A correspondence of modular forms and applications to values of L-series.

Nikolaos Diamantis, Michael Neururer, Fredrik Strömberg
(University of Nottingham)

Abstract. A new interpretation of the Rogers-Zudilin approach to the Boyd conjectures is established. This is based on a correspondence of modular forms which is of independent interest. We use the reinterpretation for two applications to values of L-series and values of their derivatives.

Key words. L-functions, derivatives of L-functions, Eisenstein series

1 Introduction

In [7], Zudilin outlines a method he developed with M. Rogers on which their proof of Boyd's conjectures is based. Here we reinterpret that method in terms of a correspondence of modular forms which is of independent interest. We use this reinterpretation to deduce some relations among values of L-functions and some relations involving values of the first derivative of an L-function.

The correspondence associates to a pair of functions F_1, F_2 and $s \in \mathbb{C}$ a new function $\Phi_s(F_1, F_2)$ which, when F_1 and F_2 are connected to modular forms, satisfies properties related to modularity for special values of s (see Proposition 2.1). In the case where F_1 and F_2 are Eisenstein series, $\Phi_s(F_1, F_2)$ is again an Eisenstein series for certain s and we make use of this fact in the two applications of our method described below.

Our main theorem, Theorem 3.2, connects the Mellin transform of the product F_1F_2 with the Mellin transform of functions associated to F_1 and F_2 via our correspondence. This is achieved using a simple "duality" relation (Lemma 3.1), which reformulates a key integration by substitution in Rogers-Zudilin's method. The content of the main theorem can be summarized as:

Theorem 1.1. *Let F_1 and F_2 be functions on the upper half-plane given by*

$$F_1(z) = \sum_{m_1, n_1 \geq 1} a_1(m_1) b_1(n_1) e^{2\pi i m_1 n_1 z} \quad \textit{and} \quad F_2(z) = \sum_{m_2, n_2 \geq 1} a_2(m_2) b_2(n_2) e^{2\pi i m_2 n_2 z},$$

where we assume that the Fourier coefficients grow at most polynomially. For $j = 1, 2$ we set

$$f_j(z) = \sum_{m_j, n_j \geq 1} b_j(n_j) e^{2\pi i m_j n_j z} \quad \textit{and} \quad g_j(z) = \sum_{m_j, n_j \geq 1} a_j(m_j) e^{2\pi i m_j n_j z}.$$

Then we have the following relation between Mellin transforms

$$\mathcal{M}(F_1 \cdot F_2|W_N)(s) = \mathcal{M}(\Phi_{s+1}(f_1, f_2) \cdot (\Phi_{-s+1}(g_2, g_1)|W_N))(s) \quad \textit{for all } s \in \mathbb{C},$$

where $\Phi_s(f, g)$ is a certain function associated to f and g as described in Section 2.

We discuss two applications of this reinterpretation: Firstly we establish a relation among critical values of L-functions of products of Eisenstein series:

Theorem 1.2. *Let χ_1, χ_2 and ψ_1, ψ_2 be pairs of non-trivial primitive Dirichlet characters modulo M_1, M_2 and N_1, N_2, respectively. Let $k \geq 1$, $l \geq 2$ such that $(\chi_1 \cdot \chi_2)(-1) = (-1)^l$ and $(\psi_1 \cdot \psi_2)(-1) = (-1)^k$. For a positive integer r we set*

$$E_k^{\psi_1,\psi_2,r}(z) = 2 \sum_{m,n \geq 1} \psi_1(m)\psi_2(n) n^{k-1} e^{2\pi i n m r z}$$

with an analogous definition for $E_l^{\chi_1,\chi_2,r}$. Then for an integer $j \in \{1, \dots, k+l-1\}$ such that $(\chi_1 \cdot \psi_1)(-1) = (-1)^{k-j}$ we have

$$\Lambda(E_l^{\chi_1,\chi_2} \cdot E_k^{\bar{\psi}_2,\bar{\psi}_1,M_1M_2}, j) = C \cdot \Lambda(E_j^{\chi_1,\psi_2} \cdot E_{k+l-j}^{\bar{\chi}_2,\bar{\psi}_1,M_1N_2}, l)$$

where C is an explicit *algebraic number given in section 4.*

This is extended to a more general class of characters in the second author's PhD thesis (in preparation). Since the space of modular forms of weight $k \geq 4$ is generated by products of such Eisenstein series when the level is prime [6], this result implies relations among critical values of L-functions for more general classes of modular forms than is at first apparent. A further extension of the main theorem of [6] to modular forms of squarefree level is implied by work in preparation by M. Dickson and the second author. Besides the independent interest of relations among critical values of different L-functions, possible applications we are currently exploring include vanishing and algebraicity results for values of L-functions, and the formulation of a general result of this nature specifically for cusp forms.

A second application exhibits a relation between derivatives of L-function of weight 2 modular forms at 1 and values of L-functions of weight 1 modular forms at 1. The precise statement will be given in Theorem 5.2 but its content can be sketched in the following form.

Theorem 1.3. *If E is in a certain subspace of the weight 2 Eisenstein space on $\Gamma_1(N)$, then*

$$\Lambda'(E, 1) = \Lambda(\tilde{E}, 1) + C$$

for an explicitly determined constant C and an explicit element $\tilde{E}$ in the weight 1 Eisenstein space.

