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'Can't you just tell us the rule?' Teaching procedures relationally

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It is now almost 40 years since Skemp's (1976) seminal division of understanding into 'instrumental' and 'relational' categories, yet the current political direction of mathematics education in the UK is decidedly towards the traditional teaching of 'standard algorithms' (DfE, 2013). In this research paper, I draw on a lively staffroom discussion about different approaches to the teaching of quadratic equations, in which one method used was derided as 'a trick'. From this, I discuss reasons why certain mathematical processes are often regarded as inherently and irretrievably 'procedural'. Informed by recent theoretical interpretations of procedural and conceptual learning in mathematics, which increasingly stress their intertwining and iterative relationship (Star, 2005; Baroody, Feil and Johnson, 2007; Star, 2007; Kieran, 2013), I make a case that stigmatising particular methods and censoring their use may deny students valuable opportunities to make sense of mathematics. I argue instead that encouraging students to take a critical stance regarding the details and the value of the procedures that they encounter can cultivate in them a deeper awareness of mathematical connections and a more empowered sense of ownership over their mathematics.

Keywords: Algorithms; Conceptual knowledge; Instrumental understanding; Procedural knowledge; Quadratic equations; Relational understanding; Student autonomy

Introduction

Don't waste time learning 'tricks of the trade'. Instead, learn the trade.

James Bennis

In a classic article, written almost 40 years ago, Skemp (1976) outlined what has become a highly-influential distinction between instrumental and relational understanding. By relational understanding, he meant "knowing both what to do and why" (p. 20), in contrast to instrumental understanding, which was merely "rules without reasons" (p. 20) – something we would not normally characterise as understanding at all. Since then, the terms procedural and conceptual learning have been widely adopted, and more recent theoretical interpretations of these in mathematics have increasingly highlighted their interweaving and iterative relationship (Star, 2005; Baroody, Feil and Johnson, 2007; Star, 2007; Kieran, 2013; Star & Stylianides, 2013). Indeed, "the wider debate is starting to move away from the opposition of conceptual understanding from factual and procedural knowledge", seeing the two as mutually reinforcing rather than antagonistic (DfE, 2011: 67). Nonetheless, there remains a wide consensus among mathematics educators that a classroom focused predominantly on the competent performance of algorithms does not offer students an authentic experience of mathematics and that the use of richer tasks is essential for developing the necessary relational understanding of the subject

(Mason and Johnston-Wilder, 2006; Watson, 2007; Sullivan, Clarke and Clarke, 2013). Teaching students to do mathematics by applying a set of memorised algorithms is viewed as hindering their mathematical development, because they are able to achieve correct answers without an understanding of the underlying mathematical principles.

Despite this, the current UK political climate shows a decided preference for the traditional teaching of 'standard algorithms', with its emphasis on practice for fluency (DfE, 2013). Indeed, the first stated aim of the new mathematics programme of study for key stage 3 in the national curriculum for England is:

> that all pupils become *fluent* in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately. (DfE, 2013: 2, original emphasis)

Here, procedural fluency is promoted as a route to conceptual understanding, yet the flavour of the prescribed curriculum as a whole is widely perceived as lying more towards the procedural side. Pope and Cotton (2013) express concern about "the heavy reliance on practice as a principal teaching approach", concluding that the "curriculum as presented will result in more attention spent on developing technical competence in outdated written methods for arithmetic at the expense of developing secure foundations for progression through mathematical concepts and skills" (p. 9).

I have previously argued that the ideological valuing of procedural knowledge has a tendency to fragment the curriculum into meaningless, bite-sized facts and skills, learned with little relational understanding (Foster, 2013a). In their most recent report on mathematics, Ofsted (2012: 18) comment that they observed few "lessons that were helping pupils to gain a better understanding of mathematics", as opposed to those with "a strong focus in teaching to the next examination". The powerful backwash effect of high-stakes assessments understandably leads many students to ask, 'Can't you just tell us the rule?' Indeed, much within the culture of the UK mathematics classroom (perhaps even the name 'exercise books') predisposes the teaching of procedures.

In this paper, I consider the potential value and dangers of teaching mathematical procedures. I base the discussion on a lively staffroom conversation about the teaching of quadratic equations and explore possible reasons why some mathematical procedures may be designated 'tricks'. Are (some) mathematical procedures inherently harmful? Are students better off not being taught standard algorithms? Or can procedures be taught in non-damaging (or less-damaging) ways?

