THE EFFECTS OF HOLES AND FREE EDGES ON THE STRESS IN LAMINATED PLATES

BY

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To My Parents

SUMMARY

This work is concerned with the study of the mechanical behaviour of elastic laminated plates subjected to different boundary conditions. For the most part, each lamina is taken to be a fibre-reinforced material which contains a family of straight, continuously distributed fibres.

When the modulus for extension in the fibre direction of each lamina is large compared with the other moduli, the laminate is termed 'highly anisotropic' and in such cases, approximate solutions can be obtained by treating the individual laminae as 'ideal' materials in the sense that they are inextensible in the fibre direction and also incompressible. In the context of the plane strain bending of a laminated cantilever, we show that the theory for ideal materials predicts the occurrence of singular fibres at the lateral surfaces of the laminate and at the interfaces between the individual laminae. In a highly anisotropic cantilever these fibres correspond to regions of high stress and accordingly a boundary layer theory is developed for these regions. The boundary layer solution, together with the ideal solution, provide a good approximation to the description of the response of the cantilever, but it is found to be inadequate near the intersections of edges and interfaces, and at corners. A separate investigation is made into the asymptotic behaviour of the stress in these regions.

The major part of this thesis is concerned with the development of a general theory for laminated plates in stretching or bending. Given a laminate subject to specified boundary conditions, we define a single homogeneous equivalent plate which has material properties obtained by an appropriate averaging of the material properties of each lamina. The equivalent plate is subjected to the same boundary conditions as the laminate and the equivalent displacements are determined by classical thin plate theory. The theory then assumes that the displacement components in each lamina can be expressed as the sum of the equivalent displacements and correction displacements. The correction solutions satisfy the conditions of displacement and traction continuity across the inter-laminar boundaries and the condition that the lateral surfaces of the plate are free from traction. In the special case of the laminae being isotropic, the solutions given by the theory exactly satisfy the full three-dimensional equations of linear elasticity. When the equivalent displacements are known, the complete solution in each lamina is readily determined and this is illustrated by examples.

At the edges of the laminated plate, the prescribed boundary conditions are satisfied only in an average sense and therefore in these regions, an additional correction is required. The deviation of the calculated boundary condition from the specified boundary condition is used to determine the magnitude of this further correction.

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CHAPTER SIX: END EFFECTS ON A LAMINATED SEMI-INFINITE STRIP

ERRATUM

1) Page 12 : Equation $(1.3.5)_4$ should read

$$\nu_T = \frac{C_{11}C_{23} - C_{12}^2}{C_{11}C_{22} - C_{12}^2} = \frac{\lambda\beta' - (\lambda+\alpha)^2}{(\lambda+2\mu_T)\beta' - (\lambda+\alpha)^2}$$

2) Page 13 : Equation (1.3.7) should read

$$\beta = \frac{E_L^2(1-\nu_T^2) + E_T E_L\{1-2\nu_L(1+\nu_T)\} - E_T^2\nu_L^2}{(1+\nu_T)\{E_L(1-\nu_T) - 2E_T\nu_L^2\}} - 4\mu_L$$

3) Page 23 : Paragraph 1, eighth word should be "discontinuity".

4) Page 26 : Equation $(2.4.5)_3$ should read

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\sigma_{xy}}{\mu_L}$$

5) Page 29 : Paragraph 1, ".....we 'stretch' the y coordinates by the scaling factor ε (defined by (2.4.6)) and if ....."

6) Page 47 : Paragraph 1, ".....in our case, they coincide at the common interface."

7) Page 54 : "....in the region of the crack tip is O(Re(1/r)) which...."

8) Page 71 : Equation $(4.3.9)_3$ should read

$$\frac{\partial\sigma_{xz-1}}{\partial x} + \frac{\partial\sigma_{yz-1}}{\partial y} + \frac{\partial\sigma_{zz0}}{\partial z} = 0$$

9) Page 72 : "For isotropic materials we have

$$Q_{13} = \frac{Q_{12}}{2}, \quad Q_{44} = Q_{66}$$

10) Page 75 : Equation (4.3.27) should read

$$\begin{matrix} u_3 \\ v_3 \end{matrix} = \frac{(Q_{11}-Q_{13})}{Q_{44}}\left(\frac{z^3}{6} + B_1\frac{z^2}{2} + B_3 z + B_4\right)\begin{matrix} \partial/\partial x \\ \partial/\partial y \end{matrix}\;\nabla^2 w_0$$

11) Page 77 : Equation (4.3.36) should read

$$\begin{matrix} \sigma_{xx3} \\ \sigma_{yy3} \\ \sigma_{xy3} \end{matrix} = \frac{2Q_{66}}{Q_{44}}(Q_{11}-Q_{13})\left(\frac{z^3}{6} + B_1\frac{z^2}{2} + B_3 z + B_4\right)\begin{matrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ \partial^2/\partial x\partial y \end{matrix}\;\nabla^2 w_0$$

12) Page 82 : Paragraph 1, Line 2 "elastic constants".

13) Page 85 : Equation (4.6.5) should read

$$\hat{N}_x = \hat{H}\int_{-1}^{1}\hat{\sigma}_{xz}\,dz = -\hat{\varepsilon}^2\frac{2}{3}\hat{Q}_{11}\frac{\partial}{\partial x}\nabla^2\hat{w}$$

14) Page 160 : Equation $(5.9.2)_{1,2}$ should read

$$\phi_1' = \frac{1}{2(\hat{s}_1^2 - \hat{s}_2^2)} - \frac{i}{2(\hat{s}_1 - \hat{s}_2)(1+i\hat{s}_1)}\left\{1 - \frac{z_1}{\sqrt{z_1^2 - (1+\hat{s}_1^2)}}\right\}$$

$$\phi_2' = \frac{-1}{2(\hat{s}_1^2 - \hat{s}_2^2)} + \frac{i}{2(\hat{s}_1 - \hat{s}_2)(1+i\hat{s}_1)}\left\{1 - \frac{z_2}{\sqrt{z_2^2 - (1+\hat{s}_2^2)}}\right.$$

15) Page 166 : Line 7, third word should be "satisfied".

INTRODUCTION

Laminated materials have received considerable attention in recent years as a result of the search for materials which combine great strength and rigidity with lightness. Examples of this can be found in the carbon-epoxy composite parts of aircraft, where fibre-reinforced layers are arranged at various orientations to each other so as to give the laminate preferred directions of strength and stiffness.

For epoxy resin reinforced by strong fibres, the modulus for extension in the fibre direction is much larger than the moduli transverse to the fibres for extension or shear as as a consequence this material is referred to as being 'highly' anisotropic. This property can be idealized by the assumption that the material cannot extend at all in the fibre direction and, together with the assumption that the material is incompressible, forms the basis of the theory of 'ideal fibre-reinforced materials' first proposed by Mulhearn, Rogers and Spencer (1967) in the context of plastic deformation, and subsequently extended by Pipkin and Rogers (1971), Spencer (1972) and others to include all material response and for various classes of boundary value problems. This theory includes the possibility of certain fibres becoming 'singular' in the sense that they support infinite stress but finite load. Such singular fibres can be interpreted as thin layers in which the stresses are large and finite; this has been illustrated by Everstine and Pipkin (1973), who examine the plane strain bending of a highly anisotropic cantilever, Everstine and Pipkin (1971) and Spencer (1974). Some of the relevant theory is outlined in Chapter 1, where we also present results which are used in subsequent parts of this thesis. In Chapter 2 we develop the work of Everstine and Pipkin (1973) to account for highly anisotropic materials which are bonded together to form a laminated cantilever. When a comparison of the solutions for the field quantities in the highly anisitropic laminate is made with those in an ideal laminate, it is found that the predictions of ideal theory are in fact a first order approximation to what actually happens in a highly anisotropic laminate. The chapter is completed with a simple scheme which can be used to detect the possible initiation of delamination.

Motivated by the results of Chapter 2, we have considered the stress singularities which arise at the interface of dissimilar materials where traction boundary conditions are specified. Bonded quarter planes composed of different isotropic materials have been treated by Bogy (1968,1970) and then extended (Bogy 1971) to include two bonded wedges of arbitrary angle. Bogy (1972) and then Kuo and Bogy (1974) have also examined anisotropic, homogeneous wedges under various boundary and loading conditions and have included the problems of two identical orthotropic materials orientated at plus/minus angles with respect to their bond line. Work in this area has also been carried out by Ting and Chou (1981) and Delale (1984). In Chapter 3 we adopt an eigenvalue approach to calculate the order of the stress singularity at the apex of a wedge of arbitrary angle and which consists of two highly anisotropic materials. The asymptotic behaviour of the stress components in the vicinity of the apex is found to be $O(1/r^{\alpha + i\beta})$ where α and β are real and in the special case of a crack between two dissimilar materials, α takes the value 0.5.

In Chapters 4 and 5 we develop a theory for the stretching and bending of laminated plates. This forms the major portion of the thesis. In both the chapters we suppose that the layers of the laminate are transversely isotropic with the distinction that in Chapter 4 the preferred directions are normal to the plane of the laminate (from which the laminate of isotropic constituents can be obtained as a special case), whilst in Chapter 5 they lie in planes parallel to the plane of the laminate (as in a laminate of unidirectionally-reinforced layers).

Probably the simplest laminate theory is Classical Laminate Theory (CLT) which assumes that each layer is in a state of plane stress and that the in-plane displacement components in the entire laminate are linear in the through thickness co-ordinate. Although this theory is widely used, it exhibits important deficiencies. The characteristic which limits its validity in the description of the laminate response is that it neglects shear deformation implied by the Kirchoff-Love hypothesis and since a state of plane stress is assumed in the constitutive equations, it is not possible to calculate the inter-laminar stresses. However, the theory is reasonably accurate for plates which have a small aspect ratio (ratio of the half width to a typical in-plane length). The theory of Srinivas (1973) improves CLT by assuming that the in-plane displacements are piecewise linear in the through thickness co-ordinate and when it is applied to laminated plates, it predicts interlaminar shear stresses. However, these stress components are independent of the through thickness co-ordinate and as a consequence of this, the theory fails to satisfy the continuity conditions at the interfaces of the layers and the boundary conditions specified on the lateral surfaces. In an attempt to ascertain the magnitude of the interlamina-shear stresses in a laminate subject to stretching, Puppo and Evensen (1970) assume that the anisotropic layers of the laminate are in a state of plane stress and that they are bonded together by isotropic adhesives. They assume that in the isotropic layers the out-of-plane displacement component is negligible in comparison to the in-plane components and that these layers can develop only interlaminar shear stresses. These shear stresses are then evaluated by forming the product of the isotropic shear modulus with the magnitude of the discontinuity in the displacement components at the interface of the layers. The theory is found to predict the interlaminar shear stress components, but it fails to give any variation in them through the thickness of the laminate. Dispensing with the isotropic layers and bonding the anisotropic layers directly together, Pipes and Pagano (1970) develop a solution for the field quantities in the laminate by assuming that stress components depend only on one of the in-plane co-ordinates. This assumption reduces the set of simultaneous differential equations to depend on two space variables instead of three, but renders the theory unsuitable for structures of complex geometry. In a subsequent paper, Pagano (1980) assumes that the in-plane stress components are linear in the thickness co-ordinate and are the same as those given in two-dimensional laminate theory. By introducing the concept of 'layer equilibrium' all the continuity conditions and the shear stress free conditions on the lateral surfaces of the laminate are satisfied. The predictions of the theory are an improvement on the previous theories, but the number of layers that can be taken in the laminate are restricted by computational limitations. Tang (1975) and Tang and Levy (1975), following Reiss and Locke (1960), develop a boundary layer theory in which the CLT solution in the interior region is matched to a boundary region solution. However, no consideration is given to the possibility of the layers being highly anisotropic and we show in Chapter 5 that, for such materials, CLT cannot be taken to be the complete solution in the interior.

We consider a laminated plate comprising of an arbitrary number of laminae, each of which is of homogeneous elastic material with different elastic constants in each layer. The plate is subject to edge loading with traction free lateral surfaces. An 'equivalent' homogeneous elastic plate is defined as having material properties which are obtained by an appropriate averaging of the material properties of the separate laminae; the averaging in the bending theory is different from that in stretching. The equivalent plate is then subjected to specified in-plane loads and the relevant equivalent displacements are found by standard plate theory. We then consider the three-dimensional solution for each layer of the laminated plate when the laminate is subjected to the same loading as the equivalent plate. The displacement in each layer is expressed as the sum of the equivalent displacement and a correction displacement and the three-dimensional solution obtained is made to satisfy continuity of displacement and traction across the inter-laminar boundaries and the conditions of traction-free lateral surfaces of the plate. Furthermore, for the laminated plates described in Chapter 4, we show that the solution obtained is an <u>exact</u> solution of the <u>full</u> equations of three-dimensional elasticity. A notable feature of the theory is that when the equivalent plate solution is known, the correction solution is readily determined from it.

Finally, in the last chapter, we examine a laminated semi-infinite plate with the aim of obtaining an insight into the behaviour of the stress field in the vicinity of an edge. The problem is taken to be one in plane elasticity and throughout the chapter we adopt the term 'laminated strip' for the configuration. The laminated strip is subjected to specified tractions on one end, whilst the lateral surfaces are kept traction free. We restrict the analysis to isotropic materials and by using a technique developed by Hess (1969) we obtain the decay rates of the stress components in the immediate vicinity of the end. The analysis can be easily extended to account for anisotropic materials and in such cases can be used to determine the decay rate of the stress components at the end of the cantilever in Chapter 2.

CHAPTER ONE

CONSTITUTIVE EQUATIONS FOR LINEAR ELASTIC MATERIALS

1.1 INTRODUCTION

The mechanical behaviour of a fibre-reinforced composite material offers two levels of study. At one level the interaction between the fibres with the surrounding matrix is important and the elastic properties of the composite are determined from the elastic properties of the constituents. At another level no distinction is made between the fibre and matrix and the composite is treated as an anisotropic continuum. The elastic constants in this case are obtained from direct measurements on a sample of the composite. In this thesis we treat the fibre-reinforced material as an anisotropic continuum.

The results presented in this chapter and the problems to which they are applied are restricted to small deformations and hence to linear constitutive equations. In Section 1.2 we review the equations of linear elasticity and present them in cartesian and cylindrical polar co-ordinates. Following Spencer (1974) we state in Section 1.3, the constitutive equations for a solid reinforced by a single family of fibres which have a preferred local direction $\underset{\sim}{a}$, where $\underset{\sim}{a}$ is a unit vector. From the equations given in Section 1.3, we obtain, as special cases, in Sections 1.4 and 1.5, the constitutive equations for an ideal and an isotropic material respectively.

1.2 EQUATIONS OF LINEAR ELASTICITY

Let U_i denote the components of displacement at the point with cartesian co-ordinates x_i. For small deformations, the components of strain e_{ij} are related to the displacement components by

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad i,j = 1,2,3. \tag{1.2.1}$$

Equations (1.2.1) referred to cylindrical polar co-ordinates (r,θ,z) become

$$\begin{aligned}
e_{rr} &= \frac{\partial U_r}{\partial r},\\
e_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial U_\theta}{\partial \theta} + U_r\right),\\
e_{r\theta} &= \tfrac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right),\\
e_{\theta z} &= \tfrac{1}{2}\left(\frac{\partial U_\theta}{\partial z} + \frac{1}{r}\frac{\partial U_z}{\partial \theta}\right),\\
e_{rz} &= \tfrac{1}{2}\left(\frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z}\right),\\
e_{zz} &= \frac{\partial U_z}{\partial z},
\end{aligned} \tag{1.2.2}$$

where (U_r,U_θ,U_z) is the displacement field referred to the cylindrical co-ordinate system.

If the problem can be reduced to one of plane strain in the (x_1,x_2) plane say, only the strain components e_{11}, e_{12} and e_{22} are considered. In such cases, these three components satisfy the following strain compatibility condition

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = 2\,\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} \, . \tag{1.2.3}$$

Denote the stress components by σ_{ij}. In cartesian co-ordinates, these stress components satisfy the following equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j = 0, \quad i,j = 1,2,3 \tag{1.2.4}$$

where f_j are the components of an external body force and ρ is the material density. In the absence of body forces, Equations (1.2.4) in cylindrical polar co-ordinates are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0, \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0,
\end{aligned} \tag{1.2.5}$$

where σ_{ij} in (1.2.5) are now the components of stress referred to the cylindrical co-ordinate system.

The relationship between the cylindrical polar components of stress and the cartesian components of stress is

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{zz} \end{pmatrix} = \begin{pmatrix} c^2 & s^2 & 2sc & 0 & 0 & 0 \\ s^2 & c^2 & -2sc & 0 & 0 & 0 \\ -sc & sc & c^2 - s^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{33} \end{pmatrix} \qquad (1.2.6)$$

where

$$c = \cos\theta$$

$$s = \sin\theta.$$

1.3 CONSTITUTIVE EQUATIONS FOR TRANSVERSELY ISOTROPIC MATERIALS

The constitutive equations for a transversely isotropic linear elastic material which has a preferred direction in the direction of the unit vector $\underset{\sim}{a}$ are given by (Spencer 1974)

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) + \ldots$$

$$+ 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta a_k a_m e_{km} a_i a_j . \qquad (1.3.1)$$

Here a_i are the components of $\underset{\sim}{a}$ referred to a cartesian co-ordinate system and may be a function of position, δ_{ij} is the Kronecker delta function. The coefficients λ, α, β, μ_L and μ_T are elastic constants and have dimensions of stress. In particular μ_L and μ_T respectively represent the shear moduli along and transverse to the fibre direction $\underset{\sim}{a}$.

If $\underset{\sim}{a} = (1,0,0)$ then the stress-strain relations given by (1.3.1), referred to cartesian co-ordinates (x,y,z), become

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta & \lambda + \alpha & \lambda + \alpha & 0 & 0 & 0 \\ & \lambda + 2\mu_T & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu_T & 0 & 0 & 0 \\ & & & \mu_T & 0 & 0 \\ & & & & \mu_L & 0 \\ \text{Symmetrical} & & & & & \mu_L \end{pmatrix} \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{yz} \\ 2e_{xz} \\ 2e_{xy} \end{pmatrix} \qquad (1.3.2)$$

Equations (1.3.2) can be expressed in the form

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{22} & 0 & 0 & 0 \\ & & & \frac{1}{2}(c_{22} - c_{23}) & 0 & 0 \\ & & & & c_{66} & 0 \\ \text{Symmetrical} & & & & & c_{66} \end{pmatrix} \begin{pmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{yz} \\ 2e_{xz} \\ 2e_{xy} \end{pmatrix} \qquad (1.3.3)$$

and a comparison of (1.3.2) and (1.3.3) gives

$$\begin{aligned} c_{11} &= \lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta, \\ c_{12} &= \lambda + \alpha, \\ c_{22} &= \lambda + 2\mu_T, \\ c_{23} &= \lambda, \\ c_{66} &= \mu_L. \end{aligned} \qquad (1.3.4)$$

The five constants c_{11}, c_{12}, c_{22}, c_{23} and c_{66} or λ, β, α, μ_T and μ_L are five independent elastic constants of the material. They give a complete description of the stiffness of the material, but they are not the most convenient forms when considering how to determine them by experiment. For this reason the engineering properties are introduced.

By considering simple states of stress and strain, it can be shown (Christensen 1979) that

$$E_L = c_{11} - \frac{2c_{12}^2}{c_{22} + c_{23}} = \frac{(\lambda + \mu_T)\beta' - (\lambda + \alpha)^2}{\lambda + \mu_T},$$

$$E_T = \frac{\{c_{11}(c_{22} + c_{23}) - 2c_{12}^2\}(c_{22} - c_{23})}{c_{11}c_{22} - c_{12}^2}.$$

$$= \frac{\{(\lambda + \mu_T)\beta' - (\lambda + \alpha)^2\}.4\mu_T}{(\lambda + 2\mu_T)\beta' - (\lambda + \alpha)^2}, \tag{1.3.5}$$

$$\nu_L = \frac{c_{12}}{c_{22} + c_{23}} = \frac{\lambda + \alpha}{2(\lambda + \mu_T)},$$

$$\nu_T = \frac{c_{11}c_{23} - c_{12}^2}{c_{11}c_{22} - c_{12}^2} = \frac{\lambda\beta' - (\lambda + \alpha)^2}{(\lambda + 2\mu_T)\beta' - (\lambda + \alpha)^2},$$

where

$$\beta' = \lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta.$$

In (1.3.5) E_L and E_T are the Youngs moduli along and transverse to the fibres, ν_L and ν_T are the corresponding Poisson ratios. Since only five of the engineering constants E_L, E_T, ν_L, ν_T, μ_L and μ_T are independent, there exists a relationship between them. From (1.3.4) and (1.3.5) it is found that

$$\mu_T = \frac{E_T}{2(1 + \nu_T)}, \tag{1.3.6}$$

which is identical to the relationship given in isotropic materials.

The elastic constants α, β and λ can be expressed in terms of the engineering constants. By inverting (1.3.5) we obtain

$$\lambda = \frac{E_T(E_L\nu_T + E_T\nu_L^2)}{(1 + \nu_T)\{E_L(1 - \nu_T) - 2E_T\nu_L^2\}}$$

$$\alpha = \frac{E_T\{E_L(\nu_L + \nu_T\nu_L - \nu_T) - E_T\nu_L^2\}}{(1 + \nu_T)\{E_L(1 - \nu_T) - 2E_T\nu_L^2\}}, \tag{1.3.7}$$

$$\beta = \frac{E_L^2(1 - \nu_T^2) + E_TE_L\{1 - 2\nu_L(1 + \nu_T)\} - E_T^2\nu_L^2}{(1 + \nu_T)\{E_L(1 - \nu_T) - 2E_T\nu_L^2\}} - 4\mu_L.$$

If Equations (1.3.3) are written in the form

$$\sigma_{ij} = [c_{ij}]e_{ij}, \tag{1.3.8}$$

then

$$e_{ij} = [s_{ij}]\sigma_{ij} \tag{1.3.9}$$

where the compliance matrix $[s_{ij}]$ is given by

$$[s_{ij}] = \frac{[\text{Cofactor Matrix of } c_{ij}]^T}{|c_{ij}|}. \tag{1.3.10}$$

When a transversely isotropic material is rotated so that $\underset{\sim}{a} = (\cos\theta, \sin\theta, 0)$, then the stress-strain relations given by (1.3.8) take the form

$$\sigma_{ij} = [c'_{ij}]e_{ij} \tag{1.3.11}$$

where

$$\begin{aligned}
c'_{11} &= c^4c_{11} + 2c^2s^2c_{12} + c^2s^2 4c_{66} + s^4c_{22},\\
c'_{12} &= c^2s^2c_{11} + (c^4 + s^4)c_{12} + c^2s^2c_{22} - c^2s^2 4c_{66},\\
c'_{16} &= -c^3sc_{11} + (c^3s - cs^3)c_{12} + cs^3c_{22} + (c^3s - cs^3)2c_{66},\\
c'_{22} &= s^4c_{11} + 2c^2s^2c_{12} + c^4c_{22} + c^2s^2 4c_{66},\\
c'_{26} &= -cs^3c_{11} + (cs^3 - c^3s)c_{12} + c^3sc_{22} + (cs^3 - c^3s)2c_{66},\\
c'_{66} &= c^2s^2c_{11} - 2c^2s^2c_{12} + c^2s^2c_{22} + (c^2 - s^2)^2c_{66},\\
c'_{13} &= c^2c_{12} + s^2c_{23},\\
c'_{23} &= s^2c_{12} + c^2c_{23},\\
c'_{36} &= -csc_{12} + csc_{23},\\
c'_{44} &= c^2c_{44} + s^2c_{66},\\
c'_{45} &= csc_{44} - csc_{66},\\
c'_{55} &= s^2c_{44} + c^2c_{66},\\
c'_{33} &= c_{22},
\end{aligned} \tag{1.3.12}$$

$$c = \cos\theta, \quad s = \sin\theta. \tag{1.3.13}$$

1.4 CONSTITUTIVE EQUATIONS FOR IDEAL MATERIALS

If the modulus for extension in the fibre direction is much greater than the elastic moduli in transverse extension or shear then, from (1.3.7), β is much greater than α and λ. In the limit of an inextensible material

$$\beta = \infty. \tag{1.4.1}$$

The condition of inextensibility is that the material will not extend in the direction of $\underset{\sim}{a}$, therefore

$$a_i a_j e_{ij} = 0. \tag{1.4.2}$$

Denoting the limit of $\beta a_i a_j e_{ij}$ as $\beta \to \infty$, $a_i a_j e_{ij} \to 0$ by

$$\beta a_i a_j e_{ij} = T, \tag{1.4.3}$$

where T represents an arbitrary fibre tension, the constitutive equations for inextensible transversely isotropic materials are obtained from (1.3.1)

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij}$$
$$+ 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + T a_i a_j. \tag{1.4.4}$$

Here the terms involving α have been absorbed into T.

If, in addition the material is incompressible then

$$e_{kk} = 0, \tag{1.4.5}$$

and the hydrostatic pressure in (1.4.4) becomes an arbitrary reaction to this constraint.

$$\sigma_{ij} = -p\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + Ta_i a_j, \tag{1.4.6}$$

where p represents the hydrostatic pressure. Evidently Equation (1.4.6) arises from (1.4.4) in the limit

$$\lambda \to \infty. \tag{1.4.7}$$

Equation (1.4.6) represents the constitutive equation of an ideal material.

1.5 CONSTITUTIVE EQUATIONS FOR ISOTROPIC MATERIALS

The constitutive equations for isotropic materials are obtained by letting $E_L = E_T$, $\nu_L = \nu_T$ and $\mu_L = \mu_T$ and from (1.3.7), this corresponds to taking $\alpha = \beta = 0$.

From (1.3.1)

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \tag{1.5.1}$$

where

$$\mu = \mu_L = \mu_T.$$

CHAPTER TWO

PLANE STRAIN BENDING OF A LAMINATED FIBRE-REINFORCED CANTILEVER

2.1 INTRODUCTION

The plane strain bending of a fibre-reinforced cantilever has been examined by Everstine and Pipkin (1973). They found that when the material is ideal, as described in Section 1.4 , singular sheets of fibres which carry infinite stress but finite force occur adjacent to the surfaces of the body. For materials with small but finite extensibility, these singular surfaces represent narrow bands of rapidly varying stress. If L is a characteristic length of a problem and E, μ the moduli for extension and shear respectively, these bands are found to have width of order $L(\mu/E)^{\frac{1}{2}}$ and the disturbance in them propagates distances of order $(E/\mu)^{\frac{1}{2}}$ without attenuation. Everstine and Pipkin gave a boundary layer analysis of the stress and deformation in the regions adjacent to the surfaces.

In this chapter the work of Everstine and Pipkin is extended to the case of the bending of a laminated cantilever plate. It is shown that if the individual layers of the cantilever are ideal materials then there exist singular surfaces at the interface of the layers besides those adjacent to the surface of the body. For a cantilever plate consisting of incompressible but nearly inextensible layers, these regions correspond to boundary layers similar to those which occur at the lateral surfaces. We investigate the stress and deformation in these layers and discuss their implications for the possible initiation of delamination.

2.2 STATEMENT OF THE PROBLEM

Consider a cantilever plate of length L and width 2h consisting of two transversely isotropic layers which have different elastic moduli. The reference co-ordinate system is taken such that the interface between the layers coincides with the x-axis as shown in Figure (2.1)

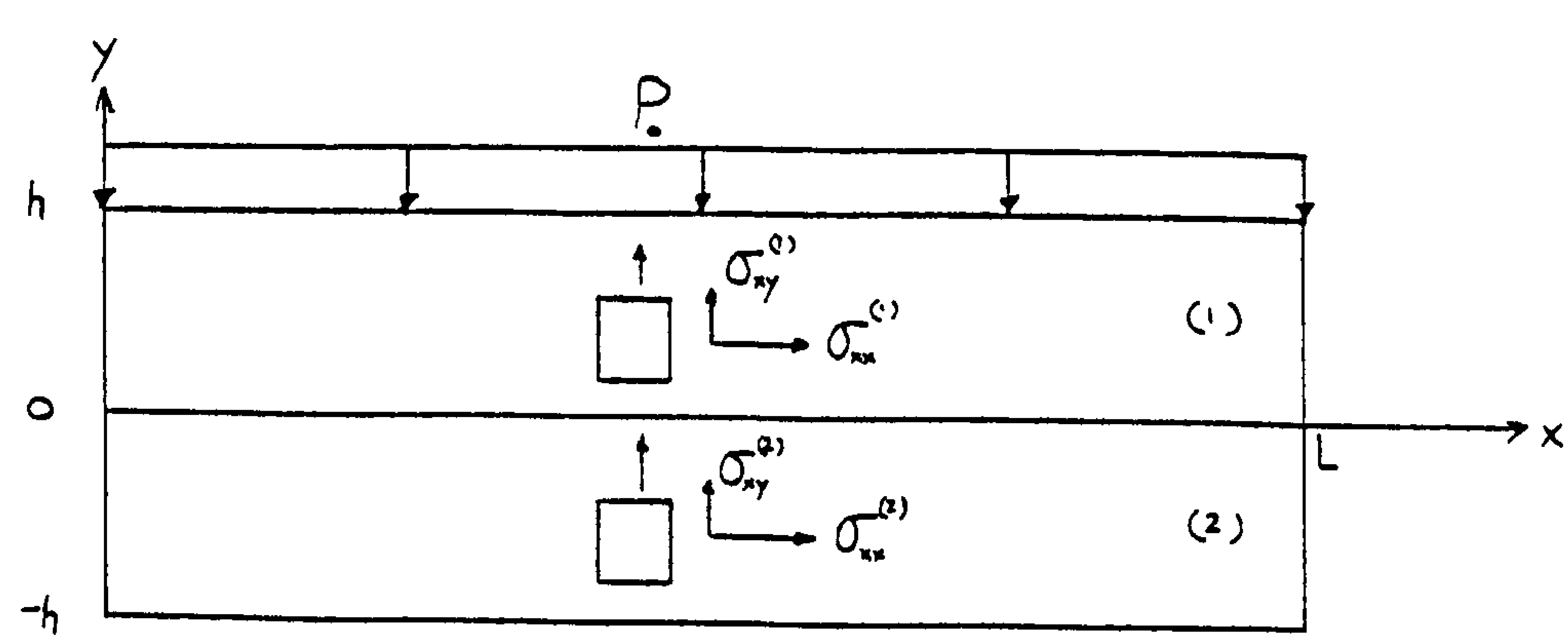


Figure (2.1). The cantilever plate.

The layers of the cantilever are distinguished by the subscripts/superscripts 1 and 2, where 1 denotes the layer $y \geq 0$, and each contains straight continuously distributed fibres which are parallel to the x-axis. In the notation of Chapter One,

$$\underset{\sim}{a} = (1,0,0), \tag{2.2.1}$$

for each layer.

The cantilever is fixed at the end $x = 0$ and is subjected to a uniform pressure P_o on the lateral surface $y = h$. In each layer we shall take

$$u = u(x,y), \quad v = v(x,y), \quad w = 0, \tag{2.2.2}$$

where (u,v,w) denote the displacement components in the x, y and z directions respectively.

The boundary conditions satisfied by the stress and displacement components are

(i) fixed end condition

$$\begin{aligned} u^{(1)} &= v^{(1)} = 0, \quad x = 0, \\ u^{(2)} &= v^{(2)} = 0, \quad x = 0, \end{aligned} \tag{2.2.3}$$

(ii) stress free conditions

$$\begin{aligned} \sigma_{xx}^{(1)} &= \sigma_{xy}^{(1)} = 0, \quad x = L, \\ \sigma_{xx}^{(2)} &= \sigma_{xy}^{(2)} = 0, \quad x = L, \\ \sigma_{xy}^{(2)} &= \sigma_{yy}^{(2)} = 0, \quad y = -h, \\ \sigma_{xy}^{(1)} &= 0, \qquad\qquad y = h. \end{aligned} \tag{2.2.4}$$

(iii) applied loads

$$\sigma_{yy}^{(1)} = -P_o, \quad y = h. \tag{2.2.5}$$

(iv) continuity conditions

$$u^{(1)} = u^{(2)}, \quad v^{(1)} = v^{(2)}, \quad y = 0,$$
$$\sigma^{(1)}_{xy} = \sigma^{(2)}_{xy}, \quad \sigma^{(1)}_{yy} = \sigma^{(2)}_{yy}, \quad y = 0. \tag{2.2.6}$$

The stress equilibrium equations (1.2.4), in the absence of body forces reduce to

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0,$$
$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0, \tag{2.2.7}$$

the third equation being satisfied identically.

2.3 IDEAL THEORY

For ideal materials which are in a state of plane strain and have a preferred direction $\underset{\sim}{a} = (1,0,0)$, the constitutive equations (1.4.6) become

$$\sigma_{xx} = -p + T + (4\mu_L - 2\mu_T)e_{xx},$$
$$\sigma_{yy} = -p + 2\mu_T e_{yy},$$
$$\sigma_{zz} = -p, \tag{2.3.1}$$
$$\sigma_{xy} = 2\mu_L e_{xy},$$
$$\sigma_{yz} = \sigma_{xz} = 0,$$

and the kinematic constraints (1.4.2,5) are now

$$e_{xx} + e_{yy} = 0,$$
$$e_{xx} = 0. \tag{2.3.2}$$

It follows from Equations (2.3.2) that the displacement components in each layer of the cantilever must take the form

$$u = u(y), \quad v = v(x). \tag{2.3.3}$$

Since the end $x = 0$ of the cantilever is fixed we have, from (2.3.3)

$$u^{(1)}(y) = u^{(2)}(y) = 0, \tag{2.3.4}$$

and for continuity of displacement components at $y = 0$

$$v^{(1)}(x) = v^{(2)}(x) = v(x). \tag{2.3.5}$$

Expressions (2.3.4) and (2.3.5) describe the displacement field in each layer of the cantilever. For a cantilever of an arbitrary number of layers these expressions generalise to

$$u^{(\ell)} = 0, \quad v^{(\ell)} = v(x), \tag{2.3.6}$$

where the superscript ℓ corresponds to the ℓth layer.

With the forms of the displacement components now determined, the constitutive equations (2.3.1) become

$$\begin{aligned}\sigma_{xx} &= -p + T, \\ \sigma_{yy} &= \sigma_{zz} = -p, \\ \sigma_{xy} &= \mu_L v'(x), \\ \sigma_{yz} &= \sigma_{xz} = 0.\end{aligned} \tag{2.3.7}$$

By considering the equilibrium of a section of the cantilever which is bounded by the planes $x = x_o$, $x = L$ and the lateral surfaces $y = \pm h$ (Figure 2.2) we obtain

$$P_o(L - x_o) + \int_o^h \sigma_{xy}^{(1)}\Big|_{x=x_o} dy + \int_{-h}^o \sigma_{xy}^{(2)}\Big|_{x=x_o} dy = 0. \tag{2.3.8}$$

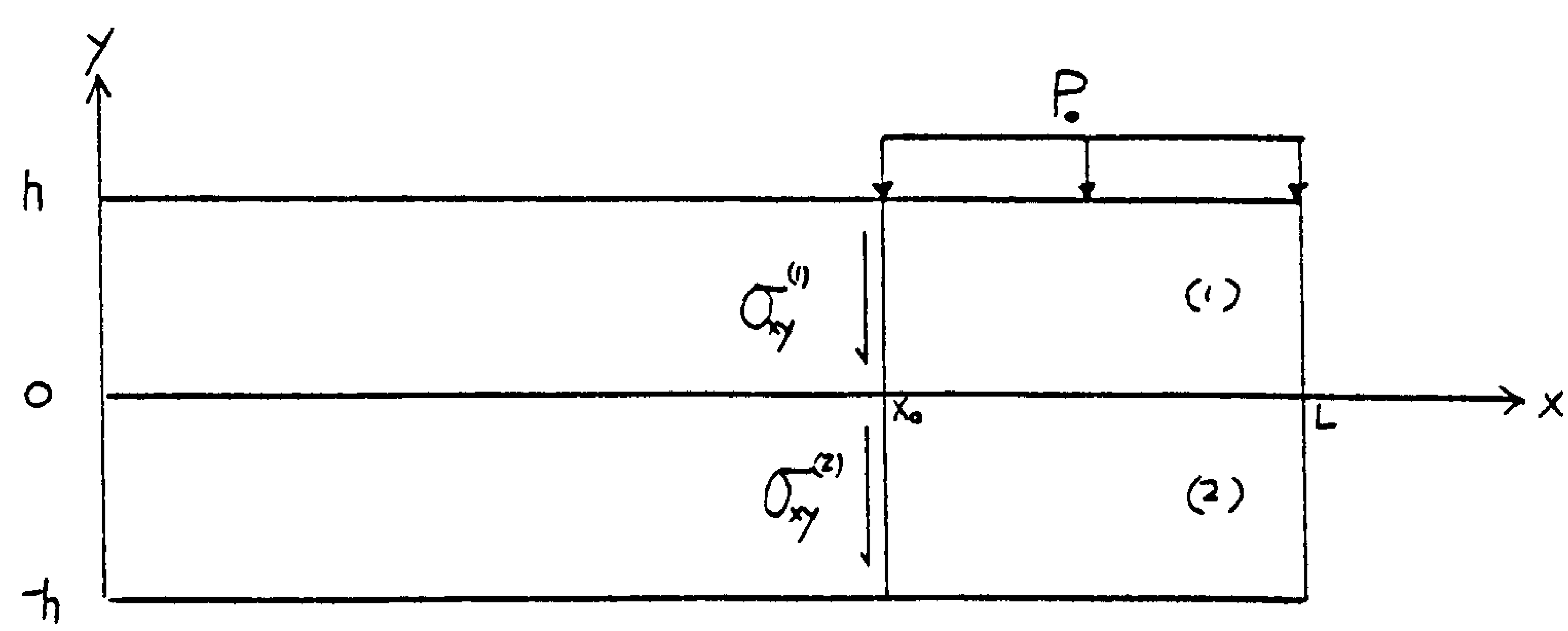


Figure (2.2).

The shear stress components given in (2.3.7) are substituted into Equation (2.3.8) and after some manipulation we find

$$v(x) = \frac{P_o}{h(\mu_L^{(1)} + \mu_L^{(2)})}\left(\frac{x^2}{2} - Lx\right), \tag{2.3.9}$$

where the constant of integration has been chosen so that the fixed end condition at x = 0 is satisfied. Hence,

$$\sigma_{xy}^{(1)} = \frac{\mu_L^{(1)} P_o}{h(\mu_L^{(1)} + \mu_L^{(2)})}(x - L) \quad 0 \leq y \leq h,$$

$$\sigma_{xy}^{(2)} = \frac{\mu_L^{(2)} P_o}{h(\mu_L^{(1)} + \mu_L^{(2)})}(x - L), \quad -h \leq y \leq 0. \qquad (2.3.10)$$

Note that if $\mu_L^{(1)} \neq \mu_L^{(2)}$ then there is a discontinuity of the shear stress components at y = 0. Furthermore, since the surfaces y = ±h are required to be shear traction free, the shear components will also be discontinuous at these surfaces. Accordingly we write

$$\sigma_{xy} = \frac{P_o(x - L)}{h(\mu_L^{(1)} + \mu_L^{(2)})}(\mu_L^{(2)} H(y + h) + (\mu_L^{(1)} - \mu_L^{(2)})H(y) - \mu_L^{(1)} H(y - h)), \quad -h \leq y \leq h, \qquad (2.3.11)$$

where H is the usual Heaviside function; unity for positive argument and zero for negative argument.

Substituting (2.3.11) in (2.2.7), and integrating gives

$$\sigma_{xx} = \frac{-P_o(x - L)^2}{2h(\mu_L^{(1)} + \mu_L^{(2)})}(\mu_L^{(2)} \delta(y + h) + (\mu_L^{(1)} - \mu_L^{(2)})\delta(y) - \mu_L^{(1)} \delta(y - h)). \qquad (2.3.12)$$

In (2.3.12) δ is the Dirac delta function and the arbitrary function of integration has been chosen such that the stress free condition on x = L is satisfied.

The tensile stress σ_{xx} is zero everywhere except at y = h,0,-h where it is infinite, but the force carried by these fibres, which is found by integrating across them, is finite and is given by

$$\text{(i)} \quad y = h; \quad \frac{P_o \mu_L^{(1)} (x - L)^2}{2h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad \text{fibre in tension}$$

$$\text{(ii)} \quad y = 0; \quad \frac{P_o (\mu_L^{(2)} - \mu_L^{(1)}) (x - L)^2}{2h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad \text{fibre in tension if } \mu_L^{(2)} - \mu_L^{(1)} > 0 \tag{2.3.13}$$

$$\text{(iii)} \quad y = -h; \quad \frac{-P_o \mu_L^{(2)} (x - L)^2}{2h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad \text{fibre in compression.}$$

The surfaces y = ±h,0 are referred to as singular surfaces. If the layers are of the same material then there are only singular surfaces at the lateral surfaces of the cantilever and this is in agreement with Everstine and Pipkin (1973).

The normal stress component in the y direction can now be obtained from the second component of the stress equilibrium equations.

$$\sigma_{yy}^{(1)} = \frac{-\mu_L^{(1)} P_o (y + A_1)}{h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad 0 \leq y \leq h,$$

$$\sigma_{yy}^{(2)} = \frac{-\mu_L^{(2)} P_o (y + A_2)}{h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad -h \leq y \leq 0, \tag{2.3.14}$$

where A_1 and A_2 are arbitrary constants of integration and are determined from the through-thickness boundary and continuity conditions.

By satisfying these conditions

$$\sigma_{yy}^{(1)} = \frac{-P_o}{h(\mu_L^{(1)} + \mu_L^{(2)})}(\mu_L^{(1)} y + h\mu_L^{(2)}), \qquad 0 \leq y \leq h,$$

$$\sigma_{yy}^{(2)} = \frac{-P_o}{h(\mu_L^{(1)} + \mu_L^{(2)})}(\mu_L^{(2)} y + h\mu_L^{(2)}), \qquad -h \leq y \leq 0. \tag{2.3.15}$$

2.4 FORMULATION OF THE BASIC EQUATIONS FOR INCOMPRESSIBLE AND NEARLY INEXTENSIBLE MATERIALS

For materials which are incompressible and 'stiff' in the fibre direction we have from (1.3.7)

$$\lambda = \infty, \quad \beta >> 1. \tag{2.4.1}$$

By substituting (2.4.1) into (1.3.1) and absorbing the α terms into the hydrostatic pressure $-p$ we obtain

$$\sigma_{ij} = -p\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki})$$

$$+ \beta a_k a_m e_{km} a_i a_j. \tag{2.4.2}$$

Suppose that the material is in a state of plane strain and that the fibre direction is given by $\underset{\sim}{a} = (1,0,0)$. From (2.4.2)

$$\sigma_{xx} = -p + (4\mu_L - 2\mu_T)e_{xx} + \beta e_{xx},$$

$$\sigma_{yy} = -p + 2\mu_T e_{yy}, \tag{2.4.3}$$

$$\sigma_{xy} = 2\mu_L e_{xy},$$

and the incompressibility condition gives

$$e_{xx} + e_{yy} = 0. \tag{2.4.4}$$

Inverting Equations (2.4.3) and using (2.4.4) produces

$$\frac{\partial u}{\partial x} = \frac{\varepsilon^2}{\mu_L}(\sigma_{xx} - \sigma_{yy}),$$

$$\frac{\partial v}{\partial y} = -\frac{\varepsilon^2}{\mu_L}(\sigma_{xx} - \sigma_{yy}), \tag{2.4.5}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\sigma_{xy}}{\mu_L},$$

where

$$\varepsilon^2 = \frac{\mu_L}{(\beta + 4\mu_L)}. \tag{2.4.6}$$

Note that if $(\beta/\mu_L) >> 1$ then $\varepsilon^2 << 1$.

2.5 INTERIOR SOLUTION

Suppose that each displacement component can be represented, in the asymptotic sense, by a power series in ε as follows

$$u(x,y;\varepsilon) = \bar{u}_o + \varepsilon\bar{u}_1 + \varepsilon^2\bar{u}_2 + O(\varepsilon^3),$$

$$v(x,y;\varepsilon) = \bar{v}_o + O(\varepsilon), \tag{2.5.1}$$

and that the stress components, symbolized by $\sigma(x,y;\varepsilon)$ have the expansion

$$\sigma(x,y;\varepsilon) = \bar{\sigma}_o + O(\varepsilon). \tag{2.5.2}$$

By substituting (2.5.1) and (2.5.2) into Equations (2.4.5) and the stress equilibrium equations (2.2.7) we obtain to lowest order

$$\begin{gathered}
\bar{u}_o = \bar{u}_o(y), \quad \bar{u}_1 = \bar{u}_1(y), \\
\frac{\partial \bar{u}_2}{\partial x} = \frac{1}{\mu_L}(\bar{\sigma}_{xxo} - \bar{\sigma}_{yyo}), \\
\frac{\partial \bar{v}_o}{\partial y} = 0, \\
\frac{\partial \bar{v}_o}{\partial x} = \frac{\bar{\sigma}_{xyo}}{\mu_L}, \\
\frac{\partial \bar{\sigma}_{xxo}}{\partial x} + \frac{\partial \bar{\sigma}_{xyo}}{\partial y} = 0, \\
\frac{\partial \bar{\sigma}_{xyo}}{\partial x} + \frac{\partial \bar{\sigma}_{yyo}}{\partial y} = 0.
\end{gathered} \tag{2.5.3}$$

Solving Equations (2.5.3) gives

$$\begin{gathered}
\bar{u}_o = \bar{u}_1 = 0, \\
\bar{v}_o = f(x), \\
\frac{\bar{\sigma}_{xyo}}{\mu_L} = f'(x), \\
\frac{\bar{\sigma}_{xxo}}{\mu_L} = g(y), \\
\frac{\bar{\sigma}_{yyo}}{\mu_L} = -y\mu_L f''(x) + h(x),
\end{gathered} \tag{2.5.4}$$

where f, g and h are arbitrary functions.

As expected, Equations (2.5.4) coincide with those of the ideal theory, except that now the order of $\dot{u}$ can be determined. The arbitrary functions are found by the boundary conditions of a given problem.

In particular, for the cantilever problem described in Section (2.2) we obtain the following stress field in each layer

$$
\begin{aligned}
&\bar{\sigma}^{(1)}_{xxo} = \bar{\sigma}^{(2)}_{xxo} = 0 \\
&\bar{\sigma}^{(1)}_{xyo} = \frac{P_o \mu_L^{(1)} (x - L)}{h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad 0 \leq y \leq h, \\
&\bar{\sigma}^{(2)}_{xyo} = \frac{P_o \mu_L^{(2)} (x - L)}{h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad -h \leq y \leq 0, \\
&\bar{\sigma}^{(1)}_{yyo} = \frac{-P_o (\mu_L^{(1)} y + h\mu_L^{(2)})}{h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad 0 \leq y \leq h, \\
&\bar{\sigma}^{(2)}_{yyo} = \frac{-P_o (\mu_L^{(2)} y + h\mu_L^{(2)})}{h(\mu_L^{(1)} + \mu_L^{(2)})} \qquad -h \leq y \leq 0.
\end{aligned}
\tag{2.5.5}
$$

2.6 BOUNDARY LAYER REGIONS

The shear components of the interior solution given by (2.5.5) do not satisfy the prescribed boundary conditions on the lateral surfaces of the cantilever and are discontinuous at the interface of the layers. Following Everstine and Pipkin (1971,1973), we expect the stress components, in regions adjacent to these surfaces, to vary rapidly in the y direction.

We will only consider regions adjacent to $y = 0$ since the analysis applied by Everstine and Pipkin remains valid at the surfaces $y = \pm h$.

By the usual boundary layer theory we 'stretch' the y co-ordinate by ~~a~~ the scaling factor ε and if η is the new stretched variable then

$$\eta = \frac{y}{\varepsilon}, \qquad \varepsilon \ll 1. \tag{2.6.1}$$

Writing Equations (2.4.5) and (2.4.6) in terms of η we obtain

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\varepsilon^2}{\mu_L}(\sigma_{xx} - \sigma_{yy}), \\
\frac{\partial v}{\partial \eta} &= -\frac{\varepsilon^3}{\mu_L}(\sigma_{xx} - \sigma_{yy}), \\
\frac{\partial u}{\partial \eta} + \varepsilon\frac{\partial v}{\partial x} &= \varepsilon\frac{\sigma_{xy}}{\mu_L}, \\
\varepsilon\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial \eta} &= 0, \\
\varepsilon\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0.
\end{aligned}
\tag{2.6.2}
$$

Suppose that the stress and displacement components can be asympotically represented by the following power series in ε,

$$
\begin{aligned}
u(x,\eta;\varepsilon) &= \varepsilon\hat{u}(x,\eta) + O(\varepsilon^2), \\
v(x,\eta;\varepsilon) &= \varepsilon^2\hat{v}(x,\eta) + O(\varepsilon^3),
\end{aligned}
\tag{2.6.3}
$$

and

$$
\begin{aligned}
\sigma_{xx}(x,\eta;\varepsilon) &= \frac{\hat{\sigma}_{xx}}{\varepsilon}(x,\eta) + \ldots \\
\sigma_{yy}(x,\eta;\varepsilon) &= \varepsilon\hat{\sigma}_{yy}(x,\eta) + \ldots \\
\sigma_{xy}(x,\eta;\varepsilon) &= \hat{\sigma}_{xy}(x,\eta) + \ldots\ .
\end{aligned}
\tag{2.6.4}
$$

Substituting Equations (2.6.3) and (2.6.4) into (2.6.2) we obtain to lowest order

$$\frac{\partial \hat{u}}{\partial x} = \frac{\hat{\sigma}_{xx}}{\mu_L},$$

$$\frac{\partial \hat{v}}{\partial \eta} = -\frac{\hat{\sigma}_{xx}}{\mu_L},$$

$$\frac{\partial \hat{u}}{\partial \eta} = \frac{\hat{\sigma}_{xy}}{\mu_L}, \tag{2.6.5}$$

$$\frac{\partial \hat{\sigma}_{xx}}{\partial x} + \frac{\partial \hat{\sigma}_{xy}}{\partial \eta} = 0,$$

$$\frac{\partial \hat{\sigma}_{xy}}{\partial x} + \frac{\partial \hat{\sigma}_{yy}}{\partial \eta} = 0,$$

and it follows from these equations that

$$\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial \eta^2} = 0. \tag{2.6.6}$$

Equations (2.6.5) and (2.6.6) are the boundary layer equations. To determine solutions from them for each layer of the cantilever we will require that

(i) $\hat{\sigma}^{(1)}(x,\infty) = 0$, $\hat{\sigma}^{(2)}(x,-\infty) = 0$
where $\hat{\sigma}$ is a generic symbol for the stress components satisfying (2.6.5) and (2.6.6)

(ii) When the shear components satisfying (2.6.5) and (2.6.6) are superposed on the interior solutions, the combined shear stress solution is continuous at the interface $\eta = 0$.

(iii) When the displacements components satisfying (2.6.5,6) are superposed on the interior solutions, the combined displacement is continuous at the interface $\eta = 0$.

To satisfy (i) we take

$$\hat{u}^{(1)} = \sum_{n=0}^{\infty} A_n^{(1)} \sin(\lambda_n x/L) \exp(-\lambda_n \eta_1/L) \qquad 0 \le \eta_1 < \infty,$$
$$\hat{u}^{(2)} = \sum_{n=0}^{\infty} A_n^{(2)} \sin(\lambda_n x/L) \exp(\lambda_n \eta_2/L) \qquad -\infty < \eta_2 \le 0, \tag{2.6.7}$$

where λ_n, $A_n^{(1)}$ and $A_n^{(2)}$ are arbitrary constants.

For continuity of u at $\eta_1 = \eta_2 = 0$ we require

$$\varepsilon_1 A_n^{(1)} = \varepsilon_2 A_n^{(2)}. \tag{2.6.8}$$

The end x = L is required to be free from normal traction and in order to maintain this we take

$$\lambda_n = (n + \tfrac{1}{2})\pi. \tag{2.6.9}$$

By Equations (2.6.5), the shear stress components in each layer now take the form

$$\hat{\sigma}_{xy}^{(1)} = -\sum \mu_L^{(1)} A_n^{(1)} (\lambda_n/L) \sin(\lambda_n x/L) \exp(-\lambda_n \eta_1/L),$$
$$\hat{\sigma}_{xy}^{(2)} = \sum \mu_L^{(2)} A_n^{(2)} (\lambda_n/L) \sin(\lambda_n x/L) \exp(\lambda_n \eta_2/L). \tag{2.6.10}$$

To avoid the problem of matching the stresses given by (2.6.5) and (2.6.6) with the interior stresses, we superpose these two solutions in each layer to obtain expressions for the stress components in the generic form

$$\sigma = \bar{\sigma} + \hat{\sigma}, \tag{2.6.11}$$

and allow σ to be the solution everywhere in the layer. Since $\hat{\sigma}(x,\eta) \to 0$ as $\eta \to \infty$, $\sigma \to \bar{\sigma}$ at the interior points. Hence, in order to satisfy the continuity condition for the shear stress components at the interface $y = 0$ ($\eta = 0$) we will require

$$\bar{\sigma}^{(1)}_{xyo} + \hat{\sigma}^{(1)}_{xy} = \bar{\sigma}^{(2)}_{xyo} + \hat{\sigma}^{(2)}_{xy}, \quad \eta_1 = \eta_2 = 0. \tag{2.6.12}$$

If $\bar{\sigma}^{(1)}_{xyo}$ and $\bar{\sigma}^{(2)}_{xyo}$, evaluated at $y = 0$, are expressed as Fourier series then

$$\begin{aligned} \bar{\sigma}^{(1)}_{xyo} &= \frac{P_o \mu_L^{(1)} (x - L)}{h(\mu_L^{(1)} + \mu_L^{(2)})} = \frac{P_o K}{h(K + 1)} \sum_0^\infty S_n \sin(\lambda_n \frac{x}{L}) \\ \bar{\sigma}^{(2)}_{xyo} &= \frac{P_o \mu_L^{(2)} (x - L)}{h(\mu_L^{(1)} + \mu_L^{(2)})} = \frac{P_o}{h(K + 1)} \sum_0^\infty S_n \sin(\lambda_n \frac{x}{L}) \end{aligned} \tag{2.6.13}$$

where $K = \mu_L^{(1)}/\mu_L^{(2)}$, λ_n is given by (2.6.9) and the Fourier coefficients S_n are

$$S_n = \frac{2L}{\lambda_n}[\frac{(-)^n}{\lambda_n} - 1]. \tag{2.6.14}$$

By substituting Equations (2.6.13) and (2.6.10) into (2.6.12) and making use of (2.6.8), we obtain

$$A_n^{(1)} \frac{\lambda_n}{L} = \frac{\varepsilon_2}{(\mu_L^{(2)}\varepsilon_1 + \mu_L^{(1)}\varepsilon_2)} P_o \frac{S_n}{h} \frac{(K - 1)}{(K + 1)} ,$$

$$A_n^{(2)} \frac{\lambda_n}{L} = \frac{\varepsilon_1}{(\mu_L^{(2)}\varepsilon_1 + \mu_L^{(1)}\varepsilon_2)} P_o \frac{S_n}{h} \frac{(K - 1)}{(K + 1)} . \tag{2.6.15}$$

The normal stress component in the x-direction is found from Equations (2.6.5) and (2.6.7) to be

$$\hat{\sigma}_{xx}^{(1)} = \sum_o^\infty \mu_L^{(1)} A_n^{(1)} (\lambda_n/L) \cos(\lambda_n x/L) \exp(-\lambda_n \eta_1/L)$$

$$\hat{\sigma}_{xx}^{(2)} = \sum_o^\infty \mu_L^{(2)} A_n^{(2)} (\lambda_n/L) \cos(\lambda_n x/L) \exp(\lambda_n \eta_2/L) , \tag{2.6.16}$$

and therefore, the resultant tensile force in the region $\eta_1 = \eta_2 = 0$ is

$$\int_o^\infty \hat{\sigma}_{xx}^{(1)} \, d\eta_1 + \int_{-\infty}^0 \hat{\sigma}_{xx}^{(2)} \, d\eta_2$$

$$= \frac{P_o(K - 1)}{2h(K + 1)} \sum_0^\infty 2S_n \cdot \frac{L}{\lambda_n} \cos(\lambda_n \frac{x}{L}) . \tag{2.6.17}$$

Carrying out the summation given in the second expression of (2.6.17), we find that the resultant load is

$$\frac{P_o(1 - K)}{2h(1 + K)}(x - L)^2 \tag{2.6.18}$$

which is identical to the load in the singular fibres $y = 0_+$ and $y = 0_-$ given by ideal theory.

2.7 RESULTS AND DISCUSSION

Superposition of the solution given in Sections (2.5,2.6) produces to leading order in ε the complete solution in each layer.

$$\sigma_{xy}^{(1)}/P_o = \frac{K(x - L)}{h(1 + K)} - \frac{\varepsilon_2 K(K - 1)}{h(\varepsilon_1 + K\varepsilon_2)(K + 1)} \times \sum_0^\infty S_n \sin(\lambda_n x/L)\exp(-\lambda_n y/\varepsilon_1 L) \qquad y \geq 0,$$

$$\sigma_{xy}^{(2)}/P_o = \frac{(x - L)}{h(1 + K)} + \frac{\varepsilon_1 (K - 1)}{h(\varepsilon_1 + K\varepsilon_2)(K + 1)} \times \sum_0^\infty S_n \sin(\lambda_n x/L)\exp(\lambda_n y/\varepsilon_2 L) \qquad y \leq 0, \tag{2.7.1}$$

and

$$\sigma_{xx}^{(1)}/P_o = \frac{\varepsilon_2 K(K - 1)}{h\varepsilon_1(\varepsilon_1 + K\varepsilon_2)(K + 1)} \times \sum_0^\infty S_n(\lambda_n/L)\cos(\lambda_n x/L)\exp(-\lambda_n y/\varepsilon_1 L) \qquad y \geq 0,$$

$$\sigma_{xx}^{(2)}/P_o = \frac{\varepsilon_1 (K - 1)}{h\varepsilon_2(\varepsilon_1 + K\varepsilon_2)(K + 1)} \times \sum_0^\infty S_n(\lambda_n/L)\cos(\lambda_n x/L)\exp(\lambda_n y/\varepsilon_2 L) \qquad y \leq 0. \tag{2.7.2}$$

σ_{yy} is given approximately by the interior solution since the contribution from the solution in Section (6) is of order ε_i $(i = 1,2)$.

