TRANSFORMATION METHODS IN THE STUDY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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CONTENTS

ABSTRACT

Transformation methods are perhaps the most powerful analytic tool currently available in the study of nonlinear partial differential equations. Transformations may be classified into two categories: category I includes transformations of the dependent and independent variables of a given partial differential equation and category II additionally includes transformations of the derivatives of the dependent variables.

In part I of this thesis our principal attention is focused on transformations of the category I, namely point transformations. We mainly deal with groups of transformations. These groups enable us to derive similarity transformations which reduce the number of independent variables of a certain partial differential equation. Firstly, we introduce the concept of transformation groups and in the analysis which follows three methods for determining transformation groups are presented and consequently the corresponding similarity transformations are derived. We also present a direct method for determining similarity transformations. Finally, we classify all point transformations for a particular class of equations, namely the generalised Burgers equation.

Bäcklund transformations belong to category II and they are investigated in part II. The first chapter is an introduction to the theory of Bäcklund transformations. Here two different classes of Bäcklund transformations are defined and appropriate example are given. These two classes are considered in the proceeding analysis, where we search for Bäcklund transformations for specific classes of partial differential equations.

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To my family

CHAPTER ONE

INTRODUCTION AND GENERAL OUTLINE

The study of nonlinear partial differential equations has had a sporadic history up to the present time. Such equations arise in many branches of applied mathematics, for example, continuum mechanics and mathematical physics. In spite of the fact that physical phenomena are crying out for the solution of the underlying nonlinear model equations, this solution (general or particular) is difficult, if not impossible, to find. Few methods of solution have been devised, but they only provide particular solutions. Nevertheless, in the words of de Tocqueville, "God does not need general theories. He knows all the special cases!"

While there is no existing general theory for nonlinear partial differential equations, many special cases have yielded to appropriate changes of variable. In fact, transformations are perhaps the most powerful tool currently available in this area. In general these transformations may be classified into two categories: category I includes transformations of the dependent and independent variables, namely *point transformations* and category II additionally includes transformations of the derivatives of the dependent variables, namely *contact transformations*. Part I of this thesis deals with point transformations and mainly with *continuous groups of transformations*, which are also known as Lie groups after the name of a Norwegian mathematician, while in part II our principal attention is focused on *Bäcklund transformations* which arose as a generalisation of contact transformations.

Transformation group methods are powerful tools because they are not based on linear operators, superposition or any other aspects of linear solution techniques and therefore, these methods are applicable to *nonlinear* partial differential equations. These groups enable us to derive a type of transformations, namely similarity transformations, which have the property of reducing the number of independent variables of a system of partial differential equations. For example, consider the potential Burgers equation

$$u_t = u_{xx} + u_x^2, \tag{1.1}$$

where $u_x = \dfrac{\partial u}{\partial x}$, $u_t = \dfrac{\partial u}{\partial t}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, a notation which will be used throughout this thesis. A similarity transformation of the form

$$u=F(\eta); \quad \eta = \frac{x}{\sqrt{t}} \tag{1.2}$$

would transform equation (1.1) to an ordinary differential equation of the form

$$\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2} + \left(\frac{\mathrm{d}F}{\mathrm{d}\eta}\right)^2 + \tfrac{1}{2}\eta\frac{\mathrm{d}F}{\mathrm{d}\eta} = 0. \tag{1.3}$$

Transformation groups which have the property of mapping a solution into a solution, can also be employed to generate a new solution from a known one.

These transformation groups could be generalised by allowing the transformations to depend upon the derivatives of the dependent variable as well as the independent and dependent variables. The associated transformations are called *Lie-Bäcklund transformations.* The Lie Bäcklund transformation approach [3,4,73,94] is not covered in this thesis.

An example of a Bäcklund transformation is a pair of partial differential relations involving two dependent variables, two independent variables and their derivatives which together imply that each one of the dependent variables satisfies separately a partial differential equation. Thus, for example, the transformation

$$u_x = \Psi(x,y,u,u',u_x',u_y'), \quad u_y = \Phi(x,y,u,u',u_x',u_y') \tag{1.4}$$

would imply that $u(x,y)$ and $u'(x,y)$ satisfy partial differential equations of the operational form

$$P(u) = 0, \quad Q(u') = 0. \tag{1.5}$$

Liouville's equation, $u_{xy} = \mathrm{e}^u$, provides a simple example of an equation for which the general solution may be obtained quite easily by means of a Bäcklund transformation. This equation is related to the linear equation $u_{xy}' = 0$, by the Bäcklund transformation

$$u_x = u_x' - a\exp[\tfrac{1}{2}(u+u')], \tag{1.6a}$$

$$u_y = -u_y' - \frac{2}{a}\exp[-\tfrac{1}{2}(u-u')], \quad a = const. \tag{1.6b}$$

When the general solution of $u_{xy}' = 0$, namely $u' = A(x)+B(y)$, is inserted into equations (1.6), the solutions of the resulting first order system readily yield the general solution of the Liouville's equation,

$$u = \ln\left[\frac{2A'B'}{(A+B)^2}\right],$$

where A' and B' are the derived functions of A and B, respectively.

The transformation

$$u = F(u', u_x', u_y') \tag{1.7}$$

is a special Bäcklund transformation which relates equations of the form (1.5). Such a case is the Hopf-Cole transformation [29,45], $u = u_x'/u'$, which relates the Burgers equation, $u_y = u_{xx} + 2uu_x$ and the heat equation, $u_y' = u_{xx}'$. Needless to say, the Bäcklund transformations (1.4) and (1.7) may depend on higher derivatives, when equations (1.5) must be appropriate higher order equations.

Recently there has been considerable mathematical interest in applying a method which is known as *inverse scattering*. It is connected with the theory of solitons [33,34,37]. The inverse scattering method was originally introduced by Gardner, Greene, Kruskal and Miura [40]. In effect, this method reduces the solution of a non-linear partial differential equation to that of a linear integral equation, and the partial differential equation is usually then said to be completely integrable. Another method which appears in today's research in nonlinear partial differential equations is the *Painlevé analysis*. This method was developed by Weiss, Tabor and Carnevale [101]. Weiss wrote a number of papers on this method [102]. These two methods are beyond our scope and therefore we will not examine them in the subsequent analysis.

The concept of continuous transformation groups is presented in chapter two. This chapter contains the theoretical background needed for the subsequent chapters in part I, and forms a basis for these chapters. We define a finite one-parameter group of transformations and we show how the corresponding infinitesimal transformations are obtained. We introduce the concept of invariance of differential equations under groups of transformations which lead to similarity transformations and to the generation of new solutions from known ones. We define the extended group of transformations and we show how it can be obtained. Finally the definition of a strong and weak symmetry group is given.

Given a partial differential equation, how does one construct groups of transformations which leave this equation invariant? Chapter three introduces the first method for determining such groups. Firstly, we show how the similarity transformations are obtained and then our main objective is to discover how one-parameter finite groups may be found such that a particular equation is invariant under these groups.

The application of infinitesimal transformation groups to the solution of partial differential equations is more widespread than the application of finite transformation groups in today's research. The *classical method* for determining infinitesimal transformations for a given equation, which was first introduced by Lie [63], is discussed in chapter four. This method derives transformations from which we are able to derive similarity transformations and also generate new solutions from known ones. Here we show how the latter can be achieved, through appropriate examples. In the same chapter we refer to computer algebra systems which are available today. The computer algebra system *REDUCE* [42] has greatly facilitated all computations involved in this research.

Bluman and Cole [13] introduced a generalisation of Lie's classical method for determining infinitesimal transformations, which was named the *nonclassical* method. The transformation groups obtained by this method do not map a solution into a solution and therefore can only be employed to derive similarity transformations. A common characteristic of the methods stated so far, for determining similarity transformations for a given partial differential equation, is the use of transformation groups. Clarkson and Kruskal [26] proposed a direct method for determining similarity transformations which involves no group theoretical techniques. Chapter five contains a detailed discussion of the Bluman-Cole and Clarkson-Kruskal methods through appropriate examples.

In the final chapter of part I, chapter six, we classify all finite point transformations between generalised Burgers equations of the form

$$u_t + uu_x + a(x,t)u_{xx} = 0. \tag{1.8}$$

These transformations necessarily include all invariant infinitesimal transformations and in addition they include a reciprocal point transformation as well as transformations relating equations with different function $a(x,t)$ [52].

In part II, our discussion relates to a transformation that had its origin in some investigations by Bäcklund [5,6]. The importance of Bäcklund transformations and their generalisations is basically twofold. Thus, one the one hand, invariance under a Bäcklund transformation (auto-Bäcklund transformation) may be used to generate an infinite sequence of solutions of certain nonlinear partial differential equations by purely algebraic superposition principles. On the other hand, Bäcklund transformations may also be used to link certain nonlinear partial differential equations to canonical forms whose properties are well known. Both kinds are presented with detailed examples in chapter seven. In the same chapter we introduce the special Bäcklund transformation (1.7) and appropriate examples are also given.

It is quite common to search for Bäcklund transformations for a class of partial differential equations instead for a single equation. In chapters eight and nine, our objective is to derive Bäcklund transformations of the form (1.4) and (1.7), respectively, for a given class of nonlinear partial differential equations. In chapter eight the Bäcklund transformations of the form

$$z_x = \Psi(z, \bar{z}, z', \overline{z'}, z'_x, \overline{z'_x}, z'_y, \overline{z'_y}), \tag{1.9a}$$

$$z_y = \Phi(z, \bar{z}, z', \overline{z'}, z'_x, \overline{z'_x}, z'_y, \overline{z'_y}) \tag{1.9b}$$

are consider for equations of the form

$$\mathrm{i}z_y + z_{xx} + f(z, \bar{z}) = 0. \tag{1.10}$$

Then nonlinear forms of (1.10) that admit such transformations are completely classified [51].

In chapter nine we consider transformations of the form (1.7) that link equations of the form

$$u_{xy} = f(u, u_x). \tag{1.11}$$

We classify all cases, where at least one of the equations of the form (1.11) is nonlinear [92]. A second example is also presented, where the transformation also depends upon second derivatives [93].

PART I

POINT TRANSFORMATIONS

CHAPTER TWO

TRANSFORMATION GROUPS

1. Introduction

In the latter part of the 19th century, Sophus Lie [61,62] introduced and developed quite extensively the theory of continuous groups of transformations in connection with the study of differential equations. In the last few decades, there has been a revival of interest in applying the techniques of transformation groups to the theory of nonlinear differential equations [1,2,14,15,35,43,73,83,89]. Group methods are fundamental to the development of systematic procedures that lead to invariant transformations. These transformations may be utilised to generate new solutions from known ones. Group invariants are used in reduction of the original system. Exact solutions to these reduced systems may, on occasion, be derived.

In the present chapter a brief discussion of continuous transformation groups is presented. Detailed discussion of the transformation group theory may be found in references [21,28,36,78]. This chapter contains the theoretical background needed for the three subsequent chapters.

In the following analysis we shall generally limit ourselves to transformations of three independent variables in establishing the fundamental propositions. These propositions may also be extended to transformations of n variables.

Consider the system of equations

$$x'=P(x,y,z), \quad y'=Q(x,y,z), \quad z'=R(x,y,z), \tag{2.1}$$

where P, Q and R are independent functions of the independent variables x, y and z. Equations (2.1) represent a point transformation. This means a point in space with coordinates (x,y,z) is transformed to another point in space with coordinates (x',y',z'). We suppose here that the coordinate axes remain unchanged. If, now, (2.1) can be solved in the form

$$x=P'(x',y',z'), \quad y=Q'(x',y',z'), \quad z=R'(x',y',z') \tag{2.2}$$

a transformation is obtained which will carry the point (x',y',z') back to the original position (x,y,z). The transformation (2.2) is thus called the inverse transformation of (2.1). The successive application of the transformations (2.2) and (2.1) will give a transformation of the form

$$x'=x, \quad y'=y, \quad z'=z.$$

The last is called the identity transformation. This transformation leaves the position of the point (x,y,z) unchanged.

In the context of transformation groups we have to consider point transformations that depend on (at least) one arbitrary parameter. In chapter 6 we classify finite point transformations of the form (2.1) between generalised Burgers equations, while in chapters 3, 4 and 5 groups of transformations will be used.

2. Finite Groups of Transformations

Let $P(x,y,z,\lambda)$, $Q(x,y,z,\lambda)$ and $R(x,y,z,\lambda)$ be a set of functions continuous in the variables x, y, z and the parameter λ. We also assume the continuity of derivatives. Now consider the family of transformations

$$x'=P(x,y,z,\lambda), \quad y'=Q(x,y,z,\lambda), \quad z'=R(x,y,z,\lambda). \tag{2.3}$$

Let

$$x''=P(x',y',z',\lambda'), \quad y''=Q(x',y',z',\lambda'), \quad z''=R(x',y',z',\lambda')$$

be a second transformation of the family (2.3). Then the transformation which results from performing these two successively evidently has the form

$$x''=P[P(x,y,z,\lambda),\, Q(x,y,z,\lambda),\, R(x,y,z,\lambda),\, \lambda'], \tag{2.4a}$$

$$y''=Q[P(x,y,z,\lambda),\, Q(x,y,z,\lambda),\, R(x,y,z,\lambda),\, \lambda'], \tag{2.4b}$$

$$z''=R[P(x,y,z,\lambda),\, Q(x,y,z,\lambda),\, R(x,y,z,\lambda),\, \lambda']. \tag{2.4c}$$

The family of transformations of (2.3) are said to form a finite continuous group if the following conditions are satisfied:

(C1) Transformation (2.4) can be written in the form:

$$x''=P[x,y,z,\mu(\lambda,\lambda')], \quad y''=Q[x,y,z,\mu(\lambda,\lambda')], \quad z''=R[x,y,z,\mu(\lambda,\lambda')],$$

where μ is a parameter depending only on λ and λ'. Expressed in words, this condition evidently is that the result of performing successively any two transformations of the family (2.3) upon the points in space must be equivalent to the result of performing a third transformation of that family upon those points.

(C2) There exists a certain value of λ (λ_0 say) such that the transformation (2.3) contains the identical transformation. That is, for $\lambda = \lambda_0$

$$x'=x, \quad y'=y, \quad z'=z.$$

(C3) For every transformation with the parameter λ, of the family (2.3), there exists a transformation with the parameter μ of the same family such that the latter transformation is the inverse of the former, μ being a function of λ only.

(C4) If T_λ, $T_{\lambda'}$ and $T_{\lambda''}$ represent the transformations from the point (x,y,z) to (x',y',z'), from the point (x',y',z') to (x'',y'',z'') and from the point (x'',y'',z'') to (x''',y''',z''') respectively, using (2.3), then

$$(T_\lambda T_{\lambda'})T_{\lambda''} = T_\lambda(T_{\lambda'} T_{\lambda''}).$$

This establishes the associative property of the transformation group.

Since the family of transformations (2.3) contains one parameter, we call it, under the above conditions, a group of one parameter.

3. Infinitesimal Groups of Transformations

We now proceed to introduce the concept of an infinitesimal transformation. Since P, Q and R are continuous functions the transformation (2.3) can be written as

$$x'=P(x,y,z,\lambda_0+\varepsilon), \quad y'=Q(x,y,z,\lambda_0+\varepsilon), \quad z'=R(x,y,z,\lambda_0+\varepsilon), \tag{2.5}$$

where λ_0 is the value of the parameter for which (2.3) gives the identical transformation, so that

$$x'=P(x,y,z,\lambda_0)=x, \quad y'=Q(x,y,z,\lambda_0)=y, \quad z'=R(x,y,z,\lambda_0)=z \tag{2.6}$$

and ε is an infinitesimal quantity which changes x, y and z by an infinitesimal amount.

Expanding in Taylor series equation (2.5) becomes

$$x'=P(x,y,z,\lambda_0)+\varepsilon\left(\frac{\partial P}{\partial \lambda}\right)_{\lambda_0}+\frac{\varepsilon^2}{2!}\left(\frac{\partial^2 P}{\partial \lambda^2}\right)_{\lambda_0}+\ldots$$

$$y'=Q(x,y,z,\lambda_0)+\varepsilon\left(\frac{\partial Q}{\partial \lambda}\right)_{\lambda_0}+\frac{\varepsilon^2}{2!}\left(\frac{\partial^2 Q}{\partial \lambda^2}\right)_{\lambda_0}+\ldots$$

$$z'=R(x,y,z,\lambda_0)+\varepsilon\left(\frac{\partial R}{\partial \lambda}\right)_{\lambda_0}+\frac{\varepsilon^2}{2!}\left(\frac{\partial^2 R}{\partial \lambda^2}\right)_{\lambda_0}+\ldots$$

Since ε is an infinitesimal quantity the above equations become

$$x'=x+\varepsilon X(x,y,z)+o(\varepsilon^2), \qquad (2.7a)$$

$$y'=y+\varepsilon Y(x,y,z)+o(\varepsilon^2), \qquad (2.7b)$$

$$z'=z+\varepsilon Z(x,y,z)+o(\varepsilon^2). \qquad (2.7c)$$

where

$$X=\left(\frac{\partial P}{\partial \lambda}\right)_{\lambda_0}, \quad Y=\left(\frac{\partial Q}{\partial \lambda}\right)_{\lambda_0}, \quad Z=\left(\frac{\partial R}{\partial \lambda}\right)_{\lambda_0}$$

and the relation for the identical transformation, equations (2.6), has been used. Equations (2.7) represent an infinitesimal group of transformations, where X, Y, Z are called the infinitesimals of the transformation.

4. Relation Between Finite and Infinitesimal Groups of Transformations

It can be shown [4,15] that every finite transformation group of one parameter contains only one infinitesimal transformation. That is, given a finite transformation we can generate an infinitesimal transformation. Also every infinitesimal transformation belongs to a finite one.

It is clear that a practical method for obtaining the infinitesimal transformation of a given finite transformation, firstly, is to assign to the parameter, in the equations of the finite transformation, a value differing only by an infinitesimal quantity from the value which gives the identical transformation. Then using Taylor's theorem the desired infinitesimal transformation is obtained.

