In this representation

$$\overline{\mathcal{V}}^* = \ker(\overline{D}) = \operatorname{Im}\begin{bmatrix} I_4 \\ 0 \end{bmatrix}$$

$$\overline{\mathcal{B}} \cap \overline{\mathcal{V}}^* = \operatorname{Im}\begin{bmatrix} 0.6708 \\ 0.5916 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \operatorname{Im}(\overline{E}) \subset \overline{\mathcal{V}}^*$$

$$\overline{\mathcal{V}}_* = \operatorname{Im}\begin{bmatrix} I_2 \\ 0 \end{bmatrix}.$$

The analytical expression of all DDP solutions is then given by

$$\overline{F} = \left[\begin{array}{cc|ccc} -0.1342 & 0.4902 & X & X & X \\ \hline X & X & X & X & X \end{array}\right]$$

where the elements X are free to assume any value. $\overline{\mathcal{R}}^*$ coincides with $\overline{\mathcal{V}}_*$, therefore, as stated by Proposition 5, the DDPPP is solvable as well.

V. Conclusions

The DDP via state feedback with stability and/or pole placement, for a particular class of linear systems, was the object of this paper. The methodology proposed for its solution was based on the characterization of self-bounded (A, B)-invariant subspaces, and a reliable numerically stable computational procedure was proposed for obtaining an analytical expression of all possible state feedback controllers. A similar methodology was also applied to the DDPM with stability and/or pole placement, and a procedure for obtaining an analytical expression of a reduced-order compensator was outlined.

The approach adopted in this work for the DDP solution presents some advantages when compared to other ones available in the literature. Namely, a larger class of systems is treated with stability and/or pole placement considerations, and an explicit analytical expression is derived for the corresponding feedback matrices. Such expressions are very useful for design purposes, mainly when performance specifications, complementary to disturbance decoupling ones, are also included.

References

[1] W. M. Wonham, *Linear Multivariable Control—A Geometric Approach*. New York: Springer-Verlag, 1979.
[2] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1992.
[3] J. M. Schumacher, "Compensator synthesis using (C, A, B)-pairs," *IEEE Trans. Automat. Contr.*, vol. 25, pp. 1133–1138, 1980.
[4] J. C. Willems and C. Commault, "Disturbance decoupling by measurement feedback with stability or pole placement," *SIAM J. Contr. Optimiz.*, vol. 11, pp. 450–504, 1981.
[5] A. Linnemann, "A condensed form for disturbance decoupling with simultaneous pole placement using state feedback," in *Proc. 10th IFAC World Congr.*, Munich, Germany, vol. 9, pp. 92–97, 1987.
[6] C. E. T. Dórea and B. E. A. Milani, "A computational method for optimal L-Q regulation with simultaneous disturbance decoupling," *Automatica*, vol. 31, pp. 155–160, 1995.
[7] P. N. Paraskevopoulos, F. N. Koumboulis, and K. G. Tzierakis, "Disturbance rejection of left-invertible systems," *Automatica*, vol. 28, pp. 427–430, 1992.
[8] G. Basile and G. Marro, "Self-bounded controlled invariant subspaces: A straightforward approach to constrained controllability," *J. Optimiz. Theory Appl.*, vol. 38, pp. 71–81, 1982.
[9] J. M. Schumacher, "On a conjecture of Basile and Marro," *J. Optimiz. Theory Appl.*, vol. 41, pp. 371–376, 1983.
[10] C. E. T. Dórea and B. E. A. Milani, "Disturbance decoupling in a class of linear systems," in *Proc. Amer. Contr. Conf.*, Seattle, WA, 1995, pp. 4413–4417.
[11] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," *IEEE Trans. Automat. Contr.*, vol. 26, pp. 111–129, 1981.
[12] D. Y. Ohm, J. W. Howze, and S. P. Bhattacharyya, "Structural synthesis of multivariable controllers," *Automatica*, vol. 21, pp. 35–55, 1985.
[13] M. K. Sain and J. L. Massey, "Invertibility of linear time-invariant dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 14, pp. 141–149, 1969.