As $\varepsilon_i \to 0$ $(i = 1,2)$ the stress components in each layer tend to those given by ideal theory provided $y \neq 0$. For $y = 0$

$$\sigma_{xx}^{(i)} \to \infty \text{ as } \varepsilon_i \to 0, \ (\underline{i} = 1,2), \tag{2.7.3}$$

and therefore the Dirac deltas given in ideal theory are interpreted as

$$-\lim_{\varepsilon \to 0}\left[\frac{\lambda \exp(-\lambda y/\varepsilon L)}{L\varepsilon}\right]. \tag{2.7.4}$$

The shear stress components at $y = 0$ can be calculated from (2.7.1) as

$$\sigma_{xy}^{(1)}/P_o = \sigma_{xy}^{(2)}/P_o = \frac{i}{h}\frac{K}{(1+K)}\frac{(\varepsilon_1+\varepsilon_2)}{(\varepsilon_1+K\varepsilon_2)}(x-L), \tag{2.7.5}$$

and since (2.7.5) has no unique limit as $\varepsilon_i \to 0$ $(\underline{i} = 1,2)$, this may justify the use of Heaviside functions in the ideal theory.

The boundary layers in the region adjacent to $y = 0$ have thicknesses of order $\varepsilon_1 L$ and $\varepsilon_2 L$ for layers 1 and 2 respectively and the tensile stress in them is of the order of the total load predicted by ideal theory divided by this thickness.

Since we need to localize the disturbances of the $\hat{\sigma}$ solutions, we will have to require that

$$h/L > \max(\varepsilon_1, \varepsilon_2), \tag{2.7.6}$$

and this imposes restrictions on the values of the slenderness of the cantilever.

The end x = L is not stress free in the region of y = 0 since $\sigma_{xy}^{(1)}$ and $\sigma_{xy}^{(2)}$ are not zero here. To assist in the numerical evaluation of $\sigma_{xy}^{(i)}$ at x = L, the value of the summation, given in (2.7.1), is initially determined for various values of y and ε. This is shown in Figure (2.3) when ε = 1/10, 1/20, 1/40 and h/L = ¼.

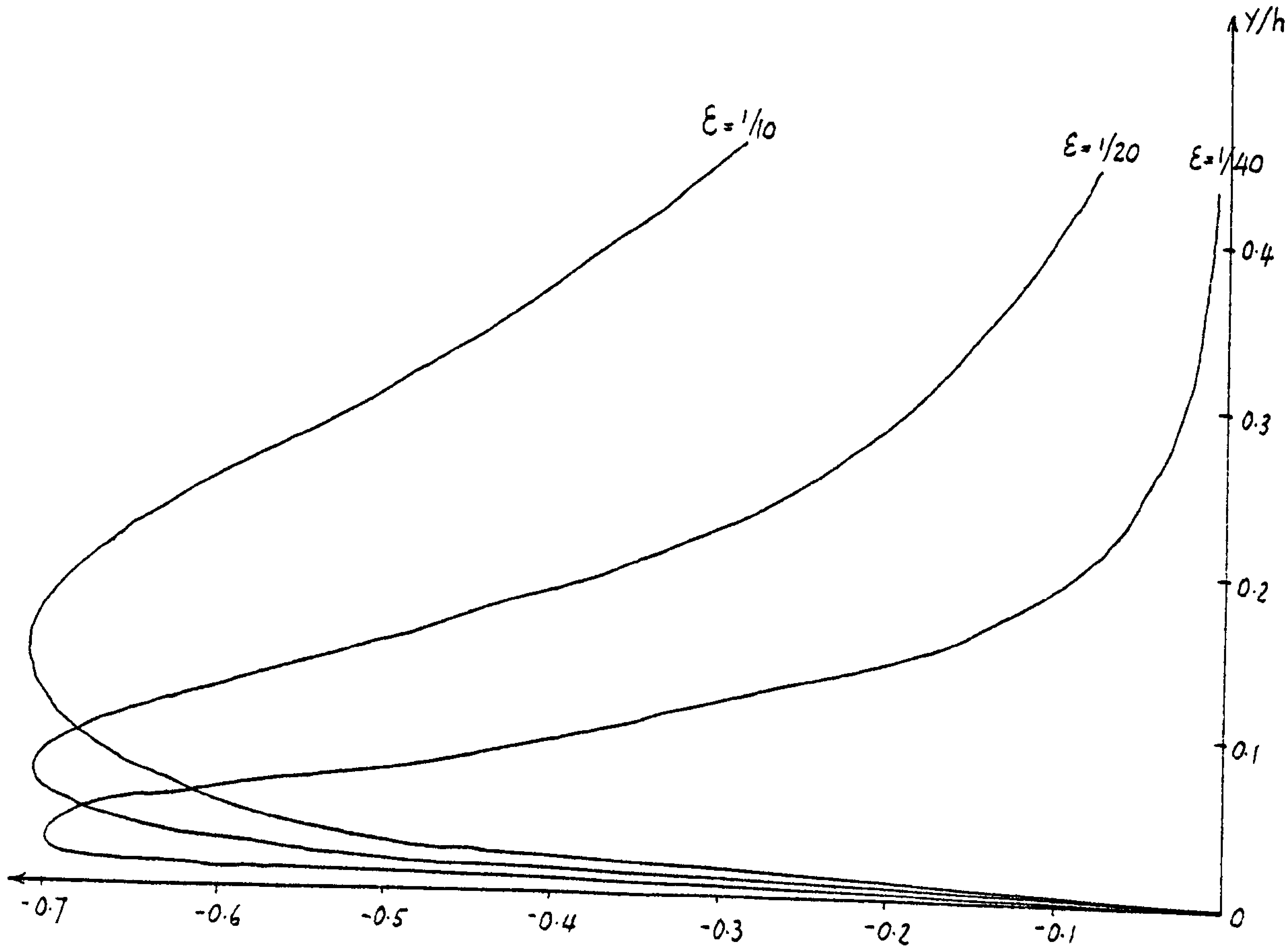


Figure 2.3. *Variation of* $\sum(S_n/h)\sin\lambda_n \exp(-\lambda_n y/\varepsilon L)$ *when* ε = 1/10, 1/20, 1/40.

Figures (2.4a,b) give the variation with K of

$$K_1 = -K\frac{(K - 1)}{(E + K)(K + 1)}, \qquad K_2 = E\,\frac{(K - 1)}{(E + K)(K + 1)} \tag{2.7.7}$$

where $E = \varepsilon_1/\varepsilon_2$. Expressions (2.7.7) are the coefficients of the summations in (2.7.1).

The value of the shear component at x = L in each layer is given by the curves (2.3) to within a multiplicative constant K_i.

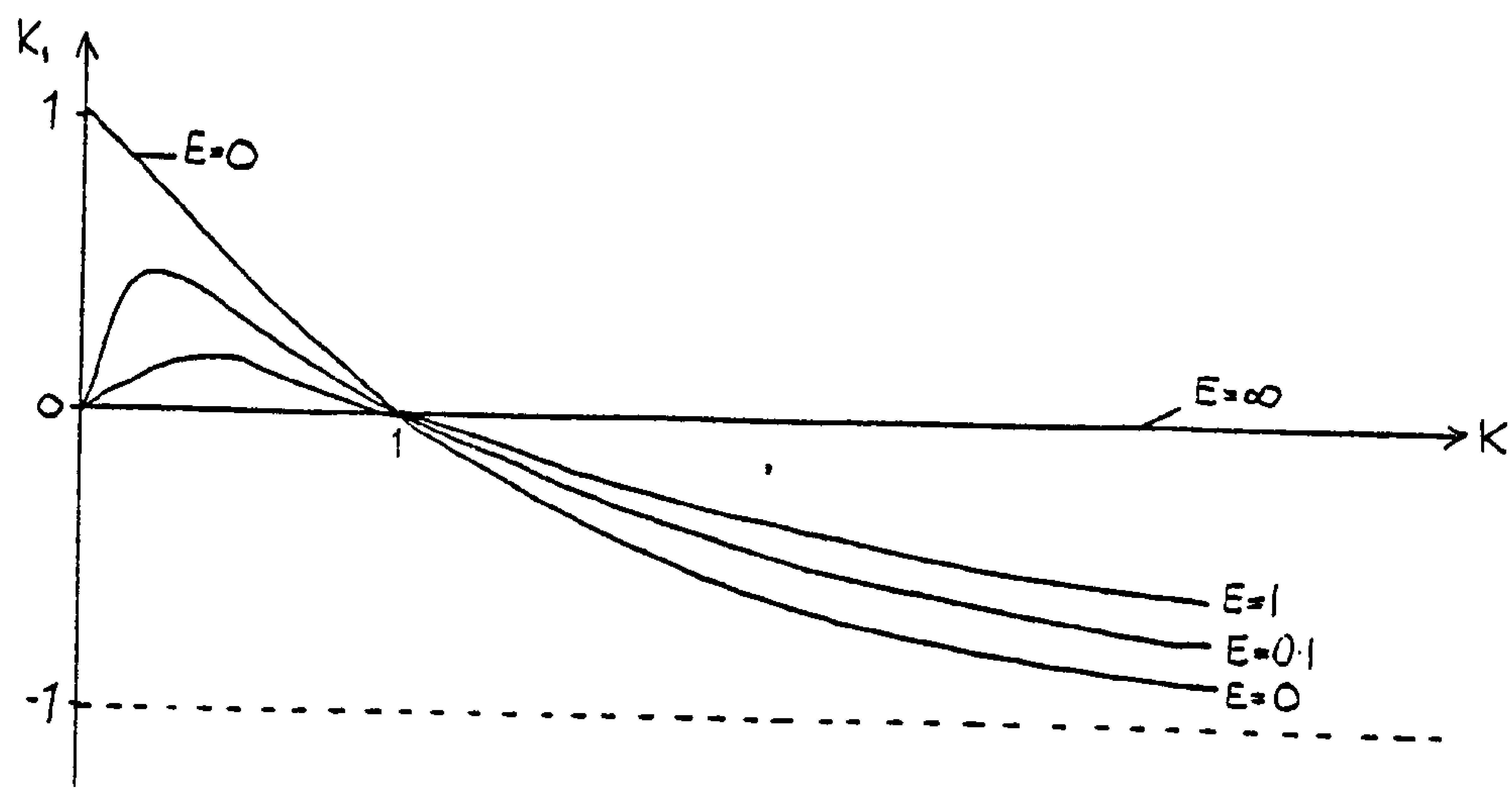


Figure (2.4a). Variation of -K(K - 1)/(E + K)(K + 1) with K for various values of E.

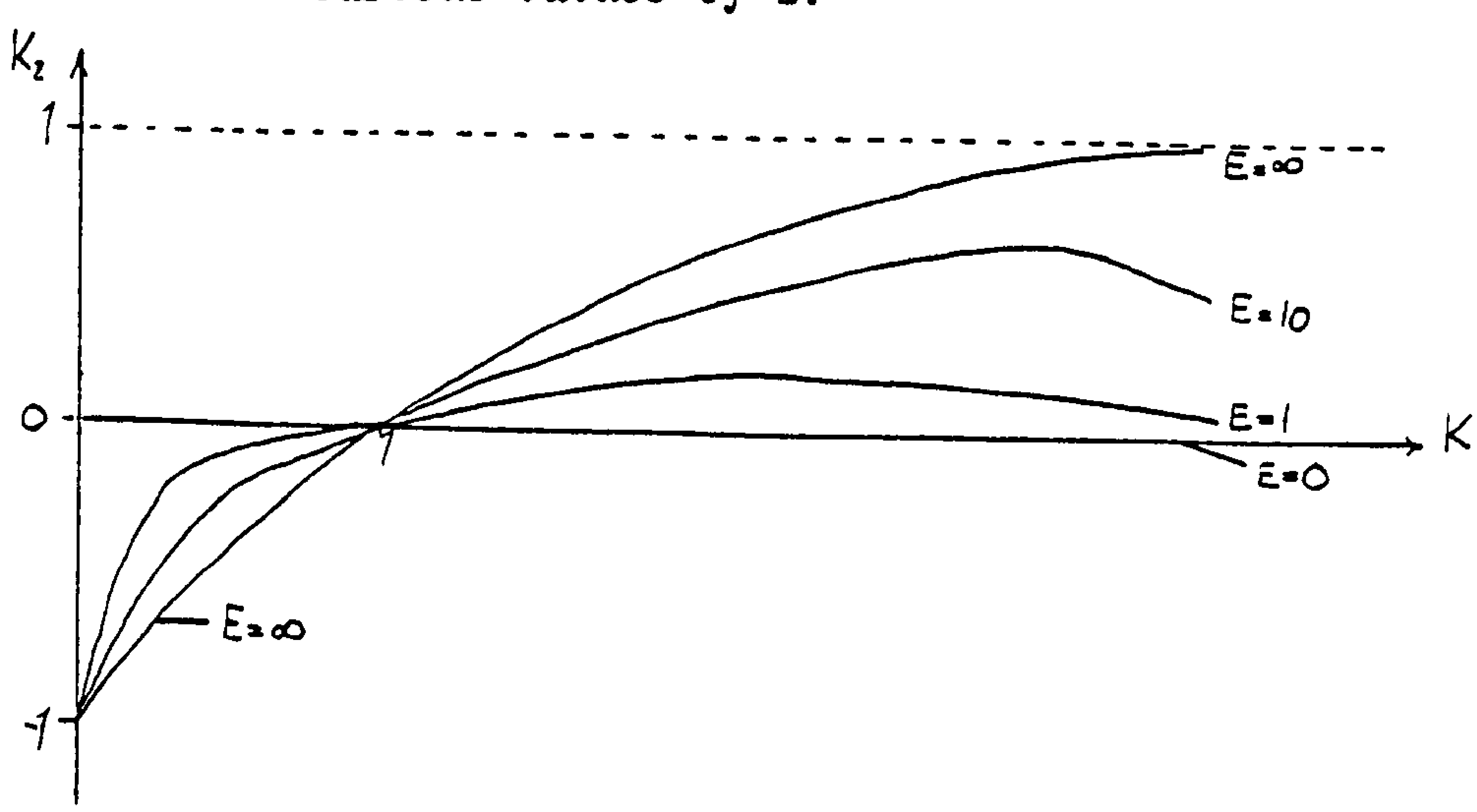
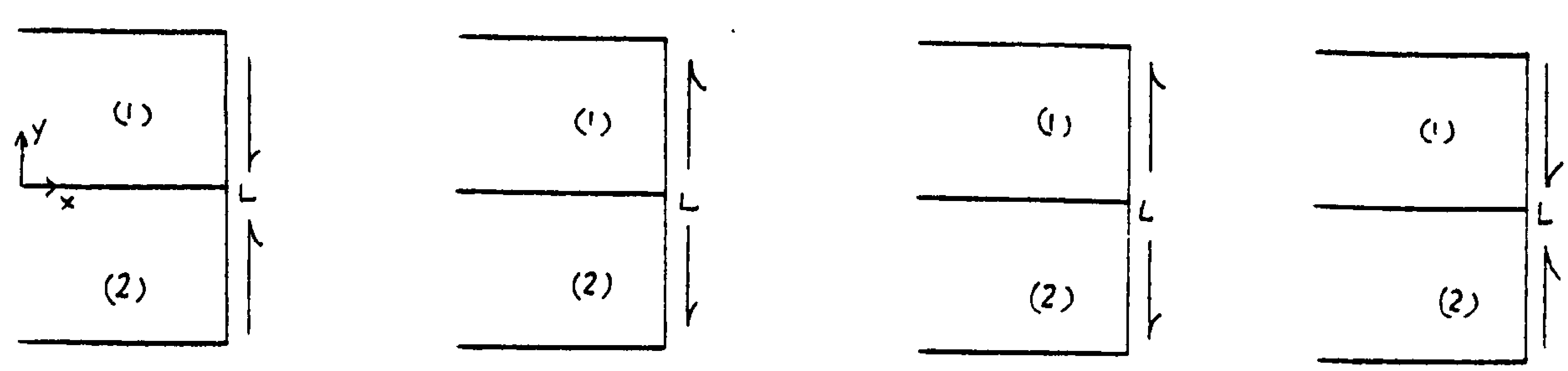


Figure (2.4b). Variation of E(K - 1)/(E + K)(K + 1) with K for various values of E.

Boundary conditions at $x = L$, $y = 0_+, 0_-$ have not been satisfied exactly. In this region an additional correction is required which, when superposed on the solutions already obtained, will correct the values of the shear components. The deviation of the magnitude of the shear components found in this chapter from the prescribed value of zero will determine the magnitude of this correction.

For $K > 1$, we have from Figures (2.3,2.4) that on the end $x = L$ the sign of the shear stress is always positive in layer 1 and always negative in layer 2 and the correction will therefore have the opposite sign. This is illustrated in Figure (2.5). Figures (2.5a,c) indicate the sign of the shear stress component obtained in this chapter and (2.5b,d) indicate the sign of the correction that would be required to restore the traction on the end $x = L$ back to the prescribed value of zero.



(a) $K < 1$ (b) $K < 1$ (c) $K > 1$ (d) $K > 1$

Figure (2.5). Sign of the shear stress.

Since the correction for $K < 1$, that is $\mu^{(1)} < \mu^{(2)}$, is to require 'pulling apart' shear tractions, the tendency for delamination is more apparent in this case.

CHAPTER THREE

THE STRESS SINGULARITY IN A BI-MATERIAL WEDGE

3.1 INTRODUCTION

In linear elasticity, infinite stress can occur at the corners of deformed structures and at points where the boundary conditions can change in type. These points, usually referred to as stress singularities, cannot be sustained by a real material and reflect the limited physical application of linearized theory. To determine whether the presence of such singularities is a consequence of the linearized nature of the equations, exact solutions must be found to the corresponding boundary value problem in finite elasticity.

When bi-material bodies are subjected to excessive loading they have a tendency to delaminate near the free edges of the common interface, and this has been attributed to the generation of high interlaminar stresses. A mathematical study of the equations of linear elasticity near these free edges reveals that the stresses have a power singularity the order of which depends, in general, on the local geometry and the elastic properties of the constituent materials. Hence, the mathematical model does not predict the exact magnitudes of the high interlamina stresses in the real material but it does give an indication of their asymptotic behaviour. This information can prove to be very useful for computational purposes and has aided the construction of special elements in finite element analysis.

Several methods of singularity analysis are available. Williams (1952), by using an Airy stress function and a separation of variables, studies the stress singularity at the apex of an isotropic wedge under different boundary conditions. Williams (1956) and England (1971) have also examined the isotropic wedge by using complex potential theory. Bi-material isotropic wedges are considered by Bogy (1968,1971) and others in which a Mellin transform technique is used. Using a complex Mellin transform, Bogy (1971,1974) extends the results for isotropic materials to include plane deformations of anisotropic wedges. The influence of anisotropy on the order of the power singularity has also been considered by Ting (1981) and Delale (1984). In each of these papers it is assumed that the displacement components are independent of the out-of-plane co-ordinate and the formulation of the problem then follows Lekhnitskii (1963).

In this chapter we consider the stress singularity arising in engineering structures which comprise of two different anisotropic materials bonded together at a common plane interface. This structure is examined with the view of modelling the stress concentrations which can occur at the free end and corners of the deformed cantilever of Chapter 2.

3.2 COMPLEX VARIABLE FORMULATION

For a transversely isotropic material which is in a state of plane strain in the x-y plane and which has a preferred direction parallel to the x-axis, the constitutive equations (1.3.9) reduce to the form

$$\begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 \\ & R22 & 0 \\ \text{sym.} & & R_{66} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}, \tag{3.2.1}$$

where the R_{ij} are related to the elastic moduli of the material by

$$R_{11} = \frac{1}{E_L}(1 - \nu_L^2 \frac{E_T}{E_L}), \qquad R_{12} = -\frac{\nu_L(1 + \nu_T)}{E_L},$$

$$R_{22} = \frac{1 - \nu_T^2}{E_T}, \qquad R_{66} = \frac{1}{\mu_L}. \tag{3.2.2}$$

When the material is isotropic, $E_T = E_L$, $\nu_T = \nu_L$ and it then follows that

$$R_{11} = R_{22} = \frac{1 - \nu^2}{E}, \qquad 2(R_{11} - R_{12}) = R_{66}. \tag{3.2.3}$$

Our main concern will be the behaviour of incompressible transversely isotropic materials. For such materials, the R_{ij} are given by

$$R_{11} = R_{22} = \frac{\varepsilon^2}{\mu_L}, \qquad R_{12} = -\frac{\varepsilon^2}{\mu_L}, \qquad R_{66} = \frac{1}{\mu_L}, \tag{3.2.4}$$

where

$$\varepsilon^2 = \frac{\mu_L}{\beta + 4\mu_L}. \tag{3.2.5}$$

It follows from (3.2.5) that if, in addition to incompressibility, the material is stiff in the fibre direction

$$\varepsilon^2 << 1. \tag{3.2.6}$$

When body forces are absent, the stress components in (3.2.1) satisfy the equilibrium equations,

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0,$$
$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0. \tag{3.2.7}$$

These can be satisfied identically by introducing the Airy stress function $\chi(x,y)$ such that

$$\sigma_{xx} = \frac{\partial^2\chi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\chi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2\chi}{\partial x\partial y}; \tag{3.2.8}$$

then by substituting (3.2.8) into (3.2.1) and using the strain compatibility relation (1.2.3), we obtain

$$(R_{22}\frac{\partial^4}{\partial x^4} + (2R_{12} + R_{66})\frac{\partial^4}{\partial x^2\partial y^2} + R_{11}\frac{\partial^4}{\partial y^4})\chi(x,y) = 0, \tag{3.2.9}$$

or equivalently

$$(\frac{\partial^2}{\partial y^2} + \lambda_1^2\frac{\partial^2}{\partial x^2})(\frac{\partial^2}{\partial y^2} + \lambda_2^2\frac{\partial^2}{\partial x^2})\chi = 0, \tag{3.2.10}$$

where λ_1, λ_2 are the roots of the characteristic equation

$$R_{11}\lambda^4 - (2R_{12} + R_{66})\lambda^2 + R_{22} = 0. \tag{3.2.11}$$

We note that for isotropic materials Equation (3.2.11) gives

$$\lambda_1^2 = \lambda_2^2 = 1, \tag{3.2.12}$$

which follows by using (3.2.3). In general, the roots of (3.2.11) have to be evaluated numerically and Milne-Thomson (1960) has shown that, for any elastic material, they will be real. If the material is incompressible then from (3.2.4) and (3.2.11) we obtain

$$\lambda_1^2\lambda_2^2 = 1, \qquad (3.2.13)$$

which will prove to be a useful relationship in the subsequent analysis. Furthermore, when the fibres are stiff we have

$$\lambda_1^2 = \frac{1}{\varepsilon^2} - 2 - \varepsilon^2 + O(\varepsilon)^4, \qquad \lambda_2^2 = \varepsilon^2 + O(\varepsilon)^4, \qquad \varepsilon^2 << 1. \qquad (3.2.14)$$

By scaling the y co-ordinate by

$$y_j = \lambda_j y \qquad j = 1,2,$$

and introducing the complex variables

$$z_j = x + i\lambda_j y, \qquad (3.2.15)$$

Equation (3.2.10) becomes

$$\frac{\partial^4 \chi}{\partial z_1 \partial \bar{z}_1 \partial z_2 \partial \bar{z}_2} = 0, \qquad (3.2.16)$$

where $(\bar{\ })$ denotes the complex conjugate. The general solution to the stress and displacement components can now be obtained by integrating (3.2.16) and using (3.2.1) and (3.2.8). Following Milne-Thomson (1960) we express these general solutions as

$$\sigma_{xx} = - \sum_j \lambda_j^2 [W_j' + \bar{W}_j'],$$

$$\sigma_{xy} = - \sum_j i\lambda_j [W_j' - \bar{W}_j'],$$

$$\sigma_{yy} = \sum_j [W_j + \bar{W}_j], \qquad j = 1,2 \qquad (3.2.17)$$

$$u = - \sum_j \lambda_j S_j [W_j + \bar{W}_j'],$$

$$v = \sum_j iT_j [W_j - \bar{W}_j],$$

where

$$z_j = x + i\lambda_j y, \qquad (3.2.18)$$

$$S_j = R_{11}\lambda_j - \frac{R_{12}}{\lambda_j}, \qquad T_j = R_{12}\lambda_j - \frac{R_{22}}{\lambda_j}, \qquad (3.2.19)$$

and $W_j = W_j(z_j)$ are arbitrary analytic functions of their arguments.

In the next section we require the stress components referred to polar co-ordinates (r,θ) and these can be obtained by substituting (3.2.17) into the stress transformation equations (1.2.6) as follows

$$\sigma_{rr} = - \sum_j [L_j^2 W_j' + \bar{L}_j^2 \bar{W}_j'],$$

$$\sigma_{r\theta} = \sum_j [L_j \bar{K}_j W_j' + \bar{L}_j K_j \bar{W}_j'], \qquad (3.2.20)$$

$$\sigma_{\theta\theta} = - \sum_j [\bar{K}_j^2 W_j' + K_j^2 \bar{W}_j'],$$

where

$$L_j = \lambda_j \cos\theta + i \sin\theta, \quad K_j = \lambda_j \sin\theta + i\cos\theta, \quad j = 1,2. \quad (3.2.21)$$

Since

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{3.2.22}$$

the z_j in (3.2.18) can be written in the 'polar' form

$$z_j = rf_j e^{i\phi_j}, \tag{3.2.23}$$

where

$$\begin{aligned} f_j &= (\cos^2\theta + \lambda_j^2 \sin^2\theta)^{\frac{1}{2}}, \\ \phi_j &= \tan^{-1}(\lambda_j \tan\theta). \end{aligned} \tag{3.2.24}$$

The convention used in (3.2.24) is that

$$\theta \in [-\pi,\pi], \quad \phi_j \in [-\pi,\pi], \qquad j = 1,2, \tag{3.2.25}$$

and when $\theta = \pm\pi/2$ then $\phi_j = \pm\pi/2$.

3.3 STATEMENT OF THE PROBLEM

A wedge subtends an angle $(\alpha + \beta)$ at the origin of a set of cartesian axes and consists of two different materials. Each material is transversely isotropic with its preferred direction parallel to the x-axis. One of the materials lies in the region $y \geq 0$ and the other lies in $y \leq 0$. The interface of the materials coincides with the x-axis (Figure 3.1).

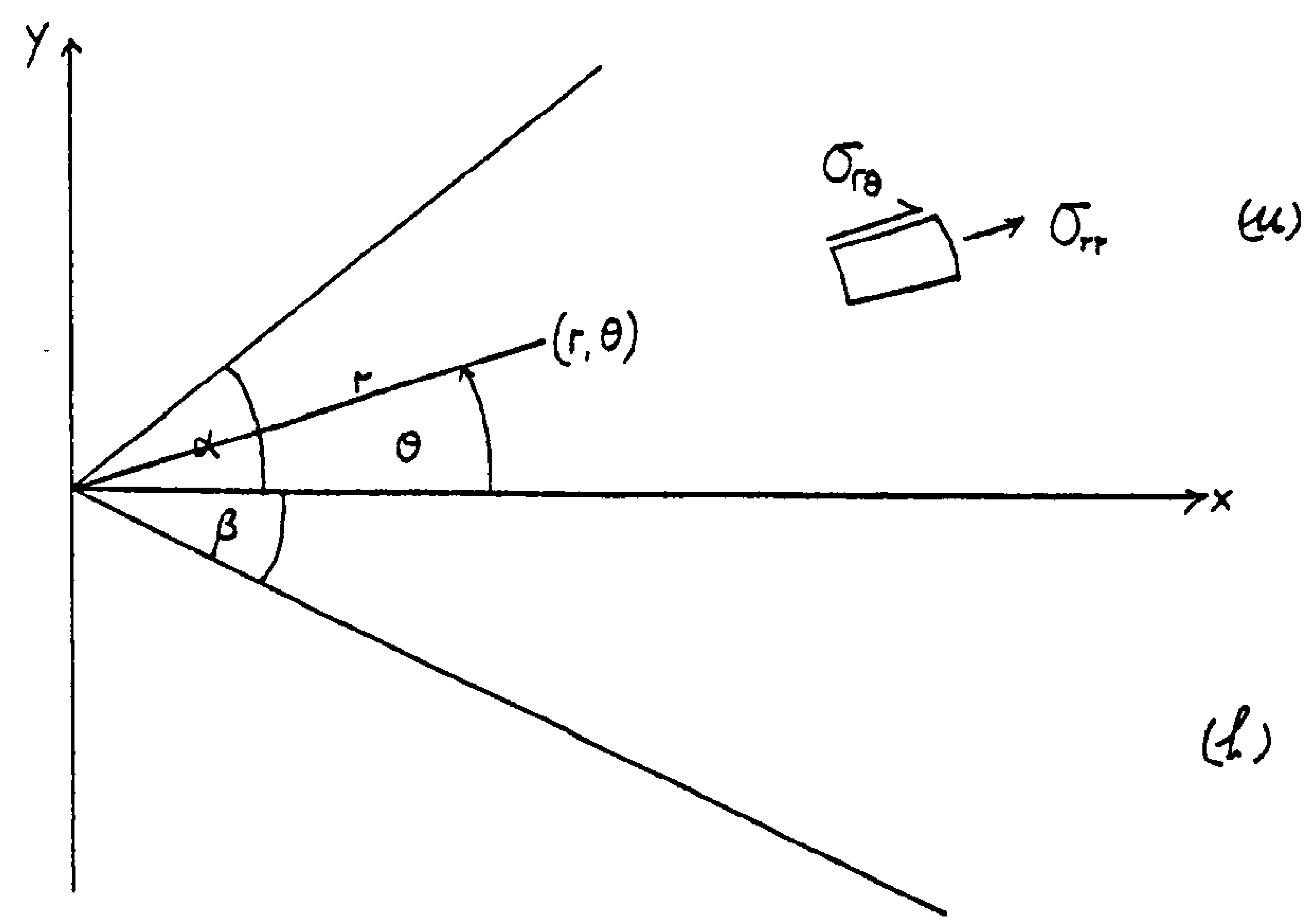


Figure 3.1. Composite Wedge

To distinguish between the two materials, superscripts/subscripts u,ℓ are introduced where u will be used to label quantities associated with the upper material $y \geq 0$ and similarly ℓ for the lower material $y \leq 0$.

If $\alpha = \pi/2$ and $\beta = \pi/2$ then the composite wedge can be used to model the stress concentration which occurs at the free end of the multi-layer cantilever described in Chapter 2.

Since we are only concerned with the singular nature of the stress components in the vicinity of the wedge apex, it is sufficient to consider the homogeneous problem, that is, when the surfaces $\theta = +\alpha, -\beta$ are traction-free. The reason for this is that particular solutions may be used to remove the specified boundary values on the wedge and the nature of the singularity is then associated with the homogeneous solution (England 1971).

The stress boundary condition gives

$$(\sigma_{r\theta})_u = (\sigma_{\theta\theta})_u = 0, \qquad \theta = \alpha,$$
$$(\sigma_{r\theta})_\ell = (\sigma_{\theta\theta})_\ell = 0, \qquad \theta = -\beta. \tag{3.3.1}$$

Continuity of the appropriate field quantities in each material at the common interface requires

$$\begin{aligned}(U_r)_u - (U_r)_\ell &= 0,\\ (U_\theta)_u - (U_\theta)_\ell &= 0,\\ (\sigma_{r\theta})_u - (\sigma_{r\theta})_\ell &= 0,\\ (\sigma_{\theta\theta})_u - (\sigma_{\theta\theta})_\ell &= 0, \qquad \theta = 0.\end{aligned} \tag{3.3.2}$$

Note that in $(3.3.2)_{1,2}$ the polar components of displacement can be replaced by the cartesian components of displacement since, in our case, they are identical at the common interface.

3.4 EIGENVALUE EQUATION

In each of the upper and lower materials we look for solutions of the form

$$\begin{aligned}W_1 &= (C_1 + iD_1)z_1^\rho ,\\ W_2 &= (C_2 + iD_2)z_2^\rho ,\end{aligned} \tag{3.4.1}$$

where C_j, D_j and ρ are arbitrary constants. It follows from (3.2.17), (3.2.20) and (3.2.23) that

$$\sigma_{rr} = -2\mathrm{Re}(\rho r^{\rho-1} \sum_j \{C_j f_j^{\rho-1}(L_j^2 e^{i(\rho-1)\phi_j} + \bar{L}_j^2 e^{-i(\rho-1)\phi_j}) + iD_j f_j^{\rho-1}(L_j^2 e^{i(\rho-1)\phi_j} - \bar{L}_j^2 e^{-i(\rho-1)\phi_j})\})$$

$$\sigma_{r\theta} = 2\mathrm{Re}(\rho r^{\rho-1} \sum_j \{C_j f_j^{\rho-1}(L_j \bar{K}_j e^{i(\rho-1)\phi_j} + \bar{L}_j K_j e^{-i(\rho-1)\phi_j}) + iD_j f_j^{\rho-1}(L_j \bar{K}_j e^{i(\rho-1)\phi_j} - \bar{L}_j K_j e^{-i(\rho-1)\phi_j})\})$$

$$\sigma_{\theta\theta} = -2\mathrm{Re}(\rho r^{\rho-1} \sum_j \{C_j f_j^{\rho-1}(\bar{K}_j^2 e^{i(\rho-1)\phi_j} + K_j^2 e^{-i(\rho-1)\phi_j}) + iD_j f_j^{\rho-1}(\bar{K}_j^2 e^{i(\rho-1)\phi_j} - K_j e^{-i(\rho-1)\phi_j})\}) \tag{3.4.2}$$

$$u = -2\mathrm{Re}(r^{\rho} \sum_j \{\lambda_j S_j f_j^{\rho} C_j (e^{i\rho\phi_j} + e^{-i\rho\phi_j}) + \lambda_j S_j f_j^{\rho} D_j (e^{i\rho\phi_j} - e^{-i\rho\phi_j})\})$$

$$v = 2\mathrm{Re}(r^{\rho} \sum_j \{i\, T_j f_j^{\rho} C_j (e^{i\rho\phi_j} - e^{-i\rho\phi_j}) + T_j f_j^{\rho} D_j (e^{i\rho\phi_j} + e^{-i\rho\phi_j})\}).$$

In (3.4.2), $j = 1,2$ and u,v are cartesian components of displacement. To obtain a non-trivial solution for the arbitrary constants when the boundary conditions (3.3.1,3.3.2) are satisfied we must have

$$\rho_u = \rho_\ell = \rho \text{ say,} \tag{3.4.3}$$

and then we require

$$
\begin{aligned}
(C_1 + C_2)_u - (C_1 + C_2)_\ell &= 0 \\
(\lambda_1 D_1 + \lambda_2 D_2)_u - (\lambda_1 D_1 + \lambda_2 D_2)_\ell &= 0 \\
(\lambda_1 S_1 C_1 + \lambda_2 S_2 C_2)_u - (\lambda_1 S_1 C_1 + \lambda_2 S_2 C_2)_\ell &= 0 \\
(T_1 D_1 + T_2 D_2)_u - (T_1 D_1 + T_2 D_2)_\ell &= 0
\end{aligned}
\tag{3.4.4}
$$

$$
\begin{aligned}
&\sum_j (C_j f_j^{\rho-1} (L_j \bar{K}_j e^{i(\rho-1)\phi_j} + \bar{L}_j K_j e^{-i(\rho-1)\phi_j}) \\
&\qquad + iD_j f_j^{\rho-1} (L_j \bar{K}_j e^{i(\rho-1)\phi_j} - \bar{L}_j K_j e^{-i(\rho-1)\phi_j})) = 0 \\
&\sum_j (C_j f_j^{\rho-1} (\bar{K}_j^2 e^{i(\rho-1)\phi_j} + K_j^2 e^{-i(\rho-1)\phi_j}) \\
&\qquad + iD_j f_j^{\rho-1} (\bar{K}_j^2 e^{i(\rho-1)\phi_j} - K_j e^{-i(\rho-1)\phi_j})) = 0
\end{aligned}
\tag{3.4.5}
$$

where the expressions (3.4.5) are evaluated on $\theta = \alpha$ and $\theta = -\beta$ separately; the appropriate subscripts have been omitted for clarity. Equations (3.4.4) and (3.4.5) can be written as

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\
\lambda_{1u}S_{1u} & 0 & \lambda_{2u}S_{2u} & 0 & -\lambda_{1\ell}S_{1\ell} & 0 & -\lambda_{2\ell}S_{2\ell} & 0 \\
a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 \\
a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 \\
0 & T_{1u} & 0 & T_{2u} & 0 & -T_{1\ell} & 0 & -T_{2\ell} \\
0 & \lambda_{1u} & 0 & \lambda_{2u} & 0 & -\lambda_{1\ell} & 0 & -\lambda_{2\ell} \\
0 & 0 & 0 & 0 & b_{31} & b_{32} & b_{33} & b_{34} \\
0 & 0 & 0 & 0 & b_{41} & b_{42} & b_{43} & b_{44}
\end{pmatrix}
\begin{pmatrix}
C_{1u} \\ D_{1u} \\ C_{2u} \\ D_{2u} \\ C_{1\ell} \\ D_{1\ell} \\ C_{2\ell} \\ D_{2\ell}
\end{pmatrix}
= \underset{\sim}{0}
\tag{3.4.6}
$$

where

$$
\begin{aligned}
a_{31} &= f_{1u}^{\rho-1}(L_{1u}\bar{K}_{1u}e^{i(\rho-1)\phi_{1u}} + \bar{L}_{1u}K_{1u}e^{-i(\rho-1)\phi_{1u}}),\\
a_{32} &= f_{1u}^{\rho-1}i(L_{1u}\bar{K}_{1u}e^{i(\rho-1)\phi_{1u}} - \bar{L}_{1u}K_{1u}e^{-i(\rho-1)\phi_{1u}}),\\
a_{33} &= f_{2u}^{\rho-1}(L_{2u}\bar{K}_{2u}e^{i(\rho-1)\phi_{2u}} + \bar{L}_{2u}K_{2u}e^{-i(\rho-1)\phi_{2u}}),\\
a_{34} &= f_{2u}^{\rho-1}i(L_{2u}\bar{K}_{2u}e^{i(\rho-1)\phi_{2u}} - \bar{L}_{2u}K_{2u}e^{-i(\rho-1)\phi_{2u}}),\\
a_{41} &= f_{1u}^{\rho-1}(\bar{K}_{1u}^{2}e^{i(\rho-1)\phi_{1u}} + K_{1u}^{2}e^{-i(\rho-1)\phi_{1u}}),\\
a_{42} &= f_{1u}^{\rho-1}i(\bar{K}_{1u}^{2}e^{i(\rho-1)\phi_{1u}} - K_{1u}^{2}e^{-i(\rho-1)\phi_{1u}}),\\
a_{43} &= f_{2u}^{\rho-1}(\bar{K}_{2u}^{2}e^{i(\rho-1)\phi_{2u}} + K_{2u}^{2}e^{-i(\rho-1)\phi_{2u}}),\\
a_{44} &= f_{2u}^{\rho-1}i(\bar{K}_{2u}^{2}e^{i(\rho-1)\phi_{2u}} - K_{2u}^{2}e^{-i(\rho-1)\phi_{2u}}).
\end{aligned}
\tag{3.4.7}
$$

The b_{ij} in (3.4.6) are obtained from (3.4.7) by replacing a by b and u by ℓ. Note that all u quantities are evaluated at $\theta = \alpha$ whilst ℓ quantities are evaluated at $\theta = -\beta$. Equations (3.4.6) represent a homogeneous system of eight equations for eight unknowns and therefore a non-trivial solution can exist only if ρ is chosen such that the determinant Δ of the coefficients vanishes, that is

$$\Delta(\rho) = 0. \tag{3.4.8}$$

Equation (3.4.8) has an infinite number of roots. By imposing the physical condition that the strain energy be bounded in a finite region, it can be shown that the real part of ρ must be greater than zero, which is equivalent to stating that the displacement components must remain bounded for small r. Furthermore, a singular stress state will only prevail if $Re(\rho) < 1$. Therefore it is sufficient to search for the root with the smallest real part and which satisfies the condition

$$0 < \mathrm{Re}(\rho) < 1. \tag{3.4.9}$$

After lengthy manipulation it may be shown that

$$\begin{aligned}
\Delta = -R^{u}_{11}R^{\ell}_{11}(\lambda^2_{1u} - \lambda^2_{2u})(\lambda^2_{2\ell} - \lambda^2_{1\ell})\{ & (\lambda_{2\ell}\lambda_{1\ell} + \lambda_{2u}\lambda_{1u})(Q_{1u}Q_{1\ell} - Q_{2u}Q_{2\ell}) \\
& + (\lambda_{2\ell}\lambda_{1\ell} - \lambda_{1u}\lambda_{2u})(Q_{2u}Q_{1\ell} - Q_{1u}Q_{2\ell}) \\
& - 16\lambda_{1u}\lambda_{2u}\lambda_{1\ell}\lambda_{2\ell}(f^{2\rho}_{2\ell} - f^{2\rho}_{1\ell})(f^{2\rho}_{2u} - f^{2\rho}_{1u}) \\
& + (\lambda_{1u}P_{1u} + K_5)(\lambda_{1\ell}P_{1\ell} + K_6) \\
& + (P_{1\ell}\lambda_{1\ell} + K_2)(P_{1u}\lambda_{1u} + K_4)\} \\
- (R^{u}_{11}\lambda^2_{1u} - R^{\ell}_{11}\lambda^2_{1\ell} - R^{u}_{12} + R^{\ell}_{12})^2 & (P_{1\ell}(\lambda_{2\ell} - \lambda_{1\ell}) - K_1)(P_{1u}(\lambda_{2u} - \lambda_{1u}) - K_3) \\
+ \lambda_{1u}R^{u^2}_{11}(\lambda^2_{1u} - \lambda^2_{2u})^2 & (P_{1\ell}(\lambda_{2\ell} - \lambda_{1\ell}) - K_1)(P_{1u} + P_{2u}) \\
+ \lambda_{1\ell}R^{\ell^2}_{11}(\lambda^2_{1\ell} - \lambda^2_{2\ell})^2 & (P_{1u}(\lambda_{2u} - \lambda_{1u}) - K_3)(P_{1\ell} + P_{2\ell}) \\
- R^{u}_{11}(\lambda^2_{1u} - \lambda^2_{2u})(R^{u}_{11}\lambda^2_{1u} - R^{\ell}_{11}\lambda^2_{1\ell} - R^{u}_{12} + R^{\ell}_{12}) & (P_{1\ell}(\lambda_{2\ell} - \lambda_{1\ell}) - K_1) \\
& \times (2P_{1u}\lambda_{1u} + K_4 + K_5) \\
+ R^{\ell}_{11}(\lambda^2_{1\ell} - \lambda^2_{2\ell})(R^{u}_{11}\lambda^2_{1u} - R^{\ell}_{11}\lambda^2_{1\ell} - R^{u}_{12} + R^{\ell}_{12}) & (P_{1u}(\lambda_{2u} - \lambda_{1u}) - K_3) \\
& \times (2P_{1\ell}\lambda_{1\ell} + K_2 + K_6)
\end{aligned} \tag{3.4.10}$$

where

$$
\begin{aligned}
Q_{1\ell} &= 2f_{1\ell}^{\rho}f_{2\ell}^{\rho}(\lambda_{2\ell} - \lambda_{1\ell})\sin\gamma_{\ell}, & P_{1\ell} &= 2f_{1\ell}^{\rho}f_{2\ell}^{\rho}(\lambda_{2\ell} - \lambda_{1\ell})\cos\gamma_{\ell},\\
Q_{2\ell} &= 2f_{1\ell}^{\rho}f_{2\ell}^{\rho}(\lambda_{2\ell} + \lambda_{1\ell})\sin\psi_{\ell}, & P_{2\ell} &= 2f_{1\ell}^{\rho}f_{2\ell}^{\rho}(\lambda_{2\ell} + \lambda_{1\ell})\cos\psi_{\ell},\\
Q_{1u} &= 2f_{1u}^{\rho}f_{2u}^{\rho}(\lambda_{2u} - \lambda_{1u})\sin\gamma_{u}, & P_{1u} &= 2f_{1u}^{\rho}f_{2u}^{\rho}(\lambda_{2u} - \lambda_{1u})\cos\gamma_{u},\\
Q_{2u} &= 2f_{1u}^{\rho}f_{2u}^{\rho}(\lambda_{2u} + \lambda_{1u})\sin\psi_{u}, & P_{2u} &= 2f_{1u}^{\rho}f_{2u}^{\rho}(\lambda_{2u} + \lambda_{1u})\cos\psi_{u},
\end{aligned}
\tag{3.4.11}
$$

$$
\gamma_k = \rho(\phi_{1k} + \phi_{2k}), \qquad \psi_k = \rho(\phi_{1k} - \phi_{2k}), \qquad k = u,\ell \tag{3.4.12}
$$

$$
\begin{aligned}
K_1 &= P_{2\ell}(\lambda_{1\ell} + \lambda_{2\ell}) - 4(\lambda_{2\ell}\lambda_{1\ell}f_{1\ell}^{2\rho} + \lambda_{1\ell}\lambda_{2\ell}f_{2\ell}^{2\rho}),\\
K_3 &= P_{2u}(\lambda_{1u} + \lambda_{2u}) - 4(\lambda_{2u}\lambda_{1u}f_{2u}^{2\rho} + \lambda_{1u}\lambda_{2u}f_{1u}^{2\rho}),\\
K_2 &= \lambda_{1\ell}(P_{2\ell} - 4\lambda_{2\ell}f_{2\ell}^{2\rho}),\\
K_4 &= \lambda_{1u}(P_{2u} - 4\lambda_{2u}f_{2u}^{2\rho}),\\
K_6 &= \lambda_{1\ell}(P_{2\ell} - 4\lambda_{2\ell}f_{1\ell}^{2\rho}),\\
K_5 &= \lambda_{1u}(P_{2u} - 4\lambda_{2u}f_{1u}^{2\rho}).
\end{aligned}
\tag{3.4.13}
$$

We note that each term in Δ contains the expression $(\lambda_{2u} - \lambda_{1u})(\lambda_{2\ell} - \lambda_{1\ell})$ and that this property will be useful when the isotropic limit is considered later in this section.

The roots of $\Delta = 0$ have to be evaluated numerically. However, considerable simplification of Δ can be achieved if the following expression can be simplified

$$
f_{ij} = (\cos^2\theta + \lambda_{ij}^2\sin^2\theta)^{\frac{1}{2}}. \tag{3.4.14}
$$

Two such cases are

$$\text{(i)} \quad \theta = \pi. \qquad\qquad \text{(ii)} \quad \lambda_{ij} = 1. \tag{3.4.15}$$

Physically case (i) will correspond to a crack between two dissimilar media and case (ii) will correspond to an isotropic wedge consisting of two different materials. These two cases are now considered.

(i) Crack Between Dissimilar Transversely Isotropic Media

If $\alpha = \beta = \pi$ in (3.4.10) then after some manipulation we obtain

$$\Delta = \cos^2\rho\pi + \Psi^2\sin^2\rho\pi = 0, \tag{3.4.16}$$

where

$$\Psi^2 = [(R^u_{11}\lambda_{1u}\lambda_{2u} + R^u_{12}) - (R^\ell_{11}\lambda_{1\ell}\lambda_{2\ell} + R^\ell_{12})]^2/[R^u_{11}\lambda_{1u}\lambda_{2u}(\lambda_{1u} + \lambda_{2u}) + R^\ell_{11}\lambda_{1\ell}\lambda_{2\ell}(\lambda_{1\ell} + \lambda_{2\ell})][R^u_{11}(\lambda_{1u} + \lambda_{2u}) + R^\ell_{11}(\lambda_{1\ell} + \lambda_{2\ell})]. \tag{3.4.17}$$

Since Δ is positive for real values of ρ, the equation $\Delta = 0$ can only be satisfied if ρ is taken to be a complex number. Let ρ_1 and ρ_2 be real numbers such that

$$\rho = \rho_1 + i\rho_2. \tag{3.4.18}$$

By substituting (3.4.18) into (3.4.16) and equating real and imaginary coefficients gives

$$\cos \rho_1\pi(\cosh \rho_2\pi \pm \Psi \sinh \rho_2\pi) = 0$$
$$\sin \rho_1\pi(\sinh \rho_2\pi \pm \Psi \cosh \rho_2\pi) = 0. \qquad (3.4.19)$$

The solutions to (3.4.19) are

$$\rho_1 = (n + \tfrac{1}{2}),$$
$$\rho_2 = \pm \frac{1}{\pi} \tanh^{-1}(\Psi),$$

or

$$\rho_1 = n,$$
$$\rho_2 = \pm \frac{1}{\pi} \coth^{-1}(\Psi), \qquad (3.4.20)$$

where n is an integer. Since ρ_1 is chosen from the range of values given in (3.4.9), the only admissible value for ρ_1 is 0.5. This means that for the case of a crack between dissimilar media, the asymptotic behaviour of the stress components in the region of the crack tip is $O(1/r^{\frac{1}{2}+i\rho_2})$ which is in agreement with Bonser (1984).

(ii) A Wedge Consisting of Two Isotropic Dissimilar Media

The characteristic equation for a wedge consisting of two isotropic materials is obtained from (3.4.10) in the limit $\lambda_{ij} = 1$. If (3.4.10) is initially divided by

$$(\lambda_{2u} - \lambda_{1u})\ (\lambda_{2\ell} - \lambda_{1\ell})\ ,$$

and the λ_{ij} is then allowed to tend to 1, the following equation is obtained.

$$\Delta = 4B^2D(\rho,\alpha)D(\rho,\beta) + 4AB\rho^2(\sin^2\beta D(\rho,\alpha) + \sin^2\alpha D(\rho,\beta))$$

$$+ A^2(\sin^2\alpha \sin^2\beta . 4\rho^2(\rho^2 - 1) + D(\rho,\alpha-\beta)) + B4\rho^2(\sin^2\alpha \sin^2\rho\beta - \sin^2\beta \sin^2\rho\alpha)$$

$$+ 2A(2\rho^2(\sin^2\beta \sin^2\rho\alpha - \sin^2\alpha \sin^2\rho\beta) + D(\rho,\beta) - D(\rho,\alpha))$$

$$+ D(\rho,\alpha+\beta) \tag{3.4.21}$$

where

$$A = \frac{R_{11}^{u} - R_{11}^{\ell}}{R_{11}^{u} + R_{11}^{\ell}}, \qquad B = \frac{(R_{11}^{u} + R_{12}^{u}) - (R_{11}^{\ell} + R_{12}^{\ell})}{2(R_{11}^{u} + R_{11}^{\ell})}, \tag{3.4.22}$$

$$D(\rho,x) = \sin^2\rho x - \rho^2\sin^2 x.$$

Equation (3.4.21) is identical to that found by Bogy (1971).

For a homogeneous isotropic wedge $A = B = 0$. In this case ρ satisfies (England 1971)

$$\sin 2\alpha\rho \pm \rho \sin 2\alpha = 0, \tag{3.4.23}$$

where 2α is the total angle at the apex of the wedge. The upper sign in (3.4.23) corresponds to the symmetric deformation of the wedge, whilst the lower corresponds to skew-symmetric deformation. In Figure 3.1 the variation of ρ is given for various values of α.

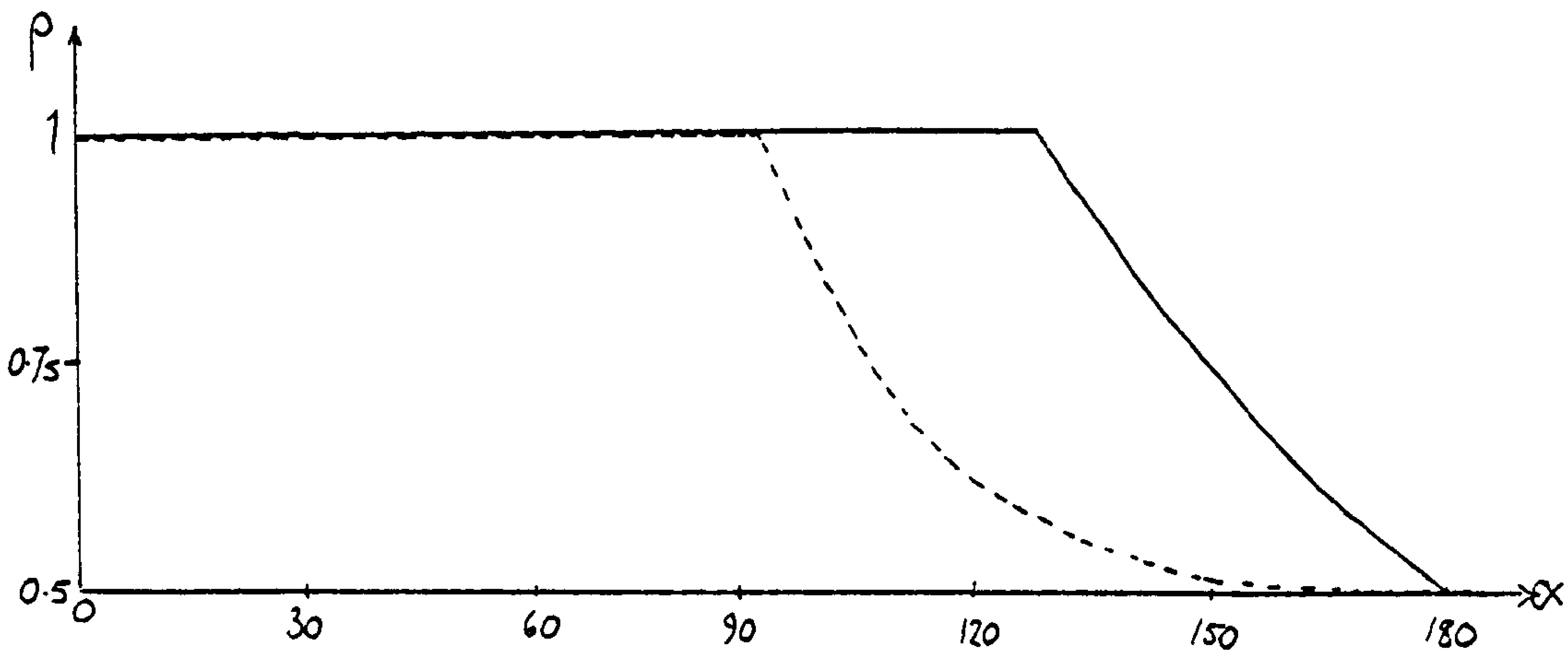


Figure 3.2. *Roots of (3.4.23)*

-------- *Symmetric Deformation*
———— *Skew-Symmetric Deformation*

The asymptotic behaviour of the stress components at a crack tip is identical to that found for an anisotropic wedge except that in the latter ρ is a complex value of which the real part is 0.5, whilst in the above ρ is 0.5 and purely real.

Another interesting case for an isotropic wedge of two dissimilar materials is when $\alpha = \beta = \pi/2$. From (3.4.21) we obtain

$$\Delta = 4B^2 D(\rho,\pi/2)D(\rho,\pi/2) + 8AB\rho^2 D(\rho,\pi/2) + 4A^2\rho^2(\rho^2 - 1) + D(\rho,\pi) = 0. \quad (3.4.24)$$

The roots of (3.4.24) have been evaluated by the bisection method and are given in Figure 3.3 for various values of E, ν_u, ν_ℓ where $E = E_u/E_\ell$ is the ratio of the Youngs moduli of each layer and ν_i are the Poisson's ratios. Note that Equation (3.4.24) is not altered by interchanging ν_u and ν_ℓ. As can be seen from Figure 3.3, the change in Poisson's ratio in each material of the wedge does not have a significant effect on the order of the singularity.

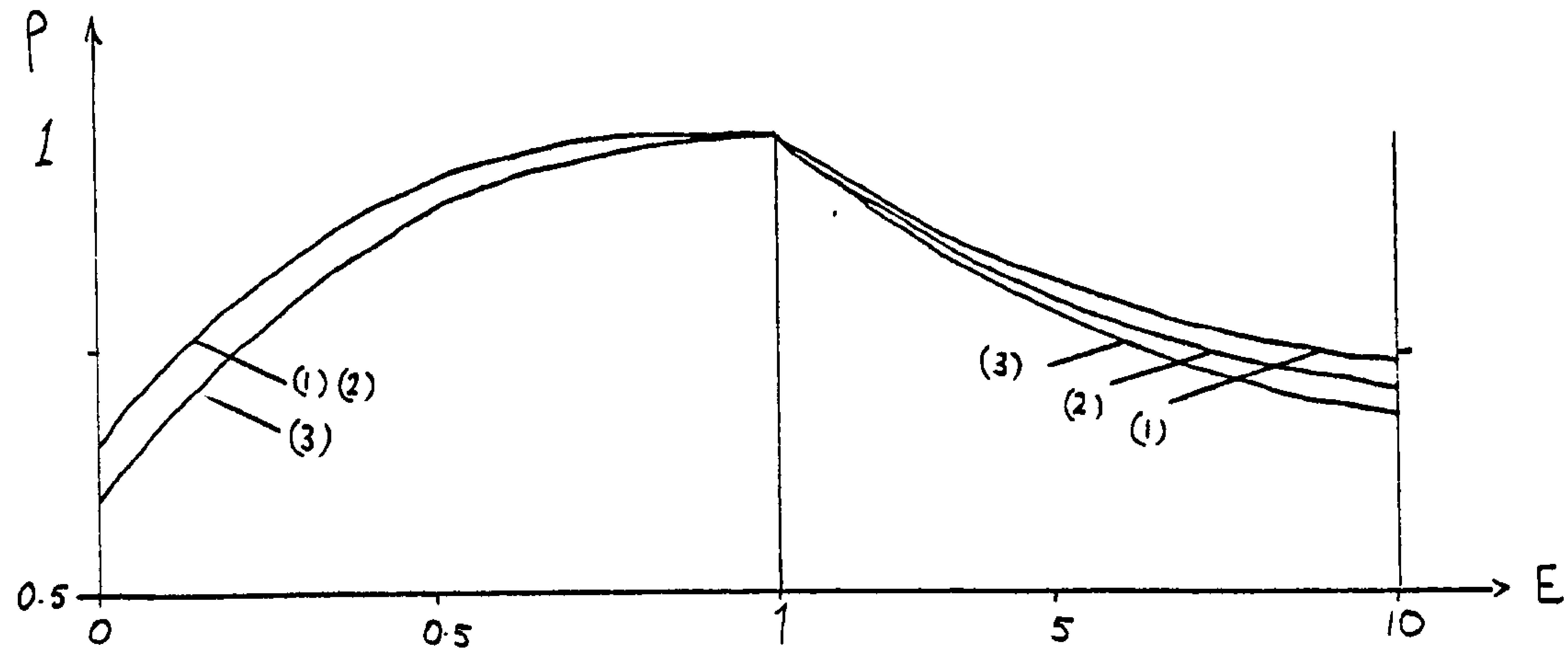


Figure 3.3. Roots of (3.4.24)

(1) $\nu_u = \nu_\ell = 0.3$, *(2)* $\nu_u = 0.3, \nu_\ell = 0.5$, *(3)* $\nu_u = \nu_\ell = 0.5$

3.5 EDGE BONDED INCOMPRESSIBLE ORTHOGONAL WEDGES

In this section we evaluate values for ρ satisfying (3.4.10) for the case when $\alpha = \beta = \pi/2$. The 'wedge' is taken to consist of two dissimilar incompressible transversely isotropic materials which have a preferred direction parallel to that of their common interface. This particular geometry is considered since it can be used to model the asymptotic behaviour of the stress components in the neighbourhood of the free end $x = L$, $y = 0$ of the cantilever described in Chapter 2.

