	0.57	0.76	0.47	0.76	0.88	0.92	0.63	0.92
1	0	0.04	0.05	0.05	0.05	0.11	0.09	0.10	0.09	0.20	0.16	0.18	0.16
	5	0.07	0.13	0.10	0.14	0.19	0.23	0.20	0.25	0.37	0.35	0.31	0.39
	10	0.14	0.33	0.22	0.33	0.39	0.55	0.37	0.56	0.68	0.73	0.51	0.75
	15	0.21	0.54	0.37	0.52	0.57	0.79	0.53	0.79	0.88	0.93	0.67	0.93
2	0	0.06	0.19	0.15	0.17	0.12	0.23	0.20	0.20	0.22	0.28	0.26	0.26
	5	0.10	0.33	0.25	0.32	0.22	0.41	0.33	0.40	0.38	0.49	0.42	0.50
	10	0.18	0.56	0.43	0.54	0.41	0.69	0.53	0.68	0.68	0.81	0.62	0.81
	15	0.24	0.73	0.57	0.70	0.57	0.87	0.68	0.86	0.88	0.95	0.76	0.95
3	0	0.10	0.50	0.39	0.45	0.16	0.52	0.42	0.47	0.24	0.55	0.47	0.51
	5	0.14	0.64	0.53	0.61	0.24	0.68	0.58	0.65	0.40	0.73	0.63	0.70
	10	0.22	0.81	0.69	0.78	0.43	0.86	0.74	0.83	0.69	0.91	0.79	0.89
	15	0.28	0.89	0.80	0.87	0.58	0.94	0.84	0.93	0.87	0.98	0.88	0.97



