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ASPHERICITY OF A LENGTH FOUR RELATIVE GROUP PRESENTATION

Abstract

We consider the relative group presentation $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ where $\boldsymbol{x} = \{x\}$ and $\boldsymbol{r} = \{xg_1xg_2xg_3x^{-1}g_4\}$. We show modulo a small number of exceptional cases exactly when $\mathcal{P}$ is aspherical. If $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle \leq G$ then the exceptional cases occur when H is isomorphic to one of C_5, C_6, C_8 or $C_2 \times C_4$.

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1 Introduction

A *relative group presentation* is a presentation of the form $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ where G is a group, $\boldsymbol{x}$ a set disjoint from G and $\boldsymbol{r}$ is a set of cyclically reduced words in the free product $G * \langle \boldsymbol{x} \rangle$ where $\langle \boldsymbol{x} \rangle$ denotes the free group on $\boldsymbol{x}$. If $G(\mathcal{P})$ denotes the group defined by $\mathcal{P}$ then $G(\mathcal{P})$ is the quotient group $G * \langle \boldsymbol{x} \rangle / N$ where N denotes the normal closure in $G * \langle \boldsymbol{x} \rangle$ of $\boldsymbol{r}$. A relative presentation is defined in [2] to be *aspherical* if every spherical picture over it contains a *dipole*. If $\mathcal{P}$ is aspherical then statements about $G(\mathcal{P})$ can be deduced and the reader is referred to [2] for a discussion of these; in particular torsion in $G(\mathcal{P})$ can easily be described.

We will consider the case when both $\boldsymbol{x}$ and $\boldsymbol{r}$ consists of a single element. Thus $\boldsymbol{r} = \{r\}$ where $r = x^{\varepsilon_1}g_1 \ldots x^{\varepsilon_k}g_k$ where $g_i \in G$, $\varepsilon_i = \pm 1$ and $g_i = 1$ implies $\varepsilon_i + \varepsilon_{i+1} \neq 0$ for $1 \leq i \leq k$, subscripts $\bmod k$. If $k \leq 3$ or if $r \in \{xg_1xg_2xg_3xg_4, xg_1xg_2xg_3xg_4xg_5\}$ then, modulo some open cases, a complete classification of when $\mathcal{P}$ is aspherical has been obtained in [1], [2], [7] and [10]. The case $r = (xg_1)^p(xg_2)^q(xg_3)^r$ for $p, q, r > 1$ was studied in [11] and $r = x^n g_1 x^{-1} g_2$ $(n \geq 4)$ was studied in [5]. The authors of [9] used results from [7] in which $r = xg_1xg_2x^{-1}g_3$ to prove asphericity for certain LOG groups. In this paper we continue the study of asphericity and consider $r = xg_1xg_2xg_3x^{-1}g_4$. Observe that $r = 1$ if and only if $x^{-1}g_2^{-1}x^{-1}g_1^{-1}x^{-1}g_4^{-1}xg_3^{-1} = 1$ so replacing x^{-1} by x it follows that we can work modulo $g_1 \leftrightarrow g_2^{-1}$ and $g_3 \leftrightarrow g_4^{-1}$. A standard approach is to make the transformation $t = xg_1$ and then consider the subgroup H of G generated by the resulting coefficients. In our case r becomes $t^2g_1^{-1}g_2tg_1^{-1}g_3g_1t^{-1}g_4$ and so $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle$. One then usually proceeds according to either when H is non-cyclic or when H is cyclic. (Note that $\langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4 \rangle$ is cyclic if and only if $\langle g_2g_1^{-1}, g_2g_4^{-1}g_2^{-1}, g_3^{-1} \rangle$ is cyclic.) The latter case seems to be the more complicated – indeed the open cases referred to in the above paragraph almost all involve H being cyclic. Our results reflect this difference in difficulty. When H is non-cyclic we obtain a complete answer except for the following case (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$) in which $H \cong C_2 \times C_4$:

(**E**) $|g_3| = 2$; $|g_4| = 4$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1g_4 = g_4g_1^{-1}g_3g_1$

where we use $|a|$ to denote the order of the element a.

Theorem 1.1 *Let $\mathcal{P}$ be the relative presentation*

$$\mathcal{P} = \langle G, x | xg_1xg_2xg_3x^{-1}g_4\rangle,$$

*where $g_i \in G$ $(1 \leq i \leq 4)$, $g_3 \neq 1$, $g_4 \neq 1$ and $x \notin G$. Let $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4\rangle$ and assume that H is non-cyclic and the exceptional case (**E**) does not hold. Then $\mathcal{P}$ is aspherical if and only if (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$) none of the following conditions holds:*

(i) $|g_4| < \infty$ *and* $g_3g_1g_2^{-1} = 1$;

(ii) $|g_1^{-1}g_2| < \infty$, $|g_3| = |g_4| = 2$ *and* $g_1^{-1}g_3g_1g_4 = g_2g_4^{-1}g_2^{-1}g_3^{-1} = 1$;

(iii) $\dfrac{1}{|g_1^{-1}g_2|} + \dfrac{1}{|g_3|} + \dfrac{1}{|g_4|} + \dfrac{1}{|g_2g_4g_1^{-1}g_3^{-1}|} > 2.$

Now let H be cyclic. Before stating the theorem we make a list of exceptions (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$).

(**E1**) $|g_4| = 5$; $g_1^{-1}g_2 = g_4^2$; $g_1^{-1}g_3g_1 = g_4^3$.

(**E2**) $|g_4| = 6$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1 = g_4^2$.

(**E3**) $|g_4| = 6$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1 = g_4^4$.

(**E4**) $|g_4| = 8$; $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1 = g_4^4$.

Observe that (**E1**) implies $H \cong C_5$; (**E2**) and (**E3**) imply $H \cong C_6$; and (**E4**) implies $H \cong C_8$.

Theorem 1.2 *Let $\mathcal{P}$ be the relative presentation*

$$\mathcal{P} = \langle G, x | xg_1xg_2xg_3x^{-1}g_4\rangle$$

*where $g_i \in G$ $(1 \leq i \leq 4)$, $g_3 \neq 1$, $g_4 \neq 1$ and $x \notin G$. Let $H = \langle g_1^{-1}g_2, g_1^{-1}g_3g_1, g_4\rangle$ be a cyclic group. Suppose that none of the exceptional conditions (**E1**)–(**E4**) holds. Then $\mathcal{P}$ is aspherical if and only if either H is infinite or H is finite and (modulo $g_1 \leftrightarrow g_2^{-1}$, $g_3 \leftrightarrow g_4^{-1}$) none of the following conditions holds:*

(i) $g_3g_1g_2^{-1} = 1$;

(ii) $g_3^{-1}g_1g_2^{-1} = g_2g_4^{-1}g_1^{-1}g_3^{-1} = 1$;

(iii) $g_3^{-1}g_1g_2^{-1} = g_4g_2^{-1}g_1 = 1$;

(iv) $|g_3| = 2$; $|g_4| = 2$;

(v) $|g_3| = 2$; $g_1^{-1}g_3g_2g_4 = g_1^{-1}g_2g_4^{-2} = 1$;

(vi) $|g_3| = 2$; $g_1^{-1}g_3g_2g_4 = (g_1^{-1}g_2)^2g_4^{-1} = 1$;

(vii) $g_1^{-1}g_2 = 1$; $g_1^{-1}g_3g_1g_4^{\pm 1} = 1$;

(viii) $g_1^{-1}g_2 = 1$; $|g_3| = 2$; $|g_4| = 3$;

(ix) $g_1^{-1}g_2 = 1$; $4 \leq |g_3| \leq 5$; $g_1^{-1}g_3^2g_1g_4$;

(x) $g_1^{-1}g_2 = 1$; $|g_3| = 6$; $g_1^{-1}g_3^3g_1g_4$.

In Section 2 we discuss pictures and curvature; in Section 3 there are some preliminary results; Theorem 1.1 and Theorem 1.2 are proved in Sections 4 and 5.

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2 Pictures

The definitions of this section are taken from [2]. The reader should consult [2] and [1] for more details.

A *picture* $\boldsymbol{P}$ is a finite collection of pairwise disjoint discs $\{\Delta_1, \ldots, \Delta_m\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1, \ldots, \alpha_n\}$ embedded in the closure of $D^2 - \bigcup_{i=1}^m \Delta_i$ in such a way that each arc meets $\partial D^2 \cup \bigcup_{i=1}^m \Delta_i$ transversely in its end points. The *boundary* of $\boldsymbol{P}$ is the circle ∂D^2, denoted $\partial \boldsymbol{P}$. For $1 \leq i \leq m$, the *corners* of Δ_i are the closures of the connected components of $\partial\Delta_i - \bigcup_{j=1}^n \alpha_j$, where $\partial\Delta_i$ is the boundary of Δ_i. The *regions* of $\boldsymbol{P}$ are the closures of the connected components of $D^2 - \left(\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j\right)$.

An *inner region* of $\boldsymbol{P}$ is a simply connected region of $\boldsymbol{P}$ that does not meet $\partial\boldsymbol{P}$. The picture $\boldsymbol{P}$ is *non-trivial* if $m \geq 1$, is *connected* if $\bigcup_{i=1}^m \Delta_i \cup \bigcup_{j=1}^n \alpha_j$ is connected, and is *spherical* if it is non-trivial and if none of the arcs meets the boundary of D^2. The number of edges in a region Δ is called the *degree* of Δ and is denoted by $d(\Delta)$. If $\boldsymbol{P}$ is a spherical picture, the number of different discs to which a disc Δ_i is connected is called the *degree* of Δ_i, denoted by $\deg(\Delta_i)$.

With $\mathcal{P} = \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$ define the following labelling: each arc α_j is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$. Each corner of $\boldsymbol{P}$ is oriented *anticlockwise* (with respect to D^2) and labelled by an element of G. If κ is a corner of a disc Δ_i of $\boldsymbol{P}$, then $W(\kappa)$ is the word obtained by reading in an anticlockwise order the labels on the arcs and corners meeting $\partial\Delta_i$ beginning with the label on the first arc we meet as we read the anticlockwise corner κ. If we cross an arc labelled x in the direction of its normal orientation, we read x, otherwise we read x^{-1}.

A picture $\boldsymbol{P}$ is called a *picture over the relative presentation* $\mathcal{P}$ if the above labelling satisfies the following conditions.

(1) For each corner κ of $\boldsymbol{P}$, $W(\kappa) \in \boldsymbol{r}^*$, the set of all cyclic permutations of the members of $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$ which begin with a member of $\boldsymbol{x}$.

(2) If $g_1, \ldots, g_l$ is the sequence of corner labels encountered in a *clockwise* traversal of the boundary of an inner region Δ of $\boldsymbol{P}$, then the product $g_1 \ldots g_l = 1$ in G. We say that $g_1 \ldots g_l$ is the *label* of Δ.

A *dipole* in a labelled picture $\boldsymbol{P}$ over $\mathcal{P}$ consists of corners κ and κ' of $\boldsymbol{P}$ together with an arc joining the two corners such that κ and κ' belong to the same region and such that if $W(\kappa) = Sg$ where $g \in G$ and S begins and ends with a member of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$, then $W(\kappa') = S^{-1}h^{-1}$. The picture $\boldsymbol{P}$ is *reduced* if it does not contain a dipole. A relative presentation $\mathcal{P}$ is called *aspherical* if every connected spherical picture over $\mathcal{P}$ contains a dipole.

The *star graph* $\mathcal{P}^{\mathrm{st}}$ of a relative presentation $\mathcal{P}$ is a graph whose vertex set is $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ and edge set is $\boldsymbol{r}^*$. For $R \in \boldsymbol{r}^*$, write $R = Sg$ where $g \in G$ and S begins and ends with a member of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$. The initial and terminal functions are given as follows: $\iota(R)$ is the first symbol of S, and $\tau(R)$ is the inverse of the last symbol of S. The labelling function on the edges is defined by $\lambda(R) = g^{-1}$ and is extended to paths in the usual way. A non-empty cyclically reduced cycle (closed path) in $\mathcal{P}^{\mathrm{st}}$ will be called *admissible* if it has trivial label in G. Each inner region of a reduced picture over $\mathcal{P}$ supports an admissible cycle in $\mathcal{P}^{\mathrm{st}}$.

As described in the introduction we will consider spherical pictures over $\mathcal{P} = \langle G, t | r \rangle$ where $r = t^2 g_1^{-1} g_2 t g_1^{-1} g_3 g_1 t^{-1} g_4$. For ease of presentation we introduce the following notation: $a = 1$, $b = g_1^{-1} g_2$, $c = g_1^{-1} g_3 g_1$ and $d = g_4$ and consider $tatbtct^{-1}d$. Exception (**E**) and conditions (i)–(iii) of Theorem 1.1 can then be re-written as

(**E**) $|c| = 2$; $|d| = 4$; $b = 1$; $cd = dc$.

(i) $|d| < \infty$ and $cab^{-1} = 1$;

(ii) $|a^{-1}b| < \infty$, $|c| = |d| = 2$ and $a^{-1}cad = bd^{-1}b^{-1}c^{-1} = 1$;

(iii) $\dfrac{1}{|a^{-1}b|} + \dfrac{1}{|c|} + \dfrac{1}{|d|} + \dfrac{1}{|bda^{-1}c^{-1}|} > 2.$

The exceptions (**E1**)–(**E4**) and conditions (i)–(x) of Theorem 1.2 can be rewritten as

(**E1**) $|d| = 5$; $b = d^2$; $c = d^3$;

(**E2**) $|d| = 6$; $b = 1$; $c = d^2$;

(**E3**) $|d| = 6$; $b = 1$; $c = d^4$;

(**E4**) $|d| = 8$; $b = 1$; $c = d^4$;

(i) $cab^{-1} = 1$;

(ii) $c^{-1}ab^{-1} = cadb^{-1} = 1$;

(iii) $c^{-1}ab^{-1} = db^{-1}a = 1$;

(iv) $|c| = |d| = 2$;

(v) $|c| = 2$; $cbda^{-1} = a^{-1}bd^{-2} = 1$;

(vi) $|c| = 2$; $cbda^{-1} = (a^{-1}b)^2 d^{-1} = 1$;

(vii) $a^{-1}b = 1$; $cad^{\pm 1}a^{-1} = 1$;

(viii) $a^{-1}b = 1$; $|c| = 2$; $|d| = 3$;

(ix) $a^{-1}b = 1$; $4 \leq |c| \leq 5$; $c^2ada^{-1} = 1$;

(x) $a^{-1}b = 1$; $|c| = 6$; $c^3ada^{-1} = 1$.

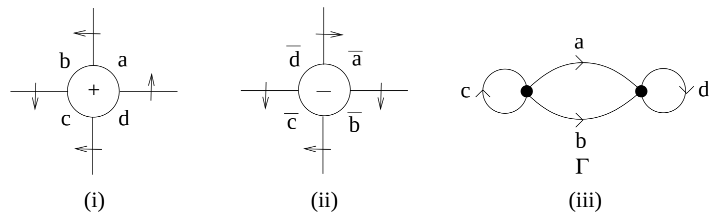


Figure 2.1: vertices and star graph

Let $\boldsymbol{P}$ be a reduced connected spherical picture over $\mathcal{P}$. Then the vertices (discs) of $\boldsymbol{P}$ are given by Figure 2.1(i) and (ii) where $\bar{x}$ denotes x^{-1} for $x \in \{a, b, c, d\}$; and the star graph Γ is given by Figure 2.1(iii).