By substituting $\alpha = \beta = \pi/2$ into (3.4.10) and making use of (3.2.13) we obtain

$$\Delta = \cos^2\rho\pi - \frac{2\cos\rho\pi(E\lambda_u - \lambda_\ell)(G_1G_2 - G_3G_4)}{(E\lambda_u + \lambda_\ell)(1 - \lambda_u^2)^2(1 - \lambda_\ell^2)^2}$$

$$- \left\{1 + \frac{2\{(EG_1 - G_3)(E\lambda_u^2G_2 - \lambda_\ell^2G_4) + (EG_4 - G_2)(E\lambda_u^2G_3 - \lambda_\ell^2G_1)\}}{(E\lambda_u + \lambda_\ell)^2(1 - \lambda_u^2)^2(1 - \lambda_\ell^2)^2}\right\}, \quad (3.5.1)$$

where

$$G_1 = (1 - \lambda_\ell^{2\rho})(\lambda_u^{2\rho+2} - 1),$$

$$G_2 = (\lambda_\ell^{2-2\rho} - \lambda_\ell^2)(1 - \lambda_u^{2-2\rho}),$$

$$G_3 = (1 - \lambda_\ell^{2\rho+2})(\lambda_u^{2\rho} - 1), \quad (3.5.2)$$

$$G_4 = (\lambda_\ell^{2-2\rho} - 1)(\lambda_u^2 - \lambda_u^{2-2\rho}),$$

$$E = \frac{\mu_\ell}{\mu_u}\frac{(1 + \lambda_\ell^2)}{(1 + \lambda_u^2)}. \quad (3.5.3)$$

In (3.5.1), λ_i^2 are the roots of the characteristic equation (3.2.11) and are given by

$$\lambda_i^2 = \frac{2\varepsilon_i^2}{(1 - 2\varepsilon_i^2) + \sqrt{(1 - 4\varepsilon_i^2)}} \qquad i = u,\ell, \quad (3.5.4)$$

where

$$\varepsilon_i^2 = 1/(\beta_i/\mu_i + 4). \qquad i = u,\ell. \quad (3.5.5)$$

Since

$$0 \le \varepsilon_i^2 \le \tfrac{1}{4}, \quad (3.5.6)$$

where the upper bound corresponds to an incompressible isotropic material and the lower bound to an ideal material, it follows from (3.5.4) that

$$0 \leq \lambda_i^2 \leq 1, \qquad i = u, \ell. \tag{3.5.7}$$

For comparison purposes we initially consider the isotropic form of (3.5.1) which is obtained by letting $\lambda_u^2 = \lambda_\ell^2 = 1$. Since this limit is not readily available, we let $\lambda_j^2 = 1 + \xi_j$ and then consider $\xi_j \to 0$. This produces

$$\Delta = \sin^2\rho\pi - 4k^2\rho^2(1 - \rho^2), \tag{3.5.8}$$

where

$$k^2 = \left(\frac{\mu_u - \mu_\ell}{\mu_u + \mu_\ell}\right)^2. \tag{3.5.9}$$

Equation (3.5.8) is identical to the expression for Δ in (3.4.24) when $\nu = \frac{1}{2}$. In Figure 3.4 the variation with ρ of $\sin^2\rho\pi$ and $4k^2\rho^2(1 - \rho^2)$ is given and the relevant root of $\Delta = 0$ is therefore given by the intersection of these curves. We note that k^2 is the maximum value attained by $4k^2\rho^2(1 - \rho^2)$ when $\rho \in [0,1]$.

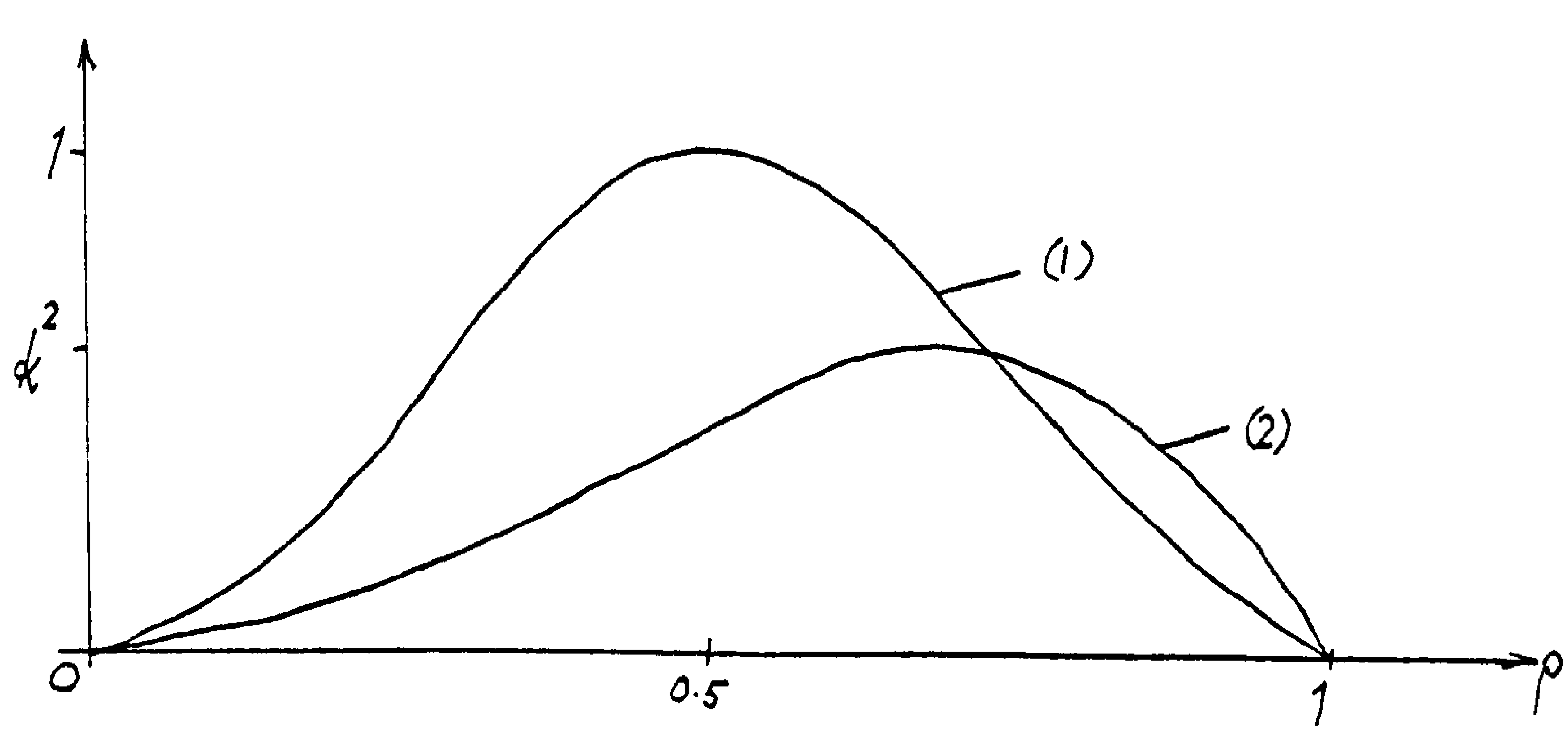


Figure 3.4. (1) $\sin^2\rho\pi$, (2) $4k^2\rho^2(1 - \rho^2)$.

It is seen from Figure 3.4 that the value of ρ is in the range

$$\tfrac{1}{2} < \rho \leq 1, \tag{3.5.10}$$

and depends on the value of the shear moduli of the materials. When the materials are identical, $k^2 = 0$ and so $\rho = 1$. The other extreme value of ρ occurs when $k^2 = 1$ and physically this corresponds to the case when one of the materials is rigid. When $k^2 = 1$

$$\rho \simeq 0.5946. \tag{3.5.11}$$

By examining (3.5.1) we find that

$$\Delta = \Delta(\lambda_u, \lambda_\ell, E, \rho) \tag{3.5.12}$$

and therefore the roots of $\Delta = 0$ are functions of the three parameters λ_u, λ_ℓ and E. Furthermore, it can be shown that

$$\Delta(0, \lambda_\ell, 0, \rho) = \Delta(\lambda_u, 0, \infty, \rho) = (\cos \rho\pi - 1)(\cos \rho\pi - 2F + 1), \tag{3.5.13}$$

where

$$F = \frac{\lambda^2}{\lambda^{2\rho}} \cdot \frac{(1 - \lambda^{2\rho})^2}{(1 - \lambda^2)^2} \, . \tag{3.5.14}$$

Physically, the eigenvalue equation given by (3.5.13) corresponds to the case when one of the materials of the wedge is rigid. The first factor of Δ yields no relevant values for ρ and therefore ρ will be determined by the second factor. In Figure 3.5 these roots are given for all the permitted values of λ. The extreme values for

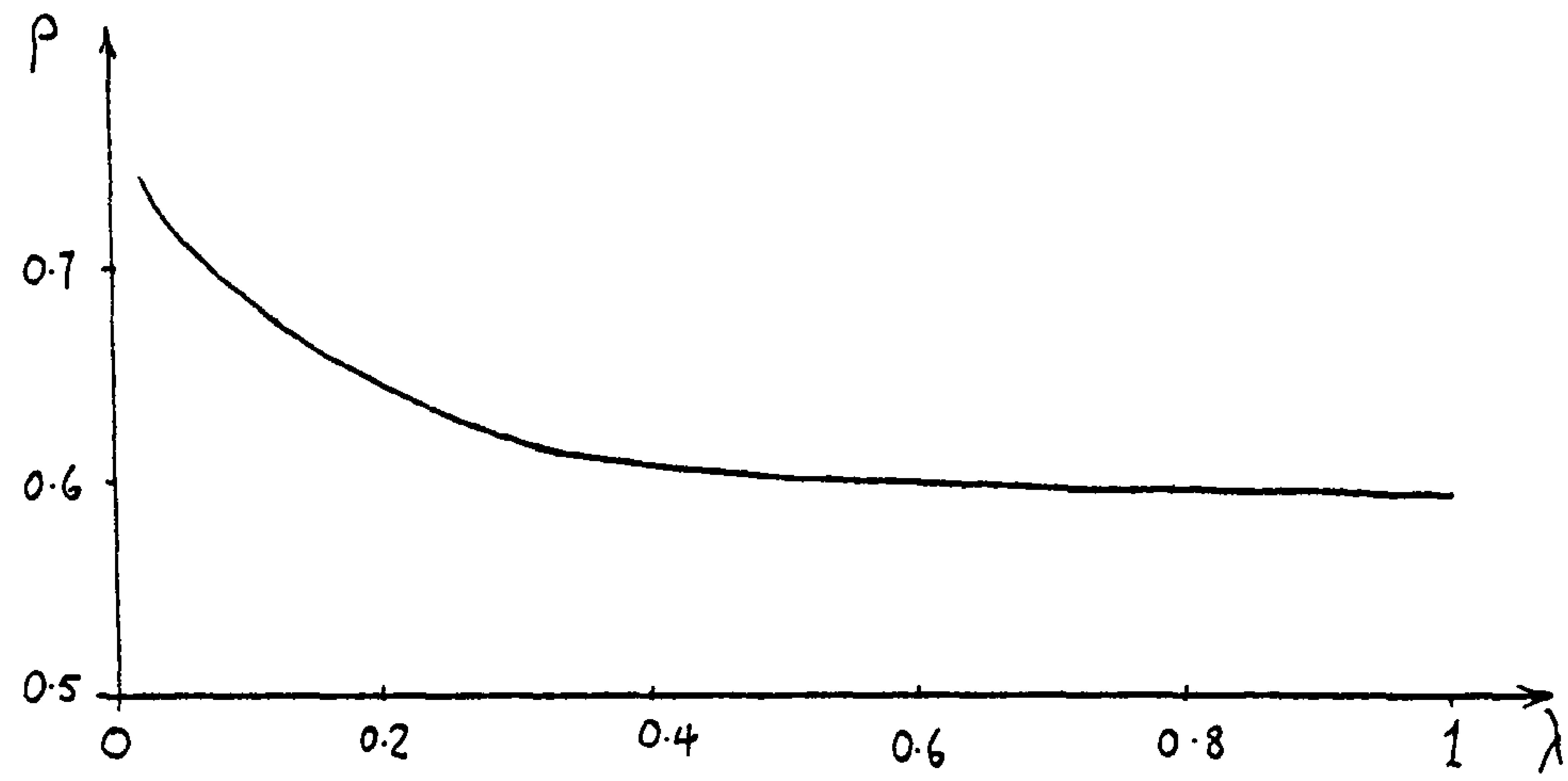


Figure 3.5. Roots of (3.5.13).

ρ, from Figure 3.5, are 1 and $\simeq 0.595$, the latter being the isotropic value and significant changes in the value of ρ from this isotropic value occur as the material becomes highly anisotropic, that is as $\lambda \to 0$.

Now consider

$$\Delta(\lambda,\lambda,1,\rho) = 0. \tag{3.5.15}$$

Equation (3.5.15) is the eigenvalue equation for a 'wedge' which has both upper and lower materials the same. The roots of this equation follow immediately from (3.5.1) to be

$$\rho = n, \tag{3.5.16}$$

where n is an integer.

The order of the singularity for the case when both the upper and lower materials are highly anisotropic is now considered. For such materials

$$\left(\frac{\beta}{\mu}\right)_i \gg 1 \qquad\qquad i = u, \ell,$$

where β and μ are elastic constants. For numerical purposes we take the following values

$$\beta_u = 240.10^9\ \text{Nm}^{-2}, \qquad \mu_u = 6.0.10^9\ \text{Nm}^{-2},$$

which are the approximate values for epoxy resin reinforced by carbon fibres. In the lower material we let

$$\mu_\ell = 3.0.10^9\ \text{Nm}^{-2}$$

and allow β_ℓ to vary from its 'isotropic' value of almost zero to the value of β_u. In Figure 3.6 the values of ρ are given to $\Delta = 0$ of (3.5.1) when the above data is used.

β_ℓ ($\times 10^{-9}$)	ρ
240	0.989
220	0.974
200	0.959
150	0.922
100	0.880
75	0.854
50	0.822
25	0.777
20	0.765
10	0.736
5	0.717
4	0.712
1	0.698
0.500	0.695
0.200	0.694
0.010	0.692

Figure 3.6. The Roots of (3.5.1).

Note from Figure 3.6 that when the materials differ only in shear moduli

$$\rho \simeq 0.989 \qquad (3.5.17)$$

and it can be easily shown, by substituting $k^2 = 1/9$ in (3.5.8), that the corresponding value for isotropic materials is

$$\rho \simeq 0.924. \qquad (3.5.18)$$

On comparing the values of (3.5.17) and (3.5.18) we see that the order of the singularity is lower in the anisotropic case than that in the isotropic case, but the difference is not very significant. For a composite of nearly isotropic and highly anisotropic materials we have

$$\rho \simeq 0.692, \qquad (3.5.19)$$

which is a more severe singularity than those given by Equations (3.5.17) and (3.5.18). In this case the presence of the highly anisotropic material is acting as an almost rigid foundation to the isotropic layer and as we found in (3.5.11) the singularity arising from such structures can give ρ as small as

$$\rho \simeq 0.5946.$$

CHAPTER FOUR

THE STRETCHING AND BENDING OF TRANSVERSELY ISOTROPIC LAMINATED PLATES

4.1 INTRODUCTION

Stretching problems in three-dimensional linear elasticity can in certain cases be solved approximataly by reducing the problem to one of two-dimensions. This happens for a wide plate when the plane strain condition applies and for a thin plate when the state of stress is that of plane stress. Filon (1903) extended the scope of two dimensional analysis by introducing the concept of 'generalised' plane stress in which the value of σ_{zz} throughout the thickness of the plate is assumed negligible and the average values of the remaining stress components are obtained. Three-dimensional bending problems can also be solved approximately (Lekhnitskii 1968). In this case it is assumed that σ_{zz} is negligible throughout the plate and that plane sections normal to the middle surface of the underformed plate remain plane and normal to this surface in the deformed state. For all the above methods, little indication is given to how near the stresses found are to the actual values found from a complete three-dimensional solution of the elastic equations.

For a homogeneous isotropic plate with stress-free lateral surfaces, an exact solution is given by Love (1927) which satisfies the full three-dimensional equations of linear elasticity. In this solution σ_{zz} is zero and the remaining stress components are expressed in terms of harmonic functions if the plate is stretched, and bi-harmonic functions if the plate is bent. These functions are otherwise completely arbitrary, but the stresses obtained from them have a specific dependence on z which implies that tractions cannot be prescribed arbitrarily at the edges of the plate. However, if the prescribed resultant tractions are equal to the resultant tractions given by Love's solution, then by Saint Venant's principle the solution approximately represents the stress state of the plate away from the edges.

The exact three-dimensional solution of Love has been generalised by Luré (1955) for both the extension and bending of an isotropic plate. Luré represents each displacement component in the form of a power series in z and shows that all the stress components are expressable in terms of the mid-plane quantities u_o, v_o and $\partial w_o/\partial z$ for stretching and $\partial u_o/\partial z$, $\partial v_o/\partial z$ and w_o for bending. These quantities are determined from the conditions imposed on the lateral surfaces of the plate and in the case when the lateral surfaces are traction-free, the solution developed reduces to the solution given by Love. Reiss (1960) produces the same results as Luré by expanding each stress component as a power series of the plate thickness. He shows that the equations of lowest order coincide with those of the two-dimensional thin plate theory. Since conditions at the plate edges cannot be prescribed arbitrarily, the expansion solution is regarded as an 'interior solution'. Near the edges of the plate a 'boundary layer' solution is developed by stretching the appropriate co-ordinates.

In this chapter, we extend these results for a single isotropic plate to the cases of stretching and bending of a laminated plate. In Section (4.4), we consider a laminated plate comprised of an arbitrary number of laminae, each of which is of homogeneous, elastic, transversely isotropic material. An 'equivalent' homogeneous elastic plate is defined in Section (4.7) as having material properties which are obtained by an appropriate averaging of thematerial properties of the individual laminae. The equivalent plate is subjected to specified edge loads and the equivalent displacements are found by the standard thin plate theory of Sections (4.5) and (4.6).

We then consider the three-dimensional solution for each layer of the laminated plate, when the laminate is subjected to the same loading as the equivalent plate. The displacements in each layer are expressed as power series in the aspect ratio ε, described in Section (4.2) and an exact three-dimensional solution which satisfies all the through thickness continuity conditions, is obtained. The leading term in the displacement components of a typical layer is identified as the 'equivalent' displacement. The complete three-dimensional solution is constructed such that the average stress through the laminate thickness is identical to that in the equivalent plate. This implies that all conditions which are satisfied by the stress at the edges of the equivalent plate are satisfied in an average sense at the edges of the laminate.

We conclude the chapter by applying the bending and stretching solutions to the case of a laminated plate containing a circular hole.

4.2 NON-DIMENSIONAL FORM OF THE EQUATIONS OF LINEAR ELASTICITY

We consider a homogeneous elastic layer of thickness 2h bounded by the planes $Z = \pm h$ where the mid-plane of the layer coincides with the (X,Y) plane. The material of the layer is transversely isotropic with a preferred direction parallel to that of the Z axis, so that in the notation of Chapter 1, $\underset{\sim}{a} = (0,0,1)$. In such cases

$$c_{ij} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ \text{symmetrical} & & & & & c_{66} \end{pmatrix} \qquad (4.2.1)$$

where

$$c_{11} - c_{12} = 2c_{66}. \qquad (4.2.2)$$

For materials which are stiff in the fibre direction,

$$\frac{c_{66}}{c_{33}} << 1. \qquad (4.2.3)$$

If the material is isotropic then

$$c_{33} = c_{11}, \quad c_{13} = c_{12}, \quad c_{44} = c_{66}, \qquad (4.2.4)$$

and in this case the elastic constants given in (4.2.1) are related to the Youngs modulus E and Poisson ratio ν or to the Lamé constants λ and μ by

$$c_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \lambda + 2\mu, \quad C_{12} = \frac{\nu E}{(1+\nu)(1-2\nu)} = \lambda. \tag{4.2.5}$$

We introduce non-dimensional displacement components

$$u = \frac{U}{a}, \quad v = \frac{V}{a}, \quad w = \frac{W}{a} \tag{4.2.6}$$

where U, V, W are the actual displacement components, and a is some characteristic length in the (X,Y) plane. We also define an aspect ratio ε and dimensionless co-ordinates (x,y,z) by

$$\varepsilon = \frac{h}{a}, \tag{4.2.7}$$

and

$$x = \frac{X}{a}, \quad y = \frac{Y}{a}, \quad z = \frac{Z}{h} . \tag{4.2.8}$$

From (4.2.7) it follows that for a thin layer

$$\varepsilon << 1. \tag{4.2.9}$$

By using Equations (4.2.6), (4.2.7), (4.2.8) the constitutive relations become

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ \text{symmetrical} & & & & & c_{66} \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{1}{\varepsilon}\, \partial w/\partial z \\ \frac{1}{\varepsilon}\, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{1}{\varepsilon}\, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \tag{4.2.10}$$

and the equations of equilibrium are

$$\varepsilon\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right) + \frac{\partial\sigma_{xz}}{\partial z} = 0,$$

$$\varepsilon\left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y}\right) + \frac{\partial\sigma_{yz}}{\partial z} = 0, \quad (4.2.11)$$

$$\varepsilon\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y}\right) + \frac{\partial\sigma_{zz}}{\partial z} = 0.$$

4.3 STRESS AND DISPLACEMENT IN A LAYER

The laminate under consideration consists of N elastic layers and the external surfaces normal to the z-axis are stress-free. In each layer, the stress and displacement components are found by solving Equations (4.2.10) and (4.2.11) in such a way that the prescribed boundary conditions are satisfied.

We shall determine an exact closed form solution, but for convenience in formulating the theory, we suppose initially that in a typical layer of the laminate the displacement components can be represented formally by a power series in ε, that is

$$u(x,y,z;\varepsilon) = u_0(x,y,z) + \varepsilon u_1(x,y,z) + \varepsilon^2 u_2(x,y,z) + \ldots),$$
$$v(x,y,z;\varepsilon) = v_0(x,y,z) + \varepsilon v_1(x,y,z) + \varepsilon^2 v_2(x,y,z) + \ldots), \quad (4.3.1)$$
$$w(x,y,z;\varepsilon) = w_0(x,y,z) + \varepsilon w_1(x,y,z) + \varepsilon^2 w_2(x,y,z) + \ldots),$$

and then from Equations (4.2.10)

$$\sigma(x,y,z;\varepsilon) = \sum_{n=-1}^{\infty} \varepsilon^n \sigma_n(x,y,z). \quad (4.3.2)$$

In (4.3.2) σ is a generic symbol for each stress component. Substituting (4.3.1) and (4.3.2) into Equations (4.2.10), (4.2.11) (and then equating coefficients of successive powers of ε) yields a system of partial differential equations which may be solved to give the coefficients in the expansions (4.3.1) and (4.3.2):

(i) Lowest Order Solution:

Equating coefficients of terms involving ε^{-1} yields

$$\sigma_{xx-1} = c_{13}\frac{\partial w_o}{\partial z}, \quad \sigma_{xz-1} = c_{44}\frac{\partial u_o}{\partial z},$$
$$\sigma_{yy-1} = c_{13}\frac{\partial w_o}{\partial z}, \quad \sigma_{yz-1} = c_{44}\frac{\partial v_o}{\partial z}, \tag{4.3.3}$$
$$\sigma_{zz-1} = c_{33}\frac{\partial w_o}{\partial z}, \quad \sigma_{xy-1} = 0,$$

$$\frac{\partial \sigma_{xz-1}}{\partial z} = 0, \quad \frac{\partial \sigma_{yz-1}}{\partial z} = 0, \quad \frac{\partial \sigma_{zz-1}}{\partial z} = 0. \tag{4.3.4}$$

From Equations (4.3.4) we find that σ_{xz-1}, σ_{yz-1} and σ_{zz-1} are independent of z. Therefore, the stress-free condition imposed on the faces of the laminate can be satisfied only if

$$\sigma_{xz-1} = \sigma_{yz-1} = \sigma_{zz-1} = 0, \tag{4.3.5}$$

and so, from (4.3.3), we obtain

$$u_o = u_o(x,y), \quad v_o = v_o(x,y), \quad w_o = w_o(x,y), \tag{4.3.6}$$

$$\sigma_{xx-1} = \sigma_{yy-1} = \sigma_{xy-1} = 0. \tag{4.3.7}$$

(ii) Terms of Order ε^{o}:

These give

$$\begin{Bmatrix} \sigma_{xxo} \\ \sigma_{yyo} \\ \sigma_{zzo} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ & c_{11} & c_{13} \\ \text{sym.} & & c_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial w_1}{\partial z} \end{Bmatrix}, \quad \begin{aligned} \sigma_{yzo} &= c_{44}(\frac{\partial v_1}{\partial z} + \frac{\partial w_o}{\partial y}), \\ \sigma_{xzo} &= c44(\frac{\partial u_1}{\partial z} + \frac{\partial w_o}{\partial x}), \\ \sigma_{xyo} &= c_{66}(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}), \end{aligned} \tag{4.3.8}$$

$$\frac{\partial \sigma_{xx-1}}{\partial x} + \frac{\partial \sigma_{xy-1}}{\partial y} + \frac{\partial \sigma_{xzo}}{\partial z} = 0, \quad \frac{\partial \sigma_{xy-1}}{\partial x} + \frac{\partial \sigma_{yy-1}}{\partial y} + \frac{\partial \sigma_{yzo}}{\partial z} = 0,$$

$$\frac{\partial \sigma_{xz-1}}{\partial x} + \frac{\partial \sigma_{yz-1}}{\partial y} + \frac{\partial \sigma_{zzo}}{\partial z} = 0. \tag{4.3.9}$$

By substituting (4.3.5) and (4.3.7) into (4.3.9), we find that σ_{xzo}, σ_{yzo} and σ_{zzo} are independent of z. Since the lateral faces of the laminate are stress-free, it follows that

$$\sigma_{xzo} = \sigma_{yzo} = \sigma_{zzo} = 0, \tag{4.3.10}$$

and, therefore, from (4.3.8)

$$u_1 = -(z + B_1)\frac{\partial w_o}{\partial x}, \quad v_1 = -(z + B_1)\frac{\partial w_o}{\partial y},$$

$$w_1 = -\frac{Q_{13}}{Q_{44}}(z + S_1)\Delta_o \tag{4.3.11}$$

where S_1, B_1 are arbitrary constants and

$$\Delta_o = \Delta_o(x,y) = \frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y}. \tag{4.3.12}$$

Substituting $(4.3.11)_3$ into (4.3.8) gives

$$\begin{Bmatrix} \sigma_{xxo} \\ \sigma_{yyo} \\ \sigma_{xyo} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q11 & 0 \\ \text{symm.} & & Q66 \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{Bmatrix}, \tag{4.3.13}$$

where the Q_{ij} are the effective elastic moduli and are given in terms of the c_{ij} by

$$Q_{11} = c_{11} - \frac{c_{13}^2}{c_{33}}, \quad Q_{12} = c_{12} - \frac{c_{13}^2}{c_{33}}, \quad Q_{13} = c_{13} \frac{c_{44}}{c_{33}},$$

$$Q_{33} = c_{33}, \quad Q_{44} = c_{44}, \quad Q_{66} = c_{66} \tag{4.3.14}$$

and

$$Q_{11} - Q_{12} = 2Q_{66}. \tag{4.3.15}$$

For isotropic materials we have

$$Q_{13} = \frac{Q_{12}}{2}, \quad Q_{44} = Q_{66}.$$

(iii) Terms of Order ε^1:

$$\begin{Bmatrix} \sigma_{xx1} \\ \sigma_{yy1} \\ \sigma_{zz1} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ & c_{11} & c_{13} \\ \text{symm.} & & c_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial v_1}{\partial y} \\ \frac{\partial w_2}{\partial z} \end{Bmatrix} \quad \begin{aligned} \sigma_{yz1} &= c_{44}\left(\frac{\partial v_2}{\partial z} + \frac{\partial w_1}{\partial y}\right), \\ \sigma_{xz1} &= c_{44}\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_1}{\partial x}\right), \\ \sigma_{xy1} &= c_{66}\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right), \end{aligned} \tag{4.3.16}$$

$$\frac{\partial\sigma_{xxo}}{\partial x} + \frac{\partial\sigma_{xyo}}{\partial y} + \frac{\partial\sigma_{xz1}}{\partial z} = 0, \quad \frac{\partial\sigma_{xyo}}{\partial x} + \frac{\partial\sigma_{yyo}}{\partial y} + \frac{\partial\sigma_{yz1}}{\partial z} = 0,$$

$$\frac{\partial\sigma_{xzo}}{\partial x} + \frac{\partial\sigma_{yzo}}{\partial y} + \frac{\partial\sigma_{zz1}}{\partial z} = 0. \qquad (4.3.17)$$

Since $\sigma_{xzo} = \sigma_{yzo} = 0$, it follows from (4.3.17) that σ_{zz1} is independent of z. For the lateral surfaces to be stress-free, we therefore have

$$\sigma_{zz1} = 0, \qquad (4.3.18)$$

and so, from (4.3.16) and $(4.3.11)_{1,2}$

$$w_2 = \frac{Q_{13}}{Q_{44}}\left(\frac{z^2}{2} + B_1 z + B_2\right)\nabla^2 w_o . \qquad (4.3.19)$$

Substituting the expressions for σ_{xz1} and σ_{yz1}, given by (4.3.16) and the expressions for σ_{xxo}, σ_{yyo} and σ_{xyo}, given by (4.3.13) into $(4.3.17)_{12}$ and then integrating with respect to z produces

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \frac{1}{Q_{44}}\left(\frac{z^2}{2} + S_2 z + S_3\right)\begin{pmatrix} (Q_{13} - Q_{11})\dfrac{\partial\Delta_o}{\partial x} + Q_{66}\dfrac{\partial\Omega_o}{\partial y} \\ (Q_{13} - Q_{11})\dfrac{\partial\Delta_o}{\partial y} - Q_{66}\dfrac{\partial\Omega_o}{\partial x} \end{pmatrix} \qquad (4.3.20)$$

where S_2, S_3 are arbitrary constants and

$$\Omega_o = \Omega_o(x,y) = \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} . \qquad (4.3.21)$$

From Equations (4.3.16) we now obtain

$$\begin{Bmatrix} \sigma_{xx1} \\ \sigma_{yy1} \\ \sigma_{xy1} \end{Bmatrix} = -(z + B_1) \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ & Q_{11} & 0 \\ \text{symm.} & & Q66 \end{bmatrix} \begin{Bmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ 2\partial^2/\partial x\partial y \end{Bmatrix} w_o, \tag{4.3.22}$$

and

$$\begin{aligned} \sigma_{yz1} &= -(Q_{11} \frac{\partial\Delta_o}{\partial y} + Q_{66} \frac{\partial\Omega_o}{\partial x}) z \\ &\quad + S_2((Q_{13} - Q_{11})\frac{\partial\Delta_o}{\partial y} - Q_{66} \frac{\partial\Omega_o}{\partial x}) - S_1 Q_{13} \frac{\partial\Delta_o}{\partial y}, \\ \sigma_{xz1} &= -(Q_{11} \frac{\partial\Delta_o}{\partial x} - Q_{66} \frac{\partial\Omega_o}{\partial y}) z \\ &\quad + S_2((Q_{13} - Q_{11})\frac{\partial\Delta_o}{\partial x} + Q_{66} \frac{\partial\Omega_o}{\partial y}) - S_1 Q_{13} \frac{\partial\Delta_o}{\partial x}. \end{aligned} \tag{4.3.23}$$

(iv) Terms of Order ε^2:

These give

$$\begin{Bmatrix} \sigma_{xx2} \\ \sigma_{yy2} \\ \sigma_{zz2} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ & c_{11} & c_{13} \\ \text{symm.} & & c_{33} \end{bmatrix} \begin{Bmatrix} \partial u_2/\partial x \\ \partial v_2/\partial y \\ \partial w_3/\partial z \end{Bmatrix}, \quad \begin{aligned} \sigma_{yz2} &= C_{44}(\frac{\partial v_3}{\partial z} + \frac{\partial w_2}{\partial y}), \\ \sigma_{xz2} &= c_{44}(\frac{\partial u_3}{\partial z} + \frac{\partial w_2}{\partial x}), \\ \sigma_{xy2} &= c_{66}(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}), \end{aligned} \tag{4.3.24}$$

$$\begin{gathered} \frac{\partial\sigma_{xx1}}{\partial x} + \frac{\partial\sigma_{xy1}}{\partial y} + \frac{\partial\sigma_{xz2}}{\partial z} = 0, \quad \frac{\partial\sigma_{xy1}}{\partial x} + \frac{\partial\sigma_{yy1}}{\partial y} + \frac{\partial\sigma_{yz2}}{\partial z} = 0, \\ \frac{\partial\sigma_{xz1}}{\partial x} + \frac{\partial\sigma_{yz1}}{\partial y} + \frac{\partial\sigma_{zz2}}{\partial z} = 0. \end{gathered} \tag{4.3.25}$$

By substituting (4.3.23) into (4.3.25), and integrating with respect to z, we obtain

$$\sigma_{zz2} = (Q_{11} \frac{z^2}{2} + (S_1 Q_{13} - S_2(Q_{13} - Q_{11}))z + S_4)\nabla^2\Delta_o, \tag{4.3.26}$$

where S_4 is an arbitrary constant. The terms u_3 and v_3 are determined by substituting σ_{yz2}, σ_{xz2}, given in (4.3.24), and σ_{xx1}, σ_{yy1}, σ_{xy1}, given by (4.3.22), into $(4.3.25)_{1,2}$ and integrating with respect to z. This produces

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \frac{(Q_{11} - Q_{13})}{Q_{44}}\left(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4\right) \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \nabla^2 w_o \qquad (4.3.27)$$

where B_3 and B_4 are arbitrary constants. Substituting (4.3.27) and (4.3.19) into (4.3.25) now gives

$$\begin{pmatrix} \sigma_{yz2} \\ \sigma_{xz2} \end{pmatrix} = \left[Q_{11}\left(\frac{z^2}{2} + B_1 z\right) + B_3(Q_{11} - Q_{13}) + B_2 Q_{13}\right] \begin{pmatrix} \partial/\partial y \\ \partial/\partial x \end{pmatrix} \nabla^2 w_o . \qquad (4.3.28)$$

In Section (4.7) it will be shown that Δ_o, Ω_o are harmonic functions of x and y, and that w_o is a bi-harmonic function of x and y. These properties have been stated here because the incorporation of them into the analysis at this stage will greatly simplify the manipulation involved in determining the higher order terms. By using these properties for Δ_o and Ω_o, Equation (4.3.26) gives

$$\sigma_{zz2} = 0$$

and it follows from (4.3.24) that

$$w_3 = w_3(x,y), \qquad (4.3.29)$$

$$\begin{pmatrix}\sigma_{xx2}\\ \sigma_{yy2}\\ \sigma_{xy2}\end{pmatrix} = \frac{2Q_{66}}{Q_{44}}\left(\frac{z^2}{2} + S_2 z + S_3\right)\begin{pmatrix}(Q_{13} - Q_{11})\frac{\partial^2\Delta_o}{\partial x^2} + Q_{66}\frac{\partial^2\Omega_o}{\partial x\partial y}\\ (Q_{13} - Q_{11})\frac{\partial^2\Delta_o}{\partial y^2} - Q_{66}\frac{\partial^2\Omega_o}{\partial x\partial y}\\ (Q_{13} - Q_{11})\frac{\partial^2\Delta_o}{\partial x\partial y} + Q_{66}\frac{\partial^2\Omega_o}{\partial y^2}\end{pmatrix}. \qquad (4.3.30)$$

(v) Terms of Order ε^3:

These give

$$\begin{pmatrix}\sigma_{xx3}\\ \sigma_{yy3}\\ \sigma_{zz3}\end{pmatrix} = \begin{pmatrix}c_{11} & c_{12} & c_{13}\\ & c_{11} & c_{13}\\ \text{symm.} & & c_{33}\end{pmatrix}\begin{pmatrix}\partial u_3/\partial x\\ \partial v_3/\partial y\\ \partial w_4/\partial z\end{pmatrix}, \quad \begin{array}{l}\sigma_{yz3} = c_{44}\left(\frac{\partial v_4}{\partial z} + \frac{\partial w_3}{\partial y}\right),\\ \sigma_{xz3} = c_{44}\left(\frac{\partial u_4}{\partial z} + \frac{\partial w_3}{\partial x}\right),\\ \sigma_{xy3} = c_{66}\left(\frac{\partial u_3}{\partial y} + \frac{\partial v_3}{\partial x}\right),\end{array} \qquad (4.3.31)$$

$$\frac{\partial\sigma_{xx2}}{\partial x} + \frac{\partial\sigma_{xy2}}{\partial y} + \frac{\partial\sigma_{xz3}}{\partial z} = 0, \quad \frac{\partial\sigma_{xy2}}{\partial x} + \frac{\partial\sigma_{yy2}}{\partial y} + \frac{\partial\sigma_{yz3}}{\partial z} = 0,$$
$$\frac{\partial\sigma_{xz2}}{\partial x} + \frac{\partial\sigma_{yz2}}{\partial y} + \frac{\partial\sigma_{zz3}}{\partial z} \equiv 0. \qquad (4.3.32)$$

Because w_o is a bi-harmonic function, the substitution of (4.3.28) into $(4.3.32)_3$ gives that σ_{zz3} is a function of x and y only. We therefore set

$$\sigma_{zz3} = 0, \qquad (4.3.33)$$

and it then follows from (4.3.31) that

$$w_4 = w_4(x,y). \qquad (4.3.34)$$

By substituting (4.3.30) in $(4.3.32)_{1,2}$ we find that σ_{xz3} and σ_{yz3} are functions of x and y. Therefore we take

$$\sigma_{xz3} = \sigma_{yz3} = 0, \tag{4.3.35}$$

and then it follows from the expression for σ_{xz3} and σ_{yz3} in (4.3.31) that u_4 and v_4 will be expressed in terms of w_3.

From (4.3.27) and (4.3.31) we obtain

$$\begin{pmatrix} \sigma_{xx3} \\ \sigma_{yy3} \\ \sigma_{xy3} \end{pmatrix} = \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})\left(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4\right) \begin{pmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ \partial^2/\partial x\partial y \end{pmatrix} \nabla^2 w_o. \tag{4.3.36}$$

By continuing the analysis to determine the higher order displacement components, we find that w_3 and w_4 remain arbitrary functions of x,y and that all subsequent terms are expressed solely in terms of them and their derivatives. Therefore, by taking

$$w_3 = w_4 = 0$$

we have that

$$\begin{aligned} w_i &= 0 \qquad i = 5, \ldots \\ u_i = v_i &= 0 \qquad i = 4, \ldots \end{aligned} \tag{4.3.37}$$

The series expansions for all the displacement components now consist of a finite number of terms and we will show that this finite series is an exact solution to the equations of linear elasticity provided

$$\nabla^2\Delta_o = 0, \quad \nabla^2\Omega_o = 0, \quad \nabla^4 w_o = 0. \tag{4.3.38}$$

(vi) Terms of Order ε^4:

As a consequence of (4.3.37), these terms give

$$\sigma_{xx4} = \sigma_{yy4} = \sigma_{xy4} = \sigma_{zz4} = \sigma_{xz4} = \sigma_{yz4} = 0 \tag{4.3.39}$$

and to this order in ε the remaining equations are

$$\frac{\partial\sigma_{xx3}}{\partial x} + \frac{\partial\sigma_{xy3}}{\partial y} = 0, \quad \frac{\partial\sigma_{xy3}}{\partial x} + \frac{\partial\sigma_{yy3}}{\partial y} = 0. \tag{4.3.40}$$

The substitution of the stress components given by (4.3.36) into (4.3.40) will show that (4.3.40) is identically satisfied. Further analysis will now only give trivial solutions.

Hence, by taking displacement components in the form

$$\begin{aligned}
\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u_o \\ v_o \end{pmatrix} - \varepsilon(z + B_1)\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} w_o \\
&+ \frac{\varepsilon^2}{Q_{44}}\left(\frac{z^2}{2} + S_2 z + S_3\right)\begin{pmatrix} (Q_{13} - Q_{11})\frac{\partial\Delta_o}{\partial x} + Q_{66}\frac{\partial\Omega_o}{\partial y} \\ (Q_{13} - Q_{11})\frac{\partial\Delta_o}{\partial y} - Q_{66}\frac{\partial\Omega_o}{\partial x} \end{pmatrix} + \ldots \\
&+ \varepsilon^3\frac{(Q_{11} - Q_{13})}{Q_{44}}\left(\frac{z^3}{6} + B_1\frac{z^2}{2} + B_3 z + B_4\right)\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix}\nabla^2 w_o
\end{aligned} \tag{4.3.41}$$

$$w = w_o - \varepsilon \frac{Q_{13}}{Q_{44}}(z + S_1)\Delta_o + \varepsilon^2 \frac{Q_{13}}{Q_{44}}(\frac{z^2}{2} + B_1 z + B_2)\nabla^2 w_o \qquad (4.3.42)$$

we obtain stress components which exactly satisfy the stress equilibrium equations. Furthermore, this is always true whatever the value of the aspect ratio ε.

The individual terms of the stress and displacement components are solely expressed in terms of u_o, v_o or w_o and their derivatives. The (u_o, v_o) system is seen to uncouple from the (w_o) system with the former representing the stretching terms whilst the latter representing the bending terms.

4.4 STRESS AND DISPLACEMENT IN A LAMINATE

The laminate consists of (2N - 1) layers, of different transversely isotropic linear elastic materials. The preferred direction of each layer is parallel to the z-axis and the width of the ℓth layer is $2h_\ell$. The construction of the laminate is mid-plane symmetric and there is no stretching/bending coupling of the laminate due to the external loads.

Any quantity related to the ℓth layer will have a subscript/superscript (ℓ). The layers are numbered according to the scheme shown in Figure (4.1) where layer $(\ell = 1)$ contains the mid-plane of the laminate and layer $(\ell = N)$ is the bounding layer.

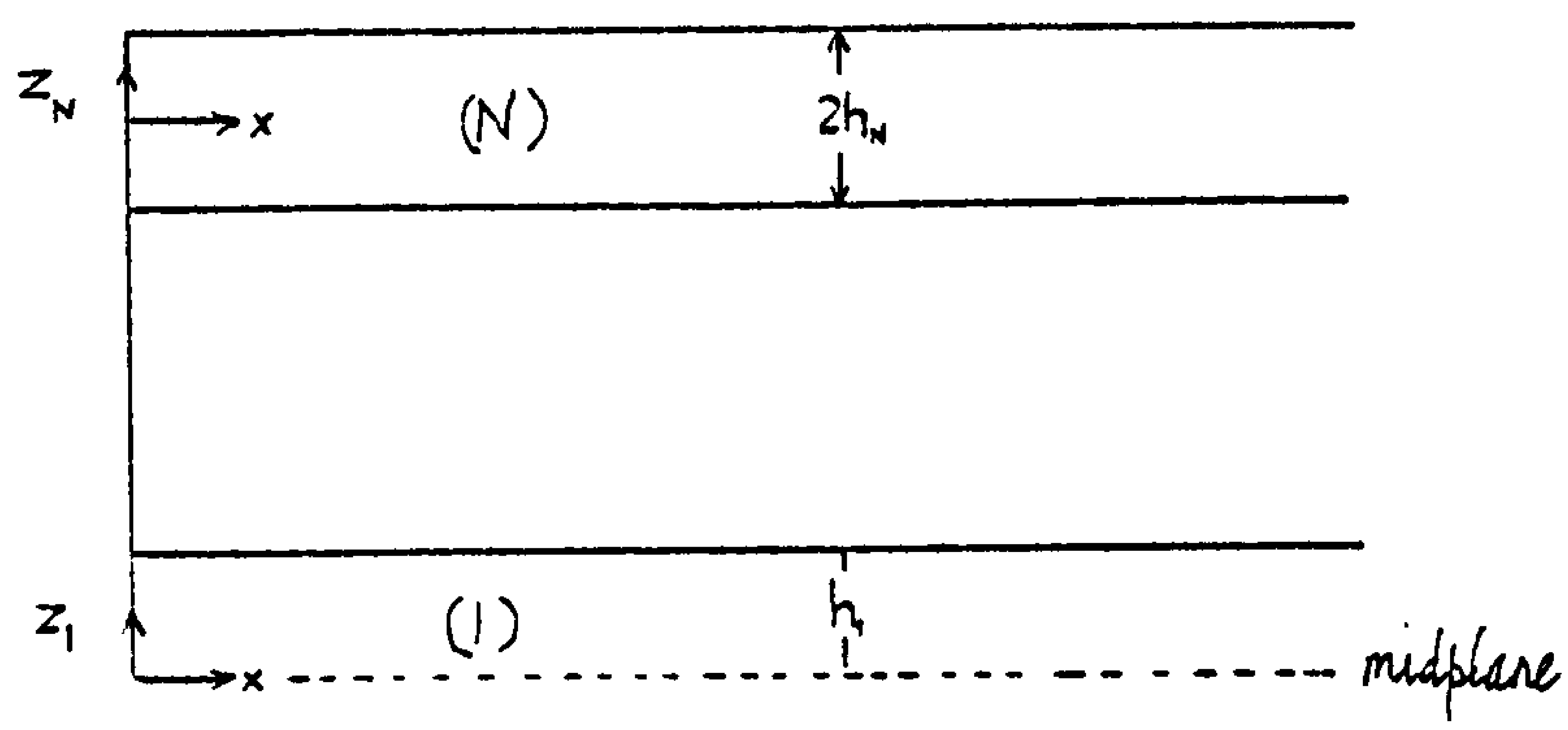


Figure 4.1. The Upper Half of the Laminate

The layers are initially referred to a rectangular co-ordinate system (X,Y,Z) and then by non-dimensionalizing X,Y by an in-plane characteristic length a and Z by the half width of the layer, each layer is referred to the local co-ordinate system (x,y,z) with the (x,y) plane coinciding with the mid-plane of the layer.

If the layers are bonded together rigidly then at each interface we will require continuity of the displacement components and the z components of stress.

$$w_\ell\Big|_{z_\ell=-1} = w_{\ell-1}\Big|_{z_{\ell-1}=1}, \quad \ell = 2 \ldots N$$

$$u_\ell\Big|_{z_\ell=-1} = u_{\ell-1}\Big|_{z_{\ell-1}=1}, \quad v_\ell\Big|_{z_\ell=-1} = v_{\ell-1}\Big|_{z_{\ell-1}=1}, \quad \ell = 2 \ldots N \tag{4.4.1}$$

$$\sigma_{jz}^{(\ell)}\Big|_{z_\ell=-1} = \sigma_{jz}^{(\ell-1)}\Big|_{z_{\ell-1}=1}, \quad \ell = 2 \ldots N, \; j = x,y,z.$$

Deformation of the laminated plate will be caused by specifying mid-plane symmetric stress or displacement conditions at the edges. The bounding surfaces of the laminate are kept traction free.

The traction-free condition gives

$$\sigma_{jz}^{(N)} = 0, \quad z_N = 1, \quad j = x,y,z, \tag{4.4.2}$$

and the symmetry about the mid-plane requires,

(i) For Stretching

$$\begin{aligned} w^{(1)} &= 0 \quad z_1 = 0, \\ \sigma_{xz}^{(1)} = \sigma_{yz}^{(1)} &= 0 \quad z_1 = 0. \end{aligned} \tag{4.4.3}$$

(ii) For Bending

$$\begin{aligned} u^{(1)} = v^{(1)} &= 0 \quad z_1 = 0, \\ \sigma_{xx}^{(1)} = \sigma_{yy}^{(1)} = \sigma_{xy}^{(1)} = \sigma_{zz}^{(1)} &= 0 \quad z_1 = 0. \end{aligned} \tag{4.4.4}$$

Before discussing the implications of the solutions found in this section, we review Classical Plate Theory since this theory will be employed at a later stage.

4.5 STRETCHING OF THIN PLATES

We consider a plate of thickness $2\hat{H}$ taken such that its mid-plane coincides with that of the (x,y) plane. The plate is thin and it is assumed that a state of generalised plane stress exists (Love 1927).

If we denote $\hat{u}$ and $\hat{v}$ as the averaged displacement components, $\hat{\sigma}_{xx}$, $\hat{\sigma}_{yy}$, $\hat{\sigma}_{xy}$ as the averaged in-plane stress components, and non-dimensionalise in accordance with (4.2.6) and (4.2.8), then it can be shown (Love 1927) (Lekhitskii 1968) that

$$\hat{\sigma}_{xx} = \hat{Q}_{11}\frac{\partial\hat{u}}{\partial x} + \hat{Q}_{12}\frac{\partial\hat{v}}{\partial y},$$

$$\hat{\sigma}_{yy} = \hat{Q}_{12}\frac{\partial\hat{u}}{\partial x} + \hat{Q}_{11}\frac{\partial\hat{v}}{\partial y}, \qquad (4.5.1)$$

$$\hat{\sigma}_{xy} = \hat{Q}_{66}\left(\frac{\partial\hat{u}}{\partial y} + \frac{\partial\hat{v}}{\partial x}\right).$$

The $\hat{Q}_{ij}$ in (4.5.1) are given by (4.3.14) with c_{ij} replaced by $\hat{c}_{ij}$, where $\hat{c}_{ij}$ are the elastic constants of the plate. The functions $\hat{u}$ and $\hat{v}$ are functions of x and y only. The substitution of (4.5.1) into the averaged stress equilibrium equations will give two partial differential equations from which $\hat{u}$ and $\hat{v}$ can be determined,

$$\hat{Q}_{11}\frac{\partial\hat{\Delta}}{\partial x} - \hat{Q}_{66}\frac{\partial\hat{\Omega}}{\partial y} = 0,$$

$$\hat{Q}_{11}\frac{\partial\hat{\Delta}}{\partial y} + \hat{Q}_{66}\frac{\partial\hat{\Omega}}{\partial x} = 0, \qquad (4.5.2)$$

where

$$\hat{\Delta} = \frac{\partial\hat{u}}{\partial x} + \frac{\partial\hat{v}}{\partial y},$$

$$\hat{\Omega} = \frac{\partial\hat{v}}{\partial x} - \frac{\partial\hat{u}}{\partial y}. \qquad (4.5.3)$$

In (4.5.2) the relationship

$$\hat{Q}_{11} - \hat{Q}_{12} = 2\hat{Q}_{66} \qquad (4.5.4)$$

has been used.

By eliminating $\hat{\Omega}$ and $\hat{\Delta}$ in turn from (4.5.2) it follows that

$$\nabla^2\hat{\Delta} = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4.5.5}$$

and

$$\nabla^2\hat{\Omega} = 0. \tag{4.5.6}$$

The resultant forces produced by the stress components in (4.5.1) on planes normal to the (x,y) plane are given by

$$\begin{aligned} \hat{T}_x &= \hat{H}\int_{-1}^{1} \hat{\sigma}_{xx}\, dz = 2\hat{H}(\hat{Q}_{11}\frac{\partial \hat{u}}{\partial x} + \hat{Q}_{12}\frac{\partial \hat{v}}{\partial y}), \\ \hat{T}_y &= \hat{H}\int_{-1}^{1} \hat{\sigma}_{yy}\, dz = 2\hat{H}(\hat{Q}_{12}\frac{\partial \hat{u}}{\partial x} + \hat{Q}_{11}\frac{\partial \hat{v}}{\partial y}), \\ \hat{T}_{xy} &= \hat{H}\int_{-1}^{1} \hat{\sigma}_{xy}\, dz = 2\hat{H}\hat{Q}_{66}(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}), \end{aligned} \tag{4.5.7}$$

and the bending moments acting on these planes are identically zero.

4.6 BENDING OF THIN PLATES

The bending theory of thin plates is based on the following assumptions (Lekhnitskii 1968)

(i) plane sections in the undeformed state normal to the middle surface remain plane and normal to this surface in the deformed state

(ii) $\hat{\sigma}_{zz}$ is small compared with $\hat{\sigma}_{xx}$, $\hat{\sigma}_{xy}$ and $\hat{\sigma}_{yy}$ where $\hat{\sigma}_{ij}$ are the stress components.

If $2\hat{H}$ is the plate thickness and $\hat{w}(x,y)$ is the deflection of the middle surface of the plate then following Lekhnitskii (1968) we obtain

$$\hat{u} = -\hat{\varepsilon}z\frac{\partial\hat{w}}{\partial x}, \quad \hat{v} = -\varepsilon z\frac{\partial\hat{w}}{\partial y}, \qquad (4.6.1)$$

and

$$\hat{\sigma}_{xx} = -\hat{\varepsilon}z(\hat{Q}_{11}\frac{\partial^2\hat{w}}{\partial x^2} + \hat{Q}_{12}\frac{\partial^2\hat{w}}{\partial y^2}),$$

$$\hat{\sigma}_{yy} = -\hat{\varepsilon}z(\hat{Q}_{12}\frac{\partial^2\hat{w}}{\partial x^2} + \hat{Q}_{11}\frac{\partial^2\hat{w}}{\partial y^2}), \qquad (4.6.2)$$

$$\hat{\sigma}_{xy} = -\hat{\varepsilon}z2\hat{Q}_{66}\frac{\partial^2\hat{w}}{\partial x\partial y}.$$

In (4.6.1) and (4.6.2) the displacement components $\hat{u}$, $\hat{v}$ and $\hat{w}$ are dimensionless in accordance with (4.2.6) and (4.2.8), and

$$\hat{\varepsilon} = \frac{\hat{H}}{a} \qquad (4.6.3)$$

where a is some characteristic in-plane length.

By substituting (4.6.2) into the first two equilibrium equations and integrating with respect to z we have

$$\hat{\sigma}_{xz} = \hat{\varepsilon}^2\hat{Q}_{11}(\frac{z^2}{2} - \tfrac{1}{2})\frac{\partial}{\partial x}\nabla^2\hat{w},$$

$$\hat{\sigma}_{yz} = \hat{\varepsilon}^2\hat{Q}_{11}(\frac{z^2}{2} - \tfrac{1}{2})\frac{\partial}{\partial y}\nabla^2\hat{w}, \qquad (4.6.4)$$

and it then follows from the third equilibrium equation that

$$\nabla^2\nabla^2\hat{w} = 0.$$

If Equations (4.6.2) are integrated through the thickness of the plate, it is found that there is no resultant force due to the in-plane stress components. The resultant shear forces are given by

$$\hat{N}_x = \hat{H}\int_{-1}^{1} \hat{\sigma}_{xz}\, dz = -\hat{\varepsilon}^2\, \frac{2}{3}\, \hat{Q}_{11}\, \frac{\partial}{\partial x}\, \nabla^2\hat{w}\ , \tag{4.6.5}$$

and a similar expression for $\hat{N}_y$. The bending moments acting on planes normal to the (x,y) plane are obtained from (4.6.2) and are

$$\hat{M}_{xx} = \hat{H}^2\int_{-1}^{1} z\hat{\sigma}_{xx}\, dz = -\hat{\varepsilon}\hat{H}^2\, \frac{2}{3}(\hat{Q}_{11}\, \frac{\partial^2\hat{w}}{\partial x^2} + \hat{Q}_{12}\, \frac{\partial^2\hat{w}}{\partial y^2}),$$

$$\hat{M}_{yy} = \hat{H}^2\int_{-1}^{1} z\hat{\sigma}_{yy}\, dz = -\hat{\varepsilon}\hat{H}^2\, \frac{2}{3}(\hat{Q}_{12}\, \frac{\partial^2\hat{w}}{\partial x^2} + \hat{Q}_{11}\, \frac{\partial^2\hat{w}}{\partial y^2}), \tag{4.6.6}$$

$$\hat{M}_{xy} = \hat{H}^2\int_{-1}^{1} z\hat{\sigma}_{xy}\, dz = -\hat{\varepsilon}\hat{H}^2\, \frac{2}{3}(2\hat{Q}_{66}\, \frac{\partial^2\hat{w}}{\partial x\partial y}).$$

4.7 THE EQUIVALENT PLATE FOR STRETCHING AND BENDING

In Section 4.3 an exact solution to the equations of linear elasticity is obtained in which the (u_o,v_o) or (Δ_o,Ω_o) system represents the stretching terms and the (w_o) system represents the bending terms. The arbitrary functions u_o, v_o and w_o will now be determined.

