Gommer, Frank and Brown, Louise P. and Wedgwood, Kyle C.A. (2015) Analytical method using gamma functions for determining areas of power elliptical shapes for use in geometrical textile models. Composites Part A: Applied Science and Manufacturing, 81 . pp. 222-224. ISSN 1359-835X

Composites: Part A 81 (2016) 222–224

Contents lists available at ScienceDirect

Composites: Part A

journal homepage: www.elsevier.com/locate/compositesa

Short communication

Analytical method using gamma functions for determining areas of power elliptical shapes for use in geometrical textile models

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ARTICLE INFO

Article history:
Received 30 September 2015
Received in revised form 2 November 2015
Accepted 18 November 2015
Available online 2 December 2015

Keywords:
A. Yarn
A. Fabrics/textiles
C. Analytical modelling
C. Computational modelling

ABSTRACT

Textile models are often assumed to have homogenous and well defined cross-sections. For these models, the use of a power elliptical cross-sectional shape has been found to be beneficial as different shapes can be created, e.g. lenticular, elliptical or rectangular, with a single function. The cross-sectional area of a power ellipse is usually determined numerically as the analytical determination of the cross-sectional area is not straightforward. This short communication presents an analytical solution for this shape.

1. Introduction

Textile reinforcements used in fibre reinforced composites are usually based on yarns which are bundles of a large number of individual filaments. These reinforcements are often modelled on the meso-(yarn) scale which assumes a homogeneous structure of the fibre bundles. For numerical models, the yarn shapes are usually assumed to be elliptical, lenticular or rectangular [1–4].

The super-ellipse (Fig. 1A) also known as Lamé curve is defined as:

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1 \quad \text{with } a,b,n>0 \tag{1}$$

with the major and minor ellipse axes, a and b [5,6]. A special form of this is the power-ellipse [7] for which the exponent, n, on the width term (x) is kept constant and defined as:

$$\left|\frac{x}{a}\right|^2 + \left|\frac{y}{b}\right|^{\frac{2}{n}} = 1 \tag{2}$$

Unlike for the super-ellipse which results, for example, in star shapes for $n>2$, the advantage of this expression is that for the entire range of exponents $n>0$ realistic yarn cross-sectional shapes can be generated (Fig. 1B). For $n=1$, the yarn shape describes an ellipse, for $n>1$ a lenticular shape and for $n<1$ a rectangle with rounded edges is resulting.

For a precise and fast determination of the yarn volume fraction, it is desirable to determine the cross-sectional area of a yarn shape analytically which is straightforward for e.g. ellipses. However, determining the area of a power ellipse is more challenging.

To derive the cross-section of an arbitrary area which can be drawn with a continuous, non-overlapping line, Green's Theorem [8] of a line integration can be used. For a practical application of this theorem, a number of equispaced points around the cross-section can be sampled and the trapezoidal rule [9] applied. This numerical technique approximates the area by assuming that it consists of a finite number of trapezoids. Its accuracy depends on the number of points sampled.

In this work, an analytical expression to determine the area of a power ellipse is derived which uses the gamma function, Γ. This function is similar to the factorial for an integer, n, but shifted by -1; hence $\Gamma(n)=(n-1)!$. The main benefit of the gamma function compared to the factorial is that it is defined for any real number.

2. Derivation of the analytical power-ellipse area

The analytical expression of the area of a power ellipse, S, can be performed in a similar way as for a super-ellipse [5,6]. A power ellipse, as used for example in TexGen [2], can be expressed as [10]:

$$x = a\sin\theta \tag{3}$$

$$y = b\cos^n\theta \tag{4}$$

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http://dx.doi.org/10.1016/j.compositesa.2015.11.031
1359-835X/

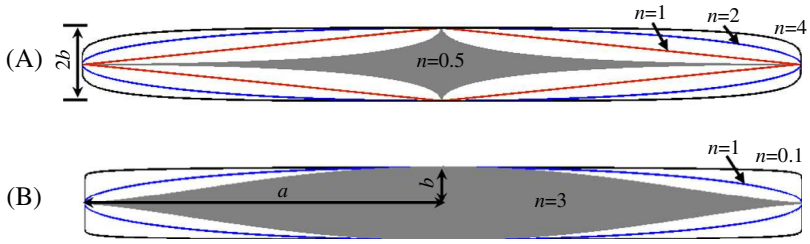

Fig. 1. Examples of (A) super-elliptical and (B) power elliptical shapes for different exponents, *n*. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

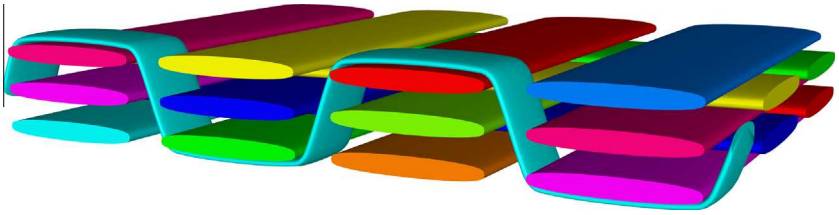

Fig. 2. Example of a geometrical textile model of a 3D weave with homogenous yarns using a power elliptical cross-sectional shape created in TexGen [1]. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

