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LIST-COLOURINGS OF NEAR-OUTERPLANAR GRAPHS

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to my father

Abstract

A list-colouring of a graph is an assignment of a colour to each vertex v from its own list $L(v)$ of colours. Instead of colouring vertices we may want to colour other elements of a graph such as edges, faces, or any combination of vertices, edges and faces. In this thesis we will study several of these different types of list-colouring, each for the class of a near-outerplanar graphs. Since a graph is outerplanar if it is both K_4-minor-free and $K_{2,3}$-minor-free, then by a near-outerplanar graph we mean a graph that is either K_4-minor-free or $K_{2,3}$-minor-free.

Chapter 1 gives an introduction to the area of graph colourings, and includes a review of several results and conjectures in this area. In particular, four important and interesting conjectures in graph theory are the List-Edge-Colouring Conjecture (LECC), the List-Total-Colouring Conjecture (LTCC), the Entire Colouring Conjecture (ECC), and the List-Square-Colouring Conjecture (LSCC), each of which will be discussed in Chapter 1. In Chapter 2 we include a proof of the LECC and LTCC for all near-outerplanar graphs. In Chapter 3 we will study the list-colouring of a near-outerplanar graph in which vertices and faces, edges and faces, or vertices, edges and face are to be coloured. The results for the case when all elements are to be coloured will prove the ECC for all near-outerplanar graphs. In Chapter 4 we will study the list-colouring of the square of a K_4-minor-free graph, and in Chapter 5 we will study the list-colouring of the square of a $K_{2,3}$-minor-free graph. In Chapter 5 we include a proof of the LSCC for all $K_{2,3}$-minor-free graphs with maximum degree at least six.

List of publications

[1] T. J. Hetherington and D. R. Woodall, Edge and total choosability of near-outerplanar graphs, *Electr. J. Combin.* **13** (2006), #R98, 7pp.

[2] T. J. Hetherington and D. R. Woodall, List-colouring the square of a K_4-minor-free graph, submitted October 2006.

[3] T. J. Hetherington and D. R. Woodall, List-colouring the square of an outerplanar graph, *Ars Combin.*, *to appear.*

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Contents

Chapter 1

Background information

1.1 Introduction

In this chapter we will give a brief history of the area of graph colourings, in particular list-colourings[1], and we will give an overview of the work contained in this thesis. This will include a review of some important results and conjectures in this area, since it is these results and conjectures that give the motivation for the work in this thesis. We will use standard terminology throughout, as can be found in the references [8, 21, 40]. However, the perhaps less well-known definitions are also included in Section 1.3 of this chapter.

1.2 Historical background

The roots of graph colouring problems can be traced back to a letter written to William Hamilton by De Morgan in 1852. The contents of this letter raised the question as to whether every map could be coloured with at most four colours so that no two countries with a border in common are given the same colour. This

[1]See Section 1.3 for definitions of graph theoretical terms.

problem is equivalent to colouring the vertices of a planar graph with at most four colours so that no two adjacent vertices are given the same colour. This problem, known as the Four Colour Theorem, was proved in 1977 by Appel, Haken and Koch [2, 3].

A year earlier, in 1976, Vizing [32] introduced the concept of a list-colouring[2], in which each vertex must be given a colour from its own list of colours so that no two adjacent vertices are given the same colour. If all lists are identical then this is equivalent to the ordinary colouring problem. Independently, in 1980, Erdős, Rubin and Taylor [11] also introduced the idea of list-colourings, and they gave examples to show that there are graphs that require more colours in each list for a list-colouring than for an ordinary colouring.

Since ordinary colourings and list-colourings were now known not to be equal, every question asked about ordinary colourings could also be asked about list-colourings. This led researchers to investigate the list-colouring analogue of the Four Colour Theorem. In 1993, Voigt [33] gave an example of a planar graph that requires more than four colours in the list of each vertex for a list-colouring. Further such examples were given by Gutner [13] and by Mirzakhani [26], both in 1996. However, it was proved by Thomassen [31] in 1994 that if each vertex of a planar graph is given a list of five colours, then each vertex can be given a colour from its list so that no two adjacent vertices are given the same colour.

An interesting problem is to investigate for which classes of graphs the number of colours needed in the list of each vertex for a list-colouring from all possible lists is the same as the number of colours needed for an ordinary colouring. Much work has been done on problems of this type, which will be reviewed in detail in Section 1.3.2. One source of information on the more recent developments in the area of graph colouring problems is [18].

[2]A more formal explanation of the ideas in this section is given in Section 1.3.

1.3 Explanation of graph theoretical terms

In this section we will give formal definitions of the terminology used throughout this thesis. A *simple graph*, or just *graph*, contains no loops or multiple edges, whereas a *multigraph* contains multiple edges and a *pseudograph* contains both loops and multiple edges. All of the work in this thesis is for simple graphs.

As usual, for a graph $G = (V, E)$, let $d_G(v)$, $\Delta(G)$, $\delta(G)$, $|V(G)|$ denote the degree of a vertex v in G, the maximum degree of G, the minimum degree of G, and the number of vertices of G respectively. Also, let K_n denote the complete graph on n vertices, and let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices. If $d_G(v) = k$ for every vertex v in G, then G is k-*regular*.

1.3.1 Basic definitions

Two vertices u, v are *adjacent* if there exists an edge $e = uv$ joining u and v, and the vertices u, v are *incident* with the edge e. Similarly, two edges are adjacent if they meet at a vertex. A graph is *planar* if it can be embedded in the plane so that no two edges intersect except at a vertex. Such an embedding is called a *plane* graph, in which two faces are adjacent if they meet at an edge, and a face is incident with the vertices and edges in its boundary. Note that a face may be adjacent to itself if there is a *cut-edge* whose removal disconnects the graph.

The *square* G^2 of a graph G has the same vertex set as G, but vertices are adjacent in G^2 if and only if they are at distance at most 2 apart in G.

Given two graphs G and H we form the *join* $G + H$ by adding an edge from each vertex of G to each vertex of H. The *union* $G \cup H$ is the graph whose components are the components of G and H. A graph is k-*connected* if the removal of fewer than k vertices does not disconnect the graph. A *block* is a 2-connected graph with at least two vertices. If a graph is 1-connected and is not 2-connected, then it contains at least two blocks and every two blocks have at most one vertex in common whose removal will disconnect the graph. Such a vertex is called a *cut-vertex*. A block that contains only one cut-vertex is called an *end-block*.

Two graphs G and H are *isomorphic*, which is denoted $G \cong H$, if there exists a one-to-one correspondence between the vertices of G and those of H such that two vertices are adjacent in G if and only if the corresponding vertices are adjacent in H. A graph H is *homeomorphic* from G if either $H = G$ or H can be obtained from G by adding vertices of degree 2 subdividing the edges of G. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, which is denoted $H \subseteq G$. If H contains all edges $uv \in E(G)$ for all $u, v \in V(H)$, then H is an *induced subgraph* of G.

To *contract* an edge $e = uv$ of a graph G, delete the edge e, identify the end-vertices u, v, and remove any multiple edges created by this operation. The resulting graph is denoted by G/e. Any graph H formed by contracting one or more edges of G is a *contraction* of G. A *subcontraction* or *minor* is a subgraph of a contraction or, equivalently, a contraction of a subgraph. If a graph G has no minor isomorphic to H, then G is *H-minor-free*.

A graph G is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the boundary of the outer face of G. Such an embedding is called an *outerplane* graph. It is well known [7] that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free. We will call a graph *near-outerplanar* if it is K_4-minor-free or $K_{2,3}$-minor-free.

1.3.2 Colourings, list-colourings and the colouring number

A *vertex colouring*, or just *colouring*, is an assignment of a colour to each vertex of a graph G. A colouring of G is *proper* if no two adjacent vertices are given the same colour. If G has a proper colouring using at most k colours, then G is *k-colourable*. The smallest integer k such that G is k-colourable is the *chromatic number* $\chi(G)$ of G.

A *list-colouring* of a graph G is an assignment of a colour to each vertex v in G from its own (unordered) list $L(v)$ of colours. We will refer to a list-colouring as simply a colouring if it is clear from the context that we mean a list-colouring. A list-colouring of G is proper if no two adjacent vertices of G are given the same colour. If $|L(v)| \geq k$ for every vertex v in G, then G is *k-choosable* if G has a proper colouring from all possible lists $L(v)$. The smallest integer k such that G is k-choosable is the *list-chromatic number* or *choosability* $\mathrm{ch}(G)$ of G.

As mentioned in Section 1.2, it is known [11] that in general the chromatic number of a graph G is not equal to the choosability of G. An easy example of this is $K_{3,3}$, which is obviously 2-colourable. If the three vertices in each partite set are given the lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, then no proper colouring exists from these lists. This shows that $K_{3,3}$ is not 2-choosable. It is also known that although every planar graph is 4-colourable [2, 3], not every planar graph is 4-choosable [33].

A graph G is *k-degenerate*, where $k \geq 0$, if every induced subgraph of G has minimum degree at most k. It follows that G can be reduced to K_1 by the successive removal of vertices whose degree is at most k, i.e., the vertices can be ordered in such a way that every vertex is preceded by at most k of its neighbours. The smallest integer k such that G is k-degenerate is the *degeneracy* of G, which is denoted $\mathrm{degeneracy}(G)$. The *colouring number* of a graph G, which is denoted $\mathrm{col}(G)$, is the least k for which the vertices can be ordered so that every vertex is preceded by fewer than k of its neighbours. It follows that $\mathrm{col}(G) = \mathrm{degeneracy}(G) + 1$.

Rather than colouring vertices, we may want to colour other elements, such as edges, faces, or any combination of vertices, edges and faces. In this thesis we will study several of these different types of list-colouring, each for the class of near-outerplanar graphs. Each of these types of colouring together with the associated chromatic numbers and list-chromatic numbers will be reviewed in Section 1.4. For example, $\chi_{\mathrm{ef}}(G)$ is the edge-face chromatic number, where the subscript denotes the elements that are to be coloured. In a proper colouring of more than one type of element, no two adjacent or incident elements can be given the same colour. By an abuse of terminology we will call two elements *neighbours* if they are adjacent or incident, since no two such elements can be given the same colour.

1.4 Review of different types of colourings and associated conjectures

In this section we will review the different types of list-colouring that are to be considered in this thesis. We will also discuss four important conjectures in graph theory that relate to these different types of colourings.

1.4.1 Edge colourings and the LECC

The *edge chromatic number* χ_{e} and the *edge choosability* ch_{e} are commonly denoted by χ' and ch' respectively. It was proposed independently by Vizing, by Gupta, and by Albertson and Collins, that the edge choosability is equal to the edge chromatic number. This was previously known as the List Colouring Conjecture [1, 8], and is now known as the List-Edge-Colouring Conjecture (LECC) [17, 21, 40].

Conjecture 1: The LECC. *For every multigraph G, $\mathrm{ch}'(G) = \chi'(G)$.*

Although the LECC has not been proved in general, several results have been proved about the LECC for special classes of graphs. In 1995, Galvin [12] proved the LECC for complete bipartite multigraphs, which had previously been conjectured by Dinitz in 1978. In 1996, Ellingham and Goddyn [10] proved the LECC for all d-regular d-edge colourable planar multigraphs. It was proved in 1997 by Borodin, Kostochka and Woodall [5] that the LECC holds for all planar graphs with maximum degree $\Delta \geq 12$.

In 1980, Erdős, Rubin and Taylor [11] proved the LECC for all graphs with maximum degree $\Delta = 2$. More recently, in 2001, Wang and Lih [37] proved the LECC for all outerplanar graphs with maximum degree $\Delta \geq 3$. This result had already been proved in 1999 by Juvan, Mohar and Thomas [20] since they proved the LECC for all K_4-minor-free graphs with maximum degree $\Delta \geq 3$, and all outerplanar graphs are K_4-minor-free. This completed the proof of the LECC for all K_4-minor-free graphs.

In 2006, Hetherington and Woodall [14] proved the LECC for all $K_{2,3}$-minor-free graphs. In fact, they replaced the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. The graph $\bar{K}_2 + (K_1 \cup K_2)$ can be obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2, or, alternatively, from K_4 by adding a vertex of degree 2 subdividing an edge. In Chapter 2 we include a proof of the LECC for all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs with maximum degree $\Delta \geq 3$. This will complete the proof of the LECC for all near-outerplanar graphs.

1.4.2 Total colourings and the LTCC

The *total chromatic number* χ_{ve} and the *total choosability* ch_{ve} are commonly denoted by χ'' and ch'' respectively. It was proposed independently by Borodin, Kostochka and Woodall [5], by Juvan, Mohar and Strekovski [19], and by Hilton and Johnson [17] that for every multigraph the total choosability is equal to the total chromatic number. This is known as the List-Total-Colouring Conjecture (LTCC).

CONJECTURE 2: The LTCC. *For every multigraph G, $\text{ch}''(G) = \chi''(G)$.*

Far less is known about the LTCC than the LECC. It was proved in 1997 by Borodin, Kostochka and Woodall [5] that the LTCC holds for all planar graphs with maximum degree $\Delta \geq 12$. A year later, in 1998, Juvan, Mohar and Skrekovski [19] proved the LTCC for all graphs with maximum degree $\Delta = 2$. In 2001, Wang and Lih [37] proved the LTCC for all outerplanar graphs with maximum degree $\Delta \geq 4$. More recently, in 2006, Woodall [41] proved the LTCC for all K_4-minor-free graphs with maximum degree $\Delta = 3$. Also in 2006, Hetherington and Woodall [14] proved the LTCC for all K_4-minor-free graphs with maximum degree $\Delta \geq 4$. This completes the proof of the LTCC for all K_4-minor-free graphs.

In the same paper, Hetherington and Woodall [14] proved the LTCC for all $K_{2,3}$-minor-free graphs also. In fact, again they replaced the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. In Chapter 2 we include a proof of the LTCC for all K_4-minor-free graphs with maximum degree $\Delta \geq 4$ and a proof of the LTCC for all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs with maximum degree $\Delta \geq 3$. This will complete the proof of the LTCC for all near-outerplanar graphs.

1.4.3 Coupled colourings

The *coupled chromatic number* and the *coupled choosability* are denoted by χ_{vf} and $\mathrm{ch}_{\mathrm{vf}}$ respectively. In 1965, Ringel [28] conjectured that $\chi_{\mathrm{vf}}(G) \leq 6$ for all plane graphs G. This was proved by Borodin [4] in 1984. In 1996, Wang and Liu [36] proved that if G is an outerplane graph, then $\chi_{\mathrm{vf}}(G) \leq 5$. In Chapter 3 we will prove that if G is a plane embedding of a K_4-minor-free graph or a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph, then $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$. This will prove that $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$ for all near-outerplane graphs G.

1.4.4 Edge-face colourings

The *edge-face chromatic number* and the *edge-face choosability* are denoted by χ_{ef} and $\mathrm{ch}_{\mathrm{ef}}$ respectively. In 1975, Melnikov [25] conjectured that $\chi_{\mathrm{ef}}(G) \leq \Delta + 3$ for all plane graphs G with maximum degree Δ. In 1997, two independent proofs of Melnikov's conjecture were published, one by Sanders and Zhao [29], the other by Waller [34]. In the paper by Sanders and Zhao it was conjectured also that $\chi_{\mathrm{ef}}(G) \leq \Delta + 2$ for all plane graphs G with maximum degree Δ, with the exception that $\chi_{\mathrm{ef}}(G) = \Delta + 3$ if $\Delta = 2$ and G has a component that is an odd cycle.

In 1995, Wang [35] proved that $\chi_{\mathrm{ef}}(G) \leq \Delta + 1$ for all outerplane graphs G with maximum degree $\Delta \geq 5$. In Chapter 3 we will prove that if G is a plane embedding of a K_4-minor-free graph or a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph, both with maximum degree Δ, then $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 2$ if $\Delta \geq 3$, $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 1$ if $\Delta \geq 5$, and $\chi_{\mathrm{ef}}(G) \leq \Delta + 1 = 5$ if $\Delta = 4$. We will also give conditions for the different values of $\mathrm{ch}_{\mathrm{ef}}(G)$ if $\Delta \leq 2$. Since $\chi_{\mathrm{ef}}(G) \leq \mathrm{ch}_{\mathrm{ef}}(G)$, these results will prove the conjecture of Sanders and Zhao for all near-outerplane graphs.

In view of the work in Chapter 3 we propose the following conjecture.

Conjecture 3. *If G is a near-outerplane graph with maximum degree $\Delta = 4$, then* $\mathrm{ch}_{\mathrm{ef}}(G) \le \Delta + 1 = 5$.

1.4.5 Entire colourings and the ECC

The *entire chromatic number* and the *entire choosability* are denoted by χ_{vef} and $\mathrm{ch}_{\mathrm{vef}}$ respectively. In 1972, Kronk and Mitchem [22] proposed the following conjecture, which is known as the Entire Colouring Conjecture (ECC).

Conjecture 4: The ECC. *For every plane graph G with maximum degree Δ,* $\chi_{\mathrm{vef}}(G) \le \Delta + 4$.

The ECC was proved for all plane graphs with maximum degree $\Delta \le 3$ by Kronk and Mitchem [23] in 1973. More recently, in 2000, Sanders and Zhao [30] proved that the ECC holds for all plane graphs with maximum degree $\Delta \ge 6$. The ECC is still an open problem if $\Delta = 4$ or 5. It was proved in 1992 by Wang and Zhang [38] that if G is an outerplane graph with maximum degree $\Delta \ge 5$, then $\chi_{\mathrm{vef}}(G) \le \Delta + 2$. In 2005, Wu and Wu [42] proved that $\chi_{\mathrm{vef}}(G) \le \max\{8, \Delta + 2\}$ for all plane embeddings of a K_4-minor-free graph G with maximum degree Δ. In Chapter 3 we will prove that if G is a plane embedding of a K_4-minor-free graph or a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph, both with maximum degree Δ, then $\mathrm{ch}_{\mathrm{vef}}(G) \le \max\{7, \Delta + 2\}$ if $\Delta \ge 3$. We will also give conditions for the different values of $\mathrm{ch}_{\mathrm{vef}}(G)$ if $\Delta \le 2$. Since $\chi_{\mathrm{vef}}(G) \le \mathrm{ch}_{\mathrm{vef}}(G)$, this improves the result of Wu and Wu and, as a special case, this proves the ECC for all near-outerplane graphs G.

In view of the work in Chapter 3 we propose the following conjecture.

Conjecture 5. *If G is a near-outerplane graph with maximum degree $\Delta = 3$, then $\mathrm{ch}_{\mathrm{vef}}(G) \le \Delta + 3 = 6$, with the exception that $\mathrm{ch}_{\mathrm{vef}}(G) = 7$ if G has K_4 as a component.*

1.4.6 The LSCC

In 2001, Kostochka and Woodall [21] proposed the following conjecture, known as the List-Square-Colouring Conjecture (LSCC), which implies the truth of the LTCC since the LTCC is a special case of the LSCC for bipartite graphs in which every vertex in one partite set has degree 2.

Conjecture 6: The LSCC. *For every graph G, $\mathrm{ch}(G^2) = \chi(G^2)$.*

If G has maximum degree $\Delta = 0$ or 1, then it is obvious that the LSCC holds. In 2000, Prowse and Woodall [27] proved that $\mathrm{ch}(G) = \chi(G)$ if G is the power of a cycle. This implies the truth of the LSCC for all graphs G with maximum degree $\Delta = 2$. In fact, for $\Delta = 2$, the situation is as follows:

$$\mathrm{ch}(G^2) = \chi(G^2) = \begin{cases} 3 & \text{if the length of every cycle in } G \text{ is divisible by 3;} \\ 5 & \text{if } G \text{ has } C_5 \text{ as a component;} \\ 4 & \text{otherwise.} \end{cases}$$

In Chapters 4 and 5 we will study, respectively, the square of a K_4-minor-free graph and the square of a $K_{2,3}$-minor-free graph, both with maximum degree $\Delta \ge 3$. More specifically, in Chapter 4, although we cannot prove that $\mathrm{ch}(G^2) = \chi(G^2)$ if G is a K_4-minor-free graph with maximum degree $\Delta \ge 3$, we can prove the same sharp upper bound for $\mathrm{ch}(G^2)$ as for $\chi(G^2)$.

In 2003, Lih, Wang and Zhu [24] proved for a K_4-minor-free graph G with maximum degree Δ that

$$\chi(G^2) \leq \begin{cases} \Delta + 3 & \text{if } \Delta = 2 \text{ or } 3; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \text{if } \Delta \geq 4; \end{cases}$$

and

$$\text{degeneracy}(G^2) \leq \begin{cases} \Delta + 2 & \text{if } \Delta = 2 \text{ or } 3; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \text{if } \Delta \geq 4. \end{cases}$$

It follows from the work of Lih, Wang and Zhu that $\text{ch}(G^2) \leq \Delta + 3$ if G is a K_4-minor-free graph with maximum degree $\Delta = 2$ or 3. In Chapter 4 we will prove that if G is a K_4-minor-free graph with maximum degree $\Delta \geq 4$, then $\text{ch}(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$. Furthermore, we will prove that G^2 is $\lceil \frac{3}{2}\Delta \rceil$-degenerate and as an immediate corollary that $\text{col}(G^2) \leq \lceil \frac{3}{2}\Delta \rceil + 1$. We will show that all these results are sharp.

In Chapter 5 we will prove that if G is a $K_{2,3}$-minor-free graph with maximum degree Δ, then $\Delta + 1 \leq \chi(G^2) \leq \text{ch}(G^2) \leq \Delta + 2$ if $\Delta \geq 3$ and $\text{ch}(G^2) = \chi(G^2) = \Delta + 1$ if $\Delta \geq 6$. We will also show that all these results are sharp. This will prove the LSCC for all $K_{2,3}$-minor-free graphs with maximum degree $\Delta \geq 6$.

1.5 Summary of the new results

In this section we will give a summary of the new results that are proved in this thesis.

Chapter 2

In Chapter 2 we will prove that the LECC and LTCC hold for all near-outerplanar graphs. The situation is summarised in the following theorem.

Theorem 1.5.1. [14] *The LECC and LTCC hold for all near-outerplanar graphs. In fact, if G is a near-outerplanar graph with maximum degree Δ, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta$ *and* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, *apart from the following exceptions*:

(*i*) *if* $\Delta = 1$ *then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 3$;

(*ii*) *if* $\Delta = 2$ *and G has a component that is an odd cycle, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta + 1 = 3$;

(*iii*) *if* $\Delta = 2$ *and G has a component that is a cycle whose length is not divisible by three, then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 4$;

(*iv*) *if* $\Delta = 3$ *and G has K_4 as a component, then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 5$.

Chapter 3

In Chapter 3 we will extend the ideas explored in Chapter 2 to prove the following theorem.

Theorem 1.5.2. *Let G be a plane embedding of a near-outerplanar graph with maximum degree Δ. Then*

(*i*) $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$;

(*ii*) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 2$ *if* $\Delta = 3$ *or* 4;

(*iii*) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 1$ *if* $\Delta \geq 5$;

(*iv*) $\chi_{\mathrm{ef}}(G) \leq \Delta + 1 = 5$ *if* $\Delta = 4$;

(*v*) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \max\{7, \Delta + 2\}$ *if* $\Delta \geq 3$.

Furthermore,

(*vi*) *if* $\Delta = 0$, *then* $\mathrm{ch}_{\mathrm{vf}}(G) = 2$, $\mathrm{ch}_{\mathrm{ef}}(G) = 1$ *and* $\mathrm{ch}_{\mathrm{vef}}(G) = 2$;

(*vii*) *if* $\Delta = 1$, *then* $\mathrm{ch}_{\mathrm{vf}}(G) = 3$, $\mathrm{ch}_{\mathrm{ef}}(G) = 2$ *and* $\mathrm{ch}_{\mathrm{vef}}(G) = 4$;

(*viii*) *if* $\Delta = 2$, *then*

$$\mathrm{ch}_{\mathrm{vf}}(G) = \mathrm{ch}_{\mathrm{ef}}(G) = \begin{cases} 5 & \textit{if } G \textit{ contains an odd cycle;} \\ 4 & \textit{if } G \textit{ contains an even cycle but no odd cycle;} \\ 3 & \textit{if } G \textit{ is cycle-free.} \end{cases}$$

and

$$\mathrm{ch}_{\mathrm{vef}}(G) = \begin{cases} 6 & \textit{if } G \textit{ has a component that is a cycle whose length} \\ & \textit{is not divisible by } 3; \\ 5 & \textit{if } G \textit{ has a component that is a cycle and the length} \\ & \textit{of every such cycle is divisible by } 3; \\ 4 & \textit{if } G \textit{ is cycle-free.} \end{cases}$$

In fact, in Chapters 2 and 3 we will replace the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs.

Chapter 4

In Chapter 4 we will study the square of a K_4-minor-free graph. We will prove the following theorem: the corollary is immediate.

Theorem 1.5.3. [15] *Let G be a K_4-minor-free graph with maximum degree $\Delta \geq 4$. Then G^2 is $\lceil \frac{3}{2}\Delta \rceil$-degenerate and* $\mathrm{ch}(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$.

Corollary 1.5.4. *Let G be a K_4-minor-free graph with maximum degree $\Delta \geq 4$. Then* $\mathrm{col}(G^2) \leq \lceil \frac{3}{2}\Delta \rceil + 1$.

Chapter 5

In Chapter 5 we will prove that the LSCC holds for all $K_{2,3}$-minor-free graphs G with maximum degree $\Delta \geq 6$. We will also give bounds for $\mathrm{ch}(G^2)$ if $\Delta \in \{3, 4, 5\}$. The situation is summarised in the following theorem.

Theorem 1.5.5. [16] *Let G be a $K_{2,3}$-minor-free graph with maximum degree Δ. Then the LSCC holds if $\Delta \geq 6$. In fact,*

(*i*) $\Delta + 1 \leq \chi(G^2) \leq \mathrm{ch}(G^2) \leq \Delta + 2$ *if* $\Delta \geq 3$;

(*ii*) $\Delta + 1 = \chi(G^2) = \mathrm{ch}(G^2)$ *if* $\Delta \geq 6$.

Chapter 2

Edge and total choosability of near-outerplanar graphs

2.1 Introduction

The List-Edge-Colouring Conjecture (LECC) and the List-Total-Colouring Conjecture (LTCC)[1] state that $\mathrm{ch}'(G) = \chi'(G)$ and $\mathrm{ch}''(G) = \chi''(G)$ for every multigraph G respectively. However, all results in this thesis are for simple graphs rather than for multigraphs.

In 1980, Erdős, Rubin and Taylor [11] proved that an even cycle is 2-choosable (or, equivalently, edge-2-choosable). This proves the LECC for all graphs G with maximum degree $\Delta = 2$; that is, that $\mathrm{ch}'(G) = \chi'(G) = 2 = \Delta$, with the exception that $\mathrm{ch}'(G) = \chi'(G) = 3 = \Delta + 1$ if G has a component that is an odd cycle. For K_4-minor-free graphs it was proved in 1999 by Juvan, Mohar and Thomas [20] that $\mathrm{ch}'(G) = \chi'(G) = \Delta$ if $\Delta \geq 3$. This completed the proof of the LECC for all K_4-minor-free graphs.

[1]For further details on the LECC and LTCC, see pages 7 and 8 respectively.

For total choosability, Juvan, Mohar and Skrekovski [19] proved in 1998 for all graphs G with maximum degree $\Delta = 2$ that $\mathrm{ch}''(G) = \chi''(G) = 3 = \Delta + 1$, with the exception that $\mathrm{ch}''(G) = \chi''(G) = 4 = \Delta + 2$ if G has a component that is a cycle whose length is not divisible by three. In 2006, Woodall [41] proved for K_4-minor-free graphs that $\mathrm{ch}''(G) = \chi''(G) = 4 = \Delta + 1$ if $\Delta = 3$. To complete the proof of the LTCC for all K_4-minor-free graphs, it remains to prove that $\mathrm{ch}''(G) = \chi''(G)$ if $\Delta \geq 4$.

In this chapter we will prove the LECC and LTCC for all near-outerplanar[2] graphs. In fact, we will replace the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. The graph $\bar{K}_2 + (K_1 \cup K_2)$ can be obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2, or, alternatively, from K_4 by adding a vertex of degree 2 subdividing an edge.

In Section 2.2 we will prove for K_4-minor-free graphs that $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$ if $\Delta \geq 4$. This will complete the proof of the LTCC for all K_4-minor-free graphs. The method of proof is based on an incomplete proof by Zhou, Matsuo and Nishizeki [43], which in turn is based on the proof by Juvan, Mohar and Thomas [20] for edge-choosability of K_4-minor-free graphs. However, it has now been brought to our attention that [44] contains a complete proof by Zhou, Matsuo and Nishizeki.

In Section 2.3, using the results in Section 2.2 and other known results, we will prove for $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs that both $\mathrm{ch}'(G) = \chi'(G) = \Delta$ and $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$ if $\Delta \geq 3$, with the exception that $\mathrm{ch}''(G) = \chi''(G) = 5$ if $\Delta = 3$ and G has K_4 as a component. This will complete the proof of the LECC and LTCC for all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs, and hence for all near-outerplanar graphs.

[2] K_4-minor-free or $K_{2,3}$-minor-free.

We will make use of the following theorem and the following lemma. Theorem 2.1.1 is a slight extension of a theorem of Dirac [9], and both parts of Lemma 2.1.2 follow from the result [10] that a d-regular edge-d-colourable planar graph is edge-d-choosable, but both parts are also easy exercises to prove directly.

Theorem 2.1.1. [39] *A K_4-minor-free graph with $|V(G)| \geq 4$ has at least two nonadjacent vertices with degree at most* 2.

Lemma 2.1.2. (i) $\mathrm{ch}'(C_4) = \chi'(C_4) = 2$. (ii) $\mathrm{ch}'(K_4) = \chi'(K_4) = 3$.

2.2 Edge and total choosability of K_4-minor-free graphs

In this section we will prove that $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$ for all K_4-minor-free graphs G with maximum degree $\Delta \geq 4$. This will complete the proof of the LECC and LTCC for all K_4-minor-free graphs. The situation is summarised in the following theorem.

Theorem 2.2.1. [14] *The LECC and LTCC hold for all K_4-minor-free graphs. In fact, if G is a K_4-minor-free graph with maximum degree Δ, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta$ *and* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, *apart from the following exceptions*:

(i) *if* $\Delta = 1$ *then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 3$;

(ii) *if* $\Delta = 2$ *and G has a component that is an odd cycle, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta + 1 = 3$;

(iii) *if* $\Delta = 2$ *and G has a component that is a cycle whose length is not divisible by three, then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 4$.

Proof. If $\Delta = 0$ or 1 then the results are obvious, and if $\Delta = 2$ the results are well known [11, 19]. Juvan, Mohar and Thomas [20] proved that $\mathrm{ch}'(G) = \chi'(G) = \Delta$ if $\Delta \geq 3$, and Woodall [41] proved that $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1 = 4$ if $\Delta = 3$.

It remains to prove that $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$ if $\Delta \geq 4$. Since $\mathrm{ch}''(G) \geq \chi''(G) \geq \Delta + 1$, it suffices to prove that $\mathrm{ch}''(G) \leq \Delta + 1$. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a K_4-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that $\mathrm{ch}''(G) > \Delta + 1$. Assume that every vertex v and every edge e of G is given a list $L(v)$ or $L(e)$ of $\Delta + 1$ colours such that G has no proper total colouring from these lists. We will prove various statements about G. Clearly G is connected.

Claim 2.2.1. *G does not contain a vertex of degree 1.*

Proof. Suppose that u is a vertex of degree 1 in G that is adjacent to v. Let $H = G - u$. By hypothesis H has a proper total colouring from its lists. The edge uv has at most Δ coloured neighbours[3] and so uv can be given a colour from its list. Since u now has two coloured neighbours u can be coloured from its list of $\Delta + 1 \geq 5$ colours. This contradiction proves Claim 2.2.1. □

Claim 2.2.2. *G does not contain two adjacent vertices of degree 2.*

Proof. Suppose that $xuvy$ is a path (or a cycle if $x = y$) where both u and v have degree 2 in G. Let $H = G - u$. By hypothesis H has a proper total colouring from its lists. Since each of the remaining elements ux, u, uv has, respectively, at most Δ, 2, 2 coloured neighbours and a list of $\Delta + 1 \geq 5$ colours, it follows that these elements can be coloured in this order. This contradiction proves Claim 2.2.2. □

[3]Recall that by neighbours we mean elements that are adjacent to or incident with each other.

