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Behaviour of the extended Toda lattice

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Abstract

We consider the first member of an extended Toda lattice hierarchy. This system of equations is differential with respect to one independent variable and differential-delay with respect to a second independent variable. We use asymptotic analysis to consider the long wavelength limits of the system. By considering various magnitudes for the parameters involved, we derive reduced equations related to the Korteweg-de Vries and potential Boussinesq equations.

Highlights:
- we analyse the behaviour of solutions of the extended Toda lattice
- we derive PDEs which are asymptotic approximations of the lattice
- we find similarity solutions of these limiting PDEs
- we show that in certain cases the PDEs can be transformed to the Boussinesq and/or KdV equations

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1. Introduction

In [1] an integrable non-isospectral (2 + 1)-dimensional extension of the Toda lattice hierarchy was constructed, this consisting of a sequence of pairs of equations in $p(n, t, y)$ and $q(n, t, y)$ with n being discrete and t and y continuous. The reductions of this hierarchy were found to include a (1 + 1)-dimensional differential-delay Toda lattice hierarchy, a sequence of evolution equations in $p(x, t)$ and $q(x, t)$ with both x and t continuous but where the equations involved derivatives with respect to x as well as shifts in x. It is the first member of this extended Toda lattice hierarchy that is the subject of the present paper.

In earlier papers [2, 3] a (1 + 1)-dimensional differential-delay Volterra lattice hierarchy had been derived. The autonomous versions of such equations were placed within a suitable modification of the usual algebraic structure associated with completely integrable evolution equations in [4]. The first member of the (1 + 1)-dimensional differential-delay Volterra lattice hierarchy was studied in [5], where we considered various amplitudes for parameters, and obtained a number of asymptotic reductions to generalizations of the Korteweg-de Vries (KdV) equation, amongst others. In the present paper, for the first member of the extended Toda lattice hierarchy, again by considering various magnitudes for the parameters involved, we derive reduced equations related to the KdV and potential Boussinesq equations.

Section 2 contains an introduction to the relevant reduction techniques we use and the equations under study. We start by using small amplitude weakly nonlinear asymptotic techniques to reduce the Toda lattice [6] to the Boussinesq equation, and outline its reduction to the KdV equation. We also reformulate the extended Toda system [1] to make it more amenable to the asymptotic techniques used subsequently.

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In sections 3 and 4, we focus on a pair of parameters in the extended system and sequentially consider their effect on small amplitude slowly-varying solutions of the extended system. We show that when these parameters are small, the system behaves as the pure Toda lattice, whilst at larger values of these parameters other phenomena are exhibited. In Section 3 we derive various generalisations of the potential Boussinesq equation, and in Section 4 we consider behaviour on the longer timescale, where the appropriate description is the KdV equation. Finally, in section 5 we summarise the main results and draw conclusions.

2. Background theory – the Toda lattice and its differential-delay extension

In this section we introduce the basic Toda lattice, and recap how, through asymptotic expansions, it can be reduced to the Boussinesq equation, and the Korteweg-de-Vries equation. Finally, we introduce and reformulate the extended Toda system (a system which is both non-isospectral and differential-delay), which is the focus of the remainder of the paper.

2.1. The pure Toda system

The Toda Lattice is usually obtained from the Hamiltonian system for the Fermi-Pasta-Ulam lattice [7], with a particular choice for the interaction potential, V, namely

$$H = \sum_n \tfrac{1}{2} g_n^2 + V(f_{n+1} - f_n), \qquad V(\phi) = \gamma_0^2 \left(\phi - 1 + \exp(-\phi)\right), \tag{2.1}$$

where $f_n(t)$ is the positions of particle n at time t and $g_n(t)$ is its momentum. The particles interact through the potential energy function $V(\cdot)$ which, in the original system studied by Fermi, Pasta and Ulam had a simple polynomial form $V(\phi) = \frac{1}{2}\phi^2 + \frac{1}{3}\alpha\phi^3$ or $V(\phi) = \frac{1}{2}\phi^2 + \frac{1}{4}\beta\phi^4$. In the Toda lattice, this potential is given by $V'(\phi) = \gamma_0^2 \left(1 - \exp(-\phi)\right)$. Hamilton's equations lead to

$$\frac{\mathrm{d}^2 f_n}{\mathrm{d}t^2} = V'(f_{n+1} - f_n) - V'(f_n - f_{n-1}) = \gamma_0^2 \exp(f_{n-1} - f_n) - \gamma_0^2 \exp(f_n - f_{n+1}). \tag{2.2}$$

The substitution $\phi(x,t) = \phi_n(t) = f_{n+1} - f_n$ with $x = n$ leads to

$$\frac{\mathrm{d}^2 \phi_n}{\mathrm{d}t^2} = V'(\phi_{n+1}) - 2V'(\phi_n) + V'(\phi_{n-1}). \tag{2.3}$$

The substitution $u = -\phi$ leads to

$$\gamma_0^{-2} u_{tt}(x,t) = \exp u(x+1,t) - 2\exp u(x,t) + \exp u(x-1,t) = \delta_x^2 \mathrm{e}^{u(x,t)}, \tag{2.4}$$

where δ_x^2 is the second central difference in x. The parameter γ_0 can be eliminated by rescaling time.

The Toda soliton is given by

$$f_n(t) = F_0 + \log\left(\frac{1 - \mathrm{e}^{-2\mu} + \eta \exp(-2\mu n + 2t \sinh\mu)}{1 - \mathrm{e}^{-2\mu} + \eta \exp(-2\mu n - 2\mu + 2t \sinh\mu)} \right). \tag{2.5}$$

which implies

$$\exp\phi_n(t) = \exp(f_{n+1}(t) - f_n(t)) = 1 + \sinh^2(\mu)\mathrm{sech}^2(t\sinh\mu - \mu n + \nu), \tag{2.6}$$

for some constant wavenumber μ, and phase shift ν in e^{ϕ}, related to the phase shift η in f_n. By symmetry, $\phi_n(-t)$ is also a solution. This sech^2 shape occurs in the KdV equation as well as the Boussinesq equation. In the limit of small amplitude, that is $\mu \ll 1$, the wave is wide and travels close to the limiting speed of $c_0 = 1$.

The pure Toda system [6] can also be derived from the system

$$\begin{aligned} \widehat{p}_t(x,t) &= \gamma_0 \left[\exp u(x+\tfrac{1}{2},t) - \exp u(x-\tfrac{1}{2},t)\right], \\ u_t(x,t) &= \gamma_0 \left[\widehat{p}(x+\tfrac{1}{2},t) - \widehat{p}(x-\tfrac{1}{2},t)\right], \end{aligned} \tag{2.7}$$

by differentiating the latter with respect to t to eliminate $\widehat{p}$, yielding (2.4).

2.2. *Small amplitude asymptotic expansion of the pure Toda system*

Equation (2.7) can be approximated using the asymptotic expansion

$$y = \epsilon x, \quad \tau = \epsilon t, \quad u(x,t) = \overline{u} + \epsilon^2 U(y,\tau), \quad \widehat{p}(x,t) = \overline{p} + \epsilon^2 P(y,\tau), \tag{2.8}$$

in which we perform a weakly nonlinear expansion of both u and p about constant solutions $u(x,t) = \overline{u}$ and $p(x,t) = \overline{p}$ to obtain

$$\epsilon^3 P_\tau = \gamma_0 e^{\overline{u}} \left[\epsilon^3 U_y + \tfrac{1}{24}\epsilon^5 U_{yyy} + \epsilon^5 U U_y \right], \tag{2.9}$$

$$\epsilon^3 U_\tau = \gamma_0 \left[\epsilon^3 P_y + \tfrac{1}{24}\epsilon^5 P_{yyy} \right]. \tag{2.10}$$

Eliminating P by differentiating (2.9) with respect to z and (2.10) with respect to τ, and simplifying, yields

$$\gamma_0^{-2} e^{-\overline{u}} U_{\tau\tau} = U_{yy} + \tfrac{1}{12}\epsilon^2 U_{yyyy} + \epsilon^2 (U U_y)_y, \tag{2.11}$$

which is the Boussinesq equation. This PDE is completely integrable, and has pulse soliton solutions of the form

$$U(y,\tau) = \frac{3(c^2 - \gamma_0^2 e^{\overline{u}})}{\epsilon 2 \gamma_0^2 e^{\overline{u}}} \operatorname{sech}^2 \left((y - (c\gamma_0^{-1} e^{-\overline{u}/2})\tau) \sqrt{\frac{3(c^2 - \gamma_0^2 e^{\overline{u}})}{\epsilon^2 \gamma_0^2 e^{\overline{u}}}} \right). \tag{2.12}$$

In order for the width and height of this soliton to be $O(1)$ in ϵ, we require $c \sim \pm\gamma_0 e^{\overline{u}/2} + O(\epsilon^2)$.