We now show how one can obtain a finite transformation from a given infinitesimal transformation. We regard the parameter λ as the time that a point

(x,y,z) takes to arrive at a new point (x',y',z'), where x', y', z' are functions of x, y, z and λ. If λ increases by $d\lambda$, then x', y' and z' will, by (2.7), receive the increments

$$dx'=X(x',y',z')d\lambda, \quad dy'=Y(x',y',z')d\lambda, \quad dz'=Z(x',y',z')d\lambda.$$

The finite transformations is then found by solving the simultaneous system of equations

$$\frac{dx'}{d\lambda} = X(x',y',z'), \quad \frac{dy'}{d\lambda} = Y(x',y',z'), \quad \frac{dz'}{d\lambda} = Z(x',y',z'),$$

under the initial conditions, $x'=x$, $y'=y$ and $z'=z$ when $\lambda=0$.

5. The Concept of Invariance

A function $f(x,y,z)$ is said to be conformal invariant under the infinitesimal transformation (2.7) [or the finite transformation (2.3)] if

$$f(x,y,z)=g(x,y,x,\lambda)f(x',y',z')$$

for some function g of the x, y, z and λ. If g is a function of λ only, then f is called constant conformal invariant and if g is identically equal to one, that is $f(x,y,z)=f(x',y',z')$, then f is said to be an absolute invariant of the transformation group.

A given function $f(x,y,z)$ is changed to $f(x',y',z')$ if it is subjected to the infinitesimal transformation (2.7). Expanding in Taylor's series, $f(x',y',z')$ becomes

$$f(x',y',z')=f(x+\varepsilon X,y+\varepsilon Y,z+\varepsilon Z)=f(x,y,z)+\varepsilon\Gamma f+\frac{\varepsilon^2}{2!}\Gamma^2 f+\ldots \tag{2.8}$$

where the operator Γ is defined by

$$\Gamma f = X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}$$

and $\Gamma^n f$ represents repeating the operator Γ n times. Equation (2.8) shows that f is invariant if $\Gamma f = \Gamma^2 f = \Gamma^3 f=\ldots=0$. However, since $\Gamma^2 f = \Gamma(\Gamma f)$, $\Gamma^3 f = \Gamma^2(\Gamma f)$,..., it follows that the condition

$$\Gamma f = X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z} = 0 \tag{2.9}$$

is both necessary and sufficient requirement for invariance of $f(x,y,z)$.

Equation (2.9) is a first order linear partial differential equation which can be solved by the method of characteristics [90]. That is,

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}. \tag{2.10}$$

Equation (2.10) has two independent solutions. These two independent solutions form the required invariant functions corresponding to the infinitesimal transformation (2.7). Therefore a one parameter group of transformations in three variables has only two independent absolute invariants. It can be shown that a one parameter group in n variables has (n-1) independent invariants [36].

Hence, for a given group of infinitesimal transformations of the form (2.7), the group invariants are found by solving (2.10). For one parameter finite transformations of the form (2.3) the following two methods for finding the group invariants are suggested. Without loss of generality let the identical transformation given when $\lambda = 0$. If one of the three equations, $P(x,y,z,\lambda) =$ const., $Q(x,y,z,\lambda)=$const., $R(x,y,z,\lambda)=$const., has a unique solution for λ, then assuming that $P=$const. is the equation which gives this unique solution for λ, the functions $Q(x,y,z,F(x,y,z))$ and $R(x,y,z,F(x,y,z))$ form the two independent absolute invariants of (2.3), where $\lambda = F$ is the unique solution of $P=$const. If a unique solution does not exist for any of these three equations, then elimination of the parameter λ from (2.3) will still give the required invariants. For proofs of these results see, for example, reference [91].

6. Invariance of Differential Equations under Groups of Transformations I: Similarity Transformations

Consider the following one parameter group of transformations

$$x_i' = P_i(x_1,...,x_m,\lambda), \quad i=1,...,m, \tag{2.11a}$$

$$y_j' = Q_j(y_1,...,y_n,\lambda), \quad j=1,...,n. \tag{2.11b}$$

We now give the following theorem due to Morgan [69].

Theorem 2.1: Consider the system of partial differential equations of order k

$$\phi_j\left(x_1,\ldots,x_m;y_1,\ldots,y_n;\frac{\partial^k y_1}{\partial x_1^k},\ldots,\frac{\partial^k y_n}{\partial x_m^k}\right) = 0, \quad j=1,\ldots,n, \tag{2.12}$$

where $y_j(j=1,\ldots,n)$ are regarded as the dependent variables and $x_i(i=1,\ldots,m)$ as independent variables. If each of the forms ϕ_j is conformally invariant under the group (2.11), then the invariant solutions of (2.12) can be expressed in terms of the new system of partial differential equations

$$\psi_j\left(\eta_1,\ldots,\eta_{m-1};F_1,\ldots,F_n;\frac{\partial^k F_1}{\partial \eta_1^k},\ldots,\frac{\partial^k F_n}{\partial \eta_{m-1}^k}\right) = 0, \tag{2.13}$$

where $\eta_i(i=1,\ldots,m-1)$ are the independent variables and $F_j(j=1,\ldots,n)$ are the dependent variables. The η_i are the $m-1$ independent absolute invariants of the subgroup (2.11a) and F_j are given by

$$F_j(\eta_1,\ldots,\eta_{m-1}) = \xi_j(x_1,\ldots,x_m;y_1,\ldots,y_n), \tag{2.14}$$

where ξ_j are the remaining n independent absolute invariants of the group (2.11).

We note that the new system (2.13) has only $(m-1)$ independent variables. Hence, the number of independent variables has been reduced by one. These transformations, equation (2.14) which reduce the number of independent variables are called similarity transformations.

7. Invariance of Differential Equations under Groups of Transformations II: Generating Solutions by Finite Transformations

By definition, an invariant transformation maps a solution into a solution. So if we already know a (particular) solution of a partial differential equation, we can apply a finite transformation to obtain a (possibly) new solution. To carry out this idea, consider the one-parameter finite transformation

$$x_i' = P_i(x_1,\ldots,x_m,y_1,\ldots,y_n,\lambda), \quad i=1,\ldots,m, \tag{2.15a}$$

$$y_j' = Q_j(x_1,\ldots,x_m,y_1,\ldots,y_n,\lambda), \quad j=1,\ldots,n, \tag{2.15b}$$

where its inverse transformation is given by

$$x_i = P_i'(x_1',...,x_m',y_1',...,y_n',\lambda), \tag{2.16a}$$

$$y_j = Q_j'(x_1',...,x_m',y_1',...,y_n',\lambda), \tag{2.16b}$$

where y_j are regarded as dependent variables and x_i as independent variables of some system of partial differential equations.

Let $y_j = f_j(x_1,...,x_m)$ be a solution of this system, then

$$y_j = f_j(x_1,...,x_m) \Leftrightarrow F_j(x_1,...,x_m,y_1,...,y_n) = 0. \tag{2.17}$$

Using (2.16), equation (2.17) becomes

$$F_j(P_1'(x_1',...,y_n',\lambda),...,Q_n'(x_1',...,y_n',\lambda))=0. \tag{2.18}$$

Equation (2.18) can be solved for y_j' to give

$$y_j' = f_j'(x_1',...,x_m',\lambda),$$

which is hopefully a new solution for the given system of partial differential equations.

8. Extended Group of Transformations

Since we examine transformations of differential equations, we need to know how derivatives are transformed under a given group of transformations. It can be shown that the transformed derivatives also form a one-parameter group of transformations [46,78], which are called extended group of transformations.

In what follows in this section and in the next four chapters we consider u to be the dependent variable and x and t the independent variables for some partial differential equation and similarly for u', x' and t'. Therefore the transformations (2.1), (2.3) and (2.7) become

$$x'=P(x,t,u), \quad t'=Q(x,t,u), \quad u'=R(x,t,u). \tag{2.19}$$

$$x'=P(x,t,u,\lambda), \quad t'=Q(x,t,u,\lambda), \quad u'=R(x,t,u,\lambda). \tag{2.20}$$

$$x'=x+\varepsilon X(x,t,u)+o(\varepsilon^2), \quad t'=t+\varepsilon T(x,t,u)+o(\varepsilon^2), \quad u'=u+\varepsilon U(x,t,u)+o(\varepsilon^2). \tag{2.21}$$

Before we proceed it is pointed out that we assume that the transformations are non-degenerate. This means that the Jacobian

$$J = \frac{\partial(P,Q,R)}{\partial(x,t,u)} \neq 0 \tag{2.22}$$

and also that

$$\delta = \frac{\partial(P(x,t,u(x,t)),Q(x,t,u(x,t)))}{\partial(x,t)} \neq 0, \tag{2.23}$$

where in the case of infinitesimal transformations $P \equiv x+\varepsilon X$, $Q \equiv t+\varepsilon T$ and $R \equiv u+\varepsilon U$. In (2.23) P and Q are regarded as functions of x and t, using the fact that $u=u(x,t)$, whereas in (2.22) P, Q and R are regarded as functions of three independent variables x,t and u.

For a function $\Psi(x,t,u,u_x,u_t)$,

$$\mathrm{d}\Psi = \begin{pmatrix} \Psi_\chi & \Psi_\tau \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}t \end{pmatrix} \tag{2.24}$$

where

$$\Psi_\chi = \Psi_x + \Psi_u u_x + \Psi_{u_x} u_{xx} + \Psi_{u_t} u_{xt}, \tag{2.25}$$

$$\Psi_\tau = \Psi_t + \Psi_u u_t + \Psi_{u_x} u_{xt} + \Psi_{u_t} u_{tt}, \tag{2.26}$$

are the total derivatives of Ψ with respect to x and t respectively. Note that the function Ψ might depend on derivatives to any required order (see [73]).

In particular, using $\Psi = P(x,t,u)$ and then $\Psi = Q(x,t,u)$, it follows that

$$\begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}t' \end{pmatrix} = \begin{pmatrix} P_\chi & P_\tau \\ Q_\chi & Q_\tau \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}t \end{pmatrix}$$

so that

$$\mathrm{d}\Psi = \frac{1}{\delta} \begin{pmatrix} \Psi_\chi & \Psi_\tau \end{pmatrix} \begin{pmatrix} Q_\tau & -P_\tau \\ -Q_\chi & P_\chi \end{pmatrix} \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}t' \end{pmatrix} \tag{2.27}$$

giving the partial derivatives $\dfrac{\partial\Psi(x',t')}{\partial x'}$ and $\dfrac{\partial\Psi(x',t')}{\partial t'}$. Setting $\Psi = u'$, the partial derivatives $u'_{x'}$ and $u'_{t'}$ may be determined in terms of x, t, u, u_x and u_t from the relation

$$\mathrm{d}u' = \frac{1}{\delta} \begin{pmatrix} R_\chi & R_\tau \end{pmatrix} \begin{pmatrix} Q_\tau & -P_\tau \\ -Q_\chi & P_\chi \end{pmatrix} \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}t' \end{pmatrix}. \tag{2.28}$$

If $\Psi = u'_{x'}$ in (2.27) then $u'_{x'x'}$ and $u'_{x't'}$ can be calculated in terms of $x,t,u,u_x,u_t,u_{xx},u_{xt}$ and u_{tt}. Again for the infinitesimal transformations one needs to set $P = x+\varepsilon X$, $Q=t+\varepsilon T$ and $R=u+\varepsilon U$ in the above identities.

To complete this introductory chapter we give the following definition:

Definition: Let Δ be a system of partial differential equations. A *strong symmetry group* of Δ is a group of transformations G of independent and dependent variables which has the following properties:

(a) The elements of G transform solutions of the system to other solutions of the system.

(b) The G-invariant solutions of the system are from a reduced system of differential equations involving a fewer number of independent variables than the original system Δ.

A *weak symmetry group* of the system Δ is a group of transformations which satisfies the reduction property (b), but no longer transforms solutions to solutions.

Clearly, for a strong symmetry group both sections 6 and 7 are applicable, but for a weak symmetry group only section 6 can be used.

Once we have established the concept of transformation groups, we turn our attention to the applications of these transformations to the study of nonlinear partial differential equations. The following two chapters deal with strong symmetry groups. In chapter 3 it is shown how Morgan's result is applied and also how to determine finite groups of transformations of the form (2.20) which leave a given equation invariant. In chapter 4 we demonstrate how to determine infinitesimal transformations of the form (2.21) for a given partial differential equation. This method, namely *classical method*, is more favourable and practical than the method introduced in the previous chapter. This method was originally introduced by Lie [63].

In [13], Bluman and Cole proposed a generalisation of Lie's method for finding infinitesimal transformations, which they named *nonclassical method.* This method and the methods in chapters 3 and 4 also appear in [2]. The nonclassical method is presented in chapter 5. The transformations obtained by these methods are weak symmetry groups. In this chapter we also present a direct method introduced recently by Clarkson and Kruskal [26], which involves no group theoretical techniques, for determining similarity transformations.

In the final chapter of Part I we classify all finite point transformations of the form (2.19) between given generalised equations. These transformations include all invariant transformations and in addition they include a reciprocal transformation which can not be obtained by the group methods.

CHAPTER THREE

APPLICATIONS OF FINITE GROUPS OF TRANSFORMATIONS

1. Introduction

In this chapter, section 4, we present the first method for determining strong symmetry groups for partial differential equations. The goal of this method is to obtain one-parameter finite groups of the form (2.20) that leave the equation under consideration invariant. The next step is to derive the corresponding similarity transformations and generate new special solutions starting from a trivial one using the finite transformations, if it is possible.

A similarity transformation of partial differential equations reduces the number of independent variables in the partial differential equations. In general, when reducing the number of independent variables using group of transformations, the invariants of the group become the new variables. For a partial differential equation with independent variables x and t and dependent variable u typically one of the invariant will be of the form $\eta(x,t)$ and the other can be expressed as an arbitrary function of η, $F(\eta)$ (see chapter 2, section 6). The functional form of the similarity solution will be

$$u = W(x, t, \eta, F(\eta(x, t))), \tag{3.1}$$

η is called the similarity variable and $F(\eta)$ becomes the new dependent variable. The function W is known explicitly and by substituting (3.1) into the given partial differential equation we obtain an ordinary differential equation for $F(\eta)$.

The general theory of Morgan [69], Michal [65] and Birkhoff [11] for developing similarity solutions of partial differential equations is also discussed in detail in [1]. In the next section we discuss the application of Morgan's theorem, introduced earlier. This application will form the foundation for the subsequent methods for determining symmetry groups (strong or weak) for a given partial differential equation. In section 3 we present two theorems due to Kingston [53] which will be very beneficial throughout part I of this thesis. In the last and main section, our principal attention will be focused on determining one-parameter continuous finite groups of transformations for the Hopf-equation $u_t + uu_x = [e^u u_x]_x$.

2. Morgan's Method

Consider the one-parameter group

$$x' = e^{\lambda} x, \tag{3.2a}$$

$$t' = e^{2\lambda} t, \tag{3.2b}$$

$$u' = u \tag{3.2c}$$

and the nonlinear diffusion equation

$$u_t = [f(u)u_x]_x, \tag{3.3}$$

which first was examined by Ovjannikov [76] and later by Bluman [12]. We proceed by showing that the equation (3.3) is invariant under the transformations (3.2). Firstly, one needs to calculate the extended groups up to the second order. Let $\Psi = u'$, $P = e^{\lambda}x$ and $Q = e^{2\lambda}t$ in (2.27) to give

$$\begin{pmatrix} u'_{x'} & u'_{t'} \end{pmatrix} = \frac{1}{\delta}\begin{pmatrix} u_x & u_t \end{pmatrix}\begin{pmatrix} e^{2\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix},$$

where $\delta = e^{3\lambda}$, from (2.23). It follows that

$$u'_{x'} = e^{-\lambda} u_x, \tag{3.4a}$$

$$u'_{t'} = e^{-2\lambda} u_t. \tag{3.4b}$$

Similarly, setting $\Psi = u'_{x'}$ and then $\Psi = u'_{t'}$ in (2.27) we find that

$$u'_{x'x'} = e^{-2\lambda} u_{xx}, \tag{3.5a}$$

$$u'_{x't'} = e^{-3\lambda} u_{xt}, \tag{3.5b}$$

$$u'_{t't'} = e^{-4\lambda} u_{tt}. \tag{3.5c}$$

Using (3.4a), (3.4b) and (3.5a), it is straightforward to show that equation (3.3) is invariant under the one-parameter group (3.2). By Morgan's theorem (chapter 2, section 6) the partial differential equation (3.3) can be reduced to ordinary differential equation, with the invariant η of the subgroup (3.2a,b) being the independent variable and $F(\eta) = \xi$ the dependent variable, where ξ is the second independent absolute invariant of the group (3.2).

Clearly, $\xi = u$ and eliminating the parameter λ from (3.2a) and (3.2b) we obtain $x' t'^{-\frac{1}{2}} = x t^{-\frac{1}{2}}$. Hence, $\eta = x t^{-\frac{1}{2}}$. Therefore using the similarity transformation

$$u = F(\eta); \quad \eta = \frac{x}{t^{\frac{1}{2}}}, \tag{3.6}$$

the nonlinear diffusion equation (3.3) becomes

$$\eta \frac{\mathrm{d}F}{\mathrm{d}\eta} + 2\frac{\mathrm{d}}{\mathrm{d}\eta}\left[f(F)\frac{\mathrm{d}F}{\mathrm{d}\eta}\right] = 0.$$

An advantage of the finite group of transformations is that it is very easy to find the group invariants of the extended groups up to any required order, using either of the methods described in chapter 2, section 5. Then any function of these invariants forms a partial differential equation which is invariant under the given transformation group of one-parameter.