Claim 2.2.3. *G does not contain a 4-cycle $xuyvx$ where both u and v have degree 2 in G.*

Proof. Suppose that G does contain a 4-cycle $xuyvx$ where both u and v have degree 2 in G. Let $H = G - \{u, v\}$. By hypothesis H has a proper total colouring from its lists. Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Lemma 2.1.2(i) that these edges can be coloured. We can now colour u and v since each has four coloured neighbours and a list of at least five colours. This contradiction proves Claim 2.2.3. □

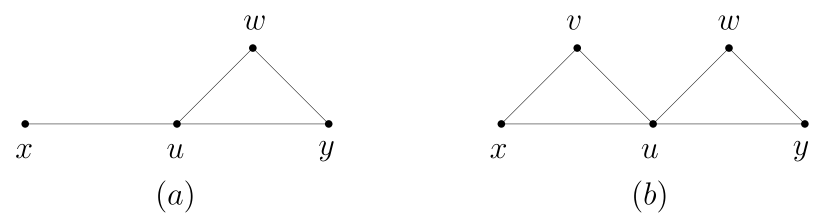


Figure 2.1

Claim 2.2.4. *G does not contain the configuration in Figure 2.1(a) where only x and y are incident with edges not shown.*

Proof. Suppose that G does contain the configuration in Figure 2.1(a) where only x and y are incident with edges not shown. Let $H = G - w$. By hypothesis H has a proper total colouring from its lists. Since each of the remaining elements wy, uw, w has, respectively, at most Δ, 3, 2 coloured neighbours and a list of $\Delta + 1 \geq 5$ colours, it follows that these elements can be coloured in this order. This contradiction proves Claim 2.2.4. □

Claim 2.2.5. *G does not contain the configuration in Figure 2.1(b) where only x and y are incident with edges not shown.*

Proof. Suppose that G does contain the configuration in Figure 2.1(b) where only x and y are incident with edges not shown. Let $H = G - \{u, v, w\}$. By hypothesis H has a proper total colouring from its lists. For each uncoloured element z, let $L'(z)$ denote the list of usable colours for z; that is, $L'(z)$ denotes $L(z)$ minus any colours already used on neighbours of z in G. Note that v and w can be coloured at the end since each has four neighbours and a list of at least five colours. So each of the remaining elements

$$vx, ux, uy, wy, u, uw, uv \tag{2.1}$$

has a list of at least 2, 2, 2, 2, 3, 5, 5 usable colours respectively. If we try to colour the remaining elements in the order (2.1) then it is only with uv that we may fail.

If $L'(uy) \cap L'(uv) = \emptyset$, then we will not fail with uv, and so we may assume that $L'(uy) \cap L'(uv) \neq \emptyset$. Similarly, by symmetry, we may assume that there is a colour $\alpha \in L'(ux) \cap L'(uw)$. If possible, give vx and uy the same colour. The remaining elements can now be coloured in the order (2.1). So we may assume that $L'(vx) \cap L'(uy) = \emptyset$. If possible, give vx and u the same colour. The remaining elements can now be coloured in the order (2.1) since the colour on u is not in $L'(uy)$. So we may assume that $L'(vx) \cap L'(u) = \emptyset$. If possible, give ux a colour that is not in $L'(vx)$. The remaining elements can now be coloured in the order (2.1) with the exception that vx is coloured last. So we may assume that $L'(ux) \subseteq L'(vx)$, which implies that $\alpha \in L'(vx)$ and that $L'(ux) \cap L'(uy) = \emptyset$. So we can give vx and uw the colour α, and then colour in order wy, uy (since $\alpha \notin L'(uy)$), ux (since the colour on uy is not in $L'(ux)$), u (since $\alpha \notin L'(u)$), and finally uv. In every case the colouring can be completed, which is the required contradiction. $\square$

If $\Delta(G) \geq 3$, then let G_1 be the graph whose vertices are the vertices of G that have degree at least 3 in G, where two vertices are adjacent in G_1 if and only if they are connected in G by an edge or by a path whose interior vertices have degree 2.

Claim 2.2.6. *G_1 is not K_4-minor-free.*

Proof. Claims 2.2.1 and 2.2.2 imply that G_1 exists and does not contain a vertex of degree 0. Furthermore, if G_1 contains a vertex of degree 1, then it follows that G contains a 4-cycle $xuyvx$ say, where both u and v have degree 2 in G. However, Claim 2.2.3 shows that this is impossible. So G_1 has no vertex of degree 1.

If G_1 contains a vertex of degree 2, then by Claims 2.2.2 and 2.2.3 it follows that any vertex of degree 2 in G_1 occurs in G as vertex u in Figure 2.1(a) or 2.1(b). However, Claims 2.2.4 and 2.2.5 show that this is impossible. So $\delta(G_1) \geq 3$, which by Theorem 2.1.1 implies that G_1 is not K_4-minor-free. $\square$

Since G_1 is a minor of G, Claim 2.2.6 implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 2.2.1. $\square$

2.3 Edge and total choosability of $(\bar{K}_2+(K_1\cup K_2))$-minor-free graphs

In this section we will use Theorem 2.2.1 to prove that the LECC and LTCC hold for all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. We will also need the following two lemmas.

Lemma 2.3.1. *Let G be a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph. Then each block of G is either K_4-minor-free or else isomorphic to K_4.*

Proof. Suppose that B is a block of G that has a K_4 minor. Since $\Delta(K_4) = 3$, it follows that B has a subgraph B' that is homeomorphic to K_4. If an edge of K_4 is subdivided, or if a path is added joining two vertices of K_4, then a $\bar{K}_2 + (K_1 \cup K_2)$ minor is formed. So $B' \cong K_4$ and $B = K_4$. □

Lemma 2.3.2. $\mathrm{ch}''(K_4) = \chi''(K_4) = 5$.

Proof. Since there are ten elements to colour (four vertices and six edges) and since no more than two elements can have the same colour, it follows that $\mathrm{ch}''(K_4) \geq \chi''(K_4) \geq 5$. It remains to prove that $\mathrm{ch}''(K_4) \leq 5$. Suppose that every vertex and every edge has a list of five colours. First colour a vertex and then its three incident edges. The remaining elements form a K_3 where each element has at least three usable colours in its list. Since $\mathrm{ch}''(K_3) = 3$ by Theorem 2.2.1, it follows that the remaining elements can be coloured. (This argument is taken from the proof of Theorem 3.1 in [19].) □

Theorem 2.3.3. [14] *The LECC and LTCC hold for all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. In fact, if G is a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree Δ, then $\mathrm{ch}'(G) = \chi'(G) = \Delta$ and $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, apart from the following exceptions*:

(*i*) *if $\Delta = 1$ then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 3$;

(*ii*) *if $\Delta = 2$ and G has a component that is an odd cycle, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta + 1 = 3$;

(*iii*) *if* $\Delta = 2$ *and* G *has a component that is a cycle whose length is not divisible by three, then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 4$;

(*iv*) *if* $\Delta = 3$ *and* G *has* K_4 *as a component, then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 5$.

Proof. If $\Delta \leq 2$, then G is K_4-minor-free and the results follow from Theorem 2.2.1. If $\Delta = 3$, then by Lemma 2.3.1 and the value of Δ, every component of G is either K_4-minor-free or else isomorphic to K_4. If G is K_4-minor-free then the results follow from Theorem 2.2.1. So we may assume that G has K_4 as a component, but since $\mathrm{ch}'(K_4) = \chi'(K_4) = 3$ by Lemma 2.1.2, and $\mathrm{ch}''(K_4) = \chi''(K_4) = 5$ by Lemma 2.3.2, again the results follow. So we may assume that $\Delta \geq 4$.

Since $\mathrm{ch}'(G) \geq \chi'(G) \geq \Delta$ and $\mathrm{ch}''(G) \geq \chi''(G) \geq \Delta + 1$, it suffices to prove that $\mathrm{ch}'(G) \leq \Delta$ and $\mathrm{ch}''(G) \leq \Delta+1$. Suppose, if possible, that G is a $(\bar{K}_2+(K_1 \cup K_2))$-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that $\mathrm{ch}'(G) > \Delta$ or $\mathrm{ch}''(G) > \Delta + 1$. Clearly G is connected.

If G is 2-connected, then by Lemma 2.3.1, G is K_4-minor-free since Δ is too large for G to be isomorphic to K_4, and the results follow from Theorem 2.2.1. So we may assume that G is not 2-connected and that G has an end-block B with cut-vertex z_0.

Claim 2.3.1. *$B \ncong K_4$.*

Proof. Suppose that $B \cong K_4$. Let $H = G - (B - z_0)$. Suppose that $\mathrm{ch}'(G) > \Delta$ and that every edge of G is given a list of Δ colours so that G has no proper edge-colouring from these lists. By hypothesis H has a proper edge-colouring from its lists. Since each edge of B has at least three usable colours in its list, and since $\mathrm{ch}'(K_4) = 3$ by Lemma 2.1.2, the remaining edges can be coloured. This contradiction shows that $\mathrm{ch}'(G) \leq \Delta$.

So suppose that $\mathrm{ch}''(G) > \Delta + 1$ and that every vertex and every edge of G is given a list of $\Delta + 1$ colours so that G has no proper total colouring from these lists. By hypothesis H has a proper total colouring from its lists. We can now colour the three edges incident with z_0 since each edge has at most $\Delta - 2$ coloured neighbours and a list of $\Delta + 1$ colours. The remaining elements form a K_3 where each element has at least three usable colours in its list. Since $\mathrm{ch}''(K_3) = 3$ by Theorem 2.2.1, it follows that the remaining elements can be coloured. This contradiction shows that $\mathrm{ch}''(G) \leq \Delta + 1$. This completes the proof of Claim 2.3.1. □

By Claim 2.3.1 and Lemma 2.3.1, it follows that B is K_4-minor-free. By the proof of Claim 2.2.1, $B \ncong K_2$, so B is 2-connected and $d_G(z_0) \geq 3$.

(Note that for the edge-colouring case of Theorem 2.2.1, Claims 2.2.1–2.2.5 were proved in [20], in which G is a K_4-minor-free graph with the smallest number of vertices and maximum degree $\Delta \geq 3$ such that $\mathrm{ch}'(G) > \Delta$. The proofs of these claims for the edge-colouring case are slightly simpler versions than those given in Theorem 2.2.1.)

If B contains a vertex with degree at least 3 in G, then let B_1 be the graph whose vertices are the vertices of B that have degree at least 3 in G, where two vertices are adjacent in B_1 if and only if they are connected in G by an edge or by a path whose interior vertices have degree 2.

Claim 2.3.2. *B_1 is not K_4-minor-free.*

Proof. Since $d_G(z_0) \geq 3$ and by the proof of Claim 2.2.2, B_1 exists and does not contain a vertex of degree 0. Furthermore, if B_1 contains a vertex of degree 1, then it follows that B contains a 4-cycle $xuyvx$ say, where both u and v have degree 2 in G. However, the proof of Claim 2.2.3 shows that this is impossible. So B_1 has no vertex of degree 1.

If B_1 contains a vertex of degree 2 that is different from z_0, then by the proofs of Claims 2.2.2 and 2.2.3 it follows that any vertex of degree 2 in B_1 occurs in B as vertex u in Figure 2.1(a) or 2.1(b), where only x and y are incident with edges not shown. (Note that w, and v if present, both have degree 2 in G and are therefore different from z_0.) However, the proofs of Claims 2.2.4 and 2.2.5 show that this is impossible. So the only possible vertex of degree 2 in B_1 is z_0, which by Theorem 2.1.1 implies that B_1 is not K_4-minor-free. □

Since B_1 is a minor of B, Claim 2.3.2 implies that B is not K_4-minor-free. This contradiction completes the proof of Theorem 2.2.1. □

Chapter 3

Coupled, edge-face and entire choosability of near-outerplane graphs

3.1 Introduction

In this chapter we will study the coupled, edge-face and entire choosability of near-outerplane graphs.

In 1984, Borodin [4] proved Ringel's conjecture [28], which states that if G is a plane graph, then $\chi_{\mathrm{vf}}(G) \leq 6$. It was proved in 1996 by Wang and Liu [36] that if G is an outerplane graph, then $\chi_{\mathrm{vf}}(G) \leq 5$. In this chapter we will prove for all near-outerplane[1] graphs G that $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$.

For an edge-face colouring of a plane graph G with maximum degree Δ, it was conjectured by Melnikov [25] in 1975 that $\chi_{\mathrm{ef}}(G) \leq \Delta + 3$. In 1997, Sanders and Zhao [29] proved Melnikov's conjecture. Moreover, they conjectured that $\chi_{\mathrm{ef}}(G) \leq \Delta + 2$ for all plane graphs with maximum degree Δ, with the exception that $\chi_{\mathrm{ef}}(G) = \Delta + 3$ if $\Delta = 2$ and G has a component that is an odd cycle. In

[1] A plane embedding of a graph that is K_4-minor-free or $K_{2,3}$-minor-free.

1995, Wang [35] proved for all outerplane graphs that $\chi_{\mathrm{ef}}(G) \le \Delta + 1$ if $\Delta \ge 5$. In this chapter we will prove for near-outerplane graphs that $\mathrm{ch}_{\mathrm{ef}}(G) \le \Delta + 2$ if $\Delta \ge 3$, $\mathrm{ch}_{\mathrm{ef}}(G) \le \Delta + 1$ if $\Delta \ge 5$, and $\chi_{\mathrm{ef}}(G) \le \Delta + 1 = 5$ if $\Delta = 4$. Using known results, we will also prove that if $\Delta \le 2$, then $\mathrm{ch}_{\mathrm{ef}}(G) \le \Delta + 2$, with the exception that $\mathrm{ch}_{\mathrm{ef}}(G) = \Delta + 3$ if G has a component that is an odd cycle. This will complete a proof of the conjecture of Sanders and Zhao for all near-outerplane graphs.

The Entire Colouring Conjecture (ECC)[2] [22] states that if G is a plane graph with maximum degree Δ, then $\chi_{\mathrm{vef}}(G) \le \Delta + 4$. It was proved by Kronk and Mitchem [23] in 1973 that the ECC holds for all plane graphs with maximum degree $\Delta \le 3$. In 2000, Sanders and Zhao [30] proved that the ECC holds for all plane graphs with maximum degree $\Delta \ge 6$. The ECC is still an open problem if $\Delta = 4$ or 5. For an outerplane graph G it was proved in 1992 by Wang and Zhang [38] that $\chi_{\mathrm{vef}}(G) \le \Delta + 2$ if $\Delta \ge 5$. Recently, in 2005, Wu and Wu [42] proved that if G is a plane embedding of a K_4-minor-free graph, then $\chi_{\mathrm{vef}}(G) \le \max\{8, \Delta + 2\}$. In this chapter we will prove that $\mathrm{ch}_{\mathrm{vef}}(G) \le \max\{7, \Delta + 2\}$ for all near-outerplane graphs. Since $\chi_{\mathrm{vef}}(G) \le \mathrm{ch}_{\mathrm{vef}}(G)$, this will improve the result of Wu and Wu, and will contain as a special case a proof of the ECC for all near-outerplane graphs.

In fact, as in Chapter 2, we will replace the class of $K_{2,3}$-minor-free graphs in each case by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. Recall that the graph $\bar{K}_2 + (K_1 \cup K_2)$ can be obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2, or, alternatively, from K_4 by adding a vertex of degree 2 subdividing an edge. Using known results, we will also give conditions for the different values of $\mathrm{ch}_{\mathrm{vf}}$, $\mathrm{ch}_{\mathrm{ef}}$ and $\mathrm{ch}_{\mathrm{vef}}$ when $\Delta \le 2$. The situation is summarised in the following theorem.

[2] See page 4 for further details.

Theorem 3.1.1. *Let G be a plane embedding of a near-outerplanar graph with maximum degree Δ. Then*

(*i*) $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$;

(*ii*) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 2$ *if* $\Delta = 3$ *or* 4;

(*iii*) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 1$ *if* $\Delta \geq 5$;

(*iv*) $\chi_{\mathrm{ef}}(G) \leq \Delta + 1 = 5$ *if* $\Delta = 4$;

(*v*) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \max\{7, \Delta + 2\}$ *if* $\Delta \geq 3$.

Furthermore,

(*vi*) *if* $\Delta = 0$, *then* $\mathrm{ch}_{\mathrm{vf}}(G) = 2$, $\mathrm{ch}_{\mathrm{ef}}(G) = 1$ *and* $\mathrm{ch}_{\mathrm{vef}}(G) = 2$;

(*vii*) *if* $\Delta = 1$, *then* $\mathrm{ch}_{\mathrm{vf}}(G) = 3$, $\mathrm{ch}_{\mathrm{ef}}(G) = 2$ *and* $\mathrm{ch}_{\mathrm{vef}}(G) = 4$;

(*viii*) *if* $\Delta = 2$, *then*

$$\mathrm{ch}_{\mathrm{vf}}(G) = \mathrm{ch}_{\mathrm{ef}}(G) = \begin{cases} 5 & \textit{if } G \textit{ contains an odd cycle;} \\ 4 & \textit{if } G \textit{ contains an even cycle but no odd cycle;} \\ 3 & \textit{if } G \textit{ is cycle-free.} \end{cases}$$

and

$$\mathrm{ch}_{\mathrm{vef}}(G) = \begin{cases} 6 & \textit{if } G \textit{ has a component that is a cycle whose length} \\ & \textit{is not divisible by } 3; \\ 5 & \textit{if } G \textit{ has a component that is a cycle and the length} \\ & \textit{of every such cycle is divisible by } 3; \\ 4 & \textit{if } G \textit{ is cycle-free.} \end{cases}$$

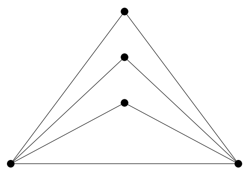


Figure 3.1

All parts of Theorem 3.1.1 are sharp, except possibly part (*ii*) when $\Delta = 4$ for which no example is known that attains the upper bound. For part (*i*) the upper bound is attained by K_3, and for part (*ii*) when $\Delta = 3$ the upper bound is attained by any graph with K_3 as a block. For parts (*iii*) and (*iv*) the upper bound is attained by $K_{1,\Delta}$. For part (*v*), if $\Delta \geq 5$ then the upper bound is $\Delta + 2$, which is attained by $K_{1,\Delta}$; otherwise the upper bound is 7, which is attained by any graph with K_4 as a block since $\mathrm{ch}_{\mathrm{vef}}(K_4) = 7$, and by both embeddings of $K_2 + \bar{K}_3$, one of which is shown in Figure 3.1. It is proved in Appendix A that $\mathrm{ch}_{\mathrm{vef}}(K_4) = 7$, and it is proved in Appendix B that $\mathrm{ch}_{\mathrm{vef}}(K_2 + \bar{K}_3) = 7$. Furthermore, the sharp results for choosability are also sharp for ordinary colourings.

It follows from the examples given that if $\Delta \geq 2$, then all but part (*ii*) when $\Delta = 4$ and part (*v*) when $\Delta = 3$ are also sharp for plane embeddings of the smaller class of K_4-minor-free graphs. Furthermore, all but part (*ii*) when $\Delta = 4$ are also sharp for plane embeddings of the smaller classes of both $K_{2,3}$-minor-free graphs and $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. It follows that part (*i*), part (*ii*) when $\Delta = 3$, parts (*iii*) and (*iv*), and part (*v*) when $\Delta \geq 5$ are also sharp for outerplane graphs.

So some unsolved problems are:

1. to determine a sharp upper bound for $\mathrm{ch}_{\mathrm{ef}}(G)$ when $\Delta = 4$ and G is a near-outerplane graph;

2. to determine a sharp upper bound for $\mathrm{ch}_{\mathrm{vef}}(G)$ when $\Delta = 3$ and G is a plane emdebbing of a K_4-minor-free graph.

For these two problems, in view of the work contained in this chapter, we propose the following conjectures.

Conjecture 7. *If G is a near-outerplane graph with maximum degree $\Delta = 4$, then $\mathrm{ch_{ef}}(G) \leq \Delta + 1 = 5$.*

Conjecture 8. *If G is a near-outerplane graph with maximum degree $\Delta = 3$, then $\mathrm{ch_{vef}}(G) \leq \Delta + 3 = 6$, with the exception that $\mathrm{ch_{vef}}(G) = 7$ if G has K_4 as a component.*

The rest of this chapter is devoted to a proof of Theorem 3.1.1. We will make use of the following two theorems, the following definition, and the following lemma. Theorem 3.1.2 is a slight extension of a theorem of Dirac [9]. Theorem 3.1.3 summarises the known results for edge and total choosability of K_4-minor-free graphs, as proved in Chapter 2. In particular we will make use of the well-known result [11, 32] that $\mathrm{ch}(C_4) = \mathrm{ch}'(C_4) = 2$, which is included in Theorem 3.1.3 since choosability and edge-choosability are equivalent when $\Delta = 2$.

Theorem 3.1.2. [39] *A K_4-minor-free graph G with $|V(G)| \geq 4$ has at least two nonadjacent vertices with degree at most 2.*

Theorem 3.1.3. [14] *If G is a K_4-minor-free graph with maximum degree Δ, then $\mathrm{ch}'(G) = \chi'(G) = \Delta$ and $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, apart from the following exceptions:*

(*i*) *if $\Delta = 1$ then $\mathrm{ch}''(G) = \chi''(G) = 3 = \Delta + 2$;*

(*ii*) *if $\Delta = 2$ and G has a component that is an odd cycle, then $\mathrm{ch}'(G) = \chi'(G) = 3 = \Delta + 1$;*

(*iii*) *if* $\Delta = 2$ *and* G *has a component that is a cycle whose length is not divisible by three, then* $\mathrm{ch}''(G) = \chi''(G) = 4 = \Delta + 2$.

Let the *bounding cycle* of a 2-connected block B of a plane graph G be the cycle of B that has the largest area inside it; that is, in a plane embedding of B the bounding cycle forms the boundary of the outer face of B.

Lemma 3.1.4. *Every component* C *of a plane graph with* $|V(C)| \geq 3$ *is either* 2*-connected or has an end-block* B *such that no interior face of* B *has a block of* C *embedded in it.*

Proof. It is clear that C is either 2-connected or has an end-block B. If $B \cong K_2$, then B has no interior face, and so we may assume that every end-block B is 2-connected. Select B so that the area inside the bounding cycle of B is as small as possible. Then no interior face of B can have another block of C embedded in it since otherwise B must contain another end-block of C, and this end-block necessarily has a smaller area inside its bounding cycle than B. □

3.2 The start of the proof of Theorem 3.1.1

If $\Delta = 0$ or 1, then the results are obvious. If $\Delta = 2$, then let f_0 be the exterior face, let F_1 be set of faces of G that are adjacent to f_0, and, recursively, let F_{k+1} be the set of faces that are adjacent to F_k $(1 \leq k \leq n-1)$ and that are not in F_j for some $j < k$. We can first colour f_0 and then, in order, each of the sets of faces $F_1, F_2, \ldots, F_n$ since no face is adjacent to more than one coloured face at the time of its colouring. It remains to colour the vertices and/or edges. Since choosability and edge-choosability are equivalent when $\Delta = 2$, the problem is reduced to edge-choosability and total choosability of paths and cycles, and these results are given in Theorem 3.1.3. If G is cycle-free, then G has only one face, and so $\mathrm{ch}_{\mathrm{vf}}(G) = \mathrm{ch}_{\mathrm{ef}}(G) = \mathrm{ch}'(G) + 1$ and $\mathrm{ch}_{\mathrm{vef}}(G) = \mathrm{ch}''(G) + 1$. If G contains a cycle, then every vertex and every edge of each cycle is incident with exactly two faces, and so $\mathrm{ch}_{\mathrm{vf}}(G) = \mathrm{ch}_{\mathrm{ef}}(G) = \mathrm{ch}'(G) + 2$ and $\mathrm{ch}_{\mathrm{vef}}(G) = \mathrm{ch}''(G) + 2$. This completes the proof of parts (vi)–$(viii)$ of Theorem 3.1.1.

It remains to prove parts (i)–(v) of Theorem 3.1.1 if $\Delta \geq 3$. We will first prove parts (i)–(v) for plane embeddings of K_4-minor-free graphs as restated in Theorem 3.3.1. We will then use Theorem 3.3.1 to prove parts (i)–(v) for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs as restated in Theorem 3.7.1. This will complete the proof of Theorem 3.1.1.

3.3 Results for plane embeddings of K_4-minor-free graphs

We will now start the proof of parts (i)–(v) of Theorem 3.1.1 for plane embeddings of K_4-minor-free graphs. These are restated in the following theorem.

Theorem 3.3.1. *Let G be a plane embedding of a K_4-minor-free graph with maximum degree Δ. Then*

(i) $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$;

(ii) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 2$ *if* $\Delta = 3$ *or* 4;

(iii) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 1$ *if* $\Delta \geq 5$;

(iv) $\chi_{\mathrm{ef}}(G) \leq \Delta + 1 = 5$ *if* $\Delta = 4$;

(v) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \max\{7, \Delta + 2\}$ *if* $\Delta \geq 3$.

The proofs of the results in Theorem 3.3.1 have been split into various sections for clarity. In Section 3.4 we will prove part (i), which is restated in Theorem 3.4.1. In Section 3.5 we will first prove part (ii), which is included in Theorem 3.5.1, and we will then prove parts (iii) and (iv), which are restated in Theorem 3.5.2. In Section 3.6 we will first prove part (v) if $\Delta = 3$, which is restated in Theorem 3.6.1, and we will then prove part (v) if $\Delta \geq 4$, which is restated in Theorem 3.6.2. We will need the following definitions and the following lemma.

Let C be a component of a plane embedding of a K_4-minor-free graph G such that no interior face of C has another component of G embedded in it. If C is 2-connected, then let $B = C$ and let z_0 be any vertex of maximum degree in C; otherwise, by Lemma 3.1.4, let B be an end-block of C with cut-vertex z_0 such that no interior face of B has a block of C embedded in it.

If B contains a vertex with degree at least 3 in G, then let B_1 be the graph whose vertices are the vertices of B that have degree at least 3 in G, where two vertices are adjacent in B_1 if and only if they are connected in G by an edge or by a path whose interior vertices have degree 2.

If $u, x \in V(B)$, then let P_{ux} be the set of paths in B of length 1 or 2 between u and x that contain no interior vertex of degree at least 3; that is, if $uvx \in P_{ux}$ then $d_G(v) = 2$. Also, let p_{ux} be the number of paths in P_{ux}.

Lemma 3.3.2. *Suppose that B does not contain a vertex of degree 1 or two adjacent vertices of degree 2 in G. Then the graph B_1 exists and does not contain a vertex of degree 0. Suppose that B_1 does not contain a vertex of degree 1. Then B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, where $p_{ux} + p_{uy} = d_G(u) \geq 3$, and where $p_{uy} \geq 2$. Moreover, no two paths in P_{uy} bound a region that has a path not in P_{uy} embedded in it, and if $p_{ux} \geq 2$, then no two paths in P_{ux} bound a region that has a path not in P_{ux} embedded in it also.*

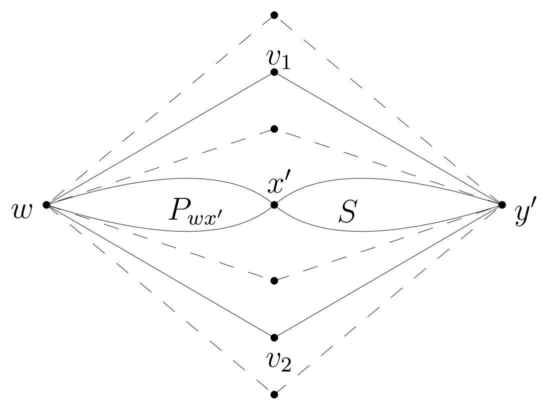


Figure 3.2

Proof. If B does not contain a vertex of degree 1, then $B \ncong K_2$, and if B does not contain two adjacent vertices of degree 2, then B is not a cycle. So B has at least two vertices with degree at least 3, and so it follows that B_1 exists and does not contain a vertex of degree 0. Since B_1 is a minor of B, it follows that B_1 is K_4-minor-free. Since, by the hypothesis of the lemma, B_1 does not contain a vertex of degree 1, it follows that $B_1 \cong K_3$, or, by Theorem 3.1.2, B_1 has at least two nonadjacent vertices with degree exactly 2.

Let w be a vertex of degree 2 in B_1 that is adjacent in B_1 to x' and y'. Then, by the definition of B_1 and since B does not contain two adjacent vertices of degree 2 in G, it follows that $p_{wx'}, p_{wy'} \geq 1$ and $p_{wx'} + p_{wy'} = d_G(w) \geq 3$. Furthermore, since $d_G(w) \geq 3$, we may assume without loss of generality that $p_{wy'} \geq 2$.

By interchanging x' and y' if necessary, we may assume that if no two paths in $P_{wy'}$ bound a region that has a path not in $P_{wy'}$ embedded in it, then no two paths in $P_{wx'}$ bound a region that has a path not in $P_{wx'}$ embedded in it also, and so the proof would be complete. So we may assume that there is a region R bounded by two paths in $P_{wy'}$ that has a path $w \dots y'$ not in $P_{wy'}$ embedded in it. Since $p_{wx'} + p_{wy'} = d_G(w)$ it follows that every such path in R must contain x', and so the bounding cycle of B consists of two paths in $P_{wy'}$. Let S be the subgraph of B obtained by deleting w and all its neighbours of degree 2 in B. An example is shown in Figure 3.2, where $R = wv_1y'v_2w$, where the dashed edges may or may not be present, and if B is an end-block, then $y' = z_0$.

Since w is adjacent in B_1 to y', and since $B_1 \cong K_3$ or has at least two nonadjacent vertices with degree exactly 2, then there is a vertex $u \neq y'$ in S such that $d_{B_1}(u) = 2$, and where possibly $u = x'$. Let u be adjacent in B_1 to x and y. Then, by what we have proved about w, the result follows since every region bounded by paths in P_{ux} or P_{uy} is inside the bounding cycle of B. This completes the proof of Lemma 3.3.2. $\square$

3.4 Coupled choosability of plane embeddings of K_4-minor-free graphs

In this section we will prove part (i) of Theorem 3.3.1, which is restated in the following theorem.

Theorem 3.4.1. *Let G be a plane embedding of a K_4-minor-free graph. Then* $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$.

Proof. Suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with the smallest number of vertices such that $\mathrm{ch}_{\mathrm{vf}}(G) > 5$. Assume that every vertex v and every face f of G is given a list $L(v)$ or $L(f)$ of five colours such that G has no proper coupled colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined at the start of Section 3.3.

Claim 3.4.1. *G does not contain a vertex of degree 1.*

Proof. Suppose that u is a vertex of degree 1 in G. Let $H = G - u$. By hypothesis H has a proper coupled colouring from its lists. Since u has two coloured neighbours[3] and a list of five colours, it follows that u can be coloured from its list. This contradiction proves Claim 3.4.1. □

Claim 3.4.2. *B does not contain a triangle $xuyx$, where $xuyx$ bounds a face in G and where u has degree 2 in G.*

Proof. Suppose that B does contain a triangle $xuyx$, where $xuyx$ bounds a face in G and where u has degree 2 in G. Let f be a face in G bounded by $xuyx$. Let $H = G - u$ where the face in H in which u was embedded is given the same list as the face in G that has xuy as part of its boundary and is different from f. By hypothesis H has a proper coupled colouring from its lists. We can now colour f and then u since each has at most four coloured neighbours at the time of its colouring. This contradiction proves Claim 3.4.2. □

Claim 3.4.3. *B does not contain two adjacent vertices of degree 2 in G.*

Proof. Suppose that $xuvy$ is a path in B where both u and v have degree 2 in G. By Claim 3.4.2, it follows that $x \neq y$. Let $H = G/uv$. By hypothesis H has a proper coupled colouring from its lists. After applying a colouring of H to G, it remains to colour u and v, which is possible since both u and v have three coloured neighbours and a list of five colours. This contradiction proves Claim 3.4.3. □

Claim 3.4.4. *B does not contain a 4-cycle $xuyvx$, where $xuyvx$ bounds a face in G and where both u and v have degree 2 in G.*

[3] Recall that by neighbours we mean elements that are adjacent to or incident with each other.

Proof. Suppose that B does contain a 4-cycle $xuyvx$, where $xuyvx$ bounds a face in G and where both u and v have degree 2 in G. Let f be a face in G bounded by $xuyvx$. Let $H = G - u$ where the face in H in which u was embedded is given the same list as the face in G that has xuy as part of its boundary and is different from f. By hypothesis H has a proper coupled colouring from its lists. First uncolour v. We can now colour in order f, u, v since each has at most four coloured neighbours at the time of its colouring. This contradiction completes the proof of Claim 3.4.4. □

Claim 3.4.1 implies that $B \not\cong K_2$ and Claim 3.4.3 implies that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined at the start of Section 3.3.