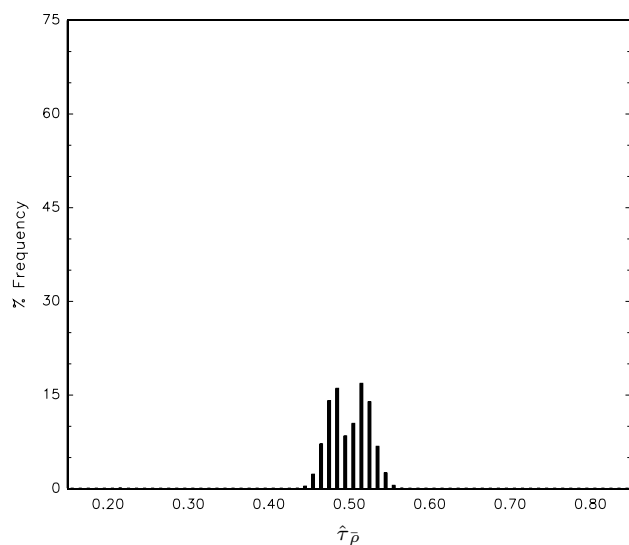
(a) $\kappa_1 = 10, \kappa_2 = 0$



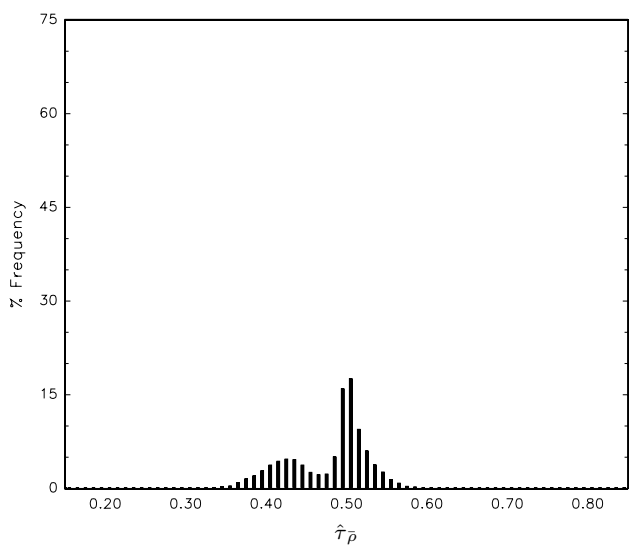
(b) $\kappa_1 = 30, \kappa_2 = 0$



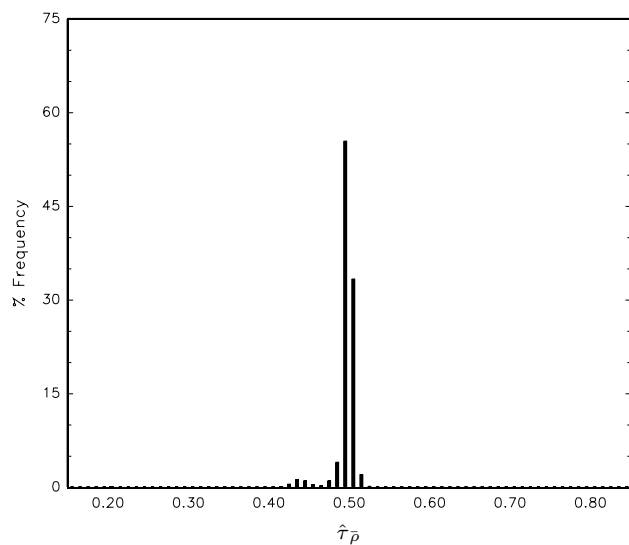
(c) $\kappa_1 = 0, \kappa_2 = 250$



(d) $\kappa_1 = 0, \kappa_2 = 750$

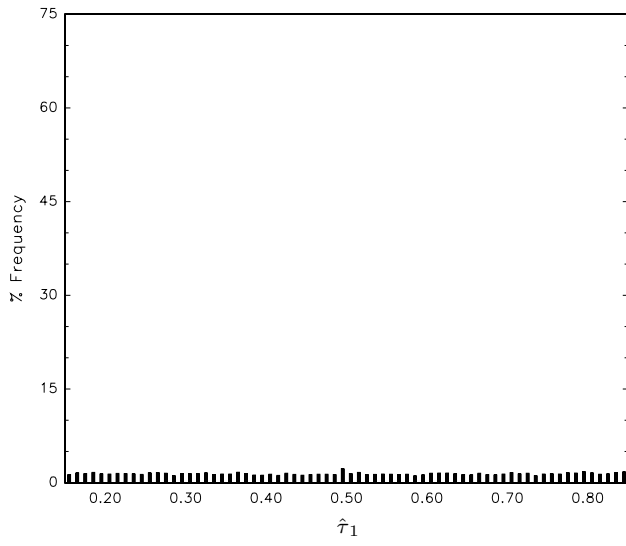


(e) $\kappa_1 = 10, \kappa_2 = 250$

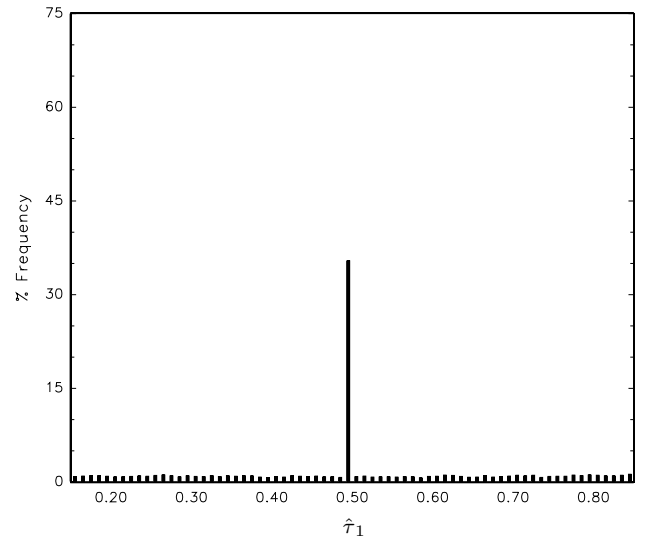


(f) $\kappa_1 = 30, \kappa_2 = 750$

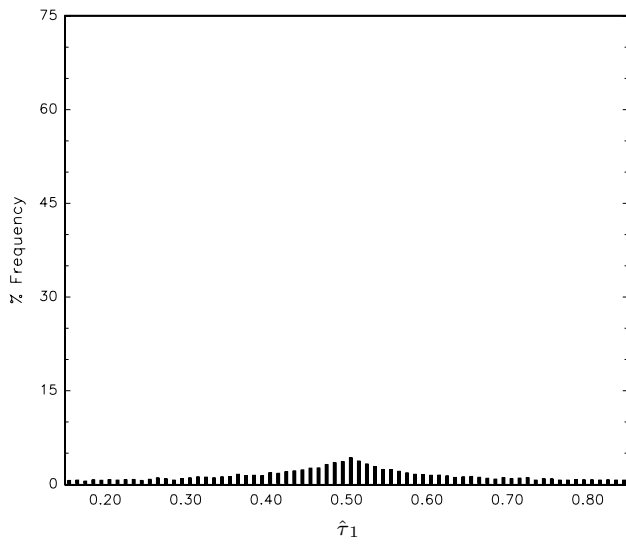
Figure 1. Histograms of limit of $\hat{\tau}_{\bar{\rho}}$, $|\bar{\rho}| < 1$ under Assumption I(0); $\tau^* = 0.5, \rho = 0, \omega_\varepsilon = 1$