For a typical layer of the laminate we have, for

(i) Stretching;

$$u = u_o + \frac{\varepsilon^2}{Q_{44}}\{(Q_{13} - Q_{11})\frac{\partial \Delta_o}{\partial x} + Q_{66}\frac{\partial \Omega_o}{\partial y}\}(\frac{z^2}{2} + S_2 z + S_3),$$

$$v = v_o + \frac{\varepsilon^2}{Q_{44}}\{(Q_{13} - Q_{11})\frac{\partial \Delta_o}{\partial y} - Q_{66}\frac{\partial \Omega_o}{\partial x}\}(\frac{z^2}{2} + S_2 z + S_3), \qquad (4.7.1)$$

$$w = -\varepsilon \frac{Q_{13}}{Q_{44}}(z + S_1)\Delta_o$$

$$\sigma_{xx} = Q_{11}\frac{\partial u_o}{\partial x} + Q_{12}\frac{\partial v_o}{\partial y}$$

$$+ \varepsilon^2 \frac{2Q_{66}}{Q_{44}}\{(Q_{13} - Q_{11})\frac{\partial^2 \Delta_o}{\partial x^2} + Q_{66}\frac{\partial^2 \Omega_o}{\partial x \partial y}\}(\frac{z^2}{2} + S_2 z + S_3),$$

$$\sigma_{yy} = Q_{12}\frac{\partial u_o}{\partial x} + Q_{11}\frac{\partial v_o}{\partial y} .$$

$$+ \varepsilon^2 \frac{2Q_{66}}{Q_{44}}\{(Q_{13} - Q_{11})\frac{\partial^2 \Delta_o}{\partial y^2} - Q_{66}\frac{\partial^2 \Omega_o}{\partial x \partial y}\}(\frac{z^2}{2} + S_2 z + S_3),$$

$$\sigma_{xy} = Q_{66}(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x})$$

$$+ \varepsilon^2 \frac{2Q_{66}}{Q_{44}}\{(Q_{13} - Q_{11})\frac{\partial^2 \Delta_o}{\partial x \partial y} + Q_{66}\frac{\partial^2 \Omega_o}{\partial y^2}(\frac{z^2}{2} + S_2 z + S_3), \qquad (4.7.2)$$

$$\sigma_{yz} = -\varepsilon\{(Q_{11}\frac{\partial \Delta_o}{\partial y} + Q_{66}\frac{\partial \Omega_o}{\partial x})z$$

$$- S_2[(Q_{13} - Q_{11})\frac{\partial \Delta_o}{\partial y} - Q_{66}\frac{\partial \Omega_o}{\partial x}] + S_1 Q_{13}\frac{\partial \Delta_o}{\partial y}\},$$

$$\sigma_{xz} = -\varepsilon\{(Q_{11}\frac{\partial \Delta_o}{\partial x} - Q_{66}\frac{\partial \Omega_o}{\partial y})z$$

$$- S_2[(Q_{13} - Q_{11})\frac{\partial \Delta_o}{\partial x} + Q_{66}\frac{\partial \Omega_o}{\partial y}] + S_1 Q_{13}\frac{\partial \Delta_o}{\partial x}\},$$

$$\sigma_{zz} = 0.$$

The stress resultants on planes normal to the mid-plane of the layer are obtained by integrating (4.7.2) with respect to z through the thickness of the layer. This gives

$$T_x = h\int_{-1}^{1} \sigma_{xx}\,dz$$

$$= 2h\,[Q_{11}\frac{\partial u_o}{\partial x} + Q_{12}\frac{\partial v_o}{\partial y} + \varepsilon^2\frac{2Q_{66}}{Q_{44}}((Q_{13} - Q_{11})\frac{\partial^2\Delta_o}{\partial x^2} + Q_{66}\frac{\partial^2\Omega_o}{\partial x\partial y}$$

$$\times (\frac{1}{3} + 2S_3)] \qquad (4.7.3)$$

and similar expressions for T_y and T_{xy}. For the resultant shear forces we have

$$N_x = h\int_{-1}^{1}\sigma_{xz}\,dz = 2h\varepsilon[S_2((Q_{13} - Q_{11})\frac{\partial\Delta_o}{\partial x} + Q_{66}\frac{\partial\Omega_o}{\partial y})$$

$$+ S_1Q_{13}\frac{\partial\Delta_o}{\partial x}], \qquad (4.7.4)$$

and a similar expression for N_y with $\partial/\partial x$, $\partial/\partial y$ replaced by $\partial/\partial y$, $\partial/\partial x$ respectively and Ω_o replaced by $-\Omega_o$.

(ii) Bending;

In this case we have, in a typical layer

$$w = w_o + \varepsilon^2 \frac{Q_{13}}{Q_{44}}(\frac{z^2}{2} + B_1 z + B_2)\nabla^2 w_o,$$

$$u = -\varepsilon(z + B_1)\frac{\partial w_o}{\partial x}$$

$$+ \varepsilon^3(\frac{Q_{11} - Q_{13}}{Q_{44}})(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4)\frac{\partial}{\partial x}\nabla^2 w_o, \qquad (4.7.5)$$

$$v = -\varepsilon(z + B_1)\frac{\partial w_o}{\partial y}$$

$$+ \varepsilon^3(\frac{Q_{11} - Q_{13}}{Q_{44}})(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4)\frac{\partial}{\partial y}\nabla^2 w_o,$$

$$\sigma_{xx} = -\varepsilon(z + B_1)(Q_{11} \frac{\partial^2 w_o}{\partial x^2} + Q_{12} \frac{\partial^2 w_o}{\partial y^2})$$

$$+ \varepsilon^3 \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4)\frac{\partial^2}{\partial x^2}\nabla^2 w_o,$$

$$\sigma_{yy} = -\varepsilon(z + B_1)(Q_{12} \frac{\partial^2 w_o}{\partial x^2} + Q_{11} \frac{\partial^2 w_o}{\partial y^2})$$

$$+ \varepsilon^3 \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4)\frac{\partial^2}{\partial y^2}\nabla^2 w_o,$$

$$\sigma_{xy} = -\varepsilon(z + B_1)2Q_{66} \frac{\partial^2 w_o}{\partial x \partial y}$$

$$+ \varepsilon^3 \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})(\frac{z^3}{6} + B_1 \frac{z^2}{2} + B_3 z + B_4)\frac{\partial^2}{\partial x \partial y}\nabla^2 w_o, \qquad (4.7.6)$$

$$\sigma_{yz} = \varepsilon^2(Q_{11}(\frac{z^2}{2} + B_1 z) + B_3(Q_{11} - Q_{13}) + B_2 Q_{13})\frac{\partial}{\partial y}\nabla^2 w_o,$$

$$\sigma_{xz} = \varepsilon^2(Q_{11}(\frac{z^2}{2} + B_1 z) + B_3(Q_{11} - Q_{13}) + B_2 Q_{13})\frac{\partial}{\partial x}\nabla^2 w_o,$$

$$\sigma_{zz} = 0.$$

The stress resultants and bending moments associated with Equations (4.7.6) are

$$T_x = h\int_{-1}^{1} \sigma_{xx}\, dz = -\varepsilon h 2B_1 (Q_{11} \frac{\partial^2 w_o}{\partial x^2} + Q_{12} \frac{\partial^2 w_o}{\partial y^2})$$

$$+ \varepsilon^3 h \frac{2Q_{66}}{Q_{44}} (Q_{11} - Q_{13}) (\frac{B_1}{3} + 2B_4) \frac{\partial^2}{\partial x^2} \nabla^2 w_o$$

$$T_y = h\int_{-1}^{1} \sigma_{yy}\, dz = -\varepsilon h 2B_1 (Q_{12} \frac{\partial^2 w_o}{\partial x^2} + Q_{11} \frac{\partial^2 w_o}{\partial y^2})$$

$$+ \varepsilon^3 h \frac{2Q_{66}}{Q_{44}} (Q_{11} - Q_{13}) (\frac{B_1}{3} + 2B_4) \frac{\partial^2}{\partial y^2} \nabla^2 w_o \qquad (4.7.7)$$

$$T_{xy} = h\int_{-1}^{1} \sigma_{xy}\, dz = -\varepsilon h 2B_1 (2Q_{66} \frac{\partial^2 w_o}{\partial x \partial y})$$

$$+ \varepsilon^3 h \frac{2Q_{66}}{Q_{44}} (Q_{11} - Q_{13}) (\frac{B_1}{3} + 2B_4) \frac{\partial^2}{\partial x \partial y} \nabla^2 w_o$$

$$N_x = h\int_{-1}^{1} \sigma_{xz}\, dz = \varepsilon^2 h (\frac{Q_{11}}{3} + 2B_3 (Q_{11} - Q_{13}) + 2B_2 Q_{13}) \frac{\partial}{\partial x} \nabla^2 w_o$$

$$(4.7.8)$$

$$N_y = h\int_{-1}^{1} \sigma_{yz}\, dz = \varepsilon^2 h (\frac{Q_{11}}{3} + 2B_3 (Q_{11} - Q_{13}) + 2B_2 Q_{13}) \frac{\partial}{\partial y} \nabla^2 w_o$$

$$M_{xx} = h^2\int_{-1}^{1} z\sigma_{xx}\, dz = -\varepsilon h^2\ \frac{2}{3}(Q_{11}\ \frac{\partial^2 w_o}{\partial x^2} + Q_{12}\ \frac{\partial^2 w_o}{\partial y^2})$$

$$+\ \varepsilon^3 h^2\ \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})(\frac{1}{15} + \frac{2B_3}{3})\frac{\partial^2}{\partial x^2}\ \nabla^2 w_o$$

$$M_{yy} = h^2\int_{-1}^{1} z\sigma_{yy}\, dz = -\varepsilon h^2\ \frac{2}{3}(Q_{12}\ \frac{\partial^2 w_o}{\partial x^2} + Q_{11}\ \frac{\partial^2 w_o}{\partial y^2})$$

$$+\ \varepsilon^3 h^2\ \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})(\frac{1}{15} + \frac{2B_3}{3})\frac{\partial^2}{\partial y^2}\ \nabla^2 w_o \qquad (4.7.9)$$

$$M_{xy} = h^2\int_{-1}^{1} z\sigma_{xy}\, dz = -\varepsilon h^2\ \frac{2}{3}(2Q_{66}\ \frac{\partial^2 w_o}{\partial x\partial y})$$

$$+\ \varepsilon^3 h^2\ \frac{2Q_{66}}{Q_{44}}(Q_{11} - Q_{13})(\frac{1}{15} + \frac{2B_3}{3})\frac{\partial^2}{\partial x\partial y}\ \nabla^2 w_o .$$

For an isotropic layer, the Q_{ij} given in (4.7.2) and (4.7.6) are given by

$$Q_{11} = \frac{E}{1 - \nu^2}, \qquad Q_{12} = \frac{\nu E}{1 - \nu^2},$$

$$Q_{13} = \frac{Q_{12}}{2}, \quad Q_{44} = Q_{66} = \tfrac{1}{2}(Q_{11} - Q_{12}), \qquad (4.7.10)$$

where E, ν are the Youngs modulus and Poisson ratios of that layer. In this case, by setting

$$S_1 = S_2 = 0, \quad S_3 = -\frac{1}{6}\ , \qquad (4.7.11)$$

in (4.7.1) we obtain the results of Lure (1955) for the stretching of a single isotropic layer. Likewise, by setting

$$B_1 = B_2 = B_4 = 0, \quad B_3 = -\frac{1}{2 - \nu}, \qquad (4.7.12)$$

in (4.7.2) we obtain the results of Love (1927) for the bending of a single isotropic layer. The constants (4.7.11) and (4.7.12) have been chosen so that the stress-free conditions on the lateral surfaces of the layer are satisfied and that

(i) in stretching, u,v are even in z, w is odd in z,

(ii) in bending, u,v are odd in z, w is even in z.

An equivalent plate, shown in Figure (4.2) is defined to be a single plate which is identical in overall geometry to the laminate and has material constants $\hat{Q}_{ij}$ which are obtained by an appropriate averaging of the material constants of the laminae. Details of the averaging are given later in this section.

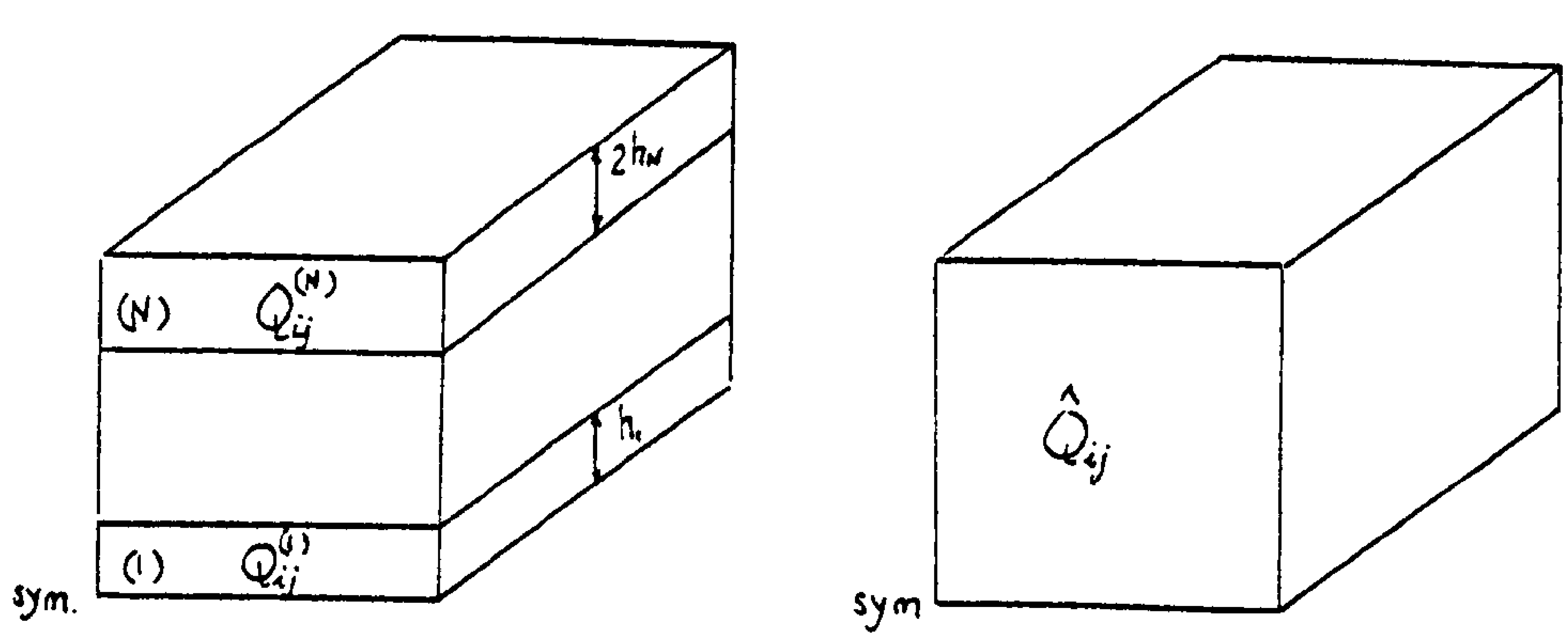


Figure (4.2)(a). The Laminate (b) The Equivalent Plate

The equivalent plate is subjected to specified loads, which are the same as those applied to the laminate, and the equivalent displacements $\hat{u}$, $\hat{v}$ and $\hat{w}$ are determined by the thin plate theory of Sections

(4.5) and (4.6). The arbitrary functions u_o, v_o and w_o in a typical layer of the laminate are now specified by supposing

$$u_o = \hat{u}, \qquad v_o = \hat{v}, \qquad w_o = \hat{w}, \tag{4.7.13}$$

and it then follows from Sections (4.5), (4.6) and Equation (4.7.13) that

$$\nabla^2 \Delta_o = \nabla^2 \hat{\Delta} = 0, \qquad \nabla^2 \Omega_o = \nabla^2 \hat{\Omega} = 0, \qquad \nabla^2 \nabla^2 w_o = \nabla^2 \nabla^2 \hat{w} = 0. \tag{4.7.14}$$

For the choice (4.7.13) of u_o and v_o we obtain from (4.5.2) the following relationships

$$\frac{\partial \Omega_o}{\partial y} = \frac{\hat{Q}_{11}}{\hat{Q}_{66}} \frac{\partial \Delta_o}{\partial x}, \qquad \frac{\partial \Omega_o}{\partial x} = -\frac{\hat{Q}_{11}}{\hat{Q}_{66}} \frac{\partial \Delta_o}{\partial y}. \tag{4.7.15}$$

Note that:

(i) If the displacement components of the equivalent plate are available from thin plate theory, an exact solution for the field quantities in each lamina are readily determined from them, as can be seen in Equations (4.7.1), (4.7.2), (4.7.5) and (4.7.6).

(ii) The lowest order terms in the field quantities in each layer represent those given by Classical Laminate Theory (Christensen 1979) and the magnitude of the higher order terms will determine the extent to which this theory is applicable.

(iii) The arbitrary constants S_i and B_i for each layer are determined by the through thickness boundary conditions of Section (4.4). In Section (4.8) we satisfy all these conditions and find that, in both the stretching and bending solutions, there remains a disposable constant. This constant can be chosen so that the resultant stresses and bending moments due to the higher order stress terms acting on cross-sections normal to the plane of the laminate are zero. This is explained further in Section (4.8).

We now determine the elastic constants $\hat{Q}_{ij}$ of the equivalent plate. The resultant stress, due to the stretching solution, on planes normal to the plane of the laminate is

$$\hat{T}_x = T_x^{(1)} + 2\sum_{\ell=2}^{N} T_x^{(\ell)}, \tag{4.7.16}$$

and similar expressions for $\hat{T}_y$ and $\hat{T}_{xy}$. If we substitute $T_x^{(\ell)}$ given by (4.7.3) into (4.7.16) we obtain

$$\begin{aligned}\hat{T}_x = {} & (2h^{(1)}Q_{11}^{(1)} + 2\sum_{\ell=2}^{N} 2h^{(\ell)}Q_{11}^{(\ell)})\frac{\partial u_o}{\partial x} \\ & + (2h^{(1)}Q_{12}^{(1)} + 2\sum_{\ell=2}^{N} 2h^{(\ell)}Q_{12}^{(\ell)})\frac{\partial v_o}{\partial y} .\end{aligned} \tag{4.7.17}$$

The resultant stress on planes normal to the plane of the equivalent plate in a state of generalised plane stress is given by (4.3.7)

$$\hat{T}_x = 2\hat{H}(\hat{Q}_{11}\frac{\partial \hat{u}}{\partial x} + \hat{Q}_{12}\frac{\partial \hat{v}}{\partial y}). \tag{4.7.18}$$

Since $u_o = \hat{u}$, $v_o = \hat{v}$ and the stress resultants in Equations (4.7.17) and (4.7.18) are required to be the same

$$\hat{Q}_{11} = (h^{(1)}Q_{11}^{(1)} + 2\sum_{\ell=2}^{N} h^{(\ell)}Q_{11}^{(\ell)})/\hat{H},$$

$$\hat{Q}_{12} = (h^{(1)}Q_{12}^{(1)} + 2\sum_{\ell=2}^{N} h^{(\ell)}Q_{12}^{(\ell)})/\hat{H}. \tag{4.7.19}$$

Equations (4.7.19) define the elastic constants of the equivalent plate when the laminate is stretched.

To determine the equivalent plate elastic constants for bending, the resultant bending moments, due to the bending solution, on planes normal to the plane of the laminate are compared with those of the equivalent plate. For the laminated plate, the resultant bending moment is given by

$$\hat{M}_{xx} = M_{xx}^{(1)} + 2\sum_{\ell=2}^{N} M_{xx}^{(\ell)} + \sum_{\ell=2}^{N}\sum_{r=2}^{\ell} 2(h_{r-1} + h_r)T_x^{(r)}, \tag{4.7.20}$$

and similar expressions for $\hat{M}_{xy}$ and $\hat{M}_{yy}$. Substituting $M_{xx}^{(\ell)}$, $T_x^{(\ell)}$ given by (4.7.7) and (4.7.9) into (4.7.20) we obtain

$$\begin{aligned}\hat{M}_{xx} = &- \tfrac{2}{3}(\varepsilon_1 h_1^2 Q_{11}^{(1)} + 2\sum_{\ell=2}^{N} \varepsilon_\ell h_\ell^2 Q_{11}^{(\ell)} \\ &\qquad + 6\sum_{\ell=2}^{N}\sum_{r=2}^{\ell}(h_r + h_{r-1})\varepsilon_\ell h_\ell B_1^{(\ell)} Q_{11}^{(\ell)})\frac{\partial^2 w_o}{\partial x^2} \\ &- \tfrac{2}{3}(\varepsilon_1 h_1^2 Q_{12}^{(1)} + 2\sum_{\ell=2}^{N} \varepsilon_\ell h_\ell^2 Q_{12}^{(\ell)} \\ &\qquad + 6\sum_{\ell=2}^{N}\sum_{r=2}^{\ell}(h_r + h_{r-1})\varepsilon_\ell h_\ell B_1^{(\ell)} Q_{12}^{(\ell)})\frac{\partial^2 w_o}{\partial y^2} .\end{aligned} \tag{4.7.21}$$

The bending moment on planes normal to the plane of the equivalent plate is given by (4.6.6)

$$\hat{M}_{xx} = -\hat{\varepsilon}\hat{H}^2 \frac{2}{3}(\hat{Q}_{11} \frac{\partial^2 w}{\partial x^2} + \hat{Q}_{12} \frac{\partial^2 w}{\partial y^2}). \tag{4.7.22}$$

If we require the bending moments in Equations (4.7.21) and (4.7.22) to be the same, then since $w_o = \hat{w}$, we must have

$$\hat{Q}_{11} = \frac{1}{\hat{\varepsilon}\hat{H}^2}\{\varepsilon_1 h_1^2 Q_{11}^{(1)} + 2\sum_{\ell=2}^{N} \varepsilon_\ell h_\ell^2 Q_{11}^{(\ell)} + 6\sum_{\ell=2}^{N}\sum_{r=2}^{\ell}(h_r + h_{r-1})\varepsilon_\ell h_\ell B_1^{(\ell)} Q_{11}^{(\ell)}\} \tag{4.7.23}$$

$$\hat{Q}_{12} = \frac{1}{\hat{\varepsilon}\hat{H}^2}\{\varepsilon_1 h_1^2 Q_{12}^{(1)} + 2\sum_{\ell=2}^{N} \varepsilon_\ell h_\ell^2 Q_{12}^{(\ell)} + 6\sum_{\ell=2}^{N}\sum_{r=2}^{\ell}(h_r + h_{r-1})\varepsilon_\ell h_\ell B_1^{(\ell)} Q_{12}^{(\ell)}\}.$$

Equations (4.7.23) define the elastic constants of the equivalent plate when the laminate is bent. Note that the expression for the equivalent elastic constants for bending contain $B_1^{(\ell)}$ which are determined by the through thickness continuity conditions.

With the elastic constants for the equivalent plate defined for bending and stretching, the resultant stress or bending moment on any plane normal to the plane of the laminate will be identical to resultant in the equivalent plate. In particular, specified in-plane boundary loads on the laminate will be the same as those applied to the equivalent plate.

4.8 SATISFACTION OF THE INTERFACE CONDITIONS

In this section the arbitrary constants S_i and B_i in each layer are evaluated from satisfying the boundary conditions given in Section 4.4. The analysis is separated into two parts: the first part deals with the stretching solutions, and the second part deals with the bending solutions.

(A) STRETCHING

Continuity in the displacement w at each interface requires

$$S_1^{(1)} = 0,$$

$$S_1^{(\ell)} = \frac{Q_{44}^{(\ell)}}{\varepsilon_\ell Q_{13}^{(\ell)}} \sum_{r=2}^{\ell} \left(\varepsilon_r \frac{Q_{13}^{(r)}}{Q_{44}^{(r)}} + \varepsilon_{r-1} \frac{Q_{13}^{(r-1)}}{Q_{44}^{(r-1)}}\right) \qquad \ell = 2 \ldots N. \tag{4.8.1}$$

By substituting relations (4.7.15) into $(4.7.2)_{4,5}$ and satisfying the conditions of continuity in the inter-laminar shear stress components we obtain

$$S_2^{(1)} = 0,$$

$$S_2^{(\ell)}(Q_{13}^{(\ell)} - T^{(\ell)}) = -\frac{1}{\varepsilon_\ell}\left\{\sum_{r=2}^{\ell} (\varepsilon_r T^{(r)} + \varepsilon_{r-1} T^{(r-1)})\right\} + S_1^{(\ell)} Q_{13}^{(\ell)}, \quad \ell = 2 \ldots N, \tag{4.8.2}$$

where

$$T^{(r)} = Q_{11}^{(r)} - Q_{66}^{(r)} \frac{\hat{Q}_{11}}{\hat{Q}_{66}}, \qquad \varepsilon_r = \frac{h_r}{a}. \tag{4.8.3}$$

From $(4.7.2)_{4,5}$, it follows that

$$\sigma_{xz}^{(1)} = -\varepsilon_1 T^{(1)} z \frac{\partial \Delta_o}{\partial x},$$

$$\sigma_{xz}^{(\ell)} = -(\varepsilon_\ell T^{(\ell)} z + \sum_{r=2}^{\ell} (\varepsilon_r T^{(r)} + \varepsilon_{r-1} T^{(r-1)})) \frac{\partial \Delta_o}{\partial x}, \tag{4.8.4}$$

and a similar expression is obtained for σ_{yz} with $\partial/\partial x$ replaced by $\partial/\partial y$. The stress-free condition at $z_N = 1$ has been satisfied by the shear stresses given in (4.8.4) and this is confirmed by direct evaluation

$$\sigma_{xz}^{(N)}\Big|_{z_N=1} = -(\varepsilon_N T^{(N)} + \sum_{r=2}^{N} (\varepsilon_r T^{(r)} + \varepsilon_{r-1} T^{(r-1)})) \frac{\partial \Delta_o}{\partial x}$$

$$= -(\varepsilon_1 T^{(1)} + 2 \sum_{r=2}^{N} \varepsilon_r T^{(r)}) \frac{\partial \Delta_o}{\partial x} = 0. \tag{4.8.5}$$

In (4.8.5), use of (4.8.3) and the definition of $\hat{Q}_{ij}$, for stretching, has been made. If all the layers are identical in material properties then

$$T^{(r)} = 0 \tag{4.8.6}$$

and in such cases the shear stress components are identically zero.

The remaining arbitrary constants are determined from the continuity conditions for u (or v). It follows from (4.7.1) that

$$S_3^{(1)} \quad \text{disposable}$$

$$S_3^{(2)} = \frac{Q_{44}^{(2)}}{\varepsilon_2^2(Q_{13}^{(2)} - T^{(2)})} \times \left\{ \varepsilon_1^2 \frac{(Q_{13}^{(1)} - T^{(1)})}{Q_{44}^{(1)}}(\tfrac{1}{2} + S_3^{(1)}) - \varepsilon_2^2 \frac{(Q_{13}^{(2)} - T^{(2)})}{Q_{44}^{(2)}}(\tfrac{1}{2} - S_2^{(2)}) \right\}, \qquad (4.8.7)$$

$$S_3^{(\ell)} = \frac{Q_{44}^{(\ell)}}{\varepsilon_\ell^2(Q_{13}^{(\ell)} - T^{(\ell)})} \left\{ \varepsilon_1^2 \frac{(Q_{13}^{(1)} - T^{(1)})}{Q_{44}^{(1)}}(\tfrac{1}{2} + S_3^{(1)}) + 2\sum_{r=2}^{\ell-1} \varepsilon_r^2 \frac{(Q_{13}^{(r)} - T^{(r)})}{Q_{44}^{(r)}} S_2^{(r)} \dots - \varepsilon_\ell^2 \frac{(Q_{13}^{(\ell)} - T^{(\ell)})}{Q_{44}^{(\ell)}}(\tfrac{1}{2} - S_2^{(\ell)}) \right\},$$

$$\ell = 3, \dots N.$$

The disposable constant in Equations (4.8.7) can be chosen so that the resultant traction, on planes normal to the plane of the laminate, from the higher order terms is zero. From (4.7.3) we obtain

$$h_1 \varepsilon_1^2 \frac{Q_{66}^{(1)}}{Q_{44}^{(1)}}(Q_{13}^{(1)} - T^{(1)})(\tfrac{1}{3} + 2S_3^{(1)}) + 2\sum_{r=2}^{N} h_r \varepsilon_r^2 \frac{Q_{66}^{(r)}}{Q_{44}^{(r)}}(Q_{13}^{(r)} - T^{(r)}) \times (\tfrac{1}{3} + 2S_3^{(r)}) = 0. \qquad (4.8.8)$$

An alternative choice of the disposable constant which is appropriate if displacement boundary conditions are specified is to choose it so that the resultant displacement in the higher order terms of the in-plane displacement components is zero.

Finally, the variation in the inter-lamina shear stress components from layer to layer, may be controlled by the stacking sequence of the individual lamina. The 'jump' in the magnitude of the shear stress on passing through the ℓth layer is, from (4.8.4)

$$\sigma_{xz}^{(\ell)}\Big|_{z_{\ell=1}} - \sigma_{xz}^{(\ell)}\Big|_{z_{\ell=-1}} = -2\varepsilon_{\ell}T^{(\ell)}\frac{\partial\Delta_o}{\partial x}, \tag{4.8.9}$$

and therefore, the manner in which the inter-laminar shear stress changes is determined by

$$\text{sign }(T^{(\ell)}). \tag{4.8.10}$$

A consequence of this result is that if two adjacent layers in the laminate both produce the same sign in quantity (4.8.10) then the inter-laminar shear stress component will increase in the same manner in both layers.

(B) BENDING

From (4.7.5), continuity of the lowest order terms in the displacement components u,v requires that

$$B_1^{(1)} = 0,$$

$$B_1^{(2)} = \frac{1}{\varepsilon_2}(\varepsilon_1 + \varepsilon_2), \tag{4.8.11}$$

$$B_1^{(\ell)} = \frac{1}{\varepsilon_\ell}(\varepsilon_1 + 2\sum_{r=2}^{\ell-1}\varepsilon_r + \varepsilon_\ell) \quad \ell = 3 \ldots N$$

and the continuity of w gives

$$B_2^{(1)} \text{ disposable,}$$

$$B_2^{(\ell)} = \frac{\varepsilon_{\ell-1}^2 Q_{44}^{(\ell)} Q_{13}^{(\ell-1)}}{\varepsilon_\ell^2 Q_{44}^{(\ell-1)} Q_{13}^{(\ell)}}(\tfrac{1}{2} + B_2^{(\ell-1)}) + B_1^{(\ell)} - \tfrac{1}{2}, \quad \ell = 2 \dots N. \tag{4.8.12}$$

Satisfying continuity of the shear stress components at each interface and the stress-free condition at $z_N = 1$ produces

$$B_3^{(1)}(Q_{11}^{(1)} - Q_{13}^{(1)}) + B_2^{(1)} Q_{13}^{(1)} = -\frac{1}{\varepsilon_1^2} \sum_{\ell=2}^{N} \varepsilon_\ell^2 2Q_{11}^{(\ell)} B_1^{(\ell)} - \frac{Q_{11}^{(1)}}{2},$$

$$B_3^{(\ell)}(Q_{11}^{(\ell)} - Q_{13}^{(\ell)}) + B_2^{(\ell)} Q_{13}^{(\ell)} = -\frac{1}{\varepsilon_\ell^2} \sum_{r=\ell+1}^{N} \varepsilon_r^2 2Q_{11}^{(r)} B_1^{(r)}$$

$$- Q_{11}^{(\ell)}(\tfrac{1}{2} + B_1^{(\ell)}), \quad \ell = 2 \dots (N - 1),$$

$$B_3^{(N)}(Q_{11}^{(N)} - Q_{13}^{(N)}) + B_2^{(N)} Q_{13}^{(N)} = -Q_{11}^{(N)}(\tfrac{1}{2} + B_1^{(N)}). \tag{4.8.13}$$

The remaining constant B_4 in each layer is determined by considering the continuity of the higher order terms in the displacement components u and v of (4.7.5). This gives

$$B_4^{(1)} = 0,$$

$$B_4^{(\ell)} = \frac{\varepsilon_{\ell-1}^3 (Q_{11}^{(\ell-1)} - Q_{13}^{(\ell-1)}) Q_{44}^{(\ell)}}{\varepsilon_\ell^3 (Q_{11}^{(\ell)} - Q_{13}^{(\ell)}) Q_{44}^{(\ell-1)}} \left(\frac{1}{6} + \frac{B_1^{(\ell-1)}}{2} + B_3^{(\ell-1)} + B_4^{(\ell-1)}\right)$$

$$+ \frac{1}{6} - \frac{B_1^{(\ell)}}{2} + B_3^{(\ell)}, \quad \ell = 2 \dots N. \tag{4.8.14}$$

If we now require that there is no resultant bending moment, due to the higher order stress components, on planes normal to the plane of the laminate then we obtain, from (4.7.7) and (4.7.9) that

$$\varepsilon_1^3 h_1^2 \frac{Q_{66}^{(1)}}{Q_{44}^{(1)}}(Q_{11}^{(1)} - Q_{13}^{(1)})(\frac{1}{15} + 2\frac{B_3^{(1)}}{3}) + 2\sum_{\ell=2}^{N} \varepsilon_\ell^3 h_\ell^2 \frac{Q_{66}^{(\ell)}}{Q_{44}^{(\ell)}}(Q_{11}^{(\ell)} - Q_{13}^{(\ell)})(\frac{1}{15} + \frac{2B_3^{(\ell)}}{3})$$

$$+ 2\sum_{\ell=2}^{N}\sum_{r=2}^{\ell}(h_{r-1} + h_r)\varepsilon_\ell^3 h_\ell \frac{Q_{66}^{(\ell)}}{Q_{44}^{(\ell)}}(Q_{11}^{(\ell)} - Q_{13}^{(\ell)})(\frac{B_1^{(\ell)}}{3} + 2B_4^{(\ell)}) = 0. \quad (4.8.15)$$

This equation is used to determine the disposable constant $B_2^{(1)}$.

4.9 ILLUSTRATION: STRETCHING OF A LAMINATED PLATE CONTAINING A CIRCULAR HOLE

We now consider the effect of a stress-free circular hole of radius a on the stress components in a laminated plate which consists of three layers (The restriction to thenumber of layers is not necessary and has been made only to simplify the manipulation.). The in-plane dimensions of the laminate are large compared with the radius of the hole and in the direction of the x-axis, the laminate is subjected to a uniform tension which is applied at large distances from the centre of the hole. The analysis is in two parts: the first part considers the stretching of an equivalent plate containing a hole and from this the functions u_o and v_o are determined. The field quantities in each lamina are then evaluated by using these expressions.

(i) Equivalent Plate Solution

We employ cylindrical polar co-ordinates and denote

$$R = r/a, \quad (4.9.1)$$

where r is the radial distance from the centre of the hole and it follows from (4.9.1) that the edges of the hole are given by R = 1. Following Timoshenko and Goodier (1970), the stress components of the equivalent plate in a state of generalised plane stress and subjected to a constant uniaxial tension P_o in the direction of the x axis are

$$\hat{\sigma}_{rr} = \frac{P_o}{2}\{(1 - \frac{1}{R^2}) + (1 + \frac{3}{R^4} - \frac{4}{R^2})\cos 2\theta\},$$

$$\hat{\sigma}_{\theta\theta} = \frac{P_o}{2}\{(1 + \frac{1}{R^2}) - (1 + \frac{3}{R^4})\cos 2\theta\}, \qquad (4.9.2)$$

$$\hat{\sigma}_{r\theta} = -\frac{P_o}{2}\{1 - \frac{3}{R^4} + \frac{2}{R^2}\}\sin 2\theta.$$

The corresponding displacement components u_{ro} and $u_{\theta o}$, referred to a cylindrical polar co-ordinate system, are

$$u_{ro} = \frac{P_o}{4\hat{Q}_{66}}\{(\frac{\hat{Q}_{11} - \hat{Q}_{12}}{\hat{Q}_{11} + \hat{Q}_{12}})R + \frac{1}{R} + \cos 2\theta(R + \frac{4\hat{Q}_{11}}{(\hat{Q}_{11} + \hat{Q}_{12})R} - \frac{1}{R^3})\},$$

$$u_{\theta o} = +\frac{P_o}{4\hat{Q}_{66}}\{R - \frac{2}{R}\frac{(\hat{Q}_{11} - \hat{Q}_{12})}{(\hat{Q}_{11} + \hat{Q}_{12})} - \frac{1}{R^3}\}\sin 2\theta, \qquad (4.9.3)$$

where the $\hat{Q}_{ij}$ have been defined in Section 4.7.

(ii) Laminated Plate Solution

The exact solution to the stress and displacement components in each lamina is given by (4.7.1) and (4.7.2) and the S_i in these expressions have been evaluated in Section 4.8. For a three-layer laminate we have

$$
\begin{aligned}
S_1^{(1)} &= 0, \\
S_1^{(2)} &= 1 + \alpha\beta_3, \\
S_2^{(1)} &= 0 \\
S_2^{(2)} &= -\frac{(\beta_1^{(2)} + \alpha\beta_2\beta_1^{(1)})}{(1 - \beta_1^{(2)})} + \frac{1 + \alpha\beta_3}{(1 - \beta_1^{(2)})}, \\
S_3^{(2)} &= \alpha^2\beta_3 \frac{(1 - \beta_1^{(1)})}{(1 - \beta_1^{(2)})}(\tfrac{1}{2} + S_3^{(1)}) - (\tfrac{1}{2} - S_2^{(2)}),
\end{aligned}
\tag{4.9.4}
$$

where

$$
\alpha = \frac{\varepsilon_1}{\varepsilon_2} = \frac{h_1}{h_2}, \quad \beta_1^{(i)} = \frac{Q_{11}^{(i)}}{Q_{13}^{(i)}} - \frac{Q_{66}^{(i)}\hat{Q}_{11}}{Q_{13}^{(i)}\hat{Q}_{66}}, \quad \beta_2 = \frac{Q_{13}^{(1)}}{Q_{13}^{(2)}},
$$

$$
\beta_3 = \frac{Q_{44}^{(2)}Q_{13}^{(1)}}{Q_{44}^{(1)}Q_{13}^{(2)}}. \tag{4.9.5}
$$

The disposable constant in (4.9.4) is determined from (4.8.8) with $N = 2$. This gives

$$
\begin{aligned}
S_3^{(1)} = &-\frac{1}{6(1 - \beta_1^{(1)})\alpha^2\beta_3(\alpha\beta_4 + 2)} \\
&\times (\beta_3\alpha^2(1 - \beta_1^{(1)})(\alpha\beta_4 + 6) - 4(1 + 2\beta_1^{(2)}) - 12\alpha\beta_2\beta_1^{(1)}, \\
&\qquad + 12(1 + \alpha\beta_3))
\end{aligned}
\tag{4.9.6}
$$

where

$$\beta_4 = \frac{Q_{66}^{(1)}}{Q_{66}^{(2)}} . \tag{4.9.7}$$

By substituting (4.9.4) and the equivalent displacements given by (4.9.3) into (4.7.2) and employing the transformation relationships (1.2.6), the stress components in each layer of the laminate are obtained as follows

$$\frac{1}{P_o}\begin{Bmatrix} \sigma_{rr}^{(\ell)} \\ \sigma_{\theta\theta}^{(\ell)} \\ \sigma_{r\theta}^{(\ell)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}\{(\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}}) - \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}.\frac{1}{R^2} + \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}} \cos 2\theta \\ \times (1 + \frac{3}{R^4} - \frac{4(Q_{11}^{(\ell)}\hat{Q}_{11} - Q_{12}^{(\ell)}\hat{Q}_{12})}{(Q_{11}^{(\ell)} - Q_{12}^{(\ell)})(\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^2})\} \\ \frac{1}{2}\{(\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}}) + \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}.\frac{1}{R^2} - \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}} \cos 2\theta \\ \times (1 + \frac{3}{R^4} + \frac{4(\hat{Q}_{11}Q_{12}^{(\ell)} - \hat{Q}_{12}Q_{11}^{(\ell)})}{(Q_{11}^{(\ell)} - Q_{12}^{(\ell)})(\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^2})\} \\ -\frac{1}{2} \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\{1 + \frac{2}{R^2} - \frac{3}{R^4}\}\sin 2\theta \end{Bmatrix} \tag{4.9.8}$$

$$+ \frac{\varepsilon_\ell^2 24 Q_{66}^{(\ell)}(Q_{13}^{(\ell)} - T^{(\ell)})}{Q_{44}^{(\ell)}(\hat{Q}_{11} + \hat{Q}_{12})R^4}(\frac{z^2}{2} + S_2^{(\ell)}z + S_3^{(\ell)}) \begin{Bmatrix} -\cos 2\theta \\ \cos 2\theta \\ -\sin 2\theta \end{Bmatrix} \tag{4.9.9}$$

$$\frac{1}{P_o}\begin{Bmatrix}\sigma_{rz}^{(\ell)}\\ \sigma_{\theta z}^{(\ell)}\end{Bmatrix} = -\varepsilon_\ell\{T^{(\ell)}z - S_2^{(\ell)}(Q_{13}^{(\ell)} - T^{(\ell)}) + S_1^{(\ell)}Q_{13}^{(\ell)}\}$$

$$\times \frac{4}{R^3(\hat{Q}_{11} + \hat{Q}_{12})}\begin{Bmatrix}\cos 2\theta\\ \sin 2\theta\end{Bmatrix}. \qquad (4.9.10)$$

Neglecting terms of $O(\varepsilon_\ell)$, that is supposing each lamina is thin compared with the radius of the hole, (4.9.8) represents the solution predicted by classical laminate theory.

4.10 NUMERICAL RESULTS

For large R, the following equivalent stress components are obtained

$$\begin{aligned}\hat{\sigma}_{rr} &= \frac{P_o}{2}(1 + \cos 2\theta) + O(\frac{1}{R^2}),\\ \hat{\sigma}_{\theta\theta} &= \frac{P_o}{2}(1 - \cos 2\theta) + O(\frac{1}{R^2}),\\ \hat{\sigma}_{r\theta} &= -\frac{P_o}{2}\sin 2\theta + O(\frac{1}{R^2}),\end{aligned} \qquad (4.10.1)$$

whereas in the ℓth layer

$$\begin{aligned}\sigma_{rr}^{(\ell)} &= \frac{P_o}{2}\{\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} + \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\cos 2\theta\} + O(\frac{1}{R})^2,\\ \sigma_{\theta\theta}^{(\ell)} &= \frac{P_o}{2}\{\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} - \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\cos 2\theta\} + O(\frac{1}{R})^2,\\ \sigma_{r\theta}^{(\ell)} &= -\frac{P_o}{2}\frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\sin 2\theta + O(\frac{1}{R})^2,\\ \sigma_{rz}^{(\ell)} &= \sigma_{\theta z}^{(\ell)} = O(\frac{1}{R})^3 .\end{aligned} \qquad (4.10.2)$$

Note that the inter-lamina shear stress components decay as $O(1/R)^3$ whereas the stress concentration due to the hole in the equivalent plate decays as $O(1/R)^2$.

At the hole boundary R = 1

$$\hat{\sigma}_{rr} = \hat{\sigma}_{r\theta} = 0 \tag{4.10.3}$$

$$\hat{\sigma}_{\theta\theta} = P_o(1 - 2\cos 2\theta)$$

and

$$\sigma_{rr}^{(\ell)} = \frac{P_o}{2}\{\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} - \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}$$

$$+ 4\cos 2\theta(\frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}} - \frac{Q_{11}^{(\ell)}\hat{Q}_{11} - \hat{Q}_{12}^{(\ell)}Q_{12}}{\hat{Q}_{11}^2 - \hat{Q}_{12}^2})\} + O(\varepsilon)^2,$$

$$\sigma_{\theta\theta}^{(\ell)} = \frac{P_o}{2}\{\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} + \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}$$

$$- 4\cos 2\theta(\frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}} + \frac{\hat{Q}_{11}Q_{12}^{(\ell)} - \hat{Q}_{12}Q_{11}^{(\ell)}}{\hat{Q}_{11}^2 - \hat{Q}_{12}^2})\} + O(\varepsilon)^2, \tag{4.10.4}$$

$$\sigma_{r\theta}^{(\ell)} = O(\varepsilon)^2,$$

$$\sigma_{rz}^{(\ell)} = \sigma_{\theta z}^{(\ell)} = O(\varepsilon).$$

The stress concentration in the equivalent plate is $3P_o$ at $\theta = \pi/2$ and it is found from (4.10.4) that the corresponding concentration in each lamina is given by

$$3P_o\left(\frac{Q_{11}^{(\ell)}\hat{Q}_{11} - Q_{12}^{(\ell)}\hat{Q}_{12}}{\hat{Q}_{11}^2 - \hat{Q}_{12}^2}\right). \qquad (4.10.5)$$

In the equivalent plate, the hole surface is stress-free and in the laminate it is stress-free in an average sense. This can be shown by integrating the appropriate stress components in (4.10.4) through the laminate thickness and using the definition of $\hat{Q}_{ij}$. Therefore, adjacent to the hole surface, a further correction will be required and the magnitude of this correction will be determined by (4.9.8), (4.9.9) and (4.9.10) evaluated at R = 1. However, as a consequence of Saint-Venant's principle, the solutions in (4.9.8), (4.9.9) and (4.9.10) do represent the stress fields in each lamina away from the free edges of the hole.

The inter-lamina shear stress components in (4.9.10) vary linearly with z and attain their maximum value at the interface of the layers. If we denote

$$S_{max} = \sqrt{(\sigma_{rz}^2 + \sigma_{\theta z}^2)}, \text{ evaluated at the interface,} \qquad (4.10.6)$$

then

$$S_{max} = -\frac{4P_o\varepsilon_1}{(\hat{Q}_{11} + \hat{Q}_{12})}\left(Q_{11}^{(1)} - Q_{66}^{(1)}\frac{\hat{Q}_{11}}{\hat{Q}_{66}}\right)\frac{1}{R^3}. \qquad (4.10.7)$$

Note that S_{max} is independent of θ. Values of S_{max} at R = 1 are given in Table 4.3 for the case of a laminate consisting of three isotropic layers of equal width. The Youngs modulus and Poisson ratio are given as E_i, ν_i respectively.

$\nu_2 \backslash \nu_1$	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
0.1	0	-0.12	-0.28	0	-0.56	-1.23	0	-0.34	-0.69
0.3	0.11	0	-0.14	0.55	0	-0.66	0.36	0	-0.37
0.5	0.22	0.12	0	1.13	0.62	0	0.82	0.43	0
	(a)			(b)			(c)		

Table 4.3. The Variation of $S_{max}/P_o\varepsilon$ with Poisson Ratio at R = 1

(a) $E_1/E_2 = 1/10$ (b) $E_1/E_2 = 1$ (c) $E_1/E_2 = 10$.

In Table 4.4, the leading term in the in-plane stress components is given when R = 1, $\theta = \pi/2$ and the layers are isotropic and of equal width. The equivalent stress components are obtained from this table as they correspond to the case when $\nu_1 = \nu_2$ and $E_1 = E_2$. The stress component σ_{rr}, in each layer, is found, in most cases, to be small compared with P_o.

The maximum hoop stress in the equivalent plate is $3P_o$ which, for certain values of Youngs modulus and Poisson ratio, is exceeded by the hoop stress in the laminate

Through the thickness of the laminate, the maximum value of the higher order in-plane stress component occurs, in general, at the midplane of the laminate. In Table 4.5 this maximum is given at R = 1 for an isotropic laminated plate of three layers of equal width.

E_1/E_2	ν_2	ν_1	0.1		0.3		0.5	
(a) Layer 2			$\sigma_{rr}^{(2)}/P_o$	$\sigma_{\theta\theta}^{(2)}/P_o$	$\sigma_{rr}^{(2)}/P_o$	$\sigma_{\theta\theta}^{(2)}/P_o$	$\sigma_{rr}^{(2)}/P_o$	$\sigma_{\theta\theta}^{(2)}/P_o$
1	0.1		0.0	3.0	-0.21	2.95	-0.46	2.83
1	0.3		0.21	3.03	0.0	3.0	-0.25	2.89
1	0.5		0.42	3.10	0.23	3.08	0.0	3.0
1/10	0.1		0.0	4.29	-0.04	4.27	-0.11	4.23
1/10	0.3		0.04	4.29	0.0	4.29	-0.05	4.26
1/10	0.5		0.08	4.30	0.05	4.30	0.0	4.29
10	0.1		0.0	0.75	-0.13	0.73	-0.26	0.71
10	0.3		0.13	0.79	0.0	0.75	-0.14	0.70
10	0.5		0.31	0.88	0.16	0.82	0.0	0.75
(b) Layer 1			$\sigma_{rr}^{(1)}/P_o$	$\sigma_{\theta\theta}^{(1)}/P_o$	$\sigma_{rr}^{(1)}/P_o$	$\sigma_{\theta\theta}^{(1)}/P_o$	$\sigma_{rr}^{(1)}/P_o$	$\sigma_{\theta\theta}^{(1)}/P_o$
1	0.1		0.0	3.0	0.42	3.10	0.92	3.34
1	0.3		-0.42	2.93	0.0	3.0	0.49	3.21
1	0.5		-0.85	2.80	-0.46	2.83	0.0	3.0
1/10	0.1		0.0	0.43	0.09	0.45	0.21	0.53
1/10	0.3		-0.08	0.42	0.0	0.43	0.11	0.48
1/10	0.5		-0.17	0.41	-0.09	0.40	0.0	0.43
10	0.1		0.0	7.50	0.25	7.53	0.51	7.59
10	0.3		-0.27	7.43	0.0	7.50	0.28	7.59
10	0.5		-0.62	7.23	-0.32	7.35	0.0	7.50

Table 4.4. Leading Term in the In-Plane Stress Components Evaluated At $R = 1$, $\theta = \pi/2$.

ν_2 \ ν_1	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
0.1	3.78	7.97	15.31	3.27	4.70	7.14	-1.36	1.97	3.58
0.3	6.94	9.61	14.40	8.14	8.31	9.18	-4.82	-3.46	2.14
0.5	9.44	10.68	13.88	13.09	12.32	12.00	-9.82	-7.31	-5.00
		(a)			(b)			(c)	

Table 4.5. Maximum Value of $-\sigma_{rr}/P_o\varepsilon^2\cos 2\theta$, $\sigma_{\theta\theta}/P_o\varepsilon^2\cos 2\theta$, $-\sigma_{r\theta}/P_o\varepsilon^2\sin 2\theta$ *at* $R = 1$

(a) $E_1/E_2 = 1/10$ *(b)* $E_1/E_2 = 1$ *(c)* $E_1/E_2 = 10$

The higher order stress components are thus negligible if ε is small such that $10\varepsilon^2$ is small. In such cases, classical laminate theory provides an adequate solution to the in-plane stress components. However, the solution given here gives, in addition to the classical solutions, information about the interlamina shear stress components.

If $\varepsilon = 0(1)$ then the contribution to the stress components from the higher order terms is significant. In such cases, the stresses in the laminate cannot be determined solely by the classical theory.

4.11 ILLUSTRATION: BENDING OF A LAMINATED PLATE CONTAINING A CIRCULAR HOLE

In this example we examine the effect of a stress-free circular hole on the stress components in a laminated plate. The radius of the hole is a and is small compared with the in-plane dimensions of the plate. The laminate is deformed by resultant bending moments

$$\hat{M}_{xx} = H^2\varepsilon P_o, \quad \hat{M}_{yy} = 0 \quad \text{as } R \to \infty \tag{4.11.1}$$

where

$$R = r/a. \tag{4.11.2}$$

The analysis to follow is given in two parts: the first part considers the bending of the equivalent plate and from this we determine the middle surface deflection w_o. Using this expression, the field quantities in each lamina are evaluated.

(i) Equivalent Plate Solution

Note first that if $\hat{w}$ and its derivatives in (4.6.6) are O(1) then

$$\hat{M}_{xx} = O(\hat{\varepsilon}\hat{H}^2), \quad \hat{\varepsilon} << 1. \tag{4.11.3}$$

Following Timoshenko and Goodier (1970) we suppose the plate is bent by bending moments

$$\hat{M}_{xx} = \hat{H}^2\hat{\varepsilon}P_o, \quad \hat{M}_{yy} = 0 \quad \text{as } R \to \infty \tag{4.11.4}$$

where P_o is constant and has dimensions of stress. The middle surface deflection is then given by

$$\hat{w} = A[\ln R + \ln a] + [B - \frac{C}{R^2}]\cos 2\theta - \frac{3P_o}{8} R^2(\frac{1}{\hat{Q}_{11} + \hat{Q}_{12}} + \frac{\cos 2\theta}{\hat{Q}_{11} - \hat{Q}_{12}}) \tag{4.11.5}$$

where

$$A = -\frac{3P_o}{4(\hat{Q}_{11} - \hat{Q}_{12})}, \quad B = -\frac{3P_o}{4(3\hat{Q}_{11} + \hat{Q}_{12})}, \quad C = \frac{B}{2}. \tag{4.11.6}$$

From (4.11.5) and (4.6.2) the equivalent stresses are determined as follows

$$\hat{\sigma}_{rr} = \hat{\varepsilon}z\,\frac{3P_o}{4}\Big(1 - \frac{1}{R^2} + \cos 2\theta\Big[1 - \frac{3(\hat{Q}_{11} - \hat{Q}_{12})}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^4}$$

$$- \frac{4\hat{Q}_{12}}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^2}\Big]\Big),$$

$$\hat{\sigma}_{\theta\theta} = \hat{\varepsilon}z\,\frac{3P_o}{4}\Big(1 + \frac{1}{R^2} - \cos 2\theta\Big[1 - \frac{3(\hat{Q}_{11} + \hat{Q}_{12})}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^4}$$

$$+ \frac{4\hat{Q}_{11}}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^2}\Big]\Big),$$

$$\hat{\sigma}_{r\theta} = -\hat{\varepsilon}z\,\frac{3P_o}{4}\Big(1 - \frac{\hat{Q}_{11} - \hat{Q}_{12}}{(3\hat{Q}_{11} + \hat{Q}_{12})}\Big[\frac{2}{R^2} - \frac{3}{R^4}\Big]\Big)\sin 2\theta, \qquad (4.11.7)$$

$$\hat{\sigma}_{rz} = -\hat{\varepsilon}^2\Big(\frac{z^2}{2} - \tfrac{1}{2}\Big)\hat{Q}_{11}\,\frac{6P_o}{(3\hat{Q}_{11} + \hat{Q}_{12})R^3}\cos 2\theta,$$

$$\hat{\sigma}_{\theta z} = -\hat{\varepsilon}^2\Big(\frac{z^2}{2} - \tfrac{1}{2}\Big)\hat{Q}_{11}\,\frac{6P_o}{(3\hat{Q}_{11} + \hat{Q}_{12})R^3}\sin 2\theta,$$

where the $\hat{Q}_{ij}$ have been defined in (4.7.23).

