On Designing H_∞ Filters With Circular Pole and Error Variance Constraints

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***Abstract*—In this paper, we deal with the problem of designing a H_∞ filter for discrete-time systems subject to error variance and circular pole constraints. Specifically, we aim to design a filter such that the H_∞ norm of the filtering error-transfer function is not less than a given upper bound, while the poles of the filtering matrix are assigned within a prespecified circular region, and the steady-state error variance for each state is not more than the individual prespecified value. The filter design problem is formulated as an auxiliary matrix assignment problem. Both the existence condition and the explicit expression of the desired filters are then derived by using an algebraic matrix inequality approach. The proposed design algorithm is illustrated by a numerical example.**

***Index Terms*—Algebraic matrix inequality, error variance constraints, H_∞ filtering, Kalman filtering, pole assignment.**

I. Introduction

The Kalman filtering theory has been widely used over the last three decades [1]. However, it is now well recognized that the discrepancies between the true and nominal systems can substantially deteriorate the performance of the Kalman filters. Thus, many different filtering approaches have been proposed to improve the robustness of the conventional Kalman filters, such as unbiased LMS filtering [17], robust adaptive filtering [2], and H_∞ filtering [11]. In particular, the H_∞ filtering approach has been developed and has gained extensive attention. The H_∞ filtering approach is concerned with the design of an estimation procedure, which ensures that the L_2-induced gain from the noise signals to the estimation error will be less than a prescribed level. It has been shown in [12] that the H_∞ filtering scheme is less sensitive than its H_2 counterpart to the uncertainty in the system parameters.

The H_∞ filtering approach itself, however, is not suitable to the case when the filtering performance objectives are expressed explicitly as upper bounds on the steady-state estimation error variances. This case is quite common in practical filtering problems, such as the tracking of a maneuvering target and the recognition of flight paths from multiple sources, where the goal is to design filters such that the estimation error variance for each state is not more than the prespecified upper bound. The traditional H_2 or H_∞ filtering theories could minimize a selected weighted scalar sum of the error variances of the state estimation, so as to indirectly achieve the steady-state error variance constraints, but minimizing a scalar sum does not ensure that the multiple variance requirements will be satisfied [10]. This situation motivated the development of a new filtering method, namely, the error covariance assignment (ECA) filtering method, see e.g., [16]. The idea of the ECA theory is to provide a closed form solution for directly assigning the specified steady-state estimation error covariance. Subsequently, the ECA theory has been generalized to the so-called variance-constrained filtering problems for parameter uncertain systems [13] and sampled-data systems [14], where a prespecified upper bound is placed onto the steady-state estimation error variance. There, the specified variance constraints may not be minimal, but should meet given engineering requirements.

As is well known, if the transient property of the error covariance approaching its steady-state value is not satisfactory, the actual filtering efficiency will be seriously influenced. The transient behavior of a linear filtering process can be guaranteed by pole assignment that has received significant attention, see e.g., [9] and references therein. On the other hand, locations of poles vary and cannot be fixed due to the variation of the operating points, parameter identification errors, etc. Hence, placing all poles of the overall system into a desired (often circular) region rather than choosing an exact assignment may be satisfactory in practice. A large amount of literature has been reported on this topic, see e.g., [6], [7] for the discrete-time case. However, the issue of *variance-constrained* H_∞ filtering with *circular pole constraints* has not been fully investigated and remains to be important and challenging. It is, therefore, our interest to develop an H_∞ filtering algorithm that can incorporate both steady-state variance (steady-state property) and circular pole (transient property) constraints.

In this paper, the H_∞ filtering problem is dealt with subject to both error variance and circular pole constraints. It is shown that this problem can be solved by using an effective algebraic matrix inequality approach. Specifically, the conditions for the existence of the desired filters are obtained, and the explicit expression of these filters is also derived. We now remark that since the desired variance-constrained filters are often a large set, the remaining design freedom provides the possibility for achieving other expected multiple objectives, while the traditional optimal filtering theories seem not to be of such an advantage.