We make the following assumptions.

($\boldsymbol{\mathcal{A}1}$) $\boldsymbol{P}$ has a minimum number of vertices.

($\boldsymbol{\mathcal{A}2}$) If $|c| = 2$ then, subject to ($\boldsymbol{\mathcal{A}1}$), $\boldsymbol{P}$ has a maximum number of regions of degree 2 with label $c^{\pm 2}$.

Observe that ($\boldsymbol{\mathcal{A}1}$) implies that $c^{\varepsilon_1} w c^{\varepsilon_2}$, $d^{\varepsilon_1} w d^{\varepsilon_2}$ where $\varepsilon_1 = -\varepsilon_2 = \pm 1$ and $w = 1$ in G cannot occur as sublabels of a region. For otherwise a sequence of bridge moves [4] can be applied to produce a dipole which can then be deleted to obtain a picture with fewer vertices. Moreover if $|c| = 2$ then ($\boldsymbol{\mathcal{A}2}$) implies that $c^{\pm 2}$ cannot be a proper sublabel and $c^{\varepsilon} w c^{\varepsilon}$ where $\varepsilon = \pm 1$, $w = 1$ in G cannot be a sublabel of a region in $\boldsymbol{P}$. For otherwise bridge moves can be applied to increase the number of regions labelled $c^{\pm 2}$ while leaving the number of vertices unchanged.

To prove asphericity we adopt the approach of [6]. Let each corner in every region of $\boldsymbol{P}$ be given an angle. The *curvature* of a vertex is defined to be 2π less the sum of the angles at that vertex. The curvature $c(\Delta)$ of a k-gonal region Δ of $\boldsymbol{P}$ is the sum of all the angles of the corners of this region less $(k-2)\pi$. Our method of associating angles is to give each corner at a vertex of degree d an angle $2\pi/d$. This way the vertices have zero curvature and we need consider only the regions. Thus if Δ is a k-gonal region of $\boldsymbol{P}$ (a k-gon), denoted by $d(\Delta) = k$, and the degree of the vertices of Δ are d_i $(1 \leq i \leq k)$ then

$$c(\Delta) = c(d_1, d_2, \ldots, d_k) = (2-k)\pi + 2\pi \sum_{i=1}^{k} (1/d_i).$$

In fact since each $d_i = 4$ $(1 \leq i \leq k)$ we obtain

$$c(\Delta) = \pi(2 - k/2)$$

so if $d(\Delta) \geq 4$ then $c(\Delta) \leq 0$.

It follows from the fundamental curvature formula that $\sum c(\Delta) = 4\pi$ is where the sum is taken over all the regions Δ of $\boldsymbol{P}$. Our strategy to show asphericity will be to show that the positive curvature that exists in $\boldsymbol{P}$ can be sufficiently compensated by the negative curvature. To this end, as a first step, we located the regions Δ of $\boldsymbol{P}$ satisfying $c(\Delta) > 0$, that is, of positive curvature. For each such Δ we distribute all of $c(\Delta)$ to regions $\hat{\Delta}$ near Δ. For such regions $\hat{\Delta}$ of $\boldsymbol{P}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives in the distribution procedure mentioned above with the understanding that if $\hat{\Delta}$ receives no positive curvature then $c^*(\hat{\Delta}) = c(\hat{\Delta})$. Observe then that the total curvature of $\boldsymbol{P}$ is at most $\sum(c^*(\hat{\Delta}))$ where the sum is taken over all regions $\hat{\Delta}$ of D that are not positively curved regions. Therefore to prove $\mathcal{P}$ is aspherical it suffices to show that $c^*(\hat{\Delta}) \leq 0$ for each $\hat{\Delta}$.

Using the star graph Γ of Figure 2.1(iii) we can list the possible labels of regions of small degree (up to cyclic permutation and inversion).

$$d(\Delta) = 2 \Rightarrow l(\Delta) \in S_2 = \{c^2, d^2, a^{-1}b\}$$

$$d(\Delta) = 3 \Rightarrow l(\Delta) \in S_3 = \{c^3, cab^{-1}, c^{-1}ab^{-1}, d^3, db^{-1}a, d^{-1}b^{-1}a\}$$

Allowing each element in $S_2 \cup S_3$ to be either trivial or non-trivial yields 512 possibilities. This number can be reduced without any loss as follows.

1. Work modulo *T-equivalence*, that is, $a \leftrightarrow b^{-1}$, $c \leftrightarrow d^{-1}$. (So, for example, the case $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$ is equivalent to $|d| = 3$, $|c| > 3$, $a^{-1}b \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$, $c^{-1}ab^{-1} = 1$, $cab^{-1} \neq 1$.
2. Delete any combination that implies $c = 1$ or $d = 1$.
3. Delete any combination that yields a contradiction (for example $c^2 = 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$.
4. Delete any combination that yields $|d| < \infty$ and $cab^{-1} = 1$ or $|c| < \infty$ and $d^{-1}b^{-1}a = 1$ (see Lemma 3.1(i)).
5. When $H = \langle b, c, d\rangle$ is cyclic it can be assumed that H is finite (see Lemma 3.4(i)).

It can be readily verified that there remain 23 cases partitioned according to the existence in $\boldsymbol{P}$ of regions of degree 2 and are listed below.

Case A: There are no regions of degree two.

(**A0**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A1**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A2**) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A3**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A4**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.

(**A5**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} = 1$, $c^{-1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A6**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A7**) $|c| = 3$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**A8**) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**A9**) $|c| = 3$, $|d| = 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**A10**) $|c| > 3$, $|d| > 3$, $a^{-1}b \neq 1$, $cab^{-1} \neq 1$, $c^{-1}ab^{-1} = 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.

Case B: Regions of degree two are possible.

(**B1**) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B2**) $|c| = 2$, $|d| = 2$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B3**) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B4**) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} = 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B5**) $|c| = 2$, $|d| > 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**B6**) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $db^{-1}a = 1$, $d^{-1}b^{-1}a \neq 1$.
(**B7**) $|c| > 3$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B8**) $|c| = 2$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B9**) $|c| = 3$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B10**) $|c| = 2$, $|d| = 2$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B11**) $|c| = 2$, $|d| = 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
(**B12**) $|c| = 3$, $|d| = 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.

3 Preliminary results

Lemma 3.1

(a) *If any of the following conditions holds then $\mathcal{P}$ fails to be aspherical:*

(i) $|d| < \infty$ *and* $cab^{-1} = 1$*;*

(ii) $|d| < \infty$ *and* $c^{-1}ab^{-1} = bd^{-1}a^{-1}c^{-1} = 1$*;*

(iii) $|d| < \infty$ and $a^{-1}b = cad^{-1}b^{-1} = 1$.

(b) *If* $bda^{-1}c^{-1} = 1$ *and* (f_1, f_2, f_3) *is any of the following then $\mathcal{P}$ fails to be aspherical.*

(i) $(2, 2, < \infty)$*;*

(ii) $(< \infty, 2, 2,)$*;*

(iii) $(2, 3, k)$ $(3 \leq k \leq 5)$*;*

(iv) $(3, 2, l)$ $(4 \leq l \leq 5)$*;*

(v) $(k, 2, 3)$ $(3 \le k \le 5)$;

where $(f_1, f_2, f_3) = (|a^{-1}b|, |c|, |d|)$.

(c) *If* $a^{-1}b = 1$ *and* (f_1, f_2, f_3) *is any of the following then* $\mathcal{P}$ *fails to be aspherical.*

(i) $(2, 2, < \infty)$;

(ii) $(2, < \infty, 2)$;

(iii) $(2, l, 3)$ $(4 \le l \le 5)$;

(iv) $(3, k, 2)$ $(3 \le k \le 5)$;

(v) $(2, 3, k)$ $(3 \le k \le 5)$,

where $(f_1, f_2, f_3) = (|c|, |d|, |bda^{-1}c^{-1}|)$.

Proof

In all cases we have found a spherical picture over $\mathcal{P}$. (The interested reader can view these at http://arxiv.org/abs/1604.00163.) $\square$

It follows from Theorem 1(2) in [1] that if $|t| < \infty$ in $G(\mathcal{P})$ then $\mathcal{P}$ fails to be aspherical. We apply this fact in the proof of the next lemma.

Lemma 3.2 *If any of the following conditions hold then* $\mathcal{P}$ *fails to be aspherical.*

(i) $|a^{-1}b| < \infty$, $|c| = |d| = 2$ *and* $a^{-1}cad = bdb^{-1}c = 1$.

(ii) $|d| < \infty$ *and* $c^{-1}ab^{-1} = db^{-1}a = 1$.

(iii) $|d| < \infty$ *and* $a^{-1}b = cada^{-1} = 1$.

(iv) $c^2 = cbda^{-1} = a^{-1}bd^{-2} = 1$.

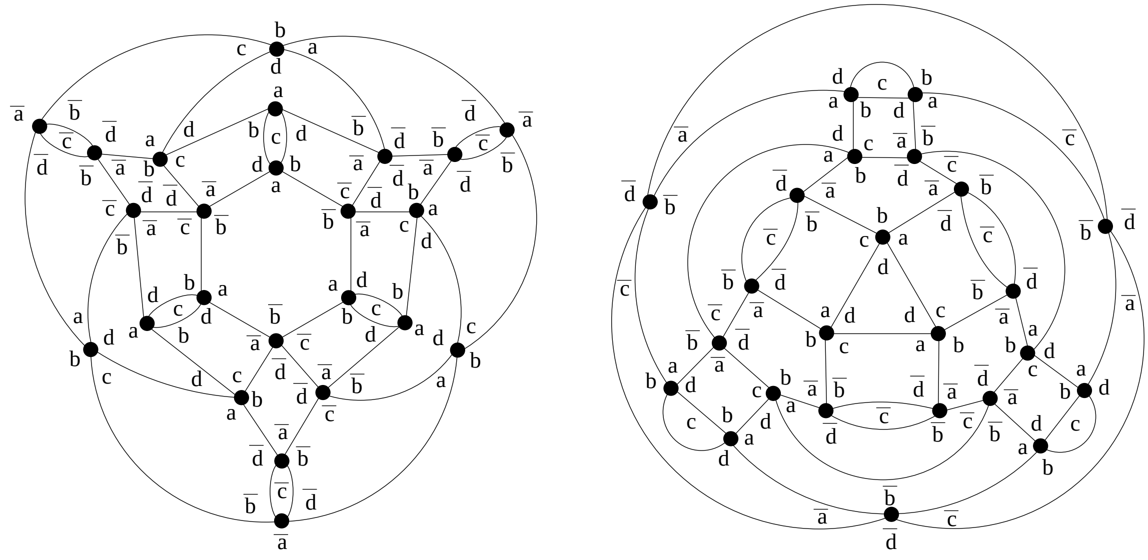
(v) $c^2 = cbda^{-1} = (a^{-1}b)^2d^{-1} = 1$.

(vi) $a^{-1}b = c^2 = d^3 = 1$ *and* $cada^{-1}cad^{-1}a^{-1} = 1$.

(vii) $a^{-1}b = c^2ada^{-1} = 1$ *and* $4 \le |c| \le 5$.

(viii) $a^{-1}b = c^3ada^{-1} = 1$ *and* $|c| = 6$.

Proof



(i) (ii)

Figure 3.1: spherical pictures

(i) It is enough to show that the group $G = \langle b, d, t \mid d^2 = b^k = 1, bd = db, t^2btdt^{-1}d = 1\rangle$ has order $2k(3^{2k} - 1)$. Now $G = \langle d, t \mid d^2, t^{-2}d^{-1}td^{-1}t^{-1}dtdt^{-1}dt^2d^{-1}, (t^3dt^{-1}d)^k\rangle$ and $G/G' = \langle d, t \mid d^2 = t^{2k} = [d, t] = 1\rangle$. Let $\mathcal{K}$ denote the covering 2-complex associated with G' [3]. Then $\mathcal{K}$ has edges $t_{0j}, t_{1j}, d_{j0}, d_{j1}$ $(1 \le j \le 2k)$ and 2-cells $d_{j0}d_{j1}$, $t_{0j}d_{1-j}t_{1j}^{-1}d_{2-j}$, $t_{i1}t_{i2}\dots t_{i2k}$ where $1 \le i \le 2$ and $1 \le j \le 2k$ and the d subscripts are mod $2k$. Collapsing the maximal subtree whose edges are d_{j0} $(1 \le j \le 2k)$, t_{0l} $(2 \le l \le 2k)$ and using the lifts of d^2 shows that $G' = \langle t_{01}, t_{1j}\ (1 \le j \le 2k)\rangle$. Using the lifts of the second relator it is easily shown that $G' = \langle t_{01}, t_{11}, t_{12}\rangle$ where $t_{11}t_{01}^{-1} = t_{12}^{-3^{2k-1}}$ and, finally, using the lift of the third relator $(t^3dt^{-1}d)^k$ one can show that $G' = \langle t_{12} \mid t_{12}^r\rangle$ where $r = \frac{1}{2}(3^{2k} - 1)$. We omit the details.

(ii) It is enough to show that $G = \langle d, x \mid d^k, t^2dtd^{-1}t^{-1}d\rangle$ has order $2k(1 + 4 + 4^2 + \dots + 4^{k-1})$. Now $G = \langle u, t \mid (ut^{-2})^k, tut^{-1}u^{-2}\rangle$ and $G/G' = \langle u, t \mid u = t^{2k} = 1\rangle$. Let $\mathcal{L}$ denote the covering complex associated with G'. Then $\mathcal{L}$ has edges t_j, u_j $(1 \le j \le 2k)$ and 2-cells $t_1t_2\dots t_{2k}, u_j$ $(1 \le j \le 2k)$. Collapsing the maximal tree whose edges are $t_1, \dots, t_{k-1}$ implies $G' = \langle t_{2k}, u_j\ (1 \le j \le 2k)\rangle$. The lifts of $tut^{-1}u^{-2}$ yield the relators $u_l = u_1^{2^{l-1}}$ for $2 \le l \le 2k$ and $t_{2k}u_1t_{2k}^{-1}u_1^{-4^k}$. The lifts of $(ut^{-2})^k$ yield the relators $t_{2k}^{-1} = \prod_{i=0}^{k-1} u_{2i+1} = \prod_{i=1}^{k} u_{2i}$. It follows that $G' = \langle u_1 \mid u_1^r\rangle$ where $r = 1 + 4 + 4^2 + \dots + 4^{k-1}$.

(iii) Here $r = t^3dt^{-1}d^{-1}$ and if $|d| = k < \infty$ then $t = d^ktd^{-k}$ which implies $t = t^{3^k}$ and so $|t| < \infty$.

(iv) – (v) A spherical picture for (iv), (v) is shown in Figure 3.1(i), (ii) (respectively). (Note that when drawing figures the discs (vertices) will often be represented by points; the edge arrows shown in Figure 2.1 will be omitted; and regions with label $c^{\pm 2}, d^{\pm 2}$ will be labelled simply by $c^{\pm 1}, d^{\pm 1}$.)