For the extended group [(3.2), (3.4), (3.5)] a set of independent absolute invariants is: $\eta = \frac{x}{t^{\frac{1}{2}}}$, $\xi = u$, $\xi_1 = \frac{u_x}{u_t^{\frac{1}{2}}}$, $\xi_2 = \frac{u_{xx}}{u_t}$, $\xi_3 = \frac{u_{xx}^2}{u_{tt}}$, $\xi_4 = \frac{u_{xt}}{u_x^3}$ and $\xi_5 = \frac{u_{tt}}{u_t^2}$, where these invariants are found by eliminating the parameter λ from (3.2), (3.4) and (3.5). For example, from (3.4a) and (3.4b) $\frac{u'_{x'}}{u'^{\frac{1}{2}}_{t'}} = \frac{u_x}{u_t^{\frac{1}{2}}}$, hence, $\xi_1 = \frac{u_x}{u_t^{\frac{1}{2}}}$. Having calculated the group invariants, any second order partial differential equation of the form

$$\Phi_2(\eta, \xi, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \text{constant}$$

is invariant under the one-parameter group (3.2). For example, letting $\Phi_2 = \xi_1^2 \xi_2 = 1$ gives the potential Burgers equation

$$u_t = u_{xx} + u_x^2. \tag{3.7}$$

Note that if we differentiate equation (3.7) with respect to x and substitute $v = u_x$, we derive the more usual form

$$v_t = v_{xx} + 2v v_x$$

of Burgers equation which represent the simplest wave equation combining both dissipative and nonlinear effects, and therefore appears in a wide variety of physical applications [19]. Using the similarity transformation (3.6), equation (3.7) becomes

$$\frac{d^2F}{d\eta^2} + \left(\frac{dF}{d\eta}\right)^2 + \tfrac{1}{2}\eta\frac{dF}{d\eta} = 0.$$

For more similarity solutions of (3.7) one can refer to [73] and [83].

Another equation which is invariant under the group (3.2) is the Boussinesq-type equation [25]

$$u_{tt} = 3u_t u_{xx} + \tfrac{9}{2}u_x^2 u_{xx} - 3u_{xxxx}.$$

This equation is obtained by choosing

$$\Phi_4 = 3\frac{\xi_2}{\xi_5} + \frac{9}{2}\frac{\xi_1^2\xi_2}{\xi_5} - 3\xi_{10} = 1,$$

where Φ_4 is any function of the group invariants of the fourth order extended groups, and $\xi_{10} = \dfrac{u_{xxxx}}{u_{tt}}$.

Therefore once the group of transformations are obtained using any of the methods described here, we firstly find the group invariants and then apply the Morgan's theorem to obtain the desired similarity transformations which reduce the number of independent variables of the equation under consideration. Hopefully, for the strong symmetry groups, the finite group of transformations will enable us to construct a sequence of particular solutions starting from a trivial one. Clearly, this is not the case for the one-parameter group (3.2).

3. On Point Transformations of Evolution Equations

In this section, we state two theorems due to Kingston [53] which will be very helpful in the next section and chapters 4, 5 and 6.

Theorem 3.1: Consider the two evolution equations

$$u_t = H_i(x, t, u, u_x, u_{xx}, \ldots) = 0, \quad i = 1,2, \tag{3.8}$$

where H_i depends on the independent variables x and t, the dependent variable $u(x,t)$ and its derivatives with respect to x up to order n($\geqslant$ 2). Suppose that the equations (3.8), with one of them ($i=1$) expressed in terms of x', t', u' instead of x, t, u, are related by the non-degenerate point transformation (2.19). Then for the point transformation (2.19) relating the two evolution equations (3.8),

$$t'=Q(t). \tag{3.9a}$$

This theorem is a generalisation of Tu result [96] who proved it for infinitesimal transformations.

Theorem 3.2: For the point transformation (2.19) relating the two equations (3.8) in which H_i are polynomials in the derivatives of u,

$$x'=P(x,t). \tag{3.9b}$$

4. Determination of Groups of One-Parameter by Finite Transformations

The goal of this method is to discover how one-parameter groups of the form (2.20) may be found such that a particular partial differential equation is invariant under these groups. This method seems to be "out of date" in today's research. Nevertheless it is a method for obtaining the proper groups for a system of partial differential equations, which was first introduced by v. Krzywoblocki and Roth [97,98,99]. The same results can also be obtained by using the classical method for determining infinitesimal transformations, described in the next chapter.

Consider the Hopf equation

$$u_t+uu_x = [f(u)u_x]_x, \tag{3.10}$$

which appears in the study of hydrodynamics of perfect fluid [47]. We shall sketch this method using the above equation, and in particular when $f(u)=e^u$. If $f(u)=1$, then equation (3.10) becomes the well-known Burgers equation. Ames [2], uses Burgers equation as an example to present this method, which also appears in detail in [104].

Now, setting $f(u) = e^u$, in equation (3.10) it results to

$$u_t+uu_x = e^u u_x^2+e^u u_{xx}. \tag{3.11}$$

If we require that (3.11) is invariant under the finite group of transformations

$$x'=P(x,t,\lambda), \quad t'=Q(t,\lambda), \quad u'=R(x,t,u,\lambda), \tag{3.12}$$

where Theorems 3.1 and 3.2 have been used, then

$$u'_{t'}+u'u'_{x'}-e^{u'}u'^2_{x'}-e^{u'}u'_{x'x'} = 0. \tag{3.13}$$

For non-degenerate transformations, since $\delta = P_x Q_t$ and $J = P_x Q_t R_u$ from (2.23) and (2.22) respectively, we must take $Q_t \neq 0, P_x \neq 0$ and $R_u \neq 0$. If we let $\Psi = u'$ in (2.27) then

$$\begin{pmatrix} u'_{x'} & u'_{t'} \end{pmatrix} = \frac{1}{\delta}\begin{pmatrix} R_x + R_u u_x & R_t + R_u u_t \end{pmatrix}\begin{pmatrix} Q_t & -P_t \\ 0 & P_x \end{pmatrix}.$$

Hence,

$$u'_{x'} = \frac{R_x + R_u u_x}{P_x}, \tag{3.14}$$

$$u'_{t'} = \frac{P_x R_t + P_x R_u u_t - P_t R_x - P_t R_u u_x}{P_x Q_t}. \tag{3.15}$$

Finally, setting $\Psi = u'_{x'}$ in (2.27) we find that

$$u'_{x'x'} = \frac{P_x R_{xx} + 2P_x R_{xu} u_x + P_x R_{uu} u_x^2 + P_x R_u u_{xx} - P_{xx} R_x - P_{xx} R_u u_x}{P_x^3}. \tag{3.16}$$

Substituting (3.14) - (3.16) into equation (3.13) and eliminating u_t, using equation (3.11), the resulting equation will depend only on the variables x, t, u, u_x and u_{xx}, which we treat as independent. The coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives of u in equation (3.13) give the following identities:

$$\mathrm{e}^u P_x^3 R_u - \mathrm{e}^R P_x Q_t R_u = 0, \tag{3.17}$$

$$\mathrm{e}^u P_x^3 R_u - \mathrm{e}^R P_x Q_t R_u^2 - \mathrm{e}^R P_x Q_t R_{uu} = 0, \tag{3.18}$$

$$-uP_x^3 R_u - P_x^2 P_t R_u + P_x^2 Q_t R R_u -$$

$$2\mathrm{e}^R P_x Q_t R_x R_u - 2\mathrm{e}^R P_x Q_t R_{xu} + \mathrm{e}^R P_{xx} Q_t R_u = 0, \tag{3.19}$$

$$P_x^3 R_t - P_x^2 P_t R_x + P_x^2 Q_t R R_x - \mathrm{e}^R P_x Q_t R_x^2 -$$

$$\mathrm{e}^R P_x Q_t R_{xx} + \mathrm{e}^R P_{xx} Q_t R_x = 0. \tag{3.20}$$

These four identities will enable us to find the functional forms of P, Q and R. From (3.17), since $R_u \neq 0$ and $P_x \neq 0$, we have

$$R = u + \ln\left(\frac{P_x^2}{Q_t}\right). \tag{3.21}$$

Identity (3.18) is also satisfied by (3.21). Upon substitution of (3.21) into (3.19) and equating coefficients of powers of u, we obtain

$$P_x = Q_t,$$

$$P_t = Q_t \ln Q_t.$$

From these last two equations, it follows that

$$P = x' = h(\lambda)x + h(\lambda)\ln[h(\lambda)]t + k(\lambda), \tag{3.22a}$$

$$Q = t' = h(\lambda)t + l(\lambda), \tag{3.22b}$$

and from (3.21)

$$R = u' = u + \ln[h(\lambda)], \tag{3.22c}$$

where h, k and l are arbitrary functions of the parameter λ. Identity (3.20) is also satisfied by equations (3.22).

Next we require that the set of transformations given by equations (3.22) form a one-parameter group. That is, the conditions C1-C4 (chapter 2, section 2) must all be satisfied. These requirements place restrictions on the parameter functions $h(\lambda)$, $k(\lambda)$ and $l(\lambda)$. Let the point (x_1, t_1, u_1) be transformed to a point (x_2, t_2, u_2), using (3.22), through a parameter λ_1. Then, construct a second transformation from (x_2, t_2, u_2) to a new point (x_3, t_3, u_3) through a parameter λ_2, using the same transformation (3.22). Hence,

$$x_3 = h(\lambda_2)x_2 + h(\lambda_2)\ln[h(\lambda_2)]t_2 + k(\lambda_2),$$

$$t_3 = h(\lambda_2)t_2 + l(\lambda_2),$$

$$u_3 = u_2 + \ln[h(\lambda_2)].$$

Using the transformation equations of x_2, t_2 and u_2 in terms of u_1, t_1 and u_1, the above equations become

$$x_3 = h(\lambda_1)h(\lambda_2)x_1 + h(\lambda_1)h(\lambda_2)\ln[h(\lambda_1)h(\lambda_2)]t_1 + h(\lambda_2)k(\lambda_1) +$$
$$l(\lambda_1)h(\lambda_2)\ln[h(\lambda_2)] + k(\lambda_2), \tag{3.23a}$$

$$t_3 = h(\lambda_1)h(\lambda_2)t_1 + l(\lambda_1)h(\lambda_2) + l(\lambda_2), \tag{3.23b}$$

$$u_3 = u_1 + \ln[h(\lambda_1)h(\lambda_2)]. \tag{3.23c}$$

Let us now perform a third transformation from the point (x_1, t_1, u_1) to the point (x_3, t_3, u_3) through a parameter λ_3 to obtain

$$x_3 = h(\lambda_3)x_1 + h(\lambda_3)\ln[h(\lambda_3)]t_1 + k(\lambda_3), \tag{3.24a}$$

$$t_3 = h(\lambda_3)t_1 + l(\lambda_3), \tag{3.24b}$$

$$u_3 = u_1 + \ln[h(\lambda_3)]. \tag{3.24c}$$

In order to satisfy condition C1 we require the system of equations (3.23) to be identical to (3.24). It follows that

$$h(\lambda_3) = h(\lambda_1)h(\lambda_2), \tag{3.25}$$

$$k(\lambda_3) = h(\lambda_2)k(\lambda_1) + l(\lambda_1)h(\lambda_2)\ln[h(\lambda_2)] + k(\lambda_2), \tag{3.26}$$

$$l(\lambda_3) = h(\lambda_2)l(\lambda_1) + l(\lambda_2). \tag{3.27}$$

From equation (3.25) we observe that interchange of λ_1 and λ_2 leaves equation (3.25) unaltered. Consequently, in order to satisfy condition C1, equations (3.26) and (3.27) must also be unaltered under the same interchange. Thus

$$h(\lambda_2)k(\lambda_1) + l(\lambda_1)h(\lambda_2)\ln[h(\lambda_2)] + k(\lambda_2) =$$

$$h(\lambda_1)k(\lambda_2) + l(\lambda_2)h(\lambda_1)\ln[h(\lambda_1)] + k(\lambda_1), \tag{3.28}$$

$$h(\lambda_2)l(\lambda_1) + l(\lambda_2) = h(\lambda_1)l(\lambda_2) + l(\lambda_1). \tag{3.29}$$

Equation (3.29) upon rearrangement become

$$l(\lambda_1)[h(\lambda_2)-1] = l(\lambda_2)[h(\lambda_1)-1]$$

a result implying that

$$l(\lambda) = h(\lambda)-1. \tag{3.30}$$

Using (3.30) and rearranging (3.28) becomes

$$[k(\lambda_1) - h(\lambda_1)\ln(h(\lambda_1))][h(\lambda_2)-1] =$$

$$[k(\lambda_2) - h(\lambda_2)\ln(h(\lambda_2))][h(\lambda_1)-1],$$

which implies that

$$k(\lambda) = h(\lambda)[\ln h(\lambda) + 1] - 1. \tag{3.31}$$

If we substitute (3.30) and (3.31) into equations (3.22) we can see by applying the transformations $x \rightarrow x-1$, $x' \rightarrow x'-1$, $t \rightarrow t-1$ and $t' \rightarrow t'-1$ that we can take without loss of generality $k=l=0$. Hence, the system (3.22) become

$$x' = h(\lambda)x + h(\lambda)\ln[h(\lambda)]t, \tag{3.32a}$$

$$t' = h(\lambda)t, \tag{3.32b}$$

$$u' = u + \ln[h(\lambda)]. \tag{3.32c}$$

It is immediately evident that the establishment of condition C1 has produced a system for which the other three conditions are easily established. Thus we need one and only one λ_0 such that $h(\lambda_0)=1$. This establishes the unique identity transformation (condition C2). Since the system (3.32) is linear, a unique inverse for each element follows immediately (condition C3) as does the associative law (condition C4). Such functions that satisfy all four conditions are, for example, $h(\lambda) = e^{\lambda}$ and $h(\lambda)=\lambda$.

Therefore the finite transformations which leaves equation (3.11) invariant is given by the system (3.32). The group invariants of (3.32) are $\eta_1 = te^{-x/t}$ and $\xi = u - x/t$. Setting $\eta = \ln\eta_1 = -x/t + \ln t$, then the similarity transformation $u = x/t + F(\eta)$ reduces (3.11) to the ordinary differential equations

$$\frac{dF}{d\eta} - F\frac{dF}{d\eta} + F = e^{-\eta}e^{F}\left[\frac{d^2F}{d\eta^2} + \left(\frac{dF}{d\eta}\right)^2 - 2\frac{dF}{d\eta} + 1\right].$$

Similarly as it was done for the one-parameter group (3.2), we can derive the absolute invariants of the extended groups of (3.32) up to any required order. Then we can easily form an infinite number of equations which are invariant under the strong symmetry group (3.32). It is obvious that the finite group (3.32) cannot be employed to generate new solutions from known ones.

CHAPTER FOUR

CLASSICAL DETERMINATION OF INFINITESIMAL TRANSFORMATIONS

1. Introduction

We have seen in the previous chapter (section 4) how to obtain finite transformation groups of one-parameter which leave a certain partial differential equation invariant. In the present chapter, in a similar way, we search for infinitesimal transformations of the form

$$x' = x + \varepsilon X(x,t,u) + o(\varepsilon^2), \tag{4.1a}$$

$$t' = t + \varepsilon T(x,t,u) + o(\varepsilon^2), \tag{4.1b}$$

$$u' = u + \varepsilon U(x,t,u) + o(\varepsilon^2), \tag{4.1c}$$

which leave a given partial differential equation invariant. This procedure for obtaining infinitesimal transformations, namely the *classical method*, was first introduced by Lie [63]. For recent descriptions of this method see, for example, [14,15,73,77,83,103].

We assume that x,t are the independent variables and $u(x,t)$ the dependent variable of the partial differential equation. If $u(x,t)$ is a solution of the equation $\Delta u = 0$, for invariance we require that $u'(x',t')$ is also a solution of $\Delta' u' = 0$, where Δ' designates Δ with the primed variables replacing the unprimed variables. Using equations (4.1) we deduce that

$$u'(x+\varepsilon X, t+\varepsilon T) = u(x,t) + \varepsilon U(x,t,u) + o(\varepsilon^2). \tag{4.2}$$

Upon expanding the left hand side of (4.2) in Taylor's series and equating coefficients of ε we obtain

$$Xu_x + Tu_t = U, \tag{4.3}$$

which is the equation of an invariant surface for u. Equation (4.3) is a first order linear partial differential equation which can be solved by the method of characteristics (Lagrange) [90]. Hence,

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}. \tag{4.4}$$

Therefore for a given equation $\Delta u = 0$, we search for those infinitesimals X, T and U for which the fact that $u(x,t)$ is a solution of $\Delta u{=}0$ implies that $u'(x',t')$ is also a solution of $\Delta' u'{=}0$. This requirement ensures that the solution is invariant. We shall present two methods for finding these infinitesimals. In this section we describe the classical method and in the next chapter we examine the nonclassical method. The first method produces strong symmetry groups, while the second gives weak symmetry groups.

The classical method only makes use of the given equation $\Delta u{=}0$ and thus involves setting $\Delta' u'$ proportional to Δu. This provides a set of conditions on X, T, U without the use of the invariant surface equation (4.3). In the next section as a vehicle to explain the classical method we use the N-dimensional radially symmetric non-linear diffusion equation of the form

$$u_t = r^{1-N}[r^{N-1}f(u)u_r]_r.$$

In section 3 we show how to generate solutions for the above equation. In section 4 we refer to the computer algebraic packages which perform a variety of analytical procedures automatically and therefore can be very handing in finding symmetry groups for a given partial differential equation. In fact, the computer algebra system *REDUCE* [42] has greatly facilitated the computations involved throughout this thesis.

2. Similarity Transformations for a Radially Symmetric Nonlinear Diffusion Equation

We consider the N-dimensional radially symmetric nonlinear diffusion equation, where for consistency with the notation we have used so far, we replace r by x. Hence,

$$u_t = x^{1-N}[x^{N-1}f(u)u_x]_x. \tag{4.5}$$

We also assume that $N \neq 1$, because otherwise equation (4.5) becomes the nonlinear diffusion equation $u_t - [f(u)u_x]_x = 0$, which has been well examined [2,12,76].