(ii) Laminated Plate Solution

The exact solution to the stress and displacement components in each lamina is given by (4.7.5), (4.7.6) and the B_i in these expressions have been evaluated in Section 4.8. For a three layer laminate we have

$$B_1^{(1)} = B_4^{(1)} = 0,$$

$$B_1^{(2)} = (1 + \alpha),$$

$$B_2^{(2)} = \tfrac{1}{2} + \alpha + \alpha^2 \beta_3 (\tfrac{1}{2} + B_2^{(1)}),$$

$$B_3^{(2)} = -\frac{\beta_1^{(2)}}{(\beta_1^{(2)} - 1)}(\frac{3}{2} + \alpha) - \frac{1}{(\beta_1^{(2)} - 1)}(\tfrac{1}{2} + \alpha + \alpha^2\beta_3(\tfrac{1}{2} + B_2^{(1)})), \qquad (4.11.8)$$

$$B_3^{(1)} = -2\,\frac{(1 + \alpha)}{\alpha^2}\,\frac{\beta_1^{(2)}\beta_2}{(\beta_1^{(1)} - 1)} - \frac{\beta_1^{(1)}}{2(\beta_1^{(1)} - 1)} - \frac{B_2^{(1)}}{(\beta_1^{(1)} - 1)},$$

$$B_4^{(2)} = \alpha^3\beta_3\,\frac{(\beta_1^{(1)} - 1)}{(\beta_1^{(2)} - 1)}(\frac{1}{6} + B_3^{(1)}) + (\frac{1}{6} + B_3^{(2)}) - \frac{(1 + \alpha)}{2}\,.$$

In (4.11.8),

$$\varepsilon = \frac{\varepsilon_1}{\varepsilon_2} = \frac{h_1}{h_2}, \quad \beta_1^{(1)} = \frac{Q_{11}^{(1)}}{Q_{13}^{(1)}}, \quad \beta_2 = \frac{Q_{13}^{(2)}}{Q_{13}^{(1)}}, \quad \beta_3 = \frac{Q_{44}^{(2)}Q_{13}^{(1)}}{Q_{44}^{(1)}Q_{13}^{(2)}}\,. \qquad (4.11.9)$$

The disposable constant $B_2^{(1)}$ in (4.11.8) is determined from (4.8.15) with N = 2. Hence

$$B_2^{(1)} = \frac{1}{\alpha^2\beta_3(\alpha^3\beta_4 + 2(3\alpha^2 + 6\alpha + 4))}\{\frac{1}{5}(40\alpha^2 - 40\alpha - 21) - \frac{\beta_3\beta_4\alpha^5}{10}(1 + 4\beta_1^{(1)}) - \ldots$$

$$\ldots - \beta_2\beta_3\beta_1^{(2)}2\alpha(1 + \alpha)(\alpha^2\beta_4 + 6(1 + \alpha))$$

$$- \beta_3\alpha^2(\alpha + 2)^2$$

$$- \beta_1^{(2)}\,\frac{4}{5}(10\alpha^2 + 25\alpha + 16)\} \qquad (4.11.10$$

where

$$\beta_4 = Q_{66}^{(1)}/Q_{66}^{(2)}\,.$$

The stress and displacement components in each layer of the laminate can now be obtained by substituting the B_i and (4.11.5) into (4.7.5), (4.7.6) as follows

$$\frac{1}{P_o}\begin{Bmatrix}\sigma_{rr}^{(\ell)}\\ \sigma_{\theta\theta}^{(\ell)}\\ \sigma_{r\theta}^{(\ell)}\end{Bmatrix} = \varepsilon_\ell \frac{3}{4}(z + B_1^{(\ell)})\begin{Bmatrix}\frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} - \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}.\frac{1}{R^2} + \cos 2\theta\left(\frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}} - \frac{3(Q_{11}^{(\ell)} - Q_{12}^{(\ell)})}{(3\hat{Q}_{11} + \hat{Q}_{12})}\frac{1}{R^4} - \frac{4Q_{12}^{(\ell)}}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^2}\right)\\ \frac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} + \frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}.\frac{1}{R^2} - \cos 2\theta\left(\frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}} - \frac{3(Q_{11}^{(\ell)} - Q_{12}^{(\ell)})}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^4} + \frac{4Q_{11}^{(\ell)}}{(3\hat{Q}_{11} + \hat{Q}_{12})}.\frac{1}{R^2}\right)\\ -\frac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\left(1 - \frac{\hat{Q}_{11} - \hat{Q}_{12}}{3\hat{Q}_{11} + \hat{Q}_{12}}\left[\frac{2}{R^2} - \frac{3}{R^4}\right]\right)\sin 2\theta\end{Bmatrix}$$

$$+ \varepsilon_\ell^3\begin{Bmatrix}\cos 2\theta\\ -\cos 2\theta\\ \sin 2\theta\end{Bmatrix}\frac{36Q_{66}^{(\ell)}(Q_{11}^{(\ell)} - Q_{13}^{(\ell)})}{Q_{44}^{(\ell)}(3\hat{Q}_{11} + \hat{Q}_{12})}\left[\frac{z^3}{6} + B_1^{(\ell)}\frac{z^2}{2} + B_3^{(\ell)}z + B_4^{(\ell)}\right]\frac{1}{R^4} \qquad (4.11.12)$$

$$\frac{1}{P_o}\begin{Bmatrix}\sigma_{rz}^{(\ell)}\\ \sigma_{\theta z}^{(\ell)}\end{Bmatrix} = -\varepsilon_\ell^2\begin{Bmatrix}\cos 2\theta\\ \sin 2\theta\end{Bmatrix}\frac{6}{(3\hat{Q}_{11} + \hat{Q}_{12})}\left[Q_{11}^{(\ell)}\left(\frac{z^2}{2} + B_1^{(\ell)}z\right) + B_3^{(\ell)}(Q_{11}^{(\ell)} - Q_{13}^{(\ell)}) + B_2^{(\ell)}Q_{13}^{(\ell)}\right]\frac{1}{R^3}. \qquad (4.11.13)$$

4.12 NUMERICAL RESULTS

When $R >> 1$, the stress components in the equivalent plate are

$$\frac{1}{P_o}\begin{Bmatrix}\hat{\sigma}_{rr}\\ \hat{\sigma}_{\theta\theta}\\ \hat{\sigma}_{r\theta}\end{Bmatrix} = \hat{\varepsilon}z\,\frac{3}{4}\begin{Bmatrix}1 + \cos 2\theta\\ 1 - \cos 2\theta\\ -\sin 2\theta\end{Bmatrix} + 0(\frac{1}{R^2}), \qquad (4.12.1)$$

$$\hat{\sigma}_{rz} = \hat{\sigma}_{\theta z} = 0(\frac{1}{R^3}),$$

which produce the prescribed bending moments

$$\hat{M}_{xx} = \hat{\varepsilon}\hat{H}^2P_o, \quad \hat{M}_{yy} = \hat{M}_{xy} = 0 \quad \text{as } R \to \infty. \qquad (4.12.2)$$

For each lamina

$$\frac{1}{P_o}\begin{Bmatrix}\sigma_{rr}^{(\ell)}\\ \sigma_{\theta\theta}^{(\ell)}\\ \sigma_{r\theta}^{(\ell)}\end{Bmatrix} = \varepsilon_\ell(z + B_1^{(\ell)})\,\frac{3}{4}\begin{Bmatrix}\dfrac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} + \dfrac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\cos 2\theta\\ \dfrac{Q_{11}^{(\ell)} + Q_{12}^{(\ell)}}{\hat{Q}_{11} + \hat{Q}_{12}} - \dfrac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\cos 2\theta\\ -\dfrac{Q_{66}^{(\ell)}}{\hat{Q}_{66}}\sin 2\theta\end{Bmatrix} + 0(\frac{1}{R^2}) \qquad (4.12.3)$$

$\ell = 1,2, \ldots$

$$\sigma_{rz}^{(\ell)} = \sigma_{\theta z}^{(\ell)} = 0(\frac{1}{R})^3.$$

As $R \to \infty$ the resultant bending moments on planes normal to the plane of the laminate are exactly those given by (4.12.2).

At the hole R = 1, the equivalent stress components satisfy

$$\hat{M}_{rr} = 0, \quad \hat{N}_r = \frac{1}{a}\frac{\partial M_{r\theta}}{\partial \theta} = 0 \tag{4.12.4}$$

which are exactly the same conditions as are satisfied by the averaged stress components of the laminate.

Therefore, by Saint Venant's principle, the solutions given by (4.11.11), (4.11.12) and (4.11.13) are valid solutions sufficiently far from the free edges of the hole. In the vicinity of the hole, a further correction will be required and the magnitude of this correction will be determined by the deviation of the values given by (4.11.11), (4.11.12) and (4.11.13) from the prescribed boundary values.

In the remainder of this section we will consider a laminate consisting of three isotropic layers of equal width. Hence

$$\varepsilon_1 = \varepsilon_2 = \varepsilon = \frac{h}{a} . \tag{4.12.5}$$

For this laminate, the equivalent material constants are, from (4.7.23)

$$\hat{Q}_{11} = \frac{E_2}{27}\left[\frac{E}{1 - \nu_1^2} + \frac{26}{1 - \nu_2^2}\right], \quad \hat{Q}_{12} = \frac{E_2}{27}\left[\frac{\nu_1 E}{1 - \nu_1^2} + \frac{26\nu_2}{1 - \nu_2^2}\right], \tag{4.12.6}$$

where

$$E = E_1/E_2 . \tag{4.12.7}$$

Note that Equations (4.12.6) are in agreement with those found by

Lehknittski (1968). The interlamina shear stress components, given by (4.11.13), are

$$\frac{1}{P_o}\begin{Bmatrix}\sigma_{rz}^{(1)}\\ \\ \sigma_{rz}^{(2)}\end{Bmatrix} = -\varepsilon^2 \frac{6\cos 2\theta}{R^3(3\hat{Q}_{11} + \hat{Q}_{12})}\begin{Bmatrix}Q_{11}^{(1)}(\frac{z^2}{2} - \frac{1}{2}) - 4Q_{11}^{(2)}\\ \\ Q_{11}^{(2)}(\frac{z^2}{2} + 2z - \frac{5}{2})\end{Bmatrix} \qquad (4.12.8)$$

with $\sigma_{\theta z}$ obtained from (4.12.8) by replacing cos 2θ by sin 2θ. The maximum through thickness value for the shear stress occurs at the midplane of the laminate and we therefore denote

$$S_{max} = \sqrt{(\sigma_{rz}^2 + \sigma_{\theta z}^2)}, \text{ evaluated at the midplane.} \qquad (4.12.9)$$

By substituting (4.12.8) into (4.12.9) we obtain

$$S_{max} = +(\frac{Q_{11}^{(2)}}{2} + 4Q_{11}^{(2)}) \frac{6\varepsilon^2 P_o}{(3\hat{Q}_{11} + \hat{Q}_{12})R^3} . \qquad (4.12.10)$$

Note that S_{max} is independent of θ. In Figure 4.6, the variation of the interlamina shear stress components is given at the hole surface $R = 1$ when $\nu_1 = \nu_2 = 0.3$.

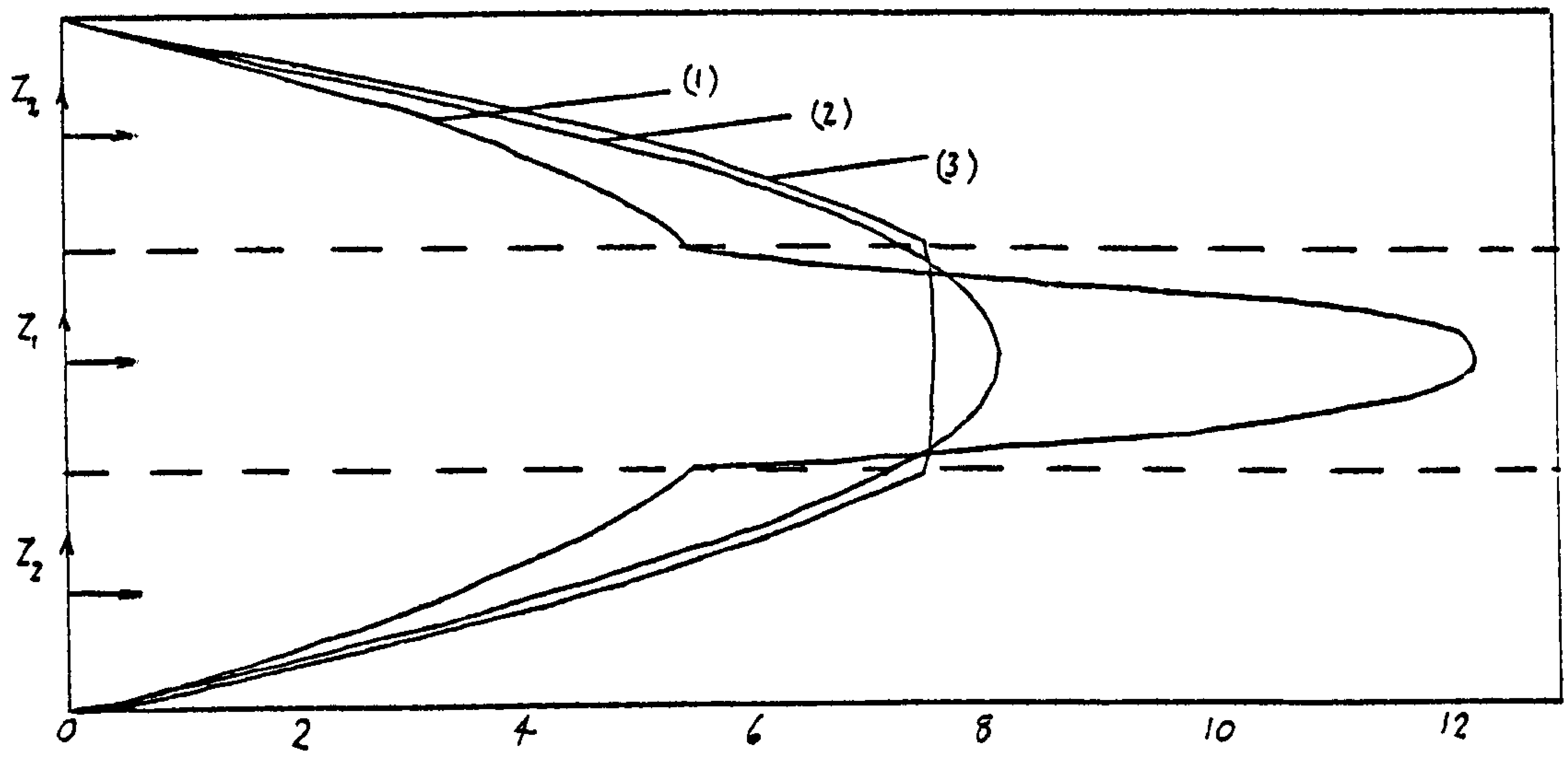


Figure 4.6. The Variation of $\sigma_{rz}/P_o\varepsilon^2\cos 2\theta$, $\sigma_{\theta z}/P_o\varepsilon^2\sin 2\theta$ with thickness at $R = 1$.

(1) $E = 10$ (2) $E = 1$ (3) $E = 1/10$.

The resultant shear traction produced by the stresses in Figure 4.7 is given in Table 4.7.

E	1/10	1	10
$N_r^{(1)}/P_o\varepsilon^2\cos 2\theta$, $N_\theta^{(1)}/P_o\varepsilon^2\sin 2\theta$	15.17	15.75	20.00
$N_r^{(2)}/P_o\varepsilon^2\cos 2\theta$, $N_\theta^{(2)}/P_o\varepsilon^2\sin 2\theta$	8.78	8.48	6.36
$\hat{N}_r/P_o\varepsilon^2\cos 2\theta$, $\hat{N}_\theta/P_o\varepsilon^2\sin 2\theta$	32.67	32.67	32.67

Table 4.7. Resultant Shear Traction in Each Layer at $R = 1$.

For comparison purposes, the equivalent plate solution corresponds to the case where E = 1 in Figure 4.6 and the bottom row of values in Table 4.7. Note that

$$\hat{N}_j = N_j^{(1)} + 2N_j^{(2)}, \quad j = r,\theta. \tag{4.12.11}$$

The variation in the shear stress components when $\nu_1 \neq \nu_2$ is given in Table 4.8. This table gives a through thickness variation at R = 1 when E = 1 and shows that the effect on the shear component values of changing Poisson's ratio is small.

		$\nu_1 = 0.2$, $\nu_2 = 0.4$	$\nu_1 = \nu_2 = 0.3$	$\nu_1 = 0.4$, $\nu_2 = 0.2$
$z_2 = 1$	Free Surface	0.0	0.0	0.0
$= 0.5$		2.44	2.5	2.56
$z_2 = 0$		4.44	4.55	4.65
$= -0.5$		6.00	6.14	6.28
$z_2 = -1$, $z_1 = 1$	Interface	7.11	7.27	7.44
$z_1 = 0.5$		7.69	7.95	8.24
$z_1 = 0$	Mid-Plane	7.88	8.18	8.5

Table 4.8. Variation in $\sigma_{rz}/P_o\varepsilon^2\cos 2\theta$, $\sigma_{\theta z}/P_o\varepsilon^2\cos 2\theta$ with Thickness at R=1, E = 1.

If $\nu_1 = \nu_2 = \nu$ say then

$$Q_{12}^{(\ell)} = Q_{11}^{(\ell)}, \quad \hat{Q}_{11} = \hat{Q}_{12} \qquad \ell = 1,2. \tag{4.12.12}$$

Substituting (4.12.12) into (4.11.11) and simplifying gives, at R = 1

$$\frac{1}{P_o}\begin{pmatrix}\sigma_{r\theta}^{(1)}\\ \\ \sigma_{r\theta}^{(2)}\end{pmatrix} = -\varepsilon\,\frac{81}{(3+\nu)}\sin 2\theta\begin{pmatrix}\dfrac{z}{(1+\frac{26}{E})}\\ \\ \dfrac{z+2}{E+26}\end{pmatrix}$$

$$\frac{1}{P_o}\begin{pmatrix}\sigma_{\theta\theta}^{(1)}\\ \\ \sigma_{\theta\theta}^{(2)}\end{pmatrix} = \varepsilon\,\frac{81}{4}(2-4\cos 2\theta)\begin{pmatrix}\dfrac{z}{1+\frac{26}{E}}\\ \\ \dfrac{z+2}{E+26}\end{pmatrix} \qquad (4.12.13)$$

$$\sigma_{rr}^{(\ell)} = 0 \qquad \ell = 1,2.$$

Equations (4.12.13) represent the leading terms of the in-plane stress components at R = 1 when $\nu_1 = \nu_2$. At the interface of the layers, the magnitude of the non-zero components in layer 1 are E times those in layer 2. Figure 4.9 shows the variation of $\sigma_{\theta\theta}$, at $\theta = \pi/2$, with thickness whilst Figure 4.10 shows the variation of $\sigma_{r\theta}$, evaluated at $\theta = \pi/4$. Note that the equivalent stresses in these figures correspond to the case when E = 1.

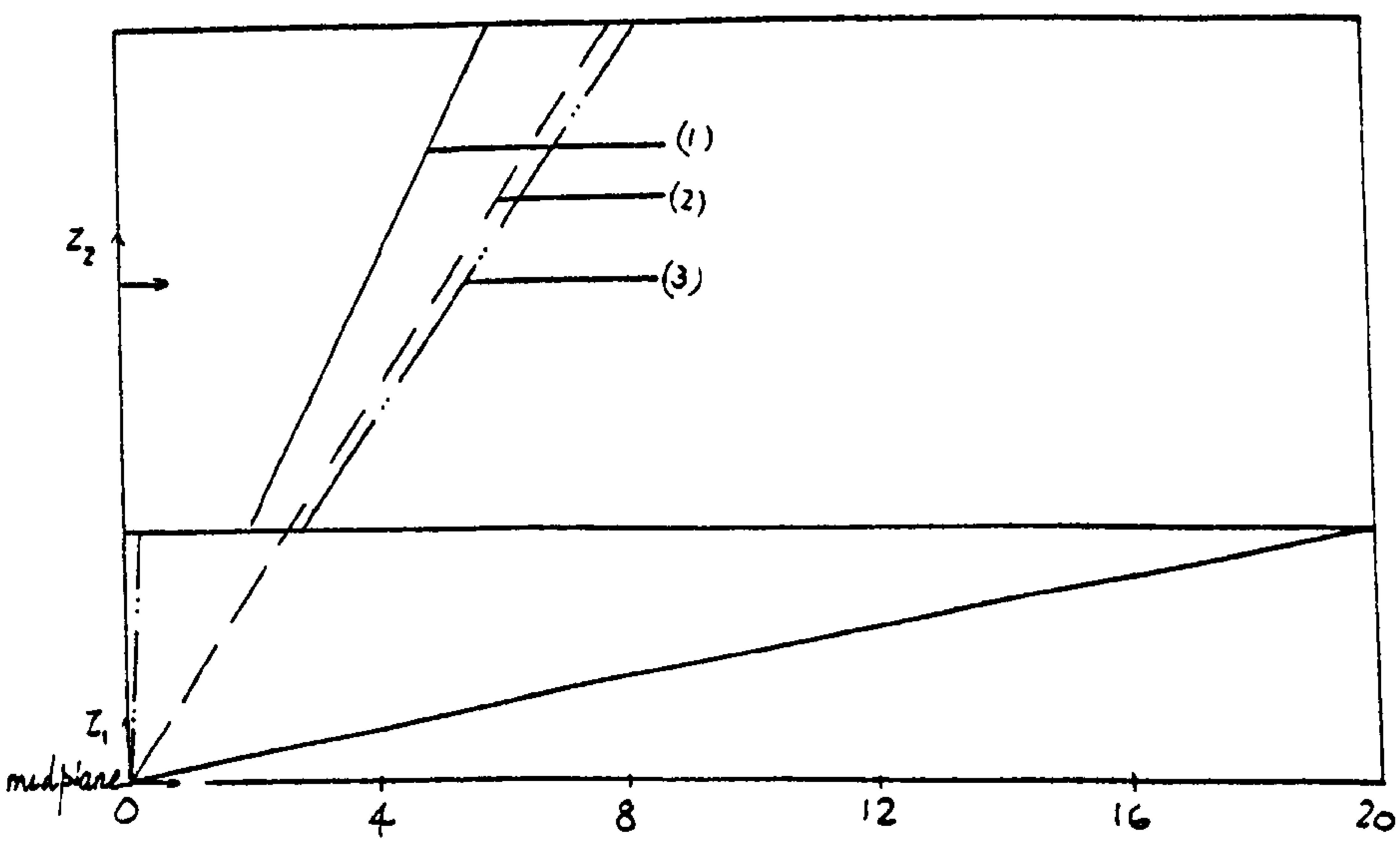


Figure 4.9. Variation of $\sigma_{\theta\theta}/P_o\varepsilon$ With Thickness At R = 1, $\theta = \pi/2$, $\nu = 0.3$.

(1) E = 10 *(2) E = 1* *(3) E = 1/10*

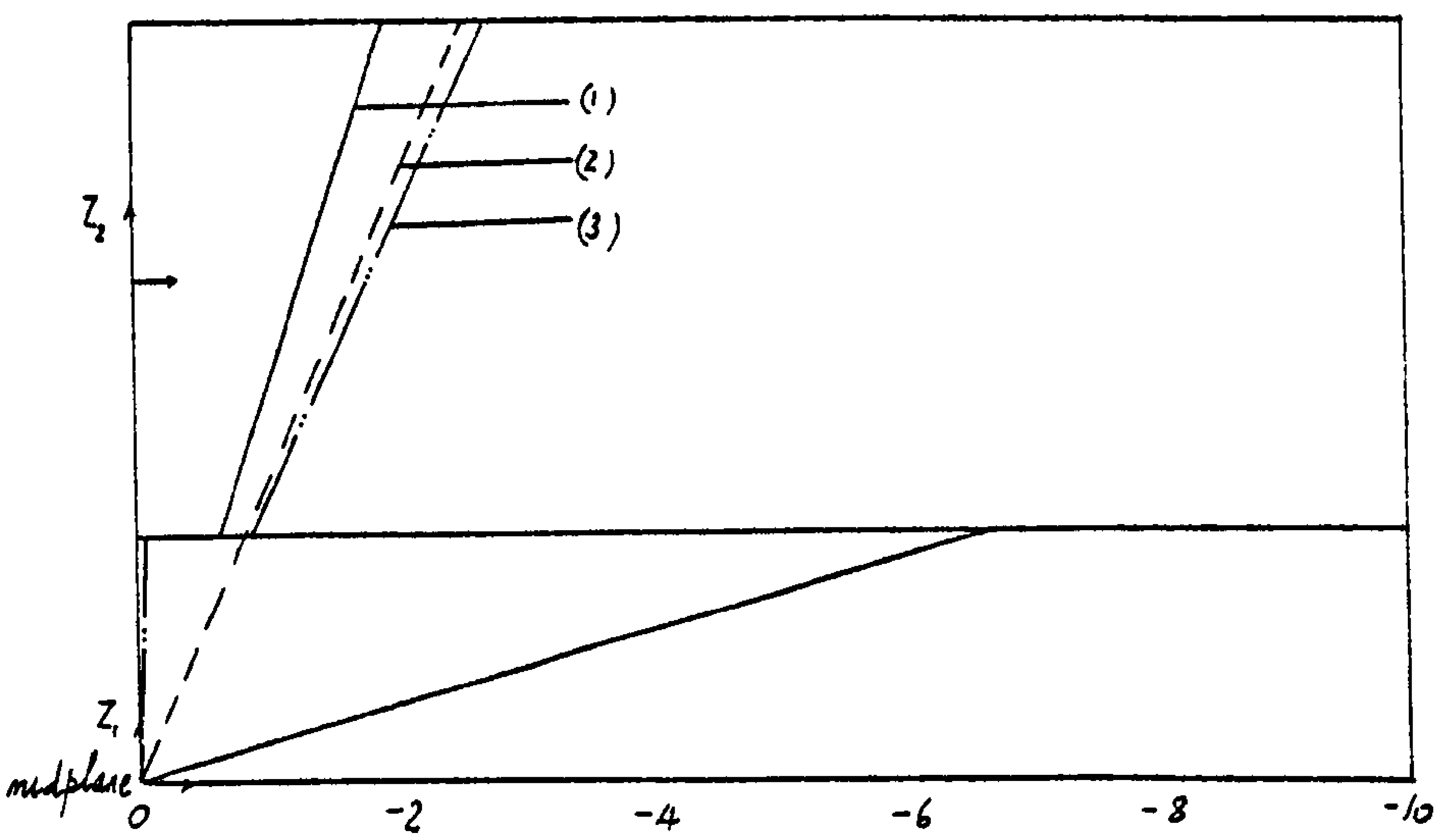


Figure 4.10. Variation of $\sigma_{r\theta}/P_o\varepsilon$ with thickness at $R = 1$, $\theta = \pi/4$, $\nu = 0.3$.

(1) $E = 10$ (2) $E = 1$ (3) $E = 1/10$

The variation with thickness of the higher order in-plane stress components when $R = 1$ is given by Table 4.11. Since these components can be discontinuous across the interface of the layers, two values are given at the interface. The extent to which these two values differ may have some bearing on the delamination of the laminate.

	ε	1/10	1	10
z_2 = 1	Free Surface	175.24	16.69	-7.85
0.5		67.33	3.28	-9.61
z_2 = 0		-34.58	-4.33	-7.01
= -0.5		-131.70	-7.30	-0.94
z_2 = -1	Interface	-225.22	-6.80	7.75
z_1 = 1	Interface	-22.52	-6.80	77.45
0.5		-11.32	-3.98	34.38
z_1 = 0	Mid-Plane	0.0	0.0	0.0

Table 4.11. Variation of the Higher Order In-Plane Stress Components with Thickness R = 1.

$\theta_{rr}/P_o\varepsilon^3 \cos 2\theta$, $-\sigma_{\theta\theta}/P_o\varepsilon^3 \cos 2\theta$, $\sigma_{r\theta}/P_o\varepsilon^3 \sin 2\theta$.

Reference to Table 4.11 shows that the higher-order terms are of order $200\varepsilon^2 P_o$. If ε is small such that these terms are negligible compared to εP_o then the leading in-plane stress components will be sufficient. In such cases Classical Laminate Theory is an adequate theory.

However if $\varepsilon = O(1)$ that is, the thickness of the plate is comparable with the radius of the hole, the numerical vlaues in Table 4.11 are much more significant than those of the leading terms shown in Figures 4.9 and 4.10. Therefore, if $\varepsilon = O(1)$ the Classical Laminate solutions cannot be taken to provide the complete state of stress in the laminated plate.

CHAPTER FIVE

THE BENDING AND STRETCHING OF LAMINATED ANISOTROPIC PLATES

5.1 INTRODUCTION

In this chapter we modify the solutions of Chapter 4 to account for a laminated plate which consists of fibre-reinforced layers having fibres orthogonal to the z-axis. By expanding the field quantities in each layer as a power series of an aspect ratio ε (= h/a) and using the techniques developed in Chapter 4, the series coefficients are determined and are given in Section 5.5. The lowest order terms are found to be those given by Classical Laminate Theory and they are in agreement with Tang (1975); however, unlike Tang's theory we also evaluate the terms of higher order.

In Sections 5.8 and 5.9 the general solutions obtained are applied to a laminated plate containing a circular hole and also to a half space subjected to sinusoidal loading. In both illustrations it is shown that when the layers contain stiff fibres the magnitude of the higher order terms for the stress components greatly exceeds those of Classical Laminate Theory and therefore for such materials this theory is not adequate.

The final section of this chapter explains why, for layers containing stiff fibres, there is a breakdown of the power series solution. By defining a second parameter k which depends on the elastic moduli of the layer, it is shown that for stiff fibres $1/k \simeq (\mu/E) << 1$ where E is the Youngs modulus in the fibre direction. By expanding the stress components in terms of k it is found that when the Classical Laminate Solution is O(1) the higher order terms are $O(\underline{\varepsilon}^2 k)$ and therefore if a power series solution is to be sought for such materials, then it must account for parameter k as well as the aspect ratio ε.

5.2 NON DIMENSIONAL FORM OF THE EQUATIONS OF LINEAR ORTHOTROPIC ELASTICITY

In Chapter 4 the non dimensional form of the equations of linear elasticity were derived for a transversely isotropic material with a preferred direction parallel to that of the Z axis. We now derive similar results for anisotropic materials; in particular we consider materials which, in the notation of Chapter 1, consist of fibres having a preferred direction of the form

$$a = (\cos \alpha, \sin \alpha, 0), \tag{5.2.1}$$

where α is a constant.

By non-dimensionalising the displacement components and the co-ordinates (X,Y,Z) in accordance with (4.2.6) and (4.2.8), the following constitutive equations are obtained from (1.3.11) for a material which has one plane of elastic symmetry,

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{16} \\ & c_{22} & c_{26} \\ \text{sym.} & & c_{66} \end{pmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{pmatrix} + \frac{1}{\varepsilon}\frac{\partial w}{\partial z}\begin{pmatrix} c_{13} \\ c_{23} \\ c_{36} \end{pmatrix}, \quad (5.2.2)$$

$$\sigma_{zz} = (c_{13}, c_{23}, c_{36}) \begin{pmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{pmatrix} + \frac{c_{33}}{\varepsilon}\frac{\partial w}{\partial z}, \quad (5.2.3)$$

$$\begin{pmatrix} \sigma_{yz} \\ \sigma_{xz} \end{pmatrix} = \begin{pmatrix} c_{44} & c_{45} \\ \text{sym.} & c_{55} \end{pmatrix} \begin{pmatrix} \partial w/\partial y \\ \partial w/\partial x \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} c_{44} & c_{45} \\ & c_{55} \end{pmatrix} \begin{pmatrix} \partial v/\partial z \\ \partial u/\partial z \end{pmatrix}, \quad (5.2.4)$$

where

$$\varepsilon = \frac{h}{a}. \quad (5.2.5)$$

In (5.2.5) a is some characteristic in-plane length and 2h represents the thickness of the material and it therefore follows that for a thin layer

$$\varepsilon << 1. \quad (5.2.6)$$

The stress components satisfy the non dimensionalised stress equilibrium equations given by (4.2.11).

Before proceeding with any detailed analysis of the above equations, a brief review of thin plate theory will be made, since the results of this theory are employed at a later stage.

5.3 STRETCHING OF THIN PLATES

A single orthotropic plate of thickness 2H is referred to the dimensionless co-ordinate system (x,y,z) such that the midplane of the plate coincides with the (x,y) plane. Suppose the plate is thin and in a state of generalised plane stress. Following Lekhnitskii (1968), the averaged displacement components are related to the averaged in-plane stress components by

$$\begin{pmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\sigma}_{xy} \end{pmatrix} = \begin{pmatrix} \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{16} \\ & \hat{Q}_{22} & \hat{Q}_{26} \\ & & \hat{Q}_{66} \end{pmatrix} \begin{pmatrix} \partial\hat{u}/\partial x \\ \partial\hat{v}/\partial y \\ \partial\hat{u}/\partial y + \partial\hat{v}/\partial x \end{pmatrix}, \qquad (5.3.1)$$

where

$$\hat{Q}_{ij} = \hat{c}_{ij} - \frac{\hat{c}_{i3}\hat{c}_{j3}}{\hat{c}_{33}}, \qquad (5.3.2)$$

and the $\hat{c}_{ij}$ are the elastic material constants of the layer. Substituting (5.3.1) into the averaged equations of equilibrium, then produces two second order partial differential equations from which the averaged displacement components can be determined:

$$\hat{Q}_{11}\frac{\partial^2\hat{u}}{\partial x^2} + 2\hat{Q}_{16}\frac{\partial^2\hat{u}}{\partial x\partial y} + \hat{Q}_{66}\frac{\partial^2\hat{u}}{\partial y^2} + \hat{Q}_{16}\frac{\partial^2\hat{v}}{\partial x^2} + (\hat{Q}_{12} + \hat{Q}_{66})\frac{\partial^2\hat{v}}{\partial x\partial y} + \hat{Q}_{26}\frac{\partial^2\hat{v}}{\partial y^2} = 0,$$

$$\hat{Q}_{66}\frac{\partial^2\hat{v}}{\partial x^2} + 2\hat{Q}_{26}\frac{\partial^2\hat{v}}{\partial x\partial y} + \hat{Q}_{22}\frac{\partial^2\hat{v}}{\partial y^2} + \hat{Q}_{16}\frac{\partial^2\hat{u}}{\partial y^2} + (\hat{Q}_{12} + \hat{Q}_{66})\frac{\partial^2\hat{u}}{\partial x\partial y} + \hat{Q}_{26}\frac{\partial^2\hat{u}}{\partial y^2} = 0. \qquad (5.3.3)$$

An alternative approach to solving generalised plane stress problems is described by Milne-Thomson (1960). By inverting (5.3.1) we obtain

$$\begin{pmatrix} \partial\hat{u}/\partial x \\ \partial\hat{v}/\partial y \\ \partial\hat{u}/\partial y + \partial\hat{v}/\partial x \end{pmatrix} = \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{16} \\ & \hat{R}_{22} & \hat{R}_{26} \\ \text{sym.} & & \hat{R}_{66} \end{pmatrix} \begin{pmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\sigma}_{xy} \end{pmatrix}, \tag{5.3.4}$$

where

$$\hat{R}_{11} = \frac{\hat{Q}_{22}\hat{Q}_{66} - \hat{Q}_{26}^2}{\Delta}, \quad \hat{R}_{12} = \frac{\hat{Q}_{16}\hat{Q}_{26} - \hat{Q}_{12}\hat{Q}_{66}}{\Delta}, \quad \hat{R}_{16} = \frac{\hat{Q}_{12}\hat{Q}_{26} - \hat{Q}_{16}\hat{Q}_{22}}{\Delta},$$

$$\hat{R}_{22} = \frac{\hat{Q}_{11}\hat{Q}_{66} - \hat{Q}_{16}^2}{\Delta}, \quad \hat{R}_{26} = \frac{\hat{Q}_{12}\hat{Q}_{16} - \hat{Q}_{11}\hat{Q}_{26}}{\Delta}, \quad \hat{R}_{66} = \frac{\hat{Q}_{11}\hat{Q}_{22} - \hat{Q}_{12}^2}{\Delta}, \tag{5.3.5}$$

$$\Delta = \hat{Q}_{11}(\hat{Q}_{22}\hat{Q}_{66} - \hat{Q}_{26}^2) - \hat{Q}_{12}(\hat{Q}_{12}\hat{Q}_{66} - \hat{Q}_{16}\hat{Q}_{26}) + \hat{Q}_{16}(\hat{Q}_{12}\hat{Q}_{26} - \hat{Q}_{16}\hat{Q}_{22}).$$

The averaged stress equilibrium equations are satisfied identically by introducing the stress function $\chi(x,y)$ such that

$$\hat{\sigma}_{xx} = \frac{\partial^2\chi}{\partial y^2}, \quad \hat{\sigma}_{yy} = \frac{\partial^2\chi}{\partial x^2}, \quad \hat{\sigma}_{xy} = -\frac{\partial^2\chi}{\partial x\partial y}, \tag{5.3.6}$$

and then from the strain compatibility equation (1.2.3) and (5.3.4) we find that $\chi(x,y)$ satisfies the fourth order partial differential equation

$$\hat{R}_{22}\frac{\partial^4\chi}{\partial x^4} - 2\hat{R}_{26}\frac{\partial^4\chi}{\partial x^3\partial y} + (2\hat{R}_{12} + \hat{R}_{66})\frac{\partial^4\chi}{\partial x^2\partial y^2} - 2\hat{R}_{16}\frac{\partial^4\chi}{\partial x\partial y^3} + \hat{R}_{11}\frac{\partial^4\chi}{\partial y^4} = 0. \tag{5.3.7}$$

These equations can be rewritten in the form

$$\left(\frac{\partial}{\partial y} - s_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - s_2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - s_3\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - s_4\frac{\partial}{\partial x}\right)\chi = 0, \tag{5.3.8}$$

where the s_i are the roots of the characteristic equation

$$\hat{R}_{11}s^4 - 2\hat{R}_{16}s^3 + (2\hat{R}_{12} + \hat{R}_{66})s^2 - 2\hat{R}_{26}s + \hat{R}_{22} = 0. \qquad (5.3.9)$$

Milne-Thomson (1960) has shown that none of the roots of (5.3.9) can be real and therefore we denote them as $\hat{s}_1, \hat{s}_2$ and their complex conjugates $\bar{\hat{s}}_1$ and $\bar{\hat{s}}_2$. We note that for isotropic materials

$$\hat{s}_1 = \hat{s}_2 = i, \qquad (5.3.10)$$

and in such cases (5.3.8) reduces to the usual bi-harmonic equation

$$\nabla^4\chi = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad (5.3.11)$$

We introduce

$$z_j = x + \hat{s}_j y, \quad \bar{z}_j = x + \bar{\hat{s}}_j y, \qquad (5.3.12)$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}, \quad \frac{\partial}{\partial y} = \hat{s}_j \frac{\partial}{\partial z_j} + \bar{\hat{s}}_j \frac{\partial}{\partial \bar{z}_j}. \qquad (5.3.13)$$

By substituting (5.3.13) into (5.3.8) we obtain

$$\frac{\partial^4\chi}{\partial z_1 \partial \bar{z}_1 \partial z_2 \partial \bar{z}_2} = 0. \qquad (5.3.14)$$

The general solution to the averaged stress and displacements can now be obtained by solving (5.3.14) and then using (5.3.4) and (5.3.6). Following Milne.Thomson, we express these general solutions as

$$\hat{u} = 2\mathrm{Re}(\hat{p}_1\phi_1 + \hat{p}_2\phi_2)$$
$$\hat{v} = 2\mathrm{Re}(\hat{q}_1\phi_1 + \hat{q}_2\phi_2) \qquad (5.3.15)$$

$$\hat{\sigma}_{xx} = 2\mathrm{Re}(\hat{s}_1^2\phi_1 + \hat{s}_2^2\phi_2),$$
$$\hat{\sigma}_{yy} = 2\mathrm{Re}(\phi_1' + \phi_2'), \qquad (5.3.16)$$
$$\hat{\sigma}_{xy} = -2\mathrm{Re}(\hat{s}_1\phi_1' + \hat{s}_2\phi_2'),$$

where, for $j = 1,2$

$$z_j = x + \hat{s}_j y,$$
$$\hat{p}_j = \hat{R}_{11}\hat{s}_j^2 + \hat{R}_{12} - \hat{R}_{16}\hat{s}_j, \qquad (5.3.17)$$
$$\hat{q}_j = \frac{\hat{R}_{12}\hat{s}_j^2 + \hat{R}_{22} - \hat{R}_{26}\hat{s}_j}{\hat{s}_j},$$

and $\phi_j(z_j)$ are analytic functions of their arguments.

The resultant forces produced by the stress components in (5.3.16) on planes normal to the (x,y) plane are given by

$$\begin{pmatrix} \hat{T}_x \\ \hat{T}_y \\ \hat{T}_{xy} \end{pmatrix} = 2\hat{H} \begin{pmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\sigma}_{xy} \end{pmatrix}, \qquad (5.3.18)$$

and the bending moments acting on these planes are identically zero.

5.4 BENDING OF THIN PLATES

The approximate bending theory for a thin plate of uniform thickness $2\hat{H}$ is based on the following two assumptions:

(i) Plane sections, which in the undeformed state are normal to the middle surface, remain plane and normal to the bent middle surface during bending;

(ii) The normal stress $\hat{\sigma}_{zz}$ in cross-sections parallel to the x-y plane is small compared with the stress components in the transverse cross-sections.

Following Lekhnitskii (1968), assumption (i) gives

$$\hat{u} = -\hat{\varepsilon}z\,\frac{\partial\hat{w}}{\partial x}, \qquad \hat{v} = -\hat{\varepsilon}z\,\frac{\partial\hat{w}}{\partial y}, \tag{5.4.1}$$

where

$$\hat{\varepsilon} = \frac{\hat{H}}{a}\,. \tag{5.4.2}$$

In (5.4.1), $(\hat{u},\hat{v},\hat{w})$ is the displacement field referred to cartesian axes (x,y,z) and non-dimensionalised in accordance with (4.2.6) and (4.2.8).

By employing (5.4.1), the constitutive equations are

$$\begin{pmatrix}\hat{\sigma}_{xx}\\ \hat{\sigma}_{yy}\\ \hat{\sigma}_{xy}\end{pmatrix} = -\hat{\varepsilon}z\begin{pmatrix}\hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{16}\\ & \hat{Q}_{22} & \hat{Q}_{26}\\ & & \hat{Q}_{66}\end{pmatrix}\begin{pmatrix}\partial^2\hat{w}/\partial x^2\\ \partial^2\hat{w}/\partial y^2\\ 2\partial^2\hat{w}/\partial x\partial y\end{pmatrix}. \tag{5.4.3}$$

Substituting (5.4.3) into the first two equilibrium equations and integrating them with respect to z gives

$$\begin{pmatrix}\hat{\sigma}_{xz}\\ \\ \hat{\sigma}_{yz}\end{pmatrix} = \hat{\varepsilon}\left(\frac{z^2}{2} - \frac{1}{2}\right)\begin{pmatrix}\hat{Q}_{11}\,\partial^3\hat{w}/\partial x^3 + 3\hat{Q}_{16}\,\partial^3\hat{w}/\partial x^2\partial y + (\hat{Q}_{12} + 2\hat{Q}_{66})\partial^3\hat{w}/\partial x\partial y^2\\ + \hat{Q}_{26}\,\partial^3\hat{w}/\partial y^3\\ \hat{Q}_{16}\,\partial^3\hat{w}/\partial x^3 + (\hat{Q}_{12} + 2\hat{Q}_{66})\partial^3\hat{w}/\partial x^2\partial y + 3\hat{Q}_{26}\,\partial^3\hat{w}/\partial x\partial y^2\\ + \hat{Q}_{22}\,\partial^3\hat{w}/\partial y^3\end{pmatrix}$$

(5.4.4)

From the third equilibrium equation, it follows that $\hat{w}$ satisfies the fourth order partial differential equation

$$\hat{Q}_{11} \frac{\partial^4 \hat{w}}{\partial x^4} + 4\hat{Q}_{16} \frac{\partial^4 \hat{w}}{\partial x^3 \partial y} + 2(\hat{Q}_{12} + 2\hat{Q}_{66}) \frac{\partial^4 \hat{w}}{\partial x^2 \partial y^2} + 4\hat{Q}_{26} \frac{\partial^4 \hat{w}}{\partial x \partial y^3} + \hat{Q}_{22} \frac{\partial^4 \hat{w}}{\partial y^4} = 0, \qquad (5.4.5)$$

which we write as

$$\left(\frac{\partial}{\partial y} - \alpha_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - \alpha_3 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y} - \alpha_4 \frac{\partial}{\partial x}\right)\hat{w} = 0. \qquad (5.4.6)$$

A comparison of (5.4.5) and (5.4.6) shows that the α_i satisfy the following characteristic equation

$$\hat{Q}_{22}\alpha^4 + 4\hat{Q}_{26}\alpha^3 + 2(\hat{Q}_{12} + 2\hat{Q}_{66})\alpha^2 + 4\hat{Q}_{16}\alpha + \hat{Q}_{11} = 0. \qquad (5.4.7)$$

Lekhnitskii (1938) has shown that (5.4.7) has no real roots and accordingly we take the roots to be $\hat{\alpha}_1, \hat{\alpha}_2$ and their complex conjugates.

The analysis is now very similar to that considered in the previous section. We let

$$z_j = x + \hat{\alpha}_j y, \qquad \bar{z}_j = x + \bar{\hat{\alpha}}_j y \qquad (5.4.8)$$

where $(\bar{\ })$ denotes the complex conjugate. Since

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j} \qquad \frac{\partial}{\partial y} = \hat{\alpha}_j \frac{\partial}{\partial z_j} + \bar{\hat{\alpha}}_j \frac{\partial}{\partial \bar{z}_j}, \qquad (5.4.9)$$

it follows from (5.4.6) that $\hat{w}$ satisfies

$$\frac{\partial^4 \hat{w}}{\partial z_1 \partial \bar{z}_1 \partial z_2 \partial \bar{z}_2} = 0. \tag{5.4.10}$$

Again we note that for isotropic materials

$$\hat{\alpha}_1 = \hat{\alpha}_2 = i, \tag{5.4.11}$$

and in such cases (5.4.6) reduces to the bi-harmonic equation

$$\nabla^4 \hat{w} = 0. \tag{5.4.12}$$

The general solution for $\hat{w}$, from (5.4.10), is

$$\hat{w} = 2\mathrm{Re}(\Phi_1 + \Phi_2), \tag{5.4.13}$$

where $\Phi_j(z_j)$ are analytic functions of their complex arguments.

If Equations (5.4.3) are integrated through the thickness of the plate, it is found that there is no resultant force due to the in-plane stress components. The resultant shear forces are given by

$$\begin{pmatrix} \hat{N}_x \\ \hat{N}_y \end{pmatrix} = \hat{H}\int_{-1}^{1} \begin{pmatrix} \hat{\sigma}_{xz} \\ \hat{\sigma}_{yz} \end{pmatrix} dz, \tag{5.4.14}$$

and the bending moments acting on planes normal to the (x,y) plane are

$$\begin{pmatrix} \hat{M}_{xx} \\ \hat{M}_{yy} \\ \hat{M}_{xy} \end{pmatrix} = \int_{-1}^{1} z \begin{pmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\sigma}_{xy} \end{pmatrix} dz. \tag{5.4.15}$$

5.5 POWER SERIES EXPANSION FOR A LAYER

We now develop a power series expansion solution for the stress and displacement components in a typical layer of the laminate. The boundary conditions satisfied by these solutions are stated in Section 5.7 and at this stage of the analysis we only require the traction free condition imposed on the lateral external surfaces of the laminate.

We assume that in a typical layer, the displacement components can be represented by a power series in the aspect ratio ε.

$$\begin{aligned} u(x,y,z;\varepsilon) &= u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \\ v(x,y,z;\varepsilon) &= v_o + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots, \\ w(x,y,z;\varepsilon) &= w_o + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots. \end{aligned} \qquad (5.5.1)$$

Then it follows from (5.2.2), (5.2.3) and (5.2.4) that the stress components will be of the same form

$$\sigma(x,y,z;\varepsilon) = \sum_{n=-1}^{\infty} \varepsilon^n \sigma_n(x,y,z), \qquad (5.5.2)$$

where σ is a generic symbol for each stress component.

Substituting (5.5.1) and (5.5.2) into (5.2.2), (5.2.3) and (5.2.4) and the stress equilibrium equations (4.2.11) we obtain, by equating coefficients of the same order in ε, a sequence of partial differential equations which involve the coefficients in the expansions (5.5.1) and (5.5.2). The equations of order $\varepsilon^{(i)}$ are

$$\begin{pmatrix} \sigma_{xxi} \\ \sigma_{yyi} \\ \sigma_{xyi} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{16} \\ & c_{22} & c_{26} \\ \text{sym.} & & c_{66} \end{pmatrix} \begin{pmatrix} \partial u_i/\partial x \\ \partial v_i/\partial y \\ \partial u_i/\partial y + \partial v_i/\partial x \end{pmatrix} + \begin{pmatrix} c_{13} \\ c_{23} \\ c_{36} \end{pmatrix} \frac{\partial w_{i+1}}{\partial z}, \qquad (5.5.3)$$

$$\frac{\partial w_{i+1}}{\partial z} = \frac{\sigma_{zzi}}{c_{33}} - \frac{1}{c_{33}}(c_{13},c_{23},c_{36})\begin{pmatrix}\partial u_i/\partial x \\ \partial v_i/\partial y \\ \partial u_i/\partial y + \partial v_i/\partial x\end{pmatrix}, \qquad (5.5.4)$$

$$\frac{\partial}{\partial z}\begin{pmatrix}u_{i+1} \\ v_{i+1}\end{pmatrix} = \begin{pmatrix}c_{55} & c_{45} \\ c_{45} & c_{44}\end{pmatrix}^{-1}\begin{pmatrix}\sigma_{xzi} \\ \sigma_{yzi}\end{pmatrix} - \begin{pmatrix}\partial w_i/\partial x \\ \partial w_i/\partial y\end{pmatrix}, \qquad (5.5.5)$$

and

$$\frac{\partial}{\partial z}\begin{pmatrix}\sigma_{xzi} \\ \sigma_{yzi}\end{pmatrix} = -\begin{pmatrix}\partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x\end{pmatrix}\begin{pmatrix}\sigma_{xx(i-1)} \\ \sigma_{yy(i-1)} \\ \sigma_{xy(i-1)}\end{pmatrix}, \qquad (5.5.6)$$

$$\frac{\partial}{\partial z}\sigma_{zzi} = -(\partial/\partial x,\ \partial/\partial y)\begin{pmatrix}\sigma_{xz(i-1)} \\ \sigma_{yz(i-1)}\end{pmatrix} \qquad i = -1, \ldots,. \qquad (5.5.7)$$

(i) Terms of Order ε^{-1}

Since $\sigma_{ijk} = 0$ when $k < -1$, we find from (5.5.6) and (5.5.7) that $\sigma_{xz(-1)}$, $\sigma_{yz(-1)}$ and $\sigma_{zz(-1)}$ are functions of x and y only. Therefore, in order that the stress-free conditions on the lateral surfaces of the laminate be satisfied, we take

$$\sigma_{xz(-1)} = \sigma_{yz(-1)} = \sigma_{zz(-1)} = 0. \qquad (5.5.8)$$

It follows from (5.5.4) and (5.5.5) that

$$u_o = u_o(x,y), \quad v_o = v_o(x,y), \quad w_o = w_o(x,y). \qquad (5.5.9)$$

In Chapter 4, the lowest order displacement components were chosen to be the displacement components of an equivalent isotropic plate. In a similar manner, we require the displacement field given by (5.5.9) to be the displacement field in a single equivalent anisotropic plate and is determined by the thin plate theory of Sections 5.3 and 5.4. The details concerning the elastic material properties of equivalent plate are deferred to Section 5.6. However, it should be emphasized that in the analysis to follow, hatted quantities will refer to the 'equivalent plate solution' so, for example $\hat{\underset{\sim}{R}}$ refers to the equivalent elastic constants.

Hence, from (5.3.15) and (5.4.13)

$$\begin{aligned} u_o &= 2\mathrm{Re} \sum \hat{p}_i \phi_i, \\ v_o &= 2\mathrm{Re} \sum \hat{q}_i \phi_i, \\ w_o &= 2\mathrm{Re} \sum \Phi_i, \end{aligned} \qquad (5.5.10)$$

where the summation is carried out for $i = 1,2$ and Re denotes 'the real part of'. In the analysis to follow, derivatives of (5.5.10) are obtained from the differential operators (5.3.13) and (5.4.9). Finally, from Equations (5.5.3) we obtain

$$\sigma_{xx(-1)} = \sigma_{yy(-1)} = \sigma_{xy(-1)} = 0 \qquad (5.5.11)$$

since $u_i = v_i = 0$ when $i < 0$.

(ii) Terms of Order ε^o

Substituting (5.5.8) and (5.5.11) into (5.5.6) and (5.5.7) shows that σ_{xzo}, σ_{yzo} and σ_{zzo} are functions of x and y only. We therefore take

$$\sigma_{xzo} = \sigma_{yzo} = \sigma_{zzo} = 0. \tag{5.5.12}$$

and it follows from integrating (5.5.4) with respect to z that

$$w_1 = -2\mathrm{Re} \sum\{a_i z + S_{1i}\}\phi_i', \tag{5.5.13}$$

where S_{1i} are arbitrary constants and

$$a_i = \frac{1}{c_{33}}(c_{13}, c_{23}, c_{36})\hat{\underset{\sim}{R}}\begin{pmatrix}\hat{s}_i^2\\ 1\\ -\hat{s}_i\end{pmatrix}. \tag{5.5.14}$$

In (5.5.14), use has been made of relationships (5.3.9), (5.3.17) and

$$\hat{\underset{\sim}{R}} = \begin{pmatrix} R_{11} & R_{12} & R_{16}\\ & R_{22} & R_{26}\\ \text{sym.} & & R_{66}\end{pmatrix}.$$

From (5.5.5) and (5.5.12) we obtain

$$\begin{pmatrix}u_1\\ v_1\end{pmatrix} = -2\mathrm{Re} \sum\{(z + B_{1i})\begin{pmatrix}1\\ \hat{\alpha}_i\end{pmatrix}\}\phi_i', \tag{5.5.15}$$

where B_{1i} are arbitrary constants. Substituting (5.5.10) and (5.5.13) into (5.5.3) gives

$$\begin{pmatrix} \sigma_{xxo} \\ \sigma_{yyo} \\ \sigma_{xyo} \end{pmatrix} = 2\mathrm{Re}\sum\{\underset{\sim}{Q}\underset{\sim}{R}\begin{pmatrix} \hat{s}_i^2 \\ 1 \\ -\hat{s}_i \end{pmatrix}\}\phi_i'. \tag{5.5.16}$$

In (5.5.16), $\underset{\sim}{Q}$ represents the reduced elastic moduli matrix of the layer. The elements of $\underset{\sim}{Q}$ are related to the c_{ij} by

$$Q_{ij} = c_{ij} - \frac{c_{i3}c_{j3}}{c_{33}}, \qquad i,j = 1,2,6. \tag{5.5.17}$$

(iii) Terms of Order ε^1

From (5.5.12) and (5.5.7) we find that σ_{zz1} is a function of x and y only. We therefore take

$$\sigma_{zz1} = 0, \tag{5.5.18}$$

and it then follows from integrating (5.5.4) that

$$w_2 = 2\mathrm{Re}\sum\{b_i(\frac{z^2}{2} + B_{1i}z) + B_{2i}\}\phi_i'', \tag{5.5.19}$$

where B_{2i} are arbitrary constants and

$$b_i = \frac{1}{c_{33}}(C_{13}, C_{23}, C_{36})\begin{pmatrix} 1 \\ \hat{\alpha}_i^2 \\ 2\hat{\alpha}_i \end{pmatrix}. \tag{5.5.20}$$

By substituting (5.5.16) into (5.5.6) and integrating with respect to z we obtain

$$\begin{pmatrix}\sigma_{xz1}\\ \sigma_{yz1}\end{pmatrix} = -2\text{Re}\sum\{\underset{\sim}{d}_i z + \begin{pmatrix}S_{2i}\\ S_{3i}\end{pmatrix}\}\phi_i'', \tag{5.5.21}$$

where S_{2i} and S_{3i} are arbitrary constants and

$$\underset{\sim}{d}_i = \begin{pmatrix}1 & 0 & \hat{s}_i\\ 0 & \hat{s}_i & 1\end{pmatrix}\underset{\sim\sim}{QR}\begin{pmatrix}\hat{s}_i^2\\ 1\\ -\hat{s}_i\end{pmatrix}. \tag{5.5.22}$$

The remaining displacement components can now be determined by substituting (5.5.21) and (5.5.13) into (5.5.5) and integrating as follows

$$\begin{pmatrix}u_2\\ v_2\end{pmatrix} = 2\text{Re}\sum\{\underset{\sim}{e}_i \frac{z^2}{2} + \underset{\sim}{f}_i z + \begin{pmatrix}S_{4i}\\ S_{5i}\end{pmatrix}\}\phi_i'', \tag{5.5.23}$$

where S_{4i}, S_{5i} are <u>arbitrary functions of x,y</u> and

$$\underset{\sim}{e}_i = \underset{\sim}{a}_i\begin{pmatrix}1\\ \hat{s}_i\end{pmatrix} - \underset{\sim\sim}{cd}_i, \tag{5.5.24}$$

$$\underset{\sim}{f}_i = S_{1i}\begin{pmatrix}1\\ \hat{s}_i\end{pmatrix} - \underset{\sim}{c}\begin{pmatrix}S_{2i}\\ S_{3i}\end{pmatrix}, \tag{5.5.25}$$

$$\underset{\sim}{c} = \begin{pmatrix}c_{55} & c_{45}\\ c_{45} & c_{44}\end{pmatrix}^{-1}.$$

From (5.5.3) and (5.5.15) we obtain

$$\begin{pmatrix}\sigma_{xx1}\\ \sigma_{yy1}\\ \sigma_{xy1}\end{pmatrix} = -2\text{Re}\sum\{(z + B_{1i})\underset{\sim}{Q}\begin{pmatrix}1\\ \hat{\alpha}_i^2\\ 2\hat{\alpha}_i\end{pmatrix}\}\phi_i''. \tag{5.5.26}$$

(iv) Terms of Order ε^2

Substituting (5.5.21) into (5.5.7) and integrating with respect to z gives

$$\sigma_{zz2} = 2\mathrm{Re}\ \sum\{(1,\hat{s}_i)\underset{\sim}{d}_i \frac{z^2}{2} + (1,\hat{s}_i)\begin{pmatrix} S_{2i} \\ S_{3i} \end{pmatrix} z + S_{6i}\}\phi_i''' , \tag{5.5.27}$$

where S_{6i} are arbitrary constants. Employing (5.5.26) and integrating Equation (5.5.6) gives

$$\begin{pmatrix} \sigma_{xz2} \\ \sigma_{yz2} \end{pmatrix} = 2\mathrm{Re}\ \sum\{\underset{\sim}{g}_i(\frac{z^2}{2} + B_{1i}z) + \begin{pmatrix} B_{3i} \\ B_{4i} \end{pmatrix}\}\phi_i''' , \tag{5.5.28}$$

where B_{3i}, B_{4i} are arbitrary constants and

$$\underset{\sim}{g}_i = \begin{pmatrix} 1 & 0 & \hat{\alpha}_i \\ 0 & \hat{\alpha}_i & 1 \end{pmatrix} \underset{\sim}{Q} \begin{pmatrix} 1 \\ \hat{\alpha}_i^2 \\ 2\hat{\alpha}_i \end{pmatrix} . \tag{5.5.29}$$

By substituting (5.5.28) and (5.5.19) into (5.5.5) and integrating with respect to z we obtain

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = 2\mathrm{Re}\ \sum\{\underset{\sim}{k}_i(\frac{z^3}{6} + B_{1i}\frac{z^2}{2}) + \underset{\sim}{m}_i z + \begin{pmatrix} B_{5i} \\ B_{6i} \end{pmatrix}\}\phi''' , \tag{5.5.30}$$

where B_{5i}, B_{6i} are arbitrary constants of integration and

$$\underset{\sim}{k}_i = \underset{\sim}{c}\underset{\sim}{g}_i - \begin{pmatrix} 1 \\ \hat{\alpha}_i \end{pmatrix} b_i , \tag{5.5.31}$$

$$\underset{\sim}{m}_i = \underset{\sim}{c}\begin{pmatrix} B_{3i} \\ B_{4i} \end{pmatrix} - \begin{pmatrix} 1 \\ \hat{\alpha}_i \end{pmatrix} B_{2i} . \tag{5.5.32}$$

The in-plane stress components can now be evaluated from (5.5.4), (5.5.5) and (5.5.23), (5.5.27) as follows.

$$\begin{Bmatrix}\sigma_{xx2}\\ \sigma_{yy2}\\ \sigma_{xy2}\end{Bmatrix} = 2\text{Re}\sum\{\underset{\sim}{A}_i \frac{z^2}{2} + \underset{\sim}{B}_i z + \underset{\sim}{C}_i + \underset{\approx}{Q}\underset{\sim}{\hat{n}}_i \begin{Bmatrix}S_{4i}\\ S_{5i}\end{Bmatrix}\}\phi_i''' , \qquad (5.5.33)$$

where

$$\underset{\sim}{\hat{n}}_i = \begin{bmatrix}1 & 0\\ 0 & \hat{s}_i\\ \hat{s}_i & 1\end{bmatrix}, \qquad \underset{\sim}{t}_i = \frac{1}{c_{33}}\begin{Bmatrix}c_{13}\\ c_{23}\\ c_{36}\end{Bmatrix},$$

$$\underset{\sim}{A}_i = \underset{\approx}{Q}\underset{\sim}{\hat{n}}_i\underset{\sim}{e}_i + \underset{\sim}{t}_i(1,\hat{s}_i)\underset{\sim}{d}_i, \qquad (5.5.34)$$

$$\underset{\sim}{B}_i = \underset{\approx}{Q}\underset{\sim}{\hat{n}}_i\underset{\sim}{f}_i + \underset{\sim}{t}_i(1,\hat{s}_i)\begin{Bmatrix}S_{2i}\\ S_{3i}\end{Bmatrix}, \qquad (5.5.35)$$

$$\underset{\sim}{C}_i = \underset{\sim}{t}_i S_{6i}. \qquad (5.5.36)$$

(v) Terms of Order ε^3

Integrating the third stress equilibrium equation (5.5.7) gives

$$\sigma_{zz3} = -2\text{Re}\sum\{(1,\hat{\alpha}_i)\underset{\sim}{g}_i(\frac{z^3}{6} + B_{1i}\frac{z^2}{2}) + (1,\hat{\alpha}_i)\begin{Bmatrix}B_{3i}\\ B_{4i}\end{Bmatrix}z + B_{7i}\}\phi_i^{IV}, \qquad (5.5.37)$$

where B_{7i} are arbitrary constants of integration. By substituting (5.5.4) into (5.5.3) and employing (5.5.37), (5.5.30) we obtain

$$\begin{pmatrix}\sigma_{xx3}\\ \sigma_{yy3}\\ \sigma_{xy3}\end{pmatrix} = 2\mathrm{Re}\sum\{\underset{\sim}{D}_i(\frac{z^3}{6} + B_{1i}\frac{z^2}{2}) + \underset{\sim}{E}_i z + \underset{\sim}{F}_i + \underset{\sim}{Q}\underset{\sim}{\hat{r}}_i\begin{pmatrix}B_{5i}\\ B_{6i}\end{pmatrix}\}\Phi_i^{IV}, \qquad (5.5.38)$$

where

$$\underset{\sim}{\hat{r}}_i = \begin{pmatrix}1 & 0\\ 0 & \hat{\alpha}_i\\ \hat{\alpha}_i & 1\end{pmatrix},$$

$$\underset{\sim}{D}_i = \underset{\sim}{Q}\underset{\sim}{\hat{r}}_i\underset{\sim}{k}_i - \underset{\sim}{t}_i(1,\hat{\alpha}_i)\underset{\sim}{g}_i, \qquad (5.5.39)$$

$$\underset{\sim}{E}_i = \underset{\sim}{Q}\underset{\sim}{\hat{r}}_i\underset{\sim}{m}_i - \underset{\sim}{t}_i(1,\hat{\alpha}_i)\begin{pmatrix}B_{3i}\\ B_{4i}\end{pmatrix}, \qquad (5.5.40)$$

$$\underset{\sim}{F}_i = -\underset{\sim}{t}_i B_{7i}. \qquad (5.5.41)$$

By continuing the analysis, the higher order terms in the stress and displacement components can be evaluated. However, for our purposes these terms are taken to be negligible. It was shown in Chapter 4 that for a transversely isotropic material these higher order terms are identically zero, but this is not the present case.