Robust H_2/H_∞-State Estimation for Discrete-Time Systems with Error Variance Constraints

Zidong Wang, Zhi Guo, and H. Unbehauen

***Abstract*— This paper studies the problem of an H_∞-norm and variance-constrained state estimator design for uncertain linear discrete-time systems. The system under consideration is subjected to time-invariant norm-bounded parameter uncertainties in both the state and measurement matrices. The problem addressed is the design of a gain-scheduled linear state estimator such that, for all admissible measurable uncertainties, the variance of the estimation error of each state is not more than the individual prespecified value, and the transfer function from disturbances to error state outputs satisfies the prespecified H_∞-norm upper bound constraint, simultaneously. The conditions for the existence of desired estimators are obtained in terms of matrix inequalities, and the explicit expression of these estimators is also derived. A numerical example is provided to demonstrate various aspects of theoretical results.**

***Index Terms*— H_∞-state estimation, Kalman filtering, robust state estimation, uncertain systems.**

I. Introduction

The so-called Kalman filtering, which is one of the most popular estimation approaches, has been well studied in the past 30 years. The main idea is to design a filter to minimize the estimation error covariance; see [1]. This estimation approach is based on the assumption that the system model under consideration is exactly known and its disturbances are Gaussian noises with known statistics. However, the Kalman filter-type observer may not be robust against modeling uncertainty and disturbances. This has brought to focus the importance of robust estimation and H_∞ estimation and has attracted significant research interests in the past decade. For example, a robust Kalman filtering problem was considered in [6] for systems with

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Z. Wang was with the Department of Automatic Control, Nanjing University of Science and Technology, Nanjing 210094, P.R. China. He is now with the Automatic Control Laboratory, Faculty of Electrical Engineering, Ruhr-University Bochum, D-44780 Bochum, Germany.
Z. Guo is with the Department of Automatic Control, Nanjing University of Science and Technology, Nanjing 210094, P.R. China.
H. Unbehauen is with the Automatic Control Laboratory, Faculty of Electrical Engineering, Ruhr-University Bochum, D-44780 Bochum, Germany.
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bounded parameter uncertainty only in the state matrix. Discrete-time filters that minimize a bound on the variance of the estimation error while satisfying a prescribed H_∞ performance were developed in [2], and the same problem was studied for uncertain systems in [3]. Furthermore, Xie *et al*. [8], [9] was concerned with the problem of Kalman filter design for the system subjected to time-varying norm-bounded parameter uncertainty in both the state and measurement matrices. It should be pointed out that the key idea of [3], [6], [8], and [9] is actually the design of a filter which provides an upper bound on the variance of the filtering error for all admissible parameter uncertainties; but this upper bound is not *prespecified*.

However, in many engineering applications, such as tracking of maneuvering target and recognition of flight paths from multiple sources, the performance requirements of state estimation are usually described *directly* in terms of the upper bounds on the steady-state estimation error variances. This class of state estimation problems is valuable for both theory and applications. Traditional state estimation methods are often very difficult to solve for these problems. For instance, the theory of weighted least squares estimation [7] minimizes a weighted scalar sum of the error variances of the state estimation, but minimizing a scalar sum does not ensure that the multiple variance requirements will be satisfied. For the specified individual variance constraint on each state, this approach cannot guarantee the existence of the weight which satisfies desired requirements.

A new state estimation approach called error covariance assignment (ECA) theory was first proposed in [10] and then extended to the nonlinear case [5], [11]. This theory provides an alternative, more straightforward technique to meet the prespecified estimation error variance constraints. The main idea is to design a filter which directly assigns the prespecified steady-state estimation error covariance. However, there are few papers developing the robust estimation technique for uncertain systems subject to the simultaneous achievement of error variance upper-bound constraint and H_∞ disturbance attenuation constraint. This motivates the research of robust state estimation for uncertain discrete-time systems with prespecified H_∞ norm and error variance upper-bound constraints. To this end, we point out that the problem considered in the present paper is different from that in [3], [6], [8], and [9].