We stress that since L-functions of Eisenstein series can be expressed in terms of Dirichlet L-functions, such relations, once identified, can be verified directly. However, the proof gives a method to determine, for each given series E, a linear combination of weight 1 Eisenstein series whose L-function at 1 satisfies an explicit relation with $L'(E, 1)$. This method is based on a combination of our reformulation of Rogers-Zudilin's method and Goldfeld's expression for derivatives of L-functions.

Furthermore the existence of an explicit, and fairly general relation between derivatives of L-functions and non-critical values of L-functions in the case of Eisenstein series, raises the question of whether analogous relations might exist in the case of cusp forms. In that case, such relations should not be possible to deduce by direct methods.

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2 A correspondence of modular forms

We begin by fixing notation will be used in the sequel. For a holomorphic function g which is exponentially decaying at 0 and ∞ we denote the Mellin transform of g by

$$\mathcal{M}g(s) := \int_0^\infty g(it)t^s \frac{dt}{t}.$$

If g is any function defined on the upper half plane $\mathfrak{H}$, $k \in \mathbf{Z}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ then we define the slash operator by

$$(g|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(z) = (cz+d)^{-k} g(\frac{az+b}{cz+d}).$$

When we omit the index k we mean the weight 0 slash operator. The Fricke involution $W_N = \begin{pmatrix} 0 & -\sqrt{N}^{-1} \\ \sqrt{N} & 0 \end{pmatrix}$ acts on modular forms of level N and weight k via the $|_k$ operator. If h is a cusp form for $\Gamma_0(N)$ then we define the completed L-function of h as:

$$\Lambda(g,s) := \Gamma(s) \left(\frac{\sqrt{N}}{2\pi}\right)^s L(g,s) = N^{s/2} \mathcal{M}(g)(s). \tag{1}$$

Let $f_1(z) = \sum_{m_1=0}^\infty \alpha(m_1) e^{2\pi i m_1 z}$ and $f_2(z) = \sum_{m_2=0}^\infty \beta(m_2) e^{2\pi i m_2 z}$ be functions on $\mathfrak{H}$. By applying the Möbius inversion formula we can re-write f_1 and f_2 as double sums:

$$f_1(z) = \sum_{m_1,n_1 \geq 0}^\infty a(m_1) e^{2\pi i m_1 n_1 z} \text{ and } f_2(z) = \sum_{m_2,n_2 \geq 0}^\infty b(m_2) e^{2\pi i m_2 n_2 z}$$

where $a(n) := \sum_{r|n} \alpha(r)\mu(n/r)$ and $b(n) := \sum_{r|n} \beta(r)\mu(n/r)$. We then define a new function $\Phi_t(f_1, f_2)$ on $\mathfrak{H}$ by the Fourier expansion

$$\Phi_t(f_1,f_2)(z) = \sum_{m=1}^\infty \sum_{d|m} a(m/d) b(d) d^{t-1} e^{2\pi m i z} = \sum_{m,n \geq 1} a(m) b(n) n^{t-1} e^{2\pi i m n z}.$$

This construction leads to a correspondence on spaces of Eisenstein series (see section 4 for definitions): Let ψ and ϕ be non-trivial primitive Dirichlet characters mod N_1 and N_2, respectively. Assume that ψ is odd and that the positive integer l satisfies $\phi(-1) = (-1)^l$. If $f_1 = E_1^{\psi,\mathbf{1}}$, $f_2 = E_l^{\phi,\mathbf{1}}$ and k is such that $k - l$ is even then the function $\Phi_k(f_1, f_2) = 2E_{k+l-1}^{\psi,\phi}$ is an Eisenstein series of weight $k + l - 1$. This fact will be used in the proof of Theorem 1.2.

An analogous construction can be carried out when f_1 and f_2 are cusp forms of weight 1, level N and Dirichlet characters v_1 and v_2, respectively. Although we do not expect $\Phi_t(f_1, f_2)$ to be a modular form, Proposition 2.1 shows that if t is even then all its twisted L-series satisfy the functional equations of a weight t cusp form of level N^2 and character $v_1 v_2$.

Specifically, for each prime $r \nmid N$, consider a primitive character ψ of conductor r such that $\psi(-1) = (-1)^u$ ($u = 0$ or 1). For convenience ψ can also stand for the trivial character $\mathbf{1} \bmod 1$. For $\mathrm{Re}(s) \gg 0$ consider

$$L(\Phi_t(f_1, f_2), \psi; s) = \sum_{m\geq 1} \frac{\psi(m)}{m^s} \left(\sum_{d|m} a\left(\frac{m}{d}\right) d^{t-1} b(d) \right) = \left(\sum_{m\geq 1} \frac{b(m)\psi(m)}{m^{s-t+1}} \right) \left(\sum_{l\geq 1} \frac{a(l)\psi(l)}{l^s} \right).$$

For the cusp forms f_1 and f_2 we define their L-series twisted by ψ in the standard way, e.g.

$$L(f_1, \psi; s) = \sum_{m\geq 1} \frac{\psi(n)}{n^s} \left(\sum_{m|n} a(m) \right) = \left(\sum_{m\geq 1} \frac{a(m)\psi(m)}{m^s} \right) \cdot L(\psi, s)$$

and similarly for f_2. From the definition of $L(\Phi_t(f_1, f_2), \psi; s)$ we immediately deduce that

$$L(\Phi_t(f_1, f_2), \psi; s) = \frac{L(f_1, \psi; s)}{L(\psi, s)} \frac{L(f_2, \psi; s-t+1)}{L(\psi, s-t+1)}. \tag{2}$$