(vi) – (viii) For these cases we use GAP [8]. For (vi), $r = t^3ct^{-1}d$ together with the conditions yields $|t| \leq 12$; for (vii) $r = t^3ct^{-1}c^{-2}$ and $|c| = 4, 5$ implies $|t| \leq 8, 10$ (respectively); and for (viii) $r = t^3ct^{-1}c^{-3}$ and $|c| = 6$ implies $|t| \leq 24$. □

Lemma 3.3 *If any of the conditions (i)–(iii) of Theorem 1.1 or (i)–(x) of Theorem 1.2 holds then $\mathcal{P}$ fails to be aspherical.*

Proof Consider Theorem 1.1. If (i) holds then $\mathcal{P}$ fails to be aspherical by Lemma 3.1(a)(i). If (ii) holds then $\mathcal{P}$ fails to be aspherical by Lemma 3.2(i). This leaves condition (iii). If $a^{-1}b \neq 1$ and $bda^{-1}c^{-1} \neq 1$ then (iii) does not hold; and if $a^{-1}b = bda^{-1}c^{-1} = 1$ then H is cyclic. Let $a^{-1}b = 1$. Since $(|c|, |d|, |bda^{-1}c^{-1}|)$ is T-equivalent to $(|d|, |c|, |bda^{-1}c^{-1}|)$ it can be assumed without any loss that $|c| \leq |d|$. The resulting ten cases are dealt with by Lemma 3.1(c). Let $bda^{-1}c^{-1} = 1$. Since $(|a^{-1}b|, |c|, |d|)$ is T-equivalent to $(|a^{-1}b|, |d|, |c|)$ it can again be assumed without any loss that $|c| \leq |d|$. The resulting ten cases are dealt with by Lemma 3.1(b). Now consider Theorem 1.2. If (i) holds then $\mathcal{P}$ is aspherical by Lemma 3.1(a)(i); if (ii) holds then by Lemma 3.1(a)(ii); if (iii) holds then by Lemma 3.2(ii); if (iv) holds then by Lemma 3.2(i); if (v) holds then by Lemma 3.2(iv); if (vi) holds then by Lemma 3.2(v); if (vii) holds then by Lemmas 3.1(a)(iii) and 3.2(iii); if (viii) holds then by Lemma 3.2(vi); if (ix) holds then by Lemma 3.2(vii); and if (x) holds then by Lemma 3.2(viii). □

A *weight function* α on the star graph Γ of Figure 2.1(iii) is a real-valued function on the set of edges of Γ. Denote the edge labelled a, b, c, d by e_a, e_b, e_c, e_d (respectively). The function α is *weakly aspherical* if the following two conditions are satisfied:

(1) $\alpha(e_a) + \alpha(e_b) + \alpha(e_c) + \alpha(e_d) \leq 2$;

(2) each admissible cycle in Γ has weight at least 2.

If there is a weakly aspherical weight function on Γ then $\mathcal{P}$ is aspherical [2].

Lemma 3.4 *If any of the following conditions holds then $\mathcal{P}$ is aspherical.*

(i) $|c| = |d| = \infty$;

(ii) $1 < |b| < \infty$ *and* $|d| = \infty$;

(iii) $|c| < \infty$, $|d| < \infty$ *and* $|b| = \infty$.

Proof The following functions α are weakly aspherical.

(i) $\alpha(e_a) = \alpha(e_b) = 1$, $\alpha(e_c) = \alpha(e_d) = 0$.

(ii) $\alpha(e_a) = \alpha(e_b) = \frac{1}{2}$, $\alpha(e_c) = 1$, $\alpha(e_d) = 0$.

(iii) $\alpha(e_a) = \alpha(e_b) = 0$, $\alpha(e_c) = \alpha(e_d) = 1$. $\square$

The following lemmas will be useful in later sections.

Lemma 3.5 *Let $d(\hat{\Delta}) = k$ where $\hat{\Delta}$ is a region of the spherical picture $\boldsymbol{P}$ over $\mathcal{P}$.*

(i) *If $\hat{\Delta}$ receives at most $\frac{\pi}{6}$ across each edge and $k \geq 6$ then $c^*(\hat{\Delta}) \leq 0$.*

(ii) *If $\hat{\Delta}$ receives at most $\frac{\pi}{4}$ across at most two-thirds of its edges, nothing across the remaining edges and $k \geq 6$ then $c^*(\hat{\Delta}) \leq 0$.*

(iii) *If $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ across at most half of its edges, nothing across the remaining edges and $k \geq 7$ then $c^*(\hat{\Delta}) \leq 0$.*

(iv) *If $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ across at most three-fifths of its edges, nothing across the remaining edges and $k \geq 8$ then $c^*(\hat{\Delta}) \leq 0$.*

Proof The statements are easy consequences of the fact that $c(\hat{\Delta}) = \pi(2 - k/2)$. $\square$

Remark We will use the above lemmas as follows. Suppose that $\hat{\Delta}$ receives positive curvature across its edge e_i. If we know that it then never receives curvatures across e_{i-1} or across e_{i+1} then we can apply the "half" results; or if we know that it receives positive curvature across at most one of e_{i-1}, e_{i+1} then we can apply the "two-thirds" results.

Let $\hat{\Delta}$ be a region of $\boldsymbol{P}$ and let e be an edge of $\hat{\Delta}$. If $\hat{\Delta}$ receives no curvature across e then e is called a *gap*; if it receives at most $\frac{\pi}{6}$ then e is called a *two-thirds gap*; and if it receives at most $\frac{\pi}{4}$ then e is called a *half gap*.

Lemma 3.6 (Four Gaps Lemma) *If $\hat{\Delta}$ has a total of at least four gaps (in particular, four edges across which $\hat{\Delta}$ does not receive any curvature) and the most curvature that crosses any edge is $\frac{\pi}{2}$ then $c^*(\hat{\Delta}) \leq 0$.*

Proof By assumption $\hat{\Delta}$ has a full gaps, b two-thirds gaps and c half-gaps where $a\frac{\pi}{2} + b\frac{\pi}{3} + c\frac{\pi}{4} \geq 2\pi$. It follows that $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{k}{2}\right) + k\frac{\pi}{2} - \left(a\frac{\pi}{2} + b\frac{\pi}{3} + c\frac{\pi}{4}\right) \leq 0$. $\square$

Checking the star graph shows that we will have the following LIST for the labels of regions of degree k where $k \in \{2, 3, 4, 5, 6, 7\}$:

If $d(\Delta) = 2$ then $l(\Delta) \in \{c^2, a^{-1}b, d^2\}$.

If $d(\Delta) = 3$ then $l(\Delta) \in \{c^3, cab^{-1}, c^{-1}ab^{-1}, db^{-1}a, d^{-1}b^{-1}a, d^3\}$.

If $d(\Delta) = 4$ then $l(\Delta) \in \{d^4, d^2a^{-1}b, d^2b^{-1}a, c^2ab^{-1}, c^2ba^{-1}, c^4, ab^{-1}ab^{-1}, d\{a^{-1}, b^{-1}\}$ $\{c, c^{-1}\}\{a, b\}\}$.

If $d(\Delta) = 5$ then $l(\Delta) \in \{d^5, d^3a^{-1}b, d^3b^{-1}a, c^3ab^{-1}, c^3ba^{-1}, c^5, cab^{-1}ab^{-1}, cba^{-1}ba^{-1}, da^{-1}ba^{-1}b, db^{-1}ab^{-1}a, d^2\{a^{-1}, b^{-1}\}\{c, c^{-1}\}\{a, b\}, c^2\{a, b\}\{d, d^{-1}\}\{a^{-1}, b^{-1}\}\}$.

If $d(\Delta) = 6$ then $l(\Delta) \in \{d^6, d^4a^{-1}b, d^4b^{-1}a, c^4ab^{-1}, c^4ba^{-1}, c^6, ab^{-1}ab^{-1}ab^{-1}, d^2a^{-1}ba^{-1}b, d^2b^{-1}ab^{-1}a, c^2ab^{-1}ab^{-1}, c^2ba^{-1}ba^{-1}, d^3\{a^{-1}, b^{-1}\}\{c, c^{-1}\}\{a, b\}, d^2\{a^{-1}, b^{-1}\}\{c^2, c^{-2}\}\{a, b\}, d\{a^{-1}, b^{-1}\}\{c^3, c^{-3}\}\{a, b\}, c\{ab^{-1}, ba^{-1}\}\{c, c^{-1}\}\{ab^{-1}, ba^{-1}\}, d\{a^{-1}b, b^{-1}a\}\{d, d^{-1}\}\{a^{-1}b, b^{-1}a\}, c\{ab^{-1}a, ba^{-1}b\}\{d, d^{-1}\}\{a^{-1}, b^{-1}\}, c\{a, b\}\{d, d^{-1}\}\{a^{-1}ba^{-1}, b^{-1}ab^{-1}\}\}$.

Where, for example $d^2\{a^{-1}, b^{-1}\}\{c, c^{-1}\}\{a, b\}$ yields the eight labels $d^2a^{-1}c^{\pm 1}a$, $d^2a^{-1}c^{\pm 1}b$, $d^2b^{-1}c^{\pm 1}a$, $d^2b^{-1}c^{\pm 1}b$.

We will use the Four Gaps Lemma and the above LIST throughout the following sections often without explicit reference.

4 Proof of Case A

In this section we consider Theorems 1.1 and 1.2 for Case A, that is, we make the following assumptions:

$$|c| > 2, \ |d| > 2 \text{ and } a^{-1}b \neq 1.$$

This implies that $d(\Delta) \geq 3$ for each region Δ of the spherical diagram $\boldsymbol{P}$. If $d(\Delta) = 3$ then we will fix the names of the fifteen neighbouring regions Δ_i $(1 \leq i \leq 15)$ of Δ as shown in Figure 4.1(i).

If **(A0)** holds (see Section 2) then $d(\Delta) > 3$ for all regions Δ. Since the degree of each vertex equals 4 it follows that $c(\Delta) = (2 - d(\Delta))\pi + d(\Delta)\frac{2\pi}{4} \leq 0$ and so $\mathcal{P}$ is aspherical. If **(A5)** holds and $|d| < \infty$ then there is a sphere by Lemma 3.1(a)(i), otherwise $|d| = \infty$ and $\mathcal{P}$ is aspherical by Lemma 3.4(ii). If **(A9)** or **(A10)** holds then H is cyclic and, since we then assume $|d| < \infty$, $\mathcal{P}$ is aspherical by Lemma 3.2(ii).

(A2) $\boldsymbol{|c| = 3}$, $\boldsymbol{|d| = 3}$, $\boldsymbol{a^{-1}b \neq 1}$, $\boldsymbol{c^{\pm 1}ab^{-1} \neq 1}$, $\boldsymbol{d^{\pm 1}b^{-1}a \neq 1}$.
If $d(\Delta) = 3$ then Δ is given by Figures 4.1(ii) and 4.1(iv). If $d(\Delta) = 4$ and $l(\Delta) \in \{bd\omega, ca\omega\}$ then $l(\Delta) \in S = \{bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}, cad^{\pm 1}a^{-1}, cad^{\pm 1}b^{-1}\}$ (see the LIST of Section 3) otherwise there is a contradiction to one of the assumptions. (Throughout this case unless otherwise stated this means one of the (**A2**) assumptions.)

The cases to be considered are (where for example case (ii) means $bda^{-1}c$ is the only member of S to equal 1):

(i) $bda^{-1}c^{\pm 1} \neq 1$, $bdb^{-1}c^{\pm 1} \neq 1$, $cad^{\pm 1}a^{-1} \neq 1$, $cadb^{-1} \neq 1$; (ii) $bda^{-1}c = 1$; (iii) $bda^{-1}c^{-1} = 1$; (iv) $bdb^{-1}c = 1$; (v) $bdb^{-1}c^{-1} = 1$; (vi) $cada^{-1} = 1$; (vii) $cad^{-1}a^{-1} = 1$; (viii) $cadb^{-1} = 1$; (ix) $bdb^{-1}c = 1$, $cada^{-1} = 1$; (x) $bdb^{-1}c = 1$, $cad^{-1}a^{-1} = 1$; (xi) $bdb^{-1}c^{-1} = 1$, $cada^{-1} = 1$; (xii) $bdb^{-1}c^{-1} = 1$, $cad^{-1}a^{-1} = 1$.

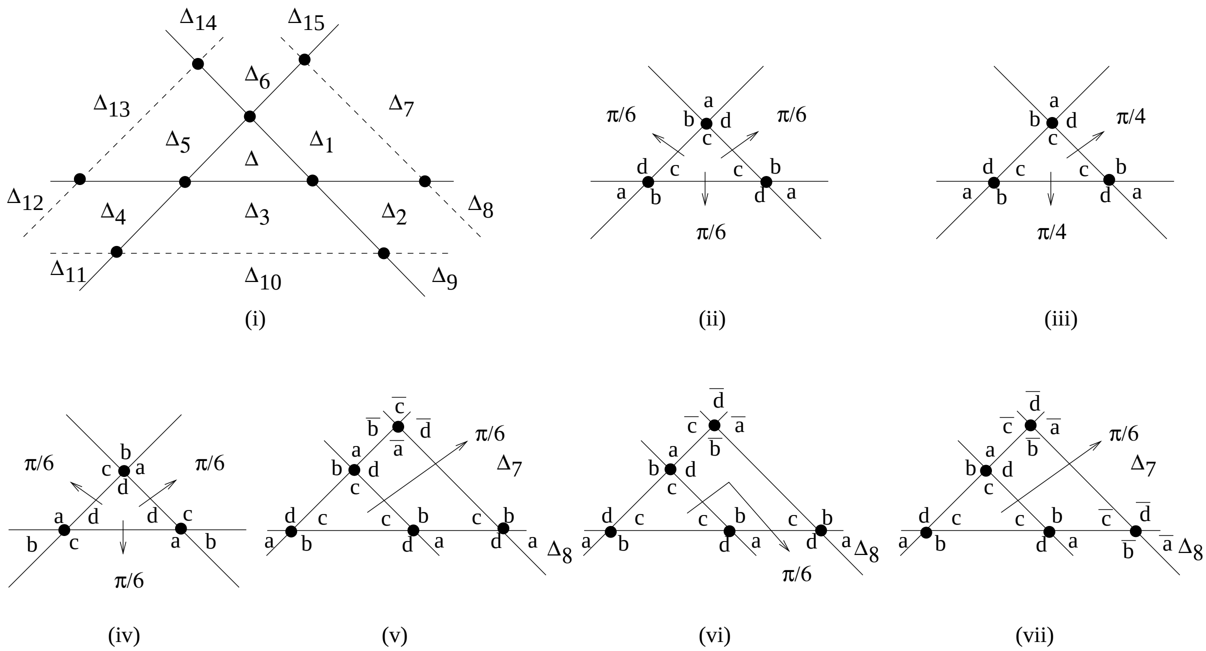


Figure 4.1: the region Δ and curvature distribution for Case (A2)

Note that any other combination of these conditions gives a contradiction to one of the assumptions. Moreover, (ii) is T-equivalent to (viii); (iv) is T-equivalent to (vi); (v) is T-equivalent to (vii); and (x) is T-equivalent to (xi). So it remains to consider (i), (ii), (iii), (iv), (v), (ix), (x) and (xii).