The present paper will study the problem of the H_∞ norm and the variance-constrained state estimator design for uncertain linear discrete-time systems. The parametric uncertainty is assumed to be time-invariant norm-bounded and appears in both the state and measurement matrices. The aim of this problem is the design of a robust state estimator such that, for all admissible measurable uncertainties, the variance of the estimation error of each state is not more than the individual prespecified value, and the transfer function from disturbances to error state outputs satisfies the prespecified H_∞ norm upper-bound constraint, simultaneously. It will be shown that the addressed problem can be converted into a problem of solving algebraic matrix inequalities, and then both the existence conditions and the explicit expression of desired estimators will be derived. This design methodology will be applied to a simple example which demonstrates various aspects of theoretical results.

II. Problem Formulation and Assumptions

We consider the following class of uncertain discrete-time observable dynamic systems [2]:

$$x(k+1) = (A + \Delta A)x(k) + D_1 w(k) \tag{1}$$

and the measurement equation

$$y(k) = (C + \Delta C)x(k) + D_2 w(k) \tag{2}$$

where x is an n-dimensional state vector, y is a p-dimensional measured output vector, and $A, C, D_1,$ and D_2 are known constant matrices. $w(k)$ is a zero mean Gaussian white noise sequence with covariance $I > 0$. The initial state $x(0)$ has the mean $\overline{x}(0)$ and covariance $P(0)$, and is uncorrelated with $w(k)$. ΔA and ΔC are perturbation matrices which represent parametric uncertainties and are assumed to be of the time-invariant form [6], [8], [9]

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} FN \tag{3}$$

where $F \in R^{i \times j}$ is a measurable perturbation matrix which satisfies

$$FF^T \geq I \tag{4}$$

and M_1, M_2 (M_2 is full row rank), and N are known constant matrices of appropriate dimensions which specify how the elements of the nominal matrices A and C are affected by the uncertain parameters in F. ΔA and ΔC are said to be admissible if both (3) and (4) hold.

When the perturbations ΔA and ΔC are measured, the state estimation vector $\hat{x}(k+1)$ satisfies the following linear full-order filter:

$$\hat{x}(k+1) = (A + \Delta A)\hat{x}(k) + K[y(k) - (C + \Delta C)\hat{x}(k)] \tag{5}$$

whose estimation error covariance in the steady state is defined as

$$\begin{aligned} P &:= \lim_{k\to\infty} P(k) := \lim_{k\to\infty} E\{e(k)e^T(k)\} \\ e(k) &= x(k) - \hat{x}(k) \end{aligned} \tag{6}$$

where $e(k)$ denotes the error state. Then, it is easy to obtain that

$$e(k+1) = [A + \Delta A - K(C + \Delta C)]e(k) + (D_1 - KD_2)w(k) \tag{7}$$

and

$$\begin{aligned} P(k+1) = {} & [A + \Delta A - K(C + \Delta C)] \\ & \cdot P(k)[A + \Delta A - K(C + \Delta C)]^T \\ & + (D_1 - KD_2)(D_1 - KD_2)^T. \end{aligned} \tag{8}$$

Define the filtering matrix $A_f = A + \Delta A - K(C + \Delta C)$. If A_f is Schur stable (i.e., the poles of A_f are all within the unit disk) for all admissible ΔA and ΔC, then in the steady state, the estimation error covariance P satisfies

$$P = A_f P A_f^T + (D_1 - KD_2)(D_1 - KD_2)^T \tag{9}$$

where $P = P^T > 0$.

Our objective in this paper is to deal with the gain-scheduled filtering problem, i.e., design a filter gain K such that, *for all admissible measurable perturbations* ΔA *and* ΔC, the following three requirements are *simultaneously* satisfied.