Claim 3.4.5. *B_1 is not K_4-minor-free.*

Proof. Since B has at least two vertices with degree at least 3, it follows that B_1 exists and has no vertex of degree 0. Suppose that x is a vertex of degree 1 in B_1. Then x is adjacent in B_1 to z_0. By the definition of B_1 and by Claim 3.4.3, it follows that $d_G(x) \geq 3$, so that $p_{xz_0} \geq 3$, and that every path in B between x and z_0 is in P_{xz_0}. So, by the definition of B, it follows that B contains a face that is bounded by a triangle xv_1z_0x or a 4-cycle $xv_1z_0v_2x$, where $d_G(v_i) = 2$ $(i = 1, 2)$. However, Claims 3.4.2 and 3.4.4 show that this is impossible. So B_1 has no vertex of degree 1.

In view of Claims 3.4.1 and 3.4.3, it follows from Lemma 3.3.2 that B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, such that there are two paths in P_{uy} that bound a face in B that is a triangle uv_1yu or a 4-cycle uv_1yv_2u, where $d_G(v_i) = 2$ $(i = 1, 2)$. However, Claims 3.4.2 and 3.4.4 show that this is impossible. This contradiction completes the proof of Claim 3.4.5. □

Since B_1 is a minor of G, Claim 3.4.5 implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 3.4.1. □

3.5 Edge-face choosability and edge-face colourability of plane embeddings of K_4-minor-free graphs

In this section we will first prove part (*ii*) of Theorem 3.3.1, which is included in Theorem 3.5.1. We will then prove parts (*iii*) and (*iv*) of Theorem 3.3.1, which are restated in Theorem 3.5.2. For each uncoloured element z in G, let $L'(z)$ denote the list of usable colours for z; that is, $L'(z)$ denotes $L(z)$ minus any colours already used on neighbours of z in G.

Theorem 3.5.1. *Let G be a plane embedding of a K_4-minor-free graph with maximum degree Δ. Then* $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 2$ *if* $\Delta \geq 3$.

Proof. Fix the value of $\Delta \geq 3$ and suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with maximum degree at most Δ such that $\mathrm{ch}_{\mathrm{ef}}(G) > \Delta + 2$. Assume that every edge e and every face f of G is given a list $L(e)$ or $L(f)$ of $\Delta + 2$ colours such that G has no proper edge-face colouring from these lists.

From the well-known result [31] that a planar graph is 5-choosable, it follows that the faces of G can be coloured from their lists since $\Delta \geq 3$. Since every edge is incident with at most two faces, it follows that every edge has at least Δ usable colours in its list. Since $\mathrm{ch}'(G) = \Delta$ by Theorem 3.1.3, it follows that these edges can be coloured. □

Note that by Theorem 3.1.1(vi)–($viii$) and Theorem 3.5.1, if G is a plane embedding of a K_4-minor-free graph with maximum degree Δ, then $\chi_{\mathrm{ef}}(G) \leq \mathrm{ch}_{\mathrm{ef}}(G) \leq 5$ if $\Delta \leq 3$ and $\mathrm{ch}_{\mathrm{ef}}(G) \leq 6$ if $\Delta = 4$.

Theorem 3.5.2. *Let G be a plane embedding of a K_4-minor-free graph with maximum degree $\Delta \geq 4$. Then*

(i) $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 1$ *if* $\Delta \geq 5$;

(ii) $\chi_{\mathrm{ef}}(G) \leq \Delta + 1 = 5$ *if* $\Delta = 4$.

Proof. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that G is a counterexample to either part. Assume that every edge e and every face f of G is given a list $L(e)$ or $L(f)$ of $\Delta + 1$ colours such that G has no proper edge-face colouring from these lists, and assume that these lists are all identical if $\Delta = 4$. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined at the start of Section 3.3.

Claim 3.5.1. *G does not contain a vertex of degree* 1*.*

Proof. Suppose that u is a vertex of degree 1 in G that is adjacent to v. Let $H = G - u$. By hypothesis H has a proper edge-face colouring from its lists. The edge uv has at most Δ coloured neighbours, and so uv can be given a colour from its list. This contradiction proves Claim 3.5.1. □

Claim 3.5.2. *B does not contain two adjacent vertices of degree 2 in G.*

Proof. Suppose that $xuvy$ is a path in B (or a cycle if $x = y$) where both u and v have degree 2 in G. If $x \neq y$, let $H = G/uv$. By hypothesis H has a proper edge-face colouring from its lists. After applying a colouring of H to G, the edge uv has four coloured neighbours, and so uv can be coloured from its list. If $x = y$, then $B \cong K_3$. Let f be the interior face of B. Let $H = G - u$ where the face in H in which v is embedded is given the same list as the exterior face of B. By hypothesis H has a proper edge-face colouring from its lists. The edge ux has at most Δ coloured neighbours and both uv and f have two coloured neighbours, so we can colour in order ux, uv, f since each has at least one usable colour in its list at the time of its colouring. This contradiction proves Claim 3.5.2. □

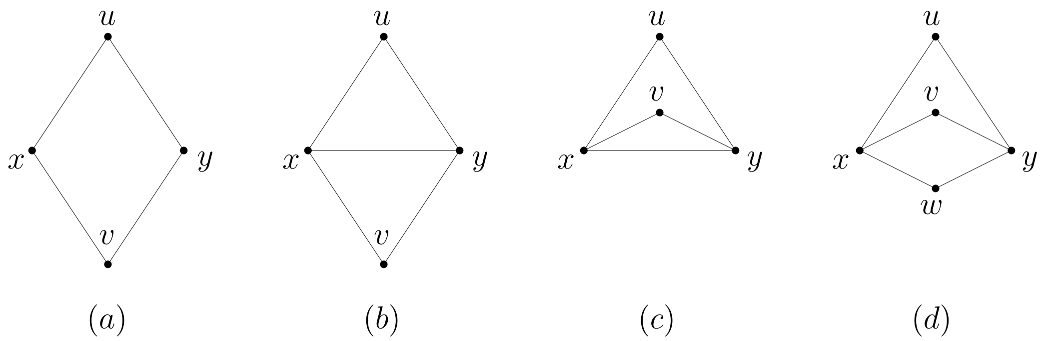


Figure 3.3

Claim 3.5.3. *If B contains the configuration in Figure 3.3(a), where $xuyvx$ is an interior face, where x is not adjacent to y, and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$.*

Proof. Suppose that B contains the configuration in Figure 3.3(a), where $xuyvx$ is an interior face, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the interior face $xuyvx$. Since, by Claim

3.5.2, both x and y have degree at least 3 in G, and if C is not 2-connected then B is an end-block by definition, it follows that f is adjacent to two different faces. Let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xvy in its boundary. Let $H = G - \{u, v\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper edge-face colouring from these lists.

(i): Suppose first that $\Delta \geq 6$. Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f since it has only six coloured neighbours. So we may assume that $\Delta = 5$ and, contrary to what we want to prove, that $d_G(x) \leq \Delta - 1 = 4$ and $d_G(y) \leq \Delta = 5$.

Now each of vy, uy, vx, ux, f has at most 4, 4, 3, 3, 2 coloured neighbours in G respectively. So each of the remaining elements

$$vy, uy, vx, ux, f \tag{3.1}$$

has a list of at least 2, 2, 3, 3, 4 usable colours[4] respectively. If we try to colour the elements in the order (3.1) then it is only with f that we may fail.

If possible, give vx and uy the same colour. The remaining elements can now be coloured in the order (3.1). So we may assume that $L'(vx) \cap L'(uy) = \emptyset$ so that $|L'(vx) \cup L'(uy)| \geq 5$. Now either $|L'(f)| \geq 5$, or else vx or uy can be given a colour that is not in $L'(f)$. In each case the remaining elements can be coloured in the order (3.1).

(ii): Let the colours in every list be the integers $1, 2, \ldots, 5$. When applying a colouring of H to G we may assume that f_1 is coloured 1, f_2 is coloured 2, and two opposite edges of the 4-cycle, say ux and vy, are coloured 3 since we may assume

[4]Recall that $L'(z)$ denotes the list of usable colours for z in G.

that 3 was on xy in H. Next, if possible, give uy the colour 2. We can now colour vx and then f since each has at most four differently coloured neighbours at the time of its colouring. So we may assume that uy cannot be coloured 2, which implies that an edge incident with y has the colour 2. Similarly, vx cannot be coloured 1. By symmetry we may assume that ux cannot be recoloured 2 and vy cannot be recoloured 1. This implies that there are exactly two edges not shown that are incident with x, one coloured 1 and the other coloured 2. The same applies to y. So we can colour vx and uy with 4, and f with 5. In every case the colouring can be completed, which is the required contradiction. □

Claim 3.5.4. *B does not contain the configuration in Figure* 3.3(b) *or* 3.3(c), *where in each case the interior faces are as shown, and where only x and y may be incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.3(b) or 3.3(c), where in each case the interior faces are as shown, and where only x and y may be incident with edges in G not shown. Let f be the face $xuyx$ or $xuyvx$ as appropriate and let f' be the face $xvyx$. Let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xvy or xy in its boundary as appropriate. (It is possible that $f_1 = f_2$, but it is only in (ii) where it is necessary to consider this separately.) Let $H = G - \{u, v\}$. Let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper edge-face colouring from these lists.

(i): Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f and then f' since each has at most five coloured neighbours at the time of its colouring.

(ii): Let the colours in every list be the integers $1, 2, \ldots, 5$. If $f_1 \neq f_2$, then each of x and y has degree $\Delta = 4$ in B and is incident with an edge that is not shown, say e_1 and e_2 respectively. When applying a colouring of H to G we may assume that f_1, f_2, xy, e_1 are coloured 1, 2, 3, 4 respectively, and that e_2 is coloured either 4 or 5. In G, recolour xy with 1, and colour f, vx, uy, ux with 2, 3, 3, 5 respectively. Next, give f' the same colour as e_2 and give vy whichever of 4 and 5 is not on e_2.

If $f_1 = f_2$, then, by the definition of B, we may assume that $d_G(y) = 3$. If $d_G(x) = 4$, then let e_1 be the edge incident with x that is not shown. When applying a colouring of H to G we may assume that f_1 and xy are coloured 1 and 3 respectively, and e_1, if it exists, is coloured 4. In G, colour f, f', uy, ux, vy with 2, 4, 4, 5, 5 respectively. If B is as in Figure 3.3(b) or 3.3(c), then vx can be coloured either 2 or 1 respectively. In every case the colouring can be completed, which is the required contradiction. $\square$

Claim 3.5.5. *B does not contain the configuration in Figure 3.3(d), where $xuyvx$ and $xvywx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.3(d), where $xuyvx$ and $xvywx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$ and let f' be the face $xvywx$. Let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xwy in its boundary. Since, by Claim 3.5.3, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper edge-face colouring from these lists.

Now each of wy, wx, ux, uy, vy, vx, f, f' has at most 3, 3, 3, 3, 2, 2, 1, 1 coloured neighbours in G respectively. So each of the remaining elements

$$wy, wx, ux, uy, vy, vx, f, f' \tag{3.2}$$

has a list of at least 3, 3, 3, 3, 4, 4, 5, 5 usable colours respectively. If we try to colour the elements in the order (3.2) then it is only with f' that we may fail.

If possible, colour both vx and vy so that vx is given a colour that is not in $L'(f')$. Next, since each edge of the 4-cycle $xuywx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f and then f' since each has at least one usable colour in its list at the time of its colouring. So we may assume that $L'(vx) \subseteq L'(f')$. If possible, give vx and wy the same colour. The remaining elements can now be coloured in the order (3.2). So we may assume that $L'(vx) \cap L'(wy) = \emptyset$ so that $|L'(vx) \cup L'(wy)| \geq 7$. Now either $|L'(f')| \geq 7$, or else wy can be given a colour that is not in $L'(f')$ since $L'(vx) \subseteq L'(f')$. In each case the remaining elements can be coloured in the order (3.2). This contradiction proves Claim 3.5.5. □

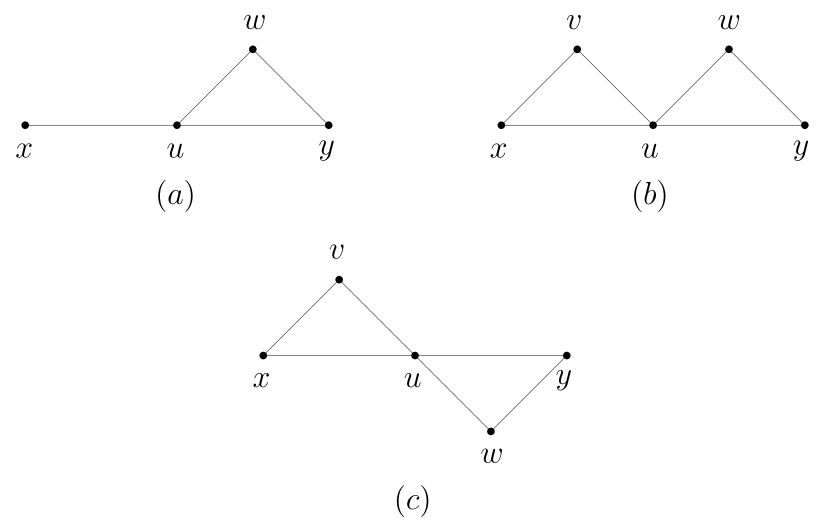


Figure 3.4

Claim 3.5.6. *B does not contain the configuration in Figure 3.4(a), where $uwyu$ is a face in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.4(a), where $uwyu$ is a face in G, and where only x and y are incident with edges in G not shown. Let f be the face $uwyu$, let f_1 be the face with $xuwy$ in its boundary and let f_2 be the face with xuy in its boundary. Since B is a block it follows that f_1 and f_2 are distinct. Let $H = G - w$ and let the faces in H that have xuy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper edge-face colouring from these lists.

(i): Each of the remaining elements wy, uw, f has a list of at least 1, 3, 3 usable colours respectively; so these elements can be coloured in this order.

(ii): Let the colours in every list be the integers $1, 2, \ldots, 5$. Suppose first that x is adjacent to y in G. Then, by Claim 3.5.4 and by the definition of B, it follows that $d_G(x) \geq 3$. When applying a colouring of H to G we may assume that f_1, f_2, ux, uy are coloured 1, 2, 4, 5 respectively.

If $d_B(y) = 3$, then xy is incident with both f_1 and f_2 and we may assume that xy is coloured 3. We can now colour wy, uw, f with 2, 3, 4 respectively. So we may assume that $d_B(y) = 4$.

Now xy is incident with either f_1 or f_2 and there is one further edge incident with y that is not shown, say e. If xy is incident with f_1, then xy is coloured either 2 or 3. If e is not coloured 4, then we can colour uw, f, wy with 2, 3, 4 respectively. If e is coloured 4, then we can give uw the same colour as xy, give wy whichever of 2 and 3 is not on xy, and colour f with 4.

If xy is incident with f_2, then xy is coloured either 1 or 3. If e is not coloured 4, then we can colour uw, f, wy with 2, 3, 4 respectively. If e is coloured 4, then we can colour uw, f, wy with 3, 4, 2 respectively.

So we may assume that x is not adjacent to y in G. Let $H_1 = H - u + xy$ and embed xy where xuy was embedded in H. By hypothesis H_1 has a proper edge-face colouring. In G, give both ux and wy the colour on xy in H_1. We can now colour in order uy, f, uw since each has at most four differently coloured neighbours at the time of its colouring. In every case the colouring can be completed, which is the required contradiction. □

Claim 3.5.7. *B does not contain the configuration in Figure 3.4(b) or 3.4(c), where in each case $xvux$ and $uwyu$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.4(b) or 3.4(c), where in each case $xvux$ and $uwyu$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xvux$ and let f' be the face $uwyu$. If G contains the configuration in Figure 3.4(b), let f_1 be the face with $xvuwy$ in its boundary and let f_2 be the face with xuy in its boundary. If G contains the configuration in Figure 3.4(c), let f_1 be the face with $xvuy$ in its boundary and let f_2 be the face with $xuwy$ in its boundary. Let $H = G - \{v, w\}$. Since, by Claim 3.5.6, both x and y have degree at least 4 in G, and since B is a block, it follows that f_1 and f_2 are distinct. Let the faces in H that have xuy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper edge-face colouring from these lists.

(i): Now each of vx, wy has at most Δ coloured neighbours in G, and each of uv, uw, f, f' has at most 3 coloured neighbours in G. So each of the remaining elements

$$vx, wy, uv, uw, f, f' \tag{3.3}$$

has a list of at least 1, 1, 3, 3, 3, 3 usable colours respectively. It follows that these elements can be coloured in the order (3.3).

(ii): Let the colours in every list be the integers $1, 2, \dots, 5$. In G, uncolour ux and uy. Suppose first that f_1 is not adjacent to f_2. Let $H_1 = G - \{u, v, w\}$. Let f_{12} be the face in H_1 formed from f_1 and f_2 in G. By hypothesis H_1 has a proper edge-face colouring. When applying a colouring of H_1 to G we give both f_1 and f_2 the colour on f_{12} in H_1. Note that since each of f and f' has three uncoloured neighbours and four usable colours in its list, it follows that both f and f' can be coloured at the end. So each of the remaining elements

$$vx, ux, uy, wy, uw, uv \tag{3.4}$$

has a list of exactly 2, 2, 2, 2, 4, 4 usable colours respectively. Furthermore, since f_1 and f_2 have the same colour, it follows that $L'(uw) = L'(uv)$, $L'(vx) = L'(ux) \subset L'(uw)$ and $L'(uy) = L'(wy) \subset L'(uw)$. If we try to colour the elements in the order (3.4) it is only with uv that we may fail. If possible, give vx and uy the same colour. The remaining elements can now be coloured in the order (3.4). So we may assume that $|L'(vx) \cap L'(uy)| = \emptyset$. So the remaining elements can be coloured in the order (3.4) where uw is given the same colour as vx, and uv is given the same colour as wy. So we may assume that f_1 is adjacent to f_2 so that f_1 and f_2 must be given different colours.

Suppose that $xy \in E(B)$. Since, by Claim 3.5.6, both x and y have degree at least 4 in G, and since $\Delta = 4$, it follows that $d_G(x) = d_G(y) = 4$. Furthermore, since B is a block, it follows that f_1 is not adjacent to f_2, which is a contradiction. So we may assume that $xy \notin E(B)$ and that x and y are connected by a path P of length at least 2 that is not shown.

Suppose that $P = xzy$. Then we may assume without loss of generality that xz separates f_1 from f_2; so there are no other paths from x to y that are not shown

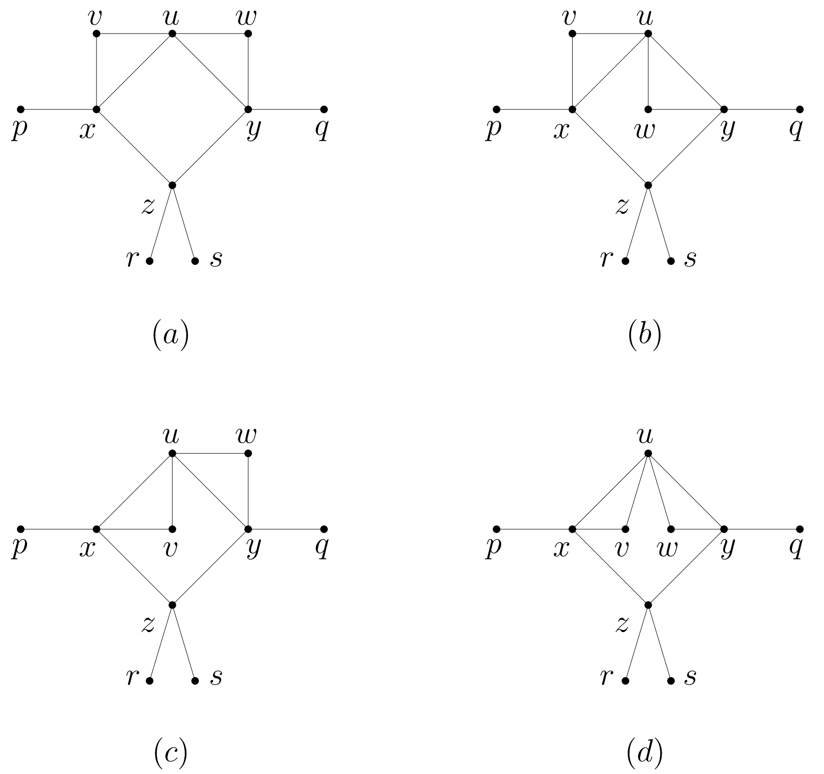


Figure 3.5

that are edge-disjoint from P. Since B is a block, and since $d_G(x) = d_G(y) = 4$, we may assume that $d_G(z) = 3$ or 4 and that yz does not separate f_1 from f_2. Let p be adjacent to x, where $p \neq u, v, z$. Then x is a cut-vertex and px is a cut-edge since xz separates f_1 from f_2. Furthermore, since x is a cut-vertex, and since in the statement of the claim $xvux$ is a face in G, then by the definition of B it follows that $xvux$ is not the bounding cycle of B.

Suppose that $d_G(z) = 4$. There are four cases to consider as shown in Figures 3.5(a)–3.5(d). Let the face with $qyzs$ in its boundary be f_3, which is distinct from f_1 and f_2 since yz does not separate f_1 from f_2. Let $H_1 = (G - \{u, v, w\})/xz$. Let f_{12} be the face in H_1 formed from f_1 and f_2 in G, and let j be the vertex formed from x and z in G. Note that j has degree $\Delta = 4$. By hypothesis H_1 has a proper edge-face colouring in which we may assume that f_{12}, jp, jy, jr are coloured 1, 2, 4, 5 respectively. Also, js is coloured either 1 or 3, f_3 is coloured either 2, 3 or 5, and qy is coloured either 2, 3 or 5. After uncolouring yz and f_{12}, this colouring of H_1 can be extended to a colouring σ of G as follows.

f_1	f_2	f	f'	vx	uv	ux	uw	uy	wy	xz	yz
1	3	4	5	3	2	5	3	4	2	4	1
1	3	4	2	3	2	5	3	4	5	4	1
1	5	3	4	5	2	4	5	1	2	3	4
1	5	2	4	5	3	4	5	1	3	3	4
1	2	3	4	5	2	4	3	1	5	3	4
1	2	3	4	5	2	4	3	5	2	3	4

Table 3.1

Case 1: $\sigma(sz) = 3$, $\sigma(f_3) = 2$ and $\sigma(qy) = 3$ or 5.

The remaining elements can be coloured as in line 1 of Table 3.1, with the exceptions that in Figure 3.5(b) put $\sigma(uw) = 1$, in Figure 3.5(c) put $\sigma(f_2) = \sigma(vx) = 1$ and $\sigma(f_1) = 3$, and in Figure 3.5(d) put $\sigma(f_2) = \sigma(vx) = \sigma(uw) = 1$ and $\sigma(f_1) = 3$.

Case 2: $\sigma(sz) = 3$, $\sigma(f_3) = 5$ and $\sigma(qy) = 2$ or 3.

The remaining elements can be coloured as in line 2 of Table 3.1, with the same exceptions as in Case 1.

Case 3: $\sigma(sz) = 1$, $\sigma(f_3) = 2$ and $\sigma(qy) = 3$ or 5.

The remaining elements can be coloured as in line 3 of Table 3.1, with the exceptions that in Figure 3.5(b) put $\sigma(uw) = 1$ and $\sigma(uy) = 3$ or 5, whichever is not on qy, in Figure 3.5(c) put $\sigma(f_2) = \sigma(vx) = 1$ and $\sigma(f_1) = 5$, and in Figure 3.5(d) put $\sigma(f_2) = \sigma(vx) = \sigma(uw) = 1$, $\sigma(f_1) = 5$ and $\sigma(uy) = 3$ or 5, whichever is not on qy.

Case 4: $\sigma(sz) = 1$, $\sigma(f_3) = 3$ and $\sigma(qy) = 2$ or 5.

The remaining elements can be coloured as in line 4 of Table 3.1, with the exceptions that in Figure 3.5(b) put $\sigma(uw) = 1$ and $\sigma(uy) = 2$ or 5, whichever is not on qy, in Figure 3.5(c) put $\sigma(f_2) = \sigma(vx) = 1$ and $\sigma(f_1) = 5$, and in Figure 3.5(d) put $\sigma(f_2) = \sigma(vx) = \sigma(uw) = 1$, $\sigma(f_1) = 5$, and $\sigma(uy) = 2$ or 5, whichever is not on qy.

Case 5: $\sigma(sz) = 1$, $\sigma(f_3) = 5$ and $\sigma(qy) = 2$.

The remaining elements can be coloured as in line 5 of Table 3.1, with the exceptions that in Figure 3.5(b) put $\sigma(uw) = 1$ and $\sigma(uy) = 3$, in Figure 3.5(c) put $\sigma(f_2) = \sigma(vx) = 1$, $\sigma(f_1) = 2$ and $\sigma(uv) = 5$, and in Figure 3.5(d) put $\sigma(f_2) = \sigma(vx) = \sigma(uw) = 1$, $\sigma(f_1) = 2$, $\sigma(uy) = 3$ and $\sigma(uv) = 5$.

Case 6: $\sigma(sz) = 1$, $\sigma(f_3) = 5$ and $\sigma(qy) = 3$.

The remaining elements can be coloured as in line 6 of Table 3.1, with the exceptions that in Figure 3.5(b) put $\sigma(wy) = 1$, in Figure 3.5(c) put $\sigma(f_2) = \sigma(uv) = 1$ and $\sigma(f_1) = 2$, and in Figure 3.5(d) put $\sigma(f_2) = \sigma(uv) = \sigma(wy) = 1$ and $\sigma(f_1) = 2$.

If $d_G(z) = 3$, then we may assume that the three vertices adjacent to z in G are x, y and s. Since in this case js and f_{12} must have different colours in H_1, it follows that in G, $\sigma(sz) = 3$ and we can colour the elements that are common with the case when $d_G(z) = 4$ as in Cases 1 and 2.

So we may assume that P is of length at least 3. Let z_1 and z_2 be the vertices of P that are adjacent to x and y in G respectively. Also, let p and q be adjacent to x and y in G respectively, as in the previous case. Let $H_1 = H/\{ux, uy\}$, which will not contain any loops or parallel edges since P is not of length 1 or 2 respectively. Let j be the vertex in H_1 formed from x and y in G. Note that j has degree $\Delta = 4$. By hypothesis H_1 has a proper edge-face colouring in which we may assume that jp, jz_1, jq, jz_2 are coloured 1, 2, 3, 4 respectively. When applying this colouring of H_1 to a colouring σ of G we may assume that f_1 is coloured either 2, 4, or 5 and f_2 is coloured either 1, 3, or 5. Recall that f_1 and f_2 must have different colours.

f	f'	vx	uv	ux	uw	uy	wy
3	3	5	1	4	5	2	1
3	3	4	1	5	4	2	1
1	1	3	5	4	3	2	5
1	1	3	2	4	3	5	2
3	4	5	1	4	3	5	2
2	3	5	1	3	4	5	2
3	4	4	1	5	3	2	5

Table 3.2

If G contains Figure 3.4(b), then every possible pair of colours for f_1 and f_2 are dealt with in Cases 7 and 8.

Case 7: f_2 is coloured either 1 or 5.

If f_1 is coloured either 2 or 4, then the remaining elements can be coloured as in line 1 of Table 3.2. If f_1 is coloured 5, and hence f_2 is coloured 1 since $\sigma(f_1) \neq \sigma(f_2)$, then the remaining elements can be coloured as in line 2 of Table 3.2.

Case 8: f_2 is coloured 3.

If f_1 is coloured either 2 or 4, then the remaining elements can be coloured as in line 3 of Table 3.2. If f_1 is coloured 5, then the remaining elements can be coloured as in line 4 of Table 3.2.

If G contains Figure 3.4(c), then, by symmetry, every possible pair of colours for f_1 and f_2 are dealt with in Cases 9–11.

Case 9: f_1 is coloured 2 and f_2 is coloured 1.

The remaining elements can be coloured as in line 5 of Table 3.2.

Case 10: f_1 is coloured 4 and f_2 is coloured 1.

The remaining elements can be coloured as in line 6 of Table 3.2.

Case 11: f_1 is coloured 5 and f_2 is coloured 1.
The remaining elements can be coloured as in line 7 of Table 3.2.

In every case we have obtained a contradiction, which proves Claim 3.5.7. □

Claim 3.5.1 implies that $B \not\cong K_2$ and Claim 3.5.2 implies that B is not a cycle; so B has at least two vertices with degree at least three and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined at the start of Section 3.3.

Claim 3.5.8. *B_1 is not K_4-minor-free.*

Proof. Since B has at least two vertices with degree at least 3, it follows that B_1 exists and has no vertex of degree 0. Suppose that x is a vertex of degree 1 in B_1. Then x is adjacent in B_1 to z_0. By the definition of B_1 and by Claim 3.5.2, it follows that $p_{xz_0} \geq 3$, and that every path between x and z_0 is in P_{xz_0}. So, by the definition of B, it follows that x must occur in B as vertex x in one of the configurations in Figures 3.3(b)–3.3(d), where the faces are as shown and where only x and y may be incident with edges in G not shown. However, Claims 3.5.4 and 3.5.5 show that this is impossible. So B_1 has no vertex of degree 1.

In view of Claims 3.5.1 and 3.5.2, it follows from Lemma 3.3.2 that B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, where $p_{ux} + p_{uy} = d_G(u) \geq 3$, where $p_{uy} \geq 2$, and where no two paths in P_{uy} bound a region that has a path not in P_{uy} embedded in it, and no two paths in P_{ux} bound a region that has a path not in P_{ux} embedded in it also.

By Claims 3.5.4 and 3.5.5, it follows that $p_{uy} = 2$ and $p_{ux} \leq 2$, and so $d_G(u) \leq 4$. By Claim 3.5.3, it follows that u must occur in B as vertex u in Figure 3.4(a), 3.4(b), or 3.4(c), where the faces are as shown and where only x and y are incident with edges in G not shown. (Note that w, and v if present, have degree 2 in G

and are therefore different from z_0.) However, Claims 3.5.6 and 3.5.7 show that this is impossible. This contradiction completes the proof of Claim 3.5.8. □

Since B_1 is a minor of G, Claim 3.5.8 implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 3.5.2. □

3.6 Entire choosability of plane embeddings of K_4-minor-free graphs

In this section we will first prove part (v) of Theorem 3.3.1 if $\Delta = 3$, which is restated in Theorem 3.6.1. We will then prove part (v) of Theorem 3.3.1 if $\Delta \geq 4$, which is restated in Theorem 3.6.2. As in Section 3.5, for each uncoloured element z in G, let $L'(z)$ denote the list of usable colours for z; that is, $L'(z)$ denotes $L(z)$ minus any colours already used on neighbours of z in G.

Theorem 3.6.1. *Let G be a plane embedding of a K_4-minor-free graph with maximum degree* 3. *Then* $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$.

Proof. Suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with maximum degree 3 such that $\mathrm{ch}_{\mathrm{vef}}(G) > 7$. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of 7 colours such that G has no proper entire colouring from these lists.

Since $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$ by Theorem 3.4.1, it follows that the vertices and faces of G can be coloured from their lists. Since every edge is incident with two vertices and at most two faces, every edge has at least 3 usable colours in its list. Since $\mathrm{ch}'(G) = 3$ by Theorem 3.1.3, it follows that these edges can be coloured. □

Note that by Theorem 3.1.1(vi)–$(viii)$ and Theorem 3.6.1, if G is a plane embedding of a K_4-minor-free graph with maximum degree $\Delta \leq 3$, then $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$.

Theorem 3.6.2. *Let G be a plane embedding of a K_4-minor-free graph with maximum degree $\Delta \geq 4$. Then*

(*i*) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \Delta + 2$ *if* $\Delta \geq 5$;

(*ii*) $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$ *if* $\Delta = 4$.

Proof. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that G is a counterexample to either part. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of $\Delta + 2$ or 7 colours as appropriate. Assume also that G has no proper entire colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined at the start of Section 3.3.

Claim 3.6.1. *G does not contain a vertex of degree* 1*.*

Proof. Suppose that u is a vertex of degree 1 in G that is adjacent to v. Let $H = G - u$. By hypothesis H has a proper entire colouring from its lists. The edge uv has at most $\Delta + 1$ coloured neighbours, and so uv can be given a colour from its list. Since u now has three coloured neighbours u can be coloured from its list. This contradiction proves Claim 3.6.1. □

Claim 3.6.2. *B does not contain two adjacent vertices of degree* 2 *in G.*

Proof. Suppose that $xuvy$ is a path in B (or a cycle if $x = y$) where both u and v have degree 2 in G. If $x \neq y$, let $H = G/uv$. By hypothesis H has a proper entire colouring from its lists. After applying a colouring of H to G, the remaining elements uv, u, v can be coloured in any order since each has at least one usable colour in its list at the time of its colouring. If $x = y$, then $B \cong K_3$. Let f be the interior face of B. Let $H = G - \{u, v\}$ where the face in H in which u and v were embedded is given the same list as the exterior face of B. By hypothesis H has a proper entire colouring from its lists.