This parametric representation needs to be integrated. As the equations are oscillating, only a quarter of the power ellipse needs to be analysed. The integration is therefore in the form:

$$\frac{S}{4} = \int_0^a y dx \tag{5}$$

Replacing dx in this equation with the first derivative of x in Eq. (3) to θ:

$$dx = a\cos\theta d\theta \tag{6}$$

and placing Eqs. (4) and (6) in (5) gives:

$$\frac{S}{4} = \int_0^{\frac{\pi}{2}} b\cos^n\theta \cdot a\cos\theta d\theta \tag{7}$$

This expression can be reduced to:

$$\frac{S}{4} = ab\int_0^{\frac{\pi}{2}} \cos^{n+1}\theta d\theta \tag{8}$$

The integral of Eq. (8) can now be evaluated considering the following trigonometric expression [5,11]:

$$\int_0^{\frac{\pi}{2}} \sin^\alpha\theta\cos^\beta\theta d\theta = \frac{1}{2}B\left(\frac{\alpha+1}{2},\frac{\beta+1}{2}\right) \quad \text{for } \alpha,\beta > -1 \tag{9}$$

where B represents the beta function. Using this for the integral in Eq. (8) multiplied with an additional term of $\sin^0\theta = 1$ to comply with the format in Eq. (9) gives:

$$\int_0^{\frac{\pi}{2}} \sin^0\theta\cos^{n+1}\theta d\theta = \frac{1}{2}B\left(\frac{0+1}{2},\frac{n+1+1}{2}\right) \tag{10}$$

The beta function can be expressed in terms of the gamma function, Γ, [12] which leads to

$$\frac{1}{2}B\left(\frac{1}{2},\frac{n+2}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n+2}{2}\right)} \tag{11}$$

Considering that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and replacing Eq. (11) for the integral in Eq. (8) and rearranging the equation results in the analytical form of the area of a power-ellipse:

$$S = 2ab\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}(n+2)\right)}{\Gamma\left(\frac{1}{2}(n+3)\right)} \tag{12}$$

3. Accuracy and time

Mathematically speaking, Eq. (12) gives the exact area of the power ellipse. However, this is given in terms of the Gamma function, which is the integral of a transcendental expression, and does not have a closed form expression, except when its argument is integer valued, in which case the Gamma function becomes the well-known factorial function. In spite of this, approximations such as those provided by Lanczos [13] and Spouge [14] provide efficient algorithms to estimate the Gamma function to arbitrary precision. Both of these approximations, themselves corrections of the Stirling algorithm, are formed by truncating a convergent series expansion of the Gamma function in terms of elementary functions at the appropriate order. The efficiency of the Lanczos approximation is demonstrated by the rapid convergence of the approach: including only 13 terms in the series expansion gives an error $O(10^{-16})$ [15] when working at fixed precision.

In contrast to the explicit formula provided by Eq. (12), the trapezium rule provides an approximation whose error scales the square of the interval size (assuming a uniform step). For a fixed error of ε, the Lanczos and Spouge algorithms require $O(-\log(\varepsilon))$ steps, whilst the trapezium rule requires $O(1/\mathrm{sqrt}(\varepsilon))$ points. There exist more efficient algorithms for numerical integration, such as Gaussian quadrature, but these will still be inefficient compared to these approximations to the Gamma function. The relevant algorithms are readily available, for example as part of Matlab's built-in functions [16] and the GNU Scientific Library [17].

Using the derived equation Eq. (12) compared to the use of the Trapezoidal rule [9], the solution accuracy is no longer dependent

of the number of points used. In addition, the area of a power ellipse can be derived significantly faster. Compared to the Trapezoidal rule with 1000 equispaced points sampled around the circumference of a power ellipse using the GetArea function in TexGen [1], the speed is 3× faster using the analytical solution in Eq. (12) as a Python implementation of Lanczos' approximation [13] on a standard desktop computer. This increased speed can be beneficial when analysing, for example, the volume fraction in geometric textile models (Fig. 2).

4. Concluding remarks

An analytical form of the area of a power ellipse has been derived. Implementing this in, for example, geometrical textile pre-processors will allow yarn volume fractions to be determined in a fast and more accurate way compared to using numerical approximations such as the Trapezoidal rule. In addition, the analytical expression of the area will make implementation of power elliptical shapes more readily accessible to developers of geometric textile models. The need for more complicated approximations, e.g. numerical estimates of areas (and volumes), which may also require determination of equispaced points at the cross-sectional circumference, is overcome.

Acknowledgements

F. Gommer and L. Brown were supported by the Engineering and Physical Sciences Research Council (Grant number: EP/IO33513/1), through the EPSRC Centre for Innovative Manufacturing in Composites (CIMComp) and K. Wedgwood was generously supported by the Wellcome Trust Institutional Strategic Support Award (Grant number: WT105618MA).

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