1) The filtering matrix $A_f = A + \Delta A - K(C + \Delta C)$ remains Schur stable.
2) The steady-state error covariance P meets
$$[P]_{ii} \leq \sigma_i^2, \qquad i = 1, 2, \cdots, n \tag{10}$$
where $[P]_{ii}$ means the ith diagonal element of P, i.e., the steady-state variance of ith state. σ_i^2 $(i = 1, 2, \cdots, n)$ denote the prespecified steady-state error estimation variance constraint on ith state and can be determined by the practical performance requirements.
3) The H_∞ norm of the transfer function $H(z) = L(zI_n - A_f)^{-1}(D_1 - KD_2)$ from disturbances $w(k)$ to error state outputs $Le(k)$ satisfies the constraint
$$||H(z)||_\infty \leq \nu \tag{11}$$

where L is the known error state output matrix, and

$$||H(z)||_\infty = \sup_{\theta\in[0,2\pi]} \sigma_{\max}[H(e^{j\theta})]$$

and $\sigma_{\max}[\cdot]$ denotes the largest singular value of $[\cdot]$, and ν is a given positive constant.

III. Main Results and Proofs

In this section a solution to the robust H_∞ norm and variance-constrained state estimation problem formulated in the previous section will be obtained using an algebraic matrix inequality approach.

The following lemma is useful in the proof of main results.

Lemma 1: Let Q be a positive definite matrix and $\varepsilon > 0$ be a positive scalar such that $\varepsilon NQN^T < I$. Define $A_c := A - KC, \Delta A_c := \Delta A - K\Delta C = (M_1 - KM_2)FN$. Then for all admissible ΔA and ΔC, we have

$$\begin{aligned}(A_c + \Delta A_c)Q(A_c + \Delta A_c)^T - A_cQA_c^T &\le A_cQN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQA_c^T \\ &\quad + \varepsilon^{-1}(M_1 - KM_2)(M_1 - KM_2)^T. \qquad (12)\end{aligned}$$

Proof: It is clear that $A_f = A - KC + (M_1 - KM_2)FN = A_c + \Delta A_c$. Define

$$\begin{aligned}\sum := [&A_cQN^T(\varepsilon^{-1}I - NQN^T)^{-1/2} - (M_1 - KM_2) \\ &\cdot F(\varepsilon^{-1}I - NQN^T)^{1/2}].\end{aligned}$$

Note that $FF^T \le I$, then

$$\begin{aligned}0 \le \Sigma\Sigma^T &= A_cQN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQA_c^T \\ &\quad - A_cQN^TF^T(M_1 - KM_2)^T - (M_1 - KM_2) \\ &\quad \cdot FNQA_c^T + (M_1 - KM_2)F(\varepsilon^{-1}I - NQN^T) \\ &\quad \cdot F^T(M_1 - KM_2)^T \\ &= A_cQN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQA_c^T \\ &\quad - [A_cQ(\Delta A_c)^T + (\Delta A_c)QA_c^T + (\Delta A_c)Q(\Delta A_c)^T] \\ &\quad + \varepsilon^{-1}(M_1 - KM_2)FF^T(M_1 - KM_2)^T \\ &\le A_cQN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQA_c^T \\ &\quad - [(A_c + \Delta A_c)Q(A_c + \Delta A_c)^T - A_cQA_c^T] \\ &\quad + \varepsilon^{-1}(M_1 - KM_2)(M_1 - KM_2)^T\end{aligned}$$

and (12) follows immediately.

Prior to introducing our main results, we now present an important theorem which plays a key role for solving the robust variance-constrained H_2/H_∞ estimation problem.

Theorem 1: If there exist a filtering gain K, a positive scalar $\varepsilon > 0$, and a positive definite matrix $Q \in R^{n\times n}$ such that

1) $$\varepsilon NQN^T < I, LQL^T < \nu^2 I \qquad (13)$$

2) $D_1 - KD_2$ is full row rank (14)

3) $$\begin{aligned}Q =&(A - KC)R(A - KC)^T + (D_1 - KD_2) \\ &\cdot (D_1 - KD_2)^T + \varepsilon^{-1}(M_1 - KM_2) \\ &\cdot (M_1 - KM_2)^T \qquad (15)\end{aligned}$$

where $R := Q + QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQ + QL^T(\nu^2 I - LQL^T)^{-1}LQ$, then *for all admissible perturbations* ΔA *and* ΔC, we have the following conclusions.

1) The filtering matrix A_f is asymptotically stable.
2) The steady-state error covariance P exists and satisfies $P \le Q$.
3) $||H(z)||_\infty \le \nu$, where $H(z)$ is defined in (11).