Now each of ux, vx, u, v, f, uv has at most Δ, Δ, 2, 2, 2, 1 coloured neighbours in G respectively. So each of the remaining elements

$$ux, vx, u, v, f, uv \tag{3.5}$$

has a list of at least 2, 2, 5, 5, 5, 6 usable colours respectively. It follows that the remaining elements can be coloured in the order (3.5). This contradiction proves Claim 3.6.2. □

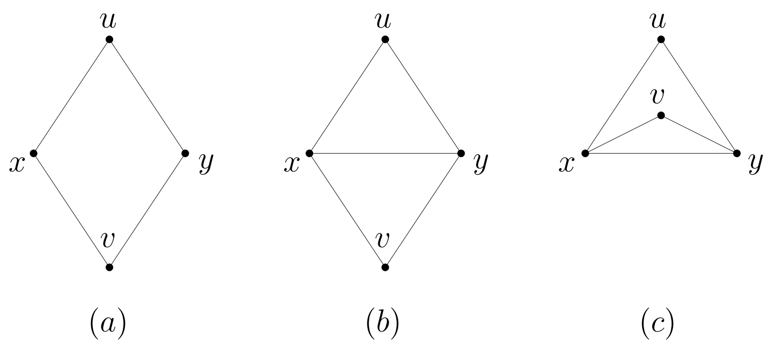


Figure 3.6

Claim 3.6.3. *If B contains the configuration in Figure 3.6(a), where $xuyvx$ is an interior face, where x is not adjacent to y, and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$ or 6.*

Proof. Suppose that B contains the configuration in Figure 3.6(a), where $xuyvx$ is an interior face, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the interior face $xuyvx$. Since, by Claim

3.6.2, both x and y have degree at least 3 in G, and if C is not 2-connected then B is an end-block by definition, it follows that f is adjacent to two different faces. Let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xvy in its boundary. Let $H = G - \{u, v\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u and v can be coloured at the end since each has six neighbours and a list of at least seven colours.

(i): Suppose first that $\Delta \geq 7$. Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f since it has only eight coloured neighbours, and then colour u and v. So we may assume that $\Delta = 5$ or 6, and contrary to what we want to prove, that $d_G(x) \leq \Delta - 1$ and that $d_G(y) \leq \Delta$.

Now each of uy, vy, f, ux, vx has at most Δ, Δ, 4, $\Delta - 1$, $\Delta - 1$ coloured neighbours in G respectively. So each of the remaining elements

$$uy, vy, f, ux, vx \tag{3.6}$$

has a list of at least 2, 2, 3, 3, 3 usable colours[5] respectively. If we try to colour the elements in the order (3.6) then it is only with vx that we may fail.

If possible, give ux and vy the same colour. The remaining elements can now be coloured in the order (3.6). So we may assume that $L'(ux) \cap L'(vy) = \emptyset$ so that $|L'(ux) \cup L'(vy)| \geq 5$. Now either $|L'(vx)| \geq 5$, or else ux or vy can be given a colour that is not in $L'(vx)$. In each case the remaining elements can be coloured in the order (3.6), using a colour that is not in $L'(vx)$ on a neighbour of vx at the first opportunity.

[5] Recall that $L'(z)$ denotes the list of usable colours for z in G.

(ii): Colour f, which is obviously possible. Next, since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. In every case the colouring can be completed, which is the required contradiction. □

Claim 3.6.4. *If B contains the configuration in Figure* 3.6(b) *or* 3.6(c), *where in each case the faces are as shown and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$.*

Proof. Suppose that B contains the configuration in Figure 3.6(b) or 3.6(c), where in each case the faces are as shown and where only x and y are incident with edges in G not shown. Let f be the face $xuyx$ or $xuyvx$ as appropriate. Let f' be the face $xvyx$. Let the other face with xuy in its boundary be f_1 and let the other face with xvy or xy in its boundary be f_2 as appropriate. (It is possible that $f_1 = f_2$ but the proof given here is still valid in this case.) Let $H = G - \{u, v\}$. Let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u and v can be coloured at the end since each has six neighbours and a list of at least seven colours.

(i): Suppose first that $\Delta \geq 6$. Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f and then f' since each has at most seven coloured neighbours at the time of its colouring. So we may assume that $\Delta = 5$, and contrary to what we want to prove, that $d_G(x) \leq \Delta - 1$ and that $d_G(y) \leq \Delta$.

If B contains the configuration in Figure 3.6(b) or 3.6(c), then each of uy, vy, f, ux, vx, f' has in Figure 3.6(b) at most 5, 5, 4, 4, 4, 4 coloured neighbours in G respectively, or in Figure 3.6(c) at most 5, 4, 3, 4, 3, 4 coloured neighbours in G

respectively. So each of the remaining elements

$$uy, vy, f, ux, vx, f' \tag{3.7}$$

has in Figure 3.6(b) a list of at least 2, 2, 3, 3, 3, 3 usable colours respectively, or in Figure 3.6(c) a list of at least 2, 3, 4, 3, 4, 3 usable colours respectively. If we try to colour the elements in the order (3.7) then it is only with f' that we may fail.

If B contains the configuration in Figure 3.6(b), then, if possible, give vy and f the same colour. The remaining elements can now be coloured in the order (3.7). So we may assume that $L'(vy) \cap L'(f) = \emptyset$ so that $|L'(vy) \cup L'(f)| \geq 5$. Now either $|L'(f')| \geq 5$, or else vy or f can be given a colour that is not in $L'(f')$. In each case the remaining elements can be coloured in the order (3.7).

If B contains the configuration in Figure 3.6(c), then either $|L'(f')| \geq 4$, or else f can be given a colour that is not in $L'(f')$. In each case the remaining elements can be coloured in the order (3.7).

(ii): Colour f and f' which is obviously possible. Next, since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. In every case the colouring can be completed, which is the required contradiction. □

Claim 3.6.5. *B does not contain the configuration in Figure* 3.7(a), *where $uwyu$ is a face in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.7(a), where $uwyu$ is a face in G, and where only x and y are incident with edges in G not shown. Let f be the face $uwyu$, let f_1 be the face with $xuwy$ in its boundary and let f_2 be the face with xuy in its boundary. Since B is a block it follows that f_1

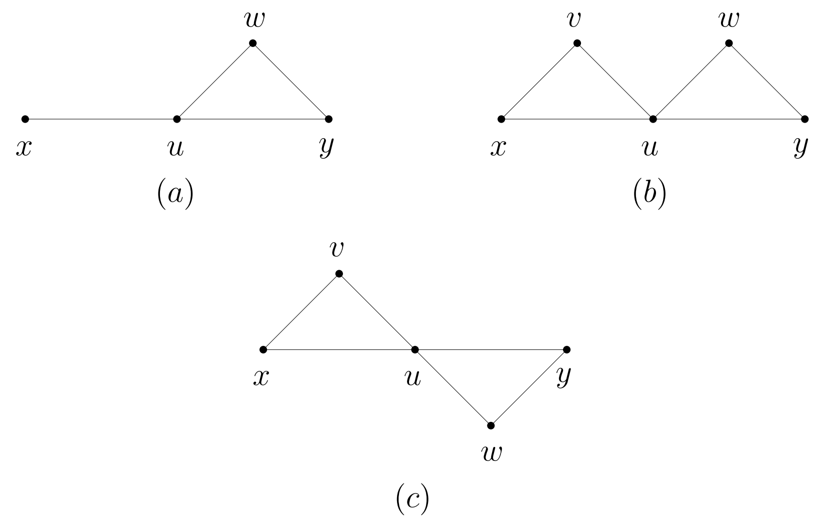


Figure 3.7

and f_2 are distinct. Let $H = G - w$ and let the faces in H that have xuy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists.

Now each of wy, f, uw, w has at most $\Delta + 1$, 5, 4, 3 coloured neighbours in G respectively, and so each has a list of at least 1, 2, 3, 4 usable colours respectively; so these elements can be coloured in this order. This contradiction completes the proof of Claim 3.6.5. □

Claim 3.6.6. *B does not contain the configuration in Figure 3.7(b) or Figure 3.7(c), where in each case $xvux$ and $uwyu$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.7(b) or Figure 3.7(c), where in each case $xvux$ and $uwyu$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xvux$ and let f' be the face $uwyu$. If G contains the configuration in Figure 3.7(b), let f_1 be the face with $xvuwy$ in its boundary and let f_2 be the face with xuy in its boundary. If

G contains the configuration in Figure 3.7(c), let f_1 be the face with $xvuy$ in its boundary and let f_2 be the face with $xuwy$ in its boundary. Let $H = G - \{v, w\}$. Since, by Claim 3.6.5, both x and y have degree at least 4 in G, and since B is a block, it follows that f_1 and f_2 are distinct. Let the faces in H that have xuy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that v and w can be coloured at the end since each has six neighbours and a list of at least seven colours.

First uncolour ux, u and uy. Now each of wy, uy, ux, vx, u, f, uv, uw, f' has at most Δ, Δ, Δ, Δ, 4, 3, 1, 1, 3 coloured neighbours in G respectively. So each of the remaining elements

$$wy, uy, ux, vx, u, f, uv, uw, f' \tag{3.8}$$

has a list of at least 2, 2, 2, 2, 3, 4, 6, 6, 4 usable colours respectively. If we try to colour the elements in the order (3.8) then it is only with f' that we may fail.

If possible, give ux and wy the same colour. The remaining elements can now be coloured in the order (3.8) with the exception that uw is coloured last. So we may assume that $L'(ux) \cap L'(wy) = \emptyset$. If possible, give u and wy the same colour. Since the colour on u is not in $L'(ux)$ the remaining elements can now be coloured in the order (3.8). So we may assume that $L'(u) \cap L'(wy) = \emptyset$ so that $|L'(u) \cup L'(wy)| \geq 5$. Now either $|L'(f')| \geq 5$, or else u or wy can be given a colour that is not in $L'(f')$. If $|L'(f')| \geq 5$, or if wy is given a colour that is not in $L'(f')$, then the remaining elements can be coloured in the order (3.8). So we may assume that u is given a colour α that is not in $L'(f')$. If $\alpha \notin L'(uy)$, then the remaining elements can be coloured in the order (3.8) with the exception that both ux and uy are coloured before wy *in that order*. If $\alpha \in L'(uy)$, then give uy the colour α and uncolour u. The remaining elements can now be coloured in the order (3.8). This contradiction proves Claim 3.6.6. $\square$

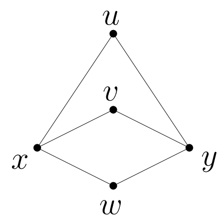


Figure 3.8

Claim 3.6.7. *If B contains the configuration in Figure 3.8, where $xuyvx$ and $xvywx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$.*

Proof. Suppose that B contains the configuration in Figure 3.8, where $xuyvx$ and $xvywx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$ and let f' be the face $xvywx$. Let the other face with xuy in its boundary be f_1 and let the other face with xwy in its boundary be f_2. Since, by Claim 3.6.3, $d_G(x) = d_G(y) = \Delta$ and $\Delta = 6$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w can be coloured at the end since each has six neighbours and a list of eight colours.

Now each of wy, wx, ux, uy, vy, vx, f, f' has at most 5, 5, 5, 5, 4, 4, 3, 3 coloured neighbours in G respectively. So each of the remaining elements

$$wy, wx, ux, uy, vy, vx, f, f' \tag{3.9}$$

has a list of at least 3, 3, 3, 3, 4, 4, 5, 5 usable colours respectively. If we try to colour the elements in the order (3.9) then it is only with f' that we may fail.

If possible, colour both vx and vy so that vx is given a colour that is not in $L'(f')$. Next, since each edge of the 4-cycle $xuywx$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f and then f' since each has at least one usable colour in its list at the time of its colouring. So we may assume that $L'(vx) \subseteq L'(f')$. If possible, give vx and wy the same colour. The remaining elements can now be coloured in the order (3.9). So we may assume that $L'(vx) \cap L'(wy) = \emptyset$ so that $|L'(vx) \cup L'(wy)| \geq 7$. Now either $|L'(f')| \geq 7$, or else wy can be given a colour that is not in $L'(f')$ since $L'(vx) \subseteq L'(f')$. In each case the remaining elements can be coloured in the order (3.9). This contradiction proves Claim 3.6.7. □

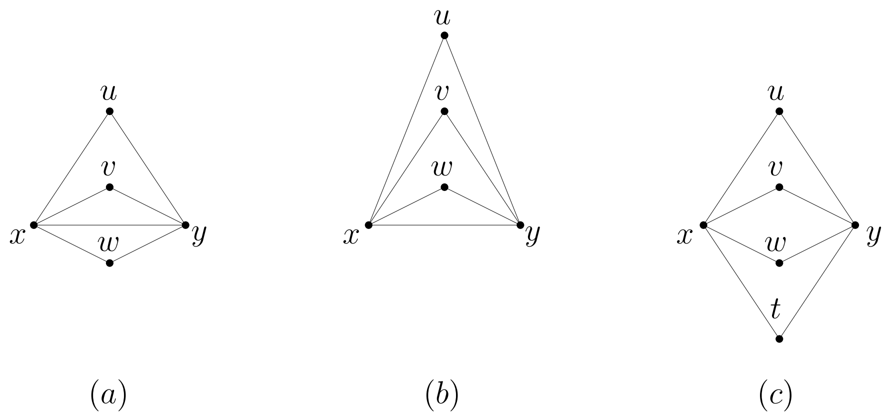


Figure 3.9

Claim 3.6.8. *B does not contain the configuration in Figure 3.9(a), where $xuyvx$, $xvyx$ and $xywx$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.9(a), where $xuyvx$, $xvyx$ and $xywx$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$, let f' be the face $xvyx$ and let f'' be the face $xywx$. Also, let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xwy in its boundary. Since, by Claim 3.6.4, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\}$ and let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w can be coloured at the end since each has six neighbours and a list of seven colours. First uncolour xy.

Now each of vy, vx, f' has 2 coloured neighbours in G, each of wy, wx, f'', ux, uy, f has 3 coloured neighbours in G, and xy has 4 coloured neighbours in G. So each of the remaining elements z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 5$ if $z \in \{vy, vx, f'\}$, $|L'(z)| \geq 4$ if $z \in \{wy, wx, f'', ux, uy, f\}$, and $|L'(xy)| \geq 3$. Now either $|L'(f)| \geq 5$, or else vy can be given a colour that is not in $L'(f)$. In each case colour vy. At this point, each of the remaining elements

$$xy, wy, wx, f'', ux, vx, uy, f, f' \tag{3.10}$$

has a list L'' of at least 2, 3, 4, 4, 4, 4, 3, 4, 4 usable colours respectively.

If possible, give f'' and vx the same colour. The remaining elements can now be coloured in the order (3.10) with the exception that if we fail at uy, then since $|L(uy)| = 7$ and at the time of its colouring uy has seven coloured neighbours in G, we can uncolour vy and give uy the colour that was on vy. We can now recolour vy since it has six coloured neighbours in G and a list of seven colours. Finally, we can give colours to f and then f'. So we may assume that $L''(f'') \cap L''(vx) = \emptyset$ so that $|L''(f'') \cup L''(vx)| \geq 8$. Now either $|L''(f')| \geq 8$, or else f'' or vx can be given a colour that is not in $L''(f')$. In each case the remaining elements can be coloured in the order (3.10), although, as above, it may be necessary to give uy the colour that is on vy and to recolour vy. This contradiction completes the proof of Claim 3.6.8. □

Claim 3.6.9. *B does not contain the configuration in Figure 3.9(b), where $xuyvx$, $xvywx$ and $xwyx$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.9(b), where $xuyvx$, $xvywx$ and $xwyx$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$, let f' be the face $xvywx$ and let f'' be the face $xwyx$. Also, let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xy in its boundary. Since, by Claim 3.6.4, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\}$ and let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w can be coloured at the end since each has six neighbours and a list of seven colours. First uncolour xy.

Now each of wy, wx, vy, vx, f' has 2 coloured neighbours in G, each of uy, ux, f, f'' has 3 coloured neighbours in G, and xy has 5 coloured neighbours in G. So each of the remaining elements z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 5$ if $z \in \{wy, wx, vy, vx, f'\}$, $|L'(z)| \geq 4$ if $z \in \{uy, ux, f, f''\}$, and $|L'(xy)| \geq 2$. Now either $|L'(f'')| \geq 5$, or else wy can be given a colour that is not in $L'(f'')$. In each case colour wy, and then colour xy. At this point, each of the remaining elements

$$uy, ux, f, vy, vx, wx, f', f'' \tag{3.11}$$

has a list L'' of at least 2, 3, 4, 3, 4, 3, 4, 3 usable colours respectively.

If possible, give f and wx the same colour. The remaining elements can now be coloured in the order (3.11). So we may assume that $L''(f) \cap L''(wx) = \emptyset$ so that $|L''(f) \cup L''(wx)| \geq 7$. Now either $|L''(f')| \geq 7$, or else f or wx can be given a colour that is not in $L''(f')$. In each case the remaining elements can be coloured in the order (3.11) with the exception that if wx is given a colour that is not in $L''(f')$ and we fail at vx, then since $|L(vx)| = 7$ and at the time of its colouring vx has seven coloured neighbours in G, we can uncolour wx and give vx the colour that was on wx. We can now recolour wx since it has six coloured neighbours in G and a list of seven colours. Finally, we can give colours to f' and then f''. This contradiction proves Claim 3.6.9. □

Claim 3.6.10. *B does not contain the configuration in Figure 3.9(c), where $xuyvx$, $xvywx$ and $xwytx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 3.9(c), where $xuyvx$, $xvywx$ and $xwytx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$, let f' be the face $xvywx$ and let f'' be the face $xwytx$. Also, let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xty in its boundary. Since, by Claim 3.6.3, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w, t\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w and t can be coloured at the end since each has six neighbours and a list of seven colours.

Now each of wy, wx, vx, vy, f' has 2 coloured neighbours in G, and each of ty, tx, ux, uy, f, f'' has 3 coloured neighbours in G. So each of the remaining elements z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 5$ if $z \in \{wy, wx, vx, vy, f'\}$, and $|L'(z)| \geq 4$ if $z \in \{ty, tx, ux, uy, f, f''\}$. Now either $|L'(f)| \geq 5$, or else vy can be given a colour that is not in $L'(f)$. Similarly, either $|L'(f'')| \geq 5$, or else wx

can be given a colour that is not in $L'(f'')$. In each case colour both vy and wx. At this point, each of the remaining elements

$$ty, tx, wy, ux, vx, uy, f', f, f'' \tag{3.12}$$

has a list L'' of at least 3, 3, 3, 3, 3, 3, 3, 4, 4 usable colours respectively.

If possible, give uy and vx the same colour. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z, where $|L'''(wy)| \geq 2$, $|L'''(tx)| \geq 2$, and $|L'''(f'')| \geq 4$. If $|L'''(wy)| = 2$ and $|L'''(tx)| = 2$, then it follows that the colour on wx was in both $L'(wy)$ and $L'(tx)$. So it is possible to give both wy and tx the colour on wx and to recolour wx. The remaining elements can now be coloured in the order (3.12). So we may assume that at least one of $L'''(wy)$ and $L'''(tx)$ has at least three colours. If possible, give wy and tx the same colour. The remaining elements can now be coloured in the order (3.12). So we may assume that $L'''(wy) \cap L'''(tx) = \emptyset$ so that $|L'''(wy) \cup L'''(tx)| \geq 5$. Now either $|L'''(f'')| \geq 5$, or else wy or tx can be given a colour that is not in $L'''(f'')$. In each case the remaining elements can be coloured in the order (3.12). So we may assume that this is not possible so that $L''(uy) \cap L''(vx) = \emptyset$, and, by symmetry, that $L''(wy) \cap L''(tx) = \emptyset$.

Since $|L''(uy) \cup L''(vx)| \geq 6$, either $|L''(f)| \geq 6$, or else uy or vx can be given a colour that is not in $L''(f)$. If $|L''(f)| \geq 6$, or uy can be given a colour that is not in $L''(f)$, then colour uy. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z. Now $|L'''(wy) \cup L'''(tx)| \geq 5$, so either $|L'''(f'')| \geq 5$, or else wy or tx can be given a colour that is not in $L'''(f'')$. In each case the remaining elements can be coloured in the order (3.12). So we may assume that vx can be given a colour that is not in $L''(f)$. Again, at this point, $|L'''(wy) \cup L'''(tx)| \geq 5$, so either $|L'''(f'')| \geq 5$, or else wy or tx can be given a colour that is not in $L'''(f'')$. In each case colour both wy and tx. The remaining elements can now be coloured in the order (3.12) with the exception that if we fail at uy, then since $|L(uy)| = 7$ and at the time of its colouring uy has seven coloured neighbours in G, we can uncolour vy and give uy the colour that was on vy. We can now recolour vy since it has six coloured neighbours in G and a list of seven colours. Finally, we can give colours to f', f, f'' *in that order.* This contradiction proves Claim 3.6.10. □

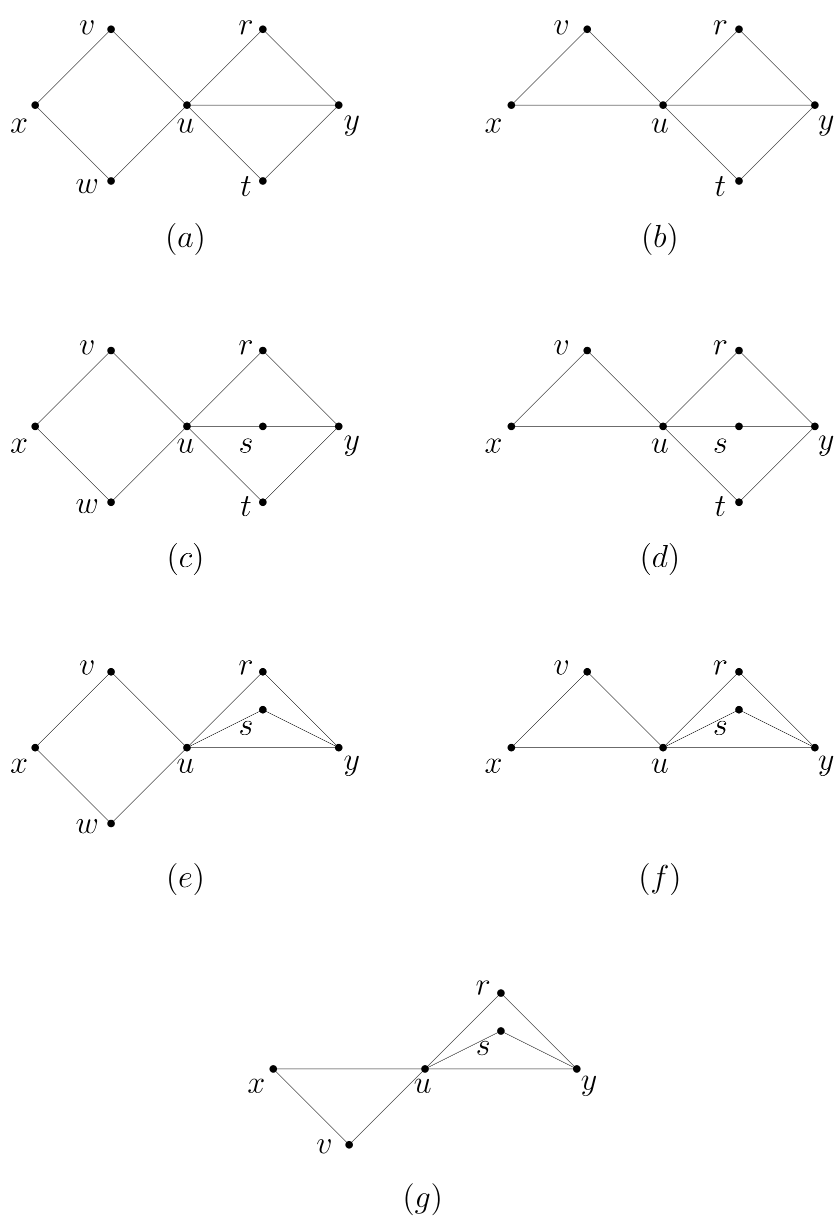


Figure 3.10

Claim 3.6.11. *B does not contain one of the configurations in Figures* 3.10(a)–3.10(d), *where the faces are as shown and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain one of the configurations in Figures 3.10(a)–3.10(d), where the faces are as shown and where only x and y are incident with edges in G not shown. Let f be the face $uryu$ or $urysu$ as appropriate. Let f' be the face $utyu$ or $utysu$ as appropriate and let f'' be the face $xvuwx$ or $xvux$ as appropriate. Also, let f_1 be the face with xvu in its boundary that is different from f'' and let f_2 be the face with uty in its boundary that is different from f'. Since B is a block it follows that both x and y are incident with edges not shown and that f_1 and f_2 are distinct. Let $H = G - r$ and let the faces in H that have xvu and uty in their boundary have the same lists as f_1 and f_2 in G respectively. By hypothesis H has a proper entire colouring from these lists. First uncolour all elements of the configuration being considered except for x, y, f_1 and f_2. Note that where present, each of v, w, r, s, t can be coloured at the end since each has six neighbours and a list of seven colours.

	vx	wx	ux	uv	uw	f''	u	ru	su	uy	tu	ry	sy	ty	f	f'
(a)	5	5		1	1	3	3	1		3	1	4		4	2	2
(b)	5		5	1		3	4	1		3	1	4		4	2	2
(c)	5	5		1	1	3	2	1	0		1	4	3	4	2	2
(d)	5		5	1		3	3	1	0		1	4	3	4	2	2
(a)	2	2		6	6	4	4	6		4	6	3		3	5	5
(b)	2		2	6		4	3	6		4	6	3		3	5	5
(c)	2	2		6	6	4	5	6	7		6	3	4	3	5	5
(d)	2		2	6		4	4	6	7		6	3	4	3	5	5

Table 3.3

For each of the configurations in Figures 3.10(a)–3.10(d) the maximum number of coloured neighbours of the remaining elements is given in the first half of Table 3.3, and the minimum number of usable colours in the list of each remaining element is given in the second half of Table 3.3.

Now either $|L'(f')| \geq 6$, or else tu can be given a colour that is not in $L'(f')$. In each case colour tu.

If B contains the configuration in Figure 3.10(a) or 3.10(c), then we can colour in order uw, wx, vx, f'', u, uv since each has at least one usable colour in its list at the time of its colouring.

If B contains the configuration in Figure 3.10(b) or 3.10(d), then either $|L'(f'')| \geq 5$, or else uv can be given a colour that is not in $L'(f'')$. In each case colour in order ux, vx, u, uv, f'' so that, where possible, at least one of these is given a colour that is not in $L'(f'')$.

At this point, if B contains the configuration in Figure 3.10(a) or 3.10(b), then each of the remaining elements

$$ru, uy, ry, ty, f, f' \tag{3.13}$$

has a list L'' of at least 2, 0, 3, 2, 4, 4 usable colours respectively.

Since $d_G(y) = \Delta = 5$ by Claim 3.6.4, it follows that uy has seven coloured neighbours. If $|L''(uy)| = 0$, then since $|L(uy)| = 7$, it follows that the colour on tu is in $L(uy)$ and is not used on any other neighbours of uy. So we can give uy the colour on tu and uncolour tu. At this point, since each edge of the 4-cycle $urytu$ has at least two usable colours in its list, it follows from Theorem 3.1.3 that these edges can be coloured. We can now colour f and then f' since each has at least one usable colour in its list at the time of its colouring.

So we may assume that $|L''(uy)| \geq 1$, and so we can colour uy. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z. If $|L'''(ty)| \geq 2$, then the remaining elements can be coloured in the order (3.13). So we may assume that $|L'''(ty)| = 1$. Since ty has six coloured neighbours and $|L(ty)| = 7$, it follows that the colour on tu is in $L(ty)$ and is not used on any other neighbour of ty. So if the colour on tu is in $L'''(ry)$, then give this colour to ry; otherwise give this colour to ty and recolour tu. In each csse the remaining elements can be coloured in the order (3.13).

So we may assume that B contains the configuration in Figure 3.10(c) or 3.10(d). Now each of the remaining elements

$$ry, ru, su, sy, ty, f, f' \tag{3.14}$$

has a list L'' of at least 3, 2, 3, 4, 2, 4, 4 usable colours respectively.

If possible, give f and ty the same colour. The remaining elements can now be coloured in the order (3.14) with the exception that ru is coloured first. So we may assume that $L''(f) \cap L''(ty) = \emptyset$ so that $|L''(f) \cup L''(ty)| \geq 6$.

Now either $|L''(f')| \geq 6$, or else f or ty can be given a colour that is not in $L''(f')$. If $|L''(f')| \geq 6$, or ty can be given a colour that is not in $L'(f')$, then colour ty. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z. If possible, give ru and sy the same colour. The remaining elements can now be coloured in the order (3.14). So we may assume that $L'''(ru) \cap L'''(sy) = \emptyset$ so that $|L'''(ru) \cup L'''(sy)| \geq 5$. Now either $|L'''(f)| \geq 5$, or else ru or sy can be given a colour that is not in $L'''(f)$. In each case the remaining elements can be coloured in the order (3.14).

So we may assume that $L''(ty) \subseteq L''(f')$. If $|L''(ty) \cap L''(ry)| \geq 1$, then we can give f' and ry the same colour. The remaining elements can now be coloured in the order (3.14) with the exception that ty is coloured first. So we may assume that

$L''(ty) \cap L''(ry) = \emptyset$. We can now give f a colour that is not in $L''(f')$ so that the remaining elements can be coloured in the order (3.14) with the exception that ru is coloured first. In every case the colouring can be completed, which is the required contradiction. $\square$

Claim 3.6.12. *B does not contain one of the configurations in Figures* 3.10(e)–3.10(g), *where the faces are as shown and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain one of the configurations in Figures 3.10(e)–3.10(g), where the faces are as shown and where only x and y are incident with edges in G not shown. Let f be the face $urysu$, let f' be the face $usyu$. Let f'' be the face $xvuwx$ or $xvux$ as appropriate. Also, let f_1 be the face with ury in its boundary that is different from f and let f_2 be the face with uy in its boundary that is different from f'. Since B is a block it follows that both x and y are incident with edges not shown and that f_1 and f_2 are distinct. Let $H = G - r$ and let the faces in H that have usy and uy in their boundary have the same lists as f_1 and f_2 in G respectively. By hypothesis H has a proper entire colouring from these lists. First uncolour all elements of the given configurations except for x, y, f_1 and f_2. Note that where present, each of v, w, r, s, can be coloured at the end since each has six neighbours and a list of seven colours.

	vx	wx	ux	uv	uw	f''	u	ru	su	uy	ry	sy	f	f'
(e)	5	5		1	1	3	3	1	0	4	4	3	2	2
(f) and (g)	5		5	1		3	4	1	0	4	4	3	2	2
(e)	2	2		6	6	4	4	6	7	3	3	4	5	5
(f) and (g)	2		2	6		4	3	6	7	3	3	4	5	5

Table 3.4

For each of the configurations in Figures 3.10(e)–3.10(g) the maximum number of coloured neighbours of the remaining elements is given in the first half of Table 3.4, and the minimum number of usable colours in the list of each remaining element is given in the second half of Table 3.4.

If B contains the configuration in Figure 3.10(e), then either $|L'(f')| \geq 7$, or else su can be given a colour that is not in $L'(f')$. In each case colour su, u, uy. At this point each of the elements

$$vx, wx, f'', uv, uw \tag{3.15}$$

has a list L'' of at least 2, 2, 3, 3, 3 usable colours respectively. If we try to colour these elements in the order (3.15) then it is only with uw that we may fail.

If possible, give uv and wx the same colour. The remaining elements can now be coloured in the order (3.15). So we may assume that $L''(uv) \cap L''(wx) = \emptyset$ so that $|L''(uv) \cup L''(wx)| \geq 5$. Now either $|L''(uw)| \geq 5$, or else uv or wx can be given a colour that is not in $L''(uw)$. In each case the remaining elements can be coloured in the order (3.15), using a colour that is not in $L''(uw)$ on a neighbour of uw at the first opportunity.