Notation: The notations in this appear are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The superscript "T" denotes the transpose. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathcal{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation and Assumptions

Consider a linear discrete time-invariant observable stochastic system [8]

$$x(k+1) = Ax(k) + D_1 w(k) \quad (1)$$

$$y(k) = Cx(k) + D_2 w(k) \quad (2)$$

where $x \in \mathbb{R}^n$ is a state vector, $y \in \mathbb{R}^p$ is a measured output vector, and A, C, D_1, and D_2 are known constant matrices. It is assumed that C is of full row rank. $w(k) \in \mathbb{R}^n$ is a zero mean Gaussian white noise sequence with covariance $I > 0$. The initial state $x(0)$ has the mean $\overline{x}(0)$ and covariance $P(0)$, and is uncorrelated with $w(k)$.

The state estimation vector $\hat{x}(k)$ satisfies the linear, full-order filter of the form

$$\hat{x}(k+1) = A\hat{x}(k) + K[y(k) - C\hat{x}(k)] \quad (3)$$

where K is a filter gain to be scheduled.

Manuscript received March 27, 2002; revised January 25, 2003. This work was supported in part by the Engineering and Physical Sciences Research Council of the U.K. under Grant GR/S27658/01 and in part by the Alexander von Humboldt Foundation of Germany. This paper was recommended by Associate Editor S.-M. Phoong.

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Digital Object Identifier 10.1109/TCSII.2003.810489

The steady-state estimation error covariance is defined by

$$P := \lim_{k\to\infty} P(k) := \lim_{k\to\infty} \mathcal{E}[e(k)e^T(k)]$$
$$e(k) = x(k) - \hat{x}(k) \tag{4}$$

where $e(k)$ stands for the error state, and is an innovation process. The propagation of the estimation error meets

$$e(k+1) = (A-KC)e(k) + (D_1 - KD_2)w(k) \tag{5}$$

and then it turns out from (3)–(5) that the evolution of (5) is

$$P(k+1) = (A-KC)P(k)(A-KC)^T + (D_1 - KD_2)(D_1 - KD_2)^T. \tag{6}$$

The filtering matrix is denoted as $A_f := A - KC$. It is well known that, if A_f is Schur stable (i.e., the poles of A_f are all located within the unit disk), then in the steady state, the estimation error covariance P exists and satisfies the discrete-time Lyapunov equation

$$P = A_f P A_f^T + (D_1 - KD_2)(D_1 - KD_2)^T \tag{7}$$

where $P = P^T > 0$.

Let $D(q, r)$ be a disk that is in the complex plane, centered at $(q, 0)$ with radius r, where $r < 1$ and $|q| + r < 1$. It is noticed that such a disk has been treated as the desired pole-region for discrete-time systems in many papers, see for example [6], [7]. Assume that the error state outputs are represented by $Le(k)$ where L is a known constant matrix of appropriate dimension. We are now in a position to state the filtering problem for linear discrete-time stochastic systems with both error variance and circular pole constraints.

H_∞ Norm, Circular Pole, and Variance-Constrained Filtering Problem (HCVFP): For the linear discrete-time stochastic system (1)–(2), design a filter gain K such that the following three performance requirements are simultaneously satisfied.

(P1) The poles of the filtering matrix $A_f = A - KC$ are placed inside a given circular region $D(q, r)$, i.e.

$$\sigma(A_f) \subset D(q, r). \tag{8}$$

where $\sigma(A_f)$ is the set of all eigenvalues of the matrix A_f.

(P2) The steady-state error covariance P exists and meets

$$[P]_{ii} \le \sigma_i^2, \qquad i = 1, 2, \ldots n \tag{9}$$

where $[P]_{ii}$ denotes the variance of the ith error state, and $\sigma_i^2 (i = 1, 2, \ldots, n)$ stands for the prespecified steady-state error variance constraint on the ith error state, which can be determined by practical engineering requirements, but should not be less than the minimal variance value obtained from traditional H_2 filtering theory.

(P3) The H_∞ norm of the transfer function $H(z) = L(zI_n - A_f)^{-1}(D_1 - KD_2)$ from disturbances $w(k)$ to error state outputs $Le(k)$ satisfies

$$\|H(z)\|_\infty \le \gamma \tag{10}$$

where L is a known error state output matrix, and

$$\|H(z)\|_\infty = \mathrm{Sup}_{\theta\in[0,2\pi]} \sigma_{\max}[H(e^{j\theta})] \tag{11}$$

and $\sigma_{\max}[\cdot]$ denotes the largest singular value of $[\cdot]$; and γ is a given positive constant.

