Evans, Sheila and Mullins, Nicola and Waring, Lucy (2014) The role of sample pupil responses in problem-solving lessons: perspectives from a design researcher and two teachers. In: 8th British Congress of Mathematics Education, 14-17 April 2014, Nottingham, UK.

The Role of Sample Pupil Responses in Problem-Solving lessons: Perspectives from a Design Researcher and Two Teachers

Sheila Evans, Nicola Mullins and Lucy Waring

University of Nottingham, The Joseph Whitaker School, Rainworth, Toot Hill School, Nottingham

The benefits of learning mathematics by comparing, reflecting on and discussing multiple approaches to a problem are well-known (Silver, 2005). However, teachers using non-routine problem-solving tasks designed to encourage multiple approaches face challenges: understanding how pupils make sense of the problem, especially when pupils use unique or unanticipated approaches and helping pupils make connections between disparate approaches and aligning these with lesson goals. In an attempt to address such challenges an extensive set of problem solving lessons were developed. Each lesson includes a range of sample solution-methods that expose pupils to multiple perspectives. A detailed teacher guide supports each lesson. This paper focuses on the use of these sample solution-methods. It explores their development from initial design to final versions. We analyse the varied interpretations and use made of sample solution-methods, in both US classrooms and by two UK teachers, and reflect on how these interpretations align with the designers' intention.

Keywords: Problem-solving, design research, multiple solutions.

Introduction

This paper arises from a study on the development and use of formative assessment lessons. Between 2010 and 2014, a team at the University of Nottingham designed over one hundred Formative Assessment Lessons (FALs) to support all grades in US High and Middle Schools implementing the new Common Core State Standards for Mathematics (http://www.corestandards.org/Math). About a third of these lessons involve non-routine, problem-solving tasks. The aim of the lessons is to assess and develop pupils' capacity to apply mathematics flexibly to unstructured problems, both from pure mathematics and from the real world. The tasks can be solved using a range of methods. The lessons are also intended to support the assessment and development of pupils' capacity to understand, evaluate and compare up to three Sample Pupil Responses (SPRs) to the same problem. During the third year of the project, the latest version of the FALs were taught by eight secondary school teachers in UK classrooms, two of whom are co-authors. Design researchers (known from this point as designers) observed the lessons, one is a co-author. This paper focuses on SPRs. It explores the development of the design of the SPRs and analyses the variation and evolution in interpretation of the use of SPRs both in trials in US classrooms and by the two UK teachers. It reflects on how these interpretations align with the designers' intentions.

Background literature

The literature is clear on the potential benefits to learning of each pupil working with multiple approaches to a problem (Silver et al., 2005). Different approaches can facilitate connections to different elements of knowledge; creating or strengthening networks of related ideas and enabling pupils to achieve ‘a coherent, comprehensive, flexible and more abstract knowledge structure’ (Seufert et al., 2007: 1056). However, this may be challenging for some teachers; to tease out and connect the important structures, representations and ideas (including misconceptions and errors) embedded in pupils’ various solution-methods (Stein et al., 1996). Teachers need to monitor pupil work with a desire to understand possibly unanticipated or nonstandard approaches; discerning the mathematical value of these approaches in order to scaffold learning; purposefully selecting solution-methods for a plenary whole class discussion; orchestrating this discussion to build on collective sense-making of pupils by intentionally ordering the work to be shared; helping pupils make connections between and among different approaches in order to advance the instructional goals of the lesson (Chazan and Ball, 1999; Lampert, 2001).

Teachers need support to develop skills to work in this way. Research has shown that lack of explicit guidance on how teachers can encourage and support mathematical learning means some teachers believe they should not press pupils for justifications or directly tell them anything, instead they should allow pupils to work things out on their own (Swan, 2006). So that whole-class presentations of pupil solution-methods may be little more than a “show-and-tell” (Ball, 2001).

This not only takes up a lot of teaching time but also precludes a major benefit of working with multiple solution-methods namely; comparing them. There are usually advantages and disadvantages of most approaches, depending on the context. For example, numerical representations are often familiar to pupils and form a convenient bridge to other representations. They may help pupils understand a problem but may miss important aspects of a problem and not lead to generalisation. Graphs may provide a visual approach and are often intuitive (at least on the surface), but may lack the accuracy necessary to solve the problem. Symbolic algebra is powerful, being both concise and general, however it may obstruct meaning for some.