If B contains the configuration in Figure 3.10(f) or 3.10(g), then first we will colour the elements

$$ux, vx, u, uv, uy, f'', su. \tag{3.16}$$

Now either $|L'(f')| \geq 7$, or else su can be given a colour that is not in $L'(f')$. If $|L'(f')| \geq 7$, then colour uy; otherwise, at the first opportunity, colour exactly one of uy, u, su using a colour that is not in $L'(f')$. At this point, let $L''(z)$ be the list of usable colours for each remaining element z. Now either $|L''(f'')| \geq 5$, or else uv can be given a colour α that is not in $L''(f'')$. In all cases the remaining elements in (3.16) can be coloured in order, using a colour that is not in $L''(f'')$ at the first opportunity, and with the exception that if it were su that was given a colour that is not in $L'(f')$, and hence not in $L'(uy)$ or $L'(u)$, then uy is coloured immediately after vx with a colour that is different from α.

At this point, if the configuration is in Figure 3.10(e), 3.10(f) or 3.10(g), then each of the remaining elements

$$ru, ry, sy, f, f' \tag{3.17}$$

has a list L''' of at least 1, 2, 2, 3, 3 usable colours respectively. If we try to colour the elements in the order (3.17) then it is only with f that we may fail.

Let β be the colour given to su. Suppose that $\beta \notin L(sy)$ or that β is used on another neighbour of sy so that $|L'''(sy)| \geq 3$. The remaining elements can now be coloured in the order (3.17) with the exception that sy is coloured immediately after f. So we may assume that $\beta \in L(sy)$ and that β is not used on any other neighbour of sy. Suppose that $\beta \notin L(ru)$ or that β is used on another neighbour of ru so that $|L'''(ru)| \geq 2$. If possible, give ru and sy the same colour. The remaining elements can now be coloured in the order (3.17). So we may assume that $L'''(ru) \cap L'''(sy) = \emptyset$ so that $|L'''(ru) \cup L'''(sy)| \geq 4$. Now either $|L'''(f)| \geq 4$, or else ru or sy can be given a colour that is not in $L'(f)$. In each case the remaining elements can be coloured in the order (3.17) with the exception that ry is coloured first. So we may assume that $\beta \in L(ru)$ and that β is not used on any other neighbour of ru. So we can give ru and sy the colour β and recolour su. The remaining elements can now be coloured in the order (3.17). In every case the colouring can be completed, which is the required contradiction. □

Claim 3.6.1 implies that $B \not\cong K_2$ and Claim 3.6.2 implies that B is not a cycle; so B has at least two vertices with degree at least three and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined at the start of Section 3.3.

CLAIM 3.6.13. *B_1 is not K_4-minor-free.*

Proof. Since B has at least two vertices with degree at least 3, it follows that B_1 exists and has no vertex of degree 0. Suppose that x is a vertex of degree 1 in B_1. Then x is adjacent in B_1 to z_0. By the definition of B_1 and by Claim 3.6.2, it follows that $p_{xz_0} \geq 3$, and that every path between x and z_0 is in P_{xz_0}. So, by the definition of B, it follows that x must occur in B as vertex x in Figure 3.6(b), 3.6(c) or 3.8, where the faces are as shown and where only x and y may be incident with edges in G not shown. Since, by Claims 3.6.4 and 3.6.7, both x and z_0 must have degree $\Delta = 5$ in G, it follows that $p_{xz_0} = 5$. So B must contain one of the configurations in Figure 3.9, where the faces are as shown and where only x and y are incident with edges in G not shown. However, Claims 3.6.8–3.6.10 show that this is impossible. So B_1 has no vertex of degree 1.

In view of Claims 3.6.1 and 3.6.2, it follows from Lemma 3.3.2 that B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, where $p_{ux} + p_{uy} = d_G(u) \geq 3$, where $p_{uy} \geq 2$, and where no two paths in P_{uy} bound a region that has a path not in P_{uy} embedded in it, and no two paths in P_{ux} bound a region that has a path not in P_{ux} embedded in it also.

By Claims 3.6.8–3.6.10, it follows that $p_{uy} \leq 3$. First suppose that $p_{uy} = 3$. Then, by Claims 3.6.4 and 3.6.7, it follows that $d_G(u) = \Delta = 5$ and that u must occur in B as vertex u in one of the configurations in Figure 3.10, where the faces are as shown and where only x and y are incident with edges in G not shown. However, Claims 3.6.11 and 3.6.12 show that this is impossible. So we may assume that $p_{uy} = 2$ and $p_{ux} \leq 2$, and so $d_G(u) \leq 4$. By Claim 3.6.3, it follows that u must occur in B as vertex u in Figure 3.7(a), 3.7(b), or 3.7(c), where the faces are as shown and where only x and y are incident with edges in G not shown. (Note that w, and v if present, have degree 2 in G and are therefore different from z_0.)

However, Claims 3.6.5 and 3.6.6 show that this is impossible. This contradiction completes the proof of Claim 3.6.13. □

Since B_1 is a minor of G, Claim 3.6.13 implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 3.6.2. □

3.7 Results for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs

We will now use Theorem 3.3.1 to prove parts (i)–(v) of Theorem 3.1.1 for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. These are restated in the following theorem.

Theorem 3.7.1. *Let G be a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree Δ. Then*

(i) $\mathrm{ch_{vf}}(G) \leq 5$;

(ii) $\mathrm{ch_{ef}}(G) \leq \Delta + 2$ *if* $\Delta = 3$ *or* 4;

(iii) $\mathrm{ch_{ef}}(G) \leq \Delta + 1$ *if* $\Delta \geq 5$;

(iv) $\chi_{\mathrm{ef}}(G) \leq \Delta + 1 = 5$ *if* $\Delta = 4$;

(v) $\mathrm{ch_{vef}}(G) \leq \max\{7, \Delta + 2\}$ *if* $\Delta \geq 3$.

The proofs of the results in Theorem 3.7.1 have been split into various sections. In Section 3.8 we will prove part (i), which is restated in Theorem 3.8.1. In Section 3.9 we will first prove part (ii), which is included in Theorem 3.9.1, and we will then prove parts (iii) and (iv), which are restated in Theorem 3.9.2. In Section

3.10 we will first prove part (v) if $\Delta = 3$, which is restated in Theorem 3.10.1, and we will then prove part (v) if $\Delta \geq 4$, which is restated in Theorem 3.10.2. We will make use of Theorem 3.7.2, which is included in Theorem 2.3.3 in Chapter 2. We will also need the following definitions and the following lemmas. Recall that $L'(z)$ denotes the list of usable colours for each uncoloured element z.

Theorem 3.7.2. [14] *Let G be a $(\bar{K}_2+(K_1\cup K_2))$-minor-free graph with maximum degree* 3. *Then* $\mathrm{ch}'(G) = \chi'(G) = 3$.

Let C be a component of a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph G such that no interior face of C has another component of G embedded in it. If C is 2-connected, then let $B = C$ and let z_0 be any vertex of maximum degree in C; otherwise, by Lemma 3.1.4, let B be an end-block of C with cut-vertex z_0 such that no interior face of B has a block of C embedded in it.

If B contains a vertex with degree at least 3 in G, then let B_1 be the graph whose vertices are the vertices of B that have degree at least 3 in G, where two vertices are adjacent in B_1 if and only if they are connected in G by an edge or by a path whose interior vertices have degree 2.

Lemma 3.7.3. *Let G be a $(\bar{K}_2+(K_1\cup K_2))$-minor-free graph. Then each block of G is either K_4-minor-free or else isomorphic to K_4.*

Proof. Suppose that B is a block of G that has a K_4 minor. Since $\Delta(K_4) = 3$, it follows that B has a subgraph B' that is homeomorphic to K_4. If an edge of K_4 is subdivided, or if a path is added joining two vertices of K_4, then a $\bar{K}_2+(K_1\cup K_2)$ minor is formed. So $B' \cong K_4$ and $B = K_4$. $\square$

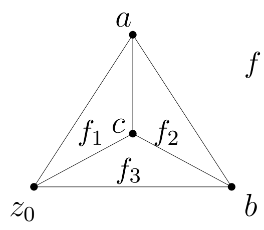


Figure 3.11

Lemma 3.7.4. *Let G be a plane embedding of K_4, as shown in Figure 3.11. If both f and z_0 are precoloured, and each of the elements a, b, c, f_1, f_2, f_3 has a list of at least 3, 3, 4, 3, 4, 3 usable colours respectively, then any given colouring of f and z_0 can be extended to the remaining vertices and faces of G.*

Proof. Each of the remaining elements

$$a, b, c, f_3, f_1, f_2 \tag{3.18}$$

has a list of at least 3, 3, 4, 3, 3, 4 usable colours respectively. Note that these elements are equivalent to a 4-cycle abf_3f_1a where c and f_2 are the interior and exterior faces.

If possible, give b and f_1 the same colour. At this point, each of the remaining elements

$$a, f_3, c, f_2 \tag{3.19}$$

has a list L'' of at least 2, 2, 3, 3 usable colours respectively. If possible, give a and f_3 the same colour. The remaining elements can now be coloured in the order (3.19). So we may assume that $L''(a) \cap L''(f_3) = \emptyset$ so that $|L''(a) \cup L''(f_3)| \geq 4$. Now either $|L''(f_2)| \geq 4$, or else a or f_3 can be given a colour that is not in $L''(f_2)$. In each case the remaining elements can be coloured in the order (3.19). So we may assume that this is not possible so that $L'(b) \cap L'(f_1) = \emptyset$, and, by symmetry, that $L'(a) \cap L'(f_3) = \emptyset$.

If $L'(f_1) = L'(f_3)$, then either $|L'(f_1)| \geq 4$, or else f_2 can be given a colour that is not in $L'(f_1)$. In each case colour f_2. Since $L'(a) \cap L'(f_1) = \emptyset$ the remaining elements can now be coloured in the order (3.18). So we may assume that $L'(f_1) \neq L'(f_3)$, and similarly that $L'(f_1) \neq L'(a)$, $L'(b) \neq L'(a)$, and $L'(b) \neq L'(f_3)$.

So give colours to c and f_2. The remaining elements are equivalent to a 4-cycle. Since $L'(a) \cap L'(f_3) = \emptyset$ and $L'(b) \cap L'(f_1) = \emptyset$, it follows that any colour given to either c or f_2 is in at most two of $L'(a)$, $L'(b)$, $L'(f_1)$, $L'(f_3)$. If each remaining element has a list of at least two usable colours, then the result follows from Theorem 3.1.3. So we may assume that at least one remaining element has only one usable colour in its list. This means that each of the colours on c and f_2 was in the list of usable colours of one remaining element.

Suppose that exactly one remaining element, say f_1, has only one usable colour in its list. Then each of a, b, f_3 has at least 2, 3, 2 usable colours in its list respectively, and so the remaining elements can be coloured in the order f_1, a, f_3, b. So we may assume that there are two remaining elements each of which has only one usable colour in its list. Since these elements are adjacent, then, by symmetry, we may assume that these elements are f_1 and f_3. Since $L'(f_1) \neq L'(f_3)$, and since each of a and b has at least three usable colours in its list, it follows that the remaining elements can be coloured in the order f_1, f_3, a, b. In every case the colouring can be completed. This completes the proof of Lemma 3.7.4. □

Lemma 3.7.5. *Let G be a plane embedding of K_4, as shown in Figure* 3.11. *If both f and z_0 are precoloured, and each of the elements az_0, bz_0, cz_0, f_1, f_3, f_2, a, b, c, ab, ac, bc has a list of at least* 3, 3, 4, 5, 5, 6, 5, 5, 6, 6, 7, 7 *usable colours respectively, then any given colouring of f and z_0 can be extended to the remaining elements of G.*

Proof. First colour in order az_0, bz_0, cz_0, f_1, f_3, which is obviously possible. Now each of the remaining elements

$$a, b, c, f_2, ab, ac, bc \tag{3.20}$$

has a list of at least 3, 3, 3, 4, 4, 4, 4 usable colours respectively.

If possible, give a and bc the same colour. At this point, each of the remaining elements

$$b, c, f_2, ab, ac \tag{3.21}$$

has a list L'' of at least 2, 2, 3, 3, 3 usable colours respectively. If possible, give b and ac the same colour. The remaining elements can now be coloured in the order (3.21). So we may assume that $L''(b) \cap L''(ac) = \emptyset$ so that $|L''(b) \cup L''(ac)| \geq 5$. Now either $|L''(ab)| \geq 5$, or else b or ac can be given a colour that is not in $L''(ab)$. In each case the remaining elements can be coloured in the order (3.21), using a colour that is not in $L''(ab)$ on either b, f_2 or ac at the first opportunity, where if ac is required to have a colour that is not in $L''(ab)$, then b and c are coloured so that this colour is not given to c. So we may assume that this is not possible so that $L'(a) \cap L'(bc) = \emptyset$, and, by symmetry, that $L'(b) \cap L'(ac) = \emptyset$ and $L'(c) \cap L'(ab) = \emptyset$.

If possible, give f_2 a colour so that each of the remaining elements has a list of at least three usable colours. Since $\mathrm{ch}''(K_3) = 3$, by Theorem 3.1.3, it follows that the remaining elements can be coloured from their lists. So we may assume that after colouring f_2, at least one of a, b, c has only two usable colours in its list. Suppose that each of a, b, c has only two usable colours in its list. Then since $|L'(f_2)| \geq 4$ we can change the colour on f_2 so that at least one of a, b, c has three usable colours in its list.

Suppose first that f_2 is given a colour that is in only one of $L'(a)$, $L'(b)$, $L'(c)$. By symmetry we may assume that this colour is in $L'(a)$, and hence not in $L'(bc)$.

At this point, let $L''(z)$ be the list of usable colours for each remaining element z, where $|L''(z)| \geq 3$ if $z \in \{b, c, ab, ac\}$, $|L''(a)| = 2$, and $|L''(bc)| \geq 4$. So both b and ac can be given a colour that is not in $L''(a)$. Note that the remaining elements are equivalent to a 4-cycle. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z, where $|L'''(a)| = 2$, $|L'''(bc)| \geq 2$, and $|L'''(c) \cup L'''(ab)| \geq 4$ since $L'(c) \cap L'(ab) = \emptyset$. If each of c and ab has at least two usable colours in its list, then it follows from Theorem 3.1.3 that the remaining elements can be coloured. So we may assume that one of c and ab has only one usable colour in its list, and so the other has at least three usable colours in its list. So, starting with whichever has only one usable colour in its list, the remaining elements can be coloured in the order c, a, bc, ab or ab, a, bc, c.

So we may assume that f_2 is given a colour that is in exactly two of $L'(a)$, $L'(b)$, $L'(c)$. By symmetry we may assume that this colour is in $L'(a)$ and $L'(b)$, and hence not in $L'(bc)$ or $L'(ac)$. At this point, let $L''(z)$ be the list of usable colours for each remaining element z, where $|L''(z)| \geq 3$ if $z \in \{c, ab\}$, $|L''(z)| \geq 4$ if $z \in \{ac, bc\}$, and $|L''(a)| = |L''(b)| = 2$. If possible, give b a colour that is in $L''(a)$ and hence not in $L''(bc)$. The remaining elements can now be coloured in the order (3.20). So we may assume that $L''(a) \cap L''(b) = \emptyset$. If possible, give c a colour that is in $L''(a)$, and hence not in $L''(bc)$ or $L''(b)$. The remaining elements can now be coloured in the order (3.20). So we may assume that $L''(a) \cap L''(c) = \emptyset$, and, by symmetry, that $L''(b) \cap L''(c) = \emptyset$. So the remaining elements can be coloured in the order (3.20) with the exception that c is coloured last. In every case the colouring can be completed. This completes the proof of Lemma 3.7.5. □

3.8 Coupled choosability of plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs

In this section we will prove part (i) of Theorem 3.7.1, which is restated in the following theorem.

Theorem 3.8.1. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph. Then* $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$.

Proof. Suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with the smallest number of vertices such that $\mathrm{ch}_{\mathrm{vf}}(G) > 5$. Assume that every vertex v and every face f of G is given a list $L(v)$ or $L(f)$ of five colours such that G has no proper coupled colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined at the start of Section 3.7.

Claim 3.8.1. $B \ncong K_4$.

Proof. Suppose that $B \cong K_4$ and let the elements of B be labelled as in Figure 3.11. Then, by hypothesis, $G - (B - z_0)$ has a proper coupled colouring from its lists in which both f and z_0 are coloured. So each of the remaining elements a, b, c, f_1, f_2, f_3 has a list of at least 3, 3, 4, 3, 4, 3 usable colours respectively, and so it follows from Lemma 3.7.4 that G can be coloured from its lists. This contradiction proves Claim 3.8.1. □

By Lemma 3.7.3 and Claim 3.8.1, it follows that B is K_4-minor-free. Claim 3.4.1 implies that $B \ncong K_2$ and Claim 3.4.3 implies that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined at the start of Section 3.7. By Claim 3.4.5 B_1 is not K_4-minor-free.

However, since B_1 is a minor of B this implies that B is not K_4-minor-free. This contradiction completes the proof of Theorem 3.8.1. $\square$

3.9 Edge-face choosability and edge-face colourability of plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs

In this section we will first prove part (*ii*) of Theorem 3.7.1, which is included in Theorem 3.9.1. We will then prove parts (*iii*) and (*iv*) of Theorem 3.7.1, which are restated in Theorem 3.9.2.

Theorem 3.9.1. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree Δ. Then $\mathrm{ch}_{\mathrm{ef}}(G) \leq \Delta + 2$ if $\Delta \geq 3$.*

Proof. Fix the value of $\Delta \geq 3$ and suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree at most Δ such that $\mathrm{ch}_{\mathrm{ef}}(G) > \Delta + 2$. Assume that every edge e and every face f of G is given a list $L(e)$ or $L(f)$ of $\Delta + 2$ colours such that G has no proper edge-face colouring from these lists.

From the well-known result [31] that a planar graph is 5-choosable, it follows that the faces of G can be coloured from their lists since $\Delta \geq 3$. Since every edge is incident with at most two faces, it follows that every edge has at least Δ usable colours in its list. Since $\mathrm{ch}'(G) = \Delta$ by Theorem 3.7.2, it follows that these edges can be coloured. $\square$

Note that by Theorem 3.1.1(vi)–($viii$) and Theorem 3.9.1, if G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree Δ, then $\chi_{\text{ef}}(G) \leq \text{ch}_{\text{ef}}(G) \leq 5$ if $\Delta \leq 3$ and $\text{ch}_{\text{ef}}(G) \leq 6$ if $\Delta = 4$.

Theorem 3.9.2. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree $\Delta \geq 4$. Then*

(i) $\text{ch}_{\text{ef}}(G) \leq \Delta + 1$ *if* $\Delta \geq 5$;

(ii) $\chi_{\text{ef}}(G) \leq \Delta + 1 = 5$ *if* $\Delta = 4$.

Proof. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that G is a counterexample to either part. Assume that every edge e and every face f of G is given a list $L(e)$ or $L(f)$ of $\Delta + 1$ colours such that G has no proper edge-face colouring from these lists, and assume that these lists are all identical if $\Delta = 4$. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined at the start of Section 3.7.

Claim 3.9.1. $B \ncong K_4$.

Proof. Suppose that $B \cong K_4$ and let the elements of B be labelled as in Figure 3.11.

(i): By hypothesis $G - (B - z_0)$ has a proper edge-face colouring from its lists in which f is coloured. Since $d_G(z_0) \leq \Delta$, there are at most $\Delta - 3$ coloured edges of $G - (B - z_0)$ incident with z_0. So each of the remaining elements

$$ac, az_0, bz_0, cz_0, f_1, f_2, f_3, bc, ab \tag{3.22}$$

has a list of at least 6, 3, 3, 4, 5, 5, 5, 6, 5 usable colours respectively. If we try to colour the remaining elements in the order (3.22) then it is only with ab that we may fail. Now either $|L'(ab)| \geq 6$, or else ac can be given a colour that is not in $L'(ab)$. In each case the remaining elements can be coloured in the order (3.22).

(ii): By hypothesis $G - (B - z_0)$ has a proper edge-face colouring in which f is coloured, and in which the edge of $G - (B - z_0)$ incident with z_0 is given a colour different from f. Since G has five colours there are a further three different colours available for each of the edges az_0, bz_0 and cz_0. It follows that the five colours can be given to the remaining elements in the following pairs: $\{ac, f\}$, $\{az_0, bc\}$, $\{cz_0, f_2\}$, $\{bz_0, f_1\}$, $\{ab, f_3\}$. This contradiction completes the proof of Claim 3.9.1. □

By Lemma 3.7.3 and Claim 3.9.1, it follows that B is K_4-minor-free. Claim 3.5.1 implies that $B \ncong K_2$ and Claim 3.5.2 implies that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined at the start of Section 3.7. By Claim 3.5.8 B_1 is not K_4-minor-free. However, since B_1 is a minor of B this implies that B is not K_4-minor-free. This contradiction completes the proof of Theorem 3.9.2. □

3.10 Entire choosability of plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs

In this section we will first prove part (v) of Theorem 3.7.1 if $\Delta = 3$, which is restated in Theorem 3.10.1. We will then prove part (v) of Theorem 3.7.1 if $\Delta \geq 4$, which is restated in Theorem 3.10.2.

Theorem 3.10.1. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree 3. Then* $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$.

Proof. Suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree 3 such that $\mathrm{ch}_{\mathrm{vef}}(G) > 7$. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of 7 colours such that G has no proper entire colouring from these lists.

Since $\mathrm{ch}_{\mathrm{vf}}(G) \leq 5$ by Theorem 3.8.1, it follows that the vertices and faces of G can be coloured from their lists. Since every edge is incident with two vertices and at most two faces, every edge has at least 3 usable colours in its list. Since $\mathrm{ch}'(G) = 3$ by Theorem 3.7.2, it follows that these edges can be coloured. □

Note that by Theorem 3.1.1(vi)–$(viii)$ and Theorem 3.10.1, if G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree $\Delta \leq 3$, then $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$.

Theorem 3.10.2. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree $\Delta \geq 4$. Then*

(i) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \Delta + 2$ *if* $\Delta \geq 5$;

(ii) $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$ *if* $\Delta = 4$.

Proof. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that G is a counterexample to either part. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of $\Delta + 2$ or 7 colours as appropriate. Assume also that G has no proper entire colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined at the start of Section 3.7.

Claim 3.10.1. $B \ncong K_4$.

Proof. Suppose that $B \cong K_4$ and let the elements of B be labelled as in Figure 3.11. Then, by hypothesis, $G-(B-z_0)$ has a proper entire colouring from its lists in which both f and z_0 are coloured. Since $d_G(z_0) \leq \Delta$, there are at most $\Delta - 3$ coloured edges of $G-(B-z_0)$ incident with z_0. So each of the remaining elements az_0, bz_0, cz_0, f_1, f_3, f_2, a, b, c, ab, ac, bc, ab has a list of at least 3, 3, 4, 5, 5, 6, 5, 5, 6, 6, 7, 7 usable colours respectively, and so it follows from Lemma 3.7.5 that G can be coloured from its lists. This completes the proof of Claim 3.10.1. $\square$

By Lemma 3.7.3 and Claim 3.10.1, it follows that B is K_4-minor-free. Claim 3.6.1 implies that $B \ncong K_2$ and Claim 3.6.2 implies that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined at the start of Section 3.7. By Claim 3.6.13 B_1 is not K_4-minor-free. However, since B_1 is a minor of B this implies that B is not K_4-minor-free. This contradiction completes the proof of Theorem 3.10.2. $\square$

Since we have now proved Theorem 3.3.1 and Theorem 3.7.1 this completes the proof of Theorem 3.1.1.

Chapter 4

List-colouring the square of a K_4-minor-free graph

4.1 Introduction

The List-Square-Colouring Conjecture (LSCC), which was proposed in 2001 by Kostochka and Woodall [21], states that $\operatorname{ch}(G^2) = \chi(G^2)$ for every graph G. The LSCC[1] is known to be true for all graphs G with maximum degree $\Delta = 0$, 1 or 2.

If G is a K_4-minor-free graph with maximum degree $\Delta \geq 3$, then although we cannot prove that $\operatorname{ch}(G^2) = \chi(G^2)$, we can prove the same sharp upper bound for $\operatorname{ch}(G^2)$ as for $\chi(G^2)$. In 2003, Lih, Wang and Zhu [24] proved the results in the following theorem. They also gave examples to show that these results are sharp, but in (4.1) for even $\Delta \geq 4$ their examples are wrong.

Theorem 4.1.1. [24] *Let G be a K_4-minor-free graph with maximum degree Δ. Then*

$$\chi(G^2) \leq \begin{cases} \Delta + 3 & \text{if } \Delta = 2 \text{ or } 3; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \text{if } \Delta \geq 4; \end{cases}$$

[1] See page 11 for further details.

and

$$\text{degeneracy}(G^2) \leq \begin{cases} \Delta + 2 & \textit{if } \Delta = 2 \textit{ or } 3; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \textit{if } \Delta \geq 4. \end{cases} \tag{4.1}$$

In this chapter we will prove that the upper bounds for $\chi(G^2)$ are sharp for $\mathrm{ch}(G^2)$ also, and we will prove a stronger form of (4.1) with $\lceil \frac{3}{2}\Delta \rceil$ in place of $\lfloor \frac{3}{2}\Delta \rfloor + 1$. By using the examples of Lih, Wang and Zhu, we will show that these results are sharp for all Δ. The situation is summarised in the following theorem.

Recall that a graph G is *k-degenerate*, where $k \geq 0$, if every induced subgraph of G has minimum degree at most k, and that degeneracy(G) is the smallest integer k for which G is k-degenerate.[2] Recall also that the *colouring number* $\mathrm{col}(G)$ is the least k for which the vertices can be ordered so that every vertex is preceded by fewer than k of its neighbours; so $\mathrm{col}(G) = \text{degeneracy}(G) + 1$.

Theorem 4.1.2. [15] *Let G be a K_4-minor-free graph with maximum degree Δ. Then*

$$\mathrm{ch}(G^2) \leq \begin{cases} \Delta + 3 & \textit{if } \Delta = 2 \textit{ or } 3; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \textit{if } \Delta \geq 4; \end{cases}$$

and

$$\text{degeneracy}(G^2) \leq \begin{cases} \Delta + 2 & \textit{if } \Delta = 2 \textit{ or } 3; \\ \lceil \frac{3}{2}\Delta \rceil & \textit{if } \Delta \geq 4. \end{cases} \tag{4.2}$$

Corollary 4.1.3. *Let G be a K_4-minor-free graph with maximum degree Δ. Then*

$$\mathrm{col}(G^2) \leq \begin{cases} \Delta + 3 & \textit{if } \Delta = 2 \textit{ or } 3; \\ \lceil \frac{3}{2}\Delta \rceil + 1 & \textit{if } \Delta \geq 4. \end{cases}$$

Proof. Since $\mathrm{col}(G) = \text{degeneracy}(G) + 1$ the results follow immediately from Theorem 4.1.2. □

[2] See page 5 for further details.

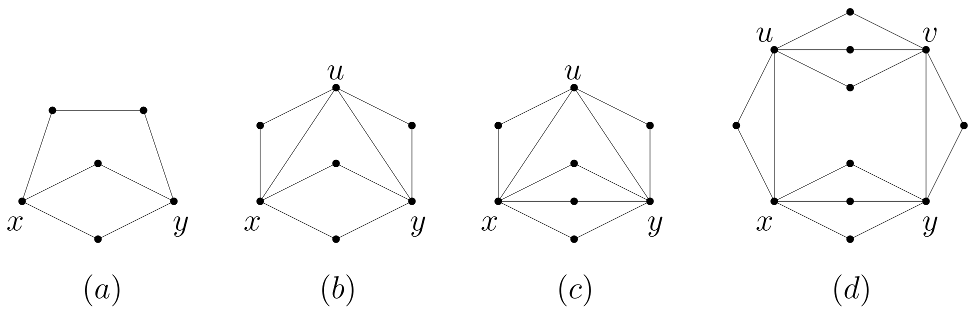


Figure 4.1

We will now show that Theorem 4.1.2 and Corollary 4.1.3 are sharp. By definition, any examples that are sharp for degeneracy(G^2) are also sharp for col(G^2). If $\Delta = 2$, then let $G = C_5$ so that degeneracy$(G^2) = 4 = \Delta + 2$ and $\mathrm{ch}(G^2) = 5 = \Delta + 3$. If $\Delta = 3$, then let G be the graph formed from a 4-cycle $xuyvx$ by adding a path of length 3 between x and y, as shown in Figure 4.1(a), so that degeneracy$(G^2) = 5 = \Delta + 2$ and $\mathrm{ch}(G^2) = 6 = \Delta + 3$. It remains to show that the results for $\Delta \geq 4$ are sharp.

If Δ is even, let $\Delta = 2k$, where $k \geq 2$. Let G_e be the graph formed from a path xuy by adding k paths of length 2 between x and y, and by adding $k-1$ paths of length 2 between both x and u, and u and y. Now $G_\mathrm{e}^2 \cong K_{3k+1}$, and so degeneracy$(G^2) = \delta(G_\mathrm{e}^2) = 3k = \lceil \frac{3}{2}\Delta \rceil$ and $\mathrm{ch}(G_\mathrm{e}^2) = 3k+1 = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Figure 4.1$(b)$ shows G_e when $\Delta = 4$.

If Δ is odd, let $\Delta = 2k+1$, where $k \geq 2$. Let G_o be the graph formed from a path xuy by adding $k+1$ paths of length 2 between x and y, and by adding $k-1$ paths of length 2 between both x and u, and u and y. Now $G_\mathrm{o}^2 \cong K_{3k+2}$, and so $\mathrm{ch}(G_\mathrm{o}^2) = 3k+2 = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Figure 4.1$(c)$ shows G_o when $\Delta = 5$.

It remains to show that (4.2) is sharp for odd Δ. Let G_{2k+1} be the graph formed from two nonadjacent edges ux and vy by adding $k+1$ paths of length 2 between both u and v, and x and y, and by adding $k-1$ paths of length 2 between both x

and u, and v and y. Now degeneracy$(G^2) = \delta(G^2_{2k+1}) = 3k+2 = \lceil \frac{3}{2}\Delta \rceil$, which is the degree in G^2 of every vertex of degree 2 in G_{2k+1}. Figure 4.1(d) shows G_{2k+1} when $\Delta = 5$. These examples show that Theorem 4.1.2 and Corollary 4.1.3 are sharp for all Δ.

The rest of this chapter is devoted to a proof of Theorem 4.1.2. We will make use of the following theorem of Dirac [9].

Theorem 4.1.4. [9] *Every K_4-minor-free graph has at least one vertex with degree at most* 2.

4.2 Proof of Theorem 4.1.2

If G is a K_4-minor-free graph with maximum degree $\Delta = 2$ or 3, then since Lih, Wang and Zhu [24] proved that degeneracy$(G^2) = \Delta + 2$, it follows that $\mathrm{ch}(G^2) \le \Delta + 3$. It remains to prove the results for $\Delta \ge 4$, which we restate in the following theorem.

Theorem 4.2.1. [15] *Let G be a K_4-minor-free graph with maximum degree $\Delta \ge 4$. Then G^2 is $\lceil \frac{3}{2}\Delta \rceil$-degenerate and* $\mathrm{ch}(G^2) \le \lfloor \frac{3}{2}\Delta \rfloor + 1$.

Proof. Fix the value of $\Delta \ge 4$ and suppose, if possible, that G_{d} and G_{s} are K_4-minor-free graphs with the smallest number of vertices and maximum degree at most Δ such that G^2_{d} is not $\lceil \frac{3}{2}\Delta \rceil$-degenerate and $\mathrm{ch}(G^2_{\mathrm{s}}) > \lfloor \frac{3}{2}\Delta \rfloor + 1$. Then

$$\delta(G^2_{\mathrm{d}}) \ge \lceil \tfrac{3}{2}\Delta \rceil + 1 \ge \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \ge \Delta + 3. \tag{4.3}$$

Assume that every vertex v of G_{s} is given a list $L(v)$ of $\lfloor \frac{3}{2}\Delta \rfloor + 1 \ge \Delta + 3$ colours such that G^2_{s} has no proper colouring from these lists. Let G denote G_{d} or G_{s}. Clearly G is connected.