Remark 1: In this paper we cope with a steady-state (or infinite horizon) filtering problem. The reason is that, we take the pole assignment constraint into account, and the pole assignment problem is only applicable for the linear time-invariant problems. Without the pole location requirement, the finite horizon (or adaptive) filtering problem should be of more practical importance.

Remark 2: It is noted that, the error variance constraint can be prespecified according to the practical engineering requirements. An example of this is the target tracking problem. Assume that the maneuvering target is accelerating with random bursts of gas from its reaction control system (RCS) thrusters, and hence the state vector could consist of the position and velocity. When tracking a maneuvering target through a radar system, we would like the target to be kept inside a designated "window" as frequently as possible. Therefore, the acceptable filtering error variance is dependent on the size of this window. A related application can be found in [18], which dealt with the pointing problem for NASA's ACES structure. As for the selection of the constant γ, for a better disturbance rejection property, we typically require $\gamma < 1$. However, if this cannot be achieved, we can increase γ as long as it meets the practical restriction, since we actually consider a multi-objective design problem in this paper. On the other hand, an interesting topic for future research would be to optimize one objective while other requirements are still kept satisfied.

III. Main Results and Proofs

We give a theorem as follows that will play a key role in the derivation of our main results.

Theorem 1: Let the positive scalar $\varepsilon > 0$ be arbitrarily small and the circular region $D(q, r)$ be given. If there exist a feedback gain K and a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ satisfying

$$LQL^T \le \gamma^2 I \tag{12}$$

$$\begin{aligned} &A_f[Q + QL^T(\gamma^2 I - LQL^T)^{-1}LQ]A_f^T \\ &\quad + (A_f - qI)Q(A_f - qI)^T \\ &\quad + (D_1 - KD_2)(D_1 - KD_2)^T + \varepsilon I \\ &= r^2 Q \end{aligned} \tag{13}$$

then we have the following conclusions:

1) the filtering matrix A_f satisfies the circular pole constraint, i.e., (8) is met;

2) $\|H(z)\|_\infty \le \gamma$ where $\|H(z)\|_\infty$ is defined in (11);

3) the steady-state error covariance P exists and satisfies $P < Q$.

Proof: 1) Assume that there exist $\varepsilon > 0$ and K such that (12) and (13) hold. For simplicity, we define $\Psi := (A_f - qI)/r$. It is easy to find that the specified circular pole constraint $\sigma(A_f) \subset D(q, r)$ is equivalent to the Schur stability of matrix Ψ, i.e., the eigenvalues of Ψ are all located inside the unit circle $D(0, 1)$ in the complex plane. It follows from the discrete-time Lyapunov stability theory that, Ψ is a Schur matrix if and only if there exists a positive definite matrix Q meeting $Q - \Psi Q \Psi^T > 0$. By means of the definition of Ψ, (13) can be equivalently expressed as

$$Q - \Psi Q \Psi^T = (1/r^2)\{A_f[Q + QL^T(\gamma^2 I - LQL^T)^{-1}LQ]A_f^T + (D_1 - KD_2)(D_1 - KD_2)^T + \varepsilon I\} > 0$$

which means that all poles of the matrix A_f should lie within the prespecified circular region $D(q, r)$, i.e., A_f satisfies the specified circular pole constraint.

2) Note that $0 < r < 1$. The equation (13) can further be rearranged as

$$Q = A_f[Q + QL^T(\gamma^2 I - LQL^T)^{-1}LQ]A_f^T + (D_1 - KD_2)(D_1 - KD_2)^T + \Omega$$

where $\Omega := (A_f - qI)Q(A_f - qI)^T + (1 - r^2)Q + \varepsilon I > 0$. The proof of $\|H(z)\|_\infty \le \gamma$ is then completely analogous to the proofs of Lemma 2.1 or Lemma 5.1 of [8].