However, it is not enough to simply suggest that solution-methods should be compared (Chazan and Ball, 1999); teachers need more guidance. There is little research in mathematics education about this, although there are some case studies of expert mathematics teachers who emphasise the value of encouraging and comparing pupils’ different solution-methods (Silver et al., 2005).

Content and Structure of the Problem-Solving Lessons

The remit for developing the FALs was that the problem-solving lessons should draw on a range of important mathematical content, be engaging and cognitively demanding. There should be multiple entry points and solution strategies, allowing different pupils to approach the task in different ways based on their own prior knowledge, leading to a sense of pupil ownership. The tasks should provide opportunities for pupils to conjecture, reflect and make connection between concepts they have already learnt and show how mathematics can help make sense of the world. Finally, the tasks should maximise opportunities for pupils to make visible to themselves and the teacher their current understanding and reasoning. In so doing teachers are able to be particularly adaptive and responsive to pupil learning needs over the course of a lesson as well as facilitate their planning of future lessons (Black and Wiliam, 1998; Swan, 2006). Each task includes both a problem and SPRs.

The structure of each problem-solving FAL is broadly consistent: pupils are given a problem to tackle on their own *before* the main lesson. The teacher reviews this work and formulates further questions for pupils to answer in order to improve their approach. The lesson starts with pupils using these questions to review their own work. Next, in small groups, pupils evaluate fellow pupils' approaches, with the aim of producing a joint approach that is better than their individual efforts. Pupils also have the opportunity to discuss strategies used by their peers and used in the SPRs. In a plenary, these responses are discussed and finally, through a questionnaire, pupils reflect on their work.

Detailed lesson guides designed to support the teacher accompany each task. The purpose of the guide is not simply to offer instructional ideas, but to serve as a catalyst for local customisation (Remillard, et al., 2004). The guide states clearly the designers' intentions; it suggests formative assessment and problem-solving strategies, and provides examples of issues pupils may face. The designers expect teachers to take account of the talents, interests and limitations of their pupils.

Lesson design: the development of Sample Pupil Work

The design of each lesson is grounded in the research literature and supported by evidence from lesson trials. The materials have been developed, from initial draft designs to latest versions, through an iterative, process of lesson design, enactment in three to five US classrooms, analysis, and redesign (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). Redesigns are based on detailed feedback from observers of the materials in use in classrooms using an instrument developed by the designers. There are at least two revision cycles per FAL. In the third year of the project, the latest version of FALs were taught in UK classrooms. UK teachers taught between six and ten of the FALs over the course of a year. Researchers observed and videoed most lessons and interviewed the teachers both before and after each lesson.

From the US trials of the FALs the designers soon discovered that within any class, although each task had the potential to be solved using a range of distinct methods, this generally did not happen. Classes tended to use one or at the most two methods per task. It is unclear as to why this was the case.

It is possible to speculate that because many classes in the US focus on one subject area for several months, e.g. algebra, pupils are highly likely to draw on the same subject area for a solution to a problem. Alternatively, pupils may choose a solution-method they assume the teacher values. This is supported by a conjecture made by a US observer:

> Due to the "traditional" approaches generally used here in the States, many teachers believe that "geometric" or sketches are NOT showing rigor or intelligence and that number is the best way and students have internalized this… I have seen teachers, especially secondary teachers scorn the use of manipulatives or actual constructions to make sense of a problem situation. (Observer 1, 2013)

The school year of pupils could also be a factor. As the problems are rich, non-routine and unstructured, the mathematics used to solve them needs thorough understanding, especially as potential methods often require the building of mathematical connections. Consequently, the lessons are aligned to school years that allow pupils to draw on mathematics from previous years. Without this alignment the breadth of potential solution-methods would not be readily available to pupils. This same issue arose in the UK classrooms. Most UK pupils preferred a numerical method, very few attempted algebraic or graphical methods.

> This presents both teachers and designers with a dilemma – how does one reveal the power of, for example, an algebraic approach to a particular problem without

> ‘forcing’ pupils to use it, in which case the lesson is no longer a true problem-solving lesson, but a mere exercise in algebra? (Malcolm Swan, lead designer)

The designers’ response was to introduce two or three pre-written SPRs for each task. These SPRs draw on a range of mathematical ideas and were to be given to pupils once they had already worked in small groups on the problem.