Now let $c(\Delta) > 0$ and so $l(\Delta) \in \{c^3, d^3\}$. In cases (i), (ii), (iv) and (v) add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as shown in Figures 4.1(ii) and 4.1(iv). If $d(\Delta_i) > 4$ then no further distribution takes place. Suppose without any loss of generality that $d(\Delta_1) = 4$. This cannot happen in case (i); in case (ii) Δ_1 is given by Figure 4.1(v) and so add the $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_7)$ across the bd and bd^{-1} edges noting that $d(\Delta_7) > 4$ otherwise $l(\Delta_7) \in \{bd^{-2}a^{-1}, bd^{-1}a^{-1}c^{\pm 1}, bd^{-1}b^{-1}c^{\pm 1}\}$ which contradicts one of the assumptions; in case (iv) Δ_1 is given by Figure 4.1(vi) and so add the $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_2)$ across the bd and ad edges noting that $d(\Delta_2) > 4$ otherwise $l(\Delta_2) \in \{ad^2b^{-1}, ada^{-1}c^{\pm 1}, ad^{-1}b^{-1}c^{\pm 1}\}$ which contradicts one of the assumptions or yields case (ix) or (x); and in case (v) Δ is given by Figure 4.1(vii) and so add the $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_7)$ across the bd and $d^{-1}a^{-1}$ edges noting $d(\Delta_7) > 4$ otherwise $l(\Delta_7) \in \{d^{-2}a^{-1}b, d^{-1}a^{-1}c^{\pm 1}a, d^{-1}a^{-1}c^{\pm 1}b\}$ which contradicts one of the assumptions or yields case (xi) or (xii). Therefore if the region $\hat{\Delta}$ receives positive curvature then it receives $\frac{\pi}{6}$ across each edge and so if $d(\hat{\Delta}) \geq 6$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(i). This leaves the case when $d(\hat{\Delta}) = 5$. After checking for vertex labels that contain the sublabels (bd), (ca), (ad) and (bd^{-1}) corresponding to the edges crossed in Figures 4.1(ii) and 4.1(iv)–(vii) we obtain $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 3.\frac{\pi}{6} = 0$. This completes cases (i), (ii), (iv) and (v).

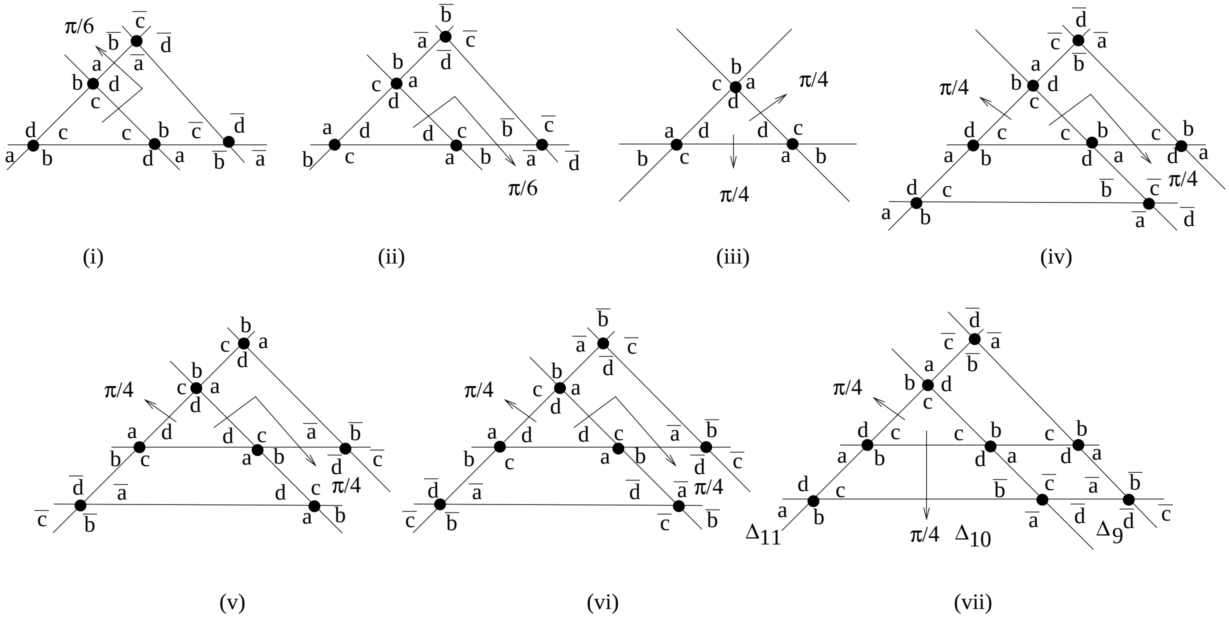


Figure 4.2: curvature distribution for Case (A2)

Consider case (iii), $bda^{-1}c^{-1} = 1$. If $b^2 = 1$ then we obtain a sphere by Lemma 3.1(b)(iii). Note also that H is non-cyclic in this case otherwise we obtain $b = 1$, a contradiction. Suppose then that $b^2 \neq 1$. First let $l(\Delta) = c^3$. Add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as in Figure 4.1(ii). If $d(\Delta_i) > 4$ then no further distribution takes place. Suppose that $d(\Delta_1) = 4$. Then add $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_6)$ across the bd and ab^{-1} edges as shown in Figure 4.2(i) noting that $d(\Delta_6) > 4$, otherwise $l(\Delta_6) \in \{b^{-1}ab^{-1}a, b^{-1}ad^{\pm 2}\}$ which is a contradiction either to $b^2 \neq 1$ or to one of the assumptions; and if $d(\Delta_3) = 4$ or $d(\Delta_5) = 4$ in Figure 4.1(ii) then similarly add $\frac{\pi}{6}$ to Δ_2 or Δ_4. Secondly, let $l(\Delta) = d^3$. Add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as in Figure 4.1(iv). If $d(\Delta_i) > 4$ then no further distribution takes place. Suppose without any loss of generality that $d(\Delta_1) = 4$. Then add $\frac{\pi}{6}$ from $c(\Delta)$ to $c(\Delta_2)$ across the ca and ba^{-1} edges as shown in Figure 4.2(ii), noting $d(\Delta_2) > 4$, otherwise $l(\Delta_2) \in \{ba^{-1}ba^{-1}, ba^{-1}c^{\pm 2}\}$ which is a contradiction either to $b^2 \neq 1$ or to one of the assumptions. If $d(\Delta_3) = 4$ or $d(\Delta_5) = 4$ then similarly add $\frac{\pi}{6}$ to Δ_4 or Δ_6. If $\hat{\Delta}$ receives positive curvature and $d(\hat{\Delta}) \geq 6$,it follows by Lemma 3.5(i) that $c^*(\hat{\Delta}) \leq 0$. It remains to check $d(\hat{\Delta}) = 5$. After checking for vertex labels that contain the sublabels (bd),(ca),$(b^{-1}a)$ and (ba^{-1}) corresponding to the edges crossed in Figures 4.1(ii), (iv) and 4.2(i), (ii) we obtain $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 3.\frac{\pi}{6} = 0$ or $l(\hat{\Delta}) \in \{bda^{-1}ba^{-1}, cab^{-1}ab^{-1}\}$ and this contradicts one of the assumptions. Therefore $c^*(\hat{\Delta}) \leq 0$.

Consider case (ix), $bdb^{-1}c = 1$ and $cada^{-1} = 1$. If $d(\Delta_i) > 4$ for at least two of Δ_i where $i \in \{1, 3, 5\}$, say Δ_1 and Δ_3,then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$ across the bd and ca edges as shown in Figures 4.1(iii) and 4.2(iii). By symmetry it can be assumed that $d(\Delta_1) = d(\Delta_3) = 4$. The two possibilities are given in Figures 4.2(iv) and 4.2(v) and

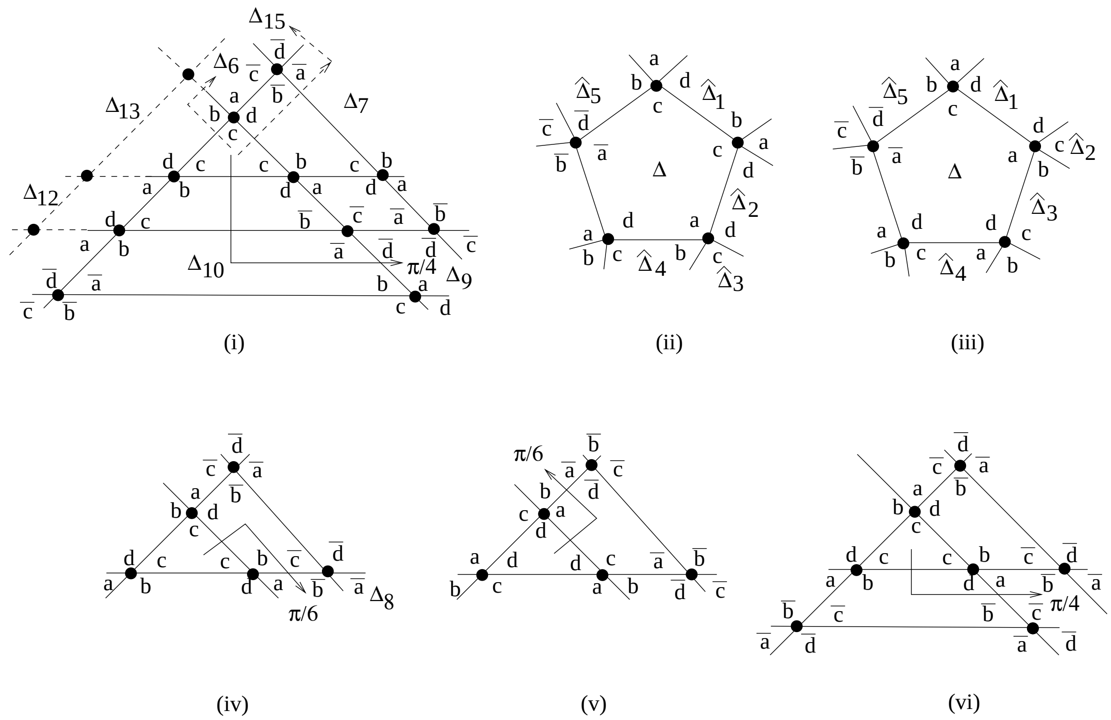


Figure 4.3: curvature distribution for Case (A2) and regions of degree 5

in both cases add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ across the bd, ad or ca, bd^{-1} edges as shown. If $d(\Delta_5) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$; or if $d(\Delta_5) = 4$ then apply the above to Δ_1 and Δ_5 to distribute the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ similarly to $c(\Delta_6)$. Now observe that if Δ_1 receives positive curvature from Δ then it does not receive positive curvature from Δ_2; and if Δ_2 receives positive curvature from Δ (as in Figures 4.2(iv) and 4.2(v)) then it does not receive positive curvature from Δ_3. It follows that if the region $\hat{\Delta}$ receives positive curvature then it does so across at most two-thirds of its edges and therefore if $\hat{\Delta}$ receives positive curvature and if $d(\hat{\Delta}) \geq 6$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(ii). Note that $d(\Delta_2) > 4$ in Figures 4.2(iv) and 4.2(v) otherwise $l(\Delta_2) \in \{c^{-1}ada^{-1}, c^{-1}adb^{-1}, cbd^{-1}a^{-1}, cbd^{-1}b^{-1}\}$ which contradicts one of the assumptions. So there remains the case $d(\hat{\Delta}) = 5$ and $l(\hat{\Delta}) \in \{bd\omega, ca\omega, c^{-1}ad\omega, cbd^{-1}\omega\}$. Checking shows that in each case $c^*(\hat{\Delta}) \leq \pi(2 - \frac{5}{2}) + 2.\frac{\pi}{4} = 0$.

Consider (x), $bdb^{-1}c = 1$ and $cad^{-1}a^{-1} = 1$. First consider $l(\Delta) = d^3$. If at least two of the Δ_i where $i \in \{1, 3, 5\}$ have degree greater than four, say Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and $c(\Delta_3)$ as shown in Figure 4.2(iii). So assume otherwise and without any loss of generality let $d(\Delta_1) = d(\Delta_3) = 4$ as shown in Figure 4.2(vi) where $d(\Delta_2) > 4$ otherwise $l(\Delta_2) = a^{-1}bd^{-2}$ which contradicts $d^{-1}b^{-1}a \neq 1$. So add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ as shown in Figure 4.2(vi). If $d(\Delta_5) > 4$ in Figure 4.2(vi) add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$ otherwise use the same argument as above for Δ_1 and Δ_5 and add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_6)$. Now consider $l(\Delta) = c^3$. If at least two of the Δ_i where $i \in \{1, 3, 5\}$ have degree > 4, say,

Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_1)$ and $c(\Delta_3)$ as in Figure 4.1(iii). Suppose exactly two of the Δ_i have degree $= 4$, say, Δ_1 and Δ_3. Add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$. If $d(\Delta_2) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ as in Figure 4.2(iv). If $d(\Delta_2) = 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_{10})$ as in Figure 4.2(vii). If now $d(\Delta_{10}) = 4$ then $l(\Delta_{10}) = ba^{-1}ba^{-1}$ and so add the $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_9)$ as in Figure 4.3(i). Observe that $l(\Delta_9) = ad^{-1}d^{-1}w$ forces $d(\Delta_9) > 4$ otherwise there is a contradiction to $d^{-1}b^{-1}a \neq 1$. Finally suppose that $d(\Delta_i) = 4$ for $i \in \{1, 3, 5\}$. Then $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ is added to either Δ_2, Δ_{10} or Δ_9 exactly as above; similarly $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ is added to Δ_6, Δ_7 or Δ_{15} as shown in Figure 4.3(i).

Now observe that in Figures 4.2(iii) and 4.1(iii) Δ_1 does not receive positive curvature from Δ_2; in Figures 4.2(vi) and 4.2(iv) Δ_2 does not receive positive curvature from Δ_3; in Figure 4.2(vii) Δ_{10} does not receive positive curvature from Δ_{11}; and in Figure 4.3(i) Δ_9 does not receive positive curvature from Δ_2. Observe that if $\hat{\Delta}$ receives positive curvature then $d(\hat{\Delta}) \geq 5$. It follows from Lemma 3.5(ii) that if $d(\hat{\Delta}) \geq 6$ then $c^*(\hat{\Delta}) \leq 0$ so let $d(\hat{\Delta}) = 5$. If $\hat{\Delta}$ receives across at most two edges then $c^*(\hat{\Delta}) \leq 0$ so it remains to check if $\hat{\Delta}$ receives curvature from more than two edges. From the above we see that positive curvature is transferred across $(ca), (bd), (bd^{-1}), (ad), (ba^{-1}), (ad^{-1})$-edges. The only two labels that contain more than two such sublabels and do not yield a contradiction are $a^{-1}ccad$ and $a^{-1}cadd$ as shown in Figures 4.3(ii)–(iii). Let $l(\Delta) = a^{-1}ccad = 1$ as in Figure 4.3(ii). Here Δ receives nothing from $\hat{\Delta}_1$ or $\hat{\Delta}_5$. If $d(\hat{\Delta}_2) > 3$ then Δ receives nothing from $\hat{\Delta}_2$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}_2) = 3$ then $d(\hat{\Delta}_3) > 3$ and Δ receives nothing from $\hat{\Delta}_3$ via $\hat{\Delta}_4$ as in Figure 4.2(iv) and again $c^*(\hat{\Delta}) \leq 0$. Let $l(\Delta) = a^{-1}cadd = 1$ as in Figure 4.3(iii). Here Δ receives nothing from $\hat{\Delta}_4$ or $\hat{\Delta}_5$. If $d(\hat{\Delta}_1) > 3$ then Δ receives nothing from $\hat{\Delta}_1$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}_1) = 3$ then $d(\hat{\Delta}_2) > 3$ and Δ receives nothing from $\hat{\Delta}_2$ via $\hat{\Delta}_3$ and again $c^*(\hat{\Delta}) \leq 0$.