Note that if f_1 and f_2 are Hecke eigenforms this equality implies that $L(\Phi_t(f_1, f_2), \psi; s)$ has an Euler product representation. If we now define the completion of $L(\Phi_t(f_1, f_2), \psi; s)$ as

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) := \frac{\Gamma(s)(Nr)^s}{(2\pi)^s} L(\Phi_t(f_1, f_2), \psi; s)$$

then we have

Proposition 2.1. *Let f_1 and f_2 be cusp forms of weight 1, level N and Dirichlet characters v_1 and v_2, respectively. Let ψ be a primitive character of prime conductor $r \nmid N$. Then for even $t > 1$ the completed L-series $\Lambda(\Phi_t(f_1, f_2), \psi; s)$ satisfies the functional equation*

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) = (-1)^{t/2} v_1(r) v_2(r) \psi(N^2) \tau(\psi) \Lambda(\Phi_t(f_2|_1 W_N, f_1|_1 W_N), \overline{\psi}; t-s),$$

where

$$\tau(\psi) := \frac{1}{\sqrt{r}} \sum_{n \bmod r} \psi(n) e^{2\pi i \frac{n}{r}}$$

is the Gauss sum of ψ.

Proof. We first express $\Lambda(\Phi_t(f_1, f_2), \psi; s)$ in terms of the completed L-series,

$$\Lambda(f, \psi; s) := \frac{\Gamma(s)(r\sqrt{N})^s}{(2\pi)^s} L(f, \psi; s) \quad \text{and} \quad \Lambda(\psi, s) := \left(\frac{r}{\pi}\right)^{s/2} \Gamma(\frac{s+u}{2}) L(\psi, s),$$

where $u = 0$ or 1 is determined by $\psi(-1) = (-1)^u$. We then have

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) = \qquad (3)$$
$$\left(\frac{Nr}{\pi}\right)^{\frac{t-1}{2}} 2^{s-t+1} \frac{\Gamma((s-t+1+u)/2)\Gamma((s+u)/2)}{\Gamma(s-t+1)} \frac{\Lambda(f_1, \psi; s)}{\Lambda(\psi; s)} \frac{\Lambda(f_2, \psi; s-t+1)}{\Lambda(\psi; s-t+1)}.$$

We recall the functional equations for the L-functions which appear in the expression above:

$$\Lambda(f_j, \psi; s) = i v_j(r) \psi(N) \tau(\psi)^2 \Lambda(f_j|_1 W_N, \overline{\psi}; 1-s), \quad (j = 1, 2) \quad \text{and}$$
$$\Lambda(\psi; s) = i^{-u} \tau(\psi) \Lambda(\overline{\psi}; 1-s).$$

By using these functional equations we can rewrite the right-hand side of (3) to obtain

$$\Lambda(\Phi_t(f_1, f_2), \psi; s) = \epsilon \cdot \Lambda(\Phi_t(f_2|_1 W_N, f_1|_1 W_N), \overline{\psi}; t-s),$$

where

$$\epsilon = 2^{2s-t} \frac{\Gamma((s-t+1+u)/2)\Gamma((s+u)/2)\Gamma(1-s)}{\Gamma((-s+1+u)/2)\Gamma((-s+t+u)/2)\Gamma(s-t+1)} (-1)^{u+1} v_1(r) v_2(r) \psi(N^2) \tau(\psi)^2.$$

The final version of the functional equation now follows from the identity

$$\frac{\Gamma((s-t+1+u)/2)\Gamma((s+u)/2)\Gamma(1-s)}{\Gamma((-s+1+u)/2)\Gamma((-s+t+u)/2)\Gamma(s-t+1)} = 2^{t-2s}(-1)^{t/2+u+1},$$

which is valid for even t and can be shown using standard properties of the Gamma function, including the reflection and duplication formulas. □

Remark. It follows immediately from (2) that $L(\Phi_t(f_1, f_2), s)$ has infinitely many poles (assuming the Grand Simplicity Hypothesis [5]) and therefore $\Phi_t(f_1, f_2)$ can not be a modular form. However, the extension of the converse theorem of [1] to general levels implies that $\Phi_t(f_1, f_2)$ is a modular *integral*.

3 A reinterpretation of the method of Rogers-Zudilin

The method of [7] relies crucially on a simple change of variables in an integral of the product of two series which leads to a product of two different functions. This part of the method can be expressed as the following simple "duality relation" involving the functions rather than their Fourier expansions. For a function h on $\mathfrak{H}$ and $x \in \mathbb{Z}$, we set $h^{(x)}$ for the function defined by $h^{(x)}(z) := h(xz)$.

Lemma 3.1. *Let $f, g : \mathfrak{H} \to \mathbb{C}$ be holomorphic functions with exponential decay at infinity and at most polynomial growth at 0. For each m, $n \in \mathbb{N}$ and $s \in \mathbb{C}$ we have*

$$\mathcal{M}(f^{(m)} \cdot (g^{(n)}|W_N))(s) = (n/m)^s \mathcal{M}(f^{(n)} \cdot (g^{(m)}|W_N))(s).$$

Proof. From the growth conditions at infinity and 0 it follows that the product $f \cdot g|W_N$ has exponential decay at both infinity and 0 and thus the Mellin transforms on both sides are well defined. By the change of variables $t \to (n/m)t$ we see that $\mathcal{M}(f^{(m)} \cdot g^{(n)}|W_N)(s)$ equals

$$\int_0^\infty f(mit)g\left(\frac{ni}{Nt}\right)t^s\frac{dt}{t} = (n/m)^s\int_0^\infty f(nit)g\left(\frac{mi}{Nt}\right)t^s\frac{dt}{t}.$$