The solution-methods were not for the teacher to present to the class, as although this approach does guarantee pupil exposure to a wide range of methods, the authority for the mathematics would remain with the teacher and the materials. Presenting the strategies as if they were written by pupils introduces a third party to the classroom, a third authority for the mathematics, an authority unknown to the pupils. This anonymity can be advantageous as pupils do not know the mathematical prowess of the author. If it is known that a pupil with an established reputation for being ‘mathematically able’ has authored the solution then it is possible to conjecture that most will assume the solution is a good one. In which case, its evaluation depends more on who is thought to have written it rather than whether or not it is a worthy one. Anonymous ‘pupils’ authoring the sample responses also reduces the emotional aspects of peer review. Feedback from the US trials indicated that sometimes pupils were reluctant to voice comments that might be perceived as negative.

Additionally, encouraging pupils to compare methods may help teachers recognise that not all mathematical ideas are equally valid, as already mentioned there are advantages and disadvantages to most approaches depending on context. This appreciation may in turn help teachers when orchestrating a whole class discussion.

Further to this, solution-methods written by ‘pupils’ rather than presented by the teacher provides designers with the opportunity to include common errors or misconceptions, write incomplete SPRs or SPRs that lack clarity. Thus, SPRs can be used for a number of interconnected and overlapping purposes, including:

- to help pupils struggling to get started or making little progress with a problem by introducing a new approach
- to help pupils understand and critique unfamiliar approaches. This may develop pupils’ capacity to flexibly solve problems, make connections between different concepts and reflect, critique and improve their own approach
- to facilitate the confrontation of a common pupil misconception by embedding it into a SPR
- to provide an opportunity for pupils to compare and critique approaches.

Discussing SPRs can make visible what pupils value and their own learning goals. For instance, do pupils regard the criteria for success the correct answer as opposed to a clear, efficient, elegant or/and generalisable method? This can provide useful feedback as to whether there is a mismatch between the teacher’s and pupils’ learning goals (Leahy et al., 2005).

Pupils working with the Sample Pupil Responses

When lessons were first trialled in the US the guide contained suggestions on how the teacher could introduce the SPRs to the class: by writing the following instructions on the board:

> Imagine you are the teacher and have to assess this work. Correct the work and write comments on the accuracy and organization of each response. Make some specific suggestions as to how the work may be improved.

Feedback from the US trials indicated to the designers that these instructions were inadequate. Teachers and pupils were not clear on what mathematics should be discussed, for example, US teachers were asking the designers questions such as:

> What is the math we want to have a conversation about? Do we want pupils to explain the method? Do we want each piece to stand-alone or should pupils compare and contrast strategies?

US observers suggested that pupils were not digging deeply into the mathematics of each SPR and unless asked a direct question by the teacher, pupils often worked in silence simply looking for errors without evaluating the overall solution strategy. Some pupils mimicked the feedback they received from the teacher, providing comments such as 'Awesome', 'Good answer' or 'Show a little more work'. A clear message came from the observers; the prompts in the guide needed to be more explicit and more focused on the mathematics of the problem; scaffolding was required. The decision was made to include more specific questions, such as:

> What is the point of figuring out the slope and intercept?

The designers hoped that such questions would also emphasise the important mathematics in the lesson. The feedback from the US observers was encouraging:

> I think the questions or prompts about each piece of student work, really focus the students on the thinking, bring out the key mathematics and are a great improvement to the original lesson…Last year students just made judgment statements, but this year the comments were focused on the mathematics. (Observer 2, 2010)

However, this contrasted with the approach taken by the two Year 9 teachers in the UK. Understanding and critiquing SPRs was new to them and their pupils. Working collaboratively was also new to some pupils. Envisaging that questions specific to each SPR could exclude some pupils from participating in discussions the teachers preferred initially to simply ask pupils to explain the approach; describe what the pupil had done well and possible improvements. The teachers noted that most pupils readily engaged with each other in answering these questions. But often, as in the US, pupils focused on finding errors or superficial features such as spelling or neatness of handwriting. The latter could be regarded as unfortunate distractors easily remedied by the designers, however these features could also serve as a useful platform for discussion: what is of value in a particular approach? The errors presented a more complex issue. The designers include them in some SPRs as a means of access into the solution, to enable most pupils to engage with the task. However, it became clear that they could also distract pupils from attempting to understand the overall strategy. As one UK teacher commented, error-free SPRs often resulted in a more holistic evaluation of a method.