Above we have obtained the standard Boussinesq equation, which by rescaling can be written as

$$u_{tt} = u_{xx} + u_{xxxx} + 6(u^2)_{xx}. \tag{2.13}$$

The reason for our derivation of the Boussinesq equation (2.13) is to understand the behaviour of the Toda system in the long wavelength limit ($k \ll 1$); we are interested in the form of slowly-varying solutions and solitary waves; we are not concerned with any ill-posedness issues caused by considering larger k.

2.3. *Reduction to the Korteweg-de-Vries equation*

We now return to equations (2.9)–(2.10) and focus on just one speed of travel and substitute

$$P(y,\tau) = \widehat{P}(z,T), \quad U(y,\tau) = \widehat{U}(z,T), \quad z = y - c\tau, \quad T = \epsilon^2 \tau, \tag{2.14}$$

so as to transform the problem to a moving coordinate frame where the speed c will be chosen strategically to simplify the ensuing analysis. The scaling of the new time variable T means that we are now considering much longer timescales than previously: taking $\tau = O(1)$ as above implies $t = O(\epsilon^{-1})$, whereas taking $T = O(1)$ means that $t = O(\epsilon^{-3})$. Hence

$$\begin{pmatrix} c & \gamma_0 \\ \gamma_0 & c e^{-\overline{u}} \end{pmatrix} \begin{pmatrix} \widehat{U}_z \\ \widehat{P}_z \end{pmatrix} = \epsilon^2 \begin{pmatrix} \widehat{U}_T - \frac{1}{24}\gamma_0 \widehat{P}_{zzz} \\ e^{-\overline{u}} \widehat{P}_T - \frac{1}{24}\gamma_0 \widehat{U}_{zzz} - \gamma_0 \widehat{U}\widehat{U}_z \end{pmatrix} \tag{2.15}$$

Here c can be viewed as an eigenvalue, and for each eigenvalue, the system has a non-trivial $O(1)$ solution. The eigenvalues are given by the determinant of the matrix being zero, namely

$$c_\pm = \pm\gamma_0 e^{\overline{u}/2}, \tag{2.16}$$

with the corresponding eigenvectors being $\mathbf{v}_+ = (\widehat{U}_z, \widehat{P}_z)^\top = (-1, e^{\overline{u}/2})^\top$ and $\mathbf{v}_- = (\widehat{U}_z, \widehat{P}_z)^\top = (1, e^{\overline{u}/2})^\top$. Since the matrix in (2.15) is singular, its range $\mathcal{R}$ is only a subset of $\mathbb{R}^2$, in particular for c_+, $\mathcal{R}_+ = \lambda(e^{\overline{u}/2}, 1)^T$ and for c_-, $\mathcal{R}_- = \lambda(-e^{\overline{u}/2}, 1)^T$.

For these particular speeds, the two equations are identical at leading order, namely $\partial_z(c\widehat{U} + \gamma_0 \widehat{P}) = 0$. Hence $P(z,T) = \mu(T) - (c/\gamma_0)U(z,T)$. Requiring the RHS of (2.15) to be in the range of the matrix implies

$$\frac{\widehat{U}_T - \frac{1}{24}\gamma_0 \widehat{P}_{zzz}}{e^{-\overline{u}} \widehat{P}_T - \frac{1}{24}\gamma_0 \widehat{U}_{zzz} - \gamma_0 \widehat{U}\widehat{U}_z} = \frac{c}{\gamma_0}, \tag{2.17}$$

which, together with $P = \mu - (c/\gamma_0)U$, yields

$$2U_T + \tfrac{1}{12}cU_{zzz} + cUU_z = \frac{c\mathrm{e}^{-\overline{u}}}{\gamma_0}\mu_T(T). \tag{2.18}$$

This has the form of a perturbed KdV equation, with a general time-dependent, although spatially uniform, forcing term. The equation (2.18) can be mapped onto the standard KdV equation by the transformation

$$U \mapsto U + g(T), \quad z \mapsto z + s(T), \quad g(T) = \frac{c\mathrm{e}^{-\overline{u}}\mu(T)}{2\gamma_0}, \quad s(T) = \frac{c}{2}\int^T g(T')\mathrm{d}T'. \tag{2.19}$$

Transformations of equations of the form (2.18) to the KdV equation have been given many times previously, for example, in [5, 8, 9].

2.4. Reformulation of the extended Toda lattice

In Gordoa *et al.* [1], an integrable generalisation of the Toda lattice is derived, given by

$$\begin{aligned}
\begin{pmatrix} p_t(x,t) \\ q_t(x,t) \end{pmatrix} + \begin{pmatrix} \beta_1 \\ 0 \end{pmatrix} \;=\; & \begin{pmatrix} -\kappa p(x,t)p_x(x,t) - q(x+1,t)s(x+2,t) + q(x,t)s(x,t) \\ -\kappa q(x,t)[p_x(x,t) + p_x(x-1,t)] - q(x,t)[p(x,t)s(x+1,t) - p(x-1,t)s(x,t)] \end{pmatrix} \\
& + \gamma_{-1}\begin{pmatrix} -p(x,t)[q(x+1,t) - q(x,t)] - q(x+1,t)p(x+1,t) + q(x,t)p(x-1,t) \\ -q(x,t)[q(x+1,t) - q(x-1,t)] - q(x,t)[p(x,t)^2 - p(x-1,t)^2] \end{pmatrix} \\
& + \gamma_0\begin{pmatrix} q(x+1,t) - q(x,t) \\ q(x,t)[p(x,t) - p(x-1,t)] \end{pmatrix} + \beta_0\begin{pmatrix} p(x,t) \\ 2q(x,t) \end{pmatrix} \\
& + \beta_{-1}\begin{pmatrix} -p(x,t)^2 - 2(x+1)q(x+1,t) + 2(x-1)q(x,t) \\ -q(x,t)[p(x,t) + p(x-1,t)] - q(x,t)[2xp(x,t) - 2(x-1)p(x-1,t)] \end{pmatrix}.
\end{aligned} \tag{2.20}$$

where $\gamma_{-1}, \gamma_0, \beta_0, \beta_1, \beta_{-1}$ are arbitrary functions of t, and the function $s(x,t)$ is determined by $s(x+1,t) - s(x,t) = \kappa q_x(x,t)/q(x,t)$.

Whilst there are many integrable evolution equations with a continuous spatial variable x (eg KdV, NLS, SG), and also many examples where the second independent variable is discrete (eg Toda, Volterra), there are few integrable evolution equations which involve both derivatives and discrete differences in space. Hence any discovery of such a system is noteworthy, and determining its relationship to other integrable systems is interesting. The basic Toda lattice can be obtained by putting $\gamma_{-1} = \kappa = 0 = \beta_1 = \beta_0 = \beta_{-1}$ and γ_0 constant, which implies $s = s_0$ (constant), thus leaving, on the right hand side of this equation, only the γ_0 term, but with γ_0 replaced by $\gamma_0 - s_0$. As we have seen above, this lattice equation is related to both the Boussinesq and the KdV equations. The purpose of this paper is to investigate whether including extra terms yields connections to other systems, and/or destroys the relationship with Bq and KdV.

The extended Toda system that we are interested in is derived by Gordoa *et al.* [1] eq (3.53) with $\gamma_{-1} = 0$ and $\beta_{-1} = 0$ and has the potential

$$s(x+1,t) - s(x,t) = \kappa q_x(x,t)/q(x,t), \tag{2.21}$$

with the other governing equations being

$$\begin{aligned}
p_t(x,t) \;=\; & \beta_0 p(x,t) - \beta_1 + q(x,t)s(x,t) - q(x{+}1,t)s(x{+}2,t) \\
& -\kappa p(x,t)p_x(x,t) + \gamma_0(q(x{+}1,t) - q(x,t)),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
q_t(x,t) \;=\; & -\kappa q(x,t)[p_x(x,t) + p_x(x-1,t)] + \gamma_0 q(x,t)[p(x,t) - p(x-1,t)] \\
& +2\beta_0 q(x,t) - q(x,t)[p(x,t)s(x+1,t) - p(x-1,t)s(x,t)].
\end{aligned} \tag{2.23}$$

These equations can be transformed to a more symmetric form by subtle changes of variables; hence we introduce the new variables

$$\widehat{s}(x,t) = s(x+\tfrac{1}{2},t), \quad u(x,t) = \log q(x,t), \quad \widehat{p}(x,t) = p(x-\tfrac{1}{2},t), \tag{2.24}$$

which are equivalent to

$$s(x,t) = \widehat{s}(x-\tfrac{1}{2},t), \qquad q(x,t) = \mathrm{e}^{u(x,t)}, \qquad p(x,t) = \widehat{p}(x+\tfrac{1}{2},t), \tag{2.25}$$

and imply

$$\widehat{s}(x+1,t) = \widehat{s}(x,t) + \kappa u_x(x+\tfrac{1}{2},t), \quad \widehat{s}(x-1,t) = \widehat{s}(x,t) - \kappa u_x(x-\tfrac{1}{2},t). \tag{2.26}$$