Since equation (4.5) is of the form (3.8) and the right hand side of (4.5) is a polynomial in the derivatives of u, Theorems 3.1 and 3.2 imply that $t'{=}t'(t)$ and

$x'=x'(x,t)$. Hence, the system (4.1) becomes

$$x'=x+\varepsilon X(x,t)+o(\varepsilon^2), \tag{4.6a}$$

$$t'=t+\varepsilon T(t)+o(\varepsilon^2), \tag{4.6b}$$

$$u'=u+\varepsilon U(x,t,u)+o(\varepsilon^2). \tag{4.6c}$$

Setting $P=x+\varepsilon X$, $Q=t+\varepsilon T$ and $\Psi = u+\varepsilon U$ in (2.23) and (2.27) we obtain

$$\delta = (1+\varepsilon X_x)(1+\varepsilon T_t) = 1+\varepsilon(X_x+T_t)+o(\varepsilon^2), \tag{4.7}$$

$$\begin{pmatrix} u'_{x'} & u'_{t'} \end{pmatrix} = \frac{1}{\delta}\begin{pmatrix} u_x+\varepsilon(U_x+U_u u_x) & u_t+\varepsilon(U_t+U_u u_t)\end{pmatrix}\begin{pmatrix} 1+\varepsilon T_t & -\varepsilon X_t \\ 0 & 1+\varepsilon X_x \end{pmatrix}, \tag{4.8}$$

respectively. Using (4.7), identity (4.8) gives

$$u'_{x'} = \frac{u_x+\varepsilon(T_t u_x+U_x+U_u u_x)+o(\varepsilon^2)}{1+\varepsilon(X_x+T_t)+o(\varepsilon^2)}$$

Hence,

$$u'_{x'} = u_x+\varepsilon[U_x+(U_u-X_x)u_x]+o(\varepsilon^2). \tag{4.9}$$

Similarly,

$$u'_{t'} = u_t+\varepsilon[U_t+(U_u-T_t)u_t-X_t u_x]+o(\varepsilon^2). \tag{4.10}$$

Also setting $\Psi = u'_{x'}$ in (2.27), straightforward calculations lead to

$$u'_{x'x'} = u_{xx}+\varepsilon[U_{xx}+(2U_{xu}-X_{xx})u_x+U_{uu}u_x^2+(U_u-2X_x)u_{xx}]+o(\varepsilon^2). \tag{4.11}$$

For invariance of equation (4.5) we demand that

$$u'_{t'}-x'^{1-N}[x'^{N-1}f(u')u'_{x'}]_{x'} = 0. \tag{4.12}$$

From Taylor's theorem,

$$f(u')=f(u+\varepsilon U)=f(u)+\varepsilon U\frac{\mathrm{d}f(u)}{\mathrm{d}u}+o(\varepsilon^2), \tag{4.13}$$

$$\frac{\mathrm{d}f(u')}{\mathrm{d}u'} = \frac{\mathrm{d}f(u)}{\mathrm{d}u}+\varepsilon U\frac{\mathrm{d}^2f(u)}{\mathrm{d}u^2}+o(\varepsilon^2). \tag{4.14}$$

Substitution of (4.9), (4.10), (4.11), (4.13) and (4.14) into equation (4.12) and also eliminating u_t from (4.5) we obtain

$$E(x,t,u,u_x,u_{xx}) = 0, \tag{4.15}$$

where E is a determined polynomial in u_x and u_{xx}. The classical method consists of imposing the condition that (4.15) is an identity in the five variables x, t, u, u_x, u_{xx} regarded as independent. Hence setting, successively, the coefficients of u_{xx}, u_x^2, u_x and the term independent of u_x and u_{xx} in (4.15) equal to zero, we are led to the relations

$$Uf_u - 2X_x f + T_t f = 0, \tag{4.16}$$

$$Uf_{uu} + U_{uu} f + U_u f_u - 2X_x f_u + T_t f_u = 0, \tag{4.17}$$

$$(2fU_{xu} + 2U_x f_u + X_t - X_{xx} f)x^2 +$$

$$(N-1)(xUf_u - xX_x f + xT_t f - Xf) = 0, \tag{4.18}$$

$$xU_t - xU_{xx} f - (N-1)U_x f = 0. \tag{4.19}$$

These four relations enable the infinitesimal transformations to be derived and ultimately impose restrictions on the functional forms of f, X, T and U.

Upon differentiating equation (4.16) with respect to u and then subtracting the resulting equation from (4.17) we deduce that

$$U = A(x,t)u + B(x,t), \tag{4.20}$$

where A and B are functions to be determined. Using (4.20), equations (4.16), (4.18) and (4.19) become

$$(Au+B)f_u + (T_t - 2X_x)f = 0, \tag{4.21}$$

$$[2fA_x + 2(A_x u + B_x)f_u + X_t - X_{xx} f]x^2 + (N-1)(xX_x - X)f = 0, \tag{4.22}$$

$$x(A_t u + B_t) - x(A_{xx} u + B_{xx})f - (N-1)(A_x u + B_x)f = 0. \tag{4.23}$$

Equation (4.21) implies that there exists a nontrivial relationship connecting uf_u, f_u and f, unless $A = B = T_t - 2X_x = 0$. This relationship is given by

$$(\lambda_1 u + \lambda_2)f_u + \lambda_3 f = 0,$$

where λ_1, λ_2 and λ_3 are constants. Redefining the constants λ_i, from the above differential equation we deduce that f must take one of the following forms

$$f = \lambda_1 e^{\lambda_2 u},$$

$$f = \lambda_1 (u + \lambda_3)^{\lambda_4}.$$

In the following analysis we let $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$ and $\lambda_4 = n$. We can therefore split this analysis into three cases:

1. $A=B=T_t-2X_x = 0$, f arbitrary;
2. $f = u^n$;
3. $f = e^u$.

Case 1. $X = \frac{1}{2}xT_t+k(t)$, $U=0$, f arbitrary

Equation (4.23) vanishes and equating coefficients of powers of x in (4.22) we obtain $T_{tt} = 0$, $k_t = 0$ and $(N-1)kf=0$. Hence,

$$X=\tfrac{1}{2}c_1x, \quad T=c_1t+c_2, \quad U=0, \tag{4.24}$$

where c_1 and c_2 are constants. We note that equations (4.21) - (4.23) are all satisfied by the system (4.24) without imposing any restrictions on the functional form of f.

Using (4.24), equation (4.4) reads

$$\frac{\mathrm{d}x}{\frac{1}{2}c_1x} = \frac{\mathrm{d}t}{c_1t+c_2} = \frac{\mathrm{d}u}{0}. \tag{4.25}$$

Assuming that $c_1 \neq 0$, then solving the first equation in (4.25) we obtain the similarity variable

$$\eta = \frac{x}{(t+c_3)^{\frac{1}{2}}},$$

where $c_3 = c_2/c_1$. Clearly, the second invariant is $\xi=u$. Hence the similarity transformation is

$$u = F(\eta)$$

which reduces the partial differential equation (4.5) to the ordinary differential equation

$$\tfrac{1}{2}\eta\frac{\mathrm{d}F}{\mathrm{d}\eta}+\eta^{1-N}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta^{N-1}f(F)\frac{\mathrm{d}F}{\mathrm{d}\eta}\right] = 0.$$

Case 2. $f = u^n$, $n\neq0$

Equation (4.5) reads

$$u_t = x^{1-N}[x^{N-1}u^nu_x]_x. \tag{4.26}$$

Equations of the form (4.26) have a large number of applications, for both $n>0$

("slow" diffusion) and $n<0$ ("fast" diffusion). See, for example, [44] and [79].

Equating coefficients of powers of u in equations (4.21) - (4.23) we obtain the following results

$$nA = 2X_x - T_t, \tag{4.27a}$$

$$B = 0, \tag{4.27b}$$

$$X_t = 0, \tag{4.27c}$$

$$\frac{4}{n}x^2X_{xx} + 3x^2X_{xx} + (N-1)(xX_x - X) = 0, \tag{4.27d}$$

$$T_{tt} = 0, \tag{4.27e}$$

$$xX_{xxx} + (N-1)X_{xx} = 0. \tag{4.27f}$$

Since $X = X(x)$ from (4.27c), equation (4.27f) becomes an ordinary differential equation. Solving this equation, X must take one of the following forms:

$$X = c_1x^{3-N} + c_2x + c_3, \quad N \neq 2,3, \tag{4.28}$$

$$X = c_1x\ln x + c_2x + c_3, \quad (N=2) \tag{4.29}$$

$$X = c_1\ln x + c_2x + c_3, \quad (N=3) \tag{4.30}$$

where c_1, c_2 and c_3 are constants.

If we substitute (4.28) into (4.27d), then the coefficient of x^{3-N} and the constant term give

$$c_3 = 0, \tag{4.31a}$$

$$c_1[(\frac{4}{n}+3)(3-N)+(N-1)] = 0. \tag{4.31b}$$

Therefore from (4.31b) we must either have (a) $c_1=0$ or (b) $c_1 \neq 0$ and $n = \frac{2(N-3)}{(4-N)}$. Thus, using (4.27a,b,e), (4.28) and (4.31) we conclude that the infinitesimals X, T, U are given by one of the following two systems:

$$X = c_1x^{3-N} + c_2x, \quad T = c_4t + c_5,$$

$$U = \frac{1}{n}[2c_1(3-N)x^{2-N} + 2c_2 - c_4]u, \quad c_1 \neq 0,\ n = \frac{2(N-3)}{(4-N)}, \tag{4.32}$$

$$X = c_2x, \quad T = c_4t + c_5, \quad U = \frac{1}{n}(2c_2 - c_4)u, \quad n \text{ arbitrary} \tag{4.33}$$

Substitution of (4.29) into (4.27d) we deduce that

$$c_3 = 0, \quad c_1\left(\frac{4}{n}+3+1\right) = 0.$$

From above we either have (a) $c_1 = 0$ or (b) $c_1 \neq 0$ and $n=-1$. Therefore when $c_1 = 0$, the infinitesimals are given by (4.33) and when $c_1 \neq 0$, are given by

$$X=c_1x\ln x+c_2x, \quad T=c_4t+c_5, \quad U=(c_4-2c_1\ln x-2c_1-2c_2)u, \quad c_1 \neq 0, \qquad (4.34)$$

Finally, using equations (4.30) and (4.27d) we obtain $c_1 = c_3 = 0$. Hence, X, T, U are given by the system (4.33).

Summarising this case we have:

(i) For n arbitrary, N arbitrary and X, T, U are given by (4.33),

(ii) For $n=\dfrac{2N-6}{4-N}$, $N \neq 2, 3, 4$ and X, T, U are given by (4.32)

(iii) For n=-1, N=2 and the infinitesimals are given by the system (4.34).

We now continue the analysis, by examining each case separately.

(i) Using (4.33), equations (4.4) read

$$\frac{\mathrm{d}x}{c_2x} = \frac{\mathrm{d}t}{c_4t+c_5} = \frac{n\,\mathrm{d}u}{(2c_2-c_4)u}. \qquad (4.35)$$

Equations (4.35) can be solved to give three different solutions, depending which constants vanish. In all cases, equations (4.35) can be solved very easily. Therefore we list the results, giving the similarity variable η, the similarity transformation and the ordinary differential equation to which the partial differential equation (4.26) is transformed by the similarity transformation.

(a) $c_4 = c_5 = 0,\ c_2 \neq 0, \quad \eta = t,$

$$u=x^{\frac{2}{n}}F(\eta), \qquad (4.36a)$$

$$(2nN+4)F^{n+1}-n^2\frac{\mathrm{d}F}{\mathrm{d}\eta} = 0. \qquad (4.36b)$$

(b) $c_4 = 0,\ c_2 \neq 0,\ c_5 \neq 0, \quad \eta = x\mathrm{e}^{-c_2t/c_5},$

$$u=\mathrm{e}^{\frac{2c_2t}{nc_5}}F(\eta), \qquad (4.37a)$$

$$c_5n\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta F^n\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]+c_5n(N-2)F^n\frac{\mathrm{d}F}{\mathrm{d}\eta}+c_2n\eta^2\frac{\mathrm{d}F}{\mathrm{d}\eta}-2c_2\eta F = 0. \qquad (4.37b)$$

(c) $$c_4 \neq 0, \quad \eta = \frac{x^{c_4}}{(c_4 t + c_5)^{c_2}},$$

$$u = (c_4 t + c_5)^{(2\frac{c_2}{c_4} - 1)/n} F(\eta), \tag{4.38a}$$

$$c_4^2 n\eta \frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta F^n \frac{\mathrm{d}F}{\mathrm{d}\eta}\right] + c_4 n(N-2)\eta F^n \frac{\mathrm{d}F}{\mathrm{d}\eta} +$$

$$c_2 c_4 n\eta\eta^{2/c_4}\frac{\mathrm{d}F}{\mathrm{d}\eta} + (c_4 - 2c_2)\eta^{2/c_4}F = 0. \tag{4.38b}$$

(ii) Using (4.32), equations (4.4) now read

$$\frac{\mathrm{d}x}{c_1 x^{3-N} + c_2 x} = \frac{\mathrm{d}t}{c_4 t + c_5} = \frac{n\,\mathrm{d}u}{[2c_1(3-N)x^{2-N} + 2c_2 - c_4]u}. \tag{4.39}$$

We list five different similarity solutions which can be obtained from equations (4.39).

(a) $$c_4 = c_5 = 0, \quad \eta = t,$$

$$u = (c_2 x + c_1 x^{3-N})^{\frac{2}{n}} F(\eta), \tag{4.40a}$$

$$\frac{(N-2)^2(4-N)c_2^2}{(N-3)^2} F F^n - \frac{\mathrm{d}F}{\mathrm{d}\eta} = 0. \tag{4.40b}$$

(b) $$c_2 = c_4 = 0,\ c_5 \neq 0, \quad \eta = e^{x^{N-2}} e^{\frac{c_1}{c_5}(2-N)t},$$

$$u = x^{N-4} F(\eta), \tag{4.41a}$$

$$(N-2)\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta F^n \frac{\mathrm{d}F}{\mathrm{d}\eta}\right] + \frac{c_1}{c_5}\frac{\mathrm{d}F}{\mathrm{d}\eta} = 0. \tag{4.41b}$$

(c) $$c_4 = 0,\ c_2 \neq 0,\ c_5 \neq 0, \quad \eta = (c_2 x^{N-2} + c_1) e^{\frac{c_2}{c_5}(2-N)t},$$

$$u = (c_2 x + c_1 x^{3-N})^{\frac{2}{n}} F(\eta), \tag{4.42a}$$

$$n^2 c_5 c_2 (N-2)^2 \eta^2 \left[F \frac{\mathrm{d}^2 F}{\mathrm{d}\eta^2} + n\left(\frac{\mathrm{d}F}{\mathrm{d}\eta}\right)^2\right] F^n + 2c_2 c_5 (Nn+2) F^n F^2 +$$

$$[n^2(N-2) + n^2 c_2 c_5 (2N^2 - 4N) F^n + n c_2 c_5 (4N-8) F^n]\eta F \frac{\mathrm{d}F}{\mathrm{d}\eta} = 0. \tag{4.42b}$$

(d) $c_2=0,\ c_4\neq 0,\quad \eta = (c_4t+c_5)^{\frac{c_1}{c_4}(2-N)}\, e^{x^{N-2}}$,

$$u=x^{N-4}e^{N'x^{N-2}}F(\eta),\quad N'=\frac{c_4}{c_1(2-N)n}, \tag{4.43a}$$

$$(N-2)^2c_1^2n^2\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta F^n\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]+(N-2)c_1n\eta\left[c_1^2\eta\eta^{\frac{c_4}{c_1(N-2)}}-2c_4(n+1)F^n\right]\frac{\mathrm{d}F}{\mathrm{d}\eta}$$

$$+(n+1)c_4^2FF^n = 0. \tag{4.43b}$$

(e) $c_2\neq 0,\ c_4\neq 0,\quad \eta = \dfrac{(c_2x^{N-2}+c_1)^{c_4}}{(c_4t+c_5)^{c_2(N-2)}}$,

$$u = x^{\frac{6-2N}{n}}(c_4t+c_5)^{\frac{2c_2(N-2)-c_4}{nc_4}}F(\eta), \tag{4.44a}$$

$$c_2^2c_4^2n(N-2)^2\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta F^n\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]-c_2^2c_4n(N-2)^2\eta F^n\frac{\mathrm{d}F}{\mathrm{d}\eta}+$$

$$c_2c_4n(N-2)\eta\eta^{2/c_4}\frac{\mathrm{d}F}{\mathrm{d}\eta}+[c_4-2c_2(N-2)]\eta^{2/c_4}F = 0. \tag{4.44b}$$

(iii) Finally, using (4.34) equations (4.4) become

$$\frac{\mathrm{d}x}{c_1x\ln x+c_2x} = \frac{\mathrm{d}t}{c_4t+c_5} = \frac{\mathrm{d}u}{[c_4-2c_1\ln x-2c_1-2c_2]u}, \tag{4.45}$$

from which we obtain the following results:

(a) $c_4 = c_5 = 0,\quad \eta=t$,

$$u=(c_1x\ln x+c_2x)^{-2}F(\eta), \tag{4.46a}$$

$$\frac{\mathrm{d}F}{\mathrm{d}\eta}-2c_1^2 = 0. \tag{4.46b}$$

(b) $c_4 = 0,\ c_5\neq 0,\quad \eta = (c_1\ln x+c_2)e^{-c_1t/c_5}$,

$$u=(c_1x\ln x+c_2x)^{-2}F(\eta), \tag{4.47a}$$

$$\eta^2\left[F\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2}-\left(\frac{\mathrm{d}F}{\mathrm{d}\eta}\right)^2\right]+\left(2+\frac{\eta}{c_1c_5}\frac{\mathrm{d}F}{\mathrm{d}\eta}\right)F^2 = 0. \tag{4.47b}$$

(c) $c_4\neq 0,\quad \eta = \dfrac{(c_1\ln x+c_2)^{c_4/c_1}}{(c_4t+c_5)}$,

$$u = \frac{(c_1 \ln x + c_2)^{\frac{c_4}{c_1} - 2}}{x^2} F(\eta), \tag{4.48a}$$

$$c_4^2 \eta \frac{d}{d\eta}\left[\eta F^{-1} \frac{dF}{d\eta}\right] + c_4 \eta (\eta F - c_1) F^{-1} \frac{dF}{d\eta} + 2c_1^2 - c_1 c_4 = 0. \tag{4.48b}$$

Case 3. $f = e^u$

Equation (4.5) becomes

$$u_t = x^{1-N}[x^{N-1} e^u u_x]_x. \tag{4.49}$$

Equating coefficients of e^u and powers of u in equations (4.21) - (4.23) we obtain

$$A=0, \tag{4.50a}$$

$$B=2X_x - T_t, \tag{4.50b}$$

$$X_t = 0, \tag{4.50c}$$

$$3x^2 X_{xx} + (N-1)(xX_x - X) = 0, \tag{4.50d}$$

$$T_{tt} = 0, \tag{4.50e}$$

$$xX_{xxx} + (N-1)X_{xx} = 0. \tag{4.50f}$$

From equation (4.50f) it follows that X must take one of the forms (4.28) - (4.30). Upon substitution of these forms, each separately, in equation (4.50d) we find that in order to satisfy this equation, the forms (4.29) and (4.30) must be restricted to $X = c_2 x$, while the first form, equation (4.28), is restricted to either (a) $X = c_2 x$ or (b) $X = \frac{c_1}{x} + c_2 x$ $(N=4)$. Therefore from equations (4.50), we deduce that the infinitesimals X, T, U are given by one of the following two systems:

(i) $\quad X = c_2 x, \; T = c_4 t + c_5, \; U = 2c_2 - c_4, \; N$ *arbitrary* $\qquad$ (4.51)

(ii) $\quad X = \frac{c_1}{x} + c_2 x, \; T = c_4 t + c_5, \; U = -2\frac{c_1}{x^2} + 2c_2 - c_4, \; c_1 \neq 0, N=4.$ $\qquad$ (4.52)

Similarly as in case 2, we state all the possible similarity transformations resulting from (4.51) and (4.52). We also give the corresponding ordinary differential equation to which equation (4.49) is transformed.