□

With the above lemma we obtain the following

Theorem 3.2. *Let $F_1, F_2 : \mathfrak{H} \to \mathbb{C}$ be given by the Fourier expansions*

$$F_1(z) = \sum_{m_1,n_1\geq 1} a_1(m_1)b_1(n_1)e^{2\pi i m_1 n_1 z} \quad \text{and} \quad F_2(z) = \sum_{m_2,n_2\geq 1} a_2(m_2)b_2(n_2)e^{2\pi i m_2 n_2 z},$$

where we assume, additionally, that the coefficients $a_j(n)$ and $b_j(n)$ grow at most polynomially in n. If we define the functions

$$f_j(z) = \sum_{m_j,n_j\geq 1} b_j(n_j)e^{2\pi i m_j n_j z} \quad \text{and} \quad g_j(z) = \sum_{m_j,n_j\geq 1} a_j(m_j)e^{2\pi i m_j n_j z} \qquad (j = 1, 2)$$

then we have the following relation between Mellin transforms

$$\mathcal{M}(F_1 \cdot F_2|W_N)(s) = \mathcal{M}(\Phi_{s+1}(f_1, f_2) \cdot (\Phi_{-s+1}(g_2, g_1)|W_N))(s) \quad \text{for all } s \in \mathbb{C}.$$

Proof. Set $h_j(z) := \sum_{n_j\geq 1} b_j(n_j)e^{2\pi i n_j z}$ for $j = 1, 2$. The growth conditions on $b_j(n)$ imply that h_1, h_2 have exponential decay at infinity and at most polynomial growth at 0. Hence Lemma 3.1 implies

$$\begin{aligned}
\mathcal{M}(h_1^{(m_1)} \cdot h_2^{(m_2)}|W_N)(s) &= \left(\frac{m_2}{m_1}\right)^s \int_0^\infty h_1(m_2 it) \cdot h_2\left(\frac{im_1}{Nt}\right) t^s \frac{dt}{t} \\
&= \left(\frac{m_2}{m_1}\right)^s \int_0^\infty \sum_{n_1,n_2\geq 1} b_1(n_1)b_2(n_2)e^{-\frac{2\pi m_1 n_2}{Nt}} e^{-2\pi n_1 m_2 t} t^s \frac{dt}{t}. \qquad (4)
\end{aligned}$$

The growth condition of b_j justifies the interchange of integration and summation, so, upon the further change of variables $t \to (n_2/m_2)t$ we deduce that

$$\begin{aligned}
\mathcal{M}(h_1^{(m_1)} \cdot h_2^{(m_2)}|W_N)(s) &= m_1^{-s} \int_0^\infty \sum_{n_1,n_2\geq 1} b_1(n_1)b_2(n_2)n_2^s e^{-\frac{2\pi m_1 m_2}{Nt}} e^{-2\pi n_1 n_2 t} t^s \frac{dt}{t} \\
&= m_1^{-s} \int_0^\infty \Phi_{s+1}(f_1, f_2)(it) e^{\frac{-2\pi m_1 m_2}{Nt}} t^s \frac{dt}{t}.
\end{aligned}$$

The desired conclusion now follows from the fact that

$$F_1 \cdot F_2|W_N(z) = \sum_{m_1,m_2\geq 1} a_1(m_1)a_2(m_2)h_1^{(m_1)}(z) \cdot (h_2^{(m_2)}|W_N)(z).$$

□

4 An application to products of Eisenstein series

We recall the weight k Eisenstein series $E_k^{\psi,\phi}$ assigned to primitive Dirichlet characters ψ mod N_1 and ϕ mod N_2 which satisfy $\psi\phi(-1) = (-1)^k$. Its Fourier expansion at infinity is given by

$$E_k^{\psi,\phi}(z) = a_{\psi,\phi} + 2\sum_{m,n\geq 1} \psi(m)\phi(n)n^{k-1}e^{2\pi i nmz},$$

where the constant term is given by

$$a_{\psi,\phi} = \begin{cases} \delta(\psi)L(1-k,\phi) \text{ if } k > 1 \text{ and} \\ \delta(\psi)L(0,\phi) + \delta(\phi)L(0,\psi) \text{ if } k = 1. \end{cases}$$

For $t \in \mathbb{R}$ we also consider the function $E_k^{\psi,\phi,t}$ given by $E_k^{\psi,\phi,t}(z) = E_k^{\psi,\phi}(tz)$.

In Section 4.8 of [2] it is shown that, for $k = 1$ and $t \in \mathbb{N}$, each $E_k^{\psi,\phi,t}$ is a weight 1 modular form for $\Gamma_0(tN_1N_2)$ and character $\psi\phi$. Furthermore, for a fixed $N \in \mathbb{N}$, these Eisenstein series give a basis of the Eisenstein subspace $\mathcal{E}_1(\Gamma_1(N))$. To be precise, if A is the set of $(\{\psi,\phi\}, t)$ such that ψ and ϕ are primitive Dirichlet characters modulo N_1 and N_2 such that $(\psi\phi)(-1) = -1$ and t is a positive integer such that $tN_1N_2|N$ then the set

$$\{E_1^{\psi,\phi,t}; (\{\psi,\phi\}, t) \in A\}$$

is a basis of $\mathcal{E}_1(\Gamma_1(N))$. For convenience we will often write E_1^{ψ} for $E_1^{\psi,\mathbf{1},1}$ and a_ψ for $a_{\psi,\mathbf{1}}$ where $\mathbf{1}$ is the trivial character modulo 1 and we also consider the subspace

$$\mathcal{E}_1'(\Gamma_1(N)) = \langle\{E_1^{\psi,\mathbf{1},1}; (\psi,\mathbf{1},1) \in A\}\rangle.$$