The problem can then be written as

$$\begin{aligned}
\widehat{p}_t(x,t) &= \beta_0\widehat{p}(x,t) - \beta_1 - \kappa\widehat{p}(x,t)\widehat{p}_x(x,t) - \kappa\partial_x\left[\exp(u(x-\tfrac{1}{2},t)) + \exp(u(x+\tfrac{1}{2},t))\right] \\
&\quad +(\gamma_0 - \widehat{s}(x,t))(\exp u(x+\tfrac{1}{2},t) - \exp u(x-\tfrac{1}{2},t)), && (2.27)\\
u_t(x,t) &= 2\beta_0 - \kappa\left[\widehat{p}_x(x+\tfrac{1}{2},t) + \widehat{p}_x(x-\tfrac{1}{2},t)\right] + \gamma_0\left[\widehat{p}(x+\tfrac{1}{2},t) - \widehat{p}(x-\tfrac{1}{2},t)\right] \\
&\quad -\left[\widehat{p}(x+\tfrac{1}{2},t)\widehat{s}(x+\tfrac{1}{2},t) - \widehat{p}(x-\tfrac{1}{2},t)\widehat{s}(x-\tfrac{1}{2},t)\right], && (2.28)\\
\kappa u_x(x,t) &= \widehat{s}(x+\tfrac{1}{2},t) - \widehat{s}(x-\tfrac{1}{2},t). && (2.29)
\end{aligned}$$

It is this form of the problem that we analyse in the following sections.

Note that taking $\kappa = 0 = \beta_0 = \beta_1$ causes (2.27)–(2.28) to be reduced to (2.7).

3. Reductions over the intermediate timescale

If β_1, β_0 and κ are small enough, then their effect will be negligible, and the extended Toda system will behave in the same manner as the basic Toda system, and we will have the same links with the Bq and KdV equations. If β_1, β_0 are much larger than γ_0 and κ we have $p_t = \beta_0 p - \beta_1$ and $u_t = 2\beta_0$, which gives linear growth in u and exponential growth or decay in p (with $p \to \beta_1/\beta_0$ as $t \to \infty$ if $\beta_0 < 0$). As we consider smaller magnitudes of β_1, β_0, we expect to find transitions in behaviour between this growth and decay and the more interesting dynamics that can be observed in Bq and KdV. Hence we analyse a sequence of magnitudes for β_1, β_0, and derive the corresponding generalised Bq, KdV equations, to see if these generalisations also exhibit integrability properties.

3.1. *The case $\beta_j = \mathcal{O}(\epsilon^3)$*

In addition to the asymptotic ansatz of (2.8), we introduce $\widehat{s}(x,t) = \overline{s} + \epsilon^2 S(y,\tau)$. We apply the same to the general system (2.27)–(2.29). Initially, we consider just (2.29), wherein (2.8) implies

$$\kappa\epsilon^3 U_y = \epsilon^2\left[S(y+\tfrac{1}{2}\epsilon,\tau) - S(y-\tfrac{1}{2}\epsilon,\tau)\right] \sim \epsilon^3 S_y + \tfrac{1}{24}\epsilon^5 S_{yyy}. \tag{3.1}$$

This equation can be integrated once straightaway; a constant of integration can be absorbed into $\overline{s}$. Whilst the constant $\overline{s}$ could be a function of time, t; for the sake of simplicity, we always choose $\overline{s}$ to be independent of time.

Since we require S in terms of U, we can approximately invert this equation using

$$S = \kappa\left(1 + \tfrac{1}{24}\epsilon^2\partial_y^2\right)^{-1} U \sim \kappa\left(1 - \tfrac{1}{24}\epsilon^2\partial_y^2\right) U = \kappa U - \tfrac{1}{24}\kappa\epsilon^2 U_{yy}. \tag{3.2}$$

Choosing the constant of integration to be zero implicitly assumes the boundary conditions that $U \to 0$ and $S \to 0$ as $y \to -\infty$.

We now consider equations (2.27), (2.28) and (2.8), writing

$$\beta_0 = \epsilon^3\beta, \qquad \beta_1 = \epsilon^3 B, \tag{3.3}$$

and hence we obtain

$$U_\tau = (\gamma_0 - 2\kappa - \overline{s})P_y + \tfrac{1}{24}\epsilon^2(\gamma_0 - 6\kappa - \overline{s})P_{yyy} + 2\beta - \kappa\overline{p}U_y - \kappa\epsilon^2(PU)_y, \tag{3.4}$$

$$P_\tau = \mathrm{e}^{\overline{u}}(\gamma_0 - 2\kappa - \overline{s})U_y + \tfrac{1}{24}\epsilon^2\mathrm{e}^{\overline{u}}(\gamma_0 - 6\kappa - \overline{s})U_{yyy} + (\beta\overline{p} - B) + \epsilon^2\mathrm{e}^{\overline{u}}(\gamma_0 - 3\kappa - \overline{s})UU_y + \beta\epsilon^2 P - \kappa\overline{p}P_y - \kappa\epsilon^2 PP_y. \tag{3.5}$$

From (3.4) and (3.5), together with the substitutions $z = y - \kappa\overline{p}\tau$, (whilst retaining τ) we obtain

$$\begin{aligned} U_\tau &= (\gamma_0-2\kappa-\overline{s})P_z + \tfrac{1}{24}\epsilon^2(\gamma_0-6\kappa-\overline{s})P_{zzz} + 2\beta - \kappa\epsilon^2(PU)_z, \qquad (3.6)\\ P_\tau &= (\gamma_0-2\kappa-\overline{s})\mathrm{e}^{\overline{u}}U_z + \tfrac{1}{24}\epsilon^2(\gamma_0-6\kappa-\overline{s})\mathrm{e}^{\overline{u}}U_{zzz} + (\beta\overline{p} - B) + \beta\epsilon^2 P - \kappa\epsilon^2 PP_z + \epsilon^2(\gamma_0-3\kappa-\overline{s})\mathrm{e}^{\overline{u}}UU_z. \qquad (3.7)\end{aligned}$$

We introduce the potential function ψ defined by $U = \psi_z$ and integrate with respect to z to find

$$\psi_\tau = (\gamma_0-2\kappa-\overline{s})P + 2\beta z + \mu(\tau) + \tfrac{1}{24}\epsilon^2(\gamma_0-6\kappa-\overline{s})P_{zz} - \kappa\epsilon^2 P\psi_z. \qquad (3.8)$$

We differentiate this with respect to τ and substitute in for P_τ from (3.7), eliminating the higher order occurrences of $\epsilon^2 P$ using the leading order approximation $(\gamma_0-2\kappa-\overline{s})P = \psi_\tau - 2\beta z - \mu(\tau)$. Redefining the unknown as $\phi = \psi - f(\tau)$ where $f'(\tau) = \mu(\tau)$ yields

$$\begin{aligned}\frac{\phi_{\tau\tau} - (\gamma_0-2\kappa-\overline{s})^2\mathrm{e}^{\overline{u}}\phi_{zz}}{\epsilon^2} &= \tfrac{1}{12}(\gamma_0-2\kappa-\overline{s})(\gamma_0-6\kappa-\overline{s})\mathrm{e}^{\overline{u}}\phi_{zzzz} + \tfrac{1}{2}(\gamma_0-2\kappa-\overline{s})(\gamma_0-3\kappa-\overline{s})\mathrm{e}^{\overline{u}}(\phi_z^2)_z\\ &\quad -\frac{\kappa(\phi_{\tau\tau}\phi_z + 2\phi_\tau\phi_{z\tau})}{(\gamma_0-2\kappa-\overline{s})} + \beta\phi_\tau - 2\beta^2 z + \frac{2\beta\kappa}{(\gamma_0-2\kappa-\overline{s})}(\phi_\tau + 2z\phi_{z\tau} - 2\beta z)\\ &\quad +(\gamma_0-2\kappa-\overline{s})(\beta\overline{p} - B). \qquad (3.9)\end{aligned}$$

In deriving this equation, we have used $\phi_{zz\tau\tau} = \phi_{zzzz}\mathrm{e}^{\overline{u}}(\gamma_0-2\kappa-\overline{s})^2 + O(\epsilon^2)$.

In equation (3.9), the first two lines correspond to terms in the standard potential Boussinesq equation, and the last three lines to the perturbations resulting from the terms in the extended Toda system.