Lack of understanding of SPRs was a feature of many UK lessons throughout the project. Pupils tended to focus on the step-by-step processes, not the underlying reasoning behind a strategy. For example, when pupils were asked to complete one particular SPR, most were able to do so, they understood the processes, and were able to work out the correct numerical answers, but then encountered difficulties interpreting the resulting figures in the context of the real-world situation. Furthermore, asking questions, such as 'Explain Harry's method' seemed to encourage pupils to concentrate their energies on the detail of each step of the SPR rather than also attempting to understand the overall strategy and its purpose. This was aptly demonstrated in another lesson when UK pupils explained the multiplication of 20 x 30 = 600 gave a correct answer, but did not think about the meaning of the answer within the context of the problem. Only when prompted by the teacher did they appreciate that the 'pupil' was calculating an area.

With encouragement from the UK teachers, many pupils noted lack of explanation in some of the SPRs, but usually with cursory suggestions for improvement, for example simply stating 'she needs to explain it better' or 'the diagrams should not be all over the place'. This could be remedied by prompting pupils to improve the SPR rather than just make suggestions on the improvements.

Number of Sample Pupil Responses (SPRs) used

The design brief was to write a problem-solving lesson that lasted one hour. Initial feedback from US observers indicated the lessons were taking a lot longer; teachers were giving out all pieces of SPRs, but there often was not sufficient time to successfully evaluate and compare the different approaches. In response to this issue, designers decided to include the generic text in all lesson guides:

> There may not be time, and it is not essential, for all groups to look at all sample responses. If this is the case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups that have struggled with a particular approach may benefit from seeing another pupil's work that uses the same strategy.

These instructions provide the potential for pupils to critique and reflect on unfamiliar approaches, to explicate a process and to compare their own work with a similar approach; this, in turn could serve as a catalyst to review and revise their own work.

However, in US trials this approach was not always successful; it turned out that comparing two SPRs or evaluating an unfamiliar method was often limited to a few pupils. This raised the question of whether pupil differentiation should be by task rather than support? Should all pupils have the opportunity to engage with all the mathematical ideas incorporated in the SPRs? If teachers limited some groups of pupils to working with, for example, just one SPR while others work with two or three, then when the teacher holds a whole class discussion about the different SPRs inevitably some pupils will be disadvantaged; it can be challenging to understand 'in the moment' of explaining, a SPR. However, giving all pupils all SPRs may not be a practical, as it could result in the task extending over several lessons.

Also, feedback from US trials highlighted the tension many teachers encountered when faced with struggling pupils. To allow pupils to struggle often conflicted with ingrained beliefs of what it meant to be a teacher. Yet the guide consistently encouraged teachers to permit pupils to persevere collaboratively with the task in hand; to struggle productively (Swan, 2006). The above instructions may inadvertently encourage teachers to do the opposite, as seen in this US observer comment:

> We have some teachers who give all the sample student work and let students choose the order and the amount they do. This might be less common. Others are very controlling and hand out certain pieces to each group….Others like a certain method to solve problems and like to use that one to model. I think this is a function of the teacher's comfort level with control and students' expectations. (observer 3, 2013)

The designers continue to struggle with this issue. It was also a concern for the UK teachers. At the start of the project they were reluctant to give pupils all SPRs at the same time, they were concerned that pupils would be overwhelmed. They believed that pupils would find it difficult to 'get into the head' of another pupil. No pupils in the study had previously had the opportunity to edit and dissect another's thinking. Also, the teachers regarded the purpose of SPRs as a catalyst for improvements of pupils' own solution and a key formative assessment tool; primarily to help pupils reflect on and critique their own work.

In a UK teacher's first lesson she gave one piece of work to each group. Those that had struggled with the problem were given what she regarded as the easier SPR. Her instructions were, as already mentioned non-SPR specific. For example: 'What has the pupil done well? What if they did it again could they do better?'. She also asked pupils to do the same for their own work. In the following whole class discussion, although limited to just a few participants, her priority was not to compare SPRs, but to bring to the attention of the class not just the answers to questions mentioned above, but how the pupils had gone about arriving at the answers. Some pupils had reworked the answer to the problem for themselves using the same method as in the SPR, others had checked the SPR line by line. Her goal here was to help struggling pupils develop strategies in order to understand and critique a piece of work whether their own or someone else's. This goal became less of a priority as the project progressed, but for both teachers using the SPRs as a tool to help solve the problem remained a fixed goal throughout the project. However, surprisingly, there was little evidence of pupils changing their work apart from when they noticed a numerical error. Some pupils acknowledged their work needed improving but did not take the next step and improve it. Observers reported the same in the US, only pupils that were stuck were likely to adapt or use a strategy from a SPR.