Claim 4.2.1. *G does not contain a vertex of degree 1.*

Proof. Suppose that u is a vertex of degree 1 in G that is adjacent in G to v. Now $d_{G^2}(u) \leq \Delta$, which by (4.3) is a contradiction if $G = G_{\mathrm{d}}$; so we may assume that $G = G_{\mathrm{s}}$. Let $H = G - u$. By hypothesis $H^2 = G^2 - u$ has a proper colouring from its lists. Now u can be given a colour from its list since it has only Δ neighbours in G^2 and a list of at least $\Delta + 3$ colours. These contradictions complete the proof of Claim 4.2.1. □

Claim 4.2.2. *G does not contain two adjacent vertices of degree 2.*

Proof. Suppose that $xuvy$ is a path in G (or a cycle if $x = y$) where both u and v have degree 2 in G. Now $d_{G^2}(u), d_{G^2}(v) \leq \Delta + 2$, which by (4.3) is a contradiction if $G = G_{\mathrm{d}}$; so we may assume that $G = G_{\mathrm{s}}$. Let $H = G - \{u, v\}$. By hypothesis $H^2 = G^2 - \{u, v\}$ has a proper colouring from its lists. Now each of u and v can be given a colour from its list since each has only $\Delta + 2$ neighbours in G^2 and a list of at least $\Delta + 3$ colours. These contradictions prove Claim 4.2.2. □

We will now consider an arbitrary vertex of degree 2 in G. Let w be such a vertex that is adjacent in G to u and x. Let M_{ux} be the set of vertices of degree 2 in G that are adjacent in G to both u and x (so that $w \in M_{ux}$), and let m_{ux} be the number of such vertices. Also, let m'_{ux} be the number of vertices of degree at least 3 in G that are adjacent in G to both u and x. Let $H = G - w$ if $ux \in E(G)$, and let $H = G - w + ux$ if $ux \notin E(G)$; so $G^2 - w \subseteq H^2$.

By (4.3), and since a colouring of H^2 can be extended to G^2 if $d_{G^2}(w) \leq \lfloor \frac{3}{2}\Delta \rfloor$, we may assume that

$$d_{G^2}(w) \geq \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \geq \Delta + 3. \tag{4.4}$$

However,

$$d_{G^2}(w) \leq d_G(u) + d_G(x) - m_{ux} + 1 - m'_{ux} - 2\varepsilon_{ux}, \tag{4.5}$$

where $\varepsilon_{ux} = 1$ if u and x are adjacent in G, and 0 otherwise. We will use this terminology throughout the rest of this chapter.

If $\Delta(G) \geq 3$, then let G_1 be the graph whose vertices are the vertices of G that have degree at least 3 in G, where two vertices are adjacent in G_1 if and only if they are connected in G by an edge or by a path whose interior vertices have degree 2.

Claim 4.2.3. *G_1 exists and does not contain a vertex of degree 0 or 1. Moreover, G_1 contains at least one vertex of degree 2.*

Proof. Claims 4.2.1 and 4.2.2 imply that G_1 exists and does not contain a vertex of degree 0. Suppose that u is a vertex of degree 1 in G_1 that is adjacent in G_1 to x. By the definition of G_1 and by Claim 4.2.2, there is a vertex w of degree 2 in G such that $w \in M_{ux}$. However, $d_{G^2}(w) \leq \Delta + 1$, which contradicts (4.4). So G_1 does not contain a vertex of degree 1. Since G_1 is a minor of G, it follows that G_1 is K_4-minor-free. By Theorem 4.1.4, G_1 has a vertex of degree 2. This completes the proof of Claim 4.2.3. $\square$

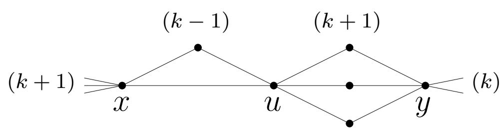


Figure 4.2

Claim 4.2.4. *Δ is odd, say $\Delta = 2k+1$, where $k \geq 2$ since $\Delta \geq 4$. Furthermore, every vertex of degree 2 in G_1 occurs in G as vertex u in Figure 4.2, where x is not adjacent to y, and where only x and y are adjacent in G to vertices not shown. Also, $d_G(u) = d_G(x) = d_G(y) = \Delta$, and $d_{G^2}(w) = \lceil \frac{3}{2}\Delta \rceil$ for every $w \in M_{ux} \cup M_{uy}$.*

Proof. G_1 has a vertex of degree 2 by Claim 4.2.3. Let u be such a vertex that is adjacent in G_1 to x and y. By the definition of G_1 and by Claim 4.2.2,

$$d_G(u) = m_{ux} + m_{uy} + \varepsilon_{ux} + \varepsilon_{uy}, \tag{4.6}$$

where m_{ux} and m_{uy} are not both 0. If $m_{ux} \neq 0$, then there exists a vertex $w \in M_{ux}$ such that (4.5) gives

$$\begin{aligned} d_{G^2}(w) &\le m_{ux} + m_{uy} + \varepsilon_{ux} + \varepsilon_{uy} + d_G(x) - m_{ux} + 1 - m'_{ux} - 2\varepsilon_{ux} \\ &\le \Delta + 1 + m_{uy} - \varepsilon_{ux} + \varepsilon_{uy}. \end{aligned} \tag{4.7}$$

Now if $m_{uy} = 0$, then (4.7) gives $d_{G^2}(w) \le \Delta + 2$, which contradicts (4.4), and so, by symmetry, m_{ux} and m_{uy} are both non-zero. Let $w \in M_{ux}$ and $w' \in M_{uy}$. Then by analogy with (4.7)

$$d_{G^2}(w') \le \Delta + 1 + m_{ux} + \varepsilon_{ux} - \varepsilon_{uy}. \tag{4.8}$$

From (4.7) and (4.8) it follows that

$$\begin{aligned} \min\{d_{G^2}(w), d_{G^2}(w')\} &\le \Delta + 1 + \tfrac{1}{2}(m_{ux} + m_{uy}) \\ &\le \Delta + 1 + \tfrac{1}{2}(d_G(u) - \varepsilon_{ux} - \varepsilon_{uy}) \\ &\le \tfrac{3}{2}\Delta + 1. \end{aligned} \tag{4.9}$$

If ε_{ux} and ε_{uy} are both 1, then $\min\{d_{G^2}(w), d_{G^2}(w')\} \le \frac{3}{2}\Delta$, which contradicts (4.4). If ε_{ux} and ε_{uy} are both 0, then

$$d_{G^2}(u) = d_G(u) + 2 \le \Delta + 2,$$

which by (4.3) is a contradiction if $G = G_{\mathrm{d}}$; so we may assume that $G = G_{\mathrm{s}}$ and without loss of generality that $d_{G^2}(w) \le d_{G^2}(w')$. Let $H = G - w$. By hypothesis $H^2 = G^2 - w$ has a proper colouring from its lists. To extend a colouring of H^2 to

G^2 first uncolour u, then colour w, and then recolour u. This contradiction shows that one of ε_{ux} and ε_{uy} equals 1, and the other equals 0.

Consequently, (4.9) implies that $\min\{d_{G^2}(w), d_{G^2}(w')\} \leq \frac{3}{2}\Delta + \frac{1}{2}$. If Δ is even this contradicts (4.4); so we may assume that Δ is odd, say $\Delta = 2k+1$, where $k \geq 2$ since $\Delta \geq 4$. So, for (4.4) to hold, $\min\{d_{G^2}(w), d_{G^2}(w')\} = \lfloor \frac{3}{2}\Delta \rfloor + 1 = \lceil \frac{3}{2}\Delta \rceil$, which implies that $d_G(u) = \Delta$, and so equality holds in (4.9). So equality holds also in (4.7) and (4.8), and so $d_{G^2}(w) = d_{G^2}(w') = \lfloor \frac{3}{2}\Delta \rfloor + 1 = 3k+2$ and $d_G(x) = d_G(y) = \Delta$. This implies that $m'_{ux} = m'_{uy} = 0$, and so x is not adjacent to y. Furthermore, we may assume without loss of generality that $\varepsilon_{ux} = 1$ and $\varepsilon_{uy} = 0$, and so $m_{ux} = k-1$ and $m_{uy} = k+1$ by (4.7) and (4.8). This completes the proof of Claim 4.2.4. □

Since $d_{G^2}(w) = \lceil \frac{3}{2}\Delta \rceil$, which contradicts (4.3) if $G = G_{\mathrm{d}}$, this completes the proof that G_{d} is $\lceil \frac{3}{2}\Delta \rceil$-degenerate. So from now on we will assume that $G = G_{\mathrm{s}}$ and that every vertex of G is given a list of $\lfloor \frac{3}{2}\Delta \rfloor + 1 = 3k+2$ colours. For each uncoloured vertex v in G^2, let $L'(v)$ denote the list of usable colours for v; that is, $L'(v)$ denotes $L(v)$ minus any colours already used on neighbours of v in G^2.

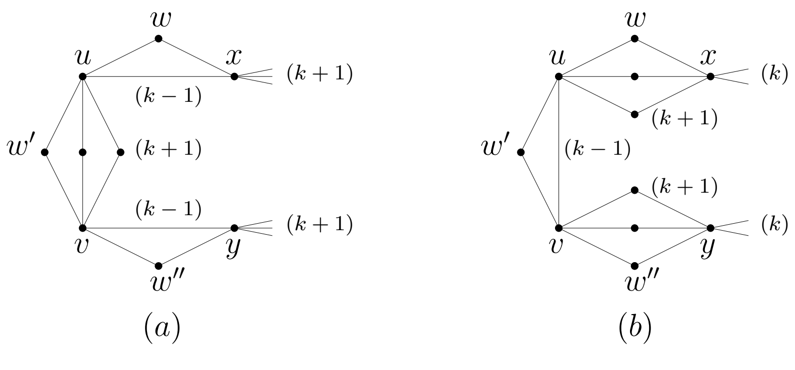


Figure 4.3

Claim 4.2.5. *G_1 does not contain two adjacent vertices of degree 2.*

Proof. Suppose that $xuvy$ is a path in G_1 where both u and v have degree 2 in G_1. By Claim 4.2.4, it follows that u and v occur in G as in Figure 4.3(a) or 4.3(b), where in each case only x and y are adjacent in G to vertices not shown. Note that $x \neq y$ since in Figure 4.3(a), by Claim 4.2.4, x and v must not be adjacent, and in Figure 4.3(b) the maximum degree of G would be exceeded. Note also that x and y may be adjacent in G, which would reduce by one the number of neighbours in G of both x and y that are not shown, but this does not affect the following argument. Let $w \in M_{ux}$, $w' \in M_{uv}$ and $w'' \in M_{vy}$. Let $H = G - w'$. By hypothesis $H^2 = G^2 - w'$ has a proper colouring from its lists. First uncolour all vertices in $M_{ux} \cup M_{uv} \cup M_{vy}$. Note that since each uncoloured vertex has degree $3k + 2 = \lfloor \frac{3}{2}\Delta \rfloor + 1$ in G^2, if we try to recolour the vertices in $M_{ux} \cup M_{uv} \cup M_{vy}$ it is only with the last vertex to be coloured that we may fail.

In Figure 4.3(a), both w and w'' have $k + 3$ coloured neighbours and w' has four coloured neighbours, and so $|L'(w)| \geq 2k - 1$, $|L'(w'')| \geq 2k - 1$ and $|L'(w')| \geq 3k - 2$. Now either $|L'(w')| \geq 4k - 2 > 3k - 2$, or else we can colour w and w'' so that they either have the same colour or one of them has a colour that is not in $L'(w')$, since if $L'(w) \cap L'(w'') = \emptyset$, then $|L'(w) \cup L'(w'')| \geq 4k - 2$. In each case the remaining vertices can be coloured with w' being coloured last.

So we may assume that u and v occur as in Figure 4.3(b). First uncolour u. Since u has two coloured neighbours, it follows that $|L'(u)| \geq 3k$. Furthermore, w'' has $k + 2$ coloured neighbours, and so $|L'(w'')| \geq 2k$. Now either $|L'(w'')| \geq 3k$, or else u can be given a colour that is not in $L'(w'')$. In each case colour u. The remaining vertices can now be coloured with w'' being coloured last. In every case the colouring can be completed, which is the required contradiction. $\square$

If $\Delta(G_1) \geq 3$, then let G_2 be the graph whose vertices are the vertices of G_1 that have degree at least 3 in G_1, where two vertices are adjacent in G_2 if and only if they are connected in G_1 by an edge or by a path whose interior vertices have degree 2.

Claim 4.2.6. *G_2 exists and does not contain a vertex of degree 0 or 1.*

Proof. Claims 4.2.3 and 4.2.5 imply that G_2 exists and does not contain a vertex of degree 0. Suppose that x is a vertex of degree 1 in G_2 that is adjacent in G_2 to y. By the definition of G_2 and by Claim 4.2.5, there are at least two vertices of degree 2 in G_1 that are adjacent in G_1 to both x and y. Let u and v be two such vertices. Then in G, by Claim 4.2.4, u contributes k to the degree of x or y and $k+1$ to the other, as does v. Since $\Delta = 2k+1$, it follows that x and y have degree 2 in G_1, which contradicts Claim 4.2.5. This contradiction completes the proof of Claim 4.2.6. □

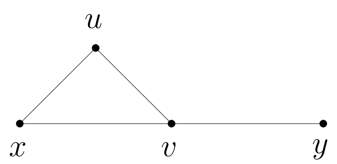


Figure 4.4

Claim 4.2.7. *G_2 does not contain a vertex of degree 2.*

Proof. Suppose that v is a vertex of degree 2 in G_2 that is adjacent in G_2 to x and y. By the definition of G_2 and by Claim 4.2.5, there is at least one vertex of degree 2 in G_1 that is adjacent in G_1 to either v and x, or v and y. In fact, there is exactly one such vertex, say u, since otherwise the maximum degree would be exceeded in G, and so we may assume without loss of generality that v occurs

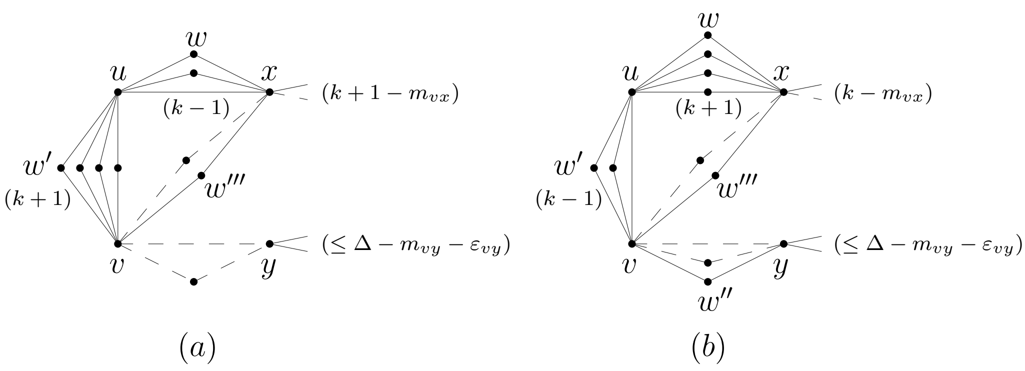


Figure 4.5

in G_1 as in Figure 4.4. So, by Claim 4.2.4, v occurs in G as in Figure 4.5(a) or 4.5(b), where in each case only x and y are adjacent in G to vertices not shown, and where $m_{vx} \geq 1$ by Claim 4.2.2 and since x is not adjacent to v in G. Note that x and y may be adjacent in G, which would reduce by one the number of neighbours in G of both x and y that are not shown, but this does not affect the following argument. Note also that x has at least one neighbour in G not shown (or is adjacent to y) since otherwise u and x are adjacent vertices of degree 2 in G_1, which contradicts Claim 4.2.5. So $1 \leq m_{vx} \leq k-1$ in Figure 4.5(b), and the same is true in Figure 4.5(a) since $d_G(v) = 2k+1$ and v is adjacent to y in G_1. Furthermore, in Figure 4.5(b), this implies that $m_{vy} \geq 1$. Note also that

$$m_{vy} = \begin{cases} k - m_{vx} - \varepsilon_{vy} & \text{in Figure 4.5}(a), \\ k + 1 - m_{vx} - \varepsilon_{vy} & \text{in Figure 4.5}(b). \end{cases} \tag{4.10}$$

Let $w \in M_{ux}$, $w' \in M_{uv}$ and $w''' \in M_{vx}$. Also, in Figure 4.5(b), let $w'' \in M_{vy}$. Let $H = G - w'$. By hypothesis $H^2 = G^2 - w'$ has a proper colouring from its lists. Note that $d_{G^2}(w') = 3k+2 = |L(w')|$, and so after applying a colouring of H^2 to G^2 we may assume that each of the colours on the neighbours of w' in G^2 are different and are in $L(w')$ since otherwise w' could be given a colour from its list. In what follows we will recolour some of the neighbours of w' in G^2 so that either two of them have the same colour, or one of them has a colour that is not in $L'(w')$. We will then colour w'.

In Figure 4.5(a), first uncolour all vertices in $\{u, v, w'''\} \cup M_{ux} \cup M_{uv}$. Now

$$\begin{aligned}
|L'(u)|, |L'(w)| &\geq (3k+2) - (k+1-m_{vx}) - (m_{vx}-1) - 1, \\
|L'(w')| &= (3k+2) - (m_{vx}-1) - m_{vy} - \varepsilon_{vy} - 1, \\
|L'(w''')| &\geq (3k+2) - (k+1-m_{vx}) - (m_{vx}-1) - m_{vy} - 1 - \varepsilon_{vy}, \\
|L'(v)| &\geq (3k+2) - (m_{vx}-1) - m_{vy} - 2 - \varepsilon_{vy}(\Delta - m_{vy} - \varepsilon_{vy}),
\end{aligned}$$

and so, by (4.10), it follows that $|L'(u)| \geq 2k+1$, $|L'(w)| \geq 2k+1$, $|L'(w')| = 2k+2$, $|L'(w''')| \geq k+1+m_{vx} \geq k+2$ and $|L'(v)| \geq k - m_{vx} + 1 \geq 2$. Since $|L'(w)| + |L'(v)| \geq |L'(w')|$, we can now colour w and v so that they either have the same colour, or one of them has a colour that is not in $L'(w')$. In each case, since $k \geq 2$, we can now colour u and w'''. Next, we can colour all the remaining vertices $z \in M_{ux} \cup M_{uv}$ ending with w', since $d_{G^2}(z) = 3k+2$ and w' has at least one usable colour in its list at the time of its colouring.

So we may assume that v occurs as in Figure 4.5(b). First uncolour all vertices in $\{u, w'', w'''\} \cup M_{ux} \cup M_{uv}$. Now

$$\begin{aligned}
|L'(u)| &\geq (3k+2) - (m_{vx}-1) - (m_{vy}-1) - 2 - \varepsilon_{vy}, \\
|L'(w)| &\geq (3k+2) - (k-m_{vx}) - (m_{vx}-1) - 2, \\
|L'(w')| &= (3k+2) - (m_{vx}-1) - (m_{vy}-1) - 1 - \varepsilon_{vy}, \\
|L'(w'')| &\geq (3k+2) - (m_{vx}-1) - (m_{vy}-1) - 2 - (\Delta - m_{vy} - \varepsilon_{vy}), \\
|L'(w''')| &\geq (3k+2) - (k-m_{vx}) - (m_{vx}-1) - (m_{vy}-1) - 2 - \varepsilon_{vy},
\end{aligned}$$

and so, by (4.10), it follows that $|L'(u)| \geq 2k+1$, $|L'(w)| \geq 2k+1$, $|L'(w')| = 2k+2$, $|L'(w'')| \geq k+1-m_{vx}+\varepsilon_{vy} \geq 2$ and $|L'(w''')| \geq k+1+m_{vx} \geq k+2$. Since $|L'(w)|+|L'(w'')| \geq |L'(w')|$, we can now colour w and w'' so that they either have the same colour, or one of them has a colour that is not in $L'(w')$. In each case, since $k \geq 2$, we can now colour u and w'''. Next, we can colour all the remaining vertices $z \in M_{ux} \cup M_{uv}$ ending with w', since $d_{G^2}(z) = 3k + 2$ and w' has at least one usable colour in its list at the time of its colouring. In every case the colouring can be completed, which is the required contradiction. □

So, by Claim 4.2.6 and Claim 4.2.7, G_2 has minimum degree at least 3, which by Theorem 4.1.4 implies that G_2 is not K_4-minor-free. Since G_2 is a minor of G this implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 4.2.1. □

Chapter 5

List-colouring the square of a $K_{2,3}$-minor-free graph

5.1 Introduction

As mentioned at the start of Chapter 4, the List-Square-Colouring Conjecture (LSCC) [21] states that $\mathrm{ch}(G^2) = \chi(G^2)$ for every graph G. The LSCC[1] is known to be true for all graphs G with maximum degree $\Delta = 0$, 1 or 2. Furthermore, it is obvious that $\mathrm{ch}(G^2) \geq \chi(G^2) \geq \chi(K_{1,\Delta}^2) = \Delta + 1$.

Recall that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free [7]. In Chapter 4 we considered the square of a K_4-minor-free graph. In this chapter we will consider the square of a $K_{2,3}$-minor-free graph. In fact, we will prove that the LSCC holds for all $K_{2,3}$-minor-free graphs with maximum degree $\Delta \geq 6$. We will also prove a sharp upper bound for $K_{2,3}$-minor-free graphs with maximum degree $\Delta \in \{3, 4, 5\}$. The situation is summarised in the following theorem, which is the same as for the slightly smaller class of outerplanar graphs.

[1] See page 11 for further details.

Theorem 5.1.1. [16] *Let G be a $K_{2,3}$-minor-free graph with maximum degree Δ. Then the LSCC holds if $\Delta \geq 6$. In fact,*

(*i*) $\Delta + 1 \leq \chi(G^2) \leq \mathrm{ch}(G^2) \leq \Delta + 2$ *if* $\Delta \geq 3$;

(*ii*) $\Delta + 1 = \chi(G^2) = \mathrm{ch}(G^2)$ *if* $\Delta \geq 6$.

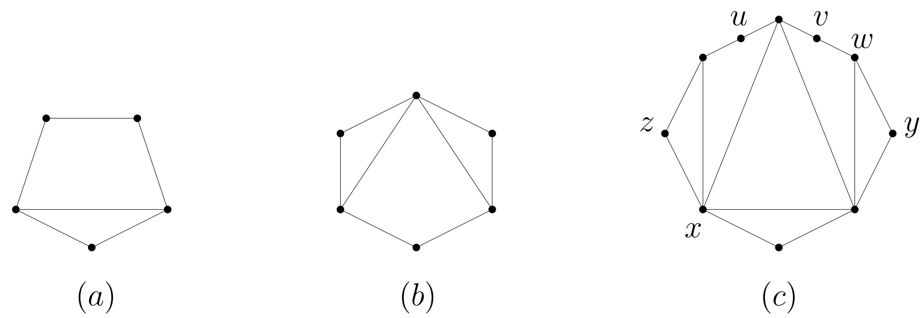


Figure 5.1

We will now show that Theorem 5.1.1 is sharp. Since $\Delta + 1 \leq \chi(G^2) \leq \mathrm{ch}(G^2)$, it remains to show that the upper bound of $\Delta + 2$ is sharp if $\Delta \in \{3, 4, 5\}$. In fact, the upper bound is sharp even for $\chi(G^2)$ and even for the slightly smaller class of outerplanar graphs. If $\Delta = 3$ or 4, then let G be the graph in Figure 5.1(a) or 5.1(b) respectively. Since $G^2 \cong K_{\Delta+2}$ it follows that $\mathrm{ch}(G^2) = \Delta + 2$. It is not difficult to see that these are the smallest extremal examples if $\Delta = 3$ or 4. If $\Delta = 5$, then let G be the graph in Figure 5.1(c), and suppose that $\chi(G^2) = \Delta + 1 = 6$. Let the six colours be the integers $1, 2, \ldots, 6$. Starting with z and continuing clockwise, colour the neighbours of x in G with $1, 2, \ldots, 5$. Now x must be coloured 6 and u must be coloured 5. This gives a contradiction since each of v, w, y must now be coloured 1 or 2 and these three vertices are adjacent to each other in G^2. From this last step it is not difficult to see that $\chi(G^2) = \Delta + 2$. This example is one of two smallest known extremal examples if $\Delta = 5$, both of which have order 10. These examples show that Theorem 5.1.1 is sharp for all Δ.

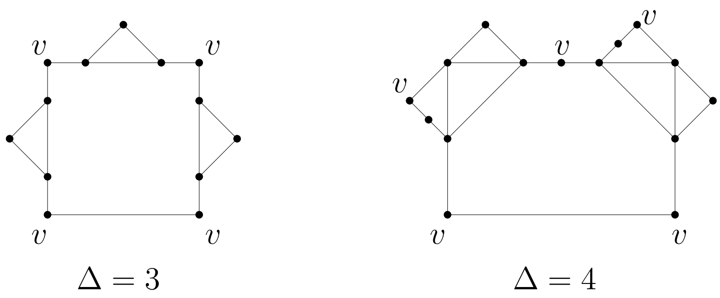


Figure 5.2

If $\Delta = 3$ or 4, then there is, in fact, an infinite family of minimal (under subgraph-inclusion) extremal examples that require $\Delta + 2$ colours. One member of each family is shown in Figure 5.2. Suppose that only $\Delta+1$ colours are available. Then every vertex labelled v must have the same colour, which gives a contradiction on the bottom edge.

The rest of this chapter is devoted to a proof of Theorem 5.1.1. We will make use of the following lemma.

Lemma 5.1.2. *Let G be a $K_{2,3}$-minor-free graph. Then each block of G is either K_4-minor-free (and hence outerplanar) or else isomorphic to K_4.*

Proof. Suppose that B is a block of G that has a K_4 minor. Since $\Delta(K_4) = 3$, it follows that B has a subgraph B' that is homeomorphic to K_4. If an edge of K_4 is subdivided, or if a path is added joining two vertices of K_4, then a $K_{2,3}$ minor is formed. So $B' \cong K_4$ and $B = K_4$. $\square$

5.2 The start of the proof of Theorem 5.1.1

Since $\Delta + 1 \leq \chi(G^2) \leq \mathrm{ch}(G^2)$, it remains to prove that $\mathrm{ch}(G^2) \leq \Delta + 2$ if $\Delta \geq 3$ and that $\mathrm{ch}(G^2) \leq \Delta + 1$ if $\Delta \geq 6$. Fix the value of $\Delta \geq 3$ and suppose, if possible, that G is a $K_{2,3}$-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that $\mathrm{ch}(G^2) > \Delta+2$ or $\Delta+1$ as appropriate. Assume that every vertex v of G is given a list $L(v)$ of $\Delta + 2$ or $\Delta + 1$ colours, as appropriate, such that G^2 has no proper colouring from these lists. Clearly G is connected and is not K_2. If G is 2-connected, then let $B = G$ and let z_0 be any vertex of G; otherwise, let B be an end-block of G with cut-vertex z_0.

Claim 5.2.1. *Not every vertex in $B - z_0$ is adjacent to z_0.*

Proof. Suppose that every vertex in $B - z_0$ is adjacent to z_0. Let u be a vertex in $B - z_0$ and let $H = G - u$. By hypothesis $H^2 = G^2 - u$ has a proper colouring from its lists. Since $d_{G^2}(u) = d_G(z_0) \leq \Delta$, it follows that u can be given a colour from its list. This contradiction proves Claim 5.2.1. □

Claim 5.2.2. *G does not contain three vertices u, v, w of degree 2 such that $uv, vw \in E(G)$.*

Proof. Suppose that G does contain three vertices u, v, w of degree 2 such that $uv, vw \in E(G)$. Let $H = G - v + uw$. By hypothesis $G^2 - v \subseteq H^2$ has a proper colouring from it lists. Since $d_{G^2}(v) = 4$, it follows that v can be given a colour from its list. This contradiction proves Claim 5.2.2. □

Claim 5.2.3. *B consists of a cycle C with at least one chord.*

Proof. By Lemma 5.1.2, each block of G is either K_4-minor-free (and hence outerplanar) or else isomorphic to K_4. By Claim 5.2.1, $B \not\cong K_2$, K_3 or K_4, and so B is a 2-connected outerplanar graph that consists of a cycle C with chords. By Claim 5.2.2, C has at least one chord. This completes the proof of Claim 5.2.3. □

Let B be embedded in the plane so that C bounds the exterior face. Let a *cap* be a region R of the plane that is bounded by a chord xy and a segment C' of C such that if $z_0 \in R$, then $z_0 = x$ or y. (This is a slight modification of the definition of a cap given in [6].) We will call x and y the *end-vertices* of R. By an abuse of terminology we will refer to an edge of C' as a 0-*cap*. For $i \geq 1$, an i-*cap* is a cap that properly contains an $(i-1)$-cap and is minimal with this property. Since B is outerplanar and consists of a cycle C with at least one chord, it follows that B contains a 1-cap.

The proof now splits into two sections. In Section 5.3 we will prove that $\mathrm{ch}(G^2) \leq \Delta+2$ if $\Delta \geq 3$. In Section 5.4 we will prove that $\mathrm{ch}(G^2) \leq \Delta+1$ if $\Delta \geq 6$, although we will postpone the proofs of Claims 5.4.7–5.4.22 until the end of Section 5.4 since these proofs are long and involved.

5.3 Proof that $\mathrm{ch}(G^2) \leq \Delta + 2$ if $\Delta \geq 3$

In this section we will assume that every vertex v of G is given a list $L(v)$ of $\Delta + 2$ colours such that G^2 has no proper colouring from these lists. For each uncoloured vertex v in G^2, let $L'(v)$ denote the list of usable colours for v; that is, $L'(v)$ denotes $L(v)$ minus any colours already used on neighbours of v in G^2.

Claim 5.3.1. *Every* 1*-cap in B is a triangle xuy where $d_G(u) = 2$, $d_G(x) \geq 4$ and $d_G(y) \geq 4$.*

Proof. By definition, a 1-cap is a region bounded by a chord xy and a segment C' of C such that $d_G(u) = 2$ for every u in $C' - \{x, y\}$. By Claim 5.2.2, C' is of length at most 3. Suppose that either $C' = xu_1u_2y$, or $C' = xu_1y$ and at least one of x, y has degree at most 3 in G. Let $H = G - u_1$. By hypothesis $H^2 = G^2 - u_1$

has a proper colouring from its lists. Since $d_{G^2}(u_1) \le d_G(x)+1 \le \Delta+1$, it follows that u_1 can be given a colour from its list. This contradiction completes the proof of Claim 5.3.1. □

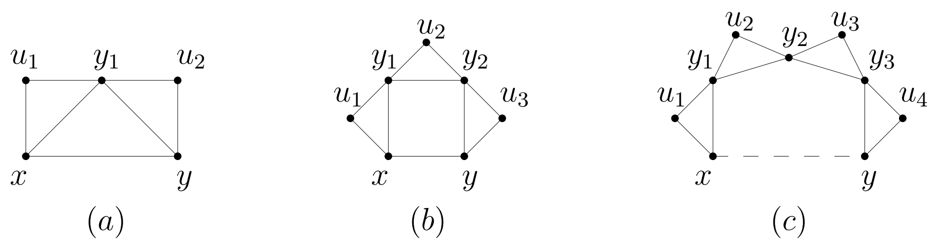


Figure 5.3

Claim 5.3.2. *B does not contain a 2-cap.*

Proof. Suppose that B does contain a 2-cap. Let R be a 2-cap in B that is bounded by a chord xy and a segment C' of C. Since R properly contains a 1-cap and is minimal with this property, it follows that there is at least one chord inside R and that every such chord bounds a 1-cap. Working around C' from x to y, let x_1y_1 be the first chord inside R, where, without loss of generality, y_1 is not an end-vertex of R and x_1 lies on the segment of C' between x and y_1, where possibly $x_1 = x$. By Claim 5.3.1, $d_G(v) \ge 4$ for every vertex v on a chord inside R; so there is another chord that is incident with y_1, say y_1y_2, and by the choice of x_1y_1 it follows that $x_1 = x$. If $y_2 = y$, then R looks like the configuration in Figure 5.3(a); otherwise there is a chord y_2y_3, and if $y_3 = y$, then R looks like the configuration in Figure 5.3(b). It is not difficult to see that every other R in this sequence will contain the configuration in Figure 5.3(c), where the dashed edge may or may not be present.

Suppose first that R is as in Figure 5.3(a). Let $H = G - u_1$. By hypothesis $H^2 = G^2 - u_1$ has a proper colouring from its lists. Since $d_{G^2}(u_1) = d_G(x) + 1 \le \Delta + 1$, it follows that u_1 can be given a colour from its list.

So suppose that R is as in Figure 5.3(b). Let $H = G - \{u_1, u_2, u_3, y_1, y_2\}$. By hypothesis $H^2 = G^2 - \{u_1, u_2, u_3, y_1, y_2\}$ has a proper colouring from its lists. Now each of u_1, u_3, y_1, y_2 has at most $\Delta - 1$ coloured neighbours in G^2, and u_2 has 2 coloured neighbours in G^2. So each of the remaining vertices

$$u_1, y_1, y_2, u_3, u_2 \tag{5.1}$$

has a list of at least 3, 3, 3, 3, 4 usable colours[2] respectively. If we try to colour the remaining vertices in the order (5.1) then it is only with u_2 that we may fail.