The complex functions ϕ and Φ are determined from the equivalent plate solution, and by examining the expansion coefficients we find that they are expressed in terms of Φ, ϕ and their derivatives. The leading terms of the displacement components have been chosen to be the displacements in the equivalent plate and therefore the lowest order stress components are those given by Classical Laminate Theory (Christensen 1979).

It is seen that there is an uncoupling of the Φ and ϕ terms. The ϕ terms are identified as the stretching terms, whilst the Φ terms represent the bending terms.

5.6 EQUIVALENT ELASTIC CONSTANTS FOR STRETCHING AND BENDING

The laminated plate under consideration consists of N different orthotropic layers and in the rest of this chapter, the superscript (ℓ) will be used to denote field quantities in the ℓth layer. In this notation, we suppose that the uniform thickness of the ℓth layer is $2h^{(\ell)}$.

An equivalent plate is defined as a single homogeneous plate which is of the same overall geometry as the laminate and has material properties which are obtained by an appropriate averaging of the material properties of the laminae. In Chapter 4 it was shown that the stretching and bending deformations each gave rise to a different set of equivalent elastic constants and therefore we again consider each of these deformations separately.

(i) Stetching

The in-plane stress resultants in a typical layer of the laminate are given by integrating (5.5.16) through the thickness of the layer

$$\begin{pmatrix} T_x \\ T_y \\ T_{xy} \end{pmatrix} = 2\mathrm{Re} \sum_i \{ 2h\underset{\sim}{Q}\hat{\underset{\sim}{R}} \begin{pmatrix} \hat{s}_i^2 \\ 1 \\ -\hat{s}_i \end{pmatrix} \} \phi_i' + O(\varepsilon)^2. \tag{5.6.1}$$

It follows that the stress resultant on planes normal to the mid-plane of the laminate is given by

$$\sum_{\ell=1}^{N} \begin{pmatrix} T_x^{(\ell)} \\ T_y^{(\ell)} \\ T_{xy}^{(\ell)} \end{pmatrix} = 2\mathrm{Re} \sum_i \sum_{\ell=1}^{N} 2h^{(\ell)} \underset{\sim}{Q}^{(\ell)} \underset{\sim}{R} \begin{pmatrix} \hat{s}_i^2 \\ 1 \\ -\hat{s}_i \end{pmatrix} \} \phi_i' + O(\varepsilon)^2. \tag{5.6.2}$$

The equivalent stress resultants, given by (5.3.18), are

$$\begin{pmatrix} \hat{T}_x \\ \hat{T}_y \\ \hat{T}_{xy} \end{pmatrix} = 2\text{Re} \sum_i \{2\hat{H} \begin{pmatrix} \hat{s}_i^2 \\ 1 \\ -\hat{s}_i \end{pmatrix} \} \phi_i' , \tag{5.6.3}$$

where $2\hat{H}$ is the uniform thickness of the equivalent plate and is given by

$$2\hat{H} = \sum_{\ell=1}^{N} 2h^{(\ell)} .$$

For the lowest order stress resultant in (5.6.2) to equal that of (5.6.3), it follows that

$$2\hat{H}\underset{\sim}{I} = \sum_{\ell=1}^{N} 2h^{(\ell)} \underset{\sim}{Q}^{(\ell)} \underset{\sim}{\hat{R}} .$$

Hence

$$\hat{R}^{-1} = \underset{\sim}{\hat{Q}} = \frac{\sum_{\ell=1}^{N} h^{(\ell)} \underset{\sim}{Q}^{(\ell)}}{\hat{H}} . \tag{5.6.4}$$

These quantities define the material properties of the equivalent plate for stretching.

(ii) Bending

The bending moments acting on planes normal to the mid-plane of a typical layer are

$$\begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = -\varepsilon 2\text{Re} \sum_i \{\frac{2}{3} h^2 \underset{\sim}{Q} \begin{pmatrix} 1 \\ \hat{\alpha}_i^2 \\ 2\hat{\alpha}_i \end{pmatrix} \} \Phi_i' + O(\varepsilon)^3 , \tag{5.6.5}$$

and the stress resultants are given by

$$\begin{pmatrix} T_x \\ T_y \\ T_{xy} \end{pmatrix} = -\varepsilon 2\mathrm{Re} \sum_i \{2hB_{1i} \underset{\sim}{Q} \begin{pmatrix} 1 \\ \hat{\alpha}_i^2 \\ 2\hat{\alpha}_i \end{pmatrix} \} \Phi_i'' + O(\varepsilon)^2. \tag{5.6.6}$$

It follows from (5.6.5) and (5.6.6) that the resultant bending moment acting on planes normal to the plane of the laminate is

$$\sum_{\ell=1}^{N} \begin{pmatrix} M_{xx}^{(\ell)} \\ M_{yy}^{(\ell)} \\ M_{xy}^{(\ell)} \end{pmatrix} + \sum_{\ell=1}^{N} \sum_{j=1}^{(\ell)} \begin{pmatrix} T_x^{(\ell)} \\ T_y^{(\ell)} \\ T_{xy}^{(\ell)} \end{pmatrix} (\underline{h}^{(j-1)} + h^{(j)})$$

$$= -2\mathrm{Re} \sum_i \{ \sum_{\ell=1}^{N} \varepsilon^{(\ell)} \frac{2}{3} h^{(\ell)^2} \underset{\sim}{Q}^{(\ell)} + \sum_{\ell=1}^{N} \sum_{j=1}^{(\ell)} \varepsilon^{(\ell)} 2h^{(\ell)} B_{1i}^{(\ell)} \underset{\sim}{Q}^{(\ell)}$$

$$\times (h^{(j-1)} + h^{(j)}) \} \begin{pmatrix} 1 \\ \hat{\alpha}_i^2 \\ 2\hat{\alpha}_i \end{pmatrix} \Phi_i'' + O(\varepsilon)^3. \tag{5.6.7}$$

The corresponding bending moment for the equivalent plate is obtained from (5.4.15) and given by

$$\begin{pmatrix} \hat{M}_{xx} \\ \hat{M}_{yy} \\ \hat{M}_{xy} \end{pmatrix} = -2\mathrm{Re} \sum_i \{\hat{\varepsilon} \frac{2}{3} \hat{H}^2 \underset{\sim}{Q}\} \begin{pmatrix} 1 \\ \hat{\alpha}_i^2 \\ 2\hat{\alpha}_i \end{pmatrix} \Phi_i'' . \tag{5.6.8}$$

If we define $\hat{\underset{\sim}{Q}}$ such that the bending moments in (5.6.7) and (5.6.8) are equal to lowest order, then

$$\hat{\underset{\sim}{Q}} = \frac{1}{\hat{\varepsilon}\hat{H}^2} \{ \sum_{\ell=1}^{N} \varepsilon^{(\ell)} h^{(\ell)^2} \underset{\sim}{Q}^{(\ell)} + 6 \sum_{\ell=1}^{N} \sum_{j=1}^{(\ell)} \varepsilon^{(\ell)} h^{(\ell)} (h^{(j)} + h^{(j-1)}) B_{1i}^{(\ell)} \underset{\sim}{Q}^{(\ell)} \} . \tag{5.6.9}$$

Equation (5.6.9) defines the material properties of the equivalent plate for bending. We note that the definition involves the arbitrary constants B_{1i} which are determined by the through thickness continuity conditions given in the next section.

5.7 SATISFACTION OF THE THROUGH THICKNESS CONDITIONS FOR A LAMINATED PLATE

We consider a laminated plate consisting of N different orthotropic layers as shown in Figure 5.1, where each layer is referred to local co-ordinates $(x,y,z^{(\ell)})$. The plane given by $z^{(\ell)} = 0$ coincides with the mid-plane of the layer

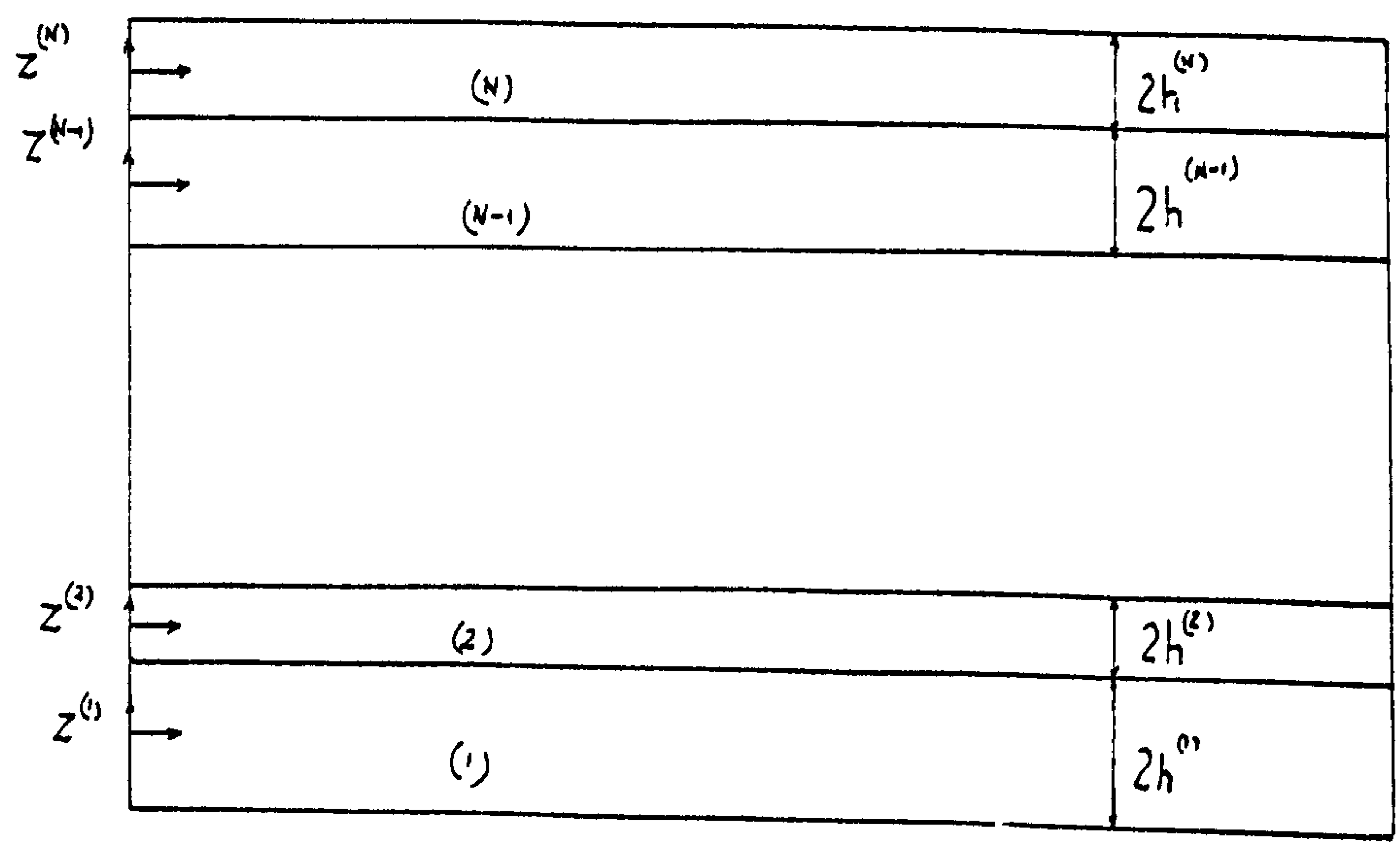


Figure 5.1. The Laminated Plate

The solution to the stress and displacement components in a typical layer has been obtained in Section 5.5 and we now state the through thickness conditions which will determine the S_i and B_i.

If the layers are perfectly bonded together then at each interface we require continuity of the displacement components and the z components of stress. Hence

$$w^{(\ell)}\Big|_{z^{(\ell)}=-1} = w^{(\ell-1)}\Big|_{z^{(\ell-1)}=+1}, \qquad \ell = 2, \ldots, N, \tag{5.7.1}$$

$$u^{(\ell)}\Big|_{z^{(\ell)}=-1} = u^{(\ell-1)}\Big|_{z^{(\ell-1)}=+1}, \qquad v^{(\ell)}\Big|_{z^{(\ell)}=-1} = v^{(\ell-1)}\Big|_{z^{(\ell)}=+1}, \tag{5.7.2}$$

$$\sigma_{jz}^{(\ell)}\Big|_{z^{(\ell)}=-1} = \sigma_{jz}^{(\ell)}\Big|_{z^{(\ell+1)}=+1}, \qquad \ell = 2, \ldots, N, \quad j = x,y,z. \tag{5.7.3}$$

The bounding lateral surfaces of the laminate are kept traction free, so

$$\sigma_{jz}^{(N)}\Big|_{z^{(N)}=1} = 0, \qquad \sigma_{jz}^{(1)}\Big|_{z^{(1)}=-1} = 0, \qquad j = x,y,z. \tag{5.7.4}$$

We note that conditions (5.7.4) were satisfied in the development of the solutions given in Section 5.5.

(A) Stretching Solutions

For a typical layer of the laminate we have, from Section 5.5, that

$$\begin{pmatrix} u \\ v \end{pmatrix} = 2\mathrm{Re}\sum_i \begin{pmatrix} \hat{p}_i \\ \hat{q}_i \end{pmatrix}\phi_i + \varepsilon^2 2\mathrm{Re}\sum_i \{\underset{\sim}{e}_i \frac{z^2}{2} + \underset{\sim}{f}_i z + \begin{pmatrix} S_{4i} \\ S_{5i} \end{pmatrix}\}\phi_i'' + O(\varepsilon)^4, \tag{5.7.5}$$

$$w = -\varepsilon 2\mathrm{Re}\sum_i \{a_i z + S_{1i}\}\phi_i' + O(\varepsilon)^3, \tag{5.7.6}$$

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = 2\mathrm{Re}\sum_i\{(\underset{\sim}{Q}\underset{\sim}{\hat{R}}\begin{pmatrix}\hat{s}_i^2\\ 1\\ -\hat{s}_i\end{pmatrix}\}\phi_i'$$

$$+ \varepsilon^2 2\mathrm{Re}\sum_i\{\underset{\sim}{A}_i\frac{z^2}{2} + \underset{\sim}{B}_i z + \underset{\sim}{C}_i + \underset{\sim}{Q}\underset{\sim}{\hat{n}}_i\begin{pmatrix}S_{4i}\\ S_{5i}\end{pmatrix}\}\phi_i''' + O(\varepsilon)^4, \qquad (5.7.7)$$

$$\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = -\varepsilon 2\mathrm{Re}\sum_i\{\underset{\sim}{d}_i z + \begin{pmatrix}S_{2i}\\ S_{3i}\end{pmatrix}\}\phi_i'' + O(\varepsilon)^3, \qquad (5.7.8)$$

$$\sigma_{zz} = \varepsilon^2 2\mathrm{Re}\sum_i\{(1,\hat{s}_i)\underset{\sim}{d}_i\frac{z^2}{2} + (1,\hat{s}_i)\begin{pmatrix}S_{2i}\\ S_{3i}\end{pmatrix}z + S_{6i}\}\phi_i''' + O(\varepsilon)^4. \qquad (5.7.9)$$

At the interface between the ℓth and $(\ell + 1)$th layer the continuity conditions (5.7.1), (5.7.2) and (5.7.3) are satisfied if

$$S_{1i}^{(\ell+1)} = \frac{\varepsilon^{(\ell)}}{\varepsilon^{(\ell+1)}}(a_i^{(\ell)} + S_{1i}^{(\ell)}) + a_i^{(\ell+1)}, \qquad \ell = 1, \ldots, N-1, \qquad (5.7.10)$$

$$\begin{pmatrix}S_{2i}^{(\ell+1)}\\ S_{3i}^{(\ell+1)}\end{pmatrix} = \frac{\varepsilon^{(\ell)}}{\varepsilon^{(\ell+1)}}\{\underset{\sim}{d}_i^{(\ell)} + \begin{pmatrix}S_{2i}^{(\ell)}\\ S_{3i}^{(\ell)}\end{pmatrix}\} + \underset{\sim}{d}_i^{(\ell+1)} \qquad \ell = 1, \ldots, N-1, \qquad (5.7.11)$$

$$S_{6i}^{(\ell)} = \frac{\varepsilon^{(\ell+1)^2}}{\varepsilon^{(\ell)^2}}\{(1,\hat{s}_i)(\frac{\underset{\sim}{d}_i^{(\ell+1)}}{2} - \begin{pmatrix}S_{2i}^{(\ell+1)}\\ S_{3i}^{(\ell+1)}\end{pmatrix}) + S_{6i}^{(\ell+1)}\}$$

$$- (1,\hat{s}_i)(\frac{\underset{\sim}{d}_i^{(\ell)}}{2} + \begin{pmatrix}S_{2i}^{(\ell)}\\ S_{3i}^{(\ell)}\end{pmatrix}), \qquad \ell = N-1, \ldots, 1, \qquad (5.7.12)$$

$$\varepsilon^{(\ell)^2}\begin{Bmatrix} S_{4i}^{(\ell)} \\ S_{5i}^{(\ell)} \end{Bmatrix} = \varepsilon^{(1)^2}\,\frac{\underset{\sim}{e}_i^{(1)}}{2} - \frac{\varepsilon^{(\ell)^2}}{2}\,\underset{\sim}{e}_i^{(\ell)}$$

$$+ \sum_{j=2}^{(\ell)}\{\varepsilon^{(j-1)^2}\underset{\sim}{f}_i^{(j-1)} + \varepsilon^{(j)^2}\underset{\sim}{f}_i^{(j)}\}$$

$$+ \varepsilon^{(1)^2}\begin{Bmatrix} S_{4i}^{(1)} \\ S_{5i}^{(1)} \end{Bmatrix} \qquad \ell = 2, \ldots, N. \qquad (5.7.13)$$

Hence, for an N layer laminate, we have 6N arbitrary functions which are chosen to satisfy the above 6N - 6 continuity conditions, together with 4 conditions which arise from the stress-free conditions specified at the bounding surfaces of the laminate. Without loss of generality, this leaves the arbitrary functions $S_{4i}^{(1)}$ and $S_{5i}^{(1)}$ in (5.7.13) disposable. At a later stage, we discuss how to choose these functions.

(B) Bending Solutions

The stress and displacement components in a typical layer, given in Section 5.5, are

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = -\varepsilon 2\mathrm{Re}\sum_i\{(z + B_{1i})\begin{Bmatrix} 1 \\ \hat{\alpha}_i \end{Bmatrix}\}\Phi_i'$$

$$+ \varepsilon^3 2\mathrm{Re}\sum_i\{k_i(\frac{z^3}{6} + B_{1i}\,\frac{z^2}{2}) + \underset{\sim}{m}_i z + \begin{Bmatrix} B_{5i} \\ B_{6i} \end{Bmatrix}\}\Phi_i''' + O(\varepsilon)^5, \qquad (5.7.14)$$

$$w = 2\mathrm{Re}\sum_i \Phi_i + \varepsilon^2 2\mathrm{Re}\sum_i\{b_i(\frac{z^2}{2} + B_{1i}z) + B_{2i}\}\Phi_i'' + O(\varepsilon)^4, \qquad (5.7.15)$$

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = -\varepsilon 2\mathrm{Re}\ \sum_i\{(z + B_{1i})\underset{\sim}{Q}\begin{pmatrix}1\\ \hat{\alpha}_i^2\\ 2\hat{\alpha}_i\end{pmatrix}\}\Phi_i''$$

$$+ \varepsilon^3 2\mathrm{Re}\ \sum_i\{\underset{\sim}{D}_i(\frac{z^3}{6} + B_{1i}\ \frac{z^2}{2}) + E_i z + \underset{\sim}{Q}\underset{\sim}{\hat{f}}_i\begin{pmatrix}B_{5i}\\ B_{6i}\end{pmatrix}\}\Phi_i'''' + O(\varepsilon)^5, \qquad (5.7.16)$$

$$\begin{pmatrix}\sigma_{xz}\\ \sigma_{yz}\end{pmatrix} = \varepsilon^2 2\mathrm{Re}\ \sum_i\{\underset{\sim}{g}_i(\frac{z^2}{2} + B_{1i}z) + \begin{pmatrix}B_{3i}\\ B_{4i}\end{pmatrix}\}\Phi_i''' + O(\varepsilon)^4, \qquad (5.7.17)$$

$$\sigma_{zz} = -\varepsilon^3 2\mathrm{Re}\ \sum_i\{(1,\hat{\alpha}_i)\underset{\sim}{g}_i(\frac{z^3}{6} + B_{1i}\ \frac{z^2}{2}) + (1,\hat{\alpha}_i)\begin{pmatrix}B_{3i}\\ B_{4i}\end{pmatrix}z + B_{7i}\}\Phi_i''''. \qquad (5.7.18)$$

The continuity conditions given by (5.7.1), (5.7.2) and (5.7.3) are satisfied at the interface of the ℓth and $(\ell + 1)$th layer if

$$B_{1i}^{(\ell+1)} = \frac{\varepsilon^{(\ell)}}{\varepsilon^{(\ell+1)}}(1 + B_{1i}^{(\ell)}) + 1 \qquad \ell = 1, \ldots, N - 1, \qquad (5.7.19)$$

$$\begin{pmatrix}B_{5i}^{(\ell+1)}\\ B_{6i}^{(\ell+1)}\end{pmatrix} = \frac{\varepsilon^{(\ell)^3}}{\varepsilon^{(\ell+1)^3}}\{k_i^{(\ell)}(\frac{1}{6} + \frac{B_{1i}^{(\ell)}}{2}) + \underset{\sim}{m}_i^{(\ell)} + \begin{pmatrix}B_{5i}^{(\ell)}\\ B_{6i}^{(\ell)}\end{pmatrix}\}$$

$$+ k_i^{(\ell+1)}(\frac{1}{6} - \frac{B_{1i}^{(\ell+1)}}{2}) + \underset{\sim}{m}_i^{(\ell+1)}, \qquad \ell = 1, \ldots, N - 1, \qquad (5.7.20)$$

$$\begin{pmatrix}B_{3i}^{(\ell)}\\ B_{4i}^{(\ell)}\end{pmatrix} = \frac{\varepsilon^{(\ell+1)^2}}{\varepsilon^{(\ell)^2}}\{\underset{\sim}{g}_i^{(\ell+1)}(\tfrac{1}{2} - B_{1i}^{(\ell+1)}) + \begin{pmatrix}B_{3i}^{(\ell+1)}\\ B_{4i}^{(\ell+1)}\end{pmatrix}\}$$

$$- \underset{\sim}{g}_i^{(\ell)}(\tfrac{1}{2} + B_{1i}^{(\ell)}), \qquad \ell = N - 1, \ldots, 1, \qquad (5.7.21)$$

$$B_{7i}^{(\ell)} = \frac{\varepsilon^{(\ell+1)^3}}{\varepsilon^{(\ell)^3}}\{(1,\hat{\alpha}_i)(\underset{\sim}{g}_i^{(\ell+1)}(-\frac{1}{6} + B_{1i}^{(\ell+1)}) - \begin{pmatrix} B_{3i}^{(\ell+1)} \\ B_{4i}^{(\ell+1)} \end{pmatrix})$$

$$- (1,\hat{\alpha}_i)(\underset{\sim}{g}_i^{(\ell)}(\frac{1}{6} + B_{1i}^{(\ell)}) + \begin{pmatrix} B_{3i}^{(\ell)} \\ B_{4i}^{(\ell)} \end{pmatrix}) + B_{7i}^{(\ell+1)}\}$$

$$\ell = N - 1, \ldots, 1, \qquad (5.7.22)$$

$$\varepsilon^{(\ell)^2} B_{2i}^{(\ell)} = \frac{\varepsilon^{(1)^2} b_i^{(1)}}{2} - \frac{\varepsilon^{(\ell)^2} b_i^{(\ell)}}{2}$$

$$+ \sum_{j=2}^{\ell} (\varepsilon^{(j)^2} b_i^{(j)} B_{1i}^{(j)} + \varepsilon^{(j-1)^2} b_i^{(j-1)} B_{1i}^{(j-1)}) + \varepsilon^{(1)^2} B_{2i}^{(1)}$$

$$\ell = 2, \ldots, N. \qquad (5.7.23)$$

For a N-layer laminate we have 7N arbitrary constants which satisfy the above 7N - 7 continuity conditions together with 6 conditions which arise from the stress-free conditions specified at the bounding surface of the laminate. Without loss of generality, this leaves $B_{2i}^{(1)}$ in (5.7.23) disposable.

From a computational point of view, it is more attractive to consider laminated plates which have the mid-plane as a plane of elastic symmetry and subjected to mid-plane symmetric boundary conditions. Accordingly, we consider a symmetric laminated plate formed from the one shown in Figure 5.1 by repeating layers 2 to N below layer 1 in such a way that the plane $z^{(1)} = 0$ becomes a plane of elastic symmetry. We now require that

(i) $u^{(1)}$ $v^{(1)}$ even functions of z, $w^{(1)}$ odd function of z in stretching, and (5.7.24)

(ii) $u^{(1)}$ $v^{(1)}$ odd functions of z, $w^{(1)}$ even function of z in bending. (5.7.25)

This means that for symmetric laminates

$$\begin{aligned} S_{1i}^{(1)} &= 0, \quad S_{2i}^{(1)} = 0, \quad S_{3i}^{(1)} = 0. \\ B_{1i}^{(1)} &= 0, \quad B_{5i}^{(1)} = 0, \quad B_{6i}^{(1)} = 0. \end{aligned} \tag{5.7.26}$$

In addition, the stress-free conditions are all satisfied if

$$S_{6i}^{(N)} = -(1,\hat{s}_i)\{\frac{\underset{\sim}{d}_i^{(N)}}{2} + \begin{pmatrix} S_{2i}^{(N)} \\ S_{3i}^{(N)} \end{pmatrix}\},$$

$$\begin{pmatrix} B_{3i}^{(N)} \\ B_{4i}^{(N)} \end{pmatrix} = -\underset{\sim}{g}_i^{(N)}(\tfrac{1}{2} + B_{1i}^{(N)}), \tag{5.7.27}$$

$$B_{7i}^{(N)} = -(1,\hat{\alpha}_i)\{\underset{\sim}{g}_i^{(N)}(\frac{1}{6} + \frac{B_{1i}^{(N)}}{2}) + \begin{pmatrix} B_{3i}^{(N)} \\ B_{4i}^{(N)} \end{pmatrix}\},$$

and with the exceptions of $B_{2i}^{(1)}$, $S_{4i}^{(1)}$ and $S_{5i}^{(1)}$, all the arbitrary quantities can now be determined directly from (5.7.10) to (5.7.13) and (5.7.19) to (5.7.23).

INTEGRAL CONDITION

The disposable quantities in both the stretching and bending solutions are determined by introducing suitable integral conditions on the field quantities in the laminate. The choice of these conditions, in general, depends on the nature of a particular problem. For example, for a displacement boundary value problem we would impose that the average displacement due to the higher order terms is zero and it would then follow that the average displacement in the laminate would be identical to that of the equivalent plate. Similarly, for a traction boundary value problem we would impose that the resultant traction due to the higher order terms is zero. As an illustration of these integral conditions, we consider a symmetric laminate being stretched by in-plane tractions. By referring to Figure 5.2 we denote $\sigma_{\xi\xi}$ and $\sigma_{\xi\eta}$ to be the normal and shear components of stress on a curve ξ = constant, where (ξ,η) are curvilinear co-ordinates.

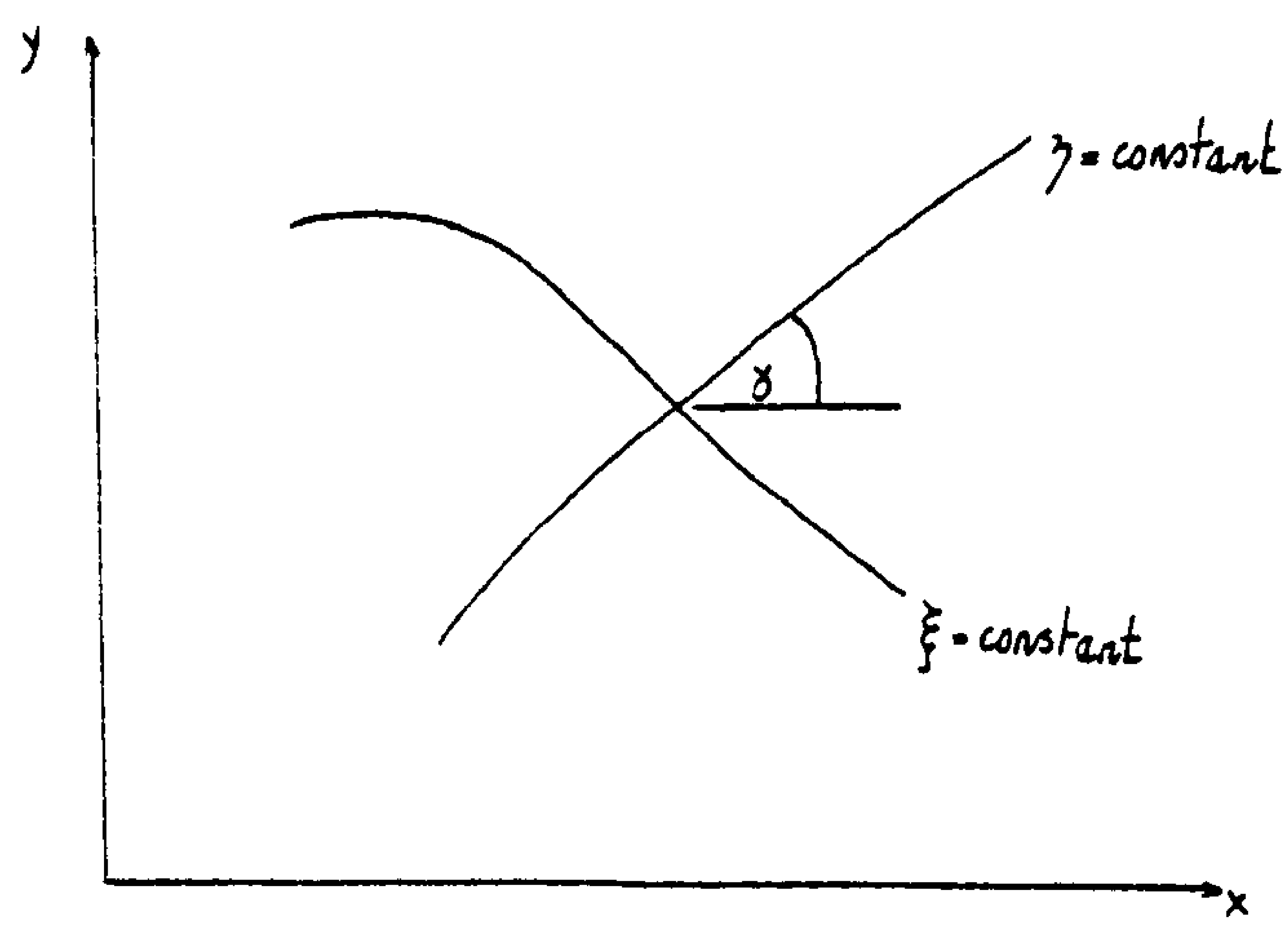


Figure 5.2.

Following Timoshenko and Goodier (1970) we have

$$\begin{Bmatrix}\sigma_{\xi\xi}\\ \sigma_{\xi\eta}\end{Bmatrix} = \underset{\sim}{\Upsilon}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} \quad (5.7.28)$$

where

$$\underset{\sim}{\Upsilon} = \begin{bmatrix}\frac{1}{2}(1+\cos 2\gamma) & \frac{1}{2}(1-\cos 2\gamma) & \sin 2\gamma\\ -\frac{1}{2}\sin 2\gamma & \frac{1}{2}\sin 2\gamma & \cos 2\gamma\end{bmatrix}. \quad (5.7.29)$$

Hence, if the ξ = constant curve is chosen to coincide with the boundary of the laminate to which tractions are being applied, then we can specify an integral condition which will result in the resultant traction from the $O(\varepsilon)^2$ terms of the in-plane stress components to be zero on this boundary. Specifically, we assume that

$$\int_{\text{laminate}} \varepsilon^2 \underset{\sim}{\Upsilon}\begin{Bmatrix}\sigma_{xx2}\\ \sigma_{yy2}\\ \sigma_{xy2}\end{Bmatrix} dz = 0. \quad (5.7.30)$$

Equation (5.7.30) represents the two conditions which will determine $S_{4i}^{(1)}$ and $S_{5i}^{(1)}$. By substituting (5.7.7) into (5.7.30) and carrying out the integration, we obtain for symmetric laminates that

$$\{2\underset{\sim}{\Upsilon}\varepsilon^{(1)^3}\underset{\sim}{Q}^{(1)}\underset{\sim}{\hat{n}}_i + 4\sum_{\ell=2}^{N}\underset{\sim}{\Upsilon}\varepsilon^{(\ell)}\varepsilon^{(1)^2}\underset{\sim}{Q}_i^{(\ell)}\underset{\sim}{\hat{n}}_i\}\begin{Bmatrix}S_{4i}^{(1)}\\ S_{5i}^{(1)}\end{Bmatrix} = -\underset{\sim}{\Upsilon}\varepsilon^{(1)^3}\left(\frac{\underset{\sim}{A}_i^{(1)}}{3} + 2\underset{\sim}{C}_i^{(1)}\right)$$

$$- 2\sum_{\ell=2}^{N}\underset{\sim}{\Upsilon}\varepsilon^{(\ell)^3}\left(\frac{\underset{\sim}{A}_i^{(\ell)}}{3} + 2\underset{\sim}{C}_i^{(\ell)}\right) - 4\sum_{\ell=2}^{N}\underset{\sim}{\Upsilon}\varepsilon^{(\ell)}\underset{\sim}{Q}_i^{(\ell)}\underset{\sim}{\hat{n}}_i$$

$$\times\{\varepsilon^{(1)^2}\frac{\underset{\sim}{e}_i^{(1)}}{2} - \varepsilon^{(\ell)^2}\frac{\underset{\sim}{e}_i^{(\ell)}}{2} + \sum_{j=2}^{(\ell)}(\varepsilon^{(j-1)^2}\underset{\sim}{f}_i^{(j-1)} + \varepsilon^{(j)^2}\underset{\sim}{f}_i^{(j)})\}. \quad (5.7.31)$$

The remaining $S_{4i}^{(\ell)}$, $S_{5i}^{(\ell)}$ now follow directly from (5.7.13).

In the remainder of this chapter, the general solutions for the stress components, given by (5.7.7) to (5.7.9), will be applied to particular examples. We restrict ourselves to the stretching of a symmetric laminate by applied in-plane tractions, though the analysis can be extended to more general cases.

The computation in these examples is carried out by the computer program given in Appendix 1. Input to the program is the

(i) number of layers in the laminate

(ii) half-width of each layer

(iii) material constants of a fibre-reinforced material with a preferred direction parallel to the x-axis

(iv) angle of orientation of each layer, given by α in (5.2.1)

(v) equation of the curve to which tractions are being applied (c.f. Equation (5.7.29)).

The program evaluates the material constants of each layer by using the transformation laws of (1.2.6) and satisfies all the necessary through thickness boundary conditions. By using the equation of the boundary to which the tractions are being applied and the integral conditions given by (5.7.30), all the stress components are evaluated.

A notable feature of the program is that if the equivalent displacements are not available analytically, but can be obtained numerically, by, for example,a two-dimensional finite element method, then the stress components in each layer can be evaluated from this two-dimensional data by forming the necessary derivatives numerically.

5.8 HALF-SPACE SUBJECTED TO SINUSOIDAL TRACTION

We consider a symmetric laminate occupying the region $y \geq 0$ and subjected to resultant tractions on the edge $y = 0$ of the form

$$\hat{\sigma}_{yy} = P_o \sin x = P_o \, \mathrm{Re}\{ie^{ix}\}, \quad \hat{\sigma}_{xy} = 0, \quad y = 0. \tag{5.8.1}$$

The stress at $y = \infty$ is zero.

In order to solve this problem, we initially consider a single equivalent plate which has the same overall geometry as the laminate and which is subjected to the boundary conditions (5.8.1). We denote the boundary values of the complex potentials ϕ_1, ϕ_2 of the equivalent plate by

$$\phi_1 = \phi_1^*, \quad \phi_2 = \phi_2^*, \quad y = 0. \tag{5.8.2}$$

By using (5.3.16) and (5.8.2) we obtain the following two linear first order ordinary differential equations for ϕ_i^*

$$\hat{s}_1 \frac{d\phi_1^*}{dx} + \hat{s}_2 \frac{d\phi_2^*}{dx} = 0, \quad \frac{d\phi_1^*}{dx} + \frac{d\phi_2^*}{dx} = -\frac{i}{2} e^{ix}. \tag{5.8.3}$$

Solving (5.8.3) and re-introducing the y-dependence gives

$$\phi_1'(z_1) = -\frac{i\hat{s}_2}{2(\hat{s}_2 - \hat{s}_1)} e^{iz_1}, \quad \phi_2'(z_2) = \frac{i\hat{s}_1}{2(\hat{s}_2 - \hat{s}_1)} e^{iz_2}, \tag{5.8.4}$$

where

$$z_j = x + s_j y. \tag{5.8.5}$$

Substitution of the potentials given in (5.8.4) into Equations (5.7.7) to (5.7.9) will now give the stress components in each layer.

NUMERICAL RESULTS

We take the elastic constants of a transversely isotropic material with a preferred direction parallel to the x-axis to be ($x\ 10^9\ Nm^{-2}$)

$$\kappa_{11} = \kappa\ 5.66,\quad \kappa_{12} = 4.37,\quad \kappa_{23} = 5.64,$$
$$\kappa_{22} = 10.57,\quad \kappa_{66} = 5.66 \qquad (5.8.6)$$

where

$$\kappa = \kappa_{11}/\kappa_{66}$$

is a specified dimensionless constant. For epoxy resin reinforced by carbon fibres $\kappa \simeq 40$.

We consider a symmetric laminate consisting of three layers of equal thickness stacked according to the scheme $[90^\circ, 0^\circ]_s$. Because of the stacking sequence we need only consider the values of the field quantities above the mid-plane of the laminate since quantities below this plane can be defined by symmetry. In Table 5.1 typical values in the variation of the lowest order stress components are given with the thickness co-ordinate when $\kappa = 10, 40$. In Table 5.2 the same variation is given for the higher order components.

	σ_{xx}/P_o	σ_{yy}/P_o	$\sigma_{xz}/P_o\varepsilon$	σ_{xx}/P_o	σ_{yy}/P_o	$\sigma_{xz}/P_o\varepsilon$
z_2 = 1.0	0.28	1.38	0.0	0.08	1.47	0.0
0.5			-0.24			-0.32
z_2 = 0.0			-0.49			-0.64
-0.5			-0.73			-0.97
z_2 = -1.0	0.28	1.38	-0.98	0.08	1.47	-1.29
z_1 = 1.0	1.75	0.24	-0.98	2.01	0.07	-1.29
0.5			-0.49			-0.64
z_1 = 0.0	1.75	0.24	0.0	2.01	0.07	0.0
	(i)			(ii)		

Table 5.1. Variation in the Lowest Order Terms with Thickness

(y = 0)

(i) κ = 10 (ii) κ = 40

When $\kappa = 10$, the magnitudes of the in-plane higher order terms in the fibre direction are of $O(10\varepsilon^2)$ compared to lowest order terms of $O(1)$ and interlaminar shear stresses of $O(\varepsilon)$. The $O(1)$ terms are from a homogeneous displacement at $y = 0$ and give rise to resultant loads given by (5.8.1). The lowest order terms therefore represent the stress field which would be obtained from Classical Laminate Theory (CLT). If ε is small such that $O(10\varepsilon^2)$ is negligible, the resulting in-plane stresses are those which would be obtained by CLT but, in addition, there is the prediction of weak interlaminar shear stresses.

As κ increases in magnitude so do the higher order terms and therefore for large values of κ the solution cannot be taken up to the boundary $y = 0$ unless $\varepsilon \ll 1$. For $\kappa = 40$ we have at the boundary, correction terms of $O(80\varepsilon^2)$ compared with lower order terms of $O(1)$. If we place a restriction on the geometry by stating that $\varepsilon \ll 1$ then

	$\sigma_{xx}/P_o\varepsilon^2$ (i)	$\sigma_{yy}/P_o\varepsilon^2$ (i)	$\sigma_{zz}/P_o\varepsilon^2$ (i)	$\sigma_{xx}/P_o\varepsilon^2$ (ii)	$\sigma_{yy}/P_o\varepsilon^2$ (ii)	$\sigma_{zz}/P_o\varepsilon^2$ (ii)
z_2 = 1.0	1.11	5.30	0.0	1.98	19.8	0.0
0.5	1.03	0.91	0.006	1.67	2.74	0.02
z_2 = 0.0	0.62	-1.51	0.02	0.94	-5.98	0.08
-0.5	-0.14	-1.95	0.05	-0.19	-6.06	0.19
z_2 = -1.0	-1.24	-0.41	0.09	-1.72	2.23	0.34
z_1 = 1.0	-9.29	-0.31	0.09	-57.1	-0.25	0.34
0.5	-12.1	0.47	0.12	-75.3	0.59	0.47
z_2 = 0.0	-13.0	0.73	0.14	-81.4	0.87	0.51

Table 5.2. *Variation in the Higher Order Terms with Thickness (y = 0)* $[90,0]_s$
(i) $\kappa = 10$ *(ii)* $\kappa = 40$

	(i)			(ii)		
z_2 = 1.0	0.77	-2.12	0.0	0.34	-2.36	0.0
0.5	1.50	-1.71	-0.02	1.54	-1.09	-0.01
z_2 = 0.0	1.65	-1.25	-0.09	1.84	-0.43	-0.04
-0.5	1.22	-0.72	-0.2	1.25	-0.38	-0.09
z_2 = -1.0	2.30	-0.14	-0.35	-0.23	-0.94	-0.17
z_1 = 1.0	1.47	1.11	-0.35	-0.37	-1.08	-0.17
0.5	2.70	2.58	-0.48	2.51	2.01	-0.23
z_1 = 0.0	3.11	3.07	-0.53	3.47	3.04	-0.26

Table 5.3. *Variation in the Higher Order Terms with Thickness (y = 0)* $[-45,45]_s$
(i) $\kappa = 10$ *(ii)* $\kappa = 40$

again the higher order terms can be neglected. But in such cases, the interlamina shear stresses become small. If the restriction on the geometry is not practical, then the solution will be valid in regions sufficiently far from $y = 0$.

The typical values given in Table 5.3 are for the higher order stress components when the stacking sequence is $[-45,45]_s$. Comparing these values with those given in Table 5.2 shows that there is a significant drop in the magnitude of the higher order terms. This is due to the fact that for the $[-45,45]_s$ stacking the fibres are not directly carrying the prescribed loads and it would therefore be reasonable to suppose that the values given by Table 5.2 represent an extreme case.

For laminates consisting of layers which are stiff in the fibre direction, the higher order terms greatly exceed the terms of lowest order which is in sharp conflict with Tang (1974) who supposes that the lowest order terms represent the 'interior' solution to sufficient accuracy. This behaviour in the higher order terms is attributed to the fact that for strong fibres $1/\kappa << 1$ and therefore we have two small parameters in the analysis. This means that the original asymptotic expansions in ε for the displacement components cannot be valid since they do not take into account the presence of the large parameter κ.

5.9 CIRCULAR HOLE IN AN INFINITE LAMINATE

We next consider an infinite laminate consisting of transversely isotropic layers and containing a circular hole of radius a. The laminate is subjected at infinity to a resultant uniaxial load P_o, in thedirection of the x-axis and there is zero resultant in-plane load applied to the surface of the hole.

Initially we consider an equivalent plate subjected to the following boundary conditions:

$$\hat{\sigma}_{xx} = P_o \qquad \hat{\sigma}_{xy} = 0, \qquad r = \sqrt{(x^2 + y^2)} \rightarrow \infty, \tag{5.9.1}$$

Hole Stress Free.

Following Savin (1961), the complex potentials of the equivalent plate are

$$\phi_1' = \frac{1}{2(\hat{s}_1^2 - \hat{s}_2^2)} - \frac{i}{2(\hat{s}_1 - \hat{s}_2)(1 + i\hat{s}_1)}\left\{1 - \frac{z_1}{\sqrt{z_1^2 - (1 + \hat{s}_1^2)}}\right\},$$

$$\phi_2' = -\frac{1}{2(\hat{s}_1^2 - \hat{s}_2^2)} + \frac{i}{2(\hat{s}_1 + \hat{s}_2)(1 + i\hat{s}_2)}\left\{1 - \frac{z_2}{\sqrt{z_2^2 - (1 + \hat{s}_2^2)}}\right\} \tag{5.9.2}$$

where

$$z_j = x + \hat{s}_j y.$$

Substitution of the complex potentials given in (5.9.2) into (5.7.7) to (5.7.9) gives the stress components in each layer of the laminate.

From Equations (5.9.2) we find that the behaviour of the potentials for large r is

$$\phi_1' \sim \frac{1}{2(\hat{s}_1^2 - \hat{s}_2^2)} + 0\left(\frac{1}{r^2}\right), \qquad \phi_2' \sim -\frac{1}{2(\hat{s}_1^2 - \hat{s}_2^2)} + 0\left(\frac{1}{r^2}\right), \tag{5.9.3}$$

and therefore the interlamina shear stress components decay as $0(1/r^3)$ whilst the higher order in-plane stresses decay as $0(1/r^4)$.

At infinity the displacement field in the laminate is homogeneous and gives rise to a resultant load given by (5.9.1). Hence, in this region the solution is the solution that would be predicted by Classical Laminate Theory.

NUMERICAL RESULTS

For numerical purposes we have taken a three-layer $[90,0]_s$ laminate, the elastic constants in each layer being derived from (5.8.6). In Table 5.4 the leading stress components are given at the hole surface when x = 0, y = 1, and for comparison the higher order stresses are given in Table 5.5; in each table κ has been taken to be 10.

	σ_{xx}/P_o	σ_{yy}/P_o	σ_{xz}/P_o
z_2 = 1.0	1.007	-0.114	0.0
0.5			0.976
z_2 = 0.0			1.952
-0.5			2.928
z_2 = -1.0	1.007	-0.114	3.904
z_1 = 1.0	8.298	0.227	3.904
0.5			1.952
z_1 = 0	8.298	0.227	0.0

Table 5.4. Variation in the Lower Order Terms with Thickness, κ = 10.

	$\sigma_{xx}/P_o\varepsilon^2$	$\sigma_{yy}/P_o\varepsilon^2$	$\sigma_{zz}/P_o\varepsilon^2$
z_2 = 1.0	120.6	266.2	0.0
0.5	108.1	217.5	0.942
z_2 = 0.0	72.95	-99.78	3.768
-0.5	15.25	-98.44	8.479
z_2 = -1.0	-65.0	25.78	15.07
z_1 = 1.0	-598.6	-7.15	15.07
0.5	-846.1	41.09	20.73
z_1 = 0	-928.6	57.18	22.61

Table 5.5. Variation in the Higher Order Terms with Thickness, κ = 10.

The resultant in-plane load on the hole surface is zero as a consequence of the definition of the equivalent elastic constants and the integral conditions (5.7.30) and therefore by Saint Venant's principle the stress solutions are valid sufficiently far from the hole. When $\kappa = 10$ the higher order terms given in Table 5.5 are $O(10^3\varepsilon^2)$ compared with leading terms of $O(10)$ and interlaminar stresses of $O(3\varepsilon)$. Hence for $\varepsilon << 1$ such that $10^3\varepsilon^2$ is negligible, the stress components at the hole surface are exactly those given by CLT with the addition of very weak interlamina shear stresses. For larger values of κ the in-plane higher order terms greatly exceed the leading terms at the boundary and for practical purposes the higher order terms cannot be made negligible by an appropriate choice of ε. This means that for $\kappa >> 1$, CLT solutions are not sufficient solutions.

5.10 THE BEHAVIOUR OF THE STRESS COMPONENTS WHEN $\kappa >> 1$

In this section we explain why there is a breakdown in the power series expansions for the stress components when $\kappa >> 1$. For a transversely isotropic material which has strong fibres

$$\kappa \simeq (\beta/\mu) >> 1, \tag{5.10.1}$$

where β and μ have dimensions of stress and are defined by (1.3.2). Expanding the stress components in (5.7.7) to (5.7.9) as power series in (β/μ) for a $[90,0]_s$ laminate we obtain

$$\sigma_{xx}^{(2)} \sim \frac{\mu}{\beta}\phi_1' + \phi_2' + \varepsilon^2\left(\frac{\mu}{\beta}\phi_1''' + \frac{\beta}{\mu}\phi_2'''\right)$$

$$\sigma_{xx}^{(1)} \sim \frac{\mu}{\beta}\phi_1' + \frac{\beta}{\mu}\phi_2' + \varepsilon^2\left(\phi_1''' + \frac{\beta^2}{\mu^2}\phi_2'''\right)$$

$$\sigma_{yy}^{(2)} \sim \phi_1' + \phi_2' + \varepsilon^2\left(\phi_1''' + \frac{\beta^2}{\mu^2}\phi_2'''\right) \tag{5.10.2}$$

$$\sigma_{yy}^{(1)} \sim \frac{\mu}{\beta}\phi_1' + \phi_2' + \varepsilon^2\left(\frac{\mu}{\beta}\phi_1''' + \frac{\beta}{\mu}\phi_2'''\right).$$

We now examine the behaviour of these stress components in relation to the illustrations of Sections 5.8 and 5.9.

HALF-SPACE SUBJECTED TO SINUSOIDAL LOADING

By expanding the complex potentials given by (5.8.4) as a power series in (μ/β) we find that at $y = 0$

$$\frac{\sigma_{xx}^{(2)}}{P_o} = \frac{\mu}{\beta}\{0(1) + 0(\varepsilon^2 \frac{\beta}{\mu})\}$$

$$\frac{\sigma_{xx}^{(1)}}{P_o} = 0(1) + 0(\varepsilon^2 \frac{\beta}{\mu})$$

$$\frac{\sigma_{yy}^{(2)}}{P_o} = 0(1) + 0(\varepsilon^2 \frac{\beta}{\mu}) \tag{5.10.3}$$

$$\frac{\sigma_{yy}^{(1)}}{P_o} = \frac{\mu}{\beta}\{0(1) + 0(\varepsilon^2 \frac{\beta}{\mu})\}.$$

For a fixed value of ε, it is seen from (5.10.3) that the higher order terms in $\sigma_{xx}^{(1)}$ and $\sigma_{yy}^{(2)}$ increase as β/μ increases, and as a result produce high tensile stresses in the fibre direction. For these terms to be small compared with the leading terms we must have

$$\varepsilon \ll \left(\frac{\mu}{\beta}\right)^{\frac{1}{2}}. \tag{5.10.4}$$

Equation (5.10.4) must be satisfied in order that Classical Laminate Theory is valid.

CIRCULAR HOLE IN AN INFINITE LAMINATE

The behaviour of stress components for large β/μ at the hole surface when $x = 0$ is obtained by expanding the complex potentials in (5.9.2) and using (5.10.2) as follows

$$\begin{aligned} \frac{\sigma_{xx}^{(2)}}{P_o} &= \left(\frac{\mu}{\beta}\right)^{\frac{1}{2}}\{O(1) + O(\varepsilon^2 \frac{\beta^2}{\mu^2})\} \\ \frac{\sigma_{xx}^{(1)}}{P_o} &= \left(\frac{\beta}{\mu}\right)^{\frac{1}{2}}\{O(1) + O(\varepsilon^2 \frac{\beta^2}{\mu^2})\}. \end{aligned} \tag{5.10.5}$$

In developing asymptotic expansions for the stress components it was assumed that each component can be written as a power series in ε and it was shown that the lowest order terms represented the stresses predicted by Classical Laminate Theory. These expansions are satisfactory provided the coefficients of ε remain reasonably well behaved. For layers containing strong fibres we have shown that the coefficients of the $O(\varepsilon^2)$ terms are much larger than the coefficients of the $O(1)$ terms. This behaviour indicates that Classical Laminate Theory is not applicable to such materials.

CHAPTER SIX

END EFFECTS ON A LAMINATED SEMI-INFINITE STRIP

6.1 INTRODUCTION

The usefulness of the theory presented in Chapters 4 and 5 is based on the validity of Saint Venant's principle which in effect states that where boundary conditions are satisfied in an average sense the stress components thus obtained represent the state of stress sufficiently far from the boundaries.

In this chapter we examine the stress field in the immediate vicinity of a boundary in a laminated plate. We consider a semi-infinite isotropic laminated strip which occupies the region $x \geq 0$, $-H \leq z \leq H$ where $2H$ is the thickness of the strip. The end $x = 0$ is subjected to prescribed conditions which are given by the theory of Chapter 4 and we assume homogeneous boundary conditions on the lateral boundaries $z = \pm H$. In the case of a homogeneous isotropic elastic strip the solution is given by a series of Fadle-Papkovitch eigenfunctions and the exponential decay of the end effects is characterized by the smallest real part of all the eigenvalues.

After a review of some basic results we consider (Section 6.4) a laminated strip consisting of an infinite number of layers. This is to ascertain whether the surfaces $z = \pm H$ have an appreciable effect on the decay rates of the stress components. In Section 6.5 we derive the characteristic equation for a three-layer laminated strip from which several limiting cases are discussed in Section 6.6. It is found that in all the cases considered, the eigenfunctions do not satisfy any known orthogonality condition and therefore the applied conditions at $x = 0$ cannot be satisfied immediately. Spence (1984) and others have observed that for a single isotropic strip there is an orthogonality condition available for 'canonical' problems (problems in which a displacement and stress component are specified at $x = 0$), but not for 'non-canonical' problems, problems in which both displacement or both stress component are specified at $x = 0$. For the latter case of problems a technique has been developed by Spence which enables the specified conditions to be satisfied, but since the method relies on a knowledge of the asymptotic behaviour of the eigenvalues it cannot readily be applied to laminated strips. Gaydon and Shepherd (1964), also solving a canonical problem, re-express each eigenfunction as a series of orthogonal functions and then satisfy conditions at $x = 0$, whilst Hess (1969) combines the eigenfunctions in such a manner so as to minimize the error involved in truncating the series expansions.

Following Hess, in Section 6.7 we formulate a least squares method with the aim of satisfying purely traction conditions prescribed at the end $x = 0$. In Section 6.8 the method is applied to a laminated strip subjected to sinusoidal normal stress at $x = 0$, for which numerical results are presented.