Finally consider case (xii), $bdb^{-1}c^{-1} = cad^{-1}a^{-1} = 1$. Then $c = d$ and so $bdb^{-1}d^2 = 1$. If now $b^2 = 1$ then we obtain $bdbd^2 = 1$ and $H = \langle bd \rangle$ is cyclic. Assume first that H is non-cyclic so, in particular, $|b| > 2$. Add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to $c(\Delta_i)$ for $i \in \{1, 3, 5\}$ as in Figures 4.1(ii) and 4.1(iv) across the bd and ca edges. If, say, $d(\Delta_1) > 4$ then no further distribution takes place. If $d(\Delta_1) = 4$ then the $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ is added to $c(\Delta_2)$ if $l(\Delta) = c^3$ across the bd and ab^{-1} edges, or to $c(\Delta_6)$ if $l(\Delta) = d^3$ across the ca and $a^{-1}b$ edges as shown in Figures 4.3(iv), (v). Observe that $d(\Delta_2) > 4$ and $d(\Delta_6) > 4$. If $\hat{\Delta}$ receives positive curvature and $d(\hat{\Delta}) \geq 6$, it follows by Lemma 3.5(i) that $c^*(\hat{\Delta}) \leq 0$. It remains to check $d(\hat{\Delta}) = 5$. After checking for vertex labels that contain the sublabels (bd),(ca),(ab^{-1}) and $(a^{-1}b)$ corresponding to the edges crossed in Figures 4.1(ii), 4.1(iv) and 4.3(iv), (v) it follows either that $\hat{\Delta}$ receives at most $3.\frac{\pi}{6}$ and so $c^*(\hat{\Delta}) \leq 0$ or $l(\hat{\Delta}) \in \{bda^{-1}ba^{-1}, cab^{-1}ab^{-1}\}$ which in each case yields a contradiction to H non-cyclic. Now assume that H is cyclic. If at least two of the Δ_i where $i \in \{1, 3, 5\}$ have degree greater than four, say Δ_1 and Δ_3, then add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to each of $c(\Delta_1)$ and $c(\Delta_3)$ as shown in Figures 4.1(iii) and 4.2(iii). By symmetry assume then that $d(\Delta_1) = d(\Delta_3) = 4$. The two possibilities are in Figures 4.3(vi) and 4.2(vi) and in each case add $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_2)$ as shown. If $d(\Delta_5) > 4$ then add the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ to $c(\Delta_5)$; or if $d(\Delta_5) = 4$ then similarly distribute the remaining $\frac{1}{2}c(\Delta) = \frac{\pi}{4}$ via Δ_5 to Δ_4 or Δ_6. Now observe that if Δ_1 receives positive curvature from Δ it does not receive curvature from Δ_2; and if Δ_2 receives positive curvature from Δ it does not receive positive curvature from Δ_1 in Figure 4.3(vi) or from Δ_3 in Figure 4.2(vi). It follows that if $d(\hat{\Delta}) \geq 6$ then $c^*(\Delta) \leq 0$ by Lemma 3.5(ii). Now $d(\Delta_2) > 4$ in Figures 4.3(vi) and 4.2(vi) so there remains the case $d(\hat{\Delta}) = 5$ and $d(\hat{\Delta}) \in \{bdw, caw, c^{-1}aw, bd^{-1}w\}$. But checking shows that in all cases $c^*(\hat{\Delta}) \leq \pi\left(2 - \frac{5}{2}\right) + 2.\frac{\pi}{4} = 0$.

In conclusion $\mathcal{P}$ fails to be aspherical in case **A2** if and only if $b^2 = bda^{-1}c^{-1} = 1$ (and H is non-cyclic).

The proofs of the remaining cases are similar to the one given for **A2** and we omit them. (For details of these proofs see http://arxiv.org/abs/1604.00163.) Indeed if **A1** holds then $\mathcal{P}$ fails to be aspherical if and only if $b^2 = bda^{-1}c^{-1} = 1$ and $|d| \in \{4, 5\}$, in particular, H is non-cyclic; if **A3** holds $\mathcal{P}$ is aspherical if and only if $|d| = \infty$; if **A4** holds then, assuming that the exceptional case **E1** does not hold, $\mathcal{P}$ fails to be aspherical if and only if $db^{-1}ca = 1$, in particular, H is cyclic; if **A6** or **A8** holds then $\mathcal{P}$ is aspherical; and if **A7** holds then $\mathcal{P}$ fails to be aspherical if and only if $bda^{-1}c = 1$, in particular, H is cyclic.

It follows from the above that either $\mathcal{P}$ is aspherical or modulo T-equivalence one of the conditions from Theorem 1.1(i), (iii) or Theorem 1.2(i), (ii), (iii) is satisfied and so Theorems 1.1 and 1.2 are proved for Case A.

5 Proof of Case B

In this section we prove Theorems 1.1 and 1.2 for Case B, that is, we make the following assumption: at least one of c^2, d^2, $a^{-1}b$ equals 1 in H.

If $d(\Delta) = 2$ then we will fix the names of the four neighbouring regions Δ_i $(1 \leq i \leq 4)$ of Δ as shown in Figure 5.1(i).

Remark Recall that if $c^2 = 1$ then $c^{\pm 2}$ cannot be a proper sublabel. This fact will be used often without explicit reference.

(B1) $\boldsymbol{|c| = 2}$, $\boldsymbol{|d| > 3}$, $\boldsymbol{a^{-1}b \neq 1}$, $\boldsymbol{c^{\pm 1}ab^{-1} \neq 1}$, $\boldsymbol{d^{\pm 1}b^{-1}a \neq 1}$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.1(ii). Observe that if $d(\Delta_i) = 4$ for $i \in \{1, 2\}$ then $l(\Delta_i) = \{bdda^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}\}$. But $bdb^{-1}c^{\pm 1} = 1$ implies $|d| = |c|$, a contradiction. Observe further that at most one of $bd^2a^{-1}, bda^{-1}c^{\pm 1}$ equals 1 otherwise there is a contradiction to $|d| > 3$. This leaves the following cases: (i) $bd^2a^{-1} \neq 1$, $bda^{-1}c^{\pm 1} \neq 1$; (ii) $bd^2a^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$; (iii)$bda^{-1}c^{\pm 1} = 1$, $bd^2a^{-1} \neq 1$.

Consider (i) $bd^2a^{-1} \neq 1$, $bda^{-1}c^{\pm 1} \neq 1$. In this case $d(\Delta_i) > 4$ for Δ_i $(1 \leq i \leq 2)$ of Figure 5.1(ii) so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_i)$ $(1 \leq i \leq 2)$. Observe from Figure 5.1(ii) that Δ_i does not receive positive curvature from Δ_j for $j \in \{3, 4\}$. It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$

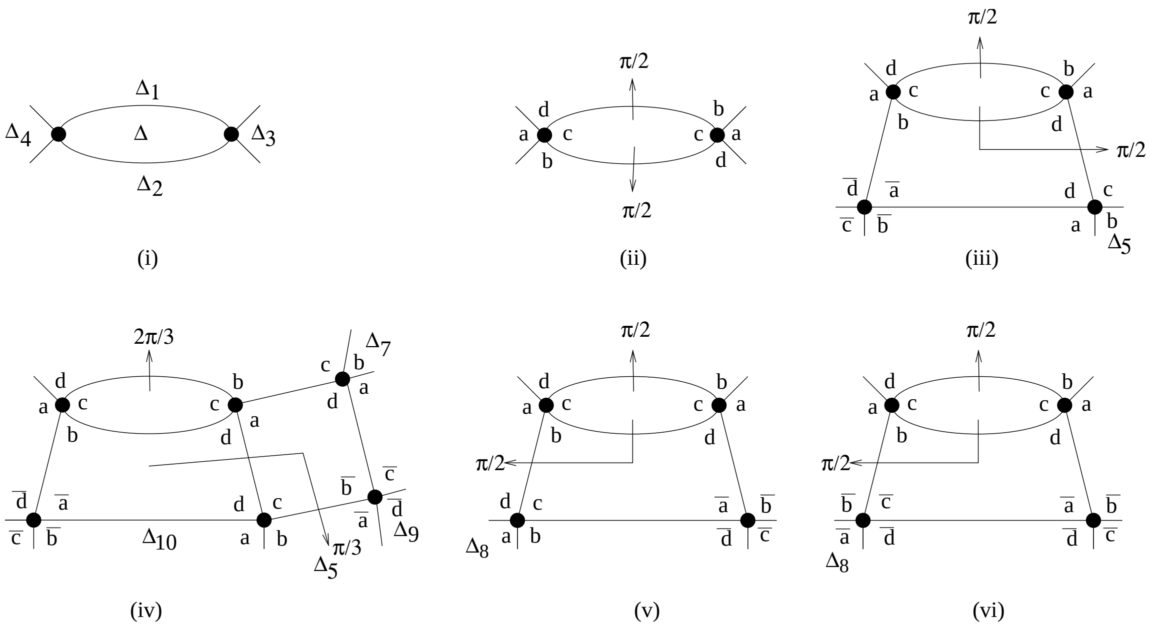


Figure 5.1: the region Δ and curvature distribution for Case (B1)

implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iii). Checking (the LIST of Section 3) shows that if $d(\hat{\Delta}) = 5$ then $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Also if $d(\hat{\Delta}) = 6$ then checking shows $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

Consider (ii) $bd^2a^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$. Suppose that $l(\Delta) = bd^2a^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$. If $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as in Figure 5.1(ii). If say $d(\Delta_2) = 4$ as in Figure 5.1(iii) then $l(\Delta_2) = bdda^{-1}$ which forces $l(\Delta_3) = ca\omega$. First assume that $cadb^{-1} \neq 1$. Then $d(\Delta_3) > 3$ and so add $\frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_2. If $d(\Delta_1) = 4$ then add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_1 in a similar way. Observe that Δ_1 does not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii); and Δ_3 does not receive positive curvature from Δ_1 or Δ_5 in Figure 5.1(iii). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iii). It remains to study $5 \leq d(\hat{\Delta}) \leq 6$. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $|c| \neq 1$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Also if $d(\hat{\Delta}) = 6$ then checking shows that $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$. Now assume that $cadb^{-1} = 1$, in which case $c = d^3$, $b = d^{-2}$ and $|d| = 6$. The distribution of curvature is exactly the same except when $d(\Delta_3) = 4$ in Figure 5.1(iii). In this case add $\frac{2}{3}c(\Delta) = \frac{2\pi}{3}$ to $c(\Delta_1)$ and $\frac{1}{3}c(\Delta) = \frac{\pi}{3}$ to $c(\Delta_5)$ via Δ_3 as shown in Figure 5.1(iv). Together with the observations above (which still hold) we also have that Δ_1 does not receive positive curvature from Δ_3, Δ_4 or Δ_7 and that Δ_5 does not receive positive curvature from Δ_9 or Δ_{10} in Figure 5.1(iv). An argument similar to those for Lemma 3.5 now shows that if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$;

and that if $l(\hat{\Delta})$ does not involve $(cbd)^{\pm 1}$ then $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$ by Lemma 3.5(iii). The conditions on b, c and d imply that if $2 < d(\hat{\Delta}) < 6$ then $l(\hat{\Delta}) \in \{d^2a^{-1}b, db^{-1}c^{\pm 1}a\}$ so if $l(\hat{\Delta})$ does not involve $(cbd)^{\pm 1}$ it remains to consider $d(\hat{\Delta}) = 6$. But checking shows that $l(\hat{\Delta})$ will then either involve at most two non-adjacent occurrences of $(bd)^{\pm 1}$, $(ca)^{\pm 1}$ or $(ba^{-1})^{\pm 1}$ and so $c^*(\hat{\Delta}) \leq 0$ or $l(\hat{\Delta}) = (ab^{-1})^3$ in which case $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + 3(\frac{\pi}{3}) = 0$. Finally if $l(\hat{\Delta}) = cbd\omega$ and $d(\hat{\Delta}) \leq 7$ then $l(\hat{\Delta}) \in \{cbda^{-1}ba^{-1}, cbdb^{-1}ab^{-1}, cbd^3b^{-1}\}$ and $c^*(\hat{\Delta}) \leq c(\hat{\Delta}) + \frac{2\pi}{3} + \frac{\pi}{3} = 0$.

Consider (iii) $bda^{-1}c^{\pm 1} = 1$, $bd^2a^{-1} \neq 1$. Now suppose that $l(\Delta) = bda^{-1}c^{\pm 1} = 1$, $bd^2a^{-1} \neq 1$. First assume that $|b| \geq 3$. Add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as in Figure 5.1(ii). If say $d(\Delta_2) = 4$ then $l(\Delta_2) = bda^{-1}c^{\pm 1}$. First let $l(\Delta_2) = bda^{-1}c$ as in Figure 5.1(v). This forces $l(\Delta_4) = ad\omega$ and so $d(\Delta_4) = 4$ forces $l(\Delta_4) = addb^{-1}$. But if $b = d^2$ then $c = d^3$ and there is a sphere by Lemma 3.2(iv), so it can be assumed that $d(\Delta_4) > 4$. So add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_2 as shown. Suppose now that $l(\Delta_2) = bda^{-1}c^{-1}$ as in Figure 5.1(vi). This forces $l(\Delta_4) = ab^{-1}\omega$ and so $d(\Delta_4) > 4$, otherwise there is a contradiction to $|b| \geq 3$. So add $\frac{\pi}{2}$ to $c(\Delta_4)$ via Δ_2 as shown. Similarly add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_3)$ if $d(\Delta_1) = 4$. Observe that Δ_1 does not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii); Δ_2 does not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii); and Δ_4 does not receive positive curvature from Δ_1 or Δ_8 in Figures 5.1(v), (vi). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iii). It remains to study $5 \leq d(\hat{\Delta}) \leq 6$.

If $|b| > 3$ then checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts one of the (**B1**) assumptions or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Checking shows that if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges or $l(\hat{\Delta}) = (ab^{-1})^3$ contradicting $|b| > 3$, and so $c^*(\hat{\Delta}) \leq 0$.