The analogous statement for $k = 2$ (Th. 4.6.2 of [2]) involves the set B of (ψ,ϕ,t) such that ψ and ϕ are primitive Dirichlet characters modulo N_1 and N_2 with $(\psi\phi)(-1) = 1$, and t is a positive integer such that $1 < tN_1N_2|N$. In this case the set

$$\{E_2^{\psi,\phi,t}; (\psi,\phi,t) \in B\} \cup \{E_2^{\mathbf{1},\mathbf{1},1} - tE_2^{\mathbf{1},\mathbf{1},t}; t|N\}$$

forms a basis of $\mathcal{E}_2(\Gamma_1(N))$. We also consider the smaller subspace

$$\mathcal{E}_2'(\Gamma_1(N)) = \langle\{E_2^{\psi,\phi,1}; (\psi,\phi,1) \in B; \phi,\psi \text{ odd}\}\rangle.$$

In the sequel we will often use the following identity

$$E_k^{\psi,\phi}|_k W_{tN_1N_2} = (-1)^k\tau(\psi_1)\tau(\psi_2)\left(\frac{N_2}{N_1}\right)^{\frac{k-1}{2}} t^{k/2}E_k^{\bar{\phi},\bar{\psi},t}, \tag{5}$$

which is valid for any $t > 0$ (see e.g. Th. 12.1 of [4] for a version with $t = 1$).

We can now use Theorem 3.2 to prove Theorem 1.2. Recall that ψ_i and χ_i $(i = 1,2)$ are non-trivial primitive characters modulo N_i and M_i such that $(\chi_1 \cdot \chi_2)(-1) = (-1)^l$ and $(\psi_1 \cdot \psi_2)(-1) = (-1)^k$. We will regard both Eisenstein series $E_l^{\chi_1,\chi_2}$ and $E_k^{\psi_1,\psi_2}$ as modular forms of level MN where $M = M_1M_2$ and $N = N_1N_2$. It follows immediately from (5) that

$$(-1)^k\tau(\psi_1)\tau(\psi_2)\left(\frac{N_2}{N_1}\right)^{\frac{k-1}{2}} M^{k/2}\Lambda(E_l^{\chi_1,\chi_2}\cdot E_k^{\bar{\psi}_2,\bar{\psi}_1,M}, j) = \Lambda(E_l^{\chi_1,\chi_2}\cdot(E_k^{\psi_1,\psi_2}|_k W_{MN}), j),$$

and if we use the definition of the completed L-function we see that the right-hand side is

$$(\sqrt{MN})^{j-k} i^{-k} \mathcal{M}(E_l^{\chi_1,\chi_2} \cdot (E_k^{\psi_1,\psi_2}|W_{MN}))(j-k).$$

We are now in a position to apply Theorem 3.2 with Fourier expansions given by

$$a_1(m_1) = \chi_2(m_1) m_1^{l-1}, b_1(n_1) = \chi_1(n_1), a_2(m_2) = \psi_1(m_2), b_2(n_2) = \psi_2(n_2) n_2^{k-1}, s = j-k,$$

and we conclude that $\mathcal{M}(E_l^{\chi_1,\chi_2} \cdot (E_k^{\psi_1,\psi_2}|W_{MN}))(j-k)$ can be expressed as

$$\begin{aligned} 4\mathcal{M}(\Phi_{j-k+1}(f_1,f_2)\cdot(\Phi_{k-j+1}(g_2,g_1)|W_{MN}))(j-k) &= \mathcal{M}(E_j^{\chi_1,\psi_2}\cdot(E_{k+l-j}^{\psi_1,\chi_2}|W_{MN}))(j-k) \\ &= (\sqrt{MN})^{k-j} i^{k+l-j} \Lambda(E_j^{\chi_1,\psi_2}\cdot(E_{k+l-j}^{\psi_1,\chi_2}|_{k+l-j}W_{MN}))(l). \end{aligned}$$

Finally, after another application of (5), we obtain the conclusion of Theorem 1.2:

$$\Lambda(E_l^{\chi_1,\chi_2}\cdot E_k^{\bar\psi_2,\bar\psi_1,M}, j) = C\cdot\Lambda(E_j^{\chi_1,\psi_2}\cdot E_{k+l-j}^{\bar\chi_2,\bar\psi_1,M_1N_2}, l), \tag{6}$$

where

$$C = (-i)^{l-j}\tau(\chi_2)\tau(\psi_2)^{-1} M_1^{\frac{l-j}{2}} M_2^{\frac{l-j-1}{2}} N_1^{-\frac{l-j}{2}} N_2^{\frac{l-j+1}{2}}.$$

5 Application to derivatives of L-functions

Let ψ and ϕ be odd, primitive Dirichlet characters modulo N_1 and N_2 respectively. Using the notation of the last section we set

$$E_1^{\psi} := E_1^{\psi,\mathbf{1}}, a_\psi := a_{\psi,\mathbf{1}}, \text{ and } f_r^{\psi,\phi} := \frac{\sqrt{N}}{4}\left(E_1^{\psi} - a_\psi\right)\cdot\left(\left(E_1^{\phi,r} - a_\phi\right)|_1 W_N\right),\ N = N_1N_2.$$