6.2 A GENERAL SOLUTION TO THE EQUATIONS OF LINEAR ELASTICITY

For problems in plane elasticity in the (x-z) plane, the stress components satisfy the stress equilibrium equations if they are written in terms of a stress function $\chi(x,z)$ as follows

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial z^2}, \quad \sigma_{xz} = -\frac{\partial^2 \chi}{\partial x \partial z}, \quad \sigma_{zz} = \frac{\partial^2 \chi}{\partial x^2}, \tag{6.2.1}$$

where for isotropic materials

$$\nabla^2 \nabla^2 \chi = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{6.2.2}$$

By seeking solutions to (6.2.2) in the form

$$\chi(x,z) = F(z) e^{-(\alpha/h)x}, \tag{6.2.3}$$

it can be shown that

$$F^{IV} + 2\left(\frac{\alpha}{h}\right)^2 F'' + \left(\frac{\alpha}{h}\right)^4 F = 0. \tag{6.2.4}$$

Here, the primes denote differentiation with respect to z and the ratio α/h is a constant that may be complex. The stress components are now given by

$$\sigma_{xx} = e^{-(\alpha/h)x} F'', \quad \sigma_{xz} = \left(\frac{\alpha}{h}\right) e^{-(\alpha/h)x} F', \quad \sigma_{zz} = \left(\frac{\alpha}{h}\right)^2 e^{-(\alpha/h)x} F. \tag{6.2.5}$$

By substituting (6.2.5) into the isotropic stress-strain relations and then integrating, we obtain the following displacement components to within a rigid body displacement;

$$u = -\frac{e^{-(\alpha/h)x}}{E}[\frac{\gamma}{(\alpha/h)}F'' + \frac{\alpha}{h}(\gamma - (1 + \nu)F],$$

$$w = -\frac{e^{-(\alpha/h)x}}{E}[\frac{\gamma}{(\alpha/h)^2}F''' + (\gamma + (1 + \nu)F'], \qquad (6.2.6)$$

where

$$\gamma = \begin{cases} 1 - \nu^2 & \text{plane strain,} \\ 1 & \text{generalised plane stress,} \end{cases} \qquad (6.2.7)$$

and E and ν denote the Youngs modulus and Poisson ratio of the material.

The general solution of Equation (6.2.4) is

$$F(z) = \frac{1}{(\alpha/h)^2}[(a_1 + a_2\frac{\alpha z}{h})\cos(\frac{\alpha z}{h}) + (a_3 + a_4\frac{\alpha z}{h})\sin(\frac{\alpha z}{h})], \qquad (6.2.8)$$

where the a_j are arbitrary constants. Hence

$$\sigma_{xx} = e^{-(\alpha/h)x}[(2a_4 - a_1 - a_2\frac{\alpha z}{h}) -(2a_2 + a_3 + a_4\frac{\alpha z}{h})\sin(\frac{\alpha z}{h})],$$

$$\sigma_{xz} = e^{-(\alpha/h)x}[(a_2 + a_3 + a_4\frac{\alpha z}{h})\cos(\frac{\alpha z}{h}) - (a_1 - a_4 + a_2\frac{\alpha z}{h})\sin(\frac{\alpha z}{h})], \qquad (6.2.9)$$

$$\sigma_{zz} = e^{-(\alpha/h)x}[(a_1 + a_2\frac{\alpha z}{h})\cos(\frac{\alpha z}{h}) + (a_3 + a_4\frac{\alpha z}{h})\sin(\frac{\alpha z}{h})],$$

$$u = \frac{e^{-(\alpha/h)x}}{E(\alpha/h)}[(a_1(1 + \nu) - 2\gamma a_4 + a_2(1 + \nu)\frac{\alpha z}{h})\cos(\frac{\alpha z}{h}) + \ldots$$
$$+ (2\gamma a_2 + (1 + \nu)a_3 + (1 + \nu)a_4\frac{\alpha z}{h})\sin(\frac{\alpha z}{h})],$$

$$(6.2.10)$$

$$w = \frac{e^{-(\alpha/h)x}}{E(\alpha/h)}[(a_2(2\gamma - 1 - \nu) - a_3(1 + \nu) - a_4(1 + \nu)\frac{\alpha z}{h})\cos(\frac{\alpha z}{h}) + \ldots$$
$$+ (a_4(2\gamma - 1 - \nu) + (1 + \nu)a_1 + (1 + \nu)a_2\frac{\alpha z}{h})\sin(\frac{\alpha z}{h})].$$

6.3 HOMOGENEOUS SEMI-INFINITE STRIP

A homogeneous semi-infinite isotropic strip, shown in Figure 6.1, is subjected to exponentially decaying normal and shear tractions on the lateral surfaces $z = \pm h$ so that

$$\begin{aligned} \sigma_{zz} &= Be^{-(\alpha x/h)}, \quad \sigma_{xz} = De^{-(\alpha x/h)}, \quad z = h, \\ \sigma_{zz} &= Ae^{-(\alpha x/h)}, \quad \sigma_{xz} = Ce^{-(\alpha x/h)}, \quad z = -h, \end{aligned} \tag{6.3.1}$$

where A, B, C and D are specified constants. These conditions are satisfied by the stress field given by (6.2.9) if

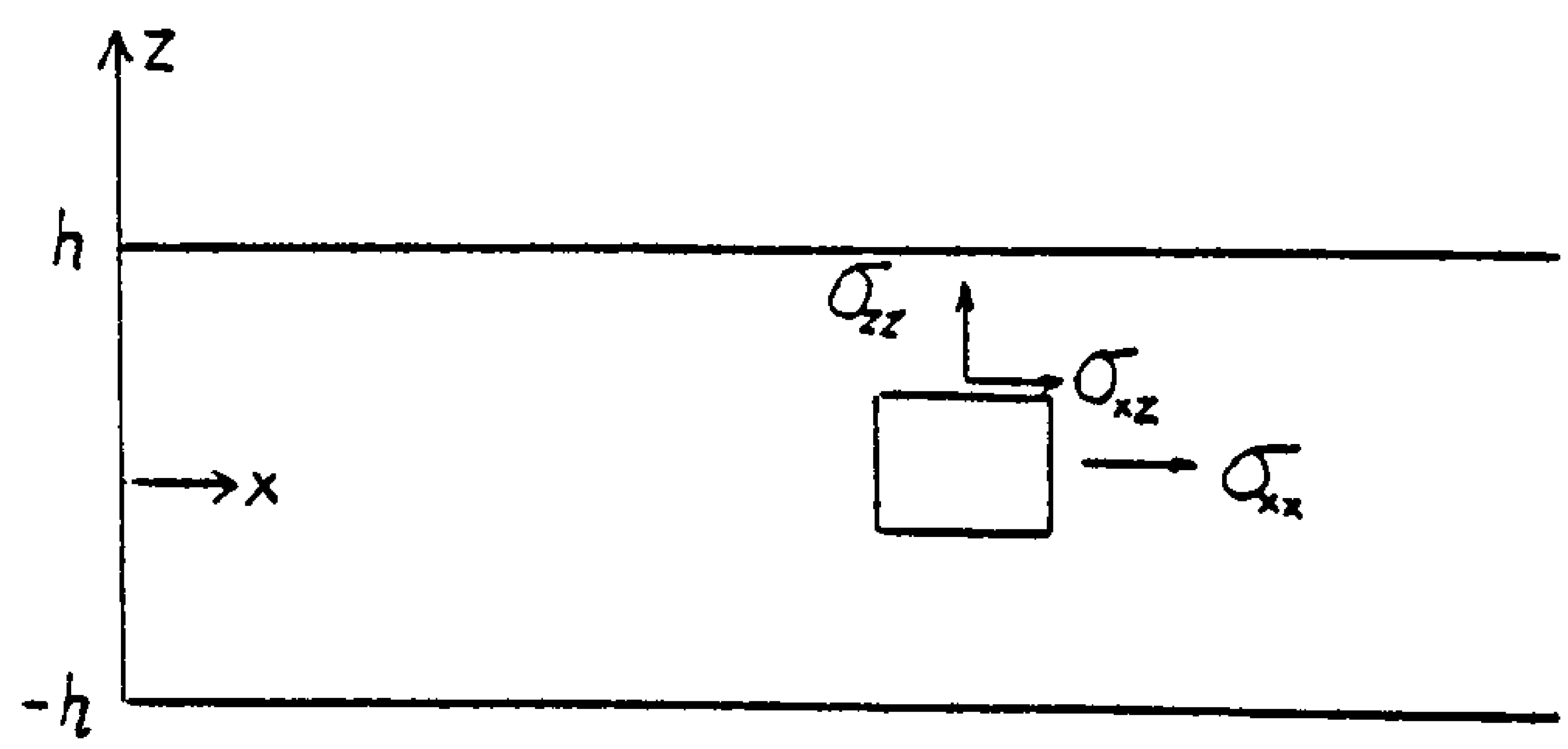


Figure 6.1. Semi-Infinite Strip.

$$\begin{aligned} a_1 &= \frac{(A + B)(\alpha \cos \alpha + \sin \alpha) + (C - D)\alpha \sin \alpha}{\sin 2\alpha + 2\alpha}, \\ a_2 &= \frac{(A - B)\cos \alpha + (C + D)\sin \alpha}{\sin 2\alpha - 2\alpha}, \\ a_3 &= \frac{(A - B)(\alpha \sin \alpha - \cos \alpha) - (C + D)\alpha \cos \alpha}{\sin 2\alpha - 2\alpha}, \\ a_4 &= \frac{(A + B)\sin \alpha - (C - D)\cos \alpha}{\sin 2\alpha + 2\alpha}. \end{aligned} \tag{6.3.2}$$

For the special case of the lateral surfaces of the strip being stress-free, A = B = C = D = 0 and in this case the coefficients a_j are identically zero unless we characterize the eigenvalues α as the non-zero roots of the transcendental equation

$$\sin 2\alpha \pm 2\alpha = 0. \tag{6.3.3}$$

This is the Fadle-Papkovitch eigenvalue equation and it gives the values of α for which a non-trivial solution of the form (6.2.9) can exist. The positive sign in (6.3.3) gives rise to eigenfunctions which are even in z and correspond to a symmetric deformation of the strip and the negative sign is associated with a skew symmetric deformation. Since the strip is now subject to applied tractions only at the end $x = 0$, the exponential rate of decay of the stresses gives a quantitative interpretation of Saint Venant's principle. We note that when α_o is a root of (6.3.3) then so are $\bar{\alpha}_o$, $-\alpha_o$ and $-\bar{\alpha}_o$, but since we require decaying solutions we consider only α_o and $\bar{\alpha}_o$.

In the analysis to follow we require the displacement of the lateral surfaces of the strip. From (6.2.10) we have that

$$\begin{aligned}
u\Big|_{z=h} &= \frac{e^{-\alpha x/h}}{(\alpha/h)}\left[\frac{2\gamma}{Ed}(g_3 - g_2\sin 2\alpha - g_4\cos 2\alpha) + \frac{(1+\nu)}{E} B\right], \\
u\Big|_{z=-h} &= \frac{e^{-\alpha x/h}}{(\alpha/h)}\left[\frac{2\gamma}{Ed}(-g_4 - g_1\sin 2\alpha + g_3\cos 2\alpha) + \frac{(1+\nu)}{E} A\right], \\
w\Big|_{z=h} &= \frac{e^{-\alpha x/h}}{(\alpha/h)}\left[\frac{2\gamma}{Ed}(g_1 + g_4\sin 2\alpha - g_2\cos 2\alpha) - \frac{(1+\nu)}{E} D\right], \\
w\Big|_{z=-h} &= \frac{e^{-\alpha x/h}}{(\alpha/h)}\left[\frac{2\gamma}{Ed}(-g_2 + g_3\sin 2\alpha + g_1\cos 2\alpha) - \frac{(1+\nu)}{E} C\right],
\end{aligned} \tag{6.3.4}$$

where

$$d = (\sin 2\alpha + 2\alpha)(\sin 2\alpha - 2\alpha),$$

$$g_1 = A\sin 2\alpha - B2\alpha,$$

$$g_2 = B\sin 2\alpha - A2\alpha,$$

$$g_3 = C\sin 2\alpha + D2\alpha,$$

$$g_4 = D\sin 2\alpha + C2\alpha.$$

6.4 A LAMINATED SEMI-INFINITE STRIP CONSISTING OF AN INFINITE NUMBER OF LAYERS

We consider a laminate which consists of layers of two alternating isotropic semi-infinite elastic strips. Each layer is referred to a local co-ordinate system (x,z) and each is labelled as shown in Figure 6.2. To distinguish the field quantities in each layer the subscripts/superscripts 1 and 2 are used.

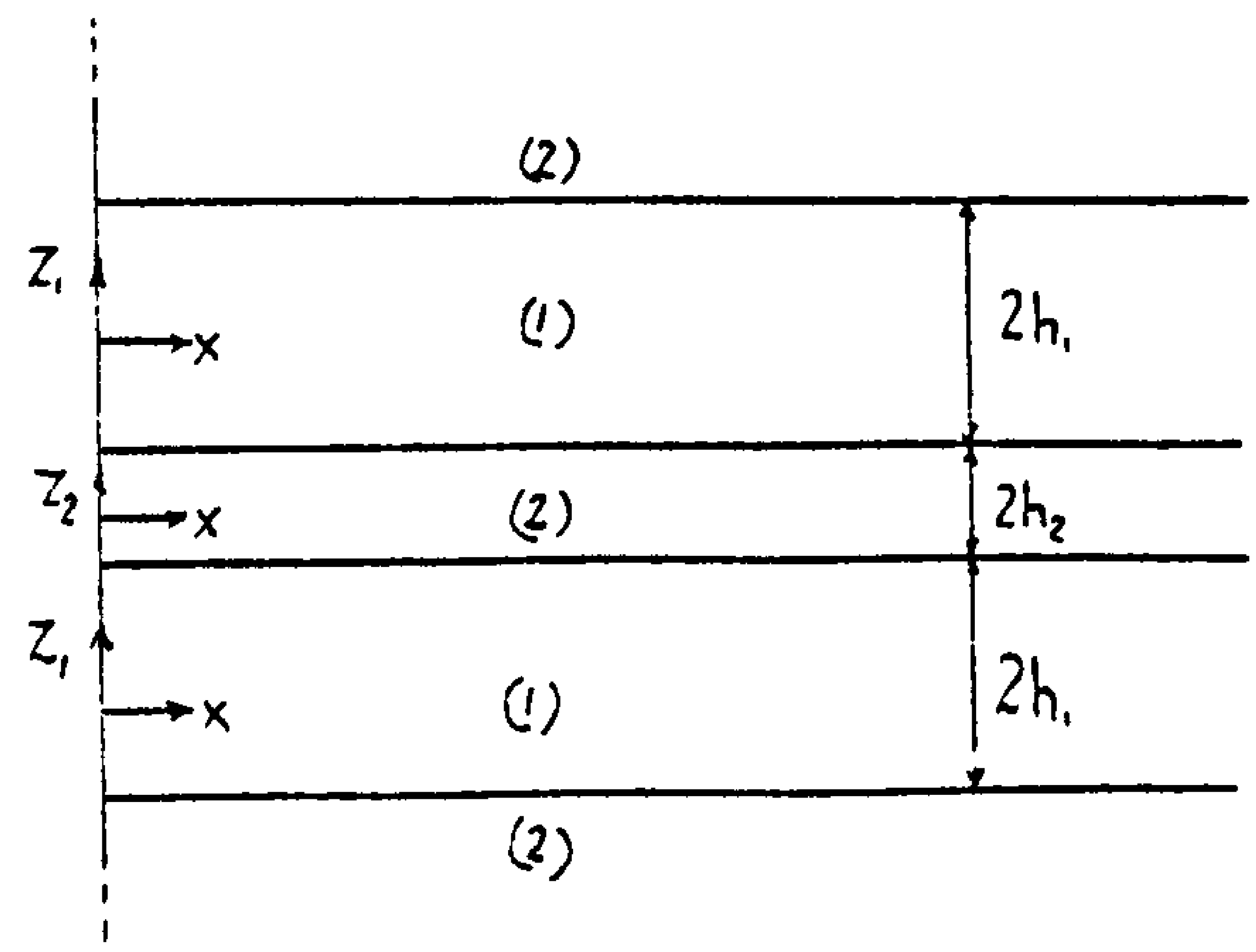


Figure 6.2.

The strips are of thickness $2h_\ell$ and are subjected to applied loads on the end x = 0 and the same load is applied to the layers which have the same material properties. Since each end load can be expressed as the sum of an even and odd function in z, the solution to the problem can be obtained by superposing the solution for the symmetric deformation problem with the solution for the skew-symmetric problem.

At each common interface we will require

(i) continuity of the displacement components,

(ii) continuity of the normal and shear stress components.

(a) Symmetric Deformation

In the case of symmetric deformation, the displacement u is an even function of z in each layer whilst w is an odd function of z. It follows from (6.3.1) that

$$A_\ell = B_\ell, \qquad C_\ell = -D_\ell, \qquad \ell = 1,2,$$

and the stress components are continuous at each common interface if

$$\frac{\alpha_1}{h_1} = \frac{\alpha_2}{h_2}, \qquad D_2 = -D_1, \qquad A_2 = A_1. \tag{6.4.1}$$

By satisfying the continuity conditions for the displacement components, it follows from (6.3.4), (6.4.1) that

$$\begin{aligned} A_1[2(\sin 2\alpha_1 + 2\alpha_1)(\sin 2\alpha_2 + 2\alpha_2) - K_2\sin 2\alpha_2 + K_1\sin 2\alpha_1] \\ + D_1[K_2(1 + \cos 2\alpha_2) + K_1(1 + \cos 2\alpha_1)] = 0, \\ A_1[K_2(1 - \cos 2\alpha_2) + K_1(1 - \cos 2\alpha_1)] \\ + D_1[2(\sin 2\alpha_1 + 2\alpha_1)(\sin 2\alpha_2 + 2\alpha_2) - K_2\sin 2\alpha_2 + K_1\sin 2\alpha_1] = 0, \end{aligned} \tag{6.4.2}$$

where

$$K_1 = \frac{4\gamma_1}{E(1 + \nu_2) - (1 + \nu_1)}, \qquad E = E_1/E_2. \tag{6.4.3}$$

$$K_2 = \frac{4\gamma_2 E}{E(1 + \nu_2) - (1 + \nu_1)},$$

For a non-trivial solution in A_1 and D_1, the determinant of the coefficients in (6.4.2) must be identically zero and therefore α_1 and α_2 satisfy

$$K_1K_2\sin^2(\alpha_1 - \alpha_2) - [(1 - K_2)\sin 2\alpha_2 + 2\alpha_2][(1 + K_1)\sin 2\alpha_1 + 2\alpha_1] = 0. \tag{6.4.4}$$

Equation (6.4.4) can be written in terms of α_1 (or α_2) alone by making use of (6.4.1).

(b) Skew-Symmetric Deformation

In skew-symmetric deformation, u is an odd function of z in each layer and w is an even function. Following the procedure given in case (a), we find that α_1 and α_2 satisfy the transcendental equation

$$K_1K_2\sin^2(\alpha_1 - \alpha_2) - [(1 - K_2)\sin 2\alpha_2 - 2\alpha_2][(1 + K_1)\sin 2\alpha_1 - 2\alpha_1] = 0. \tag{6.4.5}$$

For numerical purposes we denote

$$\frac{\alpha_1}{h_1} = \frac{\alpha_2}{h_2} = \frac{\alpha}{h_1 + h_2}, \tag{6.4.6}$$

and eliminate α_1 and α_2 from Equations (6.4.4) and (6.4.5). The values of α which satisfy (6.4.4) and (6.4.5) are given in Tables 6.1 and 6.2 for various material properties and loading conditions and in each case $h_1/h_2 = 1$. The evaluation of α involves the calculation of the expressions in (6.4.4) and (6.4.5) at discrete points in the Argand diagram and then a search for a change in sign of neighbouring points.

We note that as E increases the decay rates become smaller. When E is infinite we have, from (6.4.3), that

$$K_1 \to 0, \quad K_2 \to \frac{4\gamma_2}{1 + \nu_2}, \tag{6.4.7}$$

and by substituting (6.4.7) into (6.4.4) and (6.4.5) we obtain

$$\sin 2\alpha_2 \pm \frac{1 + \nu_2}{(1 + \nu_2) - 4\gamma_2} . 2\alpha_2 = 0, \tag{6.4.8}$$

where the upper sign is associated with symmetric deformation and the lower sign with skew-symmetric deformation. These equations are the eigenvalue equations for a homogeneous strip with built-in edges and are discussed at a later stage.

6.5 A LAMINATED SEMI-INFINITE STRIP CONSISTING OF A FINITE NUMBER OF LAYERS

We consider a semi-infinite laminated strip which consists of three layers which are bonded at their common interfaces. It is assumed that the laminate is symmetrical about its mid-plane in geometry and mechanical properties and in the analysis to follow, the subscript (1) will be used to denote the field quantities in the inner layer.

	GENERALISED PLANE STRESS		PLANE STRAIN	
E	$\nu_1 = 0.3,\ \nu_2 = 0.3$	$\nu_1 = 0.3,\ \nu_2 = 0.5$	$\nu_1 = 0.3,\ \nu_2 = 0.3$	$\nu_1 = 0.3,\ \nu_2 = 0.5$
2	2.61046	2.42739	2.55318	2.13602
5	2.23721	2.01234	2.12534	1.39829
10	2.09455	1.84696	1.95623	1.00325
10^2	1.95462	1.67993	1.78608	0.32193
10^3	1.93991	1.66204	1.76790	0.10196
10^5	1.93828	1.66005	1.76588	complex

Table 6.1. *Values of α Satisfying (6.4.4)*

	GENERALISED PLANE STRESS		PLANE STRAIN	
E	$\nu_1 = 0.3,\ \nu_2 = 0.3$	$\nu_1 = 0.3,\ \nu_2 = 0.5$	$\nu_1 = 0.3,\ \nu_2 = 0.3$	$\nu_1 = 0.3,\ \nu_2 = 0.5$
2	2.48586	2.36055	2.44324	2.31364
5	1.74305	1.64110	1.68818	1.58690
10	1.28892	1.20777	1.24005	1.16079
10^2	0.42722	0.39802	0.40793	0.38000
10^3	0.13579	0.12643	0.12955	0.12061

Table 6.2. *Values of α Satisfying (6.4.5)*

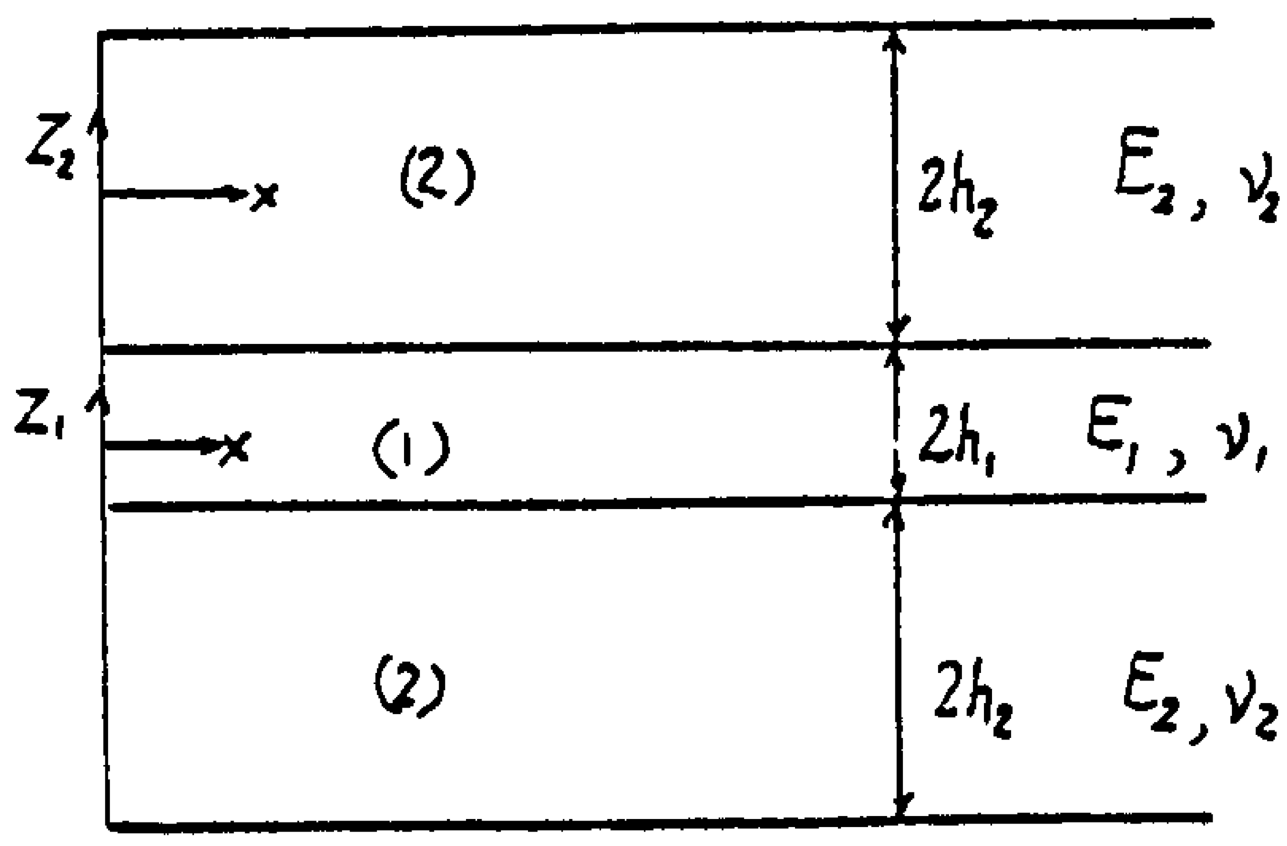


Figure 6.3. A Laminated Semi-Infinite Strip

The external lateral surfaces of the laminate are traction-free and loads are applied at the end $x = 0$ where (x, z_ℓ) is a local co-ordinate system for the ℓth layer. Assuming the applied loads to the outer layer are the same, the solution to any loading can be obtained by superposition of the solution for the symmetric deformation of the inner layer with that of the solution for skew-symmetric deformation. Since this decomposition into two separate problems is possible, we need only consider the top half of the laminate; field quantities in the lower half are defined by symmetry.

For both the symmetric and skew symmetric problems, we take the general solution (6.2.9), (6.2.10) to be the solution in each layer. The arbitrary constants a_j, in this solution, are determined by the following boundary conditions:

(i) continuity of the displacement components at each interface;

(ii) continuity of the normal and shear stress components at each interface

together with

(i) u even, w odd functions of z in the inner layer for symmetric deformation;

(ii) u odd, w even for functions of z in the inner layer for skew symmetric deformation.

From (6.3.1) we have that for the symmetric problem

$$B_2 = D_2 = 0, \quad A_2 = B_1, \quad C_2 = D_1, \quad C_1 = -D_1, \quad A_1 = B_1, \qquad (6.5.1)$$
$$\frac{\alpha_1}{h_1} = \frac{\alpha_2}{h_2},$$

and for the skew symmetric problem

$$B_2 = D_2 = 0, \quad A_2 = B_1, \quad C_2 = D_1, \quad C_1 = D_1, \quad A_1 = -B_1, \qquad (6.5.2)$$
$$\frac{\alpha_1}{h_1} = \frac{\alpha_2}{h_2}.$$

In each of (6.5.1) and (6.5.2) we have six conditions for the eight constants A_ℓ, B_ℓ, C_ℓ and D_ℓ, $\ell = 1,2$. Two additional conditions follow from considering the continuity in the displacement components. From (6.3.4) we have that

$$A_1\left(2 + \frac{K_1 \sin 2\alpha_1}{(\sin 2\alpha_1 \pm 2\alpha_1)} - \frac{K_2 \sin^2 2\alpha_2}{d_2}\right)$$
$$+ D_1\left(\frac{K_1(1 \pm \cos 2\alpha_1)}{\sin 2\alpha_1 \pm 2\alpha_1} \pm \frac{K_2(\sin 4\alpha_2 - 4\alpha_2)}{2d_2}\right) = 0$$
$$A_1\left(\frac{K_1(1 \mp \cos 2\alpha_1)}{\sin 2\alpha_1 \pm 2\alpha_1} \mp \frac{K_2(\sin 4\alpha_2 + 4\alpha_2)}{2d_2}\right) \qquad (6.5.3)$$
$$+ D_1\left(2 + \frac{K_1 \sin 2\alpha_1}{\sin 2\alpha_1 \pm 2\alpha_1} - \frac{K_2 \sin^2 2\alpha_2}{d_2}\right) = 0,$$

where K_1 and K_2 are defined by (6.4.3) and

$$d_2 = (\sin 2\alpha_2 + 2\alpha_2)(\sin 2\alpha_2 - 2\alpha_2). \tag{6.5.4}$$

For non-trivial values for A_1 and D_1, the determinant of the coefficients in (6.5.3) must vanish. Hence α_1 and α_2 satisfy

$$2K_1K_2\sin 2\alpha_2 \cos 2(\alpha_1 + \alpha_2) \pm 4K_1K_2\alpha_2 \mp 8K_1d_2\alpha_1$$

$$+ (\sin 2\alpha_1 \pm 2\alpha_1)(K_2^2 + 4d_2(1 + K_1 - K_2) - 16K_2\alpha_2^2) = 0, \tag{6.5.5}$$

where the upper sign is associated with the symmetric deformation of the inner layer and the lower sign is associated with skew symmetric deformation. The characteristic equation (6.5.5) can be written in terms of α_1 (or α_2) by using (6.5.1) and (6.5.2). The roots of this equation are located in the four quadrants of the complex plane, placed symmetrically with respect to the real and imaginary axes. For our purposes it is sufficient to confine attention to the roots in the first quadrant.

6.6 THE ROOTS OF THE CHARACTERISTIC EQUATION

Before we evaluate the roots to Equation (6.5.5) we consider the following special cases.

CASE A. $E \to 1$, $\nu_1 \to \nu_2$

From (6.4.3) we find that $K_1, K_2 \to \infty$ as $E \to 1$ and $\nu_1 \to \nu_2$. By dividing the characteristic equation by K_1K_2 and letting K_1 and K_2 tend to infinity, we obtain

$$\sin 2\alpha \pm 2\alpha = 0, \tag{6.6.1}$$

where

$$\frac{\alpha}{2h_2 + h_1} = \frac{\alpha_1}{h_1} = \frac{\alpha_2}{h_2}.$$

Equations (6.6.1) are the Fadle-Papkovitch eigenvalue equations for a homogeneous isotropic strip with traction-free lateral surfaces. The roots of these equations, which do not depend on the elastic moduli of the strip, have been tabulated extensively elsewhere and we give in Table 6.3 the first ten non-zero roots for reference purposes.

n	Re(α)	Im(α)	n	Re(α)	Im(α)
1	2.106196	1.125364	1	3.748838	1.384339
2	5.356269	1.551574	2	6.949980	1.676105
3	8.536682	1.775544	3	10.119259	1.858384
4	11.699178	1.929404	4	13.277273	1.991571
5	14.854060	2.046853	5	16.429871	2.096625
6	18.004932	2.141891	6	19.579408	2.183397
7	21.153414	2.221723	7	22.727036	2.257320
8	24.300343	2.290552	8	25.873384	2.321714
9	27.446203	2.351048	9	29.018831	2.378758
10	30.591295	2.405013	10	32.163662	2.429958
	(a)			(b)	

Table 6.3. The First Ten Non-Zero Roots of

(a) $\sin 2\alpha + 2\alpha = 0$, *(b)* $\sin 2\alpha - 2\alpha = 0$

The asymptotic location of the roots is given by

$$\alpha \sim (n \mp \tfrac{1}{4})\pi + \frac{i}{2}\ln(4n \mp 1)\pi, \quad n = 1,2, \ldots .$$

CASE B. $E \to 0$

In this case $K_1 \to -4\gamma_1/(1 + \nu_1)$ and $K_2 \to 0$ and the characteristic equation becomes

$$\sin 2\alpha_1 \pm \left(\frac{1 + \nu_1}{(1 + \nu_1) - 4\gamma_1}\right) 2\alpha_1 = 0. \tag{6.6.2}$$

This equation is identical to that found for a laminate of an infinite number of layers under the same limiting conditions. The eigenvalues given by (6.6.2) are associated with the decay rates of the stress components for a soft core with built-in lateral surfaces. For symmetric deformation of the inner layer, the leading eigenvalue is real except for when $\nu_1 = \frac{1}{2}$ and a state of plane strain exists. in Table 6.4 the values of α_1 are given for various values of the Poisson ratio ν_1.

The leading root of the skew-symmetric deformation problem is always complex since $(1 + \nu_1)/[(1 + \nu_1) - 4\gamma_1]$ is negative for all allowable values of ν_1. As $\nu_1 \to \frac{1}{2}$, there is a discontinuity in α_1 for the symmetric plane strain deformation. If ν_1 is close to zero, the leading real root can be shown to be also close to zero. Therefore, if we replace $\sin 2\alpha_1$ by the first two terms of its Taylor expansion in (6.6.2) and solve the resulting equation we obtain

$$\alpha_1 \sim \sqrt{\left(\frac{3(1 - 2\nu_1)}{(3 - 4\nu_1)}\right)}, \text{ as } \nu_1 \to \tfrac{1}{2}. \tag{6.6.3}$$

Hence, for a built-in strip which is nearly incompressible there is a slow decay rate for the stress components.

ν_1	α_1 (Plane Strain)	α_1 (G. Plane Stress)
0	1.139431	1.139431
0.05	1.111925	1.113320
0.1	1.080327	1.086420
0.15	1.043564	1.058649
0.2	1.000119	1.029917
0.25	0.947747	1.000119
0.3	0.882931	0.969133
0.35	0.779674	0.936818
0.4	0.686295	0.903003
0.45	0.513369	0.867486
0.5	COMPLEX	0.830017

Table 6.4. The Roots of (6.6.2). Symmetric Deformation of the Inner Layer.

CASE C. $E \to \infty$

From 6.4.3 we obtain $K_1 \to 0$, $K_2 \to 4\gamma_2/1 + \nu_2$. The characteristic equation in this case reduces to

$$2(K_2 - 1)\cos 4\alpha_2 + (K_2 - 1)^2 + 1 - 4(2\alpha_2)^2 = 0. \qquad (6.6.4)$$

This equation is the eigenvalue equation for a homogeneous elastic strip bonded to a rigid foundation and may be verified directly by taking the general solution for the strip and setting $u = v = 0$

on one lateral surface whilst keeping the other traction-free. In Table 6.5, the first ten non-zero roots to (6.6.4) are given for various values of the Poisson ratio ν_2.

ν_2	α_2 (Plane Strain)	α_2 (G. Plane Stress)
0	0.594808	0.594808
0.05	0.569850	0.571013
0.1	0.545930	0.550214
0.15	0.522805	0.531773
0.2	0.500280	0.515239
0.25	0.478190	0.500280
0.3	0.456392	0.486644
0.35	0.434756	0.474135
0.4	0.413155	0.462598
0.45	0.391462	0.451907
0.5	0.369543	0.441958

Table 6.5. The Roots of (6.6.4).

An approximate value for the complex roots can be obtained by an asymptotic expansion of (6.6.5).

$$\alpha_2 \sim \frac{n\pi}{2} - \frac{\ln(2n\pi) - \frac{1}{2}\ln(K_2 - 1)}{2n\pi} + \frac{i}{4}\ln\left(\frac{4n^2\pi^2}{K_2 - 1}\right). \qquad (6.6.5)$$

On comparing the values given in Tables 6.3 to 6.5, it is seen that for values of the Poisson ratio in the range 0.2-0.4, the exponential decay of the stress components in a strip bonded to a rigid foundation is about four times slower than that for a homogeneous strip and two times slower than that for a strip with built-in edges. This means that the usual approximation of neglecting end effects at distances of about one width from the end $x = 0$ is justified for the homogeneous strip, but this approximation can only be acceptable at distances of about two times larger for a strip with built-in edges and four times larger for a strip bonded to a rigid foundation.

The solution to the characteristic equation for intermediate values of E has been carried out numerically. If we write

$$\frac{\alpha_1}{h_1} = \frac{\alpha_2}{h_2} = \frac{\alpha}{2h_2 + h_1} \tag{6.6.6}$$

and eliminate α_1 and α_2 from (6.5.5) we obtain

$$D(\alpha, h_1/h_2, \nu_1, \nu_2, E) = 0. \tag{6.6.7}$$

The roots of this equation have been evaluated for the limiting cases $E \to 0$, 1 and ∞ and using these values as starting values the roots of (6.6.7) for intermediate values of E can be evaluated iteratively. In Figure 6.4 the roots are given for various values of Poisson ratio in each layer when the thickness of each layer is the same and the deformation is symmetric.

We note that when $E = 1$ and $\nu_1 = \nu_2 = 0.3$ the value of α given by Figure 6.4 characterises the decay rate in a homogeneous isotropic strip and only when $E \ll 1$ or $E \gg 1$ does there appear to be any significant change in this value of α. Therefore, apart from these extreme values of E, the decay rate for the stress components in each layer of the laminated strip can be approximated reasonably well by the decay rate for a homogeneous strip.

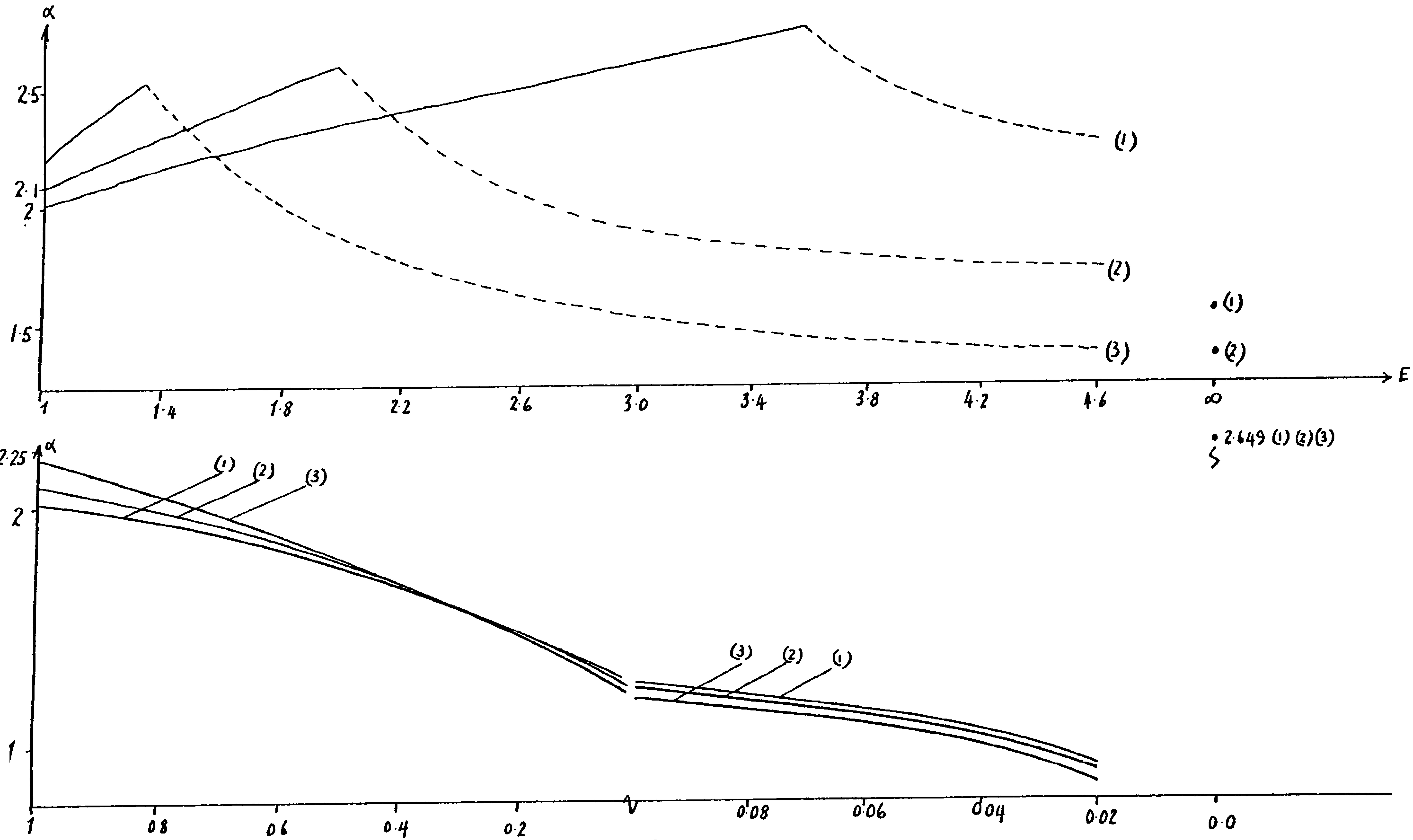


Figure 6.4

The real part of the leading root of (6.5.5) when $h_1/h_2 = 1$.

------------ real root   *————— complex root.*

(1) $\nu_1 = 0.3$ $\nu_2 = 0.1$ *(2) $\nu_1 = \nu_2 = 0.3$*   *(3) $\nu_1 = 0.3$ $\nu_2 = 0.5$*

6.7 SATISFACTION OF THE END CONDITIONS

Associated with each eigenvalue satisfying the characteristic equation (6.5.5) there is an eigenfunction F(z) in each strip which satisfies all the through thickness continuity and boundary conditions, but does not satisfy any obvious orthogonality condition. We have now to combine these eigenfunctions in such a manner that the boundary conditions at $x = 0$ are satisfied. If α is an eigenvalue then so is its complex conjugate so we can isolate the real part of the complex solution.

The stresses in each strip are given by (6.2.9) and the a_i in these expressions follow from (6.3.2).

INNER LAYER

(i) Symmetric Deformation

$$a_1 = A_1 \frac{2(\alpha_1 \cos \alpha_1 + \sin \alpha_1) - 2(D_1/A_1)\alpha_1 \sin \alpha_1}{\sin 2\alpha_1 + 2\alpha_1},$$

$$a_4 = A_1 \frac{2\sin \alpha_1 + 2(D_1/A_1)\cos \alpha_1}{\sin 2\alpha_1 + 2\alpha_1}, \qquad (6.7.1)$$

$$a_2 = a_3 = 0.$$

(ii) Skew-Symmetric Deformation

$$a_2 = A_1 \frac{2\cos \alpha_1 + 2(D_1/A_1)\sin \alpha_1}{\sin 2\alpha_1 - 2\alpha_1},$$

$$a_3 = A_1 \frac{2(\alpha_1 \sin \alpha_1 - \cos \alpha_1) - 2(D_1/A_1)\alpha_1 \cos \alpha_1}{\sin 2\alpha_1 - 2\alpha_1},$$

$$a_1 = a_4 = 0.$$

OUTER LAYER

$$a_1 = A_1 \frac{\pm(\alpha_2\cos\alpha_2 + \sin\alpha_2) + (D_1/A_1)\alpha_2\sin\alpha_2}{\sin 2\alpha_2 + 2\alpha_2},$$

$$a_2 = A_1 \frac{\pm\cos\alpha_2 + (D_1/A_1)\sin\alpha_2}{\sin 2\alpha_2 - 2\alpha_2},$$

$$a_3 = A_1 \frac{\pm(\alpha_2\sin\alpha_2 - \cos\alpha_2) - (D_1/A_1)\alpha_2\cos\alpha_2}{\sin 2\alpha_2 - 2\alpha_2}, \tag{6.7.2}$$

$$a_4 = A_1 \frac{\pm\sin\alpha_2 - (D_1/A_1)\cos\alpha_2}{\sin 2\alpha_2 + 2\alpha_2},$$

where the upper signs are associated with symmetric deformation. The ratio D_1/A_1 depends on the eigenvalue pair (α_1,α_2) and the elastic moduli as follows

$$\frac{D_1}{A_1} = \frac{(\sin 2\alpha_1 \pm 2\alpha_1)(2K_2\sin^2 2\alpha_2 - 4d_2) - 2K_1d_2\sin 2\alpha_1}{2K_1d_2(1 \pm \cos 2\alpha_1) \pm K_2(\sin 4\alpha_2 - 4\alpha_2)(\sin 2\alpha_1 \pm 2\alpha_1)}. \tag{6.7.3}$$

The end condition at x = 0 may be given as a traction/displacement condition. Suppose we confine ourselves to applied tractions of which the details are determined from the theory of Chapter 4.

For a particular eigenvalue we have, from (6.2.5), that when $x = 0$

$$\sigma_{xx}^{(1)} = F_1'', \qquad \sigma_{xz}^{(1)} = (\alpha_1/h_1)F_1',$$
$$\sigma_{xx}^{(2)} = F_2'', \qquad \sigma_{xz}^{(2)} = (\alpha_2/h_2)F_2'. \tag{6.7.4}$$

Furthermore, since the z components of stress are continuous at each interface and that the lateral surfaces of the laminate are traction-free

$$F_2(h_2) = F_2'(h_2) = 0,$$
$$F_2(-h_2) = F_1(h_1), \qquad F_2'(-h_2) = F_1'(h_1). \tag{6.7.5}$$

In addition, if the deformation of the middle layer is symmetric, then

$$F_1'(0) = 0, \tag{6.7.6}$$

and if it is skew-symmetric, then

$$F_1(0) = F_1''(0) = 0. \tag{6.7.7}$$

It follows from (6.7.5), (6.7.6) and (6.7.7) that on any plane normal to the plane of the laminate we have

$$\int \sigma_{xx} dz = \int z\sigma_{xx} dz = \int \sigma_{xz} dz = 0, \tag{6.7.8}$$

and therefore any traction applied to the end $x = 0$ must be self-equilibriating.

Our prime interest is to examine the edge effects of the applied tractions and without loss of generality we restrict ourselves to the symmetric deformation of the inner layer. Suppose that on the end $x = 0$ the following tractions are applied

$$\sigma_{xx}^{(1)} = P_1(z_1), \qquad \sigma_{xx}^{(2)} = P_2(z_2),$$
$$\sigma_{xz}^{(1)} = Q_1(z_1), \qquad \sigma_{xz}^{(2)} = Q_1(z_2), \tag{6.7.9}$$

where P_i and Q_i satisfy conditions (6.7.8). If we denote

$$A_1^{(k)} = b^{(k)} + ic^{(k)} \qquad \text{kth eigenvalue,}$$

in which $b^{(k)}$ and $c^{(k)}$ are real, then we require to choose $b^{(k)}$ and $c^{(k)}$ such that

$$\left.\begin{aligned}\sum_k (b^{(k)} n_2^{(k)} + c^{(k)} m_2^{(k)}) &= P_2(z_2),\\ \sum_k (b^{(k)} s_2^{(k)} + c^{(k)} t_2^{(k)}) &= Q_2(z_2),\end{aligned}\right\} -h_2 \leq z_2 \leq h_2,$$

$$\left.\begin{aligned}\sum_k (b^{(k)} n_1^{(k)} + c^{(k)} m_1^{(k)}) &= P_1(z_1),\\ \sum_k (b^{(k)} s_1^{(k)} + c^{(k)} t_1^{(k)}) &= Q_1(z_1),\end{aligned}\right\} 0 \leq z_1 \leq h_1 \qquad (6.7.10)$$

where the summation is over eigenvalues

$$\begin{aligned} n^{(k)} - im^{(k)} &= (2a_4^{(k)} - a_1^{(k)} - a_2^{(k)} \frac{\alpha^{(k)}z}{h})\cos(\frac{\alpha^{(k)}z}{h}) \\ &\quad - (2a_2^{(k)} + a_3^{(k)} + a_4^{(k)} \frac{\alpha^{(k)}z}{h})\sin(\frac{\alpha^{(k)}z}{h}), \\ s^{(k)} - it^{(k)} &= (a_2^{(k)} + a_3^{(k)} + \frac{\alpha^{(k)}z}{h} a_4^{(k)})\cos(\frac{\alpha^{(k)}z}{h}) \\ &\quad - (a_1^{(k)} - a_4^{(k)} + a_2^{(k)} \frac{\alpha^{(k)}z}{h})\sin(\frac{\alpha^{(k)}z}{h}), \end{aligned} \qquad (6.7.11)$$

and the $a_j^{(k)}$ are given by Equations (6.7.1) and (6.7.2).

Given N eigenfunctions there are in general 2N real coefficients to be evaluated and these have to be chosen in such a manner that Equations (6.7.10) are satisfied. It will not be possible to satisfy (6.7.10) exactly by using only a finite number of terms in the series expansions, except for special choices of P_i and Q_i, and so the 2N real coefficients are determined so as to minimize the error incurred in truncating the series.

One technique is direct collocation in which the truncated series is identically satisfied at a suitable set of points on the end $x = 0$. Suppose 2N points are chosen on the end $x = 0$, then by forming a system of 2N equations for the 2N unknowns, the coefficients can be evaluated. However, since no control is maintained on the expansions between the chosen boundary points, the resulting expressions for the stress components become highly oscillatory (Smith 1970). It is therefore necessary to overdetermine the system by choosing 2P points, $P > N$, to determine the expansion coefficients. A technique available for solving overdetermined systems is the least squares method from which the values obtained will result in the mean square error of the solution at the selected points to be minimized.

Suppose

$$\begin{aligned} M = & \{\sum_k (b^{(k)} n_2^{(k)} + c^{(k)} m_2^{(k)}) - P_2\}^2 + \{\sum_k (b^{(k)} n_1^{(k)} + c^{(k)} m_1^{(k)}) - P_1\}^2 \\ & + \{\sum_k (b^{(k)} s_2^{(k)} + c^{(k)} t_2^{(k)}) - Q_2\}^2 + \{\sum_k (b^{(k)} s_1^{(k)} + c^{(k)} t_1^{(k)}) - Q_1\}^2, \end{aligned} \tag{6.7.12}$$

then we require to choose $b^{(k)}$ and $c^{(k)}$ so that $M = 0$ for all z_1 and z_2. In practice we have to take a finite number of eigenvalues and therefore it will not be possible to achieve $M = 0$ for all z_1 and z_2. We therefore choose P points at $x = 0$ in each layer and minimize M at these points. Let

$$M^* = \sum_{j=1}^{P} M, \tag{6.7.13}$$

then a necessary condition for M^* to be a minimum is that

$$\frac{\partial M^*}{\partial b^{(k)}} = 0, \qquad \frac{\partial M^*}{\partial c^{(k)}} = 0, \qquad k = 1, \ldots, N, \tag{6.7.14}$$

which gives the following system of linear equations from which $b^{(k)}$ and $c^{(k)}$ can be determined

$$\begin{bmatrix} \sum_{j=1}^{p}\phi^{(1)}_{11}, & \ldots, & \sum_j \phi^{(1)}_{1N}, & \sum_j \phi^{(2)}_{11}, & \ldots, & \sum_j \phi^{(2)}_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ \sum_j \phi^{(1)}_{N1}, & \ldots, & \sum_j \phi^{(1)}_{NN}, & \sum_j \phi^{(2)}_{N1}, & \ldots, & \sum_j \phi^{(2)}_{NN} \\ & & & & & \\ \sum_j \phi^{(2)}_{11}, & \ldots, & \sum_j \phi^{(2)}_{N1}, & \sum_j \phi^{(3)}_{11}, & \ldots, & \sum_j \phi^{(3)}_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ \sum_j \phi^{(2)}_{1N}, & \ldots, & \sum_j \phi^{(2)}_{NN}, & \sum_j \phi^{(3)}_{N1}, & \ldots, & \sum_j \phi^{(3)}_{NN} \end{bmatrix} \begin{bmatrix} b^{(1)} \\ \vdots \\ b^{(N)} \\ \\ c^{(1)} \\ \vdots \\ c^{(N)} \end{bmatrix} = \begin{bmatrix} \sum_j d^{(1)}_1 \\ \vdots \\ \sum_j d^{(1)}_N \\ \\ \sum_j d^{(2)}_1 \\ \vdots \\ \sum_j d^{(2)}_N \end{bmatrix} \tag{6.7.15}$$

where

$$\phi^{(1)}_{rk} = n_1^{(r)} n_1^{(k)} + n_2^{(r)} n_2^{(k)} + s_2^{(r)} s_2^{(k)} + s_1^{(r)} s_1^{(k)},$$

$$\phi^{(2)}_{rk} = n_2^{(r)} m_2^{(k)} + n_1^{(r)} m_1^{(k)} + s_2^{(r)} t_2^{(k)} + s_1^{(r)} t_1^{(k)},$$

$$\phi^{(3)}_{rk} = m_1^{(r)} m_1^{(k)} + m_2^{(r)} m_2^{(k)} + t_1^{(r)} t_1^{(k)} + t_2^{(r)} t_2^{(k)}, \tag{6.7.16}$$

$$d_r^{(1)} = n_2^{(r)} P_2 + n_1^{(r)} P_1 + s_2^{(r)} Q_2 + s_1^{(r)} Q_1,$$

$$d_r^{(2)} = m_2^{(r)} P_2 + m_1^{(r)} P_1 + t_2^{(r)} Q_2 + t_1^{(r)} Q_1.$$

If any of the constants $b^{(k)}$, $c^{(k)}$ are zero (this situation arises when an eigenvalue is purely real or imaginary) then the partial derivative of M^* in (6.7.14) will not exist. In this case the row and column containing the zero constant would be excluded from the system of Equations (6.7.15).

An alternative method to the above, which will not be employed in this chapter, is to choose $c^{(k)}$ and $b^{(k)}$ so that the resultant traction given by the series expansion in a particular layer is identical to the resultant applied traction to that layer. The analysis is the same as the above except that M* in (6.7.13) is taken to be

$$M^* = \{\sum_k (b^{(k)}\hat{n}_2^{(k)} + c^{(k)}\hat{m}_2^{(k)}) - \hat{P}_2\}^2 + \{\sum_k (b^{(k)}\hat{s}_2^{(k)} + c^{(k)}\hat{t}_2^{(k)}) - \hat{Q}_2\}^2$$
$$+ \{\sum_k (b^{(k)}\hat{n}_1^{(k)} + c^{(k)}\hat{m}_1^{(k)} - \hat{P}_1\}^2 + \{\sum_k (b^{(k)}\hat{s}_1^{(k)} + c^{(k)}\hat{t}_1^{(k)}) - \hat{Q}_1\}^2, \quad (6.7.17)$$

where

$$\hat{P}_i = \int P_i \, dz, \quad \hat{Q}_i = \int Q_i \, dz, \quad (6.7.18)$$

and

$$\hat{n}_1^{(k)} - i\hat{m}_1^{(k)} = \frac{h_1}{\alpha_1^{(k)}}\left(\frac{D_1}{A_1}\right)_k, \quad \hat{n}_2^{(k)} - i\hat{m}_2^{(k)} = -\frac{h_2}{\alpha_2^{(k)}}\left(\frac{D_1}{A_1}\right)_k$$

$$\hat{s}_1^{(k)} - i\hat{t}_1^{(k)} = \frac{h_1}{\alpha_1^{(k)}}\{1 - \frac{2(\alpha_1^{(k)}\cos\alpha_1^{(k)} + \sin\alpha_1^{(k)}) - 2\,\alpha_1^{(k)}\sin\alpha_1^{(k)}(D_1/A_1)_k}{\sin 2\alpha_1^{(k)} + 2\alpha_1^{(k)}}\}$$
$$\hat{s}_2^{(k)} - i\hat{t}_2^{(k)} = -\frac{h_2}{\alpha_2^{(k)}}. \quad (6.7.19)$$

6.8 NUMERICAL RESULTS

In this section we consider the plane strain stretching of a semi-infinite laminated strip. For computational purposes we take

$$\nu_1 = \nu_2 = 0.3, \quad E = E_1/E_2 = 0.9, \quad h_1/h_2 = 1. \tag{6.8.1}$$

By substituting (6.6.6) into (6.5.5) we obtain the characteristic equation from which the first ten non-zero values for α are

$$\begin{aligned}\alpha = \; & 2.050511 + i1.169687, \\ & 5.457683 + i1.480544, \\ & 8.458945 + i1.514367, \\ & 11.610842 + i2.293294, \\ & 15.482404 + i1.712755, \\ & 17.414171 + i1.294812, \\ & 21.058802 + i2.867641, \\ & 24.443933 + i0.0, \\ & 25.691215 + i2.602456, \\ & 27.773502 + i0.0.\end{aligned} \tag{6.8.2}$$

We note that since the eigenfunctions contain terms of the form $\sin(\alpha z)$, they become large in magnitude as $\mathrm{Re}(\alpha)$ becomes large. At the end $x = 0$ we prescribe the following traction

$$P_1 = 2\cos \pi x, \quad P_2 = -\cos \pi x, \quad Q_1 = Q_2 = 0. \tag{6.8.3}$$

Since any prescribed traction can be expressed as a Fourier series, we can regard (6.8.3) as being a typical term of a Fourier expansion.

The solution vector of (6.7.15) is evaluated by the computer program of Appendix 2. The values given by (6.8.1) are taken to be the initial data for this program together with the number of eigenvalues of the truncated series solution and the number of points taken on the boundary x = 0. In Figure 6.5 we show, as the solid line, the applied tractions at x = 0 on the upper half of the laminate. The plotted points in this figure are values for the normal and shear stresses which are obtained by substituting the solution vector into the expressions for the stresses and then evaluating the stresses at discrete points.