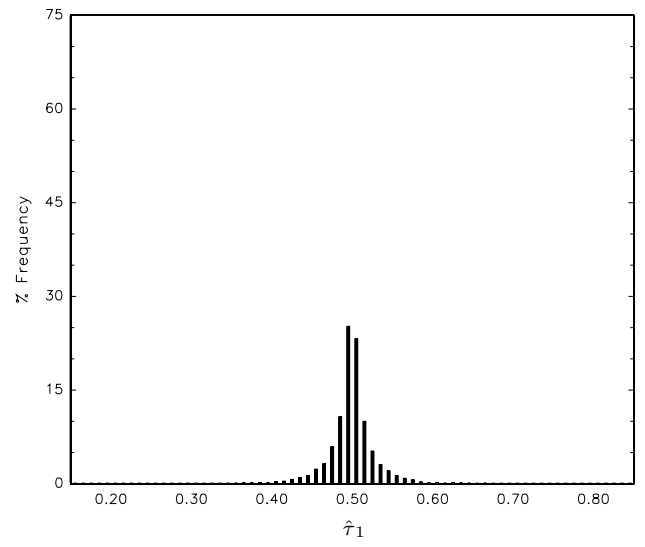
(a) $\kappa_1 = 1, \kappa_2 = 0$



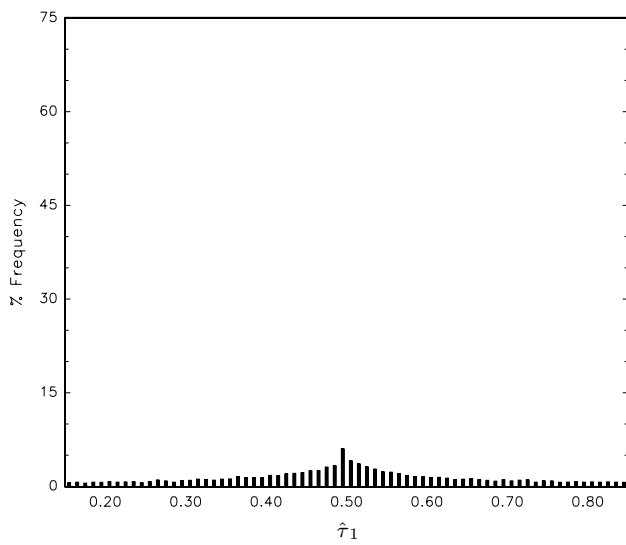
(b) $\kappa_1 = 3, \kappa_2 = 0$



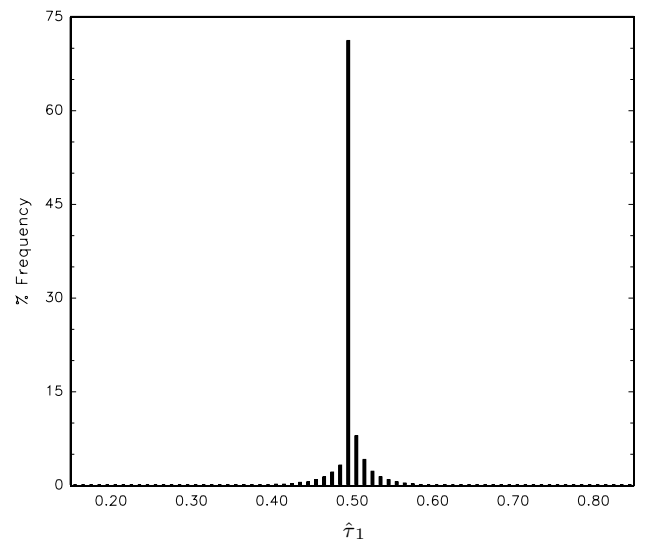
(c) $\kappa_1 = 0, \kappa_2 = 5$



(d) $\kappa_1 = 0, \kappa_2 = 15$

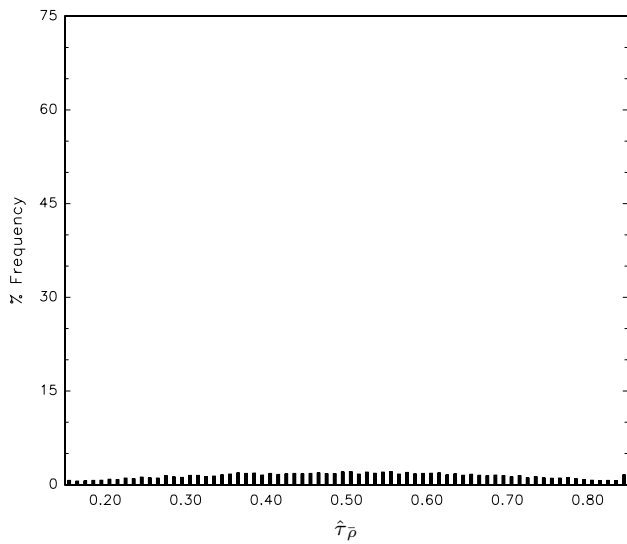


(e) $\kappa_1 = 1, \kappa_2 = 5$

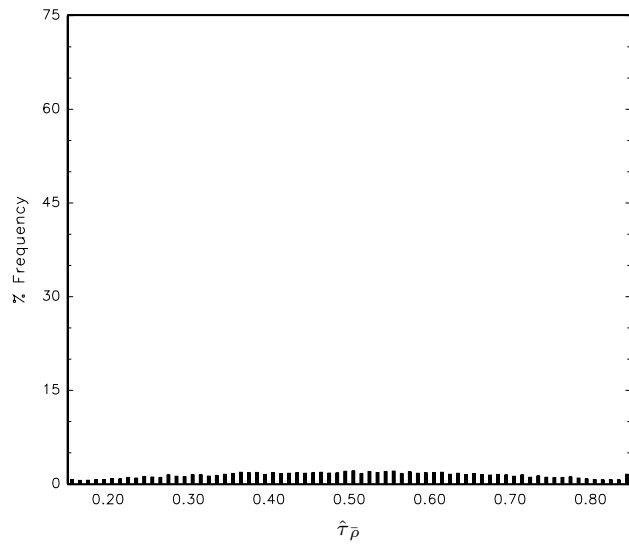


(f) $\kappa_1 = 3, \kappa_2 = 15$

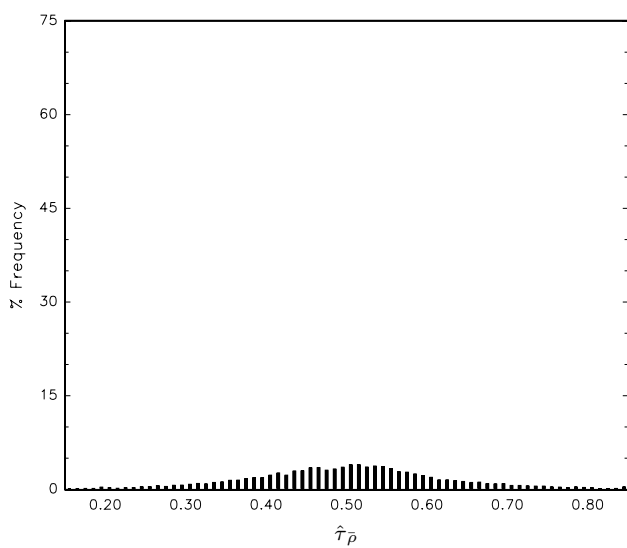
Figure 2. Histograms of limit of $\hat{\tau}_1$ under Assumption I(1); $\tau^* = 0.5, \omega_\varepsilon = 1$



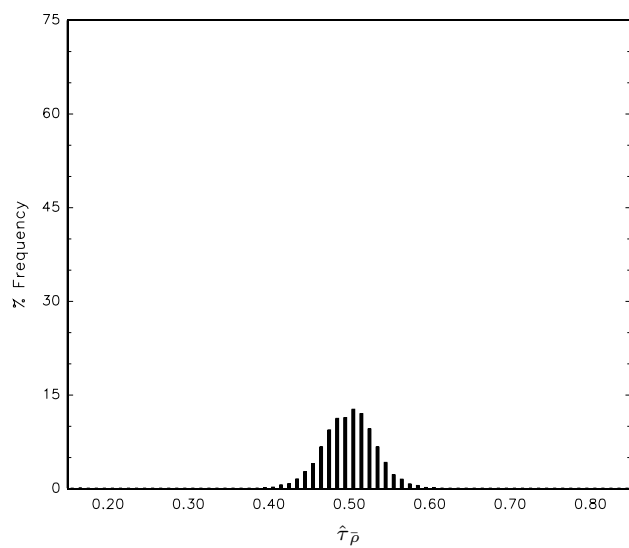
(a) $\kappa_1 = 1, \kappa_2 = 0$



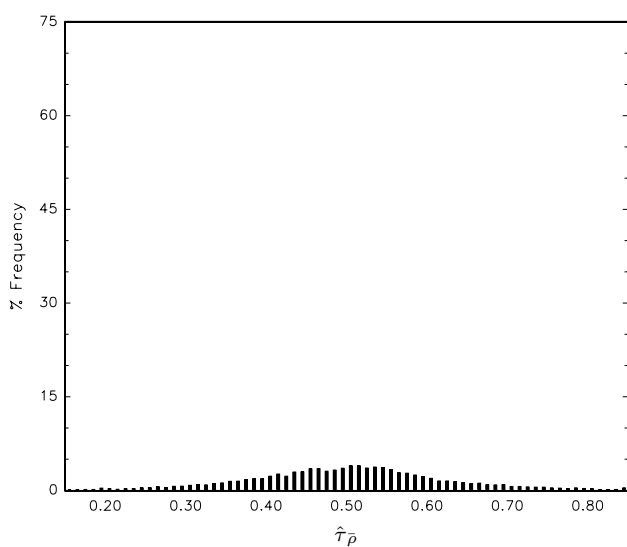
(b) $\kappa_1 = 3, \kappa_2 = 0$



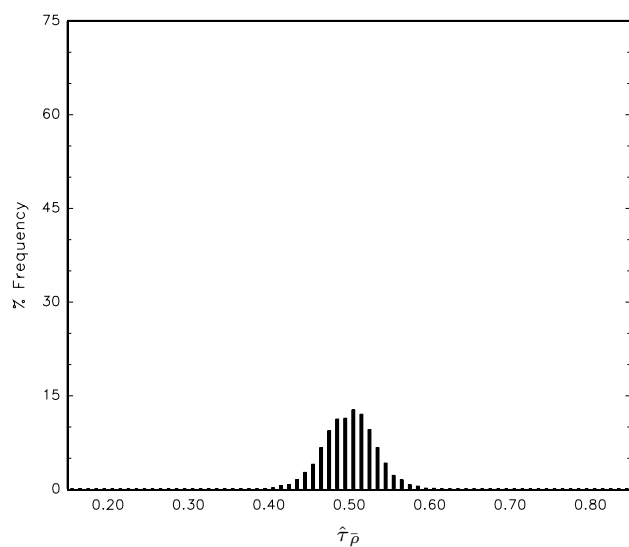
(c) $\kappa_1 = 0, \kappa_2 = 5$



(d) $\kappa_1 = 0, \kappa_2 = 15$



(e) $\kappa_1 = 1, \kappa_2 = 5$



(f) $\kappa_1 = 3, \kappa_2 = 15$

Figure 3. Histograms of limit of $\hat{\tau}_{\bar{\rho}}, |\bar{\rho}| < 1$ under Assumption I(1); $\tau^* = 0.5, \omega_\varepsilon = 1$

Appendix to “Break Date Estimation for Models with Deterministic Structural Change”

In what follows we can set $\alpha_1 = \alpha_2 = 0$ in equation (1) of the paper without any loss of generality. By way of preliminaries, in addition to $\mathbf{y}_{\bar{\rho}}$, $\mathbf{Z}_{\bar{\rho},\tau}$ and $S(\bar{\rho}, \tau)$, we also define

$$\begin{aligned}\mathbf{u}_{\bar{\rho}} &= [u_1, u_2 - \bar{\rho}u_1, \dots, u_T - \bar{\rho}u_{T-1}]', \\ \mathbf{Z}_{\bar{\rho}} &= [\mathbf{x}_1, \mathbf{x}_2 - \bar{\rho}\mathbf{x}_1, \dots, \mathbf{x}_T - \bar{\rho}\mathbf{x}_{T-1}]' \text{ with } \mathbf{x}_t = [1, t]'\end{aligned}$$

and use $\mathbf{r}_{\bar{\rho}}$ to denote the residuals from a regression of $\mathbf{y}_{\bar{\rho}}$ on $\mathbf{Z}_{\bar{\rho}}$; and $\mathbf{r}_{\bar{\rho},\tau,34}$ to denote the $T \times 2$ residuals from a regression of the final two columns of $\mathbf{Z}_{\bar{\rho},\tau}$, which we denote $\mathbf{Z}_{\bar{\rho},\tau,34}$, on $\mathbf{Z}_{\bar{\rho}}$. Define the sums of squared residuals $S(\bar{\rho}) = \mathbf{r}_{\bar{\rho}}'\mathbf{r}_{\bar{\rho}}$ and the 2×2 matrix $\mathbf{S}_{34}(\bar{\rho}, \tau) = \mathbf{r}'_{\bar{\rho},\tau,34}\mathbf{r}_{\bar{\rho},\tau,34}$. Straightforward application of the Frisch-Waugh-Lovell theorem then shows that we can write

$$S(\bar{\rho}, \tau) = S(\bar{\rho}) - (\mathbf{r}'_{\bar{\rho},\tau,34}\mathbf{r}_{\bar{\rho}})'\mathbf{S}_{34}(\bar{\rho}, \tau)^{-1}(\mathbf{r}'_{\bar{\rho},\tau,34}\mathbf{r}_{\bar{\rho}}),$$

which is a representation we shall use repeatedly in the proofs below. We will need the scaling matrices

$$\Delta_1 = \begin{bmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} T^2 & 0 \\ 0 & T^3 \end{bmatrix}, \quad \Delta_3 = \begin{bmatrix} 1 & 0 \\ 0 & T^{1/2} \end{bmatrix}.$$

The following preliminary lemmas are also used. The terms involved in Lemma 1 do not involve any stochastic components and the proofs are entirely straightforward and are therefore not presented.