Redefining the constant coefficients, this equation can be written as one of

$$\begin{aligned}\phi_{tt} &= \phi_{xx} + \phi_{xxxx} + (\phi_x^2)_x + b\phi_t - k(2\phi_t\phi_{xt} + \phi_{tt}\phi_x) + ax + cx\phi_{xt} + d, \qquad (3.10)\\ \phi_{tt} &= \phi_{xx} - \phi_{xxxx} + (\phi_x^2)_x + b\phi_t - k(2\phi_t\phi_{xt} + \phi_{tt}\phi_x) + ax + cx\phi_{xt} + d. \qquad (3.11)\end{aligned}$$

Applying the WTC Painlevé test [10] we find the conditions for integrability are $a = b = c = k = 0$, hence the only integrable cases are the potential Boussinesq equation.

Seeking similarity solutions of (3.10) leads to two cases: in the first, when $a = c = 0$ (which implies $b = 0$), we have travelling wave solutions of the form $\phi = \phi(z) + c_0 t$, $z = x - vt$ where $f = \phi'$ satisfies

$$(v^2 - 1 - 2kvc_0)f = f'' + (1 - \tfrac{3}{2}\kappa v^2)(f^2) + dz + K, \qquad (3.12)$$

with v and c_0 arbitrary. This equation has solutions in the form of elliptic functions and pulses (when $d = 0$) and the Painlevé transcendent P_I [11, 13] when $d \neq 0$. In the second case, where a and c are not both zero, there are 'stationary' solutions, where $\phi(x,t) = \phi(x) + c_0 t$, and $f = \phi'(x)$ satisfies

$$f'' + f + f^2 + \tfrac{1}{2}ax^2 + (bc_0 + d)x + K = 0. \qquad (3.13)$$

The requirement for this equation to pass the ARS Painlevé test [12] is $a = 0$. In this case (3.13) has solutions in the form of elliptic functions and pulses (when $bc_0 + d = 0$) and the Painlevé transcendent P_I [11, 13] when $bc_0 + d \neq 0$. Seeking similarity reductions of equation (3.11) leads to similar results.

The potential Boussinesq equation (3.9) describes the evolution of the extended Toda system on the $\tau = O(1)$ timescale, which corresponds to $t = O(\epsilon^{-1})$ in the original system. The expansions derived later in Section 3 will describe the evolution on significantly longer timescales, namely or $t = O(\epsilon^{-3})$ over which the behaviour is governed by the KdV equation.

3.2. The case $\beta_j = O(\epsilon^4)$

This case is very similar to the above, although we now write

$$\beta_0 = \epsilon^4\beta, \qquad \beta_1 = \epsilon^4 B. \qquad (3.14)$$

Hence, in place of (3.4) and (3.5) we obtain

$$U_\tau = (\gamma_0-2\kappa-\overline{s})P_z + \tfrac{1}{24}\epsilon^2(\gamma_0-6\kappa-\overline{s})P_{zzz} + 2\epsilon\beta - \kappa\epsilon^2(PU)_z, \tag{3.15}$$
$$P_\tau = (\gamma_0-2\kappa-\overline{s})e^{\overline{u}}U_z + \tfrac{1}{24}\epsilon^2(\gamma_0-6\kappa-\overline{s})e^{\overline{u}}U_{zzz} + \epsilon(\beta\overline{p} - B) - \kappa\epsilon^2 PP_z + \epsilon^2(\gamma_0-3\kappa-\overline{s})e^{\overline{u}}UU_z. \tag{3.16}$$

As above, we introduce the potential function $U = \psi_z$, and follow the same method as used to derive (3.9), to find

$$\frac{\phi_{\tau\tau} - (\gamma_0-2\kappa-\overline{s})^2 e^{\overline{u}}\phi_{zz}}{\epsilon^2} = \tfrac{1}{12}(\gamma_0-2\kappa-\overline{s})(\gamma_0-6\kappa-\overline{s})e^{\overline{u}}\phi_{zzzz} + \tfrac{1}{2}(\gamma_0-2\kappa-\overline{s})(\gamma_0-3\kappa-\overline{s})e^{\overline{u}}(\phi_z^2)_z - \frac{\kappa(\phi_{\tau\tau}\phi_z + 2\phi_\tau\phi_{z\tau})}{(\gamma_0-2\kappa-\overline{s})}, \tag{3.17}$$

which is identical to (3.9) with the β terms neglected. Rescaling the coefficients, this equation can be written in one of the two simpler forms

$$\phi_{tt} = \phi_{xx} + \phi_{xxxx} + (\phi_x^2)_x - k(2\phi_t\phi_{xt} + \phi_{tt}\phi_x), \tag{3.18}$$
$$\phi_{tt} = \phi_{xx} - \phi_{xxxx} + (\phi_x^2)_x - k(2\phi_t\phi_{xt} + \phi_{tt}\phi_x). \tag{3.19}$$

As with equation (3.10), applying the WTC Painlevé test [10] yields that the only integrable cases of these equations are the potential Boussinesq equations, where $k = 0$.

These equations again have travelling wave solutions. Writing $\phi = \phi(z) + c_0 t$, $z = x - vt$, $\phi' = f$, in (3.18), we find

$$(v^2 - 1 - 2kvc_0)f = f'' + (1 - \tfrac{3}{2}\kappa v^2)f^2 + K, \tag{3.20}$$

which has solutions in the form of elliptic functions and pulse-solitons for f, which correspond to kinks for ϕ. In this last case, for example when $K = 0$, the solution has the form

$$f = \frac{3(1 - v^2 + 2kvc_0)}{(3kv^2 - 2)}\operatorname{sech}^2\left(\tfrac{1}{2}z\sqrt{v^2 - 2kvc_0 - 1}\right). \tag{3.21}$$

This wave exists for v satisfying $v > kc_0 + \sqrt{1 + k^2c_0^2}$, or $v < kc_0 - \sqrt{1 + k^2c_0^2}$. Hence the presence of k modifies the range of speeds that such waves exist. Also the factor of $3kv^2 - 2$ in the denominator of the amplitude means waves of positive and negative elevation are possible. Similar results are obtained when seeking similarity reductions of (3.19).

Since equations (3.17) and (3.20) are already independent of β, considering β_0, β_1 to be smaller than $O(\epsilon^4)$ will not result in any new equations to be derived. Hence we do not consider $\beta_0, \beta_1 = O(\epsilon^5, \epsilon^6)$ and instead turn to $\beta_0, \beta_1 = O(\epsilon^2)$.

3.3. *The case $\beta_j = O(\epsilon^2)$ – multiple timescales expansion*

We now turn to consider larger magnitudes for β_0, β_1, albeit still small. We define

$$\beta_0 = \epsilon^2\beta, \qquad \beta_1 = \epsilon^2 B, \tag{3.22}$$

with $B, \beta = O(1)$; and also introduce new independent variables

$$\tau = \epsilon t, \qquad y = \epsilon x, \qquad T = \epsilon^2 t. \tag{3.23}$$

The time derivative ∂_t is replaced by $\epsilon\partial_\tau + \epsilon^2\partial_T$, and assume a multiple-scales ansatz for the dependent variables

$$\widehat{p} = \overline{p}(T) + \epsilon P(y,\tau,T), \qquad u = \overline{u}(T) + \epsilon U(y,\tau,T), \qquad \widehat{s} = \overline{s}(T) + \epsilon S(y,\tau,T). \tag{3.24}$$

Note that there are several differences between this case and that of $O(\beta^3)$ analysed in section 3.1; namely the amplitude of the spatially-dependent components of $\widehat{p}, u$ are larger, to be precise, $O(\epsilon)$ instead of $O(\epsilon^2)$; and that the background solutions, $\overline{p}, \overline{u}$ now vary on the very long timescale, T, instead of being constants.

Solving the equation for $\widehat{s}$ gives $S(y,\tau,T) = \kappa U(y,\tau,T) + O(\epsilon^2)$, in which the higher order correction terms can be neglected as they do not contribute to either the leading order balance or the first correction terms. As noted after equation (3.1), we choose $\overline{s}$ to be a constant, that is, independent of both x and t. The equation for $\widehat{p}$ yields

$$\overline{p}'(T) + P_\tau + \epsilon P_T = \beta\overline{p}(T) - B + \epsilon\beta P - \kappa\overline{p}(T)P_y - \epsilon\kappa PP_y + (\gamma_0-2\kappa-\overline{s})e^{\overline{u}}U_y + \epsilon(\gamma_0-3\kappa-\overline{s})e^{\overline{u}}UU_y, \tag{3.25}$$

while that for u gives

$$\overline{u}'(T) + U_\tau + \epsilon U_T = 2\beta + P_y(\gamma_0-2\kappa-\overline{s}) - \kappa\overline{p}(T)U_y - \epsilon\kappa(PU)_y. \tag{3.26}$$

Taking the spatially-independent parts of equations (3.25), (3.26) yields the ODEs

$$\overline{p}'(T) = \beta\overline{p}(T) - B, \qquad \overline{u}'(T) = 2\beta, \tag{3.27}$$

thus our spatially-uniform, but time-dependent background solution is

$$\overline{p}(T) = Ce^{\beta T} + B/\beta, \qquad \overline{u}(T) = 2\beta T + \overline{u}_0. \tag{3.28}$$

Since $\overline{u}_0$ can be removed by shifting the time variable, T, we take $\overline{u}_0 = 0$; however, it is important to retain both components in the solution, $\overline{p}(T)$.