A staffroom conversation

I draw on a spontaneous staffroom conversation, overheard in a UK secondary school, relating to the teaching of quadratic equations. I was a 'fly on the wall' observer for this unanticipated discussion, which I noted down afterwards. I do not present this episode as data; rather as an extract that illustrates the wider debate in a local context and is included for communicative purposes rather than as an evidential base. The names are pseudonyms.

Prior to the discussion, Jack had shown his Year 10 class (14–15-year-old students) how to solve quadratic equations by factorising, if possible, or by completing the square, if not. He was then away from school for a lesson, and a non-mathematics colleague, Jill, had taken the class in his absence. Jack had heard from his students that Jill had told them that when she was their age she always used the quadratic formula $x = (-b \pm \sqrt{b^2 - 4ac}) \div (2a)$, which they had not heard about. She had shown them the formula and some of them had remarked that they liked this method much better than Jack's methods because they found it quicker and easier, and it was one technique to remember rather than two – and they completed all of Jack's set work using Jill's formula. Jack was now back in school and Jill was telling him with some pride that she showed his class how to use the quadratic formula, because 'they didn't seem to know about it'. However, Jack was unhappy with her comment, seeming to take it as a criticism of his teaching:

Jack The quadratic formula is just a trick.

Jill What do you mean 'a trick'?

Jack They just bung numbers into a formula without thinking about what they're doing. Here's the formula; stick the numbers in. It could be any topic. It's got nothing to do with the ideas behind quadratics.

[Jill looks unsure how to respond, but other mathematics colleagues who are present begin to join in.]

Mike Completing the square is just a trick; steps you go through. Everything in maths is just a trick.

Jack No, the principles you use in completing the square are powerful and I mportant mathematically: the algebraic manipulation, solving an equation by doing the same things to both sides. And factorising is a big idea in maths that I want them to understand.

Mike Substituting into formulas is powerful – it's all they seem to do in science exams these days.

Helen [*to Jack*] Don't you teach your classes the quadratic formula then?

Jack I do, but later. If you teach it first, then they've got no motivation to learn any other method. And they have to learn completing the square first anyway, otherwise how do you derive the formula?

Helen I find that my classes are not really that interested in proofs, and the quadratic formula one is really fiddly – much too hard for them.

Mike I agree. If you asked me to sit down now and prove the quadratic formula for you, I'm not sure I could. But why do I need to? I know it and I can use it, and that's what matters.

It seems clear from this discussion that there is a significant difference between Jack's pedagogical intentions and those of his colleagues. Jack's pejorative use of the term 'trick' implies that he sees something illegitimate about the quadratic formula – that although it may be an efficient means of obtaining the correct answer it fails to expose students to 'the ideas behind quadratics'. Mike responds that the alternative methods might also be regarded as tricks, and that substituting into formulae is an important skill, but Jack maintains his position that factorising and completing the square are powerful methods which he is passionate about his students experiencing. What is it that leads to a method being derided as 'a trick'? Is Mike correct that every mathematical process is a trick?

Solving equations

Vaiyavutjamai and Clements (2006) comment on the lack of research into students' difficulties with quadratic equations, and since then a number of studies have explored this area (Kotsopoulos, 2007; Lima and Tall, 2010; Didiş, Baş and Erbaş, 2011; Olteanu and Holmqvist, 2012; Tall, Lima and Healy, 2013). Lima and Tall

(2010) report that teachers taught the methods of factorisation and completing the square but 'moved on quickly to the use of the formula in the belief that this would enable [their students] to solve any quadratic equation that would be given in a test' (p. 1). This is interesting in the light of Pólya and Szegö's (1972: viii) famous statement that "An idea that can be used only once is a trick. If one can use it more than once it becomes a method". By this definition, the quadratic formula is certainly not a trick. Indeed, Bossé and Nandakumar (2005) point out that a randomly-chosen quadratic expression with integer coefficients is extremely unlikely to be factorisable, and thus advocate completing the square and the formula as more reliable methods.