Lemma 1

(i) For $|\bar{\rho}| < 1$

$$\Delta_1^{-1}\mathbf{Z}'_{\bar{\rho}}\mathbf{Z}_{\bar{\rho}}\Delta_1^{-1} \rightarrow (1 - \bar{\rho})^2 \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}, \quad (\text{A.1})$$

$$\Delta_1^{-1}\mathbf{Z}'_{\bar{\rho}}\mathbf{Z}_{\bar{\rho},\tau,34}\Delta_1^{-1} \rightarrow (1 - \bar{\rho})^2 \begin{bmatrix} (1 - \tau) & (1 - \tau)^2/2 \\ (1 - \tau^2)/2 & (1 - \tau)^2(2 + \tau)/6 \end{bmatrix}, \quad (\text{A.2})$$

$$\Delta_1^{-1}\mathbf{S}_{34}(\bar{\rho}, \tau)\Delta_1^{-1} \rightarrow (1 - \bar{\rho})^2\mathbf{B}_2(\tau). \quad (\text{A.3})$$

where $\mathbf{B}_2(\tau)$ is as defined in Theorem 1.

(ii) For $\bar{\rho} = 1$

$$\Delta_3^{-1} \mathbf{Z}'_1 \mathbf{Z}_1 \Delta_3^{-1} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\text{A.4})$$

$$\Delta_3^{-1} \mathbf{Z}'_1 \mathbf{Z}_{1,\tau,34} \Delta_3^{-1} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 - \tau \end{bmatrix}, \quad (\text{A.5})$$

$$\Delta_3^{-1} \mathbf{S}_{34}(1, \tau) \Delta_3^{-1} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \tau(1 - \tau) \end{bmatrix}. \quad (\text{A.6})$$

Lemma 2 Under Assumption I(0),

(i) for $|\bar{\rho}| < 1$

$$T^{-1}S(\bar{\rho}) \Rightarrow \frac{\omega_\varepsilon^2}{1 - \rho^2} (1 + \bar{\rho}^2 - 2\bar{\rho}\rho), \quad (\text{A.7})$$

$$\Delta_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}} \Rightarrow (1 - \bar{\rho})^2 \omega_u \{ \mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*) \} \quad (\text{A.8})$$

where $\mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau)$, $\mathbf{B}_1(\tau)$ and $\mathbf{C}(\kappa'_1, \kappa'_2, \tau^*)$ are as defined in Theorem 1,

(ii) for $\bar{\rho} = 1$

$$T^{-1}S(1) \Rightarrow \frac{2\omega_\varepsilon^2}{1 + \rho}, \quad (\text{A.9})$$

$$\mathbf{r}'_{1,\tau,34} \mathbf{r}_1 = \begin{bmatrix} \Delta u_{\tau T+1} \\ \tau u_T - u_{\tau T} + (1 - \tau)u_1 \end{bmatrix} + o(1). \quad (\text{A.10})$$

Proof of Lemma 2

(i) Write

$$T^{-1}S(\bar{\rho}) = T^{-1} \mathbf{y}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}} - T^{-1} \mathbf{y}'_{\bar{\rho}} \mathbf{Z}_{\bar{\rho}} \Delta_1^{-1} [\Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{Z}_{\bar{\rho}} \Delta_1^{-1}]^{-1} \Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}}.$$

Now

$$\begin{aligned} T^{-1} \mathbf{y}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}} &= T^{-1} \mathbf{u}'_{\bar{\rho}} \mathbf{u}_{\bar{\rho}} + o_p(1) \\ &= T^{-1} \mathbf{u}'_0 \mathbf{u}_0 + \bar{\rho}^2 T^{-1} \mathbf{u}'_{0,-1} \mathbf{u}_{0,-1} - 2\bar{\rho} T^{-1} \mathbf{u}'_0 \mathbf{u}_{0,-1} + o_p(1) \\ &\Rightarrow \frac{\omega_\varepsilon^2}{1 - \rho^2} (1 + \bar{\rho}^2 - 2\bar{\rho}\rho), \end{aligned} \quad (\text{A.11})$$

where $\mathbf{u}_{0,-1}$ is the vector \mathbf{u}_0 lagged one period and

$$\begin{aligned}
\Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}} &= \begin{bmatrix} T^{-1/2} y_1 + (1 - \bar{\rho}) T^{-1/2} \sum_{t=2}^T (y_t - \bar{\rho} y_{t-1}) \\ T^{-3/2} y_1 + T^{-3/2} \sum_{t=2}^T [t - \bar{\rho}(t-1)] (y_t - \bar{\rho} y_{t-1}) \end{bmatrix} \\
&= \begin{bmatrix} (1 - \bar{\rho})^2 T^{-1/2} \sum_{t=2}^T y_t \\ (1 - \bar{\rho})^2 T^{-3/2} \sum_{t=2}^T t y_t \end{bmatrix} + o_p(1) \\
&= \begin{bmatrix} (1 - \bar{\rho})^2 T^{-1/2} \sum_{t=2}^T (\kappa_1 T^{-1/2} DU_t(\tau^*) + \kappa_2 T^{-3/2} DT_t(\tau^*) + u_t) \\ (1 - \bar{\rho})^2 T^{-3/2} \sum_{t=2}^T t (\kappa_1 T^{-1/2} DU_t(\tau^*) + \kappa_2 T^{-3/2} DT_t(\tau^*) + u_t) \end{bmatrix} + o_p(1) \\
\Rightarrow (1 - \bar{\rho})^2 \omega_u &\begin{bmatrix} \kappa'_1 (1 - \tau^*) + \kappa'_2 (1 - \tau^*)^2 / 2 + W(1) \\ \kappa'_1 (1 - \tau^{*2}) / 2 + \kappa'_2 (1 - \tau^*)^2 (2 + \tau^*) / 6 + \int_0^1 r dW(r) \end{bmatrix} \quad (\text{A.12})
\end{aligned}$$

These results, together with (A.1) give (A.7).