3) Since A_f satisfies the specified circular pole constraint, the steady-state error covariance P exists and satisfies (7). To establish

the relationship between the steady-state error covariance P and the positive definite matrix Q, we continue to transform (13) into

$$Q = A_f Q A_f^T + (D_1 - KD_2)(D_1 - KD_2)^T + \Theta \qquad (14)$$

where

$$\Theta := A_f[QL^T(\gamma^2 I - LQL^T)^{-1}LQ]A_f^T + (A_f - qI)Q(A_f - qI)^T + (1 - r^2)Q + \varepsilon I > 0. \quad (15)$$

Subtract (7) from (14) to give $Q - P = A_f(Q - P)A_f^T + \Theta$, which is equivalent to $Q - P = \sum_{i=0}^{\infty} A_f^i \Theta (A_f^i)^T > 0$, where $A_f^i = \overbrace{A_f \cdot A_f \cdots A_f}^{i}$ and i is a positive integer. The conclusion of $P < Q$ follows immediately. ■

Remark 3: In Theorem 1, the parameter $\varepsilon > 0$ that can be arbitrarily small is only used to guarantee that conclusion 1) holds. In the case when $D_1 - KD_2$ is of full row rank, the constant ε can be set to zero.

Remark 4: Theorem 1 reveals that the circular pole and H_∞ constraints on the filtering process are automatically enforced when a positive definite solution to (13) is known to exist, and all such solutions provide upper bounds for the actual steady-state estimation error covariance P. The upper bound obtained in Theorem 1 is not required to be minimal, as it is only needed to satisfy the given constraint. The expected filter gains, therefore, are usually nonunique, and to some extent, the resulting design freedom may explain why a multiobjective filtering problem can be addressed in this paper. On the other hand, the freedom can also be utilized to *locally* optimize one of the multiple objectives, such as minimizing the error variance or minimizing the disturbance rejection attenuation level γ. This research topic will be investigated in the future.

Definition 1: Given a matrix $Q > 0$ satisfying (12) and

$$[Q]_{ii} \le \sigma_i^2, \qquad i = 1, 2, \ldots, n. \qquad (16)$$

This specified $Q > 0$ is said to be assignable if there exists a filter gain K such that (13) holds.

It is clear from Theorem 1 that, if a positive definite matrix Q satisfying (12) and (16) is assignable, then we will have $\sigma(A_f) \subset D(q, r)$, $\|H(z)\|_\infty \le \gamma$, and $[P]_{ii} < [Q]_{ii} \le \sigma_i^2$ $(i = 1, 2, \ldots, n)$. We are now able to conclude that, if a specified $Q > 0$ satisfying (16) is assignable, the design task addressed in Section II will be accomplished, and the HCVFP problem can then be converted into an auxiliary "Q-matrix assignment" problem that focuses on the following two steps: 1) investigate the existence conditions of an assignable matrix Q and 2) characterize all filter gains achieving this assignable matrix $Q > 0$.

For technical convenience, we define the following additional notation

$$M = 2CQC^T + CQL^T(\gamma^2 I - LQL^T)^{-1}LQC^T + D_2D_2^T \quad (17)$$

$$N = 2CQA^T + CQL^T(\gamma^2 I - LQL^T)^{-1}LQA^T - qCQ + D_2D_1^T, \quad (18)$$

$$R = 2AQA^T + AQL^T(\gamma^2 I - LQL^T)^{-1}LQA^T - q(AQ + QA^T) + (q^2 - r^2)Q + D_1D_1^T + \varepsilon I. \quad (19)$$

Theorem 2: Given the desired circular pole region $D(q, r)$, the H_∞ disturbance attenuation constraint γ and the steady-state error variance constraints σ_i^2 $(i = 1, 2, \ldots, n)$. The matrix $Q > 0$ satisfying (12) and (16) is assignable if and only if the following algebraic matrix inequality

$$-R + N^T M^{-1} N \ge 0 \qquad (20)$$

holds, and the left-hand side of (20) is of maximum rank p, where M, N, R are defined in (17), (18), (19), respectively.

Proof: Note that $A_f = A - KC$, (13) can be directly rewritten as the following

$$\begin{aligned} &K[2CQC^T + CQL^T(\gamma^2 I - LQL^T)^{-1}LQC^T + D_2D_2^T]K^T \\ &- K[2CQA^T + CQL^T(\gamma^2 I - LQL^T)^{-1} \cdot LQA^T - qCQ + D_2D_1^T] \\ &- [2CQA^T + CQL^T(\gamma^2 I - LQL^T)^{-1} \cdot LQA^T - qCQ + D_2D_1^T]^T K^T \\ &+ 2AQA^T + AQL^T(\gamma^2 I - LQL^T)^{-1} \cdot LQA^T - q(AQ + QA^T) \\ &+ (q^2 - r^2)Q + D_1D_1^T + \varepsilon I = 0 \end{aligned}$$

or, for simplicity

$$KMK^T - KN - N^T K^T + R = 0. \qquad (21)$$

The matrix M is positive definite since C is of full row rank, and then (13) or (21) can be equivalently expressed as follows

$$(-KM^{1/2} + N^T M^{-1/2})(-KM^{1/2} + N^T M^{-1/2})^T = -R + N^T M^{-1} N. \quad (22)$$