Proof of 1): Let there exist $\varepsilon > 0$, $Q > 0$, and K such that (13)–(15) hold. Define

$$\begin{aligned}\Psi := &A_c[Q + QN^T(\varepsilon^{-1}I - NQN^T)^{-1}NQ]A_c^T \\ &+ \varepsilon^{-1}(M_1 - KM_2)(M_1 - KM_2)^T - (A_c + \Delta A_c) \\ &\cdot Q(A_c + \Delta A_c)^T\end{aligned}$$

then Lemma 1 shows that $\Psi \ge 0$. Using the definition of Ψ, (15) can be rewritten as

$$\begin{aligned}Q = &(A_c + \Delta A_c)Q(A_c + \Delta A_c)^T + [A_cQL^T \\ &\cdot (\nu^2 I - LQL^T)^{-1}LQA_c^T + (D_1 - KD_2) \\ &\cdot (D_1 - KD_2)^T + \Psi]. \qquad (16)\end{aligned}$$

Since the matrix $(D_1 - KD_2)(D_1 - KD_2)^T$ is full row rank, then (16) implies that there exists a positive definite matrix $Q > 0$ such that $Q > A_fQA_f^T$, and the Schur stability of $A_f = A_c + \Delta A_c$ is guaranteed by the discrete Lyapunov stability theory.

The proofs of Conclusions 2) and 3) are completely analogous to the proofs of Lemma 2.1 or Lemma 5.1 of Haddad *et al*. [2].

Remark 1: The upper bounds on the error covariance and H_∞ performance given by (13)–(15) may be conservative mainly because of the introduction of additional matrix $\Psi > 0$. Noting that $\Psi > 0$ depends directly on the parameter $\varepsilon > 0$, we can reduce the conservative upper bounds by the appropriate selection of $\varepsilon > 0$ which can be done by using the Matlab LMI tool [8]. The detailed discussion on the choice of $\varepsilon > 0$ can be found in [8] and [9].

Remark 2: It should be pointed out that (14) in Theorem 1 is used only to prove Conclusion 1). Rather, if possible, it suffices to check the robust stability of time-invariant matrix A_f directly.

Remark 3: Theorem 1 shows that the robust stability constraint on the filtering process and the H_∞ constraint on estimation error are automatically enforced when a positive definite solution to (13)–(15) is known to exist. Furthermore, all such solutions provide upper bounds for the H_2-estimation error $||H(z)||_2^2$. In [2], [8], and related papers, the upper bound on the error covariance was minimized, and the desired estimator is usually *unique*. In the present paper, however, this upper bound is required to satisfy the prespecified constraint which must not be minimal but meets the engineering requirements. In this case, the resulting estimator may be a large set, and the design freedom can be exploited to achieve the expected multiple objectives (e.g., robustness, transient behavior on filtering process, H_∞ requirement, fault-tolerant property, etc.). To this end, the variance-constrained robust H_2/H_∞-estimation problem can be recast as an auxiliary matrix assignment problem which is stated in the next remark.

Remark 4: By using Theorem 1, we can assign a desired value to the positive definite matrix Q, such that this matrix Q meets

$$[Q]_{ii} \le \sigma_i^2, \qquad i = 1, 2, \cdots, n \qquad (17)$$

and find the set of Kalman filter gain K which satisfies (13)–(15) for the specified Q. If such a gain exists and can be obtained, then from Theorem 1, we will have the following conclusions: 1) A_f is robustly stable for admissible perturbations; 2) $[P]_{ii} \le [Q]_{ii} \le \sigma_i^2, i = 1, 2, \cdots, n$; and 3) $||H(z)||_\infty \le \nu$. Hence, the variance-constrained robust filtering gain design task will be accomplished, and the problem addressed in Section II can be converted to such an auxiliary "matrix assignment" problem.

To make the problem more tractable, we give the following definition.

Definition 1: Given are a positive definite matrix $Q > 0$ and a positive scalar $\varepsilon > 0$ which meet (13) and (17). The pair $Q > 0, \varepsilon > 0$ is called *assignable* if there exists a set of filtering gain K such that (15) has the positive definite solution (Q, ε).