Next, write

$$\Delta_1^{-1} \mathbf{r}'_{\bar{\rho}, \tau, 34} \mathbf{r}_{\bar{\rho}} = \Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}, \tau, 34} \mathbf{y}_{\bar{\rho}} - \Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}, \tau, 34} \mathbf{Z}_{\bar{\rho}} \Delta_1^{-1} [\Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{Z}_{\bar{\rho}} \Delta_1^{-1}]^{-1} \Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}}.$$

Now

$$\begin{aligned}
\Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}, \tau, 34} \mathbf{y}_{\bar{\rho}} &= \begin{bmatrix} T^{-1/2} (y_{\tau T+1} - \bar{\rho} y_{\tau T}) + (1 - \bar{\rho}) T^{-1/2} \sum_{t=\tau T+2}^T (y_t - \bar{\rho} y_{t-1}) \\ T^{-3/2} (y_{\tau T+1} - \bar{\rho} y_{\tau T}) + T^{-3/2} \sum_{t=\tau T+2}^T [t - \tau T - \bar{\rho}(t - \tau T - 1)] (y_t - \bar{\rho} y_{t-1}) \end{bmatrix} \\
&= \begin{bmatrix} (1 - \bar{\rho})^2 T^{-1/2} \sum_{t=\tau T+2}^T y_t \\ (1 - \bar{\rho})^2 T^{-3/2} \sum_{t=\tau T+2}^T (t - \tau T) y_t \end{bmatrix} + o_p(1) \\
&= \begin{bmatrix} (1 - \bar{\rho})^2 T^{-1/2} \sum_{t=\tau T+2}^T (\kappa_1 T^{-1/2} DU_t(\tau^*) + \kappa_2 T^{-3/2} DT_t(\tau^*) + u_t) \\ (1 - \bar{\rho})^2 T^{-3/2} \sum_{t=\tau T+2}^T (t - \tau T) (\kappa_1 T^{-1/2} DU_t(\tau^*) + \kappa_2 T^{-3/2} DT_t(\tau^*) + u_t) \end{bmatrix} \\
&\quad + o_p(1) \\
\Rightarrow (1 - \bar{\rho})^2 \omega_u &\begin{bmatrix} \kappa'_1 G_{11}(\tau^*, \tau) + \kappa'_2 G_{21}(\tau^*, \tau) + W(1) - W(\tau) \\ \kappa'_1 G_{12}(\tau^*, \tau) + \kappa'_2 G_{22}(\tau^*, \tau) + \int_{\tau}^1 (r - \tau) dW(r) \end{bmatrix} \quad (\text{A.13})
\end{aligned}$$

where $G_{11}(\tau^*, \tau)$, $G_{21}(\tau^*, \tau)$, $G_{12}(\tau^*, \tau)$ and $G_{22}(\tau^*, \tau)$ are as defined in Theorem 1.

Combining (A.13), (A.2), (A.1) and (A.12) gives

$$\begin{aligned}
\Delta_1^{-1} \mathbf{r}'_{\bar{\rho}, \tau, 34} \mathbf{r}_{\bar{\rho}} &\Rightarrow (1 - \bar{\rho})^2 \omega_u \{ \mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) \\
&\quad - \begin{bmatrix} (1 - \tau) & (1 - \tau^2)/2 \\ (1 - \tau)^2/2 & (1 - \tau)^2(2 + \tau)/6 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*) \} \\
&= (1 - \bar{\rho})^2 \omega_u \{ \mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*) \}
\end{aligned}$$

i.e. (A.8).

(ii) Write

$$T^{-1} S(1) = T^{-1} \mathbf{y}'_1 \mathbf{y}_1 - T^{-1} \mathbf{y}'_1 \mathbf{Z}_1 \Delta_3^{-1} [\Delta_3^{-1} \mathbf{Z}'_1 \mathbf{Z}_1 \Delta_3^{-1}]^{-1} \Delta_3^{-1} \mathbf{Z}'_1 \mathbf{y}_1$$

and

$$\begin{aligned}
T^{-1}\mathbf{y}'_1\mathbf{y}_1 &= T^{-1}\mathbf{u}'_1\mathbf{u}_1 + o_p(1) \\
&= T^{-1}\mathbf{u}'_0\mathbf{u}_0 + T^{-1}\mathbf{u}'_{0,-1}\mathbf{u}_{0,-1} - 2T^{-1}\mathbf{u}'_0\mathbf{u}_{0,-1} + o_p(1) \\
&\Rightarrow \frac{\omega_\varepsilon^2}{1-\rho^2}2(1-\rho) \\
&= \frac{2\omega_\varepsilon^2}{1+\rho},
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
\mathbf{Z}'_1\mathbf{y}_1 &= \begin{bmatrix} y_1 \\ y_T \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_T + O(T^{-1/2}) \end{bmatrix}.
\end{aligned} \tag{A.15}$$

Taking (A.14), (A.4) and (A.15) together establishes (A.9).

Next, write

$$\mathbf{r}'_{1,\tau,34}\mathbf{r}_1 = \mathbf{Z}'_{1,\tau,34}\mathbf{y}_1 - \mathbf{Z}'_{1,\tau,34}\mathbf{Z}_1[\mathbf{Z}'_1\mathbf{Z}_1]^{-1}\mathbf{Z}'_1\mathbf{y}_1$$

Now

$$\begin{aligned}
\mathbf{Z}'_{1,\tau,34}\mathbf{y}_1 &= \begin{bmatrix} \Delta y_{\tau T+1} \\ y_T - y_{\tau T} \end{bmatrix} \\
&= \begin{bmatrix} \Delta u_{\tau T+1} + O(T^{-1/2}) \\ u_T - u_{\tau T} + O(T^{-1/2}) \end{bmatrix},
\end{aligned} \tag{A.16}$$

$$\mathbf{Z}'_1\mathbf{Z}_1 = \begin{bmatrix} 1 & 1 \\ 1 & T \end{bmatrix},$$

$$\mathbf{Z}'_{1,\tau,34}\mathbf{Z}_1 = \begin{bmatrix} 0 & 1 \\ 0 & T - \tau T \end{bmatrix}$$

i.e.

$$\begin{aligned}
\mathbf{r}'_{1,\tau,34}\mathbf{r}_1 &= \begin{bmatrix} \Delta u_{\tau T+1} + O(T^{-1/2}) \\ u_T - u_{\tau T} + O(T^{-1/2}) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & T - \tau T \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & T \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_T + O(T^{-1/2}) \end{bmatrix} \\
&= \begin{bmatrix} \Delta u_{\tau T+1} + O(T^{-1/2}) \\ u_T - u_{\tau T} + O(T^{-1/2}) \end{bmatrix} - \begin{bmatrix} -\frac{1}{T-1} & \frac{1}{T-1} \\ -\frac{T-T\tau}{T-1} & \frac{T-T\tau}{T-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_T + O(T^{-1/2}) \end{bmatrix} \\
&= \begin{bmatrix} \Delta u_{\tau T+1} + O(T^{-1/2}) \\ u_T - u_{\tau T} + O(T^{-1/2}) \end{bmatrix} - \begin{bmatrix} \frac{u_T + O(T^{-1/2})}{T-1} - \frac{u_1}{T-1} \\ [u_T + O(T^{-1/2})]\frac{T-T\tau}{T-1} - u_1\frac{T-T\tau}{T-1} \end{bmatrix} \\
&= \begin{bmatrix} \Delta u_{\tau T+1} \\ \tau u_T - u_{\tau T} + (1-\tau)u_1 \end{bmatrix} + o_p(1)
\end{aligned} \tag{A.17}$$

as in (A.10).