The goal of this section is to evaluate a particular linear combination of the special values $\mathcal{M}(f_r^{\psi,\phi})(2)$ in two different ways thereby obtaining a relation between values and derivatives of certain L-functions. We first observe that for a fixed positive integer r we can write

$$\mathcal{M}(f_r^{\psi,\phi})(2) = \frac{1}{4i}\mathcal{M}\left(\left(E_1^{\psi} - a_\psi\right)\cdot\left(\left(E_1^{\phi,r} - a_\phi\right)|W_N\right)\right)(1).$$

Since we now have a weight 0 action in the right-hand side we can use Theorem 3.2 with

$$s = 1, a_1(n) = 1, b_1(n) = \psi(n), a_2(n) = \delta_r(n), b_2(n) = \phi(n),$$

where $\delta_r(n) = 1$ if $r|n$ and 0 otherwise. This implies that $\mathcal{M}(f_r^{\psi,\phi})(2)$ equals

$$\frac{1}{i}\mathcal{M}\left(\Phi_2(f_1,f_2)\cdot\Phi_0(g_2,g_1)|W_N\right)(1) = \frac{1}{2i}\mathcal{M}\left(E_2^{\psi,\phi}(it)\cdot\sum_{n_1,n_2\geq 1}\frac{1}{n_1}e^{\frac{-2\pi r n_1 n_2}{Nt}}\right)(1).$$

From the following well-known expression for the logarithm of the Dedekind eta function

$$\sum_{m,n\geq 1}\frac{1}{n}e^{\frac{-2\pi rmn}{Nu}} = -\sum_{m\geq 1}\log(1-e^{\frac{-2\pi rm}{Nu}}) = -\log\left(\eta(ri/(Nu))e^{\frac{2\pi r}{24\cdot Nu}}\right),$$

we deduce that

$$\mathcal{M}(f_r^{\psi,\phi})(2) = \frac{i}{2}\int_0^\infty E_2^{\psi,\phi}(iu)\log\left(\eta(ri/(Nu))e^{\frac{2\pi r}{24\cdot Nu}}\right)du. \tag{7}$$

The integral above is well defined since $E_2^{\psi,\phi}$ decays exponentially at both ∞ and 0. The decay at infinity is immediate since ψ is not trivial and the decay at 0 follows from (5). By using (5) to rewrite $f_r^{\bar{\phi},\bar{\psi}}$ it follows from (7) that

$$\begin{aligned}\mathcal{M}(F_r^{\psi,\phi})(2) &= \frac{i}{2}\int_0^\infty (E_2^{\psi,\phi}|_2(1+W_N))(iu)\log\left(\eta(ri/(Nu))e^{\frac{2\pi r}{24\cdot Nu}}\right)du \\ &= -\frac{i}{2}\int_0^\infty (E_2^{\psi,\phi}|_2(1+W_N))(iu)\log\left(\eta(riu))e^{\frac{2\pi ru}{24}}\right)du \end{aligned}\tag{8}$$

where

$$F_r^{\psi,\phi} := f_r^{\psi,\phi} + \sqrt{\frac{N_2}{N_1}}\tau(\psi)\tau(\phi)f_r^{\bar{\phi},\bar{\psi}}.$$

It is clear from (8) that we can find a linear combination of $F_r^{\psi,\phi}$'s such that the exponentials inside the logarithm on the right-hand side are eliminated:

$$\begin{aligned}\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi}) - (1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(2) = \\ -\frac{i}{2}\int_0^\infty (E_2^{\psi,\phi}|_2(1+W_N))(iu)\log(V(iu))du,\end{aligned}\tag{9}$$

where

$$V(z) := \frac{(\eta(z)\eta(Nz))^{N_1+N_2}}{(\eta(N_1z)\eta(N_2z))^{1+N}}.$$

We will now proceed to evaluate the two sides of (9) separately.

5.1 The right-hand side of (9)

We first recall the principle behind Goldfeld's expression for derivatives of L-functions:

Proposition 5.1. *Let f and g be holomorphic functions on $\mathfrak{H}$ such that for some $N \in \mathbb{N}$:*
(i) $f|_2W_N = f$
(ii) $g|_kW_N = \pm g$, for some non-zero constant $k \in \mathbb{R}$. Then

$$\int_0^\infty f(z)dz = 0 \text{ and } 2\int_0^\infty f(iy)\log(g(iy))dy = k\int_0^\infty f(iy)\log(y)dy.$$

Proof. Condition *(i)* is equivalent to $f(W_Nz)d(W_Nz) = f(z)dz$. Therefore $\int_0^\infty f(z)dz = \int_{W_N\infty}^{W_N0} f(z)dz = \int_\infty^0 f(z)dz$ and hence $\int_0^\infty f(z)dz = 0$. Similarly, we see that

$$\begin{aligned}\int_0^\infty f(z)\log(g(z))dz &= \int_{W_N\infty}^{W_N0} f(z)\log(g(z))dz = \int_\infty^0 f(z)\log(g(W_Nz))dz \\ &= \int_\infty^0 f(z)\log(g(z))dz + ik\int_\infty^0 f(iy)\log(y)dy + c'\int_0^\infty f(z)dz\end{aligned}$$

for some $c' \in \mathbb{C}$. This equality, together with $\int_0^\infty f(z)dz = 0$, implies the conclusion. □