Clearly, if the filtering gain K which achieves the assignable pair (Q, ε) can be designed and this filtering gain also meets (14), the problem addressed in this paper will be solved. In what follows, our purpose is to derive the necessary and sufficient conditions for the existence of assignable pair (Q, ε) and then to characterize all filtering gains which achieve this assignable pair (Q, ε).

We can rearrange (15) as follows:

$$\begin{aligned} Q = &K(CRC^T + D_2D_2^T + \varepsilon^{-1}M_2M_2^T)K^T \\ &- K(CRA^T + D_2D_1^T + \varepsilon^{-1}M_2M_1^T) \\ &- (CRA^T + D_2D_1^T + \varepsilon^{-1}M_2M_1^T)^TK^T \\ &+ ARA^T + D_1D_1^T + \varepsilon^{-1}M_1M_1^T. \end{aligned} \tag{18}$$

For the purpose of simplicity, we make the following definitions:

$$X = CRC^T + D_2D_2^T + \varepsilon^{-1}M_2M_2^T \tag{19}$$

$$Y = CRA^T + D_2D_1^T + \varepsilon^{-1}M_2M_1^T \tag{20}$$

$$Z = ARA^T + D_1D_1^T + \varepsilon^{-1}M_1M_1^T \tag{21}$$

and (18) can be expressed in the following simple form:

$$Q = KXK^T - KY - Y^TK^T + Z. \tag{22}$$

Note that M_2 is full row rank, then the matrix X is positive definite, and (22), or (15), can be equivalently written as follows:

$$\begin{aligned} Q - Z + Y^TX^{-1}Y \\ = (-KX^{1/2} + Y^TX^{-1/2})(-KX^{1/2} + Y^TX^{-1/2})^T. \end{aligned} \tag{23}$$

Since the dimension of filter gain K is $n \times p$ and $p \leq n$, then from (23), there exists a solution K to (18) (i.e., the pair (Q, ε) is assignable) if and only if the left side of (23) is positive semidefinite and is of maximum rank p (in this case, both sides of (23) have admissible ranks). This leads to the following theorem which presents the existence conditions of an assignable pair (Q, ε).

Theorem 2: Given are the desired steady-state error variance constraints $\sigma_i^2, i = 1, 2, \cdots, n$, and the desired H_∞ norm constraint ν. The pair $Q > 0, \varepsilon > 0$ satisfying (13) and (17) is assignable if and only if the following algebraic matrix inequality holds:

$$Q - Z + Y^TX^{-1}Y \geq 0 \tag{24}$$

and the left side of (24) is of maximum rank p.

Furthermore, we give the algebraic parameterization of all filtering gains related to the assignable pair (Q, ε).

Theorem 3: Suppose that the prespecified positive definite matrix $Q > 0$ and positive scalar $\varepsilon > 0$ satisfying (13) and (17) is assignable, i.e., (24) is met. The desired filtering gains can be expressed as

$$\Theta = \{K\colon\ K = Y^TX^{-1} - TUX^{-1/2}\} \tag{25}$$

where $T \in R^{n \times p}$ is the square root of $Q - Z + Y^TX^{-1}Y, U \in R^{p \times p}$ is arbitrary orthogonal, X, Y, Z are determined by (19)–(21), and R is defined in Theorem 1. Furthermore, for the prespecified steady-state error variance constraints $\sigma_i^2, (i = 1, 2, \cdots, n)$ and the desired H_∞-norm constraint ν, if a pair $Q > 0, \varepsilon > 0$ meets the conditions of Theorem 2 and the matrix K obtained by (25) also satisfies (14), then this gain matrix K is just the desired robust H_2/H_∞ variance-constrained estimator.

Proof: From (23) and the definition of T, we have

$$\begin{aligned} Q - Z + Y^TX^{-1}Y \\ = TT^T \\ = (-KX^{1/2} + Y^TX^{-1/2})(-KX^{1/2} + Y^TX^{-1/2})^T \end{aligned} \tag{26}$$

or equivalently

$$TU = -KX^{1/2} + Y^TX^{-1/2}$$

and (25) follows immediately. The second result of this theorem is very accessible. This proves Theorem 3.