(i) $$\frac{dx}{c_2 x} = \frac{dt}{c_4 t + c_5} = \frac{du}{2c_2 - c_4} \tag{4.53}$$

(a) $$c_4 = c_5 = 0,\ c_2 \neq 0, \quad \eta = t,$$

$$u = 2\ln x + F(\eta), \tag{4.54a}$$

$$2Ne^F - \frac{dF}{d\eta} = 0. \tag{4.54b}$$

(b) $$c_4 = 0,\ c_5 \neq 0,\ c_2 \neq 0, \quad \eta = xe^{-\frac{c_2}{c_5}t},$$

$$u = 2\frac{c_2}{c_5}t + F(\eta), \tag{4.55a}$$

$$c_5 \frac{d}{d\eta}\left[\eta e^F \frac{dF}{d\eta}\right] + c_5(N-2)e^F \frac{dF}{d\eta} + c_2\eta^2 \frac{dF}{d\eta} - 2c_2\eta = 0. \tag{4.55b}$$

(c) $$c_4 \neq 0, \quad \eta = \frac{x^{c_4}}{(c_4 t + c_5)^{c_2}},$$

$$u = (2\frac{c_2}{c_4} - 1)\ln(c_4 t + c_5) + F(\eta), \tag{4.56a}$$

$$c_4^2\eta \frac{d}{d\eta}\left[\eta e^F \frac{dF}{d\eta}\right] + c_4(N-2)\eta e^F \frac{dF}{d\eta} +$$

$$c_2 c_4 \eta \eta^{2/c_4} \frac{dF}{d\eta} + (c_4 - 2c_2)\eta^{2/c_4} = 0. \tag{4.56b}$$

(ii) $$\frac{dx}{c_1/x + c_2 x} = \frac{dt}{c_4 t + c_5} = \frac{du}{-2c_1/x^2 + 2c_2 - c_4} \tag{4.57}$$

(a) $$c_4 = c_5 = 0, \quad \eta = t,$$

$$u = 2\ln(\frac{c_1}{x} + c_2 x) + F(\eta), \tag{4.58a}$$

$$8c_2^2 e^F - \frac{dF}{d\eta} = 0. \tag{4.58b}$$

(b) $$c_2 = c_4 = 0,\ c_5 \neq 0, \quad \eta = c_5 x^2 + 2c_1 t,$$

$$u = 2\ln\frac{c_1}{x} + F(\eta), \tag{4.59a}$$

$$2c_5^2 c_1 e^F\left[\frac{d^2F}{d\eta^2} + \left(\frac{dF}{d\eta}\right)^2\right] - \frac{dF}{d\eta} = 0. \tag{4.59b}$$

(c) $$c_4 = 0,\ c_2 \neq 0,\ c_5 \neq 0,\quad \eta = (c_2x^2+c_1)\mathrm{e}^{-2\frac{c_2}{c_5}t},$$

$$u=2\ln\left(\frac{c_1}{x}+c_2x\right)+F(\eta), \tag{4.60a}$$

$$2\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta\,\mathrm{e}^F\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]+6\eta\,\mathrm{e}^F\frac{\mathrm{d}F}{\mathrm{d}\eta}+\frac{\eta}{c_2c_5}\frac{\mathrm{d}F}{\mathrm{d}\eta}+4\mathrm{e}^F = 0. \tag{4.60b}$$

(d) $$c_2=0,\ c_4 \neq 0,\quad \eta = (c_4t+c_5)\mathrm{e}^{-\frac{c_4}{2c_1}x^2},$$

$$u = 2\ln\frac{1}{x}-\frac{c_4}{2c_1}x^2+F(\eta), \tag{4.61a}$$

$$c_4\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta\,\mathrm{e}^F\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]+2c_4\eta\,\mathrm{e}^F\frac{\mathrm{d}F}{\mathrm{d}\eta}-c_1^2\frac{\mathrm{d}F}{\mathrm{d}\eta}+c_4\mathrm{e}^F = 0. \tag{4.61b}$$

(e) $$c_2 \neq 0,\ c_4 \neq 0,\quad \eta = \frac{(c_2x^2+c_1)^{c_4/2c_2}}{(c_4t+c_5)},$$

$$u=\ln\left[\frac{(c_2x^2+c_1)^2}{x^2(c_4t+c_5)}\right]+F(\eta), \tag{4.62a}$$

$$c_4^2\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta\mathrm{e}^F\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]+6c_2c_4\eta\mathrm{e}^F\frac{\mathrm{d}F}{\mathrm{d}\eta}+c_4\eta\frac{\mathrm{d}F}{\mathrm{d}\eta}+8c_2^2\mathrm{e}^F+c_4 = 0. \tag{4.62b}$$

We note that if we write $v=\mathrm{e}^u$ then equation (4.49) becomes

$$v_t = vx^{1-N}[x^{N-1}v_x]_x.$$

Also, if we set $v=u^n$ in equation (4.26), then it becomes

$$v_t = vx^{1-N}[x^{N-1}v_x]_x+\frac{1}{n}u_x^2.$$

Therefore equation (4.49) is related to (4.26) in the limit $n \longrightarrow \infty$.

A large number of exact similarity solutions to equations (4.26) can be found in [44], [105], [48] and [49]. In [48] exact similarity solutions to (4.49) are also appeared. For example, setting $c_4=1$ and $c_2 = \dfrac{1}{nN+2}$ in (4.38b) and then multiply through by η^{N-2} gives

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta^{N-1}F^n\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]+\frac{1}{nN+2}\left[N\eta^{N-1}F+\eta^N\frac{\mathrm{d}F}{\mathrm{d}\eta}\right] = 0. \tag{4.63}$$

Integrating (4.63) gives

$$\eta^{N-1}F^{n}\frac{dF}{d\eta}+\frac{1}{nN+2}\eta^{N}F+\alpha = 0, \tag{4.64}$$

where α is an arbitrary constant. Taking $n=-1$, equation (4.64) becomes a Bernoulli equation and if we write $F=1/g$ it reads

$$\frac{dg}{d\eta}-\alpha\eta^{1-N}g = \frac{\eta}{2-N}, \quad N \neq 2. \tag{4.65}$$

When $\alpha \neq 0$, the solution of (4.65) is given by

$$g=\frac{1}{2-N}\exp\left(\frac{\alpha\eta^{2-N}}{2-N}\right)\int_{\eta_0}^{\eta} \eta\exp\left(-\frac{\alpha\eta^{2-N}}{2-N}\right)d\eta,$$

where η_0 is an arbitrary constant. Using (4.38a) we can obtain a similarity solution for equation (4.26) when $n=-1$.

3. Generating Solutions Using Strong Symmetry Groups

A completely different idea for using strong symmetry transformations groups is treated in this section. By definition, a strong symmetry transformation group maps a solution into a solution. So if we already know a special solution of a partial differential equation, we can apply a finite transformation group to obtain a (possibly) new solution. This new solution will depend on at most as many parameters as there are in the transformation we have used.

To discuss this concept, the N-dimensional radially symmetric nonlinear diffusion equation (4.5) will be used. In the previous section we have explored all the possible infinitesimals transformations for this partial differential equation, using the classical method. But to carry out this idea, we need to obtain the corresponding finite transformations of these infinitesimal transformations. In chapter 2, section 4, we demonstrated how this can be achieved.

It is not difficult to note that cases 1, 2(i) and 3(i), whose infinitesimals are given (4.24), (4.33) and (4.51) respectively, do not lead to finite transformations that can be used to generate new solutions from known ones. For example in case 2(i) the corresponding finite group of transformation is:

$$x'=xe^{c_2\lambda}, \quad t'=(t+c_5/c_4)e^{c_4\lambda}-c_5/c_4, \quad u'=ue^{A\lambda},$$

where $A=(2c_2-c_4)/n$ and c_2 and c_4 are chosen to be nonzero. Clearly, this transformation cannot generate a new special solution from a known one.

Unfortunately, in all the remaining cases $u'=u'(x,u)$, which means that starting with the trivial solution u=constant we generate a solution which is a function of x only. But this solution can be also easily obtained by setting $u_t = 0$ in the partial differential equation which then becomes an ordinary differential equation which can be solved. For example, setting $u_t = 0$ and $N=4$ in equation (4.49) becomes an ordinary differential equation. Solving this equation gives

$$u=\ln Ax^{-2}+B,$$

where A and B are constants. This solution can also be generated using the trivial solution u=constant and the finite transformation corresponding to the infinitesimal of the case 3(ii). Nevertheless the examples which are used in the following analysis show clearly how this concept is applied.

Now consider the infinitesimal transformation whose infinitesimals are given by the system (4.32) [case 2(ii)]. Setting $c_5=0$ and $c_2 = c_4=1$, the corresponding finite transformation can be found solving the system of ordinary differential equations.

$$\frac{dx'}{d\lambda} = c_1 x'^{3-N}+x', \quad \frac{dt'}{d\lambda} = t', \quad \frac{du'}{d\lambda} = \frac{1}{n}[2c_1(3-N)x'^{2-N}+1]u' \tag{4.66}$$

where $c_1 \neq 0$, $n=\dfrac{2(N-3)}{4-N}$ and λ is the parameter of the finite transformation group, under the initial conditions $x'=x$, $t'=t$, $u'=u$ when $\lambda=0$. It is straightforward to solve the system (4.66) to give

$$x'=[(x^{N-2}+c_1)\mu-c_1]^{1/N-2}, \tag{4.67a}$$

$$t'=\mu^{1/N-2}t, \tag{4.67b}$$

$$u'=(x^{N-2}+c_1-c_1/\mu)^{\frac{N-4}{N-2}}\mu^{1/n(N-2)}x^{4-N}u, \tag{4.67c}$$

where $\mu = e^{(N-2)\lambda}$. The inverse of (4.67) is

$$x=[(x'^{N-2}+c_1)/\mu-c_1]^{1/N-2}, \tag{4.68a}$$

$$t=\mu^{1/2-N}t', \tag{4.68b}$$

$$u=(x'^{N-2}+c_1-c_1\mu)^{\frac{N-4}{N-2}}\mu^{1/n(2-N)}x'^{4-N}u', \tag{4.68c}$$

simply obtained by exchanging (x,t,u) and (x',t',u') and replacing μ by $1/\mu$.

Now we can apply the finite transformation (4.67) to any solution of the equation (4.26), where $n=\frac{2(N-3)}{4-N}$, for example, to the trivial solution $u_0=A$=constant. We immediately obtain from this and (4.68c) that

$$u_1=(x'^{N-2}+c_1-c_1\mu)^{\frac{N-4}{N-2}}\mu^{1/n(2-N)}x'^{4-N}A,$$

where we have set $u \equiv u_1$ and $u' \equiv u_0$. Using (4.67a) the above equation becomes

$$u_1=[(c_1x^{2-N}+1)\mu-c_1x^{2-N}]^{\frac{4-N}{N-2}}\tau A, \tag{4.69}$$

where τ is a determined function of μ. If the above solution was a function of x and t, we could think u_1 as u_0 and replace x by x', t by t' in (4.69) and then similarly generate a second particular solution. A sequence u_m of solutions could then be generated by continued repetition of this process.

As a second and last example we employ the infinitesimal transformation whose infinitesimals are given by the system (4.35) [case 2(iii)]. Setting $c_1 = c_4=1$ and $c_2 = c_5 = 0$, the corresponding finite group of transformations is given by

$$x'=x^{\lambda}, \tag{4.70a}$$

$$t' = \lambda t, \tag{4.70b}$$

$$u' = \frac{x^{2(1-\lambda)}}{\lambda}u, \tag{4.70c}$$

with its inverse

$$x=x'^{1/\lambda}, \tag{4.71a}$$

$$t=\frac{t'}{\lambda}, \tag{4.71b}$$

$$u=\lambda x'^{2(\lambda-1)/\lambda}u', \tag{4.71c}$$

simply obtained by exchanging (x,t,u) and $(x',t'u')$ and replacing λ by $1/\lambda$.

Using the trivial solution $u_0=A$=constant of the equation (4.26), where $n=-1$ and $N=2$, the finite transformation (4.70) and its inverse (4.71) we obtain the particular solution

$$u_1 = \lambda x^{2(\lambda-1)}A \tag{4.72}$$

of the same equation. Employment of the solution (4.72), the transformation (4.70) can generate a second particular solution. Continuing this process we obtain the sequence of solutions

$$u_m = \lambda^m x^{2(\lambda-1)(\lambda^{m-1}+\lambda^{m-2}+\ldots+\lambda+1)}A, \quad m=1,2,3\ldots \tag{4.73}$$

As we stated earlier, the trivial solution u=const. was only used here to demonstrate the idea. For more interesting results, one should start with an initial solution u_0 of the equation (4.5) which is a function of t and then follow the above process.

4. The Use of Computer Algebra Systems

There is one major obstacle to the applications of group theoretical methods for obtaining analytic solutions of partial differential equations. This is the tremendous amount of algebraic computations which is usually involved in obtaining the transformation groups of the equation under consideration. In fact this the case for many problems in today's research in Applied Mathematics. For example, determining Bäcklund Transformations for a given partial differential equation requires a large amount of computations (see Part II). The introduction of computer algebra systems during the last two decades has greatly facilitated this problem. These systems now available on many machines make it feasible to perform a variety of analytical procedures automatically.

Therefore the availability of these computer algebra systems has a particularly strong influence on those areas of Applied Mathematics where large analytic manipulations are necessary for obtaining a certain result. Applying a computer algebra system means becoming accustomed to a completely new working style. Pencil and paper work is almost completely eliminated. Instead of working out a problem in the old-fasioned way over and over again with varying input, a solution strategy by applying computer algebra methods is developed. A problem which arises using such systems is that we cannot directly check the results. However, this problem can be overcome if two (or more) users work separately on the same project and at the end they verify their results.

Most of the calculations involved in this thesis have been greatly facilitated by the computer algebra system REDUCE [42]. The other most important general purpose systems currently available are MACSYMA (Math Lab Group MIT) [81], MAPLE (B. Char, University of Waterloo, Canada), Mu Math (D.R. Stoutemyer, The

Software-house, Honolulu, Hawaii), SMP (S. Wolfrom) and SCRATCHPAD I and II (R.D. Jenks and D. Yun, IBM Watson Laboratories).

Typically a computer algebra system provides modules for performing basic operations such as simplifications, differentiation, integration, factorisation, etc. These algorithms are the building blocks for packages which may be developed for special applications. For example, a package has been developed to perform almost all algebraic manipulations necessary to determine the strong symmetry groups of a given partial differential equation. Details of this package, namely SPDE, are given by Schwarz [86,87,88].

CHAPTER FIVE

NONCLASSICAL DETERMINATION OF SIMILARITY TRANSFORMATIONS

1. Introduction

In this chapter we turn our attention to weak symmetry groups of partial differential equations. Since these groups do not transform solutions to solutions, they can only be used to derive the corresponding similarity transformations. The method which is presented here to determine these transformation groups is called the *nonclassical method*. Bluman and Cole [13] proposed this method as a generalisation of the classical method which was presented in the previous chapter. The nonclassical method has been generalised by Olver and Rosenau [74,75].

A common characteristic of the classical and nonclassical methods for finding the transformation groups and then the associated similarity transformations of a given partial differential equation is the use of the group theory. In this chapter (section 3) we also present a direct method, due to Clarkson and Kruskal [26], for determining similarity transformations that makes no use of group theory. Originally this method seemed to produce results which could not be obtained by any other method, but later it appeared [60] that the same results can be obtained using the nonclassical method of Bluman and Cole. It has not been shown yet if the two methods are exactly the same. Both methods are illustrated with the use of the same example, the modified Boussinesq equation [16,17].

One of the annoying features of these methods is that all the similarity transformations obtained can sometimes be obtained more easily using the classical method. Here we use an example which this is not the case.