Lemma 3 Under Assumption I(1),

(i) for $|\bar{\rho}| < 1$

$$T^{-2}S(\bar{\rho}) \Rightarrow (1 - \bar{\rho})^2 \omega_\varepsilon^2 \{ \kappa_2''^2 (1 - \tau^*)^3 / 3 + \int_0^1 W(r)^2 dr + 2\kappa_2'' \int_{\tau^*}^1 (r - \tau^*) W(r) dr - \mathbf{K}(\kappa_2'', \tau^*)' \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} \mathbf{K}(\kappa_2'', \tau^*) \}, \quad (\text{A.18})$$

$$T^{-1} \Delta_1^{-1} \mathbf{r}'_{\bar{\rho}, \tau, 34} \mathbf{r}_{\bar{\rho}} \Rightarrow (1 - \bar{\rho})^2 \omega_\varepsilon \{ \mathbf{J}(\kappa_2'', \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{K}(\kappa_2'', \tau^*) \}, \quad (\text{A.19})$$

where $\mathbf{J}(\kappa_2'', \tau^*, \tau)$, $\mathbf{B}_1(\tau)$ and $\mathbf{K}(\kappa_2'', \tau^*)$ are as defined in Theorems 1 and 2,

(ii) for $\bar{\rho} = 1$

$$T^{-1}S(1) \Rightarrow \omega_\varepsilon^2, \quad (\text{A.20})$$

$$\Delta_3^{-1} \mathbf{r}'_{1, \tau, 34} \mathbf{r}_1 \Rightarrow \omega_\varepsilon \begin{bmatrix} \kappa_1'' H_1(\tau^*, \tau) + \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \kappa_2'' H_2(\tau^*, \tau) + \tau W(1) - W(\tau) \end{bmatrix}. \quad (\text{A.21})$$

where $H_1(\tau^*, \tau)$ and $H_2(\tau^*, \tau)$ are as defined in Theorem 2.

Proof of Lemma 3

(i) Write

$$T^{-2}S(\bar{\rho}) = T^{-2} \mathbf{y}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}} - T^{-1} \mathbf{y}'_{\bar{\rho}} \mathbf{Z}_{\bar{\rho}} \Delta_1^{-1} [\Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{Z}_{\bar{\rho}} \Delta_1^{-1}]^{-1} T^{-1} \Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}}.$$

Now

$$\begin{aligned} T^{-2} \mathbf{y}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}} &= T^{-2} y_1^2 + T^{-2} \sum_{t=2}^T (y_t - \bar{\rho} y_{t-1})^2 \\ &= (1 - \bar{\rho})^2 T^{-2} \sum_{t=2}^T y_t^2 + o_p(1) \\ &= (1 - \bar{\rho})^2 T^{-2} \sum_{t=2}^T (\kappa_1 D U_t(\tau^*) + \kappa_2 T^{-1/2} D T_t(\tau^*) + u_t)^2 + o_p(1) \\ &\Rightarrow (1 - \bar{\rho})^2 \omega_\varepsilon^2 \{ \kappa_2''^2 (1 - \tau^*)^3 / 3 + \int_0^1 W(r)^2 dr + 2\kappa_2'' \int_{\tau^*}^1 (r - \tau^*) W(r) dr \}, \end{aligned}$$

$$\begin{aligned} T^{-1} \Delta_1^{-1} \mathbf{Z}'_{\bar{\rho}} \mathbf{y}_{\bar{\rho}} &= \begin{bmatrix} T^{-3/2} y_1 + (1 - \bar{\rho}) T^{-3/2} \sum_{t=2}^T (y_t - \bar{\rho} y_{t-1}) \\ T^{-5/2} y_1 + T^{-5/2} \sum_{t=2}^T [t - \bar{\rho}(t-1)] (y_t - \bar{\rho} y_{t-1}) \end{bmatrix} \\ &= \begin{bmatrix} (1 - \bar{\rho})^2 T^{-3/2} \sum_{t=2}^T y_t \\ (1 - \bar{\rho})^2 T^{-5/2} \sum_{t=2}^T t y_t \end{bmatrix} + o_p(1) \\ &= \begin{bmatrix} (1 - \bar{\rho})^2 T^{-3/2} \sum_{t=2}^T (\kappa_1 D U_t(\tau^*) + \kappa_2 T^{-1/2} D T_t(\tau^*) + u_t) \\ (1 - \bar{\rho})^2 T^{-5/2} \sum_{t=2}^T t (\kappa_1 D U_t(\tau^*) + \kappa_2 T^{-1/2} D T_t(\tau^*) + u_t) \end{bmatrix} + o_p(1) \\ &\Rightarrow (1 - \bar{\rho})^2 \omega_\varepsilon \begin{bmatrix} \kappa_2'' (1 - \tau^*)^2 / 2 + \int_0^1 W(r) dr \\ \kappa_2'' (1 - \tau^*)^2 (2 + \tau^*) / 6 + \int_0^1 r W(r) dr \end{bmatrix} \quad (\text{A.23}) \end{aligned}$$

and using (A.1) these results give (A.18).

Next,

$$T^{-1}\Delta_1^{-1}\mathbf{r}'_{\bar{\rho},\tau,34}\mathbf{r}_{\bar{\rho}} = T^{-1}\Delta_1^{-1}\mathbf{Z}'_{\bar{\rho},\tau,34}\mathbf{y}_{\bar{\rho}} - \Delta_1^{-1}\mathbf{Z}'_{\bar{\rho},\tau,34}\mathbf{Z}_{\bar{\rho}}\Delta_1^{-1}[\Delta_1^{-1}\mathbf{Z}'_{\bar{\rho}}\mathbf{Z}_{\bar{\rho}}\Delta_1^{-1}]^{-1}T^{-1}\Delta_1^{-1}\mathbf{Z}'_{\bar{\rho}}\mathbf{y}_{\bar{\rho}}.$$

Now

$$\begin{aligned} & T^{-1}\Delta_1^{-1}\mathbf{Z}'_{\bar{\rho},\tau,34}\mathbf{y}_{\bar{\rho}} \\ &= \left[\begin{array}{c} T^{-3/2}(y_{\tau T+1} - \bar{\rho}y_{\tau T}) + (1 - \bar{\rho})T^{-3/2}\sum_{t=\tau T+2}^T(y_t - \bar{\rho}y_{t-1}) \\ T^{-5/2}(y_{\tau T+1} - \bar{\rho}y_{\tau T}) + T^{-5/2}\sum_{t=\tau T+2}^T[t - \tau T - \bar{\rho}(t - \tau T - 1)](y_t - \bar{\rho}y_{t-1}) \end{array} \right] \\ &= \left[\begin{array}{c} (1 - \bar{\rho})^2T^{-3/2}\sum_{t=\tau T+2}^T y_t \\ (1 - \bar{\rho})^2T^{-5/2}\sum_{t=\tau T+2}^T(t - \tau T)y_t \end{array} \right] + o_p(1) \\ &= \left[\begin{array}{c} (1 - \bar{\rho})^2T^{-3/2}\sum_{t=\tau T+2}^T(\kappa_1 DU_t(\tau^*) + \kappa_2 T^{-1/2}DT_t(\tau^*) + u_t) \\ (1 - \bar{\rho})^2T^{-5/2}\sum_{t=\tau T+2}^T(t - \tau T)(\kappa_1 DU_t(\tau^*) + \kappa_2 T^{-1/2}DT_t(\tau^*) + u_t) \end{array} \right] \\ &\quad + o_p(1) \\ &\Rightarrow (1 - \bar{\rho})^2\omega_\varepsilon \left[\begin{array}{c} \kappa_2''G_{21}(\tau^*, \tau) + \int_\tau^1 W(r)dr \\ \kappa_2''G_{22}(\tau^*, \tau) + \int_\tau^1(r - \tau)W(r)dr \end{array} \right], \end{aligned} \tag{A.24}$$

$$\begin{aligned} T^{-1}\Delta_1^{-1}\mathbf{r}'_{\bar{\rho},\tau,34}\mathbf{r}_{\bar{\rho}} &\Rightarrow (1 - \bar{\rho})^2\omega_\varepsilon\{\mathbf{J}(\kappa_2'', \tau^*, \tau) \\ &\quad - \left[\begin{array}{cc} (1 - \tau) & (1 - \tau^2)/2 \\ (1 - \tau)^2/2 & (1 - \tau)^2(2 + \tau)/6 \end{array} \right] \left[\begin{array}{cc} 1 & 1/2 \\ 1/2 & 1/3 \end{array} \right]^{-1} \mathbf{K}(\kappa_2'', \tau^*)\} \\ &= (1 - \bar{\rho})^2\omega_\varepsilon\{\mathbf{J}(\kappa_2'', \tau^*, \tau) - \mathbf{B}_1(\tau)\mathbf{K}(\kappa_2'', \tau^*)\} \end{aligned}$$

Combining (A.24), (A.2), (A.1) and (A.23) gives (A.19).