Notice that the dimension of the filter gain K is $n \times p$ and $p \le n$. We can see from (22) that there exists a solution K to (13) (i.e., the specified matrix Q is assignable) if and only if the right-hand side of (22) is positive semidefinite, i.e., the inequality (20) is true, and is of maximum rank p (in this case, both sides of (22) have compatible ranks). The proof of Theorem 2 is now completed. ■

Furthermore, the algebraic parameterization of all filter gains achieving the assignable matrix Q is given in the following theorem.

Theorem 3: Let the prespecified matrix $Q > 0$ satisfying (12) and (16) be assignable, i.e., the matrix $Q > 0$ satisfies the assignability condition stated in Theorem 2. The desired filter gains can be expressed as follows

$$K = N^T M^{-1} - TUM^{-1/2} \qquad (23)$$

where $T \in \mathbb{R}^{n \times p}$ is the square root of $-R + N^T M^{-1} N$, $U \in \mathbb{R}^{p \times p}$ is an arbitrary orthogonal matrix, and M, N, R are defined in (17)–(19), respectively.

Proof: It follows from (22) and the definitions of T, U that

$$\begin{aligned} -R + N^T M^{-1} N &= TT^T \\ &= (-KM^{1/2} + N^T M^{-1/2}) \\ &\quad \times (-KM^{1/2} + N^T M^{-1/2})^T \end{aligned} \qquad (24)$$

which is equivalent to $TU = -KM^{1/2} + N^T M^{-1/2}$, where $U \in \mathbb{R}^{p \times p}$ is arbitrary orthogonal. Hence, (23) follows immediately. This proves Theorem 3. ■

The following results, which are easily accessible from Theorem 2 and Theorem 3, will give the solution to the HCVFP problem addressed in this paper.

Corollary 1: Let the circular pole region $D(q, r)$, the H_∞ disturbance attenuation constraint γ and the steady-state error variance constraints σ_i^2 $(i = 1, 2, \ldots, n)$ be prespecified. If a positive definite matrix $Q > 0$ satisfies (12) and (16) as well as the condition of Theorem 2, a desired filter gain K for the HCVFP problem can be obtained by (23).

Remark 5: For a given positive matrix $Q > 0$, we can directly check whether it is assignable through the necessary and sufficient conditions for the assignability stated in Theorem 2. On the other hand, in practice, we can also construct the appropriate assignable matrix $Q > 0$ directly from the assignability condition (20) subjected to the restrictions (12) and (16), and the desired filter gains can then be immediately obtained from (23). Observe that the conditions on an assignable matrix $Q > 0$ are actually some nonlinear matrix inequalities that characterize the desired solutions. These matrix inequalities can be tackled possibly by the direct parameterized method proposed in [5], or the local numerical searching algorithms suggested in [3].

Remark 6: Notice that in recent years the linear matrix inequality (LMI) approach has become very popular because of its computational tractability and less conservatism [4], [15]. Also, in [6], [7], the robust *controller* design problems have been investigated for uncertain systems with guaranteed D-stability. Therefore, it would be more significant to extend the present results for uncertain systems within an LMI framework. This gives us one of the future research topics.

IV. Numerical Example

In this section, by means of a simple design example, we illustrate the usefulness of the approach proposed in the previous sections.

Consider a linear discrete-time stochastic system described by (1) and (2) with model parameters as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.01 & \\ 0 & 0.01 \end{bmatrix}.$$

It is desired to design filter gains such that: 1) the poles of the filtering matrix A_f are all constrained to lie inside the circular region $D(0.5, 0.45)$; 2) the transfer function $H(z)$ from disturbances $w(k)$ to error state outputs $Le(k)$ satisfies the constraint $\|H(z)\|_\infty \leq \gamma = 1$, where $L = 0.4I_2$; and 3) the steady-state covariance P exists and satisfies $[P]_{11} \leq \sigma_1^2 = 0.9083$, $[P]_{22} \leq \sigma_1^2 = 1.0075$.