2. Nonclassical Method for Determining Infinitesimal Transformations

The nonclassical method, introduced by Bluman and Cole [13], makes use of both the given equation $\Delta u=0$ and the invariant surface condition, equation (4.3). Now equation (4.3) really possesses only two independent infinitesimals since it can be divided through by X, T or U. Assuming $T \neq 0$, we set $T=1$. Equation (4.3) now reads

$$u_t = U - Xu_x. \tag{5.1}$$

The method is discussed using the modified Boussinesq equation

$$u_{tt} - u_t u_{xx} - \tfrac{1}{2} u_x^2 u_{xx} + u_{xxxx} = 0 \tag{5.2}$$

and we consider the infinitesimal transformations

$$x' = x + \varepsilon X(x,t) + o(\varepsilon^2), \tag{5.3a}$$

$$t' = t + \varepsilon + o(\varepsilon^2), \tag{5.3b}$$

$$u' = u + \varepsilon U(x,t,u) + o(\varepsilon^2). \tag{5.3c}$$

Note that equation (5.1) has implications for other derivatives, for example, if we differentiate (5.1) with respect to x and t respectively, we obtain

$$u_{xt} = U_x + U_u u_x - X_x u_x - X u_{xx}, \tag{5.4}$$

$$u_{tt} = U_t + U_u u_t - X_t u_x - X u_{xt}. \tag{5.5}$$

Using equations (5.1) and (5.4), equation (5.5) becomes

$$u_{tt} = U_t + UU_u - XU_x + (XX_x - 2XU_u - X_t) u_x + X^2 u_{xx}. \tag{5.6}$$

For invariance we demand that

$$u'_{t't'} - u'_{t'} u'_{x'x'} - \tfrac{1}{2} u'^2_{x'} u'_{x'x'} + u'_{x'x'x'x'} = 0 \tag{5.7}$$

The identities (2.23) and (2.27) enable us to determine the u'-derivatives in terms of u-derivatives. We have already calculated $u'_{x'}$, $u'_{t'}$ and $u'_{x'x'}$ [equations (4.9) - (4.11)]. It is now straightforward to work out $u'_{t't'}$ and $u'_{x'x'x'x'}$ from the identities (2.23) and (2.27), using the fact that $T=1$ and $X_u=0$:

$$u'_{t't'} = u_{tt} + \varepsilon[U_{tt} + 2U_{tu} u_t - X_{tt} u_x + U_{uu} u_t^2 + U_u u_{tt} - 2X_t u_{xt}] + o(\varepsilon^2), \tag{5.8}$$

$$\begin{aligned} u'_{x'x'x'x'} = {} & u_{xxxx} + \varepsilon[U_{xxxx} + (4U_{xxxu} - X_{xxxx}) u_x + 6U_{xxuu} u_x^2 + 4U_{xuuu} u_x^3 + \\ & U_{uuuu} u_x^4 + (6U_{xxu} - 4X_{xxx}) u_{xx} + 6U_{xuu} u_x u_{xx} + 6U_{uuu} u_x^2 u_{xx} + 3U_{uu} u_{xx}^2 + \\ & (4U_{xu} - 6X_{xx}) u_{xxx} + 4U_{uu} u_x u_{xxx} + (U_u - 4X_x) u_{xxxx}] + o(\varepsilon^2). \end{aligned} \tag{5.9}$$

Upon substitution of (4.9), (4.10), (4.11), (5.8) and (5.9) into equation (5.7) and then elimination of u_t, u_{xt}, u_{tt} and u_{xxxx} using equations (5.1), (5.4), (5.6) and (5.2) respectively, equation (5.7) takes the form

$$E(x,t,u,u_x,u_{xx},u_{xxx}) = 0, \tag{5.10}$$

where E is a determined polynomial function of x,t,u,u_x,u_{xx},u_{xxx} and derivatives of X and U. As in the classical method the infinitesimals X and U are determined by insisting that (5.10) is an identity in the variables x, t, u, u_x, u_{xx}, u_{xxx} regarded as independent.

Considering the coefficient of u_{xx}^2 in (5.10) shows that

$$U=A(x,t)u+B(x,t). \tag{5.11}$$

Coefficients of u_{xxx} and u_x^3 give respectively

$$2A_x-3X_{xx} = 0,$$

$$2A_x-X_{xx} = 0.$$

Therefore from the above equations we have $A=A(t)$ and

$$X = k(t)x+l(t). \tag{5.12}$$

Now using the coefficient of u_x^2 we obtain $A=0$ and $B=g(t)x+h(t)$. Therefore equation (5.11) becomes

$$U=g(t)x+h(t). \tag{5.13}$$

Substituting (5.12) and (5.13) back into (5.10), coefficients of $xu_{xx}u_x$, x^2u_{xx}, xu_x, u_x, u_xu_{xx}, xu_{xx}, x, u_{xx} and the term independent of u-derivatives and x lead to the following nine identities:

$$k_t+2k^2 = 0, \tag{5.14a}$$

$$2kk_t+4k^3 = 0, \tag{5.14b}$$

$$k_{tt}+2kk_t-4k^3 = 0, \tag{5.14c}$$

$$l_{tt}+2kl_t-4k^2l = 0, \tag{5.14d}$$

$$l_t+2kl-g = 0, \tag{5.14e}$$

$$-g_t+2k_tl+2kl_t-2gk+8k^2l = 0, \tag{5.14f}$$

$$g_{tt}+4g_tk-2gk_t-4gk^2 = 0, \tag{5.14g}$$

$$h_t+2kh-2ll_t-4kl^2 = 0, \tag{5.14h}$$

$$h_{tt}+4kh_t-2gl_t-4gkl = 0. \tag{5.14i}$$

From equation (5.14a),

$$k = \frac{1}{2t+c_1}, \tag{5.15}$$

where c_1 is a constant, unless k=0. We therefore split the analysis into two cases:(1) k=0 and (2) $k=\dfrac{1}{2t+c_1}$.

Case 1: Equations (5.14a,b,c) vanish and from equations (4.14d,e,h) we obtain

$$l=c_1t+c_2,$$

$$g=c_1,$$

$$h=c_1^2t^2+2c_1c_2t+c_3,$$

respectively, where c_1, c_2 and c_3 are constants. Equations (5.14f,g,i) are also satisfied by these forms of k, l, g and h. Equations (5.12) and (5.13) become

$$X=c_1t+c_2, \tag{5.16a}$$

$$U=c_1x+c_1^2t^2+2c_1c_2t+c_3. \tag{5.16b}$$

Using (5.16a), (5.16b) and that T=1, equations (4.4) become

$$\frac{dx}{c_1t+c_2} = \frac{dt}{1} = \frac{du}{c_1x+c_1^2t^2+2c_1c_2t+c_3}$$

and therefore the similarity transformation is given by

$$u=c_1xt+\tfrac{1}{2}c_1c_2t^2+c_3t+F(\eta), \tag{5.17a}$$

where the similarity variable is given by

$$\eta = x-\tfrac{1}{2}c_1t^2-c_2t. \tag{5.17b}$$

The transformation (5.17a) reduces the partial differential equation (5.2) to the ordinary differential equation

$$\frac{d^4F}{d\eta^4}-\tfrac{1}{2}\left(\frac{dF}{d\eta}\right)^2\frac{d^2F}{d\eta^2}+c_2\frac{dF}{d\eta}\frac{d^2F}{d\eta^2}-\left(c_1\eta-c_2^2+c_3\right)\frac{d^2F}{d\eta^2}-c_1\frac{dF}{d\eta}+c_1c_2=0 \tag{5.17c}$$

which upon integration becomes

$$\frac{d^3F}{d\eta^3} - \tfrac{1}{6}\left(\frac{dF}{d\eta}\right)^3 + \tfrac{1}{2}c_2\left(\frac{dF}{d\eta}\right)^2 + \left(c_2^2 - c_3 - c_1\eta\right)\frac{dF}{d\eta} + c_1c_2\eta + c_4 = 0, \qquad (5.17d)$$

where c_4 is a constant of integration.

Case 2: Using (5.15), equations (5.14d,e) can be solved to give

$$l = \frac{c_2t^2 + c_1c_2t + c_3}{2t + c_1}, \qquad (5.18)$$

$$g = c_2, \qquad (5.19)$$

respectively. Finally, from equation (5.14f),

$$h = [8c_2t^4 + 12c_1c_2^2t^3 + 6c_1^2c_2^2t^2 + 24c_2c_3t^2 +$$

$$24c_1c_2c_3t - 6c_4t + 6c_1^2c_2c_3 - 3c_1c_4]/[6(2t + c_1)^2]. \qquad (5.20)$$

All nine equations (5.14) are satisfied by (5.15), (5.18), (5.19) and (5.20). Hence, equations (5.12) and (5.13) become

$$X = \frac{x + c_2t^2 + c_1c_2t + c_3}{2t + c_1}, \qquad (5.21a)$$

$$U = c_2x + h, \qquad (5.21b)$$

where h is given by (5.20). Using (5.21a) and (5.21b) we have

$$\frac{(2t + c_1)dx}{x + c_2t^2 + c_1c_2t + c_3} = \frac{dt}{1} = \frac{du}{c_2x + h},$$

which gives the desired similarity transformation

$$u = \tfrac{1}{3}c_2x(2t + c_1) + \tfrac{1}{12}(4c_2c_3 - c_1^2c_2^2)(2t + c_1) +$$

$$\tfrac{1}{12}(c_1^3c_2^2 - 3c_4)\ln(2t + c_1) - \frac{1}{36}c_1^3c_2^2 + F(\eta), \qquad (5.22a)$$

where the similarity variable is given by

$$\eta = x(2t + c_1)^{-\frac{1}{2}} - \frac{1}{12}c_2(2t + c_1)^{\frac{3}{2}} + \tfrac{1}{4}(4c_3 - c_1^2c_2)(2t + c_1)^{-\frac{1}{2}}. \qquad (5.22b)$$

The transformation (5.22a) reduces the modified Boussinesq equation to

$$\frac{d^4F}{d\eta^4} - \tfrac{1}{2}\left(\frac{dF}{d\eta}\right)^2\frac{d^2F}{d\eta^2} + \eta\frac{d^2F}{d\eta^2}\frac{dF}{d\eta} + \eta^2\frac{d^2F}{d\eta^2} - \tfrac{1}{6}(c_1^3c_2^2 - 3c_4)\frac{d^2F}{d\eta^2} +$$

$$3\eta\frac{dF}{d\eta} - \tfrac{1}{3}(c_1^3c_2^2 - 3c_4) = 0. \qquad (5.22c)$$

As we stated earlier, we have chosen here a partial differential equation for which the nonclassical method gives different infinitesimal transformations to the ones obtained by the classical method. In fact, using the latter method, the infinitesimals of (5.2) are found to be

$$X=d_1x+d_2,\quad T=2d_1t+d_3,\quad U=d_4, \tag{5.23}$$

where d_1, d_2, d_3 and d_4 are all constants. Clearly, the infinitesimals in the two above cases differ to the ones given by (5.23).

The associated finite group of transformations of one parameter of the infinitesimal transformation whose infinitesimals are given by (5.16) and $T=1$ (case 1), can be derived by solving the system

$$\frac{dx'}{d\lambda} = c_1x',\quad \frac{dt'}{d\lambda}=1,\quad \frac{du'}{d\lambda} = c_1x'+(c_1t'+c_2)^2,$$

where, we have set $c_3 = c_2^2$. Thus,

$$x'=x+(c_1t+c_2)\lambda+\tfrac{1}{2}c_1\lambda^2, \tag{5.24a}$$

$$t'=t+\lambda, \tag{5.24b}$$

$$u'=u+[c_1x+(c_1t+c_2)^2]\lambda+\tfrac{3}{2}c_1(c_1t+c_2)\lambda^2+\tfrac{1}{2}c_1^2\lambda^3. \tag{5.24c}$$

The one parameter group (5.24) maps the modified Boussinesq equation (5.2) into equation

$$u'_{t't'}-u'_{t'}u'_{x'x'}-\tfrac{1}{2}u'^2_{x'}u'_{x'x'}+u'_{x'x'x'x'}+$$

$$2c_1\lambda[u'_{x't'}+(c_1t'+c_2)u'_{x'x'}-c_1] = 0. \tag{5.25}$$

Therefore, unless $c_1=0$, the group (5.24) does not map solutions of the modified Boussinesq equation into itself, which was expected since (5.24) is a weak symmetry group. A similar result can be derived for case 2.

3. A Direct Method for Determining Similarity Transformations

In this section we present a direct method of deriving similarity transformations of nonlinear partial differential equations, which was first introduced by Clarkson and Kruskal [26]. The unusual characteristic about this method in comparison to the others we have used so far, is that involves no use of transformation group theory. The basic idea is to seek a solution of a given partial differential equation in the form (3.1), which is the most general form for a similarity transformation. Then we require that substitution of (3.1) into the partial differential equation yields an ordinary differential equation with dependent variable F and independent variable η. For most equations it turns out that the similarity transformation takes the form

$$u = A(x,t) + B(x,t) F(\eta(x,t)), \tag{5.26}$$

where A, B and η are to be determined.

The modified Boussinesq equation

$$u_{tt} - u_t u_{xx} - \tfrac{1}{2} u_x^2 u_{xx} + u_{xxxx} = 0, \tag{5.27}$$

is used once more to demonstrate this method. It can be shown [27] that it is sufficient to seek similarity transformations of (5.27) in the form (5.26).

Before we proceed, remarks of Clarkson and Kruskal are restated here with the purpose of making the analysis simpler.

Remark 1. Substitute (5.26) into (5.27) and then demand that the resulting equation is an ordinary differential equation for $F(\eta)$. Hence, it is necessary that the ratios of different derivatives and powers of F are functions of η only. This gives an overdetermined system of equations for A, B and η, whose solutions give the desired similarity transformations.

Remark 2. The coefficient of $F''''(B\eta_x^4)$ is used, provided that $\eta_x \neq 0$, as the normalising coefficient and it is required that the other coefficients are of the form $B\eta_x^4 \Gamma(\eta)$, where Γ is a function to be determined.

Remark 3. Whenever an upper case Greek letter is used to denote a function (e.g. $\Gamma(\eta)$), then this is a function to be determined upon which any mathematical function can be performed (e.g. differentiation, exponentiate, etc) and then also call the resulting function $\Gamma(\eta)$ without loss of generality.

Remark 4. There are three freedoms in the determination of A, B, η which can be exploited, without loss of generality (these are valuable in keeping the method manageable):

(a). If $A(x,t)$ is of the form $A = A_0(x,t) + B(x,t)\Gamma(\eta)$, then it is assumed that $\Gamma = 0$ (make the transformation $F \to F - \Gamma$).

(b). If $B(x,t)$ is of the form $B = B_0(x,t)\Gamma(\eta)$ then it can be assumed that $\Gamma = 1$ (make the transformation $F \to F/\Gamma$).

(c). If $\eta(x,t)$ is defined by an equation of the form $\Gamma(\eta) = \eta_0(x,t)$, where Γ is any invertible function, then it can be assumed that $\Gamma = \eta$ (make the transformation

$\eta \rightarrow \Gamma^{-1}(\eta))$.

Substituting (5.26) into equation (5.27) yields

$$B\eta_x^4F'''' + [6B\eta_x^2\eta_{xx} + 4B_x\eta_x^3]F''' + [B(3\eta_{xx}^2 + 4\eta_x\eta_{xxx}) +$$

$$12B_x\eta_x\eta_{xx} + 6B_{xx}\eta_x^2 - \tfrac{1}{2}A_xB\eta_x^2 - A_tB\eta_x^2 + B\eta_t^2]F'' +$$

$$[B\eta_{xxxx} + 4B_x\eta_{xxx} + 6B_{xx}\eta_{xx} + 4B_{xxx}\eta_x - \tfrac{1}{2}A_x^2(2B_x\eta_x + B\eta_{xx}) -$$

$$A_xA_{xx}B\eta_x - A_t(2B_x\eta_x + B\eta_{xx}) - A_{xx}B\eta_t + B_t\eta_t + B\eta_{tt}]F' +$$

$$[B_{xxxx} - A_xA_{xx}B_x - \tfrac{1}{2}A_x^2B_{xx} - A_tB_{xx} - A_{xx}B_t + B_{tt}]F -$$

$$\tfrac{1}{2}[B\eta_x^2F'' + (2B_x\eta_x + B\eta_{xx})F' + B_{xx}F + A_{xx}][B^2\eta_x^2F'^2 + 2BB_x\eta_xFF' + B_x^2F^2] +$$

$$[B\eta_x^2F'' + (2B_x\eta_x + B\eta_{xx})F' + B_{xx}F][B_tF + B\eta_tF'] +$$

$$A_{tt} - A_tA_{xx} - \tfrac{1}{2}A_x^2A_{xx} + A_{xxxx} = 0, \qquad (5.28)$$

where $F' = \dfrac{dF}{d\eta}$, $F'' = \dfrac{d^2F}{d\eta^2}$ and so on.

Using Remark 1, coefficient of F'''' and $F''F'^2$ give the constraint

$$B\eta_x^4\Gamma(\eta) = B^3\eta_x^4. \qquad (5.29)$$

Hence, using Remark 4(b), equation (5.29) implies that

$$B = 1. \qquad (5.30)$$

The coefficient of F''' yields the constraint

$$B\eta_x^4\Gamma(\eta) = 4B_x\eta_x^3 + 6B\eta_x^2\eta_{xx},$$

which becomes

$$\eta_x\Gamma(\eta) + \frac{\eta_{xx}}{\eta_x} = 0, \qquad (5.31)$$

using (5.30) and rescaling Γ. Integrating (5.31) with respect to x gives

$$\Gamma(\eta) + \ln\eta_x = \Theta(t), \qquad (5.32)$$

where Θ is a function of integration. Using Remark 3, equation (5.32) now reads

$$\eta_x\Gamma(\eta) = \Theta(t),$$

which upon integration with respect to x becomes

$$\Gamma(\eta) = x\Theta(t) + \Phi(t), \tag{5.33}$$

where Φ is another function of integration. Using Remark 4(c), we have

$$\eta = xk(t) + l(t), \tag{5.34}$$

where k and l are functions to be determined.