Transforming (3.25)–(3.26) to a moving coordinate frame *via*

$$z = y - \kappa\overline{p}(T)\tau, \tag{3.29}$$

(whilst retaining τ as the faster time variable) yields the governing equations

$$P_\tau + \epsilon P_T = (\gamma_0-2\kappa-\overline{s})e^{\overline{u}(T)}U_z + \epsilon\beta P - \epsilon\kappa PP_z + \epsilon(\gamma_0-3\kappa-\overline{s})e^{\overline{u}(T)}UU_z, \tag{3.30}$$

$$U_\tau + \epsilon U_T = (\gamma_0-2\kappa-\overline{s})P_z - \epsilon\kappa(PU)_z. \tag{3.31}$$

We proceed to analyse this system of equations on each of the two timescales, $\tau = O(1)$, which corresponds to $t = O(\epsilon^{-1})$, and $T = O(1)$, which is relevant for $t = O(\epsilon^{-2})$.

3.4. *The case $\beta_j = O(\epsilon^2)$ – reduction to generalised Boussinesq equation*

For the shorter timescale, where $t = O(\epsilon^{-1})$ only the τ variable is relevant, and T-dependence can be ignored, leaving us with

$$P_\tau = (\gamma_0-2\kappa-\overline{s})e^{\overline{u}}U_z + \epsilon\beta P - \epsilon\kappa PP_z + \epsilon(\gamma_0-3\kappa-\overline{s})e^{\overline{u}}UU_z, \tag{3.32}$$

$$U_\tau = (\gamma_0-2\kappa-\overline{s})P_z - \epsilon\kappa(PU)_z. \tag{3.33}$$

Following the analysis of Section 3.1, we expect this to yield a form of the Boussinesq equation.

Introducing the potential ψ defined by $U = \psi_z$ and by integrating (3.33) with respect to z, we find

$$\psi_\tau = (\gamma_0-2\kappa-\overline{s})P - \epsilon\kappa P\psi_z + \mu(\tau). \tag{3.34}$$

Rearranging (3.34) we obtain

$$P = \frac{\psi_\tau - \mu(\tau)}{(\gamma_0-2\kappa-\overline{s})} + \frac{\epsilon\kappa\psi_z(\psi_\tau - \mu(\tau))}{(\gamma_0-2\kappa-\overline{s})^2} + O(\epsilon^2), \tag{3.35}$$

which we substitute into (3.32). Retaining only leading order and $O(\epsilon)$ correction terms, we eventually obtain a single equation for ψ

$$\frac{\psi_{\tau\tau} - (\gamma_0-2\kappa-\overline{s})^2e^{\overline{u}}\psi_{zz}}{\epsilon} = (\gamma_0-2\kappa-\overline{s})(\gamma_0-3\kappa-\overline{s})e^{\overline{u}}\psi_z\psi_{zz} + \beta\psi_\tau - \beta\mu(\tau) + \mu'(\tau)/\epsilon + \frac{\kappa(2\mu(\tau)\psi_{z\tau} + \mu'(\tau)\psi_z - 2\psi_\tau\psi_{z\tau} - \psi_{\tau\tau}\psi_z)}{(\gamma_0-2\kappa-\overline{s})}. \tag{3.36}$$

Writing $\psi = \phi(z,\tau) + f(\tau)$ with $f'(\tau) = \mu(\tau)$ yields the same equation for ϕ, but with all the μ terms removed. This equation has some similarities with the standard Boussinesq equation, albeit not so many as (3.10). Due to the stronger β-forcing terms, the amplitude is larger and so the nonlinearity is stronger than the dispersion and so the higher derivative terms have been relegated to higher orders in ϵ. We still have the leading-order terms from the wave equation, and nonlinearities of the form $(\psi_z^2)_z$ as one would expect in the potential Boussinesq equation. However, in addition, there are other perturbation terms which change the form of the travelling wave solutions.

Seeking travelling wave solutions of (3.36) with $\mu = 0$ and $\psi = \phi(q)$, $q = z - v\tau$ yields an equation, which can be rewritten in the form

$$\tfrac{1}{2}\phi'^2 + \phi' + \phi + aq + b = 0. \tag{3.37}$$

When $a = 1$, this equation has solutions of the form

$$\phi = \tfrac{1}{2} - b - q - \tfrac{1}{2}(q + K)^2, \qquad \phi = \tfrac{1}{2} - b - q. \tag{3.38}$$

Here, the first solution is a 'general solution', whilst the latter is a singular, or 'envelope' solution, which occurs due to the equation (3.37) not satisfying the criteria for uniqueness of solutions. In particular, $\phi' = F(\phi, q)$ is not Lipschitz continuous in ϕ.

For $a \neq 1$, equation (3.37) has general solutions of the form

$$\phi = a(1 - q) - b + (a - 1)L - \tfrac{1}{2}(a + (a - 1)L)^2, \quad \text{where } L = W\left(-\frac{1}{(a-1)}\exp\left(\frac{q - a - K}{a - 1}\right)\right), \tag{3.39}$$

where $W(x)$ is Lambert's W function [13], which satisfies $We^W = x$. The singular solution of (3.37) is $\phi = a(1 - q - a/2) - b$. There are no similarity reductions of (3.37) other than these travelling wave solutions.

3.5. The case $\beta_j = O(\epsilon^1)$

For this case we write $\beta_0 = \epsilon\beta$, $\beta_1 = \epsilon B$, with $\beta, B = O(1)$. As above, we use the scalings given by (3.23) and now generalise (3.24) to

$$\widehat{p} = \overline{p}(\tau, T) + \epsilon P(y, \tau, T), \qquad u = \overline{u}(\tau, T) + \epsilon U(y, \tau, T), \qquad \widehat{s} = \overline{s}(\tau, T) + \epsilon S(y, \tau, T), \tag{3.40}$$

Taking $\overline{s}$ to be constant, these assumptions result in

$$P_\tau + \kappa\overline{p}(\tau, T)P_y = \beta P + (\gamma_0 - 2\kappa - \overline{s})e^{\overline{u}}U_y - \overline{p}_T - \kappa\epsilon PP_y + \epsilon e^{\overline{u}}(\gamma_0 - 3\kappa - \overline{s})UU_y - \epsilon P_T \tag{3.41}$$

$$U_\tau + \kappa\overline{p}(\tau, T)U_y = (\gamma_0 - 2\kappa - \overline{s})P_y - \kappa\epsilon(PU)_y - \overline{u}_T - \epsilon U_T. \tag{3.42}$$

Firstly, we consider the behaviour on the slower of the two timescales, namely τ; hence we ignore T, and transform to a moving frame of reference given by

$$z = y - \sigma(\tau), \qquad \text{with} \quad \sigma(\tau) = \kappa B\tau/\beta + \kappa Ce^{\beta\tau}/\beta + C_2, \tag{3.43}$$

so that $\sigma'(\tau) = \kappa\overline{p}(\tau, T)$. Then

$$P_\tau = \beta P + (\gamma_0 - 2\kappa - \overline{s})e^{2\beta\tau}U_z - \kappa\epsilon PP_z + \epsilon e^{2\beta\tau}(\gamma_0 - 3\kappa - \overline{s})UU_z \tag{3.44}$$

$$U_\tau = (\gamma_0 - 2\kappa - \overline{s})P_z - \kappa\epsilon(PU)_z. \tag{3.45}$$