Now, suppose that the positive definite matrix Q has the form $Q = [q_{ij}]_{2\times 2}$, then by substituting the parameter Q into the inequality (20) subjected to (12) and (16), and using the approach discussed in previous section, we can obtain an assignable positive definite $Q > 0$ and a positive constant $\varepsilon > 0$ as follows

$$Q = \begin{bmatrix} 0.8672 & 0.0567 \\ 0.0675 & 0.9903 \end{bmatrix}, \qquad \varepsilon = 0.0018.$$

It is not difficult to calculate the matrices N, M, R, T and test that the conditions of Theorem 2 are satisfied. Then, by setting $U = I_2$, we can obtain a desired filter gain from (23) as follows

$$K = \begin{bmatrix} 0.5842 & 0.9633 \\ 0.0067 & 0.5823 \end{bmatrix}.$$

To this end, we verify that, the poles of the filtering matrix are $0.4168 \pm 0.0157i$, the steady-state error variances are $[P]_{11} = 0.7639$, $[P]_{22} = 0.7641$, and $\|H(z)\|_\infty = 0.5553$. Also, the responses of the error dynamics to the initial conditions $(20, -20)$ and $(40, 20)$ are shown in Figs. 1 and 2, respectively. Both the numerical and simulation results imply that all desired goals are achieved.

Remark 7: We can observe from the example that, because of the freedom in selecting arbitrary orthogonal matrix U, the set of the expected filters must be very large, if not empty. We may use this freedom to reduce the possible design conservatism. Compared to [13], [15], this paper considers the transient property (i.e., D-stability) of the filtering process. We can see from Figs. 1 and 2 that, as expected, the error dynamics converge to zero quickly.

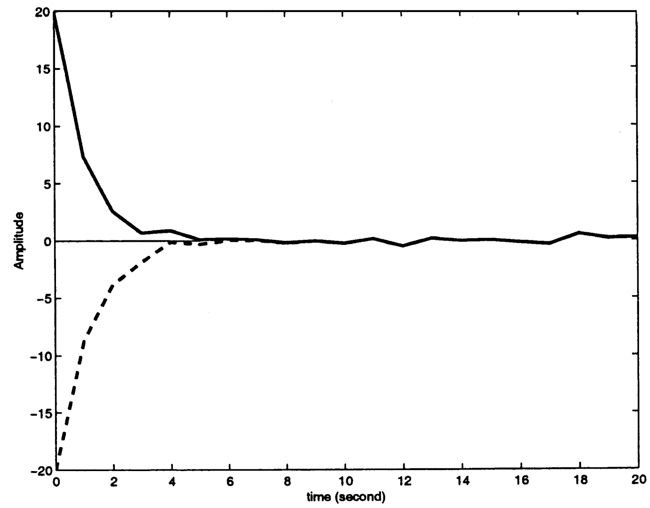

Fig. 1. $\boldsymbol{e}_1$ (solid), $\boldsymbol{e}_2$ (dashed).

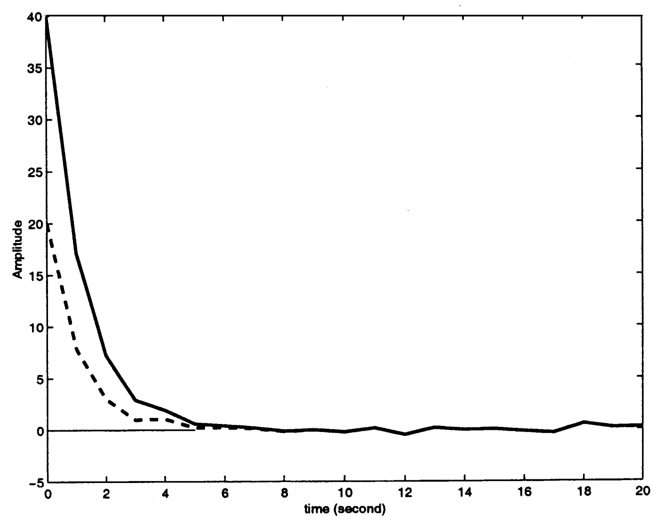

Fig. 2. $\boldsymbol{e}_1$ (solid), $\boldsymbol{e}_2$ (dashed).