Let $|b| = 3$. If H is cyclic then $bdc = 1$ implies $b = d^2$ and we obtain a sphere as before, so assume that if $bdc = 1$ then H is non-cyclic. If $|b| = 3$ and $|d| \in \{4, 5\}$ then we obtain a sphere by Lemma 3.1(b)(iv). So let $|b| = 3$, $|d| \geq 6$. Distribute curvature from Δ as shown in Figure 5.1(ii), (v) and (vi). Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $|b| = 3$ or $|d| \geq 6$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Checking shows that if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$ except when $l(\hat{\Delta}) = ba^{-1}ba^{-1}ba^{-1}$. This case is shown in Figure 5.2(i) where $c^*(\hat{\Delta}) = \frac{\pi}{2}$ and so add $\frac{1}{3}c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_i)$ for $i \in \{1, 2, 3\}$ across the edge ad^{-1}. If $d(\hat{\Delta}_i) = 4$ then $l(\hat{\Delta}_i) \in \{ad^{-1}b^{-1}c^{-1}, ad^{-1}b^{-1}c\}$. Suppose that $l(\hat{\Delta}_1) = ad^{-1}b^{-1}c^{-1}$ as in Figure 5.2(ii). Then $l(\hat{\Delta}_4) = d^2b^{-1}\omega$ and $d(\hat{\Delta}_4) > 4$ otherwise there is a contradiction to H non-cyclic. So add $\frac{\pi}{6}$ to $c(\hat{\Delta}_4)$ across the edge db^{-1}. If $l(\hat{\Delta}_1) = ad^{-1}b^{-1}c$ as in Figure 5.2(iii) then $l(\hat{\Delta}_4) = d^3\omega$ and $d(\hat{\Delta}_4) > 4$ otherwise there is a contradiction to $|d| \geq 6$. So add $\frac{\pi}{6}$ to $c(\hat{\Delta}_4)$ across the edge d^2. Observe that if $\hat{\Delta}$ receives positive curvature then it receives $\frac{\pi}{2}$ across the edges ab^{-1}, ad or bd; and receives $\frac{\pi}{6}$ across the edges ad^{-1}, db^{-1} or dd. Thus there is a gap (see Section 3) preceding $c^{\pm 1}$, b, a and a gap after $c^{\pm 1}$, b^{-1}, a^{-1} and there is a two-thirds gap across the edges ad^{-1}, db^{-1} and dd.

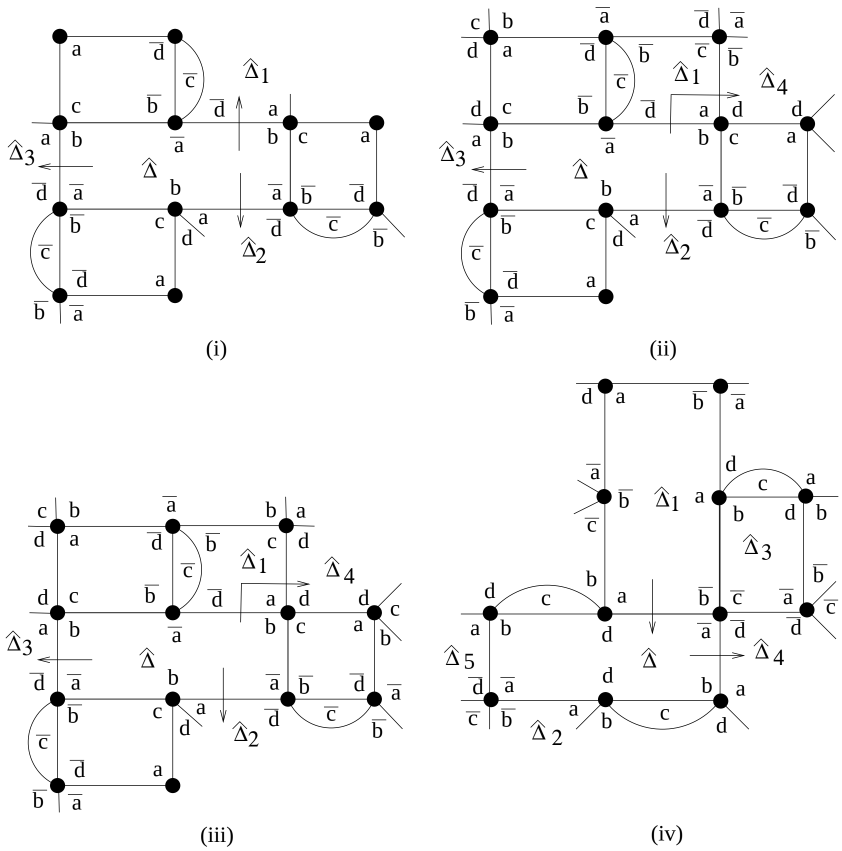


Figure 5.2: curvature distribution for Case (B1) from regions of degree 6

Also there is always a gap between two d's (other than when the subword is $d^{\pm 2}$). Now since c^2 cannot be a proper sublabel it follows that if there are at least two occurrences of $c^{\pm 1}$ then we obtain four gaps and $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6. Suppose now that there is at most one occurrence of $c^{\pm 1}$. If there is exactly one occurrence of c and either no occurrences of b or no occurrences of d then H is cyclic, a contradiction; and if there are no occurrences of c and exactly one occurrence of d or of b then again H is cyclic, a contradiction. So assume otherwise. It follows that $l(\hat{\Delta})$ contains at least four gaps or $l(\hat{\Delta}) \in \{c^{\pm 1}ad^{\pm 1}a^{-1}ba^{-1}, c^{\pm 1}bd^{\pm 1}a^{-1}ba^{-1}, c^{\pm 1}ad^{\pm 1}b^{-1}ab^{-1}, c^{\pm 1}bd^{\pm 1}b^{-1}ab^{-1},$ $d(a^{-1}b)^{\pm 1}d^{\pm 1}(a^{-1}b)^{\pm 1}\}$. But since $c = bd$ each of these labels contradicts H non-cyclic, $|d| \geq 6$ or $|b| = 3$ except when $l(\hat{\Delta}) = da^{-1}bda^{-1}b$. In this case if $c^*(\hat{\Delta}) > 0$ then it can be assumed without loss of generality that $\hat{\Delta}$ is given by Figure 5.2(iv) and $\hat{\Delta}$ receives $\frac{1}{3}c^*(\hat{\Delta}_1) = \frac{\pi}{6}$ from $c(\hat{\Delta}_1)$. This implies that $l(\hat{\Delta}_3) = a^{-1}c^{-1}bd$ and $l(\hat{\Delta}_4) = ad^{-1}d^{-1}\omega$. So add $\frac{\pi}{6}$ from $c(\hat{\Delta})$ to $c(\hat{\Delta}_4)$. Since this $\frac{\pi}{6}$ is across an ad^{-1} edge and since $l(\hat{\Delta}_4) = ad^{-2}\omega$ it follows from the above that $c^*(\hat{\Delta}_4) \leq 0$. If $l(\hat{\Delta}_2) = b^{-1}ab^{-1}ab^{-1}$ in Figure 5.2(iv) then a similar argument applies to $c(\hat{\Delta}_5)$.

Finally let $|b| = 2$. In particular, H is non-cyclic for otherwise $d^2 = 1$, a contradiction. If

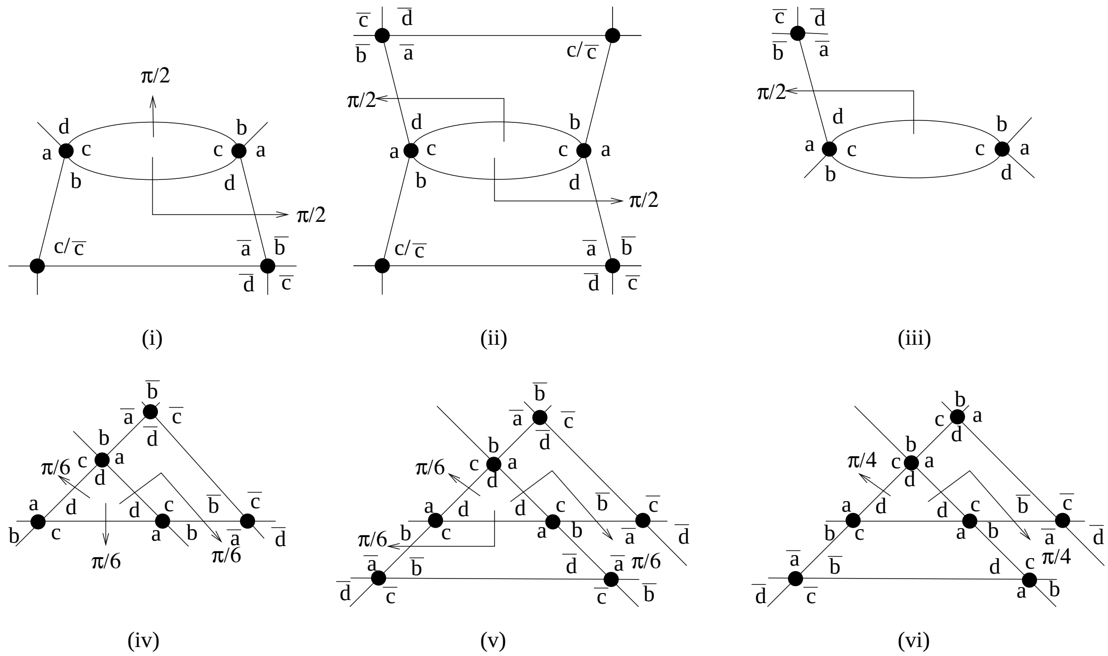


Figure 5.3: curvature distribution for Case (B3)

$|d| < \infty$ then we obtain a sphere by Lemma 3.1(b)(i) and if $|d| = \infty$ then $\mathcal{P}$ is aspherical by Lemma 3.4(ii). In conclusion $\mathcal{P}$ is aspherical in this case except when H is non-cyclic, $bda^{-1}c^{\pm 1} = 1$ and either $|b| = 3$, $|d| \in \{4, 5\}$ or $|b| = 2$, $|d| < \infty$; or when H is cyclic, $b = d^2$, $c = d^3$ and $|d| = 6$.

(B3) $|c| = 2$, $|d| = 3$, $a^{-1}b \neq 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figure 5.1(ii). If $d(\Delta) = 3$ then Δ is given by Figure 4.1(iv). Moreover, if $d(\Delta_i) = 4$ and $l(\Delta_i) = bdw$ or caw then
$l(\Delta_i) \in \{bd^2a^{-1}, bda^{-1}c^{\pm 1}, bdb^{-1}c^{\pm 1}, cad^{\pm 1}a^{-1}, cad^{\pm 1}b^{-1}\}$. But each of $bd^2a^{-1} = 1$, $bdb^{-1}c^{\pm 1} = 1$ and $cad^{\pm 1}a^{-1} = 1$ implies a contradiction to one of the (**B3**) assumptions. Thus we have the following cases: (i)$bda^{-1}c^{\pm 1} \neq 1$, $cadb^{-1} \neq 1$; (ii)$bda^{-1}c^{\pm 1} = 1$, $cadb^{-1} \neq 1$; (iii)$cadb^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$.

Consider (i) $bda^{-1}c^{\pm 1} \neq 1$, $cadb^{-1} \neq 1$. In this case $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ in Figure 5.1(ii), so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.1(iv) $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ so add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_1)$, $c(\Delta_3)$ and $c(\Delta_5)$. Observe that Δ_1 and Δ_2 do not receive positive curvature from Δ_3 or Δ_4 in Figure 5.1(ii). Also Δ_1, Δ_3 and Δ_5 do not receive positive curvature from Δ_m for $m \in \{2, 4, 6\}$ in Figure 4.1(iv). It follows that if $\hat{\Delta}$ receives positive curvature then it does so across at most half of its edges and so $d(\hat{\Delta}) \geq 7$ implies that $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iii). It remains to study $5 \leq d(\hat{\Delta}) \leq 6$. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $cab^{-1} \neq 1$ or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$. Also if $d(\hat{\Delta}) = 6$ then $\hat{\Delta}$ receives positive curvature across at most two edges and so $c^*(\hat{\Delta}) \leq 0$.

Consider (ii) $bda^{-1}c^{\pm 1} = 1$, $cadb^{-1} \neq 1$. In this case the labels $bda^{-1}c^{\pm 1}$ can occur. If H is cyclic then $d = b^2$, $c = b^3$ and there is a sphere by Lemma 3.2(v), so assume that H is non-cyclic. If $|b| \in \{2,3,4,5\}$ then we obtain spheres by Lemma 3.1(b)(i), (v), so assume that $|b| \geq 6$. In Figure 5.1(ii) if $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ then add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$ as shown. If say $d(\Delta_1) > 4$ and $d(\Delta_2) = 4$ as in Figure 5.3(i) then add $\frac{\pi}{2}$ to $c(\Delta_1)$. This implies that $l(\Delta_2) = bda^{-1}c^{\pm 1}$ as shown. This forces $l(\Delta_3) = b^{-1}a\omega$ and so $d(\Delta_3) > 4$, otherwise there is a contradiction to $|b| \geq 6$ so add $\frac{\pi}{2}$ to $c(\Delta_3)$ via Δ_2 as shown. If $d(\Delta_1) = d(\Delta_2) = 4$ then add $\frac{\pi}{2}$ to $c(\Delta_j)$ for $j \in \{3,4\}$ as shown in Figure 5.3(ii). The one exception to the above is when $l(\Delta_1) = bda^{-1}\omega$ and $d(\Delta_1) > 4$. Then $d(\Delta_4) > 4$ and in this situation add the $\frac{\pi}{2}$ from $c(\Delta)$ to $c(\Delta_4)$ via Δ_1 as shown in Figure 5.3(iii). The same applies to Δ_2. In Figure 4.1(iv) if $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ then add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_1)$, $c(\Delta_2)$ and $c(\Delta_5)$. If say $d(\Delta_1) = 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ then $l(\Delta_2) = ba^{-1}\omega$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|b| \geq 6$ so add $\frac{\pi}{6}$ to $c(\Delta_2)$, $c(\Delta_3)$ and $c(\Delta_5)$ as shown in Figure 5.3(iv). Now suppose that $d(\Delta_1) = 4$ and $d(\Delta_3) = 4$. This implies that $l(\Delta_2) = l(\Delta_4) = ba^{-1}\omega$ as shown in Figure 5.3(v). So add $\frac{\pi}{6}$ to $c(\Delta_2)$, $c(\Delta_4)$ and $c(\Delta_5)$. If $d(\Delta_1) = d(\Delta_3) = d(\Delta_5) = 4$ then similarly add $\frac{\pi}{6}$ to $c(\Delta_m)$ for $m \in \{2,4,6\}$.

We now see that if $\hat{\Delta}$ receives positive curvature then it receives at most $\frac{\pi}{2}$ across $(bd)^{\pm 1}$ and $(b^{-1}a)^{\pm 1}$; and it receives at most $\frac{\pi}{6}$ across $(ca)^{\pm 1}$ and $(ab^{-1})^{\pm 1}$. Thus there is always a gap immediately preceding c and d^{-1}; and there is a gap immediately after c^{-1} and d. This implies that if there are at least four occurrences of $c^{\pm 1}$ or $d^{\pm 1}$ then $l(\hat{\Delta})$ contains at least four gaps and so $c^*(\hat{\Delta}) \leq 0$. Suppose that there are at most three occurrences of $c^{\pm 1}$ or $d^{\pm 1}$ in $l(\hat{\Delta})$. Observe that in addition to the four gaps mentioned above the following sublabels yield gaps: $(cb)^{\pm 1}$ and $(ad)^{\pm 1}$ each yields a gap; $(bda^{-1})^{\pm 1}$ yields two gaps (see Figure 5.3(iii)); and $(ca)^{\pm 1}$ and $(ab^{-1})^{\pm 1}$ each yields the equivalent of a two-thirds gap. If $l(\hat{\Delta}) = (b^{-1}a)^{\pm n}$ where $n \geq 1$ then $l(\hat{\Delta})$ obtains at least four gaps since $|b| \geq 6$. If $l(\hat{\Delta}) \in \{d^{\pm 1}(b^{-1}a)^{\pm n}, (ab^{-1})^{\pm n}c^{\pm 1}\}$ then H is cyclic so it can be assumed that $l(\hat{\Delta})$ involves either two or three occurrences of $c^{\pm 1}$ or $d^{\pm 1}$. It follows that if there are three occurrences then $c^*(\hat{\Delta}) \leq 0$; or if exactly two occurrences then either $c^*(\hat{\Delta}) \leq 0$ or $l(\hat{\Delta}) \in \{d^{\pm 1}b^{-1}ada^{-1}b, d^{\pm 1}b^{-1}ada^{-1}ba^{-1}b, d^{\pm 1}b^{-1}ab^{-1}ada^{-1}b, cba^{-1}bd^{\pm 1}b^{-1}, cbd^{\pm 1}b^{-1}ab^{-1}\}$. But each of these labels forces H cyclic or $|b| < 6$ or a (**B3**) contradiction, therefore $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.6.