We now introduce a potential function defined by $\psi_z = U$, whereupon including terms of $O(\epsilon)$, equation (3.45) implies

$$P = \frac{\psi_\tau - \mu(\tau)}{(\gamma_0 - 2\kappa - \overline{s}) - \epsilon\kappa\psi_z} \sim \frac{\psi_\tau - \mu(\tau)}{(\gamma_0 - 2\kappa - \overline{s})} + \frac{\epsilon\kappa\psi_z(\psi_\tau - \mu(\tau))}{(\gamma_0 - 2\kappa - \overline{s})^2}, \tag{3.46}$$

the last expression being due to $\epsilon \ll 1$. Writing $\psi = \phi + f(\tau)$ with $f' = \mu$, and introducing $\widehat{\tau} = e^{\beta\tau}$ and $\widehat{z} = \beta z/(\gamma_0 - 2\kappa - \overline{s})$, yields, to $O(\epsilon)$,

$$\phi_{\widehat{\tau}\widehat{\tau}} - \phi_{\widehat{z}\widehat{z}} = \frac{\beta\epsilon}{(\gamma_0 - 2\kappa - \overline{s})^2}\left[(\gamma_0 - 3\kappa - \overline{s})\phi_{\widehat{z}}\phi_{\widehat{z}\widehat{z}} - \kappa\left(2\phi_{\widehat{\tau}}\phi_{\widehat{z}\widehat{\tau}} + \phi_{\widehat{z}}\phi_{\widehat{\tau}\widehat{\tau}}\right)\right]. \tag{3.47}$$

For ease of writing, we recast this equation as

$$u_{tt} = u_{xx} + au_xu_{xx} + 2bu_tu_{xt} + bu_xu_{tt}. \tag{3.48}$$

This equation only admits two Lie symmetry reductions, namely (i) the travelling wave $u(x,t) = v(z)$ where $z = x - ct$, which yields either

$$v''(z) = 0, \qquad \text{or} \quad v'(z) = \frac{c^2 - 1}{a + 3bc^2}; \tag{3.49}$$

or, (ii) the similarity solution $u = tv(z)$ where $z = x/t$, which yields either

$$v''(z) = 0, \qquad \text{or} \quad v'(z) = \frac{z^2 - 1}{a^2 + 3b^2z^2 - 2bz}. \tag{3.50}$$

All the ODEs in (3.49)–(3.50) are easily integrated.

3.6. Summary

We have considered a variety of magnitudes for the β-parameters, and for each scale, we have shown that the system can be reduced to a generalised wave equation. For smaller β values, this has the form of a (generalised) Boussinesq equation, whilst at larger magnitudes, the dispersion term is dropped and other forcing terms become significant as the amplitude of the resulting evolving disturbance increases from $O(\epsilon^2)$ to $O(\epsilon)$.

In many cases the systems (3.9), (3.17), (3.36) permits travelling waves, whose shape evolves over longer timescales, as we shall investigate next. However, in the case $\beta = O(\epsilon)$, the equation (3.47) retains a strong dependence on the slower timescale, and preventing analysis. Although at $\beta = O(\epsilon^4)$, the β terms have no influence on the evolution of disturbances on the $\tau = O(1)$ timescale, which corresponds to $t = O(\epsilon^{-1})$, we find that when longer timescales are considered, that is $t = O(\epsilon^{-2})$ or longer, the β terms are significant.

4. Reductions over the long timescale

We now consider the longer timescales and, using asymptotic techniques, we show that the evolution of small amplitude excitations in the extended Toda lattice reduce to the KdV equation. As in the section above, we make assumptions on the sizes of the β_j parameters in the extended Toda system, and show how they affect the resulting reduced equations.

4.1. The case $\beta_j = O(\epsilon^3)$

We now transform (3.4)–(3.5) to a travelling wave coordinate (z), and a longer timescale (T) *via* (2.14), which yields

$$\begin{pmatrix} c - \overline{p}\kappa & e^{\overline{u}}(\gamma_0 - \overline{s} - 2\kappa) \\ \gamma_0 - \overline{s} - 2\kappa & c - \overline{p}\kappa \end{pmatrix} \begin{pmatrix} P_z \\ U_z \end{pmatrix} = \begin{pmatrix} B - \beta\overline{p} \\ -2\beta \end{pmatrix} + \epsilon^2 \begin{pmatrix} P_T - \beta P + \kappa P P_z - \frac{1}{24} e^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa) U_{zzz} - e^{\overline{u}}(\gamma_0 - \overline{s} - 3\kappa) U U_z \\ U_T - \frac{1}{24}(\gamma_0 - \overline{s} - 6\kappa) P_{zzz} + \kappa (PU)_z \end{pmatrix}. \tag{4.1}$$

As in the case of the reduction of the pure Toda system in Section 2.3, in order to obtain a single scalar equation describing the system, we now require the matrix on the LHS of (4.1) to be singular. For this matrix above, which we refer to as $\mathbf{M}$, to be singular, we require the speed c to take one of two values, namely

$$c_\pm = \overline{p}\kappa \pm e^{\overline{u}/2}(\gamma_0 - \overline{s} - 2\kappa). \tag{4.2}$$

Given a value for c, the singular matrix represents a projection, and its image being a one-dimensional subspace of $\mathbb{R}^2$. That the RHS of (4.1) has to lie in the image subspace is one condition relating the functions U and P (from the $O(\epsilon^2)$ terms), and a condition relating the constants $B, \beta, \overline{p}$ (from the $O(1)$ terms).

For this given RHS, since the system (4.1) is singular, it is in effect a single equation, which we then aim to solve to give a second relationship between P and U, which provides the reduction of the Toda system to a single scalar equation.

4.1.1. The case $\beta_j = O(\epsilon^3)$, $c = c_+$

In this case the range of the matrix in (4.1) is $\lambda(\mathrm{e}^{\overline{u}/2}, 1)^T$ for any $\lambda \in \mathbb{R}$. Requiring the RHS of (4.1) to lie in this subspace yields the condition

$$\overline{p} = \frac{B}{\beta} + 2\mathrm{e}^{\overline{u}/2}. \tag{4.3}$$

We rewrite the unknowns P, U in terms of the eigenvector of the matrix and a vector orthogonal to it, that is, we put

$$\begin{pmatrix} P_z \\ U_z \end{pmatrix} = \phi(z,T)\begin{pmatrix} \mathrm{e}^{\overline{u}/2} \\ -1 \end{pmatrix} + \psi(z,T)\begin{pmatrix} 1 \\ \mathrm{e}^{\overline{u}/2} \end{pmatrix}, \tag{4.4}$$

where ϕ, ψ are introduced to describe the z and T dependence of the system in place of P and U. At leading order, and using (4.3), both components of equation (4.1) become

$$(\gamma_0 - \overline{s} - 2\kappa)(1 + \mathrm{e}^{\overline{u}})\psi(z,T) = -2\beta, \tag{4.5}$$

and ϕ remains undetermined. From (4.5)

$$\frac{P_z + \mathrm{e}^{\overline{u}/2}U_z}{(1+\mathrm{e}^{\overline{u}})} = \psi = \frac{-2\beta}{(\gamma_0 - \overline{s} - 2\kappa)(1+\mathrm{e}^{\overline{u}})}, \tag{4.6}$$

and integrating with respect to z yields

$$P(z,T) = \mu(T) - \mathrm{e}^{\overline{u}/2}U(z,T) - \frac{2\beta z}{(\gamma_0 - \overline{s} - 2\kappa)}. \tag{4.7}$$

Now we require that the $O(\epsilon^2)$ terms on the RHS of (4.1) lie in the range of the matrix, that is,

$$\mathrm{e}^{\overline{u}/2}\left[U_T - \tfrac{1}{24}(\gamma_0 - 6\kappa - \overline{s})P_{zzz} + \kappa(PU)_z\right] = P_T - \beta P + \kappa P P_z - \tfrac{1}{24}\mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 3\kappa)UU_z, \tag{4.8}$$

into which (4.7) can be inserted to obtain a final reduced governing equation

$$\begin{aligned} &2\mathrm{e}^{\overline{u}/2}U_T + \tfrac{1}{12}\mathrm{e}^{\overline{u}}(\gamma_0 - 6\kappa - \overline{s})U_{zzz} + \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)UU_z \\ &= \mu_T(T) - \frac{\beta\mu(T)(\gamma_0 - \overline{s})}{(\gamma_0 - 2\kappa - \overline{s})} + \frac{2\beta^2 z(\gamma_0 - \overline{s})}{(\gamma_0 - 2\kappa - \overline{s})^2} + \frac{\beta\mathrm{e}^{\overline{u}/2}U(\gamma_0 - \overline{s} + 2\kappa)}{(\gamma_0 - \overline{s} - 2\kappa)} + \frac{4\beta\kappa\mathrm{e}^{\overline{u}/2}zU_z}{(\gamma_0 - 2\kappa - \overline{s})} - 2\kappa\mathrm{e}^{\overline{u}/2}\mu(T)U_z. \end{aligned} \tag{4.9}$$

The LHS of this equation is the standard KdV equation, whilst the terms on the RHS are all perturbations. However, and somewhat surprisingly, it is possible to transform the above equation onto the KdV equation. The transformation described by Popovych and Vaneeva [8], shows that equations of the form eqn (4) of [8], namely

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)x + l(t) = 0, \tag{4.10}$$

can be mapped onto the KdV equation only if their condition (6) holds; this is

$$s_t = 2gs^2 - 3qs + \frac{fk}{g}, \quad \text{where} \quad s := \frac{2q - h}{g} + \frac{f_t g - f g_t}{f g^2}. \tag{4.11}$$

In our case the condition on s is met, so (4.9) is mapped onto KdV.