Since Proposition 5.1 holds for $f = E_2^{\psi,\phi}|_2(1+W_N)$ and $g = V$ with $k = N_1 + N_2 - 1 - N$, we deduce that

$$\int_0^\infty f(iu)\log(V(iu))du = \frac{k}{2}\int_0^\infty f(iu)\log(u)du = \frac{k}{2}(\mathcal{M}f)'(1). \tag{10}$$

By using Proposition 5.1 together with (1) we can express the the right-hand side of (10) as

$$\frac{(N_1-1)(1-N_2)}{2\sqrt{N}}\Lambda'(E_2^{\psi,\phi}|_2(1+W_N),1).$$

If h is a modular form of weight 2 and level N it is easy to see from the functional equation of $\Lambda(h,s)$ that $\Lambda'(h|_2(1+W_N),1) = 2\Lambda'(h,1)$. It follows that the right-hand side of (9) equals

$$\frac{i(N_1-1)(N_2-1)}{2\sqrt{N}}\Lambda'(E_2^{\psi,\phi},1). \tag{11}$$

5.2 The left-hand side of (9)

To compute the left-hand side of (9) we first express $\mathcal{M}(f_r^{\psi,\phi})(s)$ in terms of completed L-functions by applying the Riemann trick:

$$\mathcal{M}(f_r^{\psi,\phi})(s) = \int_{1/\sqrt{N}}^\infty f_r^{\psi,\phi}(it)t^s\frac{dt}{t} + N^{-s}\int_{1/\sqrt{N}}^\infty f_r^{\psi,\phi}\left(\frac{i}{Nt}\right)t^{-s}\frac{dt}{t}. \tag{12}$$

We can rewrite $f_r^{\psi,\phi}(it)$ as a linear combination of cuspidal functions as follows

$$f_r^{\psi,\phi}(it) = \frac{\sqrt{N}}{4}\Big((E_1^\psi(it)-a_\psi)\frac{a_\phi i}{\sqrt{N}t} - \Big(\Big(\Big((E_1^\psi|_1W_N)\cdot E_1^{\phi,r}\Big)|_2W_N\Big)(it) + a_\psi b_\phi\Big) \\ - a_\psi((E_1^{\phi,r}|_1W_N)(it) - b_\phi)\Big), \tag{13}$$

where b_ϕ is the constant term of the Fourier expansion of $E_1^{\phi,r}|W_N$. Here we have used the fact that $E_1^\psi\cdot(E_1^{\phi,r}|_1W_N) = -\Big((E_1^\psi|_1W_N)\cdot E_1^{\phi,r}\Big)|_2W_N$ and we have a similar expression for $f_r^{\psi,\phi}|_2W_N$. By substituting (13) and the analogue for $f_r^{\psi,\phi}|_2W_N$ into (12) and using the integral representation of the completed L-series of a, not necessarily cuspidal, modular form (e.g. Th. 7.3. of [4]), we deduce that

$$\mathcal{M}(f_r^{\psi,\phi})(s) = \frac{N^{(1-s)/2}}{4}\Big(\Lambda((E_1^\psi|_1W_N)\cdot E_1^{\phi,r},2-s) + a_\phi i\Lambda(E_1^\psi,s-1) + a_\psi i\Lambda(E_1^{\phi,r},1-s)\Big) = \\ \frac{N^{(1-s)/2}}{4}\Big(-\tau(\psi)\sqrt{N_2}\Lambda(E_1^{\bar\psi,N_2}E_1^{\phi,r},2-s) + a_\phi i\Lambda(E_1^\psi,s-1) + a_\psi i\Lambda(E_1^{\phi,r},1-s)\Big). \tag{14}$$

For the last equality we again used (5) and we have an analogous expression for $\mathcal{M}(f_r^{\bar\phi,\bar\psi})(s)$.

We will now compute the value of the linear combination

$$\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi}) - (1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(s) \tag{15}$$

at $s=2$ by considering each of the three summands of (14) and the analogue for $f_r^{\bar\phi,\bar\psi}$.

First we treat the contributions from L-functions associated to products of Eisenstein series. In $\mathcal{M}(F_1^{\psi,\phi}+F_N^{\psi,\phi})(s)$ they are

$$\frac{N^{(1-s)/2}}{4}\tau(\psi)\sqrt{N_2}\left[\Lambda(-E_1^{\bar\psi,N_2}E_1^{\phi,1}+E_1^{\phi,N_1}E_1^{\bar\psi,1}-E_1^{\bar\psi,N_2}E_1^{\phi,N}+E_1^{\phi,N_1}E_1^{\bar\psi,N},2-s)\right].$$

By using the trivial fact

$$\Lambda(f,s)=a^s\Lambda(f^{(a)},s), \tag{16}$$

combined with $(E_1^{\phi,N_1}E_1^{\bar\psi,1})^{N_2}=E_1^{\phi,N}E_1^{\bar\psi,N_2}$ and $(E_1^{\phi,1}E_1^{\bar\psi,N_2})^{N_1}=E_1^{\phi,N_1}E_1^{\bar\psi,N}$, we obtain

$$\frac{N^{(1-s)/2}}{4}\tau(\psi)\sqrt{N_2}\left[\Lambda(E_1^{\bar\psi,N_2}E_1^{\phi,1},2-s)(N_1^{s-2}-1)+\Lambda(E_1^{\phi,N_1}E_1^{\bar\psi,1},2-s)(1-N_2^{s-2})\right]. \tag{17}$$