Jack's use of the word 'just' suggests that he may see a trick as something utilised in a thoughtless, reductive way, such as is implied with the jingle reported by Wu (2011: 375) for dividing fractions: "Ours is not to reason why, just invert and multiply". If students 'just' use the quadratic formula to obtain the answer, without any deeper sense of what is going on, their understanding would rightly be described as 'instrumental'.

The factorising method might be promoted on the grounds of developing students' understanding of the zero-product property, but Didiş, Baş and Erbaş (2011) found that students took only an instrumental approach to factorising. Solving quadratic equations by factorising can be reduced to finding two numbers which multiply to give a certain amount and add to give another, and then putting them inside brackets after writing 'x +'. This would seem to be just as instrumental as substituting into the formula. The third method, completing the square, is a demanding process, involving careful algebraic manipulation. It forms the basis for the derivation of the quadratic formula, yet Wu (2011) warns that:

> When students see the technique of completing the square merely as a trick to get the quadratic formula rather than as the central idea underlying the study of quadratic functions, their understanding of the technique is superficial. (p. 380)

Anecdotal evidence suggests that although students may prefer 'the formula', and think that it is easier and more reliable, they frequently make errors in using it (such as miscalculating b^2 as $-b^2$ when b is negative) and obtain incorrect answers. Even when a quadratic equation is already in factorised form, students will sometimes expand the brackets, simplify and use the formula, leading to multiple opportunities for error and demonstrating a lack of appreciation of mathematical structure.

However, there is more for the teacher to consider than the efficiency of obtaining a solution to a given equation. Giving students a formula, especially if they are hazy about where it comes from, may position them as recipients rather than authors of mathematics. For instance, it would be perfectly possible to construct a formula for the solution of *linear* equations:

> The solution to the equation $ax + b = cx + d$ is given by $x = (d - b) \div (a - c)$, $a \neq c$.

But it is very unusual to see linear equations taught in this way, presumably because the pedagogical purpose in this topic in not so much to find out as efficiently as possible what x is, as to learn about algebraic equality and solving equations at a more conceptual level. Instead of this formula, students are more likely to be told to 'do the same operations to both sides'. This itself might be regarded as a procedure, yet one arguably giving much greater scope for students to experiment and explore, and thus not, by most definitions, an 'algorithm' (something requiring no judgment [MacCormick, 2012]). However, students may be taught to apply balancing algorithmically, 'dividing by the multiplier', etc. I have seen a student who always carried out this step, even if the multiplier was 1, so that she would convert '$1x = 5$'

into '$x = 5$' by dividing both sides of the equation by 1. When questioned about this, she said that she knew that the value would not change but believed that she had been taught that this was the 'formal' way to do it (see Feynman, 1999: 5-6, for a similar account).

The value of procedures

It cannot be denied that mathematical procedures have considerable instrumental value. No one would want to have to differentiate from first principles every time or derive every formula on each occasion that it was used. A mathematician who wishes to divide fractions will almost certainly 'invert and multiply', but without the 'ours is not to reason why' prohibition quoted above. Yet 'reasoning why' every time would doubtless get in the way of fluent performance of the operation and distract from the wider purpose for which it is being done. So, while it is necessary for the mathematician to retain awareness of the conditions under which procedures are valid, facility with an appropriate algorithm tends to preclude conscious awareness of the details.

However, the best procedures are more than a pragmatic means to a calculational end. Indeed, even algorithms – the most rigid and prescribed of procedures – can be said to have mathematical beauty (MacCormick, 2012). As Crary and Stephen Wilson (2013) put it, "At the heart of the discipline of mathematics is a set of the most efficient – and most elegant and powerful – algorithms for specific operations". There is something neat about the careful construction of an effective algorithm, and many would regard Euclid's algorithm or Dijkstra's algorithm, for instance, as possessing considerable mathematical beauty. Algorithms, like proofs, consist of a series of prescribed steps with a clearly-designated outcome, so if proofs can be beautiful, why not algorithms too (MacCormick, 2012)?