The coefficient of $F'F''$ gives

$$B\eta_x^4\Gamma(\eta) = B^2\eta_x^2(\eta_t + A_x\eta_x). \tag{5.35}$$

Using (5.30) and (5.34), equation (5.35) becomes

$$k\Gamma(\eta) = \frac{1}{k}\left(x\frac{dk}{dt} + \frac{dl}{dt}\right) + A_x,$$

which upon integration with respect to x becomes

$$A = \Gamma(\eta) - \frac{1}{2k}\left(x^2\frac{dk}{dt} + 2x\frac{dl}{dt}\right) + h(t),$$

where $h(t)$ is a function of integration to be determined. Using Remark 4(a), we have

$$A = -\frac{1}{2k}\left(x^2\frac{dk}{dt} + 2x\frac{dl}{dt}\right) + h(t). \tag{5.36}$$

Using equations (5.30), (5.34) and (5.36), equation (5.28) simplifies to

$$k^4F'''' + \tfrac{1}{2}\left(x^2k\frac{d^2k}{dt^2} + 2xk\frac{d^2l}{dt^2} + \left(\frac{dk}{dt}\right)^2 - 2k^2\frac{dh}{dt}\right)F'' +$$

$$\left(x\frac{d^2k}{dt^2} + \frac{d^2l}{dt^2}\right)F' - \tfrac{1}{2}k^4F'^2F'' + \tfrac{1}{2}k\frac{dk}{dt}F'^2 +$$

$$A_{tt} - A_tA_{xx} - \tfrac{1}{2}A_x^2A_{xx} = 0. \tag{5.37}$$

Demanding that (5.37) is an ordinary differential equation for $F(\eta)$, we must have

$$k^3\Gamma_1(\eta) = \frac{dk}{dt}, \tag{5.38a}$$

$$k^4\Gamma_2(\eta) = x\frac{d^2k}{dt^2} + \frac{d^2l}{dt^2}, \tag{5.38b}$$

$$k^4\Gamma_3(\eta) = x^2k\frac{d^2k}{dt^2} + 2xk\frac{d^2l}{dt^2} + \left(\frac{dl}{dt}\right)^2 - 2k^2\frac{dh}{dt}, \tag{5.38c}$$

$$k^4\Gamma_4(\eta) = A_{tt} - A_tA_{xx} - \tfrac{1}{2}A_x^2A_{xx}, \tag{5.38d}$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are functions to be determined. From (5.38a),

$$\Gamma_1 = \frac{1}{k^3}\frac{\mathrm{d}k}{\mathrm{d}t}.$$

Since the right hand side is a function of t, so must be the left hand side. But η is a function of x and t and therefore $\Gamma_1 = c_1$, where c_1 is a constant. Hence, equation (5.38a) becomes

$$\frac{\mathrm{d}k}{\mathrm{d}t} = c_1 k^3. \tag{5.39}$$

If now we use (5.39), equation (5.38b) becomes

$$\Gamma_2 = 3xc_1^2 k + \frac{1}{k^2}\frac{\mathrm{d}^2 l}{\mathrm{d}t^2}. \tag{5.40}$$

Differentiating (5.40) with respect to x and then integrate with respect to η we obtain

$$\Gamma_2 = 3c_1^2\eta + c_2, \tag{5.41}$$

where c_2 is a constant of integration. Using (5.40) and (5.41) we deduce that

$$\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} = (3c_1^2 l + c_2)k^4. \tag{5.42}$$

Similarly from (5.38c) we obtain

$$\Gamma_3 = 3c_1^2\eta^2 + 2c_2\eta + c_3 \tag{5.43}$$

and

$$\left(\frac{\mathrm{d}l}{\mathrm{d}t}\right)^2 - 2k^2\frac{\mathrm{d}h}{\mathrm{d}t} = k^4(3c_1^2 l^2 + 2c_2 l + c_3), \tag{5.44}$$

where c_3 is another constant of integration. Finally, using (5.37) and (5.38d) we obtain

$$A = -\tfrac{1}{2}c_1 x^2 k^2 - \frac{x}{k}\frac{\mathrm{d}l}{\mathrm{d}t} + h, \tag{5.45}$$

$$\Gamma_4 = -\tfrac{9}{2}c_1^3\eta^2 - 3c_1 c_2\eta - \tfrac{3}{2}c_1 c_3, \tag{5.46}$$

respectively.

Hence, the similarity transformation (5.26) becomes

$$u = -\tfrac{1}{2}c_1 x^2 k^2 - \frac{x}{k}\frac{\mathrm{d}l}{\mathrm{d}t} + h + F(\eta), \tag{5.47}$$

where the similarity variable η, is given by (5.34) and the functions k, l and h are given by (5.39), (5.42) and (5.44) respectively. Also the ordinary differential equation (5.37) now reads

$$G''' - \tfrac{1}{2}G^2G' + \tfrac{1}{2}c_1G^2 + (3c_1^2\eta + c_2)G + \tfrac{1}{2}(3c_1^2\eta^2 + 2c_2\eta + c_3)G' -$$

$$(\tfrac{9}{2}c_1^3\eta^2 + 3c_1c_2\eta + \tfrac{3}{2}c_1c_3) = 0, \tag{5.48}$$

where $G=F'$.

We consider two cases: (1) $c_1 = 0$ and (2) $c_1 \neq 0$.

Case 1: From (5.39), k is a constant and without loss of generality

$$k=1. \tag{5.49}$$

Using (5.49), equations (5.42) and (5.44) give respectively,

$$l=\tfrac{1}{2}c_2t^2 + c_4t + c_5, \tag{5.50}$$

$$h=\tfrac{1}{2}(c_4^2 - 2c_1c_5 - c_3)t + c_6, \tag{5.51}$$

where c_4, c_5 and c_6 are constants of integration. Using (5.49) - (5.51), the similarity variable and the similarity transformation (5.47) become

$$\eta = x + \tfrac{1}{2}c_2t^2 + c_4t + c_5, \tag{5.52a}$$

$$u=-x(c_2t+c_4) + \tfrac{1}{2}(c_4^2 - 2c_2c_5 - c_3)t + c_6 + F(\eta) \tag{5.52b}$$

and the ordinary differential equation (5.48) now reads

$$G''' - \tfrac{1}{2}G^2G' + c_2G + c_2\eta G' + \tfrac{1}{2}c_3G' = 0 \tag{5.52c}$$

which upon integration becomes

$$G'' - \tfrac{1}{6}G^3 + c_2\eta G + \tfrac{1}{2}c_3G = c_7, \tag{5.52d}$$

where c_7 is another constant of integration.

Case 2: Without loss of generality we set $c_1 = -\tfrac{1}{2}$ and $c_2 = 0$. Solving equations (5.39), (5.42) and (5.44) we obtain

$$k=(t+c_4)^{-\frac{1}{2}}, \tag{5.53}$$

$$l=c_5(t+c_4)^{\frac{3}{2}} + c_6(t+c_4)^{-\frac{1}{2}}, \tag{5.54}$$

$$h=\tfrac{1}{4}c_5^2(t+c_4)^3+\tfrac{1}{4}c_6^2(t+c_4)^{-1}-\frac{c_3}{2}\ln(t+c_4), \qquad (5.55)$$

respectively. Using (5.53) - (5.55), the similarity variable and the similarity transformation (5.47) now read

$$\eta = x(t+c_4)^{-\frac{1}{2}}+c_5(t+c_4)^{\frac{3}{2}}+c_6(t+c_4)^{-\frac{1}{2}}, \qquad (5.56a)$$

$$u = \tfrac{1}{4}x^2(t+c_4)^{-1}-\tfrac{3}{2}c_5x(t+c_4)+\tfrac{1}{2}c_6x(t+c_4)^{-1}+$$

$$\tfrac{1}{4}c_5^2(t+c_4)^3+\tfrac{1}{4}c_6^2(t+c_4)^{-1}-\frac{c_3}{2}\ln(t+c_4)+F(\eta) \qquad (5.56b)$$

and the equation (5.48) becomes

$$G'''-\tfrac{1}{2}G^2G'-\tfrac{1}{4}G^2+\tfrac{3}{4}\eta G+\tfrac{1}{2}(\tfrac{3}{4}\eta^2+c_3)G'+\tfrac{9}{16}\eta^2+\tfrac{3}{4}c_3 = 0. \qquad (5.56c)$$

If we make the transformation $\eta \rightarrow \frac{1}{\sqrt{2}}\eta_1$ and $F \rightarrow \tfrac{1}{4}\eta_1^2+\tfrac{1}{2}c_4\ln 2+F_1$ to the similarity transformation (5.22) and redefine the constants, it is straightforward to show that the similarity transformations (5.22) and (5.56) are identical. Similarly, if the constants in (5.17) are redefined, it can be shown that the similarity transformations (5.17) and (5.52) are the same. This confirms the earlier statement that the method described in this section and the nonclassical method presented in the previous section, give the same results.

For exact solutions of the ordinary differential equations (5.52d) and (5.56c) one can refer to [27], where more references are also given.

CHAPTER SIX

ON POINT TRANSFORMATIONS OF A GENERALISED BURGERS EQUATION

1. Introduction

Existence of groups has natural advantage in the study of nonlinear partial differential equations. However all invariant transformations are important. In this chapter we consider the problem of classifying all point transformations of the form (2.19) which relate two equations belonging to a given class of equations. Such transformations would necessarily include all the invariant transformations and in addition they might include transformations that cannot be found using transformation groups. A survey of methods for determining transformation groups has already been examined in the previous chapters.

Here we will carry out this classification when the two equations to be related are both of the class known as the generalised Burgers equation

$$u_t + uu_x + a(x,t)\,u_{xx} = 0. \tag{6.1}$$

When a is a function of t only, equation (6.1) is well established [30] as a model nonlinear equation for the study of acoustics and shock waves. It also arises as an alternative form for the ordinary Burgers equation [19] in the cases of cylindrical and spherical symmetry in which the function a is respectively proportional to t and e^t [59,85]. Cates [22] discovered a point transformation which led to new exact solutions of equation (6.1). More recently Doyle and Englefield [32] explored the range of functions $a(t)$ for which infinitesimal invariant transformations of (6.1) exist.

This chapter is based on research work carried out by Kingston and Sophocleous [52]. In section 2, we derive all finite point transformations relating any two generalised Burgers equations. In section 3 the one-parameter transformation groups, which are included in the finite point transformations are given. We also state the corresponding infinitesimal transformations. In addition to the finite point transformation of Cates, a reciprocal point transformation is found which is presented in section 4.

2. Classification of Point Transformations of a Generalised Burgers Equation

We consider the point transformation (2.19) which relates the equation (6.1) to the equation

$$u'_{t'} + u' u'_{x'} + b(x',t') u'_{x'x'} = 0, \tag{6.2}$$

where we assume that $a(x,t)$ and $b(x',t')$ are nonzero. In view of the Theorems 3.1 and 3.2 transformation (2.19) now reads

$$x' = P(x,t), \tag{6.3a}$$

$$t' = Q(t), \tag{6.3b}$$

$$u' = R(x,t,u). \tag{6.3c}$$

For non-degenerate transformations, $J \neq 0$ [see equation (2.22)] and $\delta \neq 0$ [see equation (2.23)]. To achieve this we must take $Q_t \neq 0$, $P_x \neq 0$ and $R_u \neq 0$. Let us now work out the partial derivatives $u'_{t'}$, $u'_{x'}$ and $u'_{x'x'}$ in terms of the u-derivatives using the transformations (6.3). From (2.23),

$$\delta = P_x Q_t. \tag{6.4}$$

Using identity (2.28) we find that

$$u'_{x'} = \frac{R_\chi Q_\tau}{\delta}.$$

Hence, using (6.4), (2.25) and (2.26)

$$u'_{x'} = \frac{R_x + R_u u_x}{P_x}. \tag{6.5}$$

Similarly, using (2.27)

$$u'_{t'} = \frac{P_x R_t + P_x R_u u_t - P_t R_x - P_t R_u u_x}{P_x Q_t}. \tag{6.6}$$

Finally, setting $\Psi = u'_{x'}$ in (2.27), it is straightforward to work out that

$$u'_{x'x'} = \frac{P_x R_{xx} + 2P_x R_{xu} u_x + P_x R_{uu} u_x^2 + P_x R_u u_{xx} - P_{xx} R_x - P_{xx} R_u u_x}{P_x^3}. \tag{6.7}$$

Upon substitution of (6.5) - (6.7) into equation (6.2) and eliminating u_{xx} from equation (6.1), we obtain

$$E(x,t,u,u_x,u_t) = 0, \tag{6.8}$$

for some function E which may be explicitly determined in terms of R and the derivatives of P, Q, R and u. We now insist that equation (6.8) depends only on x, t, u, u_x and u_t, with these five variables treated as being independent.

Multiplying through by $aQ_tP_x^3$, then the coefficient of u_x^2, u_t, u_x and the term independent of u-derivatives give the following four identities:

$$abP_xQ_tR_{uu} = 0, \tag{6.9}$$

$$aP_x^3R_u - buP_xQ_tR_u = 0, \tag{6.10}$$

$$-aP_x^2P_tR_u + aP_x^2Q_tRR_u + 2abP_xQ_tR_{xu} - abP_{xx}Q_tR_u - buP_xQ_tR_u = 0, \tag{6.11}$$

$$a(P_x^3R_t - P_x^2P_tR_x + P_x^2Q_tRR_x + bP_xQ_tR_{xx} - bP_{xx}Q_tR_x) = 0. \tag{6.12}$$

These four identities enable us to derive the functional forms of P, Q and R and consequently, the desired point transformations are obtained.

Since $a \neq 0$, $b \neq 0$, $P_x \neq 0$ and $Q_t \neq 0$, from equation (6.9) we deduce that

$$R = A(x,t)u + B(x,t), \tag{6.13}$$

where $A \neq 0$ for non-degenerate point transformations. Now equation (6.10) yields

$$b = a\frac{P_x^2}{Q_t}. \tag{6.14}$$

Using (6.15) and (6.14), coefficient of u^2 in identity (6.12) is $AA_xP_x^2Q_t$. Hence,

$$A = A(t). \tag{6.15}$$

Coefficient of u in (6.11) is $aAP_x^2(AQ_t - P_x)$. Thus,

$$P = xAQ_t + k(t), \tag{6.16}$$

where k is a function of t. Coefficient of u in (6.12) is $B_xAP_x^2Q_t + P_x^3A_t$. Therefore using (6.16) we have

$$B = -xA_t + l(t), \tag{6.17}$$

where l is also a function of t. Using (6.16) and (6.17) identity (6.11) becomes

$$2A_tQ_tx + AQ_{tt}x + k_t - lQ_t = 0.$$

Equating coefficients of powers of x, it follows that

$$l = \frac{k_t}{Q_t}, \tag{6.18}$$

$$A = \frac{c_1}{Q_t^{\frac{1}{2}}}, \tag{6.19}$$

where c_1 is a nonzero constant. Using (6.16) and (6.17) the term independent of x in equation (6.12) becomes $AQ_t l_t$. Hence, $l = c_2$ where c_2 is a constant. Now it follows from (6.18) that

$$k = c_2 Q + c_3, \tag{6.20}$$

where c_3 is another constant. Using equations (6.16) - (6.20) identity (6.12) [or equation (6.8)] now reads

$$\tfrac{1}{4} x c_1^2 Q_t^{-2}(2Q_{ttt}Q_t - 3Q_{tt}^2) = 0.$$

Therefore the above equation is satisfied if the terms in the brackets vanish. That is, if $Q(t)$ satisfies the ordinary differential equation

$$2Q_{ttt}Q_t - 3Q_{tt}^2 = 0. \tag{6.21}$$

Now, using (6.13) - (6.20) the forms of P and R are given respectively by

$$P = c_1 x Q_t^{\frac{1}{2}} + c_2 Q + c_3, \tag{6.22}$$

$$R = c_1 Q_t^{-\frac{1}{2}} u + \tfrac{1}{2} c_1 x Q_{tt} Q_t^{-\frac{3}{2}} + c_2, \tag{6.23}$$

where Q satisfies equation (6.21).

We consider two cases:(1) $Q_{tt} = 0$ and (2) $Q_{tt} \neq 0$.

Case 1: $Q = c_4^2 t + c_5$, where c_4 and c_5 are constants and the square of c_4 is used to avoid square roots in the forms of P and R. Therefore using this form of Q, equation (6.22) and equation (6.23), the desired point transformation is given by

$$x' = c_1 c_4 x + c_2 c_4^2 t + c_3, \tag{6.24a}$$

$$t' = c_4^2 t + c_5, \tag{6.24b}$$

$$u' = \frac{c_1}{c_4} u + c_2, \tag{6.24c}$$

where in (6.24a), $c_2 c_5$ has been absorbed in c_3.

Case 2: Solving the ordinary differential equation (6.21) gives

$$Q = c_4 + \frac{1}{c_5 - c_6^2 t}.$$

Hence, the point transformation is given by

$$x' = \frac{c_1 c_6 x - c_2}{c_6^2 t - c_5} + c_3, \tag{6.25a}$$

$$t' = c_4 - \frac{1}{c_6^2 t - c_5}, \tag{6.25b}$$

$$u' = c_1 c_6 (ut - x) - \frac{c_1 c_5}{c_6} u + c_2. \tag{6.25c}$$

In fact transformation (6.25) includes transformation (6.24), the latter being obtained from (6.25) by setting

$$c_2 = (\alpha^2 b_3 + b_2 b_4^2)/\beta,$$

$$c_3 = -b_2 b_4^2/\alpha^2,$$

$$c_1 c_6 = -\alpha^2 b_1 b_4/\beta,$$

$$c_5 = \alpha^2/\beta,$$

$$c_4 = -b_4^2/\alpha^2,$$

$$c_6^2 = \alpha^4/\beta,$$

where $\beta = \alpha^2 b_5 + b_4^2$. Letting $\alpha \to 0$ then leads to (6.24) with b_i replacing c_i. Thus (6.24) contains no additional transformations to those already in (6.25), although its form is more convenient for that specific five-parameter class of transformations.