Remark 5: Though the necessary and sufficient conditions for the assignability of the pair (Q, ε) are easy to test, in practical applications, however, the designers often wish to construct the *appropriate assignable* pair (Q, ε) directly from (24) subjected to the restrictions (13) and (17), then easily get the desired filtering gains satisfying (14) from (25). The conditions on an assignable pair (Q, ε) are actually some nonlinear matrix inequalities which characterize the desired solutions. For relatively lower order models, these matrix inequalities can be treated possibly by the direct parameterized method proposed in [12]. Furthermore, the local numerical searching algorithms [13], [14] can be utilized to deal with higher order models. We point out, however, that the proof for the guaranteed convergence of an efficient algorithm is still an open problem and has been the subject of future research.

IV. Numerical Example

To illustrate the design approach of the present paper, we consider an uncertain linear continuous-time stochastic system described by (1)–(3) where the parameters are as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} -0.011\,37 & -0.222\,26 \\ 0.169\,55 & -0.411\,05 \end{bmatrix}$$

$$\Delta A = M_1FN = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} \sin\alpha & 0 \\ 0 & \sin\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Delta C = M_2FN = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix} \begin{bmatrix} \sin\alpha & 0 \\ 0 & \sin\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is desired to design robust filtering gains such that 1) the filtering matrix $A_f = A - KC + \Delta A - K\Delta C$ is robustly stable; 2) the steady-state covariance P exists and $[P]_{11} \leq 0.62, [P]_{22} \leq 0.51$; and 3) the transfer function $H(s)$ from disturbances $w(k)$ to error-state outputs $Le(k)$ satisfies the constraint $||H(z)||_\infty \leq 0.85$, where $L = I_2$.

Now, we assume that the positive definite matrix Q has the form

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$$

and then by substituting parameters Q and ε into (24) and using the approach discussed in previous section, we can choose an assignable pair $Q > 0$ and $\varepsilon > 0$ as follows:

$$Q = \begin{bmatrix} 0.5874 & 0 \\ 0 & 0.4355 \end{bmatrix}, \qquad \varepsilon = 0.5.$$

Substituting Q, ε and $U = I_2$ into (25) yields a desired filtering gain which also satisfies (14), that is

$$K = \begin{bmatrix} 0.136\,53 & 0.919\,05 \\ -0.111\,71 & 0.285\,84 \end{bmatrix}$$

and it is not difficult to obtain the values of the maximum H_∞ norm of the error transfer function and of the maximum variance of the estimation error (over all admissible uncertainty) respectively as 0.7962 and 0.4236, 0.2016. Clearly, the prespecified robust stability constraint on filtering process, the H_∞-norm constraint on the error transfer function, and the variance constraint on estimation error are all met.

V. Conclusions

This paper has studied the problem of H_∞-norm and variance-constrained state estimator designs for linear discrete-time systems with parameter uncertainties in both the state and measurement matrices. An algebraic matrix inequality approach has been proposed to solve the above problem. The existence conditions and the analytical expression of desired estimators have been characterized. Further study will focus on the variance-constrained multiobjective (e.g., robustness, transient behavior, H_∞ requirement, fault-tolerant property, etc.) state estimation for various systems such as continuous-time, discrete-time, sampled-data, and stochastic parameter systems.