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Linear Phase Filter Bank Design Using LMI-based H_∞ Optimization

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***Abstract*—This paper is concerned with the design of nearly linear phase two-channel filter banks. Exactly linear phase finite-impulse response or nearly linear phase infinite-impulse response (IIR) filters are employed in analysis filter banks, and nearly linear phase IIR synthesis filter banks are designed such that the overall filter banks are also nearly linear phase. The filter bank design problem is formulated as a H_∞ norm minimization of the difference between the pure delay and the products of some polyphase components. The H_∞ norm optimizing problem is converted to a series of linear matrix inequalities (LMIs) and solved using semidefinite programming. The magnitude, phase and aliasing distortions are all incorporated in the design procedure.**

***Index Terms*—Infinite-impulse response (IIR) filter banks, linear phase filter banks, multirate filter banks, near linear phase, quadrature mirror filter (QMF), two-channel filter banks.**

I. Introduction

Perfect-reconstruction (PR) or nearly PR (NPR) two-channel filter banks have found to be useful in subband image coding [14]. They can be constructed using either finite-impulse response (FIR) or infinite-impulse response (IIR) filters. The linear phase property is desired in subband coding applications, since the symmetric extension method [4], [23] which have shown to achieve good compression results can be employed. Various design methods for filter banks with linear phase FIR subband filters are proposed in the literature [4], [6]. However, it is not practical to design filter banks with linear phase IIR subband filters, since an exactly linear phase property will render the IIR subband filter unstable. Conversely, IIR filter banks are very attractive because of their low computational complexity and considerably lower delay than those of FIR subband filters with a comparable frequency response. As a result, a stable and nearly linear phase IIR synthesis filter bank with a given linear phase IIR or FIR analysis filter bank is desirable (the nearly linear property considered in this paper is defined similar to that in [24]). Such filter bank systems are used in subband coding systems, where subband quantization noise statistics are usually not known in advance.

In practice, unstable IIR analysis filters are applied in applications such as image coding [5] since the input signal can be completely controlled before being fed into the filter bank. In comparison, the stability of the synthesis filters is much more important with respect to the uncertainty of the quantization noise which is inherited in encoded subband signals. The stability of the synthesis filters is guaranteed by the proposed design method without additional constraints. The key is to convert the filter bank design problem into a model-matching problem where the optimal solution is searched within the H_∞ space only and, hence, the result is a stable filter. The detail formulation will be derived in Section III.

The idea of multirate filter bank design using induced norm optimization was first proposed in [10], [11], where discrete-time periodic system design problems are formulated as ℓ_2 induced norm model-matching problems. [2] studied the design of multirate filter banks by optimizing the H_∞ norm of the aliasing component matrix. The solution of the full-order H_∞ model matching problem can be obtained by solving a Riccati equation. However, the standard H_∞ approaches result in filters of order equal to the order of the system. As a result, the simulation results presented in [2] are extremely high order IIR synthesis filters. On the other hand, reduced-order filters (filters of order lower than the order of the systems) are often desirable to reduce the complexity and computational burden of the real-time processing. Order reduction techniques have employed in [2] to decrease the order of the synthesis filters. Undoubtedly, new distortions will be introduced by order reduction, and the signal reconstruction error will be increased.

In comparison with those obtained by other methods [2], a substantial decrease in the order of the synthesis filters can be obtained, without order reduction, by imposing structures onto the filter banks. Besides, imposing structures onto the filter banks will also lower the design and implementation complexity significantly. We, therefore, propose a structure constrained filter bank design method in this paper. Other distinct advantages of the proposed method are that the magnitude, phase and aliasing distortions can be minimized simultaneously. It will be shown that the multirate filter bank design problem can be formulated as two H_∞ norm optimization problems of the difference between the delay function and the products of the polyphase components of the filter bank. Moreover, we considered to solve the H_∞ optimization problem by converting the H_∞ norm optimization problem to a series of linear matrix inequalities (LMIs) [12], which are solved by a

Manuscript received November 18, 2001; revised January 24, 2003. This work was supported by the Research Grants Council of Hong Kong under Grant HKUST6173/00E. This paper was recommended by Associate Editor T. Nguyen.

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Digital Object Identifier 10.1109/TCSII.2003.810490