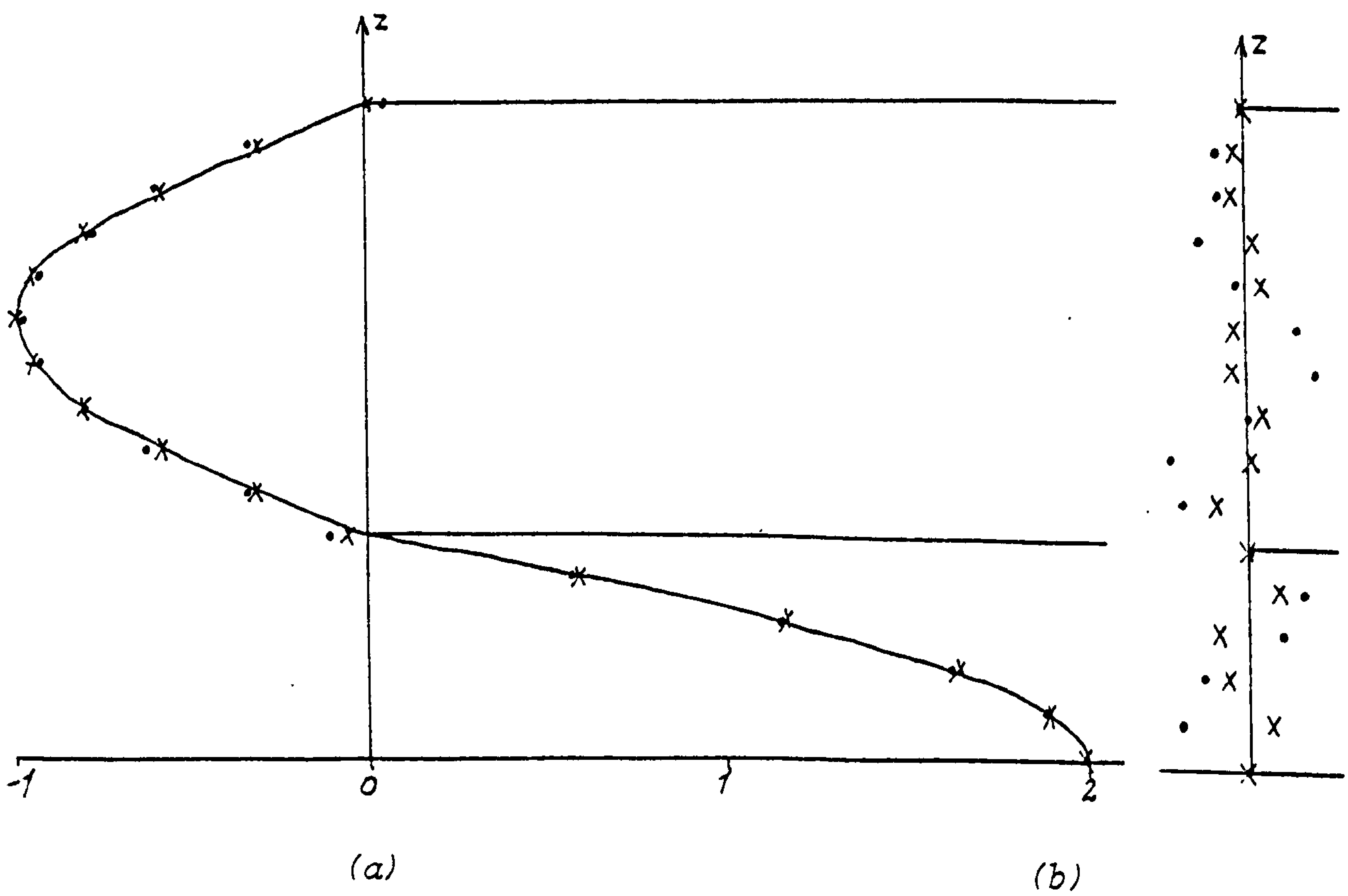


Figure 6.5. *The Stress Components at $x = 0$*
(a) σ_{xx}, (b) σ_{xz}

. - 5 Eigenvalues *x - 10 Eigenvalues*
20 Points in Each Layer

We note that intermediate values of the plotted points have been evaluated to check that the values given here represent a global picture of the computed stresses. It is seen that the variation in the computed normal traction with the number of eigenvalues is small compared with the variation in the shear traction and that as the number of eigenvalues is increased, the shear stress approaches the values of the applied shear stress. The worst 'fit' in the computed normal stress is at the interface of the layers and this is a direct consequence of the fact that the eigenfunctions in σ_{xx} are discontinuous here. In Table 6.6 the solution vector is given when 10 eigenvalues and 20 points in each layer are taken. Increasing the number of points results in about 5% change in the values of this solution vector.

{1.03, -1.52, 0.61, 0.17, -0.48, 0.28, 0.15, 0.02, -0.02, 0.03
-1.52, 1.02, 3.07, 0.30, -0.36, 2.48, 0.04, 0.00, 0.00, 0.00}

Table 6.6. The Solution Vector for 10 Eigenvalues and 20 Points in Each Layer

For the above example the normal stress σ_{zz} on the interface is plotted in Figure 6.6 by taking the first ten eigenfunctions of the series expansion

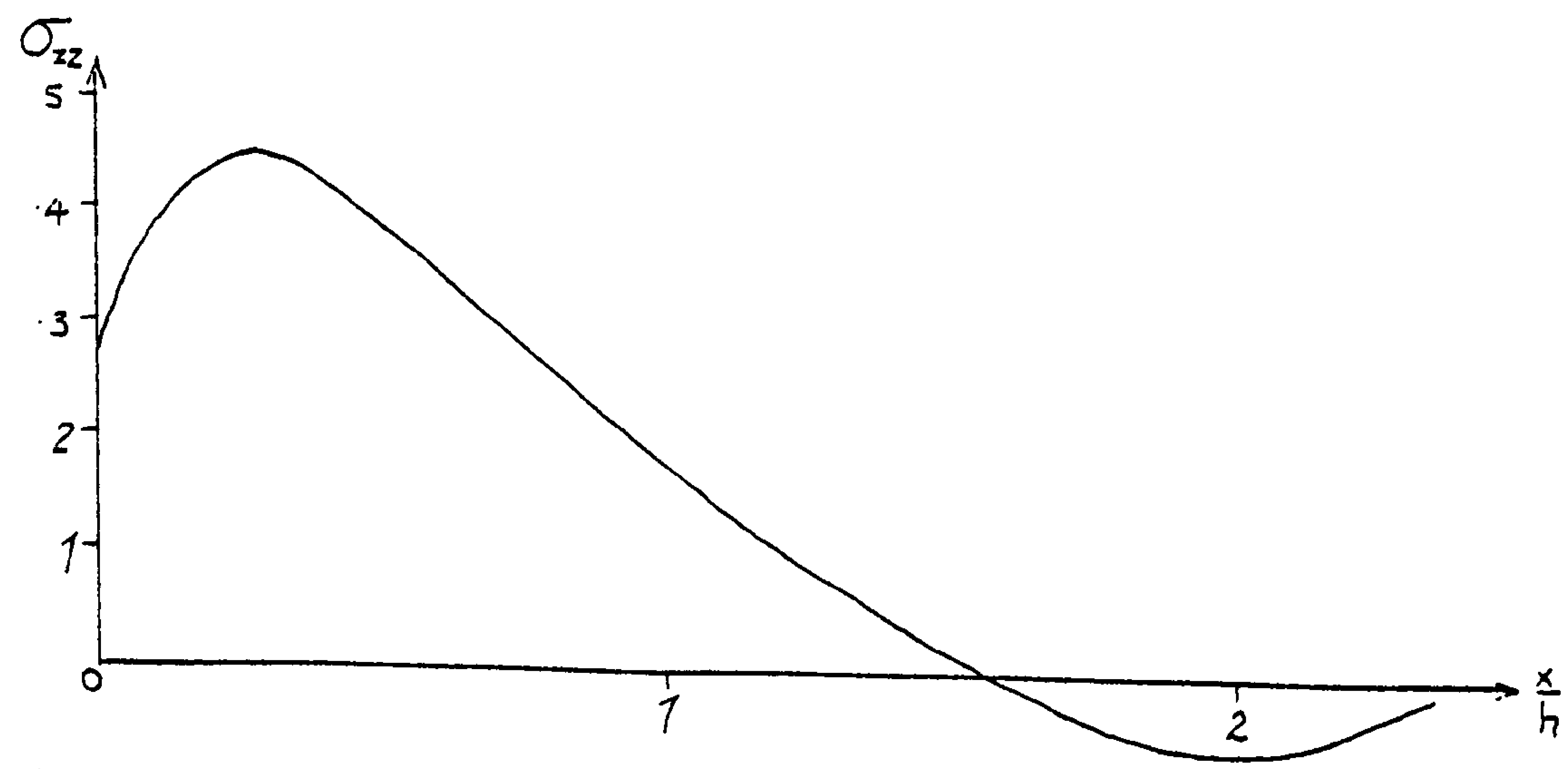


Figure 6.6. σ_{zz} at the Interface of the Layers.

When (x/h) is less than 1.6 the stress is tensile, whereas for (x/h) greater than 1.6 it becomes compressive; reaching a peak at about x/h = 1.8. The 'oscillatory' behaviour of the graph is because the solution contains terms of the form $e^{-\alpha x}$ and α, for the most part of the series, is complex.

The singularity which has been shown to exist on the interface of two quarter planes at x = 0 (c.f. Chapter 3) is not present in this analysis and this may be because the higher order eigenfunctions have not been accounted for. Furthermore, it is not known that if the higher order eigenfunctions are included there will be a reduction in the error to the extent that the series converge at $x = 0$. However, this method can be easily applied to any end problem, it does characterize the decay rates of the stress components, and can be modified to account for more than three layers.

APPENDIX 1

PROGRAM LAMINATE

This program is written for the stretching of an N layer laminated plate by assuming that the general solution in each layer is of the form given in Section 5.5. By satisfying the through thickness conditions of Section 5.7, the stress components in each layer are evaluated.

Input to the program is the

(i)	number of layers in the laminate	N
(ii)	half-width of each layer	H(N)
(iii)	characteristic in-plane length	CL
(iv)	material constants of a fibre reinforced material which has a preferred direction parallel to the x-axis	K11,K12,K23 K22,K66
(v)	angle made by the fibres of each layer with the x-axis	ANGLE (N)
(vi)	equation of the curve to which tractions are applied	ALPHA
(vii)	complex potentials of the equivalent plate and its first and second derivatives.	FI1 FI2,FI3

In the following, a description of the subroutines used in the program is given.

SUBROUTINE DATA (ANGLE, H, EP, CL, HH, N)

Input to this routine is the in-plane characteristic length, number of layers and the orientation of the fibres in each layer. On exit

EP(N) contains the aspect ratio of each layer

HH is the half-width of the equivalent plate.

SUBROUTINE MATCONST (ANGLE, N, Q, QH, RH, CT, CK, H, HH)

Using the data in this routine for an 0 degree layer and the orientation of the fibres in each layer, the elastic constants c_{ij} of each layer are evaluated. On exit

Q(I,J,K) contains the Q matrix for the Kth layer

CT(I,K),CK(I,K) are certain combinations of the c_{ij} in the Kth layer

QH(I,J) contains the $\hat{Q}$ matrix

RH(I,J) contains the $\hat{R}$ matrix.

SUBROUTINE EIGROOTS (RH, SH)

On exit SH(I) contains the complex roots of (5.3.9).

SUBROUTINE BOUNDARY (GAMMA, X, Y)

This routine evaluates the transformation relations for the stress components given the equation of the boundary ALPHA to which tractions are applied. For example, for a circular boundary

ALPHA = ATAN(Y/X).

Following the listing of the program, a sample of the output is shown.

FORTRAN77

```
      PROGRAM LAMINATE
      INTEGER NOLAYERS

      READ(5,*) NOLAYERS
      CALL MAINPROGRAM(NOLAYERS)

      STOP
      END
      SUBROUTINE MAINPROGRAM(N)
      INTEGER N,I,L,IQ,IX,IY,IZ,IPOINT
      REAL ANGLE(50),H(50),EP(50),Q(3,3,50)
      REAL QH(3,3),RH(3,3)
      REAL CT(3,50),CK(2,2,50),CL,HH,X,Y,Z,GM(2,3),PI
      REAL SXXL(50),SXXC(50),SXYL(50),SXYC(50)
      REAL SXZ(50),SYZ(50),SZZ(50),SYYL(50),SYYC(50)
      COMPLEX AL(2,50),D1(2,50),D2(2,50),E1(2,50),E2(2,50)
      COMPLEX F1(2,50),F2(2,50),S1(2,50),S2(2,50),S3(2,50)
      COMPLEX V1(2,50),V2(2,50),V3(2,50),V4(2,50),V5(2,50)
      COMPLEX V6(2,50),T4(2,50),T5(2,50),S6(2,50),A1(2,50)
      COMPLEX A2(2,50),A3(2,50),B1(2,50),B2(2,50),B3(2,50)
      COMPLEX C1(2,50),C2(2,50),C3(2,50),W1(2,50),W2(2,50)
      COMPLEX T6(2,50),T7(2,50),M1(2,50),M2(2,50),M3(2,50)
      COMPLEX P1(2,50),P2(2,50),P3(2,50),P4(2,50)
      COMPLEX W3(2,50),W4(2,50)
      COMPLEX TT1(2),TT2(2),DD(2),S4(2,50),S5(2,50),ZTEMP(2)
      COMPLEX FI1(2),FI2(2),FI3(2),CZ(2),CC(2),CI,CE,CA(2)
      COMPLEX TM1(2),TM2(2),TM3(2),TM4(2),TV1(2),TV2(2)
      COMPLEX SH(2),SH2(2),T1(2),T2(2),T3(3),PPH(2),QQH(2)

      CALL DATA(ANGLE,H,EP,CL,HH,N)
      CALL MATCONST(ANGLE,N,Q,QH,RH,CT,CK,H,HH)
      CALL EIGROOTS(RH,SH)
C ***********************************************************
C *                     Evaluate constants.                 *
C ***********************************************************
      DO 50 I=1,2
      SH2(I)=SH(I)*SH(I)
      T1(I)=RH(1,1)*SH2(I)+RH(1,2)-RH(1,3)*SH(I)
      T2(I)=RH(1,2)*SH2(I)+RH(2,2)-RH(2,3)*SH(I)
      T3(I)=RH(1,3)*SH2(I)+RH(2,3)-RH(3,3)*SH(I)

      PPH(I)=T1(I)
      QQH(I)=T2(I)/SH(I)

      DO 51 L=1,N
      AL(I,L)=CT(1,L)*T1(I)+CT(2,L)*T2(I)+CT(3,L)*T3(I)
      V1(I,L)=Q(1,1,L)+SH(I)*Q(1,3,L)
      V2(I,L)=Q(1,2,L)+SH(I)*Q(2,3,L)
      V3(I,L)=Q(1,3,L)+SH(I)*Q(3,3,L)
      V4(I,L)=Q(1,2,L)*SH(I)+Q(1,3,L)
      V5(I,L)=Q(2,2,L)*SH(I)+Q(2,3,L)
      V6(I,L)=Q(2,3,L)*SH(I)+Q(3,3,L)

      M1(I,L)=Q(1,1,L)*T1(I)+Q(1,2,L)*T2(I)+Q(1,3,L)*T3(I)
      M2(I,L)=Q(1,2,L)*T1(I)+Q(2,2,L)*T2(I)+Q(2,3,L)*T3(I)
      M3(I,L)=Q(1,3,L)*T1(I)+Q(2,3,L)*T2(I)+Q(3,3,L)*T3(I)

      D1(I,L)=V1(I,L)*T1(I)+V2(I,L)*T2(I)+V3(I,L)*T3(I)
```

```
      D2(I,L)=V4(I,L)*T1(I)+V5(I,L)*T2(I)+V6(I,L)*T3(I)

      E1(I,L)=AL(I,L)-CK(1,1,L)*D1(I,L)-CK(1,2,L)*D2(I,L)
      E2(I,L)=AL(I,L)*SH(I)-CK(2,1,L)*D1(I,L)
     +                                   -CK(2,2,L)*D2(I,L)
   51 CONTINUE
      IQ=1
      S1(I,IQ)=CMPLX(0.0,0.0)
      S2(I,IQ)=CMPLX(0.0,0.0)
      S3(I,IQ)=CMPLX(0.0,0.0)

      DO 52 L=1,N-1
      S1(I,L+1)=EP(L)*(AL(I,L)+S1(I,L))/EP(L+1)+AL(I,L+1)
      S2(I,L+1)=EP(L)*(D1(I,L)+S2(I,L))/EP(L+1)+D1(I,L+1)
      S3(I,L+1)=EP(L)*(D2(I,L)+S3(I,L))/EP(L+1)+D2(I,L+1)
   52 CONTINUE

      DO 53 L=1,N
      F1(I,L)=S1(I,L)-CK(1,1,L)*S2(I,L)-CK(1,2,L)*S3(I,L)
      F2(I,L)=S1(I,L)*SH(I)-CK(2,1,L)*S2(I,L)
     +                                   -CK(2,2,L)*S3(I,L)

      T4(I,L)=D1(I,L)+D2(I,L)*SH(I)
      A1(I,L)=V1(I,L)*E1(I,L)+V4(I,L)*E2(I,L)+CT(1,L)*T4(I,L)
      A2(I,L)=V2(I,L)*E1(I,L)+V5(I,L)*E2(I,L)+CT(2,L)*T4(I,L)
      A3(I,L)=V3(I,L)*E1(I,L)+V6(I,L)*E2(I,L)+CT(3,L)*T4(I,L)

      T5(I,L)=S2(I,L)+S3(I,L)*SH(I)
      B1(I,L)=V1(I,L)*F1(I,L)+V4(I,L)*F2(I,L)+CT(1,L)*T5(I,L)
      B2(I,L)=V2(I,L)*F1(I,L)+V5(I,L)*F2(I,L)+CT(2,L)*T5(I,L)
      B3(I,L)=V3(I,L)*F1(I,L)+V6(I,L)*F2(I,L)+CT(3,L)*T5(I,L)

      T6(I,L)=D1(I,L)+SH(I)*D2(I,L)
      T7(I,L)=S2(I,L)+SH(I)*S3(I,L)
   53 CONTINUE

      S6(I,N)=-0.5*T6(I,N)-T7(I,N)
      DO 54 L=N-1,1,-1
      S6(I,L)=(0.5*T6(I,L+1)-T7(I,L+1)
     +                   +S6(I,L+1))*(EP(L+1)/EP(L))**2
     +                                   -0.5*T6(I,L)-T7(I,L)
   54 CONTINUE
      DO 55  L=1,N
      C1(I,L)=CT(1,L)*S6(I,L)
      C2(I,L)=CT(2,L)*S6(I,L)
      C3(I,L)=CT(3,L)*S6(I,L)
   55 CONTINUE
   50 CONTINUE
C *************************************************************
C *             Select in-plane coordinates X,Y.              *
C *************************************************************
      PI=4.0*ATAN(1.0)
      DO 60  IX=1,2
      DO 70  IY=1,1
      X=(IX-1.0)*0.5*PI
      Y=(IY-1.0)
      CALL BOUNDARY(GM,X,Y)

      DO 80  I=1,2
      DO 81  L=1,N
```

```
      P1(I,L)=GM(1,1)*V1(I,L)+GM(1,2)*V2(I,L)+GM(1,3)*V3(I,L)
      P2(I,L)=GM(2,1)*V1(I,L)+GM(2,2)*V2(I,L)+GM(2,3)*V3(I,L)
      P3(I,L)=GM(1,1)*V4(I,L)+GM(1,2)*V5(I,L)+GM(1,3)*V6(I,L)
      P4(I,L)=GM(2,1)*V4(I,L)+GM(2,2)*V5(I,L)+GM(2,3)*V6(I,L)
      W1(I,L)=(GM(1,1)*A1(I,L)+GM(1,2)*A2(I,L)
     +                                +GM(1,3)*A3(I,L))/3.0
      W2(I,L)=(GM(2,1)*A1(I,L)+GM(2,2)*A2(I,L)
     +                                +GM(2,3)*A3(I,L))/3.0
      W3(I,L)=(GM(1,1)*C1(I,L)+GM(1,2)*C2(I,L)
     +                                +GM(1,3)*C3(I,L))*2.0
      W4(I,L)=(GM(2,1)*C1(I,L)+GM(2,2)*C2(I,L)
     +                                +GM(2,3)*C3(I,L))*2.0
   81 CONTINUE
   80 CONTINUE
C **************************************************************
C *   Apply the integral condition on the curve                *
C *          zeta=constant.                                    *
C **************************************************************
      DO 83 I=1,2
      TM1(I)=CMPLX(0.0,0.0)
      TM2(I)=CMPLX(0.0,0.0)
      TM3(I)=CMPLX(0.0,0.0)
      TM4(I)=CMPLX(0.0,0.0)
      T1(I)=CMPLX(0.0,0.0)
      T2(I)=CMPLX(0.0,0.0)

      DO 84 L=2,N
      TM1(I)=TM1(I)+EP(L)*P1(I,L)
      TM2(I)=TM2(I)+EP(L)*P2(I,L)
      TM3(I)=TM3(I)+EP(L)*P3(I,L)
      TM4(I)=TM4(I)+EP(L)*P4(I,L)
      T1(I)=T1(I)+(W1(I,L)+W3(I,L))*EP(L)**3
      T2(I)=T2(I)+(W2(I,L)+W4(I,L))*EP(L)**3
   84 CONTINUE
      TM1(I)=(2.0*TM1(I)+EP(1)*P1(I,1))*2.0*EP(1)**2
      TM2(I)=(2.0*TM2(I)+EP(1)*P2(I,1))*2.0*EP(1)**2
      TM3(I)=(2.0*TM3(I)+EP(1)*P3(I,1))*2.0*EP(1)**2
      TM4(I)=(2.0*TM4(I)+EP(1)*P4(I,1))*2.0*EP(1)**2
      T1(I)=-2.0*T1(I)-(W1(I,1)+W3(I,1))*EP(1)**3
      T2(I)=-2.0*T2(I)-(W2(I,1)+W4(I,1))*EP(1)**3
   83 CONTINUE

      DO 90 I=1,2
      DO 91 L=2,N
      TV1(I)=CMPLX(0.0,0.0)
      TV2(I)=CMPLX(0.0,0.0)
      TT1(I)=CMPLX(0.0,0.0)
      TT2(I)=CMPLX(0.0,0.0)
      DO 92 J=2,L
      TT1(I)=TT1(I)+F1(I,J-1)*EP(J-1)**2+F1(I,J)*EP(J)**2
      TT2(I)=TT2(I)+F2(I,J-1)*EP(J-1)**2+F2(I,J)*EP(J)**2
   92 CONTINUE
      S4(I,L)=TT1(I)
      S5(I,L)=TT2(I)
      TT1(I)=TT1(I)+0.5*(E1(I,1)*EP(1)**2-E1(I,L)*EP(L)**2)
      TT2(I)=TT2(I)+0.5*(E2(I,1)*EP(1)**2-E2(I,L)*EP(L)**2)
      TV1(I)=EP(L)*(P1(I,L)*TT1(I)+P3(I,L)*TT2(I))+TV1(I)
      TV2(I)=EP(L)*(P2(I,L)*TT1(I)+P4(I,L)*TT2(I))+TV2(I)
   91 CONTINUE
      TV1(I)=T1(I)-4.0*TV1(I)
```

```
      TV2(I)=T2(I)-4.0*TV2(I)
   90 CONTINUE

      DO 100 I=1,2
      DD(I)=TM1(I)*TM4(I)-TM2(I)*TM3(I)
      S4(I,1)=(TV1(I)*TM4(I)-TV2(I)*TM3(I))/DD(I)
      S5(I,1)=(TV2(I)*TM1(I)-TV1(I)*TM2(I))/DD(I)

      DO 110 L=2,N
      S4(I,L)=(S4(I,L)+0.5*(E1(I,1)*EP(1)**2
     +                   -E1(I,L)*EP(L)**2)
     +                      +S4(I,1)*EP(1)**2)/EP(L)**2
      S5(I,L)=(S5(I,L)+0.5*(E2(I,1)*EP(1)**2
     +                   -E2(I,L)*EP(L)**2)
     +                      +S5(I,1)*EP(1)**2)/EP(L)**2
  110 CONTINUE
  100 CONTINUE
C ***************************************************
C *         Evaluate the equivalent complex potential *
C * and its derivatives.                            *
C ***************************************************
      DO 120 I=1,2
      CZ(I)=X+SH(I)*Y
      CI=CMPLX(0.0,1.0)
      CC(1)=-(CI*SH(2))/(2.0*(SH(2)-SH(1)))
      CC(2)= (CI*SH(1))/(2.0*(SH(2)-SH(1)))
      CE=CEXP(CI*CZ(I))
      FI1(I)=CC(I)*CE
      FI2(I)=CI*FI1(I)
      FI3(I)=-FI1(I)
  120 CONTINUE
C ***************************************************
C * Evaluate the leading in-plane stress components.  *
C ***************************************************
      WRITE(6,123)
  123 FORMAT(/,2X,1HL,3X,3Hx/a,3X,3Hy/a,3X,3Hz/h,4X,
     +7HSxxl/Po,5X,7HSxyl/Po,5X,7HSyyl/Po,5X,7HSxxc/Po,
     +5X,7HSxyc/Po,5X,
     +7HSyyc/Po,5X,7HSxzl/Po,5X,7HSyzl/Po,5X,7HSzzl/Po)

      DO 130 L=1,N
      IPOINT=1
      IF(L.EQ.1) IPOINT=3
      SXXL(L)=0.0
      SXYL(L)=0.0
      SYYL(L)=0.0
      DO 131 I=1,2
      SXXL(L)=SXXL(L)+2.0*REAL(M1(I,L)*FI1(I))
      SYYL(L)=SYYL(L)+2.0*REAL(M2(I,L)*FI1(I))
      SXYL(L)=SXYL(L)+2.0*REAL(M3(I,L)*FI1(I))
  131 CONTINUE

      WRITE(6,9998) L,X,Y,SXXL(L),SXYL(L),SYYL(L)
 9998 FORMAT(I4,2F6.2,6X,3E12.3)
C ***************************************************
C *    Select values for the Z coordinate and evaluate*
C * and remaining stress components.                 *
C ***************************************************
      DO 132 IZ=IPOINT,5
      SXXC(L)=0.0
```

```
      SXYC(L)=0.0
      SYYC(L)=0.0
      SXZ(L)=0.0
      SYZ(L)=0.0
      SZZ(L)=0.0
      Z=(IZ-3.0)/2.0
      DO 133 I=1,2
      SXZ(L)=SXZ(L)
     +      -EP(L)*2.0*REAL((D1(I,L)*Z+S2(I,L))*FI2(I))
      SYZ(L)=SYZ(L)
     +      -EP(L)*2.0*REAL((D2(I,L)*Z+S3(I,L))*FI2(I))
      SZZ(L)=SZZ(L)
     +    +EP(L)**2*2.0*REAL((T6(I,L)*0.5*Z*Z+T7(I,L)*Z
     +                                  +S6(I,L))*FI3(I))
      SXXC(L)=SXXC(L)
     +    +EP(L)**2*2.0*REAL((A1(I,L)*0.5*Z*Z+B1(I,L)*Z
     +                         +C1(I,L)+S4(I,L)*V1(I,L)
     +                        +S5(I,L)*V4(I,L))*FI3(I))
      SYYC(L)=SYYC(L)
     +    +EP(L)**2*2.0*REAL((A2(I,L)*0.5*Z*Z+B2(I,L)*Z
     +                         +C2(I,L)+S4(I,L)*V2(I,L)
     +                        +S5(I,L)*V5(I,L))*FI3(I))
      SXYC(L)=SXYC(L)
     +    +EP(L)**2*2.0*REAL((A3(I,L)*0.5*Z*Z+B3(I,L)*Z
     +                         +C3(I,L)+S4(I,L)*V3(I,L)
     +                        +S5(I,L)*V6(I,L))*FI3(I))
  133 CONTINUE
      WRITE(6,9997) L,X,Y,Z,SXXC(L),SXYC(L),SYYC(L),
     +                       SXZ(L),SYZ(L),SZZ(L)
 9997 FORMAT(I4,3F6.2,36X,6E12.3)

  132 CONTINUE
  130 CONTINUE
   70 CONTINUE
   60 CONTINUE
      RETURN
      END
      SUBROUTINE DATA(ANGLE,H,EP,CL,HH,N)
      INTEGER L,N
      REAL ANGLE(N),H(N),EP(N),CL,HH
C ********************************************************
C * Read    CL- characteristic in-plane length,          *
C *         ANGLE(L)-orientation of the Lth layer,       *
C *         H(L)-half-width of the Lth layer.            *
C * Form    HH-half-width of the equivalent plate,       *
C *         EP(L)-epsilon of Lth layer.                  *
C ********************************************************
      READ(5,*) CL
      WRITE(6,5100) CL
 5100 FORMAT(1X,25HCHARACTERISTIC LENGTH IS ,F6.2)
      HH=0.0
      DO 10 L=1,N
      READ(5,*) ANGLE(L),H(L)
      WRITE(6,701)L,ANGLE(L),H(L)
  701 FORMAT(2X,30HANGLE OF ORIENTATION OF LAYER ,I3,
     +4H IS ,F7.2,2X,27HDEGREES. HALF-THICKNESS IS ,F6.2)
      HH=HH+2.0*H(L)
      ANGLE(L)=ANGLE(L)*ATAN(1.0)/45.0
      EP(L)=H(L)/CL
   10 CONTINUE
```

```
      HH=HH-H(1)
      RETURN
      END
      SUBROUTINE MATCONST(ANGLE,N,Q,QH,RH,CT,CK,H,HH)
      INTEGER N,I
      REAL Q(3,3,N),QH(3,3),RH(3,3),CT(3,N),CK(2,2,N)
      REAL ANGLE(N),H(N),D,HH
      REAL K11,K12,K22,K23,K44,K66,C,S,C2,S2,C4,S4,CS2

C ****************************************************
C *       The elastic constants of a 0 degree layer. *
C ****************************************************
      K11=226.40
      K12=4.370
      K22=10.57
      K23=5.64
      K44=0.5*(K22-K23)
      K66=5.66
C ****************************************************
C *              Evaluate Qij in each layer.         *
C ****************************************************
      DO 11 I=1,N
      C=COS(ANGLE(I))
      S=SIN(ANGLE(I))
      C2=C**2
      S2=S**2
      C4=C**4
      S4=S**4
      CS=C*S
      CS2=CS**2
      C11=K11*C4+2.0*(K12+2.0*K66)*CS2+K22*S4
      C22=K11*S4+2.0*(K12+2.0*K66)*CS2+K22*C4
      C16=-CS*(K11*C2-K22*S2-(C2-S2)*(K12+2.0*K66))
      C26=-CS*(K11*S2-K22*C2+(C2-S2)*(K12+2.0*K66))
      C12=CS2*(K11+K22-4.0*K66)+(C4+S4)*K12
      C66=CS2*(K11+K22-2.0*K12)+K66*(C2-S2)**2
      C13=C2*K12+S2*K23
      C23=S2*K12+C2*K23
      C36=CS*(K23-K12)
      C45=CS*(K44-K66)
      C44=C2*K44+S2*K66
      C55=S2*K44+C2*K66
      C33=K22
      WRITE(6,702)I,C11,C12,C13,C16,C22,C23,C26,C33,C36,
     +                C44,C45,C55,C66
  702 FORMAT(//,2X,23HTHE C MATRIX FOR LAYER ,
     +I2,5H IS:-/20X,3E13.4,
     +26X,E13.4/33X,2E13.4,26X,E13.4/14X,6Hx1.0E9,
     +26X,E13.4,26X,E13.4/
     +59X,2E13.4/72X,E13.4/85X,E13.4)

      Q(1,1,I)=C11-(C13*C13)/C33
      Q(1,2,I)=C12-(C13*C23)/C33
      Q(1,3,I)=C16-(C13*C36)/C33
      Q(2,1,I)=Q(1,2,I)
      Q(2,2,I)=C22-(C23*C23)/C33
      Q(2,3,I)=C26-(C23*C36)/C33
      Q(3,1,I)=Q(1,3,I)
      Q(3,2,I)=Q(2,3,I)
      Q(3,3,I)=C66-(C36*C36)/C33
```

```
      WRITE(6,703)I
  703 FORMAT(//,2X,23HTHE Q MATRIX FOR LAYER ,I2
     +                                ,14H IS (x1.0E9):-)
      WRITE(6,704) Q(1,1,I),Q(1,2,I),Q(1,3,I),Q(2,2,I),
     +             Q(2,3,I),Q(3,3,I)
  704 FORMAT(20X,3E13.4/33X,2E13.4/46X,E13.4,////)

      CT(1,I)=C13/C33
      CT(2,I)=C23/C33
      CT(3,I)=C36/C33
      D=C44*C55-C45*C45
      CK(1,1,I)=C44/D
      CK(1,2,I)=-C45/D
      CK(2,1,I)=CK(1,2,I)
      CK(2,2,I)=C55/D
   11 CONTINUE
C ***************************************************************
C *       Evaluate the equivalent elastic constants.            *
C ***************************************************************
      DO 12 I=1,3
      DO 13 J=1,3
      TEMP=0.0
      DO 14 L=2,N
      TEMP=TEMP+2.0*H(L)*Q(I,J,L)
   14 CONTINUE

      QH(I,J)=(H(1)*Q(I,J,1)+TEMP)/HH
   13 CONTINUE
   12 CONTINUE
      WRITE(6,705)
  705 FORMAT(//,2X,
     +    41HTHE Q MATRIX FOR THE EQUIVALENT PLATE IS ,
     +                                    10H(x1.0E9):-)
      WRITE(6,704)QH(1,1),QH(1,2),QH(1,3),QH(2,2)
     +                                   ,QH(2,3),QH(3,3)
      D=QH(1,1)*(QH(2,2)*QH(3,3)-QH(2,3)*QH(2,3))
     + -QH(1,2)*(QH(1,2)*QH(3,3)-QH(1,3)*QH(2,3))
     + +QH(1,3)*(QH(1,2)*QH(2,3)-QH(1,3)*QH(2,2))
      RH(1,1)=(QH(2,2)*QH(3,3)-QH(2,3)*QH(2,3))/D
      RH(1,2)=(QH(1,3)*QH(2,3)-QH(1,2)*QH(3,3))/D
      RH(1,3)=(QH(1,2)*QH(2,3)-QH(1,3)*QH(2,2))/D
      RH(2,1)=RH(1,2)
      RH(2,2)=(QH(1,1)*QH(3,3)-QH(1,3)*QH(1,3))/D
      RH(2,3)=(QH(1,2)*QH(1,3)-QH(1,1)*QH(2,3))/D
      RH(3,1)=RH(1,3)
      RH(3,2)=RH(2,3)
      RH(3,3)=(QH(1,1)*QH(2,2)-QH(1,2)*QH(1,2))/D
      WRITE(6,706)
  706 FORMAT(//,2X
     +   ,41HTHE R MATRIX FOR THE EQUIVALENT PLATE IS ,
     +                                   11H(x1.0E-9):-)
      WRITE(6,704) RH(1,1),RH(1,2),RH(1,3),RH(2,2)
     +                                   ,RH(2,3),RH(3,3)
      RETURN
      END
      SUBROUTINE EIGROOTS(RH,SH)
      INTEGER IFAIL,K,ID
      REAL RH(3,3),AR(5),REZ(5),IMZ(5),TOL
      COMPLEX SH(2)
      EXTERNAL X02AAF,C02AEF
```

```
C ***************************************************
C *   Initialize quantities for C02AEF(NAG)(routine    *
C * which locates the roots of a polynomial.           *
C ***************************************************
      ID=5
      TOL=X02AAF(0.1)
      IFAIL=0
      AR(1)=RH(1,1)
      AR(2)=-2.0*RH(1,3)
      AR(3)=2.0*RH(1,2)+RH(3,3)
      AR(4)=-2.0*RH(2,3)
      AR(5)=RH(2,2)
      WRITE(6,707) ID-1
  707 FORMAT(///,
     +  41H ORDER OF THE CHARACTERISTIC EQUATION IS ,I4
     +          /1X,12HCOEFFICIENTS/24X,9HREAL PART,4X
     +                              ,14HIMAGINARY PART,/)
      DO 41 J=1,ID
      WRITE(6,708) AR(J)
  708 FORMAT(21X,E13.4)
   41 CONTINUE

      CALL C02AEF(AR,ID,REZ,IMZ,TOL,IFAIL)
C ***************************************************
C *         The roots are REZ(I)+iIMZ(I),I=1,4          *
C ***************************************************
      WRITE(6,709)
  709 FORMAT(//,10H ROOTS ARE/24X,9HREAL PART,4X
     +                              ,14HIMAGINARY PART)
      K=1
      DO 40 J=1,4
      IF(IMZ(J).GT.0.0) THEN
                        SH(K)=CMPLX(REZ(J),IMZ(J))
                        WRITE(6,9999) SH(K)
 9999                   FORMAT(22X,E13.4,1X,E13.4)
                        K=K+1
                        ENDIF
   40 CONTINUE
      WRITE(6,517)
  517 FORMAT(///)
      RETURN
      END
      SUBROUTINE BOUNDARY(GAMMA,X,Y)
      REAL GAMMA(2,3),X,Y,ALPHA,TALPHA,CTALPHA,STALPHA

      ALPHA=2.0*ATAN(1.0)
      TALPHA=2.0*ALPHA
      CTALPHA=COS(TALPHA)
      STALPHA=SIN(TALPHA)

      GAMMA(1,1)=0.5*(1.0+CTALPHA)
      GAMMA(1,2)=0.5*(1.0-CTALPHA)
      GAMMA(1,3)=STALPHA
      GAMMA(2,1)=-0.5*STALPHA
      GAMMA(2,2)=-GAMMA(2,1)
      GAMMA(2,3)=CTALPHA
      RETURN
      END
C ***************************************************
```

```
****************************************************************
* The following is a sample of the output produced             *
* by PROGRAM LAMINATE for a three-layer laminate               *
* containing a circular hole and subjected to uniaxial         *
* tension Po in the x direction.                               *
* Lay-up of the laminate is [90,0]s.                           *
****************************************************************
CHARACTERISTIC LENGTH IS 1.0
LAYER   ANGLE OF ORIENTATION(DEGREES)     HALF-THICKNESS
  1             0.0                             1.0
  2            90.0                             1.0

THE C MATRIX FOR LAYER 1 IS:-
       0.32D+2 0.44D+1 0.44D+1                     0.00D+0
               0.11D+2 0.56D+1                     0.00D+0
                       0.11D+2                     0.00D+0
x1.0E9                         0.25D+1 0.00D+0
                                       0.57D+1
       symm.                                       0.57D+1

THE Q MATRIX FOR LAYER 1 IS:-
       0.30D+2 0.20D+1 0.00D+0
x1.0E9         0.76D+1 0.00D+1
       symm.           0.57D+1

THE C MATRIX FOR LAYER 2 IS:-
       0.11D+2 0.44D+1 0.56D+1                     0.00D+0
               0.32D+2 0.44D+1                     0.00D+0
x1.0E9                 0.11D+2                     0.00D+0
                               0.57D+1 0.00D+0
                                       0.25D+1
       symm.                                       0.57D+1

THE Q MATRIX FOR LAYER 2 IS:-
       0.76D+1 0.20D+1 0.00D+0
x1.0E9         0.30D+2 0.00D+1
       symm.           0.57D+1

THE Q MATRIX FOR THE EQUIVALENT PLATE IS
       0.15D+2 0.20D+1 0.00D+0
x1.0E9         0.22D+2 0.00D+0
       symm.           0.57D+1

ROOTS OF THE CHARACTERISTIC EQUATION
               REAL PART       IMAGINARY PART
                  0.0             0.1457D+1
                  0.0             0.5611D+0
**********************************************************************
                ....LOWEST ORDER......   ....HIGHER ORDER......
L x/a y/a z/h   Sxx/Po  Sxy/Po  Syy/Po  Sxx/Po  Syz/Po  Szz/Po
1 1.6 0.0 0.0   1.55    0.00    0.40    -4.16   0.00    -0.09
1 1.6 0.0  .5                           -3.90   0.07    -0.08
1 1.6 0.0  1.                           -3.12   0.15    -0.06
2 1.6 0.0 -1.   0.45    0.00    1.30    -0.83   0.15    -0.06
2 1.6 0.0 -.5                           -0.09   0.11    -0.03
2 1.6 0.0 0.0                            0.36   0.07    -0.01
2 1.6 0.0  .5                            0.52   0.04    -0.00
2 1.6 0.0 1.0                            0.39   0.00     0.00
**********************************************************************
```

APPENDIX 2

PROGRAM FIT

This program calculates the solution vector to the system of Equation (6.7.15) given the material properties and thickness of each layer and the applied tractions at the end x = 0 to each layer. Note that the eigenvalues are not calculated by this program and that this data must be supplied by the user. By using the solution vector the stress components in each layer of the laminated strip are determined at x = 0. The routines used by the program are.

SUBROUTINE ELASCON (K1,K2,E,ZNU1,XNU2)

On entry the Youngs modulus ratio E and the Poisson ratio XNU1, XNU2 in each layer must be specified. On exit, the values of K_1 and K_2 given by (6.4.3) are held in K1, K2.

SUBROUTINE APPTRAC (P,Q,LAYER,Z)

This routine contains the expressions for the applied tractions P_i and Q_i given by (6.7.9).

SUBROUTINE SCOF (XN,XM,XS,XT,A1,A2,A3,A4,EIG,ZOH,LAYER,WR)

This routine evaluates the expressions for n, m, s and t in (6.7.11). On entry an eigenvalue (EIG), the value of z/h (ZOH) and the layer number (LAYER) must be specified

SUBROUTINE CONST (A1,A2,A3,A4,EIG,EL1,EL2,LAYER,WR)

This routine evaluates the arbitrary constants a_i given by (6.7.1) and (6.7.2) and returns the values in A1, A2, A3 and A4.

SUBROUTINE DECOMP (A,XU,XL,N)

Given a matrix A(N x N), this routine decomposes it into an upper (XU) and lower (XL) triangular matrix such that

$$A = (XU)(XL).$$

SUBROUTINE MATMULT (A,B,C,N)

This routine multiplies B(N x N) with C(N x N) and returns the result in A. That is

$$A = BC.$$

SUBROUTINE INVERSE (A,AI,N,IFLAG)

This routine calculates the inverse of the triangular matrix A(N x N) and returns the result in AI. Note that

IFLAG = 0 when A is upper triangular,

IFLAG = 1 when A is lower triangular.

SUBROUTINE MATCOL (A,B,C,N)

This routine calculates the product of matrix B(N x N) with the vector C(N x 1) and returns the result in A(N x 1). That is

$$A = BC.$$

FORTRAN77

```
      PROGRAM FIT
      INTEGER N,R,K,PTS,P,L,M,ICONT,IFLAG(50),MK,D,IPTS
      REAL A(50,50),AU(50,50),AL(50,50),AUI(50,50),ALI(50,50)
      REAL REIG,IEIG,E,XNU1,XNU2,WR,H1,H2,K1,K2,XNR,XNK
      REAL XMR,XMK,XSR,XSK,XTR,XTK,TEMP1,TEMP2,TEMP3,ZOH
      REAL TEMP4,TEMP5,AP,AQ,TRAC(50),SOL(50),SXX,SXZ
      COMPLEX EIG(50),A1R,A2R,A3R,A4R,A1K,A2K,A3K,A4K
C ***********************************************************
C *                   Set elastic constants.                *
C *    E=E1/E2.....  Ratio of youngs moduli                 *
C *    XNU1,XNU2...  Poisson ratio in each layer            *
C *    H1,H2.......  Half-width of each layer               *
C ***********************************************************
      E=0.9
      XNU1=0.3
      XNU2=0.3
      H1=1.0
      H2=1.0
      WR=H1/H2
      WRITE(6,9998)
 9998 FORMAT(36X,35HPLANE STRAIN:SYMMETRIC DEFORMATION:,///)
      WRITE(6,9999) WR,E,XNU1,XNU2
 9999 FORMAT(36X,19HWIDTH RATIO(h1/h2)=,F6.2,5X
     +                                  ,14HYOUNGS MODULUS,
     + 14H RATIO(E1/E2)=,F6.2/36X,21HPOISSON RATIO(INNER)=,
     +              F6.2,5X,21HPOISSON RATIO(OUTER)=,F6.2,/)
      CALL ELASCON(K1,K2,E,XNU1,XNU2)
C ***********************************************************
C *   N.... Number of non-zero eigenvalues                  *
C *   PTS.. Number of points taken at x=0 in each layer     *
C ***********************************************************
      READ(5,*) N,PTS
      WRITE(6,9997) N
 9997 FORMAT(36X,5HFIRST,I4,26H NON ZERO EIGENVALUES ARE:)
C ***********************************************************
C *   ICONT... Number of eigenvalues which are real         *
C *   IFLAG... Marker for the real eigenvalues              *
C ***********************************************************
      ICONT=0
      DO 10 I=1,N
      READ(5,*) REIG,IEIG
      IF(ABS(IEIG).LT.1.0E-8)THEN
                             ICONT=ICONT+1
                             IFLAG(ICONT)=I
                             ENDIF
      EIG(I)=CMPLX(REIG,IEIG)
      WRITE(6,9996) REIG,IEIG
 9996 FORMAT(62X,F10.6,2H+i,F10.6)
   10 CONTINUE
C ***********************************************************
C *   Set a marker for the next complex eigenvalue          *
C ***********************************************************
      IFLAG(ICONT+1)=0
      WRITE(6,9995) PTS
 9995 FORMAT(1H1,///,10X,I4,26H POINTS TAKEN IN A LAYER.  ,///)
      DO 20 R=1,N
      TEMP4=0.0
      TEMP5=0.0
```

```
      DO 30 K=1,N
      TEMP1=0.0
      TEMP2=0.0
      TEMP3=0.0
      DO 40 L=1,2
      CALL CONST(A1R,A2R,A3R,A4R,EIG(R),K1,K2,L,WR)
      CALL CONST(A1K,A2K,A3K,A4K,EIG(K),K1,K2,L,WR)
      DO 50 P=1,PTS
      IF(L.EQ.1) ZOH=(P-1.0)/(PTS-1.0)
      IF(L.EQ.2) ZOH=(2.0*P-(1.0+PTS))/(PTS-1.0)
      CALL SCOF(XNR,XMR,XSR,XTR,A1R,A2R,A3R,A4R,EIG(R),ZOH,L,WR)
      CALL SCOF(XNK,XMK,XSK,XTK,A1K,A2K,A3K,A4K,EIG(K),ZOH,L,WR)
      TEMP1=XNR*XNK+XSR*XSK+TEMP1
      TEMP2=XNR*XMK+XSR*XTK+TEMP2
      TEMP3=XMR*XMK+XTR*XTK+TEMP3
      IF(K.EQ.1) THEN
                 CALL APPTRAC(AP,AQ,L,ZOH)
                 TEMP4=XNR*AP+XSR*AQ+TEMP4
                 TEMP5=XMR*AP+XTR*AQ+TEMP5
                 ENDIF
   50 CONTINUE
   40 CONTINUE
C *************************************************************
C * Store the system of equations in A(2N,2N) and TRAC(2N).  *
C *************************************************************
      A(R,K)=TEMP1
      A(R,K+N)=TEMP2
      A(K+N,R)=TEMP2
      A(R+N,K+N)=TEMP3
   30 CONTINUE
      TRAC(R)=TEMP4
      TRAC(R+N)=TEMP5
   20 CONTINUE
      WRITE(6,8000)
 8000 FORMAT(2X,33HTHE A MATRIX WITHOUT ADJUSTMENTS:)
C     DO 7 I=1,2*N
C     WRITE(6,11)(A(I,J),J=1,2*N),TRAC(I)
   11 FORMAT(50F7.2)
C   7 CONTINUE
      IF(ICONT.EQ.0) THEN
                     WRITE(6,8001)
 8001 FORMAT(2X,24HNO ADJUSTMENTS REQUIRED.)
                     GOTO 105
                     ENDIF
      WRITE(6,8002)
 8002 FORMAT(2X,18HADJUSTED A MATRIX:)
      MK=1
      DO 100 J=N+1,2*N
      IF((IFLAG(MK)+N).EQ.J) THEN
                             MK=MK+1
                             GOTO 100
                             ENDIF
      DO 101 I=1,2*N
      A(I,J+1-MK)=A(I,J)
      A(J+1-MK,I)=A(I,J)
  101 CONTINUE
      TRAC(J+1-MK)=TRAC(J)
  100 CONTINUE
```

```
C ***************************************************************
C *                Define  D to be 2N-ICONT                     *
C ***************************************************************
      D=2*N-ICONT
C     DO 103 I=1,D
C     WRITE(6,11)(A(I,J),J=1,D),TRAC(I)
C 103 CONTINUE
      WRITE(6,13)
   13 FORMAT(///,2X,21HDECOMPOSITION BEGINS:)
  105 D=2*N-ICONT
      DO 5 I=1,D
      DO 6 J=1,D
      AU(I,J)=0.0
      AL(I,J)=0.0
      AUI(I,J)=0.0
      ALI(I,J)=0.0
    6 CONTINUE
    5 CONTINUE
      CALL DECOMP(A,AU,AL,D)
      CALL INVERSE(AU,AUI,D,0)
      CALL INVERSE(AL,ALI,D,1)
C     WRITE(6,111)
C     DO 104 I=1,D
C     WRITE(6,11)(AU(I,J),J=1,D)
C 104 CONTINUE
C     WRITE(6,111)
C     DO 107 I=1,D
C     WRITE(6,11)(AL(I,J),J=1,D)
C 107 CONTINUE
      CALL MATMULT(AU,AUI,ALI,D)
C     CALL MATMULT(AUI,A,AU,D)
C     WRITE(6,111)
C     DO 108 I=1,D
C     WRITE(6,11)(AUI(I,J),J=1,D)
C 108 CONTINUE
  111 FORMAT(///)
      CALL MATCOL(SOL,AU,TRAC,D)
      WRITE(6,8007)
 8007 FORMAT(2X,16HSOLUTION VECTOR:)
      WRITE(6,11)(SOL(I),I=1,D)
      WRITE(6,111)
C ***************************************************************
C *        Re-introduce zeros in the solution vector.           *
C ***************************************************************
      IF(ICONT.EQ.0) GOTO 9111
      WRITE(6,8009)
 8009 FORMAT(2X,38HRE-INTRODUCE ZEROS IN SOLUTION VECTOR:)
      MK=1
      DO 200 J=N+1,2*N
      IF((IFLAG(MK)+N).EQ.J) THEN
                             MK=MK+1
                             GOTO 200
                             ENDIF
      SOL(J)=SOL(J+1-MK)
  200 CONTINUE
      DO 201 I=1,ICONT
      SOL(IFLAG(I)+N)=0.0
  201 CONTINUE
      WRITE(6,11)(SOL(I),I=1,2*N)
      WRITE(6,111)
```

```
 9111 WRITE(6,187)
  187 FORMAT(4X,3Hz/h,6X,3HSxx,6X,8HSxx(app),7X,3HSxz,
     +        6X,8HSxz(app))
C ***********************************************************
C * IPTS..... Controls the number of output points in       *
C *           each layer.                                   *
C ***********************************************************
      IPTS=2*PTS
      DO 180 L=2,1,-1
      DO 181 P=IPTS,1,-1
      IF(L.EQ.1) ZOH=(P-1.0)/(IPTS-1.0)
      IF(L.EQ.2) ZOH=(2.0*P-(1.0+IPTS))/(IPTS-1.0)
      TEMP1=0.0
      TEMP2=0.0
      DO 182 K=1,N
      CALL CONST(A1K,A2K,A3K,A4K,EIG(K),K1,K2,L,WR)
      CALL SCOF(XNK,XMK,XSK,XTK,A1K,A2K,A3K,A4K,EIG(K),ZOH,L,WR)
      TEMP1=SOL(K)*XNK+SOL(K+N)*XMK+TEMP1
      TEMP2=SOL(K)*XSK+SOL(K+N)*XTK+TEMP2
  182 CONTINUE
      SXX=TEMP1
      SXY=TEMP2
      CALL APPTRAC(AP,AQ,L,ZOH)
      WRITE(6,185) ZOH,SXX,AP,SXY,AQ
  185 FORMAT(2X,F6.3,4F12.4)
  181 CONTINUE
  180 CONTINUE
      STOP
      END
      SUBROUTINE MATCOL(A,B,C,N)
      INTEGER N,I
      REAL A(50),B(50,50),C(50),TEMP
      DO 715 I=1,N
      TEMP=0.0
      DO 716 J=1,N
      TEMP=B(I,J)*C(J)+TEMP
  716 CONTINUE
      A(I)=TEMP
  715 CONTINUE
      RETURN
      END
      SUBROUTINE ELASCON(K1,K2,E,XNU1,XNU2)
      REAL K1,K2,E,XNU1,XNU2,GAMMA1,GAMMA2,TP
      TP=E*(1.0+XNU2)-(1.0+XNU1)
C ***********************************************************
C *   GAMMA =....  1-nu**2 for plane strain                 *
C *                1       for generalised plane stress     *
C ***********************************************************
      GAMMA1=1.0-XNU1*XNU1
      GAMMA2=1.0-XNU2*XNU2
      K1=(4.0*GAMMA1)/TP
      K2=(4.0*GAMMA2*E)/TP
      RETURN
      END
      SUBROUTINE APPTRAC(P,Q,LAYER,Z)
C ***********************************************************
C *   Define the applied tractions at x=0 on each layer.    *
C ***********************************************************
      INTEGER LAYER
      REAL P,Q,Z
      IF(LAYER.EQ.2) GOTO 650
```

```
      P=2.0
      Q=0.0
      GOTO 651
  650 P=-1.0
      Q=0.0
  651 RETURN
      END
      SUBROUTINE SCOF(XN,XM,XS,XT,A1,A2,A3,A4,EIG,ZOH,LAYER,WR)
      INTEGER LAYER
      REAL ZOH,WR,XN,XM,XS,XT
      COMPLEX EIG,A1,A2,A3,A4,T1,T2,T3,T4,AL(2),AZH
      AL(1)=(EIG*WR)/(2.0+WR)
      AL(2)=EIG/(2.0+WR)
      AZH=AL(LAYER)*ZOH
      T1=(2.0*A4-A1-A2*AZH)*COS(AZH)
      T2=(2.0*A2+A3+A4*AZH)*SIN(AZH)
      T3=(A2+A3+A4*AZH)*COS(AZH)
      T4=(A1-A4+A2*AZH)*SIN(AZH)
      XN= REAL(T1-T2)
      XM=-AIMAG(T1-T2)
      XS= REAL(T3-T4)
      XT=-AIMAG(T3-T4)
      RETURN
      END
      SUBROUTINE CONST(A1,A2,A3,A4,EIG,EL1,EL2,LAYER,WR)
      INTEGER LAYER
      REAL WR,EL1,EL2
      COMPLEX A1,A2,A3,A4,EIG,AL1,AL2,TA1,TA2,C2A1,C2A2,S2A1
      COMPLEX C1,C2,C3,C4,D1,D2,RT1,RT2,RAT,S2A2
      AL1=(EIG*WR)/(2.0+WR)
      AL2= EIG/(2.0+WR)
      TA1=2.0*AL1
      TA2=2.0*AL2
      C2A1=COS(TA1)
      C2A2=COS(TA2)
      S2A1=SIN(TA1)
      S2A2=SIN(TA2)
      C1=S2A1+TA1
      C2=1.0+COS(TA1)
      C3=S2A2+TA2
      C4=S2A2-TA2
      D1=2.0*(S2A2*C2A2-TA2)
      D2=C3*C4
      RT1=C1*(2.0*EL2*S2A2*S2A2-4.0*D2)-2.0*EL1*D2*S2A1
      RT2=2.0*EL1*D2*C2+EL2*D1*C1
      RAT=RT1/RT2
      IF(LAYER.EQ.1) GOTO 550
      A1=(AL2*COS(AL2)+SIN(AL2)+RAT*AL2*SIN(AL2))/C3
      A2=(COS(AL2)+RAT*SIN(AL2))/C4
      A3=(AL2*SIN(AL2)-COS(AL2)-RAT*AL2*COS(AL2))/C4
      A4=(SIN(AL2)-RAT*COS(AL2))/C3
      GOTO 560
  550 A1=2.0*(AL1*COS(AL1)+SIN(AL1)-RAT*AL1*SIN(AL1))/C1
      A2=0.0
      A3=0.0
      A4=2.0*(SIN(AL1)+RAT*COS(AL1))/C1
  560 RETURN
      END
      SUBROUTINE DECOMP(A,XU,XL,N)
      INTEGER N,I,J,K
      REAL A(50,50),XU(50,50),XL(50,50),TEMP1,TEMP2
```

```
      DO 500 I=1,N
      XU(1,I)=A(1,I)
      XL(I,1)=A(I,1)/XU(1,1)
  500 CONTINUE
      DO 501 I=2,N
      DO 502 J=I,N
      TEMP1=0.0
      TEMP2=0.0
      DO 503 K=1,I-1
      TEMP1=XL(I,K)*XU(K,J)+TEMP1
  503 CONTINUE
      XU(I,J)=A(I,J)-TEMP1
      DO 504 K=1,I-1
      TEMP2=XL(J,K)*XU(K,I)+TEMP2
  504 CONTINUE
      XL(J,I)=(A(J,I)-TEMP2)/XU(I,I)
  502 CONTINUE
  501 CONTINUE
      RETURN
      END
      SUBROUTINE INVERSE(A,AI,N,IFLAG)
      INTEGER N,I,J,K,IFLAG
      REAL A(50,50),AI(50,50),TEMP1,TEMP2
      DO 600 I=1,N
      AI(I,I)=1.0/A(I,I)
  600 CONTINUE
      DO 601 L=1,N-1
      DO 602 I=1,N-L
      J=I+L
      IF(I.EQ.J) GOTO 602
      IF(IFLAG.EQ.0) GOTO 604
      TEMP1=0.0
      DO 603 K=I,J-1
      TEMP1=-A(J,K)*AI(K,I)+TEMP1
  603 CONTINUE
      AI(J,I)=TEMP1/A(J,J)
      GOTO 602
  604 TEMP2=0.0
      DO 607 K=I+1,J
      TEMP2=-A(I,K)*AI(K,J)+TEMP2
  607 CONTINUE
      AI(I,J)=TEMP2/A(I,I)
  602 CONTINUE
  601 CONTINUE
      RETURN
      END
      SUBROUTINE MATMULT(A,B,C,N)
      INTEGER N,I,J,K
      REAL A(50,50),B(50,50),C(50,50),TEMP
      DO 700 I=1,N
      DO 701 J=1,N
      TEMP=0.0
      DO 703 K=1,N
      TEMP=B(I,K)*C(K,J)+TEMP
  703 CONTINUE
      A(I,J)=TEMP
  701 CONTINUE
  700 CONTINUE
      RETURN
      END
C ***************************************************************
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