(ii) Write

$$T^{-1}S(1) = T^{-1}\mathbf{y}'_1\mathbf{y}_1 - T^{-1}\mathbf{y}'_1\mathbf{Z}_1\Delta_3^{-1}[\Delta_3^{-1}\mathbf{Z}'_1\mathbf{Z}_1\Delta_3^{-1}]^{-1}\Delta_3^{-1}\mathbf{Z}'_1\mathbf{y}_1.$$

Now

$$\begin{aligned} T^{-1}\mathbf{y}'_1\mathbf{y}_1 &= T^{-1}\mathbf{u}'_1\mathbf{u}_1 + o_p(1) \\ &\Rightarrow \omega_\varepsilon^2, \end{aligned}$$

$$\Delta_3^{-1}\mathbf{Z}'_1\mathbf{y}_1 \Rightarrow \left[\begin{array}{c} u_1 \\ \omega_\varepsilon\{\kappa_2''(1 - \tau^*) + W(1)\} \end{array} \right]$$

and the result of (A.20) follows given (A.4).

Next

$$\begin{aligned}
\Delta_3^{-1} \mathbf{r}'_{1,\tau,34} \mathbf{r}_1 &= \Delta_3^{-1} \mathbf{Z}'_{1,\tau,34} \mathbf{y}_1 - \Delta_3^{-1} \mathbf{Z}'_{1,\tau,34} \mathbf{Z}_1 \Delta_3^{-1} [\Delta_3^{-1} \mathbf{Z}'_1 \mathbf{Z}_1 \Delta_3^{-1}]^{-1} \Delta_3^{-1} \mathbf{Z}'_1 \mathbf{y}_1 \\
&= \Delta_3^{-1} \begin{bmatrix} \Delta y_{\tau T+1} \\ y_T - y_{\tau T} \end{bmatrix} - \Delta_3^{-1} \begin{bmatrix} 0 & 1 \\ 0 & T - \tau T \end{bmatrix} \times \\
&\quad \Delta_3^{-1} \left(\Delta_3^{-1} \begin{bmatrix} 1 & 1 \\ 1 & T \end{bmatrix} \Delta_3^{-1} \right)^{-1} \Delta_3^{-1} \begin{bmatrix} y_1 \\ y_T \end{bmatrix} \\
&= \begin{bmatrix} \kappa_1 1(\tau T = \tau^* T) + \Delta u_{\tau T+1} + O(T^{-1/2}) \\ T^{-1/2} \kappa_1 + \kappa_2 T^{-1}(T - \tau^* T) - T^{-1/2} \kappa_1 1(\tau T > \tau^* T) - \\ \kappa_2 T^{-1}(\tau T - \tau^* T) 1(\tau T > \tau^* T) + T^{-1/2}(u_T - u_{\tau T}) \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & T^{-1/2} \\ 0 & 1 - \tau \end{bmatrix} \begin{bmatrix} 1 & T^{-1/2} \\ T^{-1/2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ T^{-1/2} \kappa_1 + \kappa_2 T^{-1}(T - \tau^* T) + T^{-1/2} u_T \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} \kappa_1 1(\tau = \tau^*) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} \\ \kappa_2(1 - \tau^*) - \kappa_2(\tau - \tau^*) 1(\tau > \tau^*) + \omega_\varepsilon [W(1) - W(\tau)] \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & 0 \\ 0 & 1 - \tau \end{bmatrix} \begin{bmatrix} u_1 \\ \kappa_2(1 - \tau^*) + \omega_\varepsilon W(1) \end{bmatrix} \\
&= \begin{bmatrix} \kappa_1 1(\tau = \tau^*) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} \\ \kappa_2(1 - \tau^*) - \kappa_2(\tau - \tau^*) 1(\tau > \tau^*) + \omega_\varepsilon [W(1) - W(\tau)] \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 \\ (1 - \tau)[\kappa_2(1 - \tau^*) + \omega_\varepsilon W(1)] \end{bmatrix} \\
&= \omega_\varepsilon \begin{bmatrix} \kappa_1'' 1(\tau = \tau^*) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \tau \kappa_2''(1 - \tau^*) - \kappa_2''(\tau - \tau^*) 1(\tau > \tau^*) + \tau W(1) - W(\tau) \end{bmatrix} \\
&= \omega_\varepsilon \begin{bmatrix} \kappa_1'' H_1(\tau^*, \tau) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \kappa_2'' H_2(\tau^*, \tau) + \tau W(1) - W(\tau) \end{bmatrix}.
\end{aligned}$$

Proof of Theorem 1

(i) We have

$$\begin{aligned}
\hat{\tau}_{\bar{\rho}} &= \arg \min_{\tau \in \Lambda} S(\bar{\rho}, \tau) \\
&= \arg \min_{\tau \in \Lambda} [S(\bar{\rho}) - (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' \mathbf{S}_{34}(\bar{\rho}, \tau)^{-1} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})] \\
&= \arg \max_{\tau \in \Lambda} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' \mathbf{S}_{34}(\bar{\rho}, \tau)^{-1} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}}) \\
&= \arg \max_{\tau \in \Lambda} (\Delta_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' [\Delta_1^{-1} \mathbf{S}_{34}(\bar{\rho}, \tau) \Delta_1^{-1}]^{-1} (\Delta_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}}).
\end{aligned}$$

Then using (A.8) and (A.3) we obtain

$$\begin{aligned}
\hat{\tau}_{\bar{\rho}} &\Rightarrow \arg \sup_{\tau \in \Lambda} (1 - \bar{\rho})^2 \omega_u^2 [\mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*)]' \mathbf{B}_2(\tau)^{-1} \times \\
&\quad [\mathbf{A}(\kappa'_1, \kappa'_2, \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{C}(\kappa'_1, \kappa'_2, \tau^*)].
\end{aligned}$$

Since $(1 - \bar{\rho})^2 \omega_u^2$ appears only as a scaling constant in the limit function to which $\arg \sup_{\tau \in \Lambda}$ is applied, the result follows.