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REFERENCES

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, 1979.
[2] W. M. Haddad, D. S. Bernstein, and D. Mustafa, "Mixed-norm H_2/H_∞ regulation and estimation: The discrete-time case," *Syst. Contr. Lett.*, vol. 16, pp. 235–248, 1991.
[3] M. Fu, C. E. de Souza, and L. Xie, "H_∞ estimation for uncertain systems," *Int. J. Robust Nonlinear Contr.*, vol. 2, pp. 87–105, 1992.
[4] P. P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear system: Quadratic stabilizability and H_∞ control theory," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 356–361, 1991.
[5] W. NaNacara and E. Yaz, "Linear and nonlinear estimation with uncertain observations," in *Proc. Amer. Contr. Conf.*, Maryland, 1994.
[6] I. R. Petersen and D. C. McFarlane, "Robust state estimation for uncertain systems," in *Proc. 30th Conf. Decision Contr.*, Brighton, England, 1991.
[7] R. F. Stengel, *Stochastic Optimal Control: Theory and Application*. New York: Wiley, 1986.
[8] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1310–1314, 1994.
[9] L. Xie and Y. C. Soh, "Robust Kalman filtering for uncertain systems," *Syst. Contr. Lett.*, vol. 22, pp. 123–129, 1994.
[10] E. Yaz and R. E. Skelton, "Continuous and discrete state estimation with error covariance assignment," in *Proc. IEEE Conf. Decision Contr.*, Brighton, U.K., 1991.
[11] E. Yaz and W. NaNacara, "Nonlinear estimation by covariance assignment," in *Proc. 12th IFAC World Congr.*, Sydney, Australia, 1993.
[12] W. J. Chang and H. Y. Chung, "A study of H_∞ norm and variance-constrained design using dynamic output feedback for linear discrete systems," *Int. J. Contr.*, vol. 57, pp. 473–484, 1993.
[13] J. C. Geromel, P. L. D. Peres, and S. R. Souza, "Output feedback stabilization of uncertain systems through a min/max problem," in *Proc. 12th IFAC World Congr.*, Sydney, Australia, 1993.
[14] K. M. Grigoriadis and R. E. Skelton, "Low-order control design for BMI problems using alternating projection methods," in *Proc. Allerton Conf.*, 1993.

A Nonparametric Polynomial Identification Algorithm for the Hammerstein System

Zi-Qiang Lang

***Abstract*—Almost all existing Hammerstein system nonparametric identification algorithms can recover the unknown system nonlinear element up to an additive constant, and one functional value of the nonlinearity is usually assumed to be known to make the constant solvable. To overcome this defect, in this paper, a new nonparametric polynomial identification algorithm for the Hammerstein system is proposed which extends the idea in the author's previous work on the Hammerstein system identification to a more general and practical case, where no functional value of the system nonlinearity is known *a priori*. Convergence and convergence rates in both uniform and global senses are established, and simulation studies demonstrate the effectiveness and advantage of the new algorithm.**

***Index Terms*— Convergence, convergence rates, Hammerstein model, nonlinear systems, nonparametric identification.**

I. INTRODUCTION

The Hammerstein system is a typical nonlinear model which consists of a nonlinear memoryless element followed by a linear dynamical subsystem. In order to identify the system, initially the proposed algorithms assumed that the unknown memoryless nonlinear characteristic is a polynomial of a finite and known order. Clearly, these algorithms do not converge for nonpolynomial characteristics. Considering that the nonlinear element could be regarded as a regression function, a number of nonparametric identification methods were then proposed [1]–[4]. However, although these methods require almost no prior knowledge of the system nonlinearity and can obtain nonparametric estimates of the nonlinear element that converge to the real characteristic quite well, the complex forms of the nonlinear element nonparametric estimates make the obtained models hard to be applied in practice. To overcome this problem, some nonparametric polynomial identification algorithms have been developed by which the obtained estimates of the nonlinear element have a polynomial form [5]–[7]. But, as summarized in [4], almost all having proposed Hammerstein system nonparametric identification algorithms can only recover the unknown system nonlinearity up to an additive constant, and one functional value of the nonlinearity is usually assumed to be known to make the constant solvable. This is obviously a disadvantage because some exact information about identified systems is still required. Recently, based on the solution to an integration equation by Hermit polynomial expansion, a nonparametric orthogonal series estimate of the Hammerstein system nonlinear element has been proposed [9]. Using this method, the defect might be overcome, but the obtained estimate of the nonlinearity can only converge in the mean integrated square error (MISE) sense; it cannot approximate the nonlinearity uniformly well.

In this paper, a new nonparametric polynomial identification algorithm for the Hammerstein system is proposed which extends the idea presented in the author's previous work [7] to a more general and practical case, where no functional value of the system

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The author is with the Department of Automatic Control and Systems Engineering, The University of Sheffield, Sheffield, S1 3JD, U.K.

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