If possible, give u_1 and u_3 the same colour. The remaining vertices can now be coloured in the order (5.1). So we may assume that $L'(u_1) \cap L'(u_3) = \emptyset$ so that $|L'(u_1) \cup L'(u_3)| \geq 6$. Now either $|L'(u_2)| \geq 6$, or else u_1 or u_3 can be given a colour that is not in $L'(u_2)$. In each case the remaining vertices can be coloured in the order (5.1), using a colour that is not in $L'(u_2)$ at the first opportunity.

So suppose that R contains the configuration in Figure 5.3(c), where the dashed edge may or may not be present. Let $H = G - \{u_1, u_2, u_3, u_4, y_1, y_2, y_3\}$. By hypothesis $H^2 = G^2 - \{u_1, u_2, u_3, u_4, y_1, y_2, y_3\}$ has a proper colouring from its lists. Now each of u_1, u_4, y_1, y_3 has at most $\Delta - 1$ coloured neighbours in G^2, each of u_2, u_3 has 1 coloured neighbour in G^2, and y_2 has 2 coloured neighbours in G^2. So each of the remaining vertices

$$u_1, y_3, y_1, y_2, u_4, u_3, u_2 \tag{5.2}$$

has a list of at least 3, 3, 3, 4, 3, 5, 5 usable colours respectively. If we try to colour the remaining vertices in the order (5.2) then it is only with u_2 that we may fail.

If possible, give u_1 and y_3 the same colour. The remaining vertices can now be coloured in the order (5.2). So we may assume that $L'(u_1) \cap L'(y_3) = \emptyset$ so that

[2] Recall that $L'(v)$ denotes the list of usable colours for v in G^2.

$|L'(u_1) \cup L'(y_3)| \geq 6$. Now either $|L'(u_2)| \geq 6$, or else u_1 or y_3 can be given a colour that is not in $L'(u_2)$. In each case the remaining vertices can be coloured in the order (5.1), using a colour that is not in $L'(u_2)$ at the first opportunity. In every case the colouring can be completed, which is the required contradiction. $\square$

Claim 5.3.3. *B does not contain a 1-cap.*

Proof. Suppose that B does contain a 1-cap. By Claim 5.3.2, every chord in B bounds a 1-cap. Working clockwise around C from z_0, let xy be the first chord in B that bounds a 1-cap, where, without loss of generality, $y \neq z_0$. If $x = z_0$, then z_0y bounds two 1-caps, and so $d_G(y) = 3$, which contradicts Claim 5.3.1. So we may assume that $x \neq z_0$. Since $d_G(x) \geq 4$ by Claim 5.3.1, it follows that there is another chord xy_1, where y_1 is on the segment of C between y and z_0 by the choice of xy. However, xy_1 bounds a cap that is not a 1-cap since it properly contains the chord xy. This contradiction completes the proof of Claim 5.3.3. $\square$

Since Claim 5.3.3 contradicts Claim 5.2.3, this completes the proof that $\mathrm{ch}(G^2) \leq \Delta + 2$ if $\Delta \geq 3$.

5.4 Proof that $\mathrm{ch}(G^2) \leq \Delta + 1$ if $\Delta \geq 6$

In this section we will assume that every vertex v of G is given a list $L(v)$ of $\Delta + 1 \geq 7$ colours such that G^2 has no proper colouring from these lists.

Claim 5.4.1. *Every vertex of degree 2 in G has degree at least $\Delta + 1$ in G^2.*

Proof. Suppose that u is a vertex of degree 2 in G whose neighbours in G are x and y such that $d_{G^2}(u) \leq \Delta$. Let $H = G - u$ if $xy \in E(G)$ and let $H = G - u + xy$ if $xy \notin E(G)$. By hypothesis $G^2 - u \subseteq H^2$ has a proper colouring from its lists.

Since $d_{G^2}(u) \leq \Delta$, it follows that u can be given a colour from its list. This contradiction proves Claim 5.4.1. $\square$

Figure 5.4

Claim 5.4.2. *Every* 1*-cap in* B *looks like the configuration in Figure* 5.4(a) *or* 5.4(b), *where* $d_G(x) + d_G(y) \geq \Delta + 3$ *in Figure* 5.4(a), *and* $d_G(x) = d_G(y) = \Delta$ *in Figure* 5.4(b).

Proof. By the definition of a 1-cap and by Claim 5.2.2, it follows that every 1-cap in B looks like the configuration in Figure 5.4(a) or 5.4(b). By Claim 5.4.1, it follows that in Figure 5.4(a)

$$\Delta + 1 \leq d_{G^2}(u) \leq d_G(x) + d_G(y) - 2,$$

and in Figure 5.4(b)

$$\Delta + 1 \leq \begin{cases} d_{G^2}(u_1) \leq d_G(x) + 1, \\ d_{G^2}(u_2) \leq d_G(y) + 1. \end{cases}$$

This completes the proof of Claim 5.4.2. $\square$

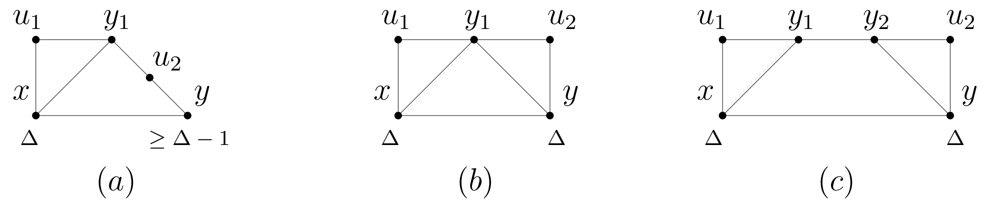


Figure 5.5

Claim 5.4.3. *Every* 2*-cap in* B *looks like the configuration in Figure* 5.5(a), 5.5(b) *or* 5.5(c), *where the degree of both* x *and* y *are restricted as specified.*

Proof. Let R be a 2-cap in B that is bounded by a chord xy and a segment C' of C. Since R properly contains a 1-cap and is minimal with this property, it follows that there is at least one chord inside R and that every such chord bounds a 1-cap. So $d_G(v) \leq 4$ for every vertex v in $C' - \{x, y\}$ since otherwise there would be a chord inside R that bounds a cap that is not a 1-cap. By the degree restrictions in Claim 5.4.2, it follows that every 1-cap in R looks like the configuration in Figure 5.4(a). Moreover, since $\Delta + 3 \geq 9 > 4 + 4$, every 1-cap in R has x or y as an end-vertex.

If R contains only one 1-cap, then R is as in Figure 5.5(a) (or its reflection), where the degree restrictions follow from Claim 5.4.1 since $d_{G^2}(u_1) = d_G(x) + 1$ and $d_{G^2}(u_2) = d_G(y) + 2$. Note that if u_2 were not present, just an edge y_1y, then $d_{G^2}(u_1) = d_G(x) \leq \Delta$, and if the edge u_2y were subdivided then $d_{G^2}(u_2) = 5 < \Delta$, which contradicts Claim 5.4.1 in each case.

If R contains two 1-caps, then R is as in Figure 5.5(b) or 5.5(c), where the degree restrictions follow from Claim 5.4.1 since $d_{G^2}(u_1) = d_G(x) + 1$ and $d_{G^2}(u_2) = d_G(y) + 1$ in each case. Note that if the edge y_1y_2 in Figure 5.5(c) were subdivided by a vertex v, then $d_{G^2}(v) = 6 \leq \Delta$, which contradicts Claim 5.4.1. This completes the proof of Claim 5.4.3. $\square$

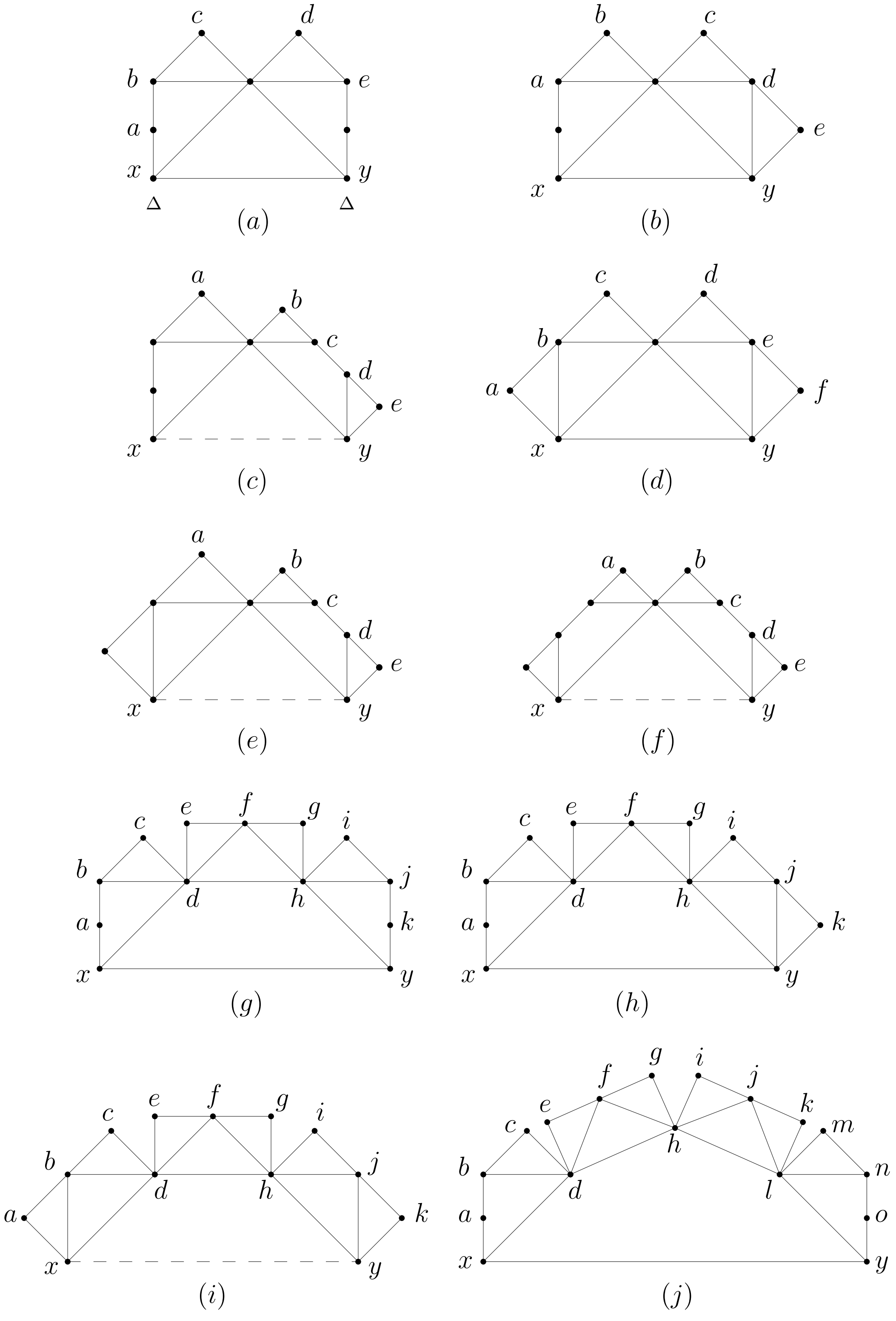

c d b e a x y Δ Δ (a)
b c a d e x y (b)
a b c d e x y (c)
c d b e a f x y (d)
a b c d e x y (e)
a b c d e x y (f)
e f g c i b j d h a k x y (g)
e f g c i b j d h a k x y (h)
e f g c i b j d h a k x y (i)
g i f j e k c m h b n d l a o x y (j)

Figure 5.6

Claim 5.4.4. *Every* 3*-cap in B looks like the configuration in Figure* 5.6(a), *where* $d_G(x) = d_G(y) = \Delta = 6$.

Proof. Let R be a 3-cap in B that is bounded by a chord xy and a segment C' of C. Since R properly contains a 2-cap and is minimal with this property, it follows that there is at least one chord inside R that bounds a 2-cap, and that every chord inside R bounds a 1-cap or a 2-cap. Let v be an end-vertex of a 2-cap R_1 in R, where $v \neq x, y$. Then it follows from Claim 5.4.3 that $d_{R_1}(v) \leq 3$, and so $d_G(v) \leq 6 \leq \Delta$ since otherwise there would be a chord inside R that bounds a cap that is not a 1-cap or a 2-cap. Furthermore, $d_G(v) = \Delta - 1$ or Δ by the degree restrictions in Figure 5.5. If $d_G(v) = \Delta - 1$, then $d_{R_1}(v) = 2$ and since $\Delta - 1 \geq 5$, it follows that v is the end-vertex of another cap R_2 that does not contain R_1 such that $d_{R_2}(v) \geq 3$. However, this implies that R_2 is a 2-cap, and that $d_G(v) = \Delta$ by the degree restrictions in Figure 5.5. This contradiction shows that $d_G(v) \neq \Delta - 1$. So $d_G(v) = \Delta = 6$ for each end-vertex v of a 2-cap R_1 in R, where $v \neq x, y$. Furthermore, $d_{R_1}(v) = 3$ and so v is the end-vertex another 2-cap R_2 in R, where $d_{R_2}(v) = 3$. Working around C' from x to y, let x_1y_1 be the first chord inside R that bounds a 2-cap R_1, where, without loss of generality, y_1 is not an end-vertex of R and x_1 lies on the segment of C' between x and y_1, where possibly $x_1 = x$. In fact, by the choice of x_1y_1 it follows that $x_1 = x$, and there is a path $xy_1 \ldots y_ny$ in R each edge of which bounds a 2-cap, where $d_G(y_i) = \Delta = 6$ for all i.

If $n = 1$, then R contains exactly two 2-caps. Since $d_G(y_1) = \Delta = 6$, it follows that R is one of the configurations in Figures 5.6(a)–5.6(f) (or their reflections). However, Claims 5.4.7–5.4.11 show that R is as in Figure 5.6(a), where both x and y have degree $\Delta = 6$.

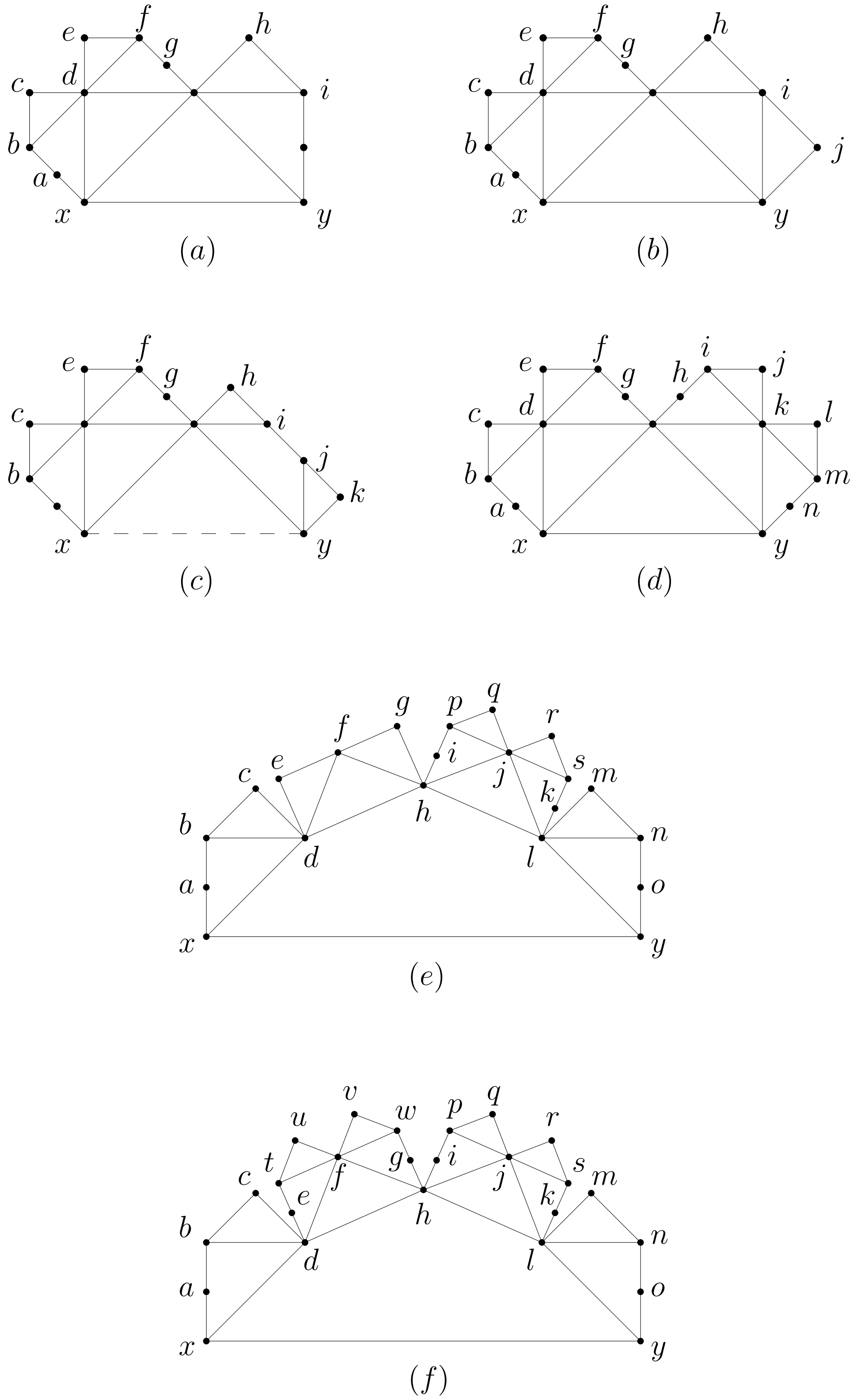
If $n = 2$, then R contains exactly three 2-caps. Let R_1, R_2, R_3 be the 2-caps bounded by xy_1, y_1y_2, y_2y respectively. Since $d_G(y_1) = d_G(y_2) = \Delta = 6$ and since, by Claims 5.4.9 and 5.4.11, R does not contain the configurations in Figures 5.6(c), 5.6(e) and 5.6(f), where in each case the dashed edge is not present, it follows that R_2 is a 2-cap of the type in Figure 5.5(b). Moreover, any 2-cap in R that does not have x or y as an end-vertex is of the type in Figure 5.5(b). So R is one of the configurations in Figures 5.6(g)–5.6(i) (or their reflections). However, Claims 5.4.12 and 5.4.13 show that this is impossible.

If $n \geq 3$, then R contains at least four 2-caps. Since any 2-cap in R that does not have x or y as an end-vertex is of the type in Figure 5.5(b), and since R does not contain the configurations in Figures 5.6(e) and 5.6(i), where in each case the dashed edge is not present, it follows that R contains exactly four 2-caps and that R is as in Figure 5.6(j). However, Claim 5.4.14 shows that this is impossible. This completes the proof of Claim 5.4.4. □

Claim 5.4.5. *B does not contain a 4-cap. So every cap in B looks like one of the configurations in Figures 5.4, 5.5 and 5.6(a), and so every end-vertex v of a cap R has $d_R(v) \leq 3$.*

Proof. Suppose that B does contain a 4-cap. Let R be a 4-cap in B that is bounded by a chord xy and a segment C' of C. Since R properly contains a 3-cap and is minimal with this property, it follows that there is at least one chord inside R that bounds a 3-cap, and that every chord inside R bounds a 1-cap, a 2-cap or a 3-cap.

Let v be an end-vertex of a 2-cap or a 3-cap R_1 in R, where $v \neq x, y$. As in the proof of Claim 5.4.4, by the degree restrictions for v, it follows that $d_G(v) = \Delta = 6$ and $d_{R_1}(v) = 3$, and so there is a path $xy_1 \ldots y_ny$ in R each edge of which bounds a 2-cap or a 3-cap, where $d_G(y_i) = \Delta = 6$ for all i.


e f g h
c d i
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k
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(c)
e f g h i j
c d k l
b m
a n
x y
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g p q
f r
c e i s
j m
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Figure 5.7

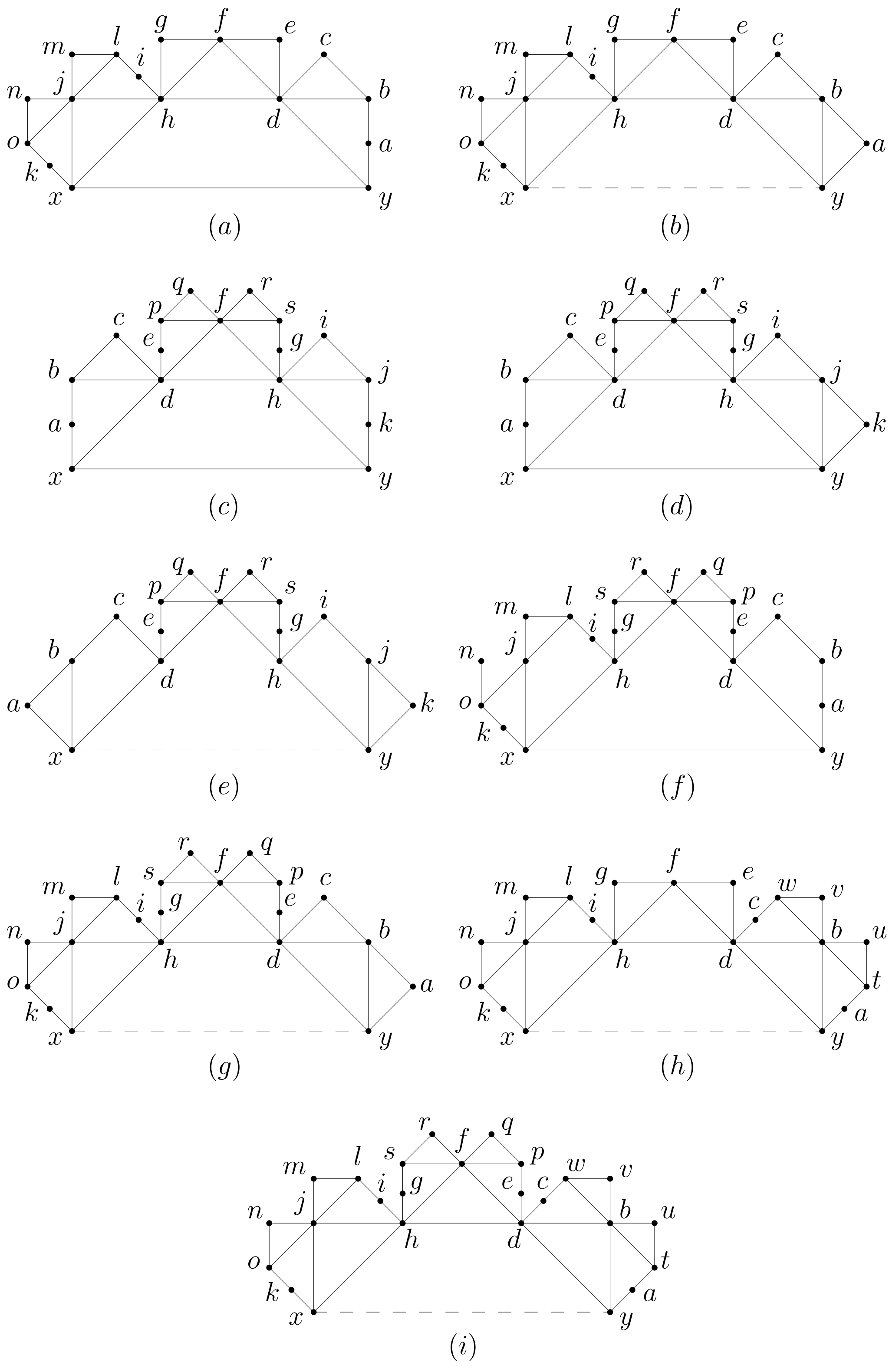

Figure 5.8

If $n = 1$, then since $d_G(y_1) = \Delta = 6$ and at least one of xy_1 and y_1y bounds a 3-cap, it follows that R is one of the configurations in Figures 5.7(a)–5.7(d) (or their reflections). However, Claims 5.4.15–5.4.18 show that this is impossible.

If $n = 2$, then let R_1, R_2, R_3 be the caps in R bounded by xy_1, y_1y_2, y_2y respectively. Since $d_G(y_1) = d_G(y_2) = \Delta = 6$ and since, by Claims 5.4.9, 5.4.11 and 5.4.17, R does not contain the configurations in Figures 5.6(c), 5.6(e), 5.6(f) and 5.7(c), where in each case the dashed edge is not present, it follows that R_2 looks like the configuration in Figure 5.5(b) or 5.6(a). Moreover, any 2-cap in R that does not have x or y as an end-vertex is of the type in Figure 5.5(b). So R is one of the configurations in Figures 5.8(a)–5.8(i) (or their reflections). However, Claims 5.4.19–5.4.20 show that this is impossible.

If $n \geq 3$, then let R_1, R_2, R_3, R_4 be the caps in R bounded by xy_1, y_1y_2, y_2y_3, y_3y_j respectively, where possibly $y_j = y$. Since each of R_2 and R_3 looks like the configuration in Figure 5.5(b) or 5.6(a), and since R does not contain the configurations in Figures 5.6(e), 5.7(c), 5.8(b), 5.8(e), 5.8(g)–5.8(i), where in each case the dashed edge is not present, it follows that $y_j = y$, and that R is as in Figure 5.7(e) or 5.7(f) (or their reflections). However, Claims 5.4.21 and 5.4.22 show that this is impossible. In every case we have obtained a contradiction, which proves Claim 5.4.5. □

Claim 5.4.6. *B does not contain a chord.*

Proof. Suppose that B does contain a chord. Then working clockwise around C from z_0, let xy be the first chord in B that bounds a cap R_1, where, without loss of generality, $y \neq z_0$, and if x is incident with more than one chord then choose xy so that $d_{R_1}(x)$ is as large as possible. If $x = z_0$, then z_0y bounds two caps R_1 and R_2. Since $d_{R_i}(y) \leq 3$ $(i = 1, 2)$ by Claim 5.4.5, it follows that $d_G(y) \leq 5 < \Delta$.

In order to satisfy the degree restrictions in Figures 5.4, 5.5 and 5.6(a), it follows that $d_G(y) = 3$ and that each of R_1 and R_2 is a 1-cap of the type in Figure 5.4(a). However, every vertex of B is adjacent to z_0, which contradicts Claim 5.2.1. So we may assume that $x \neq z_0$.

Since $d_{R_1}(x) \leq 3$, then by the choice of xy it follows that $d_G(x) \leq 4$. So in order to satisfy the degree restrictions in Figures 5.4, 5.5 and 5.6(a), it follows that R_1 is a 1-cap of the type in Figure 5.4(a). Since $d_G(x) = 3$, then by Claim 5.4.2 it follows that $d_G(y) = \Delta$. So y is incident with another chord, say yy_1, that bounds a cap R_2. By the choice of xy, it follows that y_1 lies on the segment of C between z_0 and y that does not contain x. Since $d_G(y) = \Delta \geq 6$, we may assume that $d_{R_2}(y) \geq 4$, which contradicts Claim 5.4.5. This contradiction completes the proof of Claim 5.4.6. $\square$

Since Claim 5.4.6 contradicts Claim 5.2.3, this completes the proof that $\mathrm{ch}(G^2) \leq \Delta + 1$ if $\Delta \geq 6$. This completes the proof of Theorem 5.1.1.

5.4.1 Proof of Claims 5.4.7–5.4.22

In this section we will consider many different caps R in B. By definition, if $z_0 \in R$, then $z_0 = x$ or y. We will prove various statements about R. In the proof of each claim let S be the set of labelled vertices in R, as shown in the corresponding figure. Let $H = G - (S \setminus \{x, y\})$. By hypothesis $H^2 = G^2 - (S \setminus \{x, y\})$ has a proper colouring from its lists. In each of Claims 5.4.8–5.4.22 we will extend a proper colouring of H^2 to a proper colouring of G^2, which will prove that B does not contain R if R looks like one of the configurations in Figures 5.6(b)–5.8, where in each case the dashed edge may or may not be present. Recall that $L'(v)$ denotes the list of usable colours for each uncoloured vertex v in G^2.

Claim 5.4.7. *If R looks like the configuration in Figure 5.6(a), then $d_G(x) = d_G(y) = \Delta$.*

Proof. Suppose that R is as in Figure 5.6(a) such that $d_G(x) \leq \Delta - 1$ and $d_G(y) \leq \Delta$. Let $H = G - \{a, b, c, d, e\}$. By hypothesis $H^2 = G^2 - \{a, b, c, d, e\}$ has a proper colouring from its lists. Now each of b, c has at most 3 coloured neighbours in G^2, each of d, e has at most 4 coloured neighbours in G^2, and a has at most $\Delta - 1$ coloured neighbours in G^2. So each of the remaining vertices

$$c, a, b, e, d \tag{5.3}$$

has a list of at least 4, 2, 4, 3, 3 usable colours respectively. Now either $|L'(d)| \geq 4$, or else c can be given a colour that is not in $L'(d)$. In each case the remaining vertices can be coloured in the order (5.3). This contradiction completes the proof of Claim 5.4.7. □

Claim 5.4.8. *R does not contain the configuration in Figure 5.6(b).*

Proof. Suppose that R does contain the configuration in Figure 5.6(b). Then after applying a colouring of H^2 to G^2, each of the remaining vertices

$$e, d, a, b, c \tag{5.4}$$

has a list of at least 2, 2, 3, 3, 4 usable colours respectively. If we try to colour the remaining vertices in the order (5.4) then it is only with c that we may fail.

If possible, give a and e the same colour. The remaining vertices can now be coloured in the order (5.4). So we may assume that $L'(a) \cap L'(e) = \emptyset$ so that $|L'(a) \cup L'(e)| \geq 5$. Now either $|L'(c)| \geq 5$, or else a or e can be given a colour that is not in $L'(c)$. In each case the remaining vertices can be coloured in the order (5.4), using a colour that is not in $L'(c)$ at the first opportunity. This contradiction proves Claim 5.4.8. □

Claim 5.4.9. *R does not contain the configuration in Figure* 5.6(c), *where the dashed edge may or may not be present.*

Proof. Suppose that R does contain the configuration in Figure 5.6(c), where the dashed edge may or may not be present. Then after applying a colouring of H^2 to G^2, each of the remaining vertices

$$d, e, c, a, b \tag{5.5}$$

has a list of at least 2, 2, 3, 2, 3 usable colours respectively. If we try to colour the remaining vertices in the order (5.5) then it is only with b that we may fail.

If possible, give a and d the same colour. The remaining vertices can now be coloured in the order (5.5). So we may assume that $L'(a) \cap L'(d) = \emptyset$ so that $|L'(a) \cup L'(d)| \geq 4$. Now either $|L'(b)| \geq 4$, or else a or d can be given a colour that is not in $L'(b)$. In each case the remaining vertices can be coloured in the order (5.4), using a colour that is not in $L'(b)$ on either d, c or a at the first opportunity. This contradiction proves Claim 5.4.9. □

Claim 5.4.10. *R does not contain the configuration in Figure* 5.6(d).

Proof. Suppose that R does contain the configuration in Figure 5.6(d). Then after applying a colouring of H^2 to G^2, each of the remaining vertices

$$a, b, e, f, d, c \tag{5.6}$$

has a list of at least 2, 2, 2, 2, 4, 4 usable colours respectively. If we try to colour the remaining vertices in the order (5.6) then it is only with c that we may fail.