It would seem then that procedures, even strict algorithms, are not inherently harmful in and of themselves. Their rigidity does not have to be experienced as oppressive and destructive to original thought; indeed the affordance of automation may simultaneously open up greater opportunities for originality within a wider context. For Brousseau (1997), "It is the didactical function and didactical presentation which retain or remove the value of a procedure. More exactly, it is the nature of the contract which takes shape on their behalf" (p. 40). If the teacher implies that there is a standard known method for solving a particular problem, this can block the 'devolution of the problem' to the student. Where preferred methods are privileged by their presentation as 'best', without the student coming to see their value for themselves, this is indeed likely to be disempowering. As Gutiérrez (2013) comments:

> when schools demarcate which algorithms are valid when learners are asked to show their work, the practice can lead to immigrant students discounting the knowledge of their parents who have learned mathematics in other countries, even if those 'foreign' algorithms are correct. (p. 44)

It seems essential that students are introduced to useful procedures and acquire facility in their use while at the same time feeling ownership and control over them, but how can this be achieved?

Conclusion

Farmelo (2009: 300) describes how Paul Dirac regarded a mathematician as someone who "plays a game in which he invents the rules". Inventing rules and taking ownership over them are critical elements of doing mathematics, but in order to experience this students' attention must be freed up from the minutiae of incidental procedures. Boaler and Greeno (2000: 185) describe how students (particularly girls) became alienated from mathematics when understanding was side-lined:

> [The students] were capable of practicing the procedures they were given and gaining success in the classroom and on tests, but they desired a more connected understanding that included consideration of 'why' the procedures they used were effective. (p. 185)

But that does not mean that procedures must never be used without consciously thinking about the details. Fully internalising an important procedure so as to develop an intuitive 'expert-induced amnesia' about it shifts the process into implicit memory. This enables the student to operate faster than would be possible with conscious thought and frees up working memory for other things (Syed, 2011).

So I suggest that an algorithm has two possible legitimate roles to play in the teaching of school mathematics:

1. *An object of focus in its own right*: students develop an algorithm to achieve a particular end or take a critical approach to given algorithms, comparing, modifying, inventing and evaluating;
2. *An incidental tool for pursuing a wider mathematical problem*: here the students' attention is deliberately on a larger problem and the algorithm is merely a means to an end.

On the one hand, the algorithm can be probed analytically, and on the other it can be utilised for a grander purpose, where the goal of procedural fluency may be embedded in a richer more worthwhile problem – what I have described elsewhere as a 'mathematical étude' (Foster, 2013b). Burying procedural practice within a more interesting problem may have the advantage of taking attention away from the algorithm, perhaps aiding the development of fluent mastery. What must be avoided is the all-too-common situation where the focus is on the algorithm, but not in order to probe and understand its workings, or to fulfil some greater purpose, but simply in order to perfect its performance.

Viewed in this way, none of the methods mentioned in the staffroom discussion is ruled out *per se*. The quadratic formula is taught, but students attend to its construction and interrogate its components by considering questions such as:

- What happens if $a = 0$? or if $b^2 - 4ac = 0$? or if $b^2 - 4ac < 0$? Why?
- What happens if a, b and c are all multiplied by the same factor k? Why?
- How does the formula compare with an alternative such as $x = 2c \div (-b \pm \sqrt{b^2 - 4ac})$?
- What values of a, b and c will make *both* values of the formula positive/negative/zero? Or lead to one positive and one negative value?

Exploring such questions takes students well beyond simply being on the receiving end of a proof. Similarly, with other methods, such as factorising, critical thought can be encouraged with questions such as:

- How can you solve an equation like $(x - 2)(x + 3) = 8$ for integer x?
- What is the value of $(x - a)(x - b)(x - c)\ldots(x - z)$?

Even more important than posing these questions is encouraging students to ask their own questions about the mathematics (Foster, 2011). If students are to regard methods such as the quadratic formula, 'invert and multiply' for fractions, 'cross multiplying' and so on as more than a trick, they need opportunities to probe and question those methods in order to gain insight into how and why they work.

Injunctions to adhere rigidly to somebody else's rules may be perceived by students as disempowering. The sense of not being trusted to work things out for themselves can lead students into learning to accept rules that make no sense to them. As Noyes (2007: 11) puts it, "Many children are trained to do mathematical calculations rather than being educated to think mathematically". I argue in this paper that the answer is not to eschew procedures wholesale but to ensure that students encounter them in a critical, questioning spirit and then, when convinced of their value, internalise them to the point that they regard them as useful tools with which to pursue more interesting mathematical problems.

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