Consider (iii) $cadb^{-1} = 1$, $bda^{-1}c^{\pm 1} \neq 1$. In this case the label $cadb^{-1}$ can occur. First assume that H is non-cyclic. In Figure 5.1(ii) $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$ otherwise there is a contradiction to $|c| = 2$ or $|d| = 3$, so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 4.1(iv) if $d(\Delta_1) > 4$, $d(\Delta_3) > 4$ and $d(\Delta_5) > 4$ add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each of $c(\Delta_1)$, $c(\Delta_2)$ and $c(\Delta_5)$. If say $d(\Delta_1) = 4$ only then add $\frac{\pi}{4}$ to $c(\Delta_3)$ and $c(\Delta_5)$. Now suppose that $d(\Delta_1) = d(\Delta_3) = 4$. This implies that their label is $cadb^{-1}$ which forces $l(\Delta_2) = cba^{-1}\omega$ as shown in Figure 5.3(vi). So add $\frac{\pi}{4}$ to $c(\Delta_2)$ via Δ_1 and to $c(\Delta_5)$. Finally if $d(\Delta_1) = d(\Delta_3) = d(\Delta_5) = 4$ then in a similar way add $\frac{\pi}{6}$ to $c(\Delta_m)$ for $m \in \{2,4,6\}$. Observe that Δ_1 does not receive positive curvature from Δ_3 and Δ_2 does not receive positive curvature from Δ_4 in Figure 5.1(ii). In Figure 4.3(i) Δ_1 does not receive positive curvature from Δ_2. In Figure 5.3(vi)

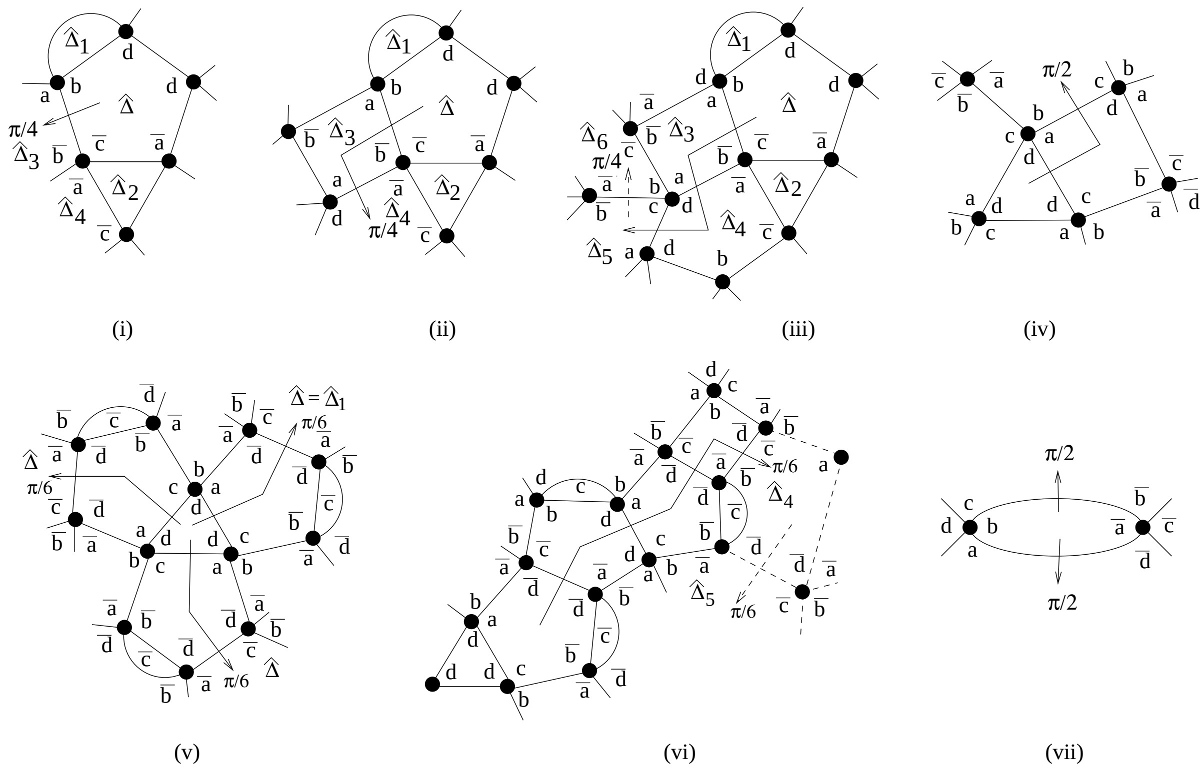


Figure 5.4: curvature distribution for Case (B3)

Δ_2 does not receive positive curvature from Δ_3. Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ across the bd edge and $\frac{\pi}{4}$ across the ca, ba^{-1} edges it follows that $\hat{\Delta}$ receives an average of $\frac{\pi}{4}$ across each of its edges, so $d(\hat{\Delta}) \geq 8$ implies that $c^*(\hat{\Delta}) \leq 0$. It remains to study $5 \leq d(\hat{\Delta}) \leq 7$. Checking shows that if $d(\hat{\Delta}) = 5$ then either the label contradicts $|d| = 3$ or H non-cyclic or $\hat{\Delta}$ receives positive curvature across at most one edge and so $c^*(\hat{\Delta}) \leq 0$, except when $l(\hat{\Delta}) = bdda^{-1}c^{-1}$ as in Figure 5.4(i). In this case $\hat{\Delta}$ receives $\frac{\pi}{2}$ from $c(\hat{\Delta}_1)$ and $\frac{\pi}{4}$ from $c(\hat{\Delta}_2)$. If $|b| > 2$ then this implies that $d(\hat{\Delta}_3) > 4$ and so add $\frac{\pi}{4}$ to $c(\hat{\Delta}_3)$ noting that this is a similar edge to the one crossed in Figure 5.3(vi) so there is no change to the above argument and $c^*(\hat{\Delta}) \leq 0$ in this case. Suppose now that $|b| = 2$ and $l(\hat{\Delta}_3) = ab^{-1}ab^{-1}$ as in Figure 5.4(ii). If $d(\hat{\Delta}_4) > 5$ then add $\frac{\pi}{4}$ to $c(\hat{\Delta}_4)$ across the da^{-1} edge. If $d(\hat{\Delta}_4) = 5$ then $l(\hat{\Delta}_4) = da^{-1}c^{-1}bd$ which implies that $l(\hat{\Delta}_5) = ca\omega$ and so if $d(\hat{\Delta}_5) > 5$ then add $\frac{\pi}{4}$ to $c(\hat{\Delta}_5)$ as in Figure 5.4(iii). If $d(\hat{\Delta}_5) \in \{4, 5\}$ then $l(\hat{\Delta}_5) \in \{cadb^{-1}, cad^{-2}b^{-1}\}$ and this forces $l(\hat{\Delta}_6) = c^{-1}ba^{-1}\omega$ and so $d(\hat{\Delta}_6) > 5$ otherwise there is a contradiction to $|c| \neq 1$. So add $\frac{\pi}{4}$ to $c(\hat{\Delta}_6)$ again as shown in Figure 5.4(iii).

Observe that $\hat{\Delta}_4$ in Figure 5.4(ii) can now receive $\frac{\pi}{4}$ from $c(\hat{\Delta})$, however it receives no

positive curvature from $\hat{\Delta}_3$ or any other region across the da^{-1} edge. Moreover, it is clear from Figure 5.4(iii) that $\hat{\Delta}_5$ receives only the $\frac{\pi}{4}$ from $\hat{\Delta}_4$ across its ca edge; and $\hat{\Delta}_6$ receives only the $\frac{\pi}{4}$ from $\hat{\Delta}_5$ across its ba^{-1} edge. Finally observe that Figures 5.4(ii)–(iii) do not alter the fact that Δ_1 does not receive positive curvature from Δ_3 and Δ_2 does not receive positive curvature from Δ_4 in Figure 5.1(ii). Therefore the average positive curvature that $\hat{\Delta}$ receives across each edge is still $\frac{\pi}{4}$ and so if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$. It remains to check $6 \leq d(\hat{\Delta}) \leq 7$ for the sublabels $(bd)^{\pm 1}(\pi/2)$ and $(ca)^{\pm 1}$, $(ab^{-1})^{\pm 1}$, $(da^{-1}c^{-1})^{\pm 1}(\pi/4)$. Checking shows that if $d(\hat{\Delta}) = 6$ then the most curvature that $\hat{\Delta}$ can receive is either $2(\frac{\pi}{2})$ or $\frac{\pi}{2} + 2(\frac{\pi}{4})$ or $4(\frac{\pi}{4})$ and so $c^*(\hat{\Delta}) \leq 0$. If $d(\hat{\Delta}) = 7$ then the most curvature received is $3(\frac{\pi}{2})$ or $2(\frac{\pi}{2}) + 2(\frac{\pi}{4})$ or $\frac{\pi}{2} + 4(\frac{\pi}{4})$ or $6(\frac{\pi}{4})$ and $c^*(\hat{\Delta}) \leq 0$ except for $l(\hat{\Delta}) = da^{-1}c^{-1}bda^{-1}b$; but this implies $cd = 1$, a contradiction.

Now let H be cyclic. Then $d = b^4$ and $c = b^3$. Again add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to each of $c(\Delta_1)$, $c(\Delta_2)$ as in Figure 5.1(ii). In Figure 4.1(iv) if say $d(\Delta_1) > 5$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and so it can be assumed that $d(\Delta_i) \leq 5$ for $i \in \{1,3,5\}$ in which case $l(\Delta_i) \in \{cadb^{-1}, cad^{-2}b^{-1}\}$. If say $d(\Delta_1) = 4$ then add $c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_6)$ via Δ_1 as shown in Figure 5.4(iv). It can be assumed then that $d(\Delta_i) = 5$ for $i \in \{1,3,5\}$ in which case add $\frac{1}{3}c(\Delta) = \frac{\pi}{6}$ to each $c(\hat{\Delta})$ via Δ_i where $i \in \{1,3,5\}$ as shown in Figure 5.4(v). If say $\hat{\Delta} = \hat{\Delta}_1$ and $d(\hat{\Delta}_1) = 5$ then repeat the above, that is, add the $\frac{\pi}{6}$ from $c(\Delta)$ across another ca edge and continue in this way until $\frac{\pi}{6}$ is eventually added to a region $\hat{\Delta}_k$ where either $d(\hat{\Delta}_k) > 5$ (and so the process terminates) or $d(\hat{\Delta}_k) = 4$ in which case the $\frac{\pi}{6}$ from $c(\Delta)$ is added to $c(\hat{\Delta}_{k+1})$ as shown in Figure 5.4(vi), where $k = 3$. If $d(\hat{\Delta}_{k+1}) > 5$ then the process terminates (and note that $l(\hat{\Delta}_{k+1}) = d^{-1}b^{-1}c^{-1}w$); otherwise $l(\hat{\Delta}_{k+1}) = d^{-1}b^{-1}c^{-1}ad^{-1}$ and the $\frac{\pi}{6}$ from $c(\Delta)$ is added to $c(\hat{\Delta}_{k+2})$ where $\hat{\Delta}_{k+2}$ is the region shown in Figure 5.4(vi) with $k = 3$. Observe that $l(\hat{\Delta}_{k+2}) = ba^{-1}c^{-1}w$ so $d(\hat{\Delta}_{k+2}) > 5$ and the process terminates. This completes the distribution of curvature that occurs. It follows that if $\hat{\Delta}$ receives positive curvature across an edge e_i say then $\hat{\Delta}$ does not receive any curvature across the adjacent edges e_{i-1}, e_{i+1} except when $\hat{\Delta}$ is given by $\hat{\Delta}_{k+1} = \hat{\Delta}_4$ in Figure 5.4(vi). Therefore if $l(\hat{\Delta})$ does not involve $(cbd)^{\pm 1}$ then Lemma 3.5(iii) applies and $c^*(\hat{\Delta}) \leq 0$ for $d(\hat{\Delta}) \geq 7$; and if $d(\hat{\Delta}) = 6$ then checking for $(bd)^{\pm 1}$, $(ca)^{\pm 1}$ and $(cba^{-1})^{\pm 1}$ shows that $\hat{\Delta}$ receives positive curvature across at most two edges and $c^*(\hat{\Delta}) \leq 0$. Finally if $l(\hat{\Delta}) = cbdw$ then we see from Figure 5.4(vi) that the maximum amount $\hat{\Delta}$ receives is on average $\frac{\pi}{3}$ across $\frac{2}{3}$ of its edges and so if $d(\hat{\Delta}) \geq 8$ then $c^*(\hat{\Delta}) \leq 0$ by Lemma 3.5(iv). Checking shows that if $6 \leq d(\hat{\Delta}) \leq 7$ then $l(\hat{\Delta}) \in \{cbdb^{-1}ab^{-1}, cbda^{-1}bdb^{-1}, cbda^{-1}c^{-1}ba^{-1}, cbda^{-1}cba^{-1}\}$ and so if $d(\hat{\Delta}) = 6, 7$ then $\hat{\Delta}$ receives curvature across at most $2, 3$ edges (respectively) and $c^*(\hat{\Delta}) \leq 0$.

In conclusion $\mathcal{P}$ fails to be aspherical in this case when H is non-cyclic, $bda^{-1}c^{-1} = 1$ and $|b| \in \{2,3,4,5\}$; or when H is cyclic and $bda^{-1}c^{\pm 1} = 1$.

(B8) $|c| = 2$, $|d| > 3$, $a^{-1}b = 1$, $c^{\pm 1}ab^{-1} \neq 1$, $d^{\pm 1}b^{-1}a \neq 1$.
If $d(\Delta) = 2$ then Δ is given by Figures 5.1(ii) and 5.4(vii). In Figure 5.4(vii) $l(\Delta_1) = b^{-1}c\omega$ and $l(\Delta_2) = ad^{-1}\omega$. This implies that $d(\Delta_1) > 4$,$d(\Delta_2) > 4$, otherwise there is a contradiction to $|d| > 3$ so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. In Figure 5.1(ii) $l(\Delta_1) = l(\Delta_2) = bd\omega$. This similarly implies that $d(\Delta_1) > 4$ and $d(\Delta_2) > 4$, so add $\frac{1}{2}c(\Delta) = \frac{\pi}{2}$ to $c(\Delta_1)$ and $c(\Delta_2)$. Observe that in Figure 5.4(vii) Δ_1 does not receive

positive curvature from Δ_4; and Δ_2 does not receive positive curvature from Δ_4. Observe also that if $\hat{\Delta}$ receives positive curvature then it does so across the edges $b^{-1}c$, ad^{-1} or bd. Thus there is always a gap immediately preceding c^{-1} and a; and there is a gap after c and a^{-1}. This implies that if there are at least four occurrences of $c^{\pm 1}$ then $l(\hat{\Delta})$ contains at least four gaps and so $c^*(\hat{\Delta}) \leq 0$. We will proceed according to the number of occurrences of $c^{\pm 1}$ in $l(\hat{\Delta})$. If there are no occurrences of $c^{\pm 1}$ then either $l(\hat{\Delta}) = (ab^{-1})^k$ where $|k| \geq 2$ and there are four gaps, or $l(\hat{\Delta}) = d(ab^{-1})^{k_1} \ldots d(ab^{-1})^{k_m}$ where $k_i \in \mathbb{Z}$ $(1 \leq i \leq m)$. But since there is always at least one gap between any two occurrences of $d^{\pm 1}$ it follows that again there are four gaps or $|d| \leq 3$, a contradiction. So $c^*(\hat{\Delta}) \leq 0$ in this case.