If $L'(e) \cap L'(c) = \emptyset$, then the vertices can be coloured in the order (5.6). So we may assume that $L'(e) \subseteq L'(c)$. If possible, give a and e the same colour. The remaining vertices can now be coloured in the order (5.6). So we may assume that $L'(a) \cap L'(e) = \emptyset$. If possible, give b a colour that is not in $L'(a)$. The remaining vertices can now be coloured in the order (5.6) with the exception that a is coloured last. So we may assume that $L'(b) \subseteq L'(a)$ and, by symmetry, that $L'(e) \subseteq L'(f)$. So after colouring a and b, we can give c and f the same colour since $L'(e) \subseteq L'(c)$ and $L'(e) \subseteq L'(f)$. We can now colour e since $L'(b) \subseteq L'(a)$ and $L'(a) \cap L'(e) = \emptyset$, and finally d. In every case the colouring can be completed, which is the required contradiction. □

Claim 5.4.11. *R does not contain the configuration in Figure 5.6(e) or 5.6(f), where in each case the dashed edge may or may not be present.*

Proof. Suppose that R does contain the configuration in Figure 5.6(e) or 5.6(f), where in each case the dashed edge may or may not be present. Then after applying a colouring of H^2 to G^2, each of the remaining vertices d, e, c, a, b has a list of at least 2, 2, 3, 2, 3 usable colours respectively. The remaining vertices can be coloured as in the proof of Claim 5.4.9. This contradiction completes the proof of Claim 5.4.11. □

Claim 5.4.12. *R does not contain the configuration in Figure 5.6(g) or 5.6(h).*

Proof. Suppose that R does contain the configuration in Figure 5.6(g) or 5.6(h). Then after applying a colouring of H^2 to G^2, in Figure 5.6(g) each of the remaining vertices

$$h, d, a, f, k, j, i, g, e, b, c \tag{5.7}$$

has a list of at least 2, 2, 2, 5, 2, 6, 6, 6, 6, 6, 6 usable colours respectively, and in Figure 5.6(h) each of the remaining vertices

$$h, d, a, j, f, k, i, g, e, b, c \tag{5.8}$$

has a list of at least 3, 2, 2, 3, 5, 3, 6, 6, 6, 6, 6 usable colours respectively. If we try to colour the remaining vertices in the order (5.7) or (5.8), as appropriate, then it is only with c that we may fail.

1. If h can be given a colour that is not in $L'(c)$, then we will not fail with c. So we may assume that $L'(h) \subseteq L'(c)$.

2. If possible, give a and h the same colour. The remaining vertices can now be coloured in the order (5.7) or (5.8), as appropriate. So we may assume that $L'(a) \cap L'(h) = \emptyset$.

3. If possible, give a and f the same colour, which is not in $L'(h)$. The remaining vertices can now be coloured in the order (5.7) or (5.8), as appropriate, with the exception that d is coloured before h in each case, and in Figure 5.6(h), j is also coloured before h, but after d. So we may assume that $L'(a) \cap L'(f) = \emptyset$ so that $|L'(a) \cup L'(f)| \geq 7$.

4. Now either $|L'(c)| \geq 7$, or else a or f can be given a colour that is not in $L'(c)$. In each case the remaining vertices can be coloured in the order (5.7) or (5.8), as appropriate, using a colour that is not in $L'(c)$ at the first opportunity, with the exceptions that in Figure 5.6(h), h is coloured immediately after j, and if f is required to have a colour that is not in $L'(c)$, then this colour is not given to j.

In every case the colouring can be completed, which is the required contradiction. This completes the proof of Claim 5.4.12. □

Claim 5.4.13. *R does not contain the configuration in Figure 5.6(i), where the dashed edge may or may not be present.*

Proof. Suppose that R does contain the configuration in Figure 5.6(i), where the dashed edge may or may not be present. Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{d, h\}$, $|L'(z)| \geq 3$ if $z \in \{a, b, j, k\}$, $|L'(z)| \geq 6$ if $z \in \{c, e, g, i\}$, and $|L'(f)| \geq 5$. If we try to colour the remaining vertices in the order

$$h, d, b, j, f, k, i, g, e, c, a \tag{5.9}$$

or

$$d, h, b, j, f, a, k, i, g, e, c \tag{5.10}$$

then it is only with the last vertex to be coloured that we may fail.

1. If possible, give d a colour that is not in $L'(a)$. The remaining vertices can now be coloured in the order (5.9). So we may assume that $L'(d) \subseteq L'(a)$.

2. If possible, give b a colour that is not in $L'(a)$, and hence not in $L'(d)$. The remaining vertices can now be coloured in the order (5.9). So we may assume that $L'(b) \subseteq L'(a)$.

3. If possible, give a and h the same colour. The remaining vertices can now be coloured in the order (5.10). So we may assume that $L'(a) \cap L'(h) = \emptyset$, and so $L'(b) \cap L'(h) = \emptyset$ and $L'(d) \cap L'(h) = \emptyset$.

4. If possible, give a and f the same colour. The remaining vertices can now be coloured in the order (5.10) with the exception that h is coloured immediately after j, which is possible since the colours on b, d, f are not in $L'(h)$. So we may assume that $L'(a) \cap L'(f) = \emptyset$ so that $|L'(a) \cup L'(f)| \geq 8$.

5. Now either $|L'(c)| \geq 8$, or else a or f can be given a colour that is not in $L'(c)$. In each case the remaining vertices can be coloured in the order (5.10), using a colour that is not in $L'(c)$ at the first opportunity, with the exceptions that h is coloured immediately after j, and if f is required to have a colour that is not in $L'(c)$, then this colour is not given to j.

In every case the colouring can be completed, which is the required contradiction. This completes the proof of Claim 5.4.13. □

Claim 5.4.14. *R does not contain the configuration in Figure* 5.6(j).

Proof. Suppose that R does contain the configuration in Figure 5.6(j). Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{a, d, l, o\}$, $|L'(z)| \geq 6$ if $z \in \{b, c, e, f, j, k, m, n\}$, $|L'(z)| \geq 7$ if $z \in \{g, i\}$, and $|L'(h)| \geq 5$. If we try to colour the remaining vertices in the order

$$d, a, l, o, h, n, m, j, k, i, f, g, e, b, c \tag{5.11}$$

then it is only with c that we may fail.

1. If $|L'(c)| \geq 7$, then we will not fail with c, and so we may assume that $|L'(c)| = 6$.

2. If possible, give a or d a colour that is not in $L'(c)$. In each case the remaining vertices can now be coloured in the order (5.11). So we may assume that $L'(a) \subseteq L'(c)$ and $L'(d) \subseteq L'(c)$.

3. If possible, give h a colour that is not in $L'(c)$, and hence not in $L'(d)$. We can now colour in order l, o, d, a, and then the remaining vertices can be coloured in the order (5.11). So we may assume that $L'(h) \subseteq L'(c)$.

4. Since $|L'(a)| + |L'(h)| \geq 7$ and since $|L'(c)| = 6$, $L'(a) \subseteq L'(c)$ and $L'(h) \subseteq L'(c)$, it follows that $|L'(a) \cap L'(h)| \geq 1$. So we can give a and h the same colour. If the remaining vertices cannot now be coloured in the order (5.11), even with l coloured first, then $|L'(d)| = |L'(l)| = 2$ and $L'(d) = L'(l)$, and so $L'(l) \subseteq L'(c)$.

5. If possible, colour a and f so that either f is given a colour that is not in $L'(c)$, and hence not in $L'(l)$, or a and f are given the same colour and this colour in not in $L'(l)$. We can now colour in order d, l, o, h. At this point, each of the remaining vertices

$$k, j, n, m, i, g, e, b, c \tag{5.12}$$

has a list L'' of at least 4, 2, 3, 3, 3, 3, 3, 3 usable colours respectively. If we try to colour the remaining vertices in the order (5.12) then it is only with m that we may fail.

   (a) If $|L''(m)| \geq 4$, then we will not fail with m, and so we may assume that $|L''(m)| = 3$.

   (b) Since $|L''(k)| \geq 4$ we can give k a colour that is not in $L''(m)$. The remaining vertices can now be coloured in the order (5.12). So we may assume that $L'(f) = L'(c)$, and so $L'(a), L'(d), L'(l) \subseteq L'(f)$, and also $L'(a) \subseteq L'(l)$.

6. If possible, give a, e and l the same colour. The remaining vertices can now be coloured in the order (5.11). So we may assume that $L'(a) \cap L'(e) = \emptyset$ so that $|L'(a) \cup L'(e)| \geq 8$.

7. Since $L'(a) \subseteq L'(c)$ and $|L'(c)| = 6$, it follows that there are at least two colours $\alpha, \beta \in L'(e)$ that are not in $L'(c)$, and hence not in $L'(f)$. So we can give e the colour α. The remaining vertices can now be coloured in the order

(5.11) with the exception that if we fail at g, then since $|L(g)| = 7$ and g has seven coloured neighbours in G^2, we can uncolour e and give g the colour α. We can now recolour e with β since the coloured neighbours of e in G^2 are d, f, g, h, each of which is given a colour that is in $L'(c)$. Finally, we can give colours to b and then c.

In every case the colouring can be completed, which is the required contradiction. This completes the proof of Claim 5.4.14. □

Claim 5.4.15. *R does not contain the configuration in Figure 5.7(a).*

Proof. Suppose that R does contain the configuration in Figure 5.7(a). Then after applying a colouring of H^2 to G^2, each of the remaining vertices

$$a, d, i, h, g, e, f, b, c \tag{5.13}$$

has a list of at least 2, 2, 3, 3, 4, 5, 5, 5, 5 usable colours respectively. If we try to colour the remaining vertices in the order (5.13) then it is only with c that we may fail.

If possible, give a or d a colour that is not in $L'(c)$. In each case the remaining vertices can now be coloured in the order (5.13). So we may assume that $L'(a) \subseteq L'(c)$ and $L'(d) \subseteq L'(c)$. If possible, give d a colour that is not in $L'(h)$, and then colour a. At this point, each of the remaining vertices

$$c, g, i, h, f, b, e \tag{5.14}$$

has a list L'' of at least 3, 3, 2, 3, 4, 3, 4 usable colours respectively. If possible, give c and g the same colour. The remaining vertices can now be coloured in the order (5.14). So we may assume that $L''(c) \cap L''(g) = \emptyset$ so that $|L''(c) \cup L''(g)| \geq 6$. Now either $|L''(e)| \geq 6$, or else c or g can be given a colour that is not in $L''(e)$. In each case the remaining vertices can be coloured in the order (5.14). So we may assume that $L'(d) \subseteq L'(h)$.

Since $|L'(g)| \geq 4$ and $|L'(h)| \geq 3$, we can assume without loss of generality that g can be given a colour that is not in $L'(h)$, and hence not in $L'(d)$. At this point, each of the remaining vertices

$$a, e, d, i, h, f, b, c \tag{5.15}$$

has a list L'' of at least 2, 4, 2, 2, 3, 4, 5, 5 usable colours respectively. If possible, give a and e the same colour. The remaining vertices can now be coloured in the order (5.15). So we may assume that $L''(a) \cap L''(e) = \emptyset$ so that $|L''(a) \cup L''(e)| \geq 6$. Now either $|L''(c)| \geq 6$, or else e can be given a colour that is not in $L'(c)$. In each case the remaining vertices can be coloured in the order (5.15). This contradiction proves Claim 5.4.15. $\square$

Claim 5.4.16. *R does not contain the configuration in Figure 5.7(b).*

Proof. Suppose that R does contain the configuration in Figure 5.7(b). Then after applying a colouring of H^2 to G^2, each of the remaining vertices

$$d, a, i, j, h, g, e, f, b, c \tag{5.16}$$

has a list of at least 2, 2, 2, 2, 4, 4, 5, 5, 5, 5 usable colours respectively. If we try to colour the remaining vertices in the order (5.16) then it is only with c that we may fail. If $|L'(c)| \geq 6$, then we will not fail with c, and so we may assume that $|L'(c)| = 5$.

If possible, give a a colour that is not in $L'(c)$. The remaining vertices can now be coloured in the order (5.16). So we may assume that $L'(a) \subseteq L'(c)$. If possible, give d and j the same colour, and then colour a and i. Next, we may assume without loss of generality that e can be given a colour that is not in $L'(c)$, since at this point each of c, e has at least 3, 4 usable colours in its list respectively. The remaining vertices can now be coloured in the order (5.16) with the exception that g is coloured first. So we may assume that $L'(d) \cap L'(j) = \emptyset$.

If $L'(i) \neq L'(j)$, then give i a colour that is not in $L'(j)$, and then colour d and a. Next, we may assume without loss of generality that e can be given a colour that is not in $L'(c)$, since at this point each of c, e has at least 3, 4 usable colours in its list respectively. The remaining vertices can now be coloured in the order (5.16) with the exception that g and h are coloured before j *in that order*. So we may assume that $L'(i) = L'(j)$.

So we may colour in order a, d, e so that either a and e are given the same colour, or else e is given a colour that in not in $L'(c)$, which is possible since $L'(a) \subseteq L'(c)$ and $|L'(a)| + |L'(e)| \geq 7$. The remaining vertices can now be coloured in the order (5.16) with the exception that if we fail at g, then since $|L(g)| = 7$ and at the time of its colouring g has seven coloured neighbours in G^2, and since $L'(d) \cap L'(j) = \emptyset$, we can swap the colours on i and j so that both g and j now have the colour that was on i. In every case the colouring can be completed, which is the required contradiction. $\square$

Claim 5.4.17. *R does not contain the configuration in Figure 5.7(c), where the dashed edge may or may not be present.*

Proof. Suppose that R does contain the configuration in Figure 5.7(c), where the dashed edge may or may not be present. Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{j,k\}$, $|L'(z)| \geq 3$ if $z \in \{b,c,g,h,i\}$, and $|L'(z)| \geq 4$ if $z \in \{e,f\}$. So we may assume without loss of generality that e can be given a colour that is not in $L'(c)$. At this point, each of the remaining vertices

$$j, k, i, g, h, f, b, c \tag{5.17}$$

has a list L'' of at least 2, 2, 3, 2, 3, 3, 2, 2 usable colours respectively.

If possible, give g and j the same colour. The remaining vertices can now be coloured in the order (5.17). So we may assume that $L''(g) \cap L''(j) = \emptyset$ so that $|L''(g) \cup L''(j)| \geq 4$. Now either $|L''(h)| \geq 4$, or else g or j can be given a colour that is not in $L''(h)$. In each case the remaining vertices can be coloured in the order (5.17), using a colour that is not in $L''(h)$ on either j, i or g at the first opportunity. This contradiction proves Claim 5.4.17. $\square$

Claim 5.4.18. *R does not contain the configuration in Figure 5.7(d).*

Proof. Suppose that R does contain the configuration in Figure 5.7(d). Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{a, d, k, n\}$, $|L'(z)| \geq 4$ if $z \in \{g, h\}$, and $|L'(z)| \geq 5$ if $z \in \{b, c, e, f, i, j, l, m\}$. If we try to colour the remaining vertices in the order

$$a, d, k, n, j, h, g, i, m, l, e, f, b, c \tag{5.18}$$

then it is only with l and/or c that we may fail. If possible, colour both a and d so that one of them is given a colour that is not in $L'(c)$, and then colour k and n. Next, we may assume without loss of generality that j can be given a colour that is not in $L'(l)$, since at this point each of j, l has at least 4, 3 usable colours in its list respectively. The remaining vertices can now be coloured in the order (5.18). So we may assume that $L'(a) \subseteq L'(c)$ and $L'(d) \subseteq L'(c)$.

Since $L'(a) \subseteq L'(c)$ and $|L'(a)| + |L'(e)| \geq 7$, and since we may assume without loss of generality that $|L'(c)| = 5$, it follows that we have three cases to consider:

(i) $L'(e)$ has at least two colours that are not in $L'(c)$;

(ii) $L'(e)$ has one colour that is not in $L'(c)$ and $|L'(a) \cap L'(e)| = 1$;

(*iii*) $L'(a) \subseteq L'(e)$.

Case (*i*) *and* (*ii*): If possible, give a and e the same colour; otherwise colour a and e so that e is given a colour that is not in $L'(c)$. The remaining vertices can now be coloured in the order (5.18), where, as above, j is given a colour that is not in $L'(l)$, and with the exception that if we fail at g, then since $|L(g)| = 7$ and at the time of its colouring g has seven coloured neighbours in G^2, we can uncolour e and give g the colour that was on e. We can now recolour e with a colour that is not in $L'(c)$, and then continue in the order (5.18).

Case (*iii*): Give a and e the same colour. If this colour is not in $L'(g)$, or if k can be given the same colour as a and e, then the remaining vertices can now be coloured in the order (5.18), where, as above, j is given a colour that is not in $L'(l)$. So we may assume that $L'(a) \subseteq L'(g)$ and $L'(a) \cap L'(k) = \emptyset$.

If $|L'(a) \cap L'(d)| \leq 1$, then give a and e the same colour, say α, so that, if possible, $\alpha \in L'(d)$. The remaining vertices can now be coloured in the order (5.18), where, as above, j is given a colour that is not in $L'(l)$, and with the exception that if we fail at g, then since $|L(g)| = 7$ and at the time of its colouring g has seven coloured neighbours in G^2, we can uncolour a and e and give g the colour α. We can now recolour both a and e with another colour that is in $L'(a)$, and then continue in the order (5.18). So we may assume without loss of generality that $|L'(a)| = |L'(d)| = 2$ and $L'(a) = L'(d)$.

If $|L'(a) \cap L'(h)| \geq 1$, then give a, e, and h the same colour, and then colour d. At this point, each of the remaining vertices

$$n, k, g, i, m, l, j, f, b, c \tag{5.19}$$

has a list L'' of at least 2, 2, 2, 4, 5, 5, 4, 3, 3, 3 usable colours respectively. If possible, give k a colour that is not in $L'(j)$. The remaining vertices can now be coloured in the order (5.19). So we may assume that $L'(k) \subseteq L'(j)$. So we may assume without loss of generality that l can be given a colour that is not in $L'(j)$, and hence not in $L'(k)$. The remaining vertices can now be coloured in the order (5.19). So we may assume that $L'(a) \cap L'(h) = \emptyset$.

So we can give a and e the same colour. The remaining vertices can now be coloured in the order (5.18), where, as above, j is given a colour that is not in $L'(l)$, and with the exception that h is coloured immediately after g, which is possible since $L'(d) \cap L'(h) = \emptyset$. In every case the colouring can be completed, which is the required contradiction. □

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
(a)	2	6	6	2	6	5	6	3	6	3	3	6	6	6	6				
(b)	3	3	6	2	6	5	6	2	6	3	3	6	6	6	6				
(c)	2	6	6	2	6	5	6	2	6	6	2					7	7	7	7
(d)	2	6	6	2	6	5	6	3	6	3	3					7	7	7	7
(e)	3	3	6	2	6	5	6	3	6	3	3					7	7	7	7
(f)	2	6	6	2	6	5	6	3	6	3	3	6	6	6	6	7	7	7	7
(g)	3	3	6	2	6	5	6	2	6	3	3	6	6	6	6	7	7	7	7

Table 5.1

Claim 5.4.19. *R does not contain one of the configurations in Figures* 5.8(a)–5.8(g) (*or their reflections*), *where in each case the dashed edge may or may not be present.*

Proof. Suppose that R does contain one of the configurations in Figures 5.8(a)–5.8(g) (or their reflections), where in each case the dashed edge may or may not be present. Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours and $|L'(z)| \geq n$, where n is the number shown in the appropriate line of Table 5.1.

Due to the similarities between Figures 5.8(a)–5.8(g) and Figures 5.6(g)–5.6(i), we can extend the arguments used in Claims 5.4.12 and 5.4.13.

If R is as in Figure 5.8(a) or 5.8(b), then the remaining vertices can be coloured as in the proof of Claim 5.4.12 for Figure 5.6(h), or Claim 5.4.13 respectively, with the exception in each case that immediately after colouring k, we may assume without loss of generality that m can be given a colour that is not in $L'(n)$, since at this point each of m, n has at least 4, 3 usable colours in its list respectively. We can now colour in order i, o, l, n, and then continue as before.

If R is as in Figure 5.8(c), 5.8(d), 5.8(e), 5.8(f) or 5.8(g), then the remaining vertices can be coloured as in the proof of Claim 5.4.12 for Figure 5.6(g), Claim 5.4.12 for Figure 5.6(h), Claim 5.4.13, or as above for Figure 5.8(a) or 5.8(b) respectively, with the exception in each case that immediately after colouring g, we may assume without loss of generality that q can be given a colour that is not in $L'(r)$, since at this point each of q, r has at least 4, 3 usable colours in its list respectively. We can now colour in order p, s, r, and then continue as before. In every case the colouring can be completed, which is the required contradiction. □

Claim 5.4.20. *R does not contain the configuration in Figure 5.8(h) or 5.8(i), where in each case the dashed edge may or may not be present.*

Proof. Suppose that R does contain the configuration in Figure 5.8(h) or 5.8(i), where in each case the dashed edge may or may not be present. Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{d, h\}$, $|L'(z)| \geq 3$ if $z \in \{a, b, j, k\}$, $|L'(z)| \geq 6$ if $z \in \{c, e, g, i, l, m, n, o, t, u, v, w\}$, and $L'(f) \geq 5$, and in Figure 5.8(i),

$|L'(z)| \geq 7$ if $z \in \{p, q, r, s\}$. The methods of the previous arguments fail since a is not adjacent to d in G^2.

If R is as in Figure 5.8(h), then we can colour the remaining vertices as follows.

1. If possible, give d a colour that is not in $L'(a)$. Now colour in order h, b, j, k, f. Next, we may assume without loss of generality that m can be given a colour that is not in $L'(n)$, since at this point each of m, n has at least 4, 3 usable colours in its list respectively. Now colour i, o, l, n, g, e, c. Now we may assume without loss of generality that u can be given a colour that is not in $L'(v)$, since at this point each of u, v has at least 4, 3 usable colours in its list respectively. The remaining vertices can now be coloured in the order a, t, w, v. So we may assume that $L'(d) \subseteq L'(a)$.

2. If possible, give b a colour that is not in $L'(a)$, and hence not in $L'(d)$. After colouring first h and then d, the remaining vertices can now be coloured as in step 1. So we may assume that $L'(b) \subseteq L'(a)$.

3. Suppose that there is a colour $\alpha \in L'(a)$ that is not in at least one of $L'(t)$, $L'(u)$, $L'(v)$ or $L'(w)$ so that we can give α to d (or b if $\alpha \notin L'(d)$). The remaining vertices can now be coloured in the order

$$d \text{ (or } b), h, b \text{ (or } d), a, j, k, f, m, o, i, l, n, g, e, c, (u, v, w, t), \tag{5.20}$$

   where, as in step 1, m can be given a colour that is not in $L'(n)$. Also, u, v, w, t are in any order with the exception that the last vertex must not have α in its list. So we may assume that $L'(a) \subseteq L'(z)$, where $z \in \{u, v, w, t\}$.

4. If possible, colour t and u so that at least one of them is given a colour that is not in $L'(v)$ (or $L'(w)$). The remaining vertices can be now coloured in the order (5.20), where m can be given a colour that is not in $L'(n)$, and

where v (or w) is coloured last. So we may assume without loss of generality that $L'(t) = L'(u) = L'(v) = L'(w)$.

5. If possible, give a and h the same colour. Since $L'(a) \subseteq L'(w)$ the same colour can be given to w also. Now colour d and b. The remaining vertices can now be coloured in the order (5.20), where m can be given a colour that is not in $L'(n)$, and where t is coloured last. So we may assume that $L'(h) \cap L'(a) = \emptyset$. So b and d are definitely given colours that are not in $L'(h)$.

6. If possible, give b a colour that is not in $L'(c)$. The remaining vertices can now be coloured in the order

$$d, a, h, j, k, f, m, o, i, l, n, g, e, v, c, t, w, u \tag{5.21}$$

where, as before, m is given a colour that is not in $L'(n)$ and where v receives the same colour as a. So we may assume that $L'(b) \subseteq L'(c)$.

7. If possible, give h a colour that is not in $L'(c)$. Now colour in order d, b, a. The remaining vertices can now be coloured in the order (5.21), as in step 6. So we may assume that $L'(h) \subseteq L'(c)$.

8. If possible, give a, f and w the same colour. Now colour in order d, b, j, h. The remaining vertices can now be coloured in the order (5.20), as in step 5. So we may assume that $L'(a) \cap L'(f) = \emptyset$.

9. Since $|L'(a) \cup L'(f)| \geq 8$ and $L'(a) \subseteq L'(c)$ and since we may assume without loss of generality that $|L'(c)| = 6$, it follows that there are at least two colours in $L'(f)$ that are not in $L'(c)$. So first colour in order d, b, a, h, j, k. We can now give f a colour that is not in $L'(c)$ since there are at least two such colours. The remaining vertices can now be coloured in the order (5.21), as in step 6.

If R is as in Figure 5.8(i), then the remaining vertices can be coloured as above, with the exception that immediately after colouring g, we may assume without loss of generality that q can be given a colour that is not in $L'(r)$, since at this point each of q, r has at least 4, 3 usable colours in its list respectively. We can now colour in order p, s, r, and then continue as before. In every case the colouring can be completed, which is the required contradiction. □

Claim 5.4.21. *R does not contain the configuration in Figure* 5.7(e).

Proof. Suppose that R does contain the configuration in Figure 5.7(e). Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{a, d, l, o\}$, $|L'(z)| \geq 6$ if $z \in \{b, c, e, f, j, k, m, n\}$, $|L'(z)| \geq 7$ if $z \in \{g, i, p, q, r, s\}$, and $|L'(h)| \geq 5$.

Due to the similarities between Figure 5.7(e) and Figure 5.6(j), the remaining vertices can be coloured as in the proof of Claim 5.4.14 on page 123, with the exception that immediately after colouring k, we may assume without loss of generality that q can be given a colour that is not in $L'(r)$, since at this point each of q, r has at least 4, 3 usable colours in its list respectively. We can now colour in order i, p, s, r, and then continue as before. In every case the colouring can be completed, which is the required contradiction. □

Claim 5.4.22. *R does not contain the configuration in Figure* 5.7(f).

Proof. Suppose that R does contain the configuration in Figure 5.7(f). Then after applying a colouring of H^2 to G^2, each of the remaining vertices z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 2$ if $z \in \{a, d, l, o\}$, $|L'(z)| \geq 6$ if $z \in \{b, c, e, f, j, k, m, n\}$, $|L'(z)| \geq 7$ if $z \in \{g, i, p, q, r, s, t, u, v, w\}$, and $|L'(h)| \geq 5$.

Due to the similarities between Figure 5.7(e) and Figure 5.6(f), we can extend the argument used in Claim 5.4.21, which in turn is based on the proof of Claim 5.4.14.

1. The first five steps are as in the proof of Claim 5.4.21 with the exception that immediately after colouring g, we may assume without loss of generality that u can be given a colour that is not in $L'(v)$, since at this point each of u, v has at least 4, 3 usable colours in its list respectively. We can now colour in order t, w, v, and then continue as before. So we may assume that $|L'(c)| = 6$, $L'(a) \subseteq L'(c)$, $L'(d) \subseteq L'(c)$, $L'(h) \subseteq L'(c)$, $L'(f) = L'(c)$ and $L'(d) = L'(l) = L'(a)$.

2. Suppose that there is a colour $\alpha \in L'(f)$ that is not in at least one of $L'(u)$, $L'(v)$, $L'(w)$, $L'(t)$ so that we can give α to d (or f if $\alpha \notin L'(d)$). The remaining vertices can now be coloured in the order

$$d, a, l, o, f, h, n, j, m, k, q, i, p, s, r, g, b, c, e, (u, v, w, t) \tag{5.22}$$

where, as before, q can be given a colour that is not in $L'(r)$. Also, u, v, w, t are in any order with the exception that the last vertex must not have α in its list. So we may assume that $L'(f) \subseteq L'(z)$, where $z \in \{u, v, w, t\}$.

3. If possible, give a, e and l the same colour. Since $L'(a) \subseteq L'(f)$, it follows that the same colour can be given to v also. The remaining vertices can now be coloured in the order (5.22), where q can be given a colour that is not in $L'(r)$, and where t is coloured last. So we may assume that $L'(a) \cap L'(e) = \emptyset$ so that $|L'(a) \cup L'(e)| \geq 8$.

4. Since $|L'(c)| = 6$ and since $L'(a) \subseteq L'(c)$, it follows that there are at least two colours in $L'(e)$ that are not in $L'(c)$, and hence not in $L'(f)$. Since we may assume without loss of generality that $|L'(f)| = 6$ and $|L'(t)| = 7$, and since $L'(f) \subseteq L'(t)$, it follows that at least one of the colours in $L'(e)$ that is not in $L'(c)$ is not in $L'(t)$ also. After giving this colour to e the remaining vertices can now be coloured in the order (5.22), where q is given a colour that is not in $L'(r)$, and where t is coloured last.

In every case the colouring can be completed, which is the required contradiction. This completes the proof of Claim 5.4.22. $\square$

Appendix A

In this appendix we will prove that $\mathrm{ch}_{\mathrm{vef}}(K_4) = \chi_{\mathrm{vef}}(K_4) = 7$. We will make use of Lemma 5.4.1, which follows from the result of Ellingham and Goddyn [10] that a d-regular edge-d-colourable planar graph is edge-d-choosable, and it is also an easy exercise to prove directly.

Lemma 5.4.1. $\mathrm{ch}'(K_4) = \chi'(K_4) = 3$.

Lemma 5.4.2. $\mathrm{ch}_{\mathrm{vef}}(K_4) = \chi_{\mathrm{vef}}(K_4) = 7$.

Proof. Since there are fourteen elements to colour (four vertices, six edges and four faces) and since no more than two elements can have the same colour, it follows that $\mathrm{ch}_{\mathrm{vef}}(K_4) \geq \chi(K_4) \geq 7$. It remains to prove that $\mathrm{ch}_{\mathrm{vef}}(K_4) \leq 7$. Suppose that every element has a list of seven colours. First colour the four vertices and four faces, which is possible since at the time of its colouring each has at most six coloured neighbours. Now since each edge is incident with two vertices and two faces, it follows that each edge has at least 3 usable colours in it list. Since $\mathrm{ch}'(K_4) = 3$, by Lemma 5.4.1, it follows that the edges can be coloured. This completes the proof of Lemma 5.4.2. $\square$

Appendix B

In this appendix we will prove that $\mathrm{ch}_{\mathrm{vef}}(K_2 + \bar{K_3}) = \chi_{\mathrm{vef}}(K_2 + \bar{K_3}) = 7$, using a case by case argument.

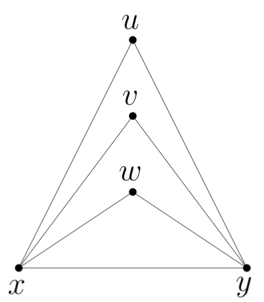

Figure 5.9

Lemma 5.4.3. $\mathrm{ch}_{\mathrm{vef}}(K_2 + \bar{K_3}) = \chi_{\mathrm{vef}}(K_2 + \bar{K_3}) = 7$.

Proof. Let $K_2 + \bar{K_3}$ be embedded as in Figure 5.9. (A similar argument works for the other embedding.) Let the vertices be labelled as in Figure 5.9 and let f, f', f'', f''' be the faces $xuyvx$, $xvywx$, $xwyx$, $xuyx$ respectively. Since $\chi_{\mathrm{vef}}(K_2+\bar{K_3}) \leq \mathrm{ch}_{\mathrm{vef}}(K_2 + \bar{K_3}) \leq 7$ by Theorem 3.1.1, it remains to prove that $\chi_{\mathrm{vef}}(K_2 + \bar{K_3}) \geq 7$.

Suppose that $\chi_{\mathrm{vef}}(K_2 + \bar{K_3}) \leq 6$ so that $K_2 + \bar{K_3}$ has a proper entire colouring from the colours $1, 2, \ldots, 6$. In this colouring we may assume that f''', x, y, xy, f'' are coloured 1, 2, 3, 4, 5 respectively. This implies that wx is coloured either

1, 3 or 6, and wy is coloured either 1, 2 or 6. By symmetry there are four cases to consider, each of which admits a contradiction.

Case 1: wx is coloured 1 and wy is coloured 2. Now f' is coloured either 4 or 6. Suppose that f' is coloured 4. Now both f and uy are coloured either 5 or 6, which implies that vy is coloured 1 and ux is coloured 3. This implies that both v and vx are coloured either 5 or 6, which is impossible since f is also coloured either 5 or 6 and is incident with both v and vx. So we may assume that f' is coloured 6. Now vx is coloured either 3 or 5. If vx is coloured 5, then vy is coloured 1. This means that both v and f must be coloured 4, which is impossible. So vx is coloured 3, which implies that both ux and uy are coloured either 5 or 6. This means that both u and f must be coloured 4, which is impossible.

Case 2: wx is coloured 1 and wy is coloured 6. This means that both w and f' must be coloured 4, which is impossible.

Case 3: wx is coloured 3 and wy is coloured 2. Now both ux and uy are coloured either 5 or 6. This means that both u and f must be coloured 4, which is impossible.

Case 4: wx is coloured 3 and wy is coloured 6. Now f' is coloured either 1 or 4. Suppose that f' is coloured 1. Now both ux and vx are coloured either 5 or 6, which implies that f is coloured 4. This implies that u is coloured either 5 or 6. So uy is coloured 2, which implies that vy is coloured 5. This means that both v and vx must be coloured 6, which is impossible. So we may assume that f' is coloured 4. Now both ux and f are coloured either 5 or 6, which implies that uy is coloured 2 and vx is coloured 1. This implies that vy is coloured 5. This means that both v and f must be coloured 6, which is impossible.

In every case we have obtained a contradiction, which proves Lemma 5.4.3. $\square$

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