4.1.2. The case $\beta_j = O(\epsilon^3)$, $c = c_-$

In this case the range of the matrix in (4.1) is $\lambda(\mathrm{e}^{\overline{u}/2}, -1)^\top$ for any λ. Requiring the RHS of (4.1) to lie in this subspace yields

$$\overline{p} = \frac{B}{\beta} - 2\mathrm{e}^{\overline{u}/2}, \tag{4.12}$$

We write

$$\begin{pmatrix} P_z \\ U_z \end{pmatrix} = \phi(z,T)\begin{pmatrix} \mathrm{e}^{\overline{u}/2} \\ 1 \end{pmatrix} + \psi(z,T)\begin{pmatrix} -1 \\ \mathrm{e}^{\overline{u}/2} \end{pmatrix}. \tag{4.13}$$

At leading order, from (4.1) we obtain

$$\psi(\gamma_0 - \overline{s} - 2\kappa)(1 + \mathrm{e}^{\overline{u}})\begin{pmatrix} \mathrm{e}^{\overline{u}}/2 \\ -1 \end{pmatrix} = 2\beta\begin{pmatrix} \mathrm{e}^{\overline{u}}/2 \\ -1 \end{pmatrix}, \tag{4.14}$$

hence $\psi = 2\beta/(1 + \mathrm{e}^{\overline{u}})(\gamma_0 - \overline{s} - 2\kappa)$. From (4.13), we obtain another equation for ψ, in terms of P, U, which can be integrated to show

$$P(z, T) = \mu(T) + \mathrm{e}^{\overline{u}/2}U(z, T) - \frac{2\beta z}{(\gamma_0 - \overline{s} - 2\kappa)}. \tag{4.15}$$

The final equation comes from substituting this expression for P into the condition that the $O(\epsilon^2)$ terms in (4.1) lie in the range of the matrix. This calculation yields

$$\begin{aligned} &2\mathrm{e}^{\overline{u}/2}U_T - \tfrac{1}{12}\mathrm{e}^{\overline{u}}(\gamma_0 - 6\kappa - \overline{s})U_{zzz} - \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)UU_z \\ = \quad & -\mu_T(T) + \frac{\beta\mu(T)(\gamma_0 - \overline{s})}{(\gamma_0 - 2\kappa - \overline{s})} - \frac{2\beta^2 z(\gamma_0 - \overline{s})}{(\gamma_0 - 2\kappa - \overline{s})^2} + \frac{\beta \mathrm{e}^{\overline{u}/2}U(\gamma_0 - \overline{s} + 2\kappa)}{(\gamma_0 - \overline{s} - 2\kappa)} + \frac{4\beta\kappa \mathrm{e}^{\overline{u}/2} z U_z}{(\gamma_0 - 2\kappa - \overline{s})} - 2\kappa \mathrm{e}^{\overline{u}/2}\mu(T)U_z. \end{aligned} \tag{4.16}$$

As with equation (4.9), this equation can be mapped onto KdV using the transformation of Popovich and Vaneeva [8].

4.2. The case $\beta_j = O(\epsilon^4)$

This case is very similar to the above, noting (3.14), in place of (4.1) we obtain

$$\begin{aligned} \begin{pmatrix} c - \overline{p}\kappa & \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 2\kappa) \\ \gamma_0 - \overline{s} - 2\kappa & c - \overline{p}\kappa \end{pmatrix}\begin{pmatrix} P_z \\ U_z \end{pmatrix} &= \epsilon\begin{pmatrix} B - \beta\overline{p} \\ -2\beta \end{pmatrix} \\ &+ \epsilon^2\begin{pmatrix} P_T + \kappa P P_z - \frac{1}{24}\mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 3\kappa)UU_z \\ U_T - \frac{1}{24}(\gamma_0 - \overline{s} - 6\kappa)P_{zzz} + \kappa(PU)_z \end{pmatrix}. \end{aligned} \tag{4.17}$$

Hence we have the same conditions for c, namely (4.2), and the same equations for $\overline{p}$, namely (4.3) and (4.12). The final KdV-type equations are slightly simpler than those quoted above, since smaller values for the parameters β_0, β_1 mean that some terms are small enough to be ignored at $O(\epsilon^2)$ in (4.1).

Hence for the larger speed, c_+, in place of (4.9), we obtain

$$2\mathrm{e}^{\overline{u}/2}U_T + \tfrac{1}{12}\mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} + \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)UU_z = \mu_T(T) - 2\kappa \mathrm{e}^{\overline{u}/2}\mu(T)U_z; \tag{4.18}$$

and for the smaller speed, c_-, in place of (4.16) we have

$$0 = 2U_T - \tfrac{1}{12}\mathrm{e}^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - (\gamma_0 - \overline{s} - 6\kappa)\mathrm{e}^{\overline{u}/2}UU_z + 2\kappa\mu(T)U_z + \mathrm{e}^{-\overline{u}/2}\mu_T(T). \tag{4.19}$$

As in the case of $\beta = O(\epsilon^3)$, both these equations are perturbed forms of KdV, although here, the perturbations are simpler. Both (4.18) and (4.19) can be mapped onto the standard KdV equation using transformations such as those described by Popovych and Vaneeva [8] and Pickering *et al.* [5].

4.3. The case $\beta_j = O(\epsilon^5)$

To obtain the relevant results in this case, we now write

$$\beta_0 = \epsilon^5\beta, \qquad \beta_1 = \epsilon^5 B, \tag{4.20}$$

alternatively, we can apply the transformation $\beta \mapsto \epsilon^2\beta$ and $B \mapsto \epsilon^2 B$ in (4.1). With this transformation, the entire RHS of (4.1) is $O(\epsilon^2)$. Hence, in place of (4.1), we obtain

$$\begin{pmatrix} c - \overline{p}\kappa & \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 2\kappa) \\ \gamma_0 - \overline{s} - 2\kappa & c - \overline{p}\kappa \end{pmatrix}\begin{pmatrix} P_z \\ U_z \end{pmatrix} = \epsilon^2\begin{pmatrix} B - \beta\overline{p} + P_T + \kappa P P_z - \frac{1}{24}\mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - \mathrm{e}^{\overline{u}}(\gamma_0 - \overline{s} - 3\kappa)UU_z \\ U_T - \frac{1}{24}(\gamma_0 - \overline{s} - 6\kappa)P_{zzz} + \kappa(PU)_z - 2\beta \end{pmatrix}. \tag{4.21}$$

We have the same solutions for c as in (4.2), and the leading order solutions for P, U are

$$P(z,T) = \mu(T) - \frac{(\gamma_0 - \overline{s} - 2\kappa)e^{\overline{u}}}{(c - \overline{p}\kappa)} U(z,T). \tag{4.22}$$

For c_+, requiring the RHS of (4.21) to be parallel to the range of the matrix, namely $(e^{\overline{u}/2}, 1)^T$ yields

$$\begin{aligned} 2U_T + \tfrac{1}{12}e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} + e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)UU_z \\ = \mu_T(T)e^{-\overline{u}/2} - 2\kappa\mu(T)U_z + 2\beta - e^{-\overline{u}/2}(\beta\overline{p} - B). \end{aligned} \tag{4.23}$$

This can be mapped onto the KdV equation by a suitable change of variables. For c_-, the range of the matrix is $(e^{\overline{u}/2}, -1)^T$, and the corresponding equation is

$$\begin{aligned} 2U_T - \tfrac{1}{12}e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)UU_z \\ = -\mu_T(T)e^{-\overline{u}/2} - 2\kappa\mu(T)U_z + 2\beta + e^{-\overline{u}/2}(\beta\overline{p} - B). \end{aligned} \tag{4.24}$$

Since the only differences between these two equations are sign changes, this last equation can also be mapped on to the KdV equation.