Both $\Lambda(E_1^{\bar\psi,N_2}E_1^{\phi,1},2-s)$ and $\Lambda(E_1^{\phi,N_1}E_1^{\bar\psi,1},2-s)$ have a simple pole at $s=2$ with residue $-a_{\bar\psi}a_\phi$. Therefore (17) is equal to $\tau(\psi)a_{\bar\psi}a_\phi\log(N_1/N_2)/4\sqrt{N_1}$ at $s=2$. It is easy to verify that the contribution of products of Eisenstein series in $\mathcal{M}(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi})(2)$ is exactly the same as that in $\mathcal{M}(F_1^{\psi,\phi}+F_N^{\psi,\phi})(2)$ and hence the products of Eisenstein series contribute

$$\frac{\tau(\psi)a_{\bar\psi}a_\phi(N_1+N_2-1-N)}{4\sqrt{N_1}}\log\left(\frac{N_1}{N_2}\right)$$

to $\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi})-(1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(s)$.

Secondly, to compute the contribution of the terms coming from $E_1^{\phi,r}$ and $E_1^{\bar\psi,r}$ we apply (16) to $\Lambda(E_1^{\phi,r},1-s)$ and $\Lambda(E_1^{\bar\psi,r},1-s)$. Thus their contribution to $\mathcal{M}(F_r^{\phi,\psi})(s)$ is

$$\frac{N^{(1-s)/2}}{4}r^{s-1}\Big(a_\psi i\Lambda(E_1^\phi,1-s)+a_{\bar\phi}i\sqrt{\frac{N_2}{N_1}}\tau(\psi)\tau(\phi)\Lambda(E_1^{\bar\psi},1-s)\Big),$$

which implies that the contribution of these terms to (15) at $s=2$ is 0. We are now left with

$$\begin{aligned}\mathcal{M}((N_1+N_2)(F_1^{\psi,\phi}+F_N^{\psi,\phi})-(1+N)(F_{N_1}^{\psi,\phi}+F_{N_2}^{\psi,\phi}))(2)=&\\ \frac{(N_1+N_2-1-N)}{4\sqrt{N_1}}\left[\tau(\psi)a_{\bar\psi}a_\phi\log\left(\frac{N_1}{N_2}\right)+\frac{i}{\sqrt{N_2}}\Lambda(\mathcal{E}^{\psi,\phi},1)\right]&\end{aligned} \tag{18}$$

where $\mathcal{E}^{\psi,\phi}$ is given by $\mathcal{E}^{\psi,\phi}:=L(\phi,0)E_1^\psi+\sqrt{\frac{N_2}{N_1}}\tau(\psi)\tau(\phi)L(\bar\psi,0)E_1^{\bar\phi}$. We note that the last term of (18) is well-defined because the residues of $\Lambda(E_1^{\bar\phi},s)$ and $\Lambda(E_1^\psi,s)$ at 1 cancel when we take the linear combination giving $\mathcal{E}^{\psi,\phi}$. Equations (11) and (18) together give

$$i\sqrt{N_2}\tau(\psi)a_{\bar\psi}a_\phi\log\left(\frac{N_1}{N_2}\right)-\Lambda(\mathcal{E}^{\psi,\phi},1)=2\Lambda'(E_2^{\psi,\phi},1). \tag{19}$$

In the following theorem we extend the identity (19) to the spaces spanned by the Eisenstein series that occur in it. We denote by j the map from $\mathcal{E}_2'(\Gamma_1(N))$ to $\mathcal{E}_1'(\Gamma_1(N))\otimes\mathcal{E}_1'(\Gamma_1(N))$ induced by $E_2^{\psi,\phi}\mapsto(E_1^\psi,E_1^\phi)$.

Theorem 5.2. *Let* $\alpha : \mathcal{E}_2'(\Gamma_1(N)) \to \mathbb{C}$ *and* $\beta : j(\mathcal{E}_2'(\Gamma_1(N))) \to \mathbb{C}$ *be given by*

$$\alpha(E) = 2\Lambda'(E,1) \qquad \textit{and}$$
$$\beta(E_1^{\psi}, E_1^{\phi}) = -\Lambda(\mathcal{E}^{\psi,\phi,1}, 1) + i\sqrt{N_2}\tau(\psi)a_{\bar{\psi}}a_{\phi}\log\left(\frac{N_1}{N_2}\right).$$

where ψ *and* ϕ *are odd, primitive Dirichlet characters modulo* N_1 *and* N_2 *with* $N_1|N$ *and* $N_2|N$ *and* β *is extended linearly. Then*

$$\alpha = \beta \circ j.$$

References

[1] A. Daughton, *A Hecke correspondence theorem for automorphic integrals with infinite log-polynomial sum period functions* Int. J. Number Theory, 10, 1857 (2014)

[2] F. Diamond, J. Shurman *A First Course in Modular Forms* Springer 2005

[3] D. Goldfeld *Special values of derivatives of L-functions* Number theory (Halifax, NS, 1994), 159173, CMS Conf. Proc., 15, Amer. Math. Soc., Providence, RI, 1995.

[4] H. Iwaniec *Topics in Classical Automorphic Forms* AMS 2005

[5] M. Rubinstein, P. Sarnak *Chebyshev's bias.* Experiment. Math. Volume 3, Issue 3 (1994), 173-197.

[6] W. Kohnen, Y. Martin, *Products of two Eisenstein series and spaces of cusp forms of prime level*, J. Ramanujan Math. Soc., 23, 2008,

[7] W. Zudilin *Period(d)ness of L-values.* Number theory and related fields, 381395, Springer Proc. Math. Stat., 43, Springer, New York, 2013.