Assume first that H is non-cyclic. If there is exactly one occurrence of $c^{\pm 1}$ in $l(\hat{\Delta})$ then H is cyclic so suppose that there are either two or three occurrences of $c^{\pm 1}$. Then either the label contains at least four gaps or it contradicts one of the (**B8**) assumptions or one of the following cases $\hat{\Delta}_i$ $(1 \leq i \leq 9)$ occurs:

(1) $cad^{-1}b^{-1}cad^{-1}b^{-1}$;

(2) $cad^{-1}b^{-1}cbd^{-1}b^{-1}$;

(3) $cad^{-1}b^{-1}c^{-1}ad^{-1}b^{-1}$;

(4) $cad^{-1}b^{-1}c^{-1}bd^{-1}b^{-1}$;

(5) $cad^{-1}b^{-1}cbda^{-1}$;

(6) $cad^{-1}b^{-1}cbdb^{-1}$;

(7) $cad^{-1}b^{-1}c^{-1}bda^{-1}$;

(8) $cad^{-1}b^{-1}c^{-1}bdb^{-1}$;

(9) $(bda^{-1}c^{-1})^3$.

If any of (1)–(4) occurs with any of (5)–(8) or with (9) then $|d| = 2$, a contradiction. Also if any of (5)–(8) occurs with (9) then $c = d^3$ and H is cyclic, so assume otherwise.

Consider (1)–(4). These yield the relator $(cd)^2$ and it follows that $cd^k = d^{-k}c$ for $k \in \mathbb{Z}$. Moreover if $|d| < \infty$ then there is a sphere by Lemma 3.1(c)(ii) so it can be assumed that $|d| = \infty$. In case (1) $\hat{\Delta}_1$ is given by Figure 5.5(i) where, given that $c^*(\hat{\Delta}_1) > 0$, it can be assumed that $d(\Delta_1) = d(\Delta_2) = 2$ and at least one of $d(\Delta_3)$, $d(\Delta_4)$ equals 2. Add $\frac{\pi}{2}$ from $c(\hat{\Delta}_1)$ to $c(\hat{\Delta}_{10})$ as shown in Figure 5.5(i); and if $d(\Delta_3) = d(\Delta_4) = 2$ add a further $\frac{\pi}{2}$ of $c(\hat{\Delta}_1)$ to $c(\hat{\Delta}_{12})$ as shown. In cases (2)–(4) $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$ where $\hat{\Delta} \in \{\hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4\}$ and $\frac{\pi}{2}$ is added from $c(\hat{\Delta})$ to $c(\hat{\Delta}_{10})$ as shown in Figure 5.5(ii). Observe that $x \neq b$ in Figure 5.5(i), (ii) for otherwise c^2 would be a proper sublabel, and so $x \in \{a, d\}$. If $x = a$ then the sublabel ad yields a gap so let $x = d$. Then either dd yields a gap or $\hat{\Delta}_{11} \in \{\hat{\Delta}_i \colon 1 \leq i \leq 4\}$ and $\frac{\pi}{2}$ is added to $c(\hat{\Delta}_{10})$ from $c(\hat{\Delta}_{11})$. Continuing this way, since $|d| = \infty$, eventually we

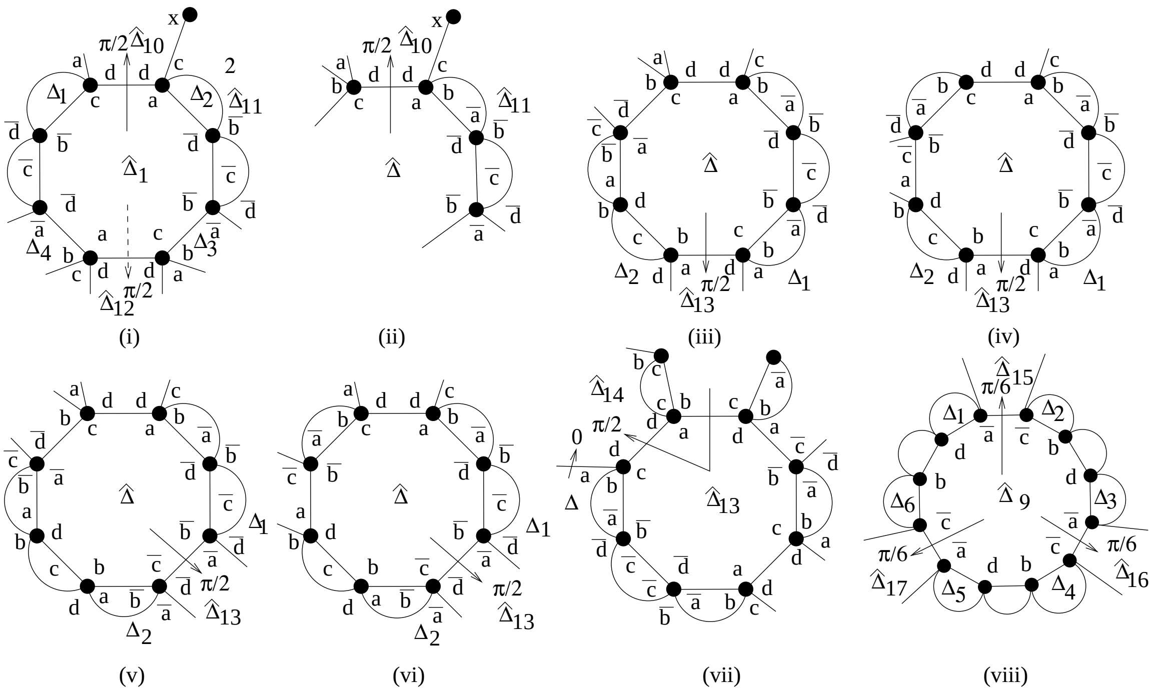


Figure 5.5: curvature distribution for Case (B8)

get a sublabel ad or dd which contributes a gap. Consider $l(\hat{\Delta}_{10})$. If it contains an odd number of occurrences of c then $cd^k = d^{-k}c$ implies that $c \in \langle d \rangle$ and H is cyclic. This leaves the case when there are exactly two occurrences of c and $cd^{\alpha_1}cd^{\alpha_2} = 1$ for $\alpha_1, \alpha_2 \in \mathbb{Z}\backslash\{0\}$. If $|\alpha_1|, |\alpha_2| > 1$ then there are four gaps and $c^*(\hat{\Delta}_{10}) \leq 0$; and if $|\alpha_1| > 1$, $|\alpha_2| = 1$ this implies $|d| < \infty$, a contradiction.

Consider (5)–(8). These yield the relator $cdcd^{-1}$ and H is Abelian. Observe that $|d| = 4$ yields (**E**), so assume otherwise. In each case add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{13})$ as shown in Figure 5.5(iii)–(vi). Observe that $\hat{\Delta}_{13}$ receives no curvature from Δ_1 or Δ_2; that $l(\hat{\Delta}_{13}) = adw$ implies $d(\hat{\Delta}_{13}) > 4$ otherwise there is a contradiction to $|d| > 3$; and there is still a gap between each pair of occurrences of d. If $l(\hat{\Delta}_{13})$ contains an odd number of occurrences of c then H is cyclic so it can be assumed that $l(\hat{\Delta}_{13})$ yields the relator $cd^{\beta_1}cd^{\beta_2}$. If $|\beta_1| > 1$ and $|\beta_2| > 1$ then there are four gaps and if $(\beta_1, \beta_2) \in \{(2,1), (2,-1), (1,1)\}$ then $|d| \leq 3$, so this leaves the case $\beta_1 = 1$, $\beta_2 = -1$. Again there are four gaps except when $l(\hat{\Delta}_{13}) = adb^{-1}cad^{-1}b^{-1}c$ and this is shown in Figure 5.5(vii): add $c^*(\hat{\Delta}_{13}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{14})$ and observe that $\hat{\Delta}_{14}$ does not receive positive curvature from Δ. Consider $l(\hat{\Delta}_{14}) = bddw$. If there are at least four occurrences of c then $c^*(\hat{\Delta}_4) \leq 0$; and if there is an odd number of occurrences then H is cyclic. Suppose firstly that there are no occurrences of c in $l(\hat{\Delta}_{14})$. Since $|d| \geq 5$, if there is one occurrence of b then $l(\hat{\Delta}_4) = a^{-1}bd^k$ $(k \geq 5)$ and there are four gaps; and since each $(a^{-1}b)^{\pm 1}$ yields a gap and each $(bd^l)^{\pm 1}$ $(l \geq 2)$ yields a gap it follows that if there are at least two occurrences of b then again $c^*(\hat{\Delta}_{14}) \leq 0$. Suppose finally that there are two occurrences of c and so $cd^{\beta_1}cd^{\beta_2} = 1$ where $\beta_1 \geq 2$ and $|\beta_2| \geq 0$. If $|\beta_2| > 1$ then there are four gaps; and if $|\beta_2| = 1$ then $\beta_1 \geq 4$, otherwise there is a contradiction to $|d| > 4$, and again there are four gaps, so $c^*(\hat{\Delta}_{14}) \leq 0$.

Finally consider case (9). In this case $\hat{\Delta}_9$ is given by Figure 5.5(viii). Suppose that $c^*(\hat{\Delta}_9) > 0$. Then it can be assumed that $d(\Delta_i) = 2$ for $1 \leq i \leq 6$ and $c^*(\hat{\Delta}) = \frac{\pi}{2}$ so add $\frac{1}{3}c^*(\hat{\Delta}) = \frac{\pi}{6}$ to $c(\hat{\Delta}_l)$ for $l \in \{15, 16, 17\}$. In this case if $|d| \in \{4, 5\}$ then we obtain a sphere by Lemma 3.1(c)(iii). Now if $|d| \geq 6$ then as shown in Figure 5.5(viii) $l(\Delta_l) = d^{-2}\omega$ and d^{-2} will contribute two-thirds of a gap. If there are now at least two occurrences of c then either $|d| < 6$ or H is cyclic, a contradiction, or there are four gaps; if there is exactly one occurrence of c then this contradicts H non-cyclic; and if there are no occurrences of c then $l(\Delta_l) = d^{k_1}(b^{-1}a)^{m_1} \ldots d^{k_n}(b^{-1}a)^{m_n}$ where $m_i \in \mathbb{Z}, k_i \geq 1$. Since $k_1 + \ldots + k_n \geq 6$ it follows that there are at least four gaps and $c^*(\hat{\Delta}_l) \leq 0$.

Now let H be cyclic. If $c = d^2$ or $c = d^3$ then there is a sphere by T-equivalence and Lemma 3.2(vii), (viii); and $c = d^4$ is (**E4**), so assume otherwise. In particular, $|d| > 4$. We follow the same argument as above and so if $l(\hat{\Delta})$ contains no occurrences or at least four occurrences of c then, as before, $c^*(\hat{\Delta}) \leq 0$; and if $l(\hat{\Delta})$ contains an odd number of occurrences of c then $c = d^k$ for some $k \geq 4$ which implies there are at least four gaps and $c^*(\hat{\Delta}) \leq 0$. Suppose then that $l(\hat{\Delta})$ involves c exactly twice. Subcases (1)–(4) imply $d^2 = 1$ and (9) implies $c = d^3$, a contradiction. This leaves subcases (5)–(8).

Add $c^*(\hat{\Delta}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{13})$ as in Figure 5.5(iii)-(vi). Since there is still a gap between each pair of occurrences of d it follows from the above paragraph and the previous argument that $c^*(\hat{\Delta}_{13}) \leq 0$ except when $l(\hat{\Delta}_{13}) = adb^{-1}cad^{-1}b^{-1}c$. Again add $c^*(\hat{\Delta}_{13}) = \frac{\pi}{2}$ to $c(\hat{\Delta}_{14})$ as shown in Figure 5.5(vii). If $l(\hat{\Delta}_{14}) = bddw$ involves at least three occurrences of c then, since $\hat{\Delta}_4$ does not receive positive curvature from Δ in Figure 5.5(vii), there are at least four gaps and $c^*(\hat{\Delta}_4) \leq 0$. Otherwise checking the possible labels for $l(\hat{\Delta}_4) = bddw$ shows that there are four gaps or a contradiction to $|d| > 4$ or $c \notin \{d^3, d^4\}$.

In conclusion $\mathcal{P}$ is aspherical except when $|cd| = 2$, $|d| < \infty$ or when $|cd| = 3$, $|d| \in \{4, 5\}$ or when H is cyclic and $c = d^2$ or d^3.

If **B4** holds then either $|d| < \infty$ and there is a sphere by Lemma 3.1(a)(i) or $|d| = \infty$ and $\mathcal{P}$ is aspherical by Lemma 3.4(ii). The proofs for the remaining cases are similar to those given above so we omit them. (Again for full details see http://arxiv.org/abs/1604.00163.) Indeed if **B2** holds then $\mathcal{P}$ fails to be aspherical either when H is cyclic or when H is non-cyclic, $|b| < \infty$ (by Lemma 3.4(iii)) and either $bda^{-1}c^{-1} = 1$ or $bdb^{-1}c^{-1} = a^{-1}cad = 1$; if **B5** holds then $\mathcal{P}$ is aspherical if and only if $bda^{-1}c \neq 1$; if **B6** holds then $\mathcal{P}$ is aspherical; if **B7** holds then $\mathcal{P}$ is aspherical except when either $c = d^{\pm 1}$ or $c^5 = 1$ and $c = d^2$ or $d^5 = 1$ and $d = c^2$; if **B9** holds then, assuming that the exceptional cases **E2** and **E3** do not hold, $\mathcal{P}$ is aspherical except when H is non-cyclic, $|c^{-1}d| = 2$ and $|d| \in \{4, 5\}$; if **B10** holds then $\mathcal{P}$ is aspherical if and only if $|cd| = \infty$; if **B11** holds then $\mathcal{P}$ is aspherical except when H is non-cyclic and $|cd| \in \{2, 3, 4, 5\}$ or when H is cyclic; and if **B12** holds then $\mathcal{P}$ is aspherical except when H is cyclic or when H is non-cyclic and $|cd| = 2$. It follows that either $\mathcal{P}$ is aspherical or modulo T-equivalence one of the conditions of Theorem 1.1 (i)–(iii) or Theorem 1.2 (i), (ii), (iv)–(x) is satisfied and so Theorems 1.1 and 1.2 are proved for Case B.

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