(ii) We have

$$\begin{aligned}\hat{\tau}_1 &= \arg \min_{\tau \in \Lambda} S(1, \tau) \\ &= \arg \max_{\tau \in \Lambda} (\mathbf{r}'_{1,\tau,34} \mathbf{r}_1)' \mathbf{S}_{34}(1, \tau)^{-1} (\mathbf{r}'_{1,\tau,34} \mathbf{r}_1).\end{aligned}$$

Now

$$\begin{aligned}\mathbf{S}_{34}(1, \tau) &= \mathbf{Z}'_{1,\tau,34} \mathbf{Z}_{1,\tau,34} - \mathbf{Z}'_{1,\tau,34} \mathbf{Z}_1 (\mathbf{Z}'_1 \mathbf{Z}_1)^{-1} \mathbf{Z}'_1 \mathbf{Z}_{1,\tau,34} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & T - \tau T \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & T - \tau T \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & T \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & T - \tau T \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{1}{T-1} & 1 - \frac{T-T\tau}{T-1} \\ 1 - \frac{T-T\tau}{T-1} & T - T\tau - \frac{(T-T\tau)^2}{T-1} \end{bmatrix},\end{aligned}$$

and so, using (A.17)

$$\begin{aligned}\hat{\tau}_1 &= \arg \max_{\tau \in \Lambda} \begin{bmatrix} \Delta u_{\tau T+1} + o_p(1) \\ \tau u_T - u_{\tau T} + (1 - \tau)u_1 + o_p(1) \end{bmatrix}' \begin{bmatrix} -\frac{T-T\tau}{T\tau-T+1} & \frac{1}{\frac{1}{T\tau-T+1}} \\ \frac{1}{T\tau-T+1} & -\frac{1}{\frac{1}{T\tau-T+1}} \frac{T-2}{T-1} \end{bmatrix} \times \\ &\quad \begin{bmatrix} \Delta u_{\tau T+1} + o_p(1) \\ \tau u_T - u_{\tau T} + (1 - \tau)u_1 + o_p(1) \end{bmatrix} \\ &= \arg \max_{\tau \in \Lambda} [\Delta u_{\tau T+1}^2 + o_p(1)] \\ &\Rightarrow \arg \sup_{\tau \in \Lambda} \lim_{T \rightarrow \infty} \Delta u_{\tau T+1}^2.\end{aligned}$$

Proof of Theorem 2

(i) We have

$$\begin{aligned}\hat{\tau}_{\bar{\rho}} &= \arg \max_{\tau \in \Lambda} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' \mathbf{S}_{34}(\bar{\rho}, \tau)^{-1} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}}) \\ &= \arg \max_{\tau \in \Lambda} T^{-1} \mathbf{\Delta}_1^{-1} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' [\mathbf{\Delta}_1^{-1} \mathbf{S}_{34}(\bar{\rho}, \tau) \mathbf{\Delta}_1^{-1}]^{-1} T^{-1} \mathbf{\Delta}_1^{-1} (\mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}}).\end{aligned}$$

Then using (A.19) and (A.3) we obtain

$$\hat{\tau}_{\bar{\rho}} \Rightarrow \arg \sup_{\tau \in \Lambda} (1 - \bar{\rho})^2 \omega_\varepsilon^2 [\mathbf{J}(\kappa_2'', \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{K}(\kappa_2'', \tau^*)]' \mathbf{B}_2(\tau)^{-1} [\mathbf{J}(\kappa_2'', \tau^*, \tau) - \mathbf{B}_1(\tau) \mathbf{K}(\kappa_2'', \tau^*)].$$

Since $(1 - \bar{\rho})^2 \omega_\varepsilon^2$ appears only as a scaling constant in the limit function to which $\arg \sup_{\tau \in \Lambda}$ is applied, the result follows.

(ii) We have

$$\begin{aligned}\hat{\tau}_1 &= \arg \max_{\tau \in \Lambda} (\mathbf{r}'_{1,\tau,34} \mathbf{r}_1)' \mathbf{S}_{34}(1, \tau)^{-1} (\mathbf{r}'_{1,\tau,34} \mathbf{r}_1) \\ &= \arg \max_{\tau \in \Lambda} \mathbf{\Delta}_3^{-1} (\mathbf{r}'_{1,\tau,34} \mathbf{r}_1)' [\mathbf{\Delta}_3^{-1} \mathbf{S}_{34}(1, \tau) \mathbf{\Delta}_3^{-1}]^{-1} \mathbf{\Delta}_3^{-1} (\mathbf{r}'_{1,\tau,34} \mathbf{r}_1)\end{aligned}$$

and using (A.21) and (A.6) we find

$$\begin{aligned}\hat{\tau}_1 \Rightarrow \arg \sup_{\tau \in \Lambda} \omega_\varepsilon^2 &\begin{bmatrix} \kappa_1'' H_1(\tau^*, \tau) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \kappa_2'' H_2(\tau^*, \tau) + \tau W(1) - W(\tau) \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & \tau(1 - \tau) \end{bmatrix}^{-1} \times \\ &\begin{bmatrix} \kappa_1'' H_1(\tau^*, \tau) + \lim_{T \rightarrow \infty} \varepsilon_{\tau T+1} / \omega_\varepsilon \\ \kappa_2'' H_2(\tau^*, \tau) + \tau W(1) - W(\tau) \end{bmatrix}.\end{aligned}$$

The result follows since ω_ε^2 appears only as a scaling constant.

Proof of Theorem 3

For $|\bar{\rho}| < 1$ write

$$\begin{aligned}T^{-1}S(\bar{\rho}, \tau) &= T^{-1}S(\bar{\rho}) - T^{-1}(\mathbf{\Delta}_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' \{ \mathbf{\Delta}_1^{-1} \mathbf{S}_{34}(\bar{\rho}, \tau) \mathbf{\Delta}_1^{-1} \}^{-1} (\mathbf{\Delta}_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}}) \\ &= T^{-1}S(\bar{\rho}) + o_p(1) \\ &\Rightarrow \frac{\omega_\varepsilon^2}{1 - \rho^2} (1 + \bar{\rho}^2 - 2\bar{\rho}\rho)\end{aligned}$$

using (A.7), (A.8) and (A.3).

For $\bar{\rho} = 1$,

$$\begin{aligned}T^{-1}S(1, \tau) &= T^{-1}S(1) - T^{-1}(\mathbf{\Delta}_3^{-1} \mathbf{r}'_{1,\tau,34} \mathbf{r}_1)' \{ \mathbf{\Delta}_3^{-1} \mathbf{S}_{34}(1, \tau) \mathbf{\Delta}_3^{-1} \}^{-1} (\mathbf{\Delta}_3^{-1} \mathbf{r}'_{1,\tau,34} \mathbf{r}_1) \\ &= T^{-1}S(1) + o_p(1) \\ &\Rightarrow \frac{2\omega_\varepsilon^2}{1 + \rho}\end{aligned}$$

using (A.9), (A.10) and (A.6). Since the second limit is simply the first limit evaluated at $\bar{\rho} = 1$, the result is shown.

Proof of Theorem 4

(i) For $|\bar{\rho}| < 1$ write

$$T^{-2}S(\bar{\rho}, \tau) = T^{-2}S(\bar{\rho}) - T^{-2}(\mathbf{\Delta}_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})' \{ \mathbf{\Delta}_1^{-1} \mathbf{S}_{34}(\bar{\rho}, \tau) \mathbf{\Delta}_1^{-1} \}^{-1} (\mathbf{\Delta}_1^{-1} \mathbf{r}'_{\bar{\rho},\tau,34} \mathbf{r}_{\bar{\rho}})$$

which has a limit distribution given by combining the results in (A.18), (A.19) and (A.3). This distribution is non-degenerate for all τ .

(ii) For $\bar{\rho} = 1$,

$$\begin{aligned}T^{-1}S(1, \tau) &= T^{-1}S(1) - T^{-1}(\mathbf{\Delta}_3^{-1} \mathbf{r}'_{1,\tau,34} \mathbf{r}_1)' \{ \mathbf{\Delta}_3^{-1} \mathbf{S}_{34}(1, \tau) \mathbf{\Delta}_3^{-1} \}^{-1} (\mathbf{\Delta}_3^{-1} \mathbf{r}'_{1,\tau,34} \mathbf{r}_1) \\ &= T^{-1}S(1) + o_p(1) \\ &\Rightarrow \omega_\varepsilon^2\end{aligned}$$

using (A.20), (A.21) and (A.6).