4.4. The case $\beta_j = O(\epsilon^6)$

This case is similar to the above, only now the β, B terms are even smaller, we put

$$\beta_0 = \epsilon^6\beta, \qquad \beta_1 = \epsilon^6 B, \tag{4.25}$$

The end result of this is that terms involving B, β can be completely neglected since they do not enter the leading order equations. Hence we observe KdV equations with fewer perturbing terms; however, the terms due to $\overline{s}$ and κ are still present.

$$\begin{pmatrix} c - \overline{p}\kappa & e^{\overline{u}}(\gamma_0 - \overline{s} - 2\kappa) \\ \gamma_0 - \overline{s} - 2\kappa & c - \overline{p}\kappa \end{pmatrix} \begin{pmatrix} P_z \\ U_z \end{pmatrix} = \epsilon^2 \begin{pmatrix} P_T + \kappa PP_z - \frac{1}{24}e^{\overline{u}}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - e^{\overline{u}}(\gamma_0 - \overline{s} - 3\kappa)UU_z \\ U_T - \frac{1}{24}(\gamma_0 - \overline{s} - 6\kappa)P_{zzz} + \kappa(PU)_z \end{pmatrix}. \tag{4.26}$$

For c_+, requiring the RHS of (4.26) to be parallel to the range of the matrix, namely $(e^{\overline{u}/2}, 1)^T$ yields

$$2U_T + \tfrac{1}{12}e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} + e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)UU_z = \mu_T(T)e^{-\overline{u}/2} - 2\kappa\mu(T)U_z. \tag{4.27}$$

For c_-, the corresponding equation is

$$2U_T - \tfrac{1}{12}e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)U_{zzz} - e^{\overline{u}/2}(\gamma_0 - \overline{s} - 6\kappa)UU_z = -\mu_T(T)e^{-\overline{u}/2} - 2\kappa\mu(T)U_z. \tag{4.28}$$

Again, the only differences between these two equations are subtle changes in sign. Both these equations can be mapped onto the KdV. These final equations are similar to those in Section 4.3, the only differences being the removal of β, B. Hence, taking β_0, β_1 to be smaller than $O(\epsilon^6)$ will simply result in these expressions again.

4.5. The case $\beta_j = O(\epsilon^2)$

We return to the full equations (3.30)–(3.31), namely

$$P_\tau + \epsilon P_T = (\gamma_0 - 2\kappa - \overline{s})e^{\overline{u}(T)}U_z + \epsilon\beta P - \epsilon\kappa PP_z + \epsilon(\gamma_0 - 3\kappa - \overline{s})e^{\overline{u}(T)}UU_z, \tag{4.29}$$

$$U_\tau + \epsilon U_T = (\gamma_0 - 2\kappa - \overline{s})P_z - \epsilon\kappa(PU)_z, \tag{4.30}$$

where it is now important to consider both the τ- and the T-dependence. We treat this as a system and the evolution on the long timescale is obtained from the Fredholm consistency criteria.

We seek travelling wave solutions of the form

$$U(z,\tau,T) = U(w,T), \qquad P(z,\tau,T) = P(w,T), \qquad w = z - v\tau, \tag{4.31}$$

which transforms (4.29)–(4.30) into a system of the form $\mathbf{Mu} = \epsilon\mathbf{b}$, specifically,

$$\begin{pmatrix} v & (\gamma_0 - 2\kappa - \overline{s})e^{\overline{u}} \\ (\gamma_0 - 2\kappa - \overline{s}) & v \end{pmatrix} \begin{pmatrix} P_w \\ U_w \end{pmatrix} = \epsilon \begin{pmatrix} P_T - (\gamma_0 - 3\kappa - \overline{s})e^{\overline{u}}UU_w - P(\beta - \kappa P_w) \\ U_T + \kappa(PU)_w \end{pmatrix}. \tag{4.32}$$

At leading order, where $\mathbf{Mu} = \mathbf{0}$, this system only has nontrivial solutions if $v = \pm(\gamma_0 - 2\kappa - \overline{s})e^{\overline{u}/2}$. We consider each case in turn.

4.5.1. The case $\beta_j = O(\epsilon^2)$, long timescale, and $v = +\mathrm{e}^{\overline{u}/2}(\gamma_0-2\kappa-\overline{s})$.

When $v = (\gamma_0 - 2\kappa - \overline{s})\mathrm{e}^{\overline{u}/2}$, solutions of $\mathbf{Mu} = \mathbf{0}$ lie in the kernel of the matrix $\mathbf{M}$, so must have the form $\mathbf{u} = \lambda(\mathrm{e}^{\overline{u}/2}, -1)^T$.

For nonzero RHSs of the matrix equation (4.32), we write the solution as

$$\begin{pmatrix} P_w \\ U_w \end{pmatrix} = \psi_w(w,T)\begin{pmatrix} \mathrm{e}^{\overline{u}/2} \\ -1 \end{pmatrix} + \phi_w(w,T)\begin{pmatrix} 1 \\ \mathrm{e}^{\overline{u}/2} \end{pmatrix}, \tag{4.33}$$

where ψ, ϕ are assumed to be $O(1)$. Here, the first vector is the zero-eigenvector of the matrix $\mathbf{M}$ so when $\mathbf{M}$ acts on (P_w, U_w) the product is the zero matrix. The second vector is simply the vector orthogonal to the zero-eigenvector; when $\mathbf{M}$ acts on this, the product is nonzero. Since the RHS of (4.32) is $O(\epsilon)$, we introduce a coefficient of ϵ in front of ϕ, but no such coefficient is needed in front of ψ. Hence, to leading order we take

$$P(w,T) = \mu(T) - \mathrm{e}^{\beta T} U. \tag{4.34}$$

We note that $\overline{u} = \overline{u}(T) = 2\beta T$, from (3.28).

The range of the matrix $\mathbf{M}$ is $\lambda(\mathrm{e}^{\overline{u}/2}, 1)^T$ for arbitrary parameter λ. Since (4.32) is a singular equations, the condition that (4.32) has nontrivial solutions is that the RHS lies in the range of the matrix $\mathbf{M}$. This condition implies

$$P_T + \kappa P P_w - \beta P - \mathrm{e}^{2\beta T}(\gamma_0-3\kappa-\overline{s})UU_w = \mathrm{e}^{\beta T} U_T + \kappa \mathrm{e}^{\beta T}(UP)_w, \tag{4.35}$$

which, using (4.34) and $U = \mathrm{e}^{-\beta T} Q$, can be simplified to

$$0 = 2Q_T + (\gamma_0-6\kappa-\overline{s})QQ_w + 2\kappa\mu(T)Q_w + 2\beta Q - \mu'(T) + \beta\mu(T). \tag{4.36}$$

4.5.2. The case $\beta_j = O(\epsilon^2)$, long timescale, and $v = -\mathrm{e}^{\overline{u}/2}(\gamma_0-2\kappa-\overline{s})$.

When $v = -(\gamma_0-2\kappa-\overline{s})\mathrm{e}^{\overline{u}/2}$, solutions of $\mathbf{Mu} = \mathbf{0}$ have the form $\mathbf{u} = \lambda(\mathrm{e}^{\overline{u}/2}, 1)^T$. For nonzero RHSs of the matrix equation (4.32), we write the solution as

$$P_w = -\mathrm{e}^{\beta T} U_w, \quad \text{hence} \quad P(w,T) = \mu(T) + \mathrm{e}^{\beta T} U(w,T). \tag{4.37}$$

The condition that the RHS of (4.32) is in the range of $\mathbf{M}$ implies

$$P_T - \beta P + \kappa P P_z - (\gamma_0-3\kappa-\overline{s})\mathrm{e}^{2\beta T} UU_w + \mathrm{e}^{\beta T} U_T + \mathrm{e}^{\beta T}\kappa(PU)_w = 0. \tag{4.38}$$

Using (4.37), together with $U = \mathrm{e}^{-\beta T} Q$, this equation can be reduced to

$$0 = 2Q_T - 2\beta Q - (\gamma_0-6\kappa-\overline{s})QQ_w + 2\kappa\mu(T)Q_w + \mu'(T) - \beta\mu(T). \tag{4.39}$$

Equation (4.39) can be solved by the method of characteristics; for example, given initial data of $Q(w,0) = Q_0(w)$, in the case $\mu(\tau) = 0$, the solution is given by the implicit form

$$Q(w,T) = \mathrm{e}^{2\beta T} Q_0\left(w - \frac{(\gamma_0-6\kappa-\overline{s})(1-\mathrm{e}^{-2\beta T})Q(w,T)}{2\beta}\right). \tag{4.40}$$

Equation (4.36) is closely related to (4.39), and can be solved by the same methods.

5. Conclusions

In this paper we have considered the first member of the extended (non-isospectral and differential-delay) Toda hierarchy. We have analysed the evolution of small amplitude disturbances around a spatially uniform solution using asymptotic techniques. We have considered a wide range of magnitudes for the β_0, β_1 parameters, and found that when these are small, the system is governed, to leading order, by the Boussinesq equation. On increasing these parameters the governing PDE, changes through a sequence of increasingly generalised Boussinesq equations, losing the highest derivative term, and gaining forcing/damping terms. We have outlined the forms of solutions of these equations.

Over the longer timescale, solutions of the extended Toda system are governed by the KdV equation. Although for some magnitudes the equation initially derived has many perturbing terms, we have shown that transformations exist which map the equation back onto the KdV.

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