Conclusion

Using SPRs was a new and challenging experience for both UK teachers and pupils. However, at the end of the project both teachers noted that the SPRs acted as professional development; serving to improve their own flexible knowledge of mathematics and of pupil's thinking when solving problems. This in turn helped them resist insisting or explaining the "one" way for pupils to "get" the answer.

In the study it became clear that not all of the designers' purposes for introducing SPRs were enacted in the classroom. Teachers had their own reasons for using SPRs and these reasons evolved as the project progressed. One of the designers' purposes was to help pupils struggling with the problem, yet there was little pupil written evidence of this happening. Pupils were able to acknowledge mistakes in their own work, but generally did not take the next step and write it down. It is possible to speculate this was due to timing, most pupils are given the SPRs at the same time, after they have worked on the problem for some time, so had 'moved on'.

Another purpose for SPRs was to help pupils understand and critique unfamiliar methods. The UK teachers did consider their pupils' capacity to do so improved a little and they anticipated with more experience of SPRs pupils would move beyond explaining the detail of the step-by-step processes within a SPR to also explaining the overall reasoning. Revisions to the materials may support this move.

A third purpose for the introduction of SPRs was to provide an opportunity for pupils to compare methods. The UK teachers rarely overtly encouraged pupils to compare SPRs and it may not have been appropriate to do so, especially as, as already mentioned, pupils found understanding another, unfamiliar, pre-written solution very challenging. UK pupils were consistently given the opportunity to compare their own work with one piece of SPR but not to compare SPRs. This was also the case in the US trials, although pupils often received all SPRs there was little evidence that they compared them. However, the UK teachers now plan, once pupils are accustomed to analysing another person's solution-method, to use the powerful tool of comparison, especially when pupils are critiquing work.

More exploration is also required into how the use of SPRs affects pupils' own capacity to solve problems. For example, one might have expected that the range of distinct methods used to solve a problem would increase as the project progressed; there was scant evidence of this. One teacher did note that some of her pupils now write fuller explanations when solving the problems for themselves.

There are no major studies that focus on how teachers work with a range of pre-written solution-methods for a range of rich problems. This one-year study raises many issues and in so doing acts as a launch pad for further more detailed studies

We would like to thank our funder, the Bill & Melinda Gates Foundation.

References

Ball, D. L. (2001) Teaching with respect to mathematics and students. In Wood, T., Nelson, B.S. & Warfield, J. (Eds.) *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 11–22). Mahwah, NJ: Lawrence Erlbaum Associates.

Black, P. & Wiliam D. (1998) Assessment and Classroom Learning. *Assessment in Education: Principles, Policy & Practice*. 5(1), 7-74.

Chazan, D. & Ball, D. L. (1999) Beyond being told not to tell. *For the Learning of Mathematics* 19(2), 2–10.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003) Design Experiments in Educational Research. *Educational Researcher*. 32(1), 9-13.

Common Core Standards,

Lampert, M. (2001) *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Leahy, S., Lyon, C., Thompson, M., & Wiliam, D. (2005) Classroom Assessment:Minute by Minute, Day by Day, *Educational Leadership*. 63(3), 19 -24.

Remillard, J.T. & Bryans, M.B. (2004) Teachers' Orientations Toward Mathematics Curriculum Materials: Implications for Teacher Learning, *Journal for Research in Mathematics Education* 35(5) 352–388.

Seufert, T., Janen, I., & Brunken, R. (2007) The impact of intrinsic cognitive load on the effectiveness of graphical help for coherence formation. *Computers in Human Behavior*. 23, 1055–1071.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C. & Font Strawhun, B.T. (2005) Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior*. 24, 287–301.

Stein, M., Grover, B. W. & Henningsen, M. (1996). Building Student Capacity for Mathematical Thinking and Reasoning: An Analysis of Mathematical Tasks Used in Reform Classrooms. *American Educational Research Journal*. 33(2), 455-488.

Swan, M. (2006) *Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices*. London: NIACE/NRDC.