Note that in both cases, from equation (6.14)

$$b(x', t') = c_1^2 a(x, t). \tag{6.26}$$

Hence, for invariance of equation (6.1) it is necessary for the function a to satisfy the functional equation

$$a(x', t') = c_1^2 a(x, t). \tag{6.27}$$

3. Transformation Groups

The five-parameter class of point transformation (6.24) contains five one-parameter groups of finite transformations which are associated with five independent infinitesimal transformations. Setting $c_1 = c_4 = 1$ and $c_2 = c_4 = 0$, transformation (6.24) becomes

(i) $$x'=x+c_3,\quad t'=t,\quad u'=u. \tag{6.28a}$$

Let $c_1 = c_4 = 1$ and $c_2 = c_3 = 0$ in (6.24) to give

(ii) $$x'=x,\quad t'=t+c_5,\quad u' = u. \tag{6.28b}$$

Similarly the following three one-parameter groups can be obtained:

(iii) $$x'=x+c_2t,\quad t'=t,\quad u'=u+c_2, \tag{6.28c}$$

(iv) $$x'=c_4x,\quad t'=c_4^2t,\quad u' = \frac{u}{c_4}, \tag{6.28d}$$

(v) $$x'=c_1x,\quad t'=t,\quad u'=c_1u. \tag{6.28e}$$

From (6.26), in (6.28a) - (6.28d) $b=a$ and in (6.28e) $b = c_1^2a$. A sixth one-parameter group of finite transformations corresponding to a sixth independent infinitesimal transformation may be obtained from (6.25) by choosing $c_2 = c_3 = 0$, $c_1 = 1$, $c_5 = -c_6$, $c_4 = 1/c_6$. Then (6.25) gives

(vi) $$x' = \frac{x}{1+c_6t},\quad t'=\frac{t}{1+c_6t},\quad u'=u+c_6(ut-x). \tag{6.28f}$$

Here $b=a$. The transformation (6.25), of course, also includes (6.28a) - (6.28e)

We now proceed to construct the infinitesimal transformations corresponding to the finite transformations (6.28). Setting $c_4=1+\varepsilon$ in (6.28d) we have

$$x'=x+\varepsilon x,\quad t'=t+2\varepsilon t+o(\varepsilon^2),\quad u'=u-\varepsilon u+o(\varepsilon^2)$$

which is the infinitesimal transformation generated by the one-parameter finite group (6.28d). Let $c_6 = -\varepsilon$ in (6.28f) to give

$$x' = x+\varepsilon xt+o(\varepsilon^2),\quad t'=t+\varepsilon t^2+o(\varepsilon^2),\quad u'=u+\varepsilon(x-ut)+o(\varepsilon^2)$$

Similarly, we can generate the infinitesimal transformations for the other four one-parameter groups (6.28a,b,c,e). Therefore the six linearly independent infinitesimal transformations which generate the six groups (6.28) have generators with components

$$X=1,\quad T=0,\quad U=0, \tag{6.29a}$$

$$X=0,\quad T=1,\quad U=0, \tag{6.29b}$$

$$X=t, \quad T=0, \quad U=1, \tag{6.29c}$$

$$X=x, \quad T=2t, \quad U=-u, \tag{6.29d}$$

$$X=x, \quad T=0, \quad U=u, \tag{6.29e}$$

$$X=xt, \quad T=t^2, \quad U=x-ut, \tag{6.29f}$$

respectively. Any infinitesimal transformation must have a generator which is a linear combination of these six.

In fact using the classical method for determining infinitesimal transformations described in chapter 4, it can be shown that the equation (6.1) is invariant under the infinitesimal transformation

$$x'=x+\varepsilon(d_6xt+d_1'x+d_4'x+d_2t+d_3)+o(\varepsilon^2), \tag{6.30a}$$

$$t'=t+\varepsilon(d_6t^2+d_4'+d_5)+o(\varepsilon^2), \tag{6.30b}$$

$$u' = u+\varepsilon(d_6(x-ut)+d_1'u+d_2)+o(\varepsilon^2), \tag{6.30c}$$

where d_1', d_2, d_3, d_4', d_5 and d_6 are constants and the function $a(x,t)$ satisfies the first order linear partial differential equation

$$Xa_x+Ta_t = (2d_1'+d_4')a, \tag{6.31}$$

where X and T are the infinitesimals of (6.30a) and (6.30b) respectively. Let $d_4' = 2d_4$ and $d_1' = d_1-d_4$, so that (6.30) and (6.31) now read

$$x'=x+\varepsilon(d_6xt+d_1x+d_4t+d_2t+d_3)+o(\varepsilon^2), \tag{6.32a}$$

$$t'=t+\varepsilon(d_6t^2+2d_4t+d_5)+o(\varepsilon^2), \tag{6.32b}$$

$$u'=u+\varepsilon[d_6(x-ut)+d_1u-d_4u+d_2]+o(\varepsilon^2) \tag{6.32c}$$

and

$$Xa_x+Ta_t = 2d_1a, \tag{6.33}$$

respectively. Clearly, multiplying each component in (6.29) by a different constant and then adding them all up, gives the infinitesimals in (6.32).

For (6.28a) - (6.28f) to leave (6.1) invariant the functional form of $a(x,t)$ must be restricted as in (6.27). For example, (i) is an invariant transformation if and only if $a(x+c_3,t)=a(x,t)$, for all x, t, c_3. Hence, a must be a function of t only, f(t) say. For (ii), $a(x,t+c_5)=a(x,t)$. Therefore $a=f(x)$. Similarly, cases (iii) - (vi) leave equation

(6.1) invariant if and only if $a=f(t)$, $f(t/x^2)$, $x^2f(t)$ and $f(x/t)$ respectively.

Alternatively, another way of finding these functional forms of $a(x,t)$ is to use the corresponding infinitesimals for each group (i) - (vi) [equations (6.29)] and the equation (6.33). For example, for (iv) $X=x$, $T=2t$, $d_1 = 0$. Hence, solving equation (6.33) with the method of characteristics we have

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{da}{0}$$

from which it follows that $a=f(t/x^2)$. For (v), $X=x$, $T=0$, $d_1 = 1$. From equation (6.33)

$$\frac{dx}{x} = \frac{dt}{0} = \frac{da}{2a},$$

which gives the solution $a=x^2f(t)$. For (vi)

$$\frac{dx}{xt} = \frac{dt}{t^2} = \frac{da}{0}$$

which gives $a=f(x/t)$. Similarly for cases (i)-(iii) the forms that obtained earlier can be verified using (6.29) and (6.33).

Other one-parameter groups may be constructed from the basis of the infinitesimal transformations (6.29). Doyle and Englefield [32] analysed those for which $a(x,t)$ was a function of t only.

The point transformation of Cates [22] lies on the orbit through (x, t, u) defined by (vi), which is the path of the point (x', t', u') in $\mathbb{R}^3$ as c_6 varies in $\mathbb{R}$. Cates transformation corresponds to $c_6=1$. The reciprocal point transformation from (x, t, u) to (x',t',u') which is described in the next section differs in that no one parameter group of transformations exists whose orbit passes through both (x, t, u) and (x',t',u').

4. Reciprocal Point Transformation

Making the choice $c_1 = c_6 = i$, $c_2 = c_3 = c_4 = c_5 = 0$ in (6.25) gives the real reciprocal transformation:

$$x' = \frac{x}{t}, \quad t' = \frac{1}{t}, \quad u'=-(ut-x). \tag{6.34}$$

From (6.26), $b=-a$ so that, in particular, when $a=a(t)$, $b(t)=-a(1/t)$.

This transformation is reciprocal in the sense that a double application of (6.34) is the identity transformation. All finite transformations may in fact be obtained by combining this reciprocal transformation with the five-parameter family (6.24).

To show that this transformation does not lie on the orbit of one-parameter group of transformations consider the infinitesimal transformations (6.32). From chapter 2, section 4, we know that in order to find the corresponding finite transformation group one needs to solve the system

$$\frac{dx'}{d\lambda} = d_6x't' + d_1x' + d_4x' + d_2t' + d_3,$$

$$\frac{dt'}{d\lambda} = d_6t'^2 + 2d_4t' + d_5,$$

$$\frac{du'}{d\lambda} = d_6(x'-u't') + d_1u' - d_4u' + d_2.$$

under the initial conditions $x'(x,t,u,\lambda)=x$, $t'(x,t,u,\lambda)=t$ and $u'(x,t,u,\lambda)=u$ when $\lambda=0$, where λ is the parameter of the transformation group. Whatever the values of d_1, d_2, d_3, d_4, d_5 and d_6 it is found impossible to choose λ in such a way that you obtain the reciprocal transformation (6.34).

However it is possible to construct a one-parameter family (not a group) of transformations whose orbit connects (x,t,u) and (x',t',u'). Indeed a continuous transformation exists connecting (x,t,u) to any (x',t',u') given by the transformation (6.25). To see this consider the members of (6.25) corresponding to choosing $c_i = b_i$, $i=1,...,6$, which transform (x,t,u) to (x_1,t_1,u_1) say. Now consider the one-parameter (α) family of transformations given by (6.25) when $c_i = c_i(\alpha)$, where

$$c_2 = \alpha^2 b_2, \quad c_3 = \alpha^2 b_3, \quad c_1c_6 = \alpha(\alpha - 1 + \alpha b_1 b_6),$$

$$c_6^2 = \alpha^2(1-\alpha+\alpha b_6^2), \quad c_5 = \alpha(1-\alpha^2+\alpha^2 b_5),$$

$$c_4 = (\alpha^2 - 1 + \alpha^2 b_4)/\alpha.$$

It is straightforward to check that, as α varies continuously between 0 and 1, (x',t',u') varies continuously between (x,t,u) and (x_1,t_1,u_1).

For equation (6.1) to be invariant under the reciprocal transformation (6.34), from (6.27) the function $a(x,t)$ must be such that

$$a(x/t, 1/t) = -a(x,t). \tag{6.35}$$

Making the transformations $x \to e^x$ and $t \to e^t$ equation (6.35) becomes

$$-a(e^x, e^t) = a(e^{x-t}, e^{-t}). \tag{6.36}$$

Also making the transformations $x \to x+t/2$ and $t \to t$ we obtain

$$-a(e^{x+t/2}, e^t) = a(e^{x-t/2}, e^{-t}).$$

Hence, $a(e^{x+t/2}, e^t)$ is an odd function for t, with general form

$$a(e^{x+t/2}, e^t) = f(x,t) - f(x,-t), \tag{6.37}$$

where f is arbitrary. Using (6.36) and (6.35), (6.37) now reads

$$a(x,t) = f(x/\sqrt{t}, t) - f(x/\sqrt{t}, -t). \tag{6.38}$$

Therefore (6.1) is invariant under (6.34) if $a(x,t)$ is given by (6.38). For cases of current practical interest, where $a=a(t)$, this invariance condition becomes $a(x,t)=f(t)-f(1/t)$.

Another feature of the reciprocal transformation (6.34) is that if an expression involving u', $u'_{x'}$, $u'_{t'}$, $u'_{x'x'}$ (no explicit x' and t' transforms to $L(u, u_x, u_t, u_{xx})$, then L must be a function of $(u_t + uu_x)/u_{xx}$, and in fact, $(u'_{t'} + u'u'_{x'})/u'_{x'x'} = -(u_t + uu_x)/u_{xx}$. Thus, for example, the equation $u_t + uu_x + a(x,t,u)u_{xx} = 0$ retains its form under the transformation (6.34). The corresponding result when third order spatial derivatives are included is that $L = f((u_t + uu_x)/u_{xx}, u_{xxx}^3/u_{xx}^4)$ so that, for example, $u_t + uu_x = a(x,t,u)(u_{xxx})^{3/4}$ retains its form under (6.34).

PART II

BÄCKLUND TRANSFORMATIONS

CHAPTER SEVEN

INTRODUCTION TO BÄCKLUND TRANSFORMATIONS

1. Introduction

In the following three chapters our discussion relates to a transformation that had its origin in some investigations by Bäcklund [5,6] concerning simultaneous equations of the first order arising in differential geometry. Bäcklund transformation (BT) provides a method of constructing various classes of "equivalent" equations, thereby leading to the integrals of the original equation.

The modern interest in BTs lies in that they may be used in one of two important ways in connection with integral surfaces of certain nonlinear partial differential equations. Thus, invariance under BTs may be used under appropriate circumstances, to generate an infinite sequence of solutions of such equations by a purely algebraic superposition principle. On the other hand, BTs may sometimes be used to link nonlinear equations to canonical forms whose properties are well established. Both kinds of BTs have important applications in mathematical physics and continuum mechanics [68,82] and, accordingly, examples for each will be given in this chapter.

In what follows, we shall denote by $x, y, u, p=u_x, q=u_y$ an element of any surface, and by x', y', u', p', q' an element of any other surface. To connect the two surface elements completely, though not uniquely, it is necessary to have five distinct equations relating the two sets of variables. However, since each set of five variables defines a surface element, they are related by the total differentials

$$du=p\,dx+q\,dy, \quad du'=p'dx'+q'dy'. \tag{7.1}$$

Thus if due regard is paid to equations (7.1), only four independent equations

$$F_n(x,y,u,p,q,x',y',u',p',q') = 0, \quad n=1,2,3,4, \tag{7.2}$$

are required to connect the surface elements. When these correspondences are used, there are certain cases when the variable u (or u') is an integral of an equation of Monge-Ampère form,

$$Rv_{xx}+Sv_{xy}+Tv_{xy}+U(v_{xx}v_{yy}-v_{xy}^2) = V, \tag{7.3}$$

where R, S, T, U and V are functions of x, y, v, v_x and v_y. When the variables u and u' separately satisfy equations of the second order, these equations can be regarded as transformable into one another by the four relations, equations (7.2), usually called a Bäcklund Transformation. As the transformations change, it is important to ascertain the limitations or restrictions upon the allowable equations, other than (7.3). First one must ask whether a given Monge-Ampère equation admits a BT, and second how does one construct such a transformation. These are lengthy questions for the general case and are beyond the scope of this work. Nevertheless, the general treatment is available in the work of Forsyth [ref 38, vol 6, pp 432-454]. For recent descriptions of BTs see, for example, [82]. Herein we shall demonstrate a development of the transformation equations, due to Clairin [23,24].

Probably the simplest Bäcklund case arises when there are two simultaneous equations involving two dependent variables u and u'. We may take the equations in the form

$$f(x,y,u,u',p,p',q,q')=0, \quad g(x,y,u,u',p,p',q,q')=0.$$

When these two equations can be solved algebraically for (say), p and q, under the condition that the Jacobian, $\dfrac{\partial(f,g)}{\partial(p,q)}$, does not vanish identically, we find

$$p=\psi(x,y,u,u',p',q'), \quad q=\phi(x,y,u,u',p',q'). \tag{7.4}$$

The integrability condition $\dfrac{dp}{dy} = \dfrac{dq}{dx}$ generates the relationship

$$\frac{\partial\psi}{\partial y}+\phi\frac{\partial\psi}{\partial u}+q'\frac{\partial\psi}{\partial u'}+s'\frac{\partial\psi}{\partial p'}+t'\frac{\partial\psi}{\partial q'} =$$

$$\frac{\partial\phi}{\partial x}+\psi\frac{\partial\phi}{\partial u}+p'\frac{\partial\phi}{\partial u'}+r'\frac{\partial\phi}{\partial p'}+s'\frac{\partial\phi}{\partial q'}, \tag{7.5}$$

where, with the usual notation, $r'=u'_{xx}$, $s'=u'_{xy}$ and $t'=u'_{yy}$. Equation (7.5) is linear in r', s' and t' and in general depends upon x, y, u, u', p' and q'.

Suppose our initial concern is with two simultaneous equations (7.4). When u occurs in (7.5), we can think of that equation as solved so as to express u in terms of x, y, u', p', q', r', s' and t'. When the value of u so obtained is substituted into the first order equations (7.4), they become two equations of the third order for the determination of u'. From general theory it is known that unless the original equations cannot be solved with respect p and p', or with respect to q and q', they possess common integrals. Consequently, the two third order equations which are satisfied by

u' must be compatible. They must therefore lead to values of u' that involve arbitrary functions. If the original equations are second order, then the equations for u' will be of the fourth order in the proceeding argument.

If the integrability condition (7.5) is free of u, then it becomes a single second order equation for u'. Upon solving this, the u' so obtained is substituted into (7.4), and a quadrature of those equations leads to a value of u containing an arbitrary constant. An exceptional case arises when (7.5) does not contain r', s' and t'. This happens when

$$\frac{\partial \psi}{\partial q'} = 0, \quad \frac{\partial \phi}{\partial p'} = 0, \quad \frac{\partial \psi}{\partial p'} = \frac{\partial \phi}{\partial q'}.$$

Here if u is involved, then u' satisfies two equations of the second order. If u is not present, then u' satisfies a single equation of the first order.

The exceptional case in which $\psi = 0$ and $\phi = 0$ cannot be solved for p and q because the Jacobian is zero is discussed by Forsyth [ref 38, vol 6, p. 452].

In the two succeeding sections the theory of BTs is illustrated with a number of examples. In section 2 we concentrate on BTs which leave a certain partial differential equation invariant, while in section 3 we derive BTs which relate different equations. In the final section a special class of BTs is defined and corresponding examples are given.

2. Examples of Bäcklund Invariance Transformations

We proceed now to give the detailed construction of Bäcklund invariance transformations, known as auto-BTs (or self BTs). The approach which will be used here differs to the one given by Ames [2] (or Anderson and Ibragimov [3]). Ames makes use of the partial differential equation in u' and the integrability condition. In his analysis, many arbitrary functions appear, and educated choices of them simplify the analysis while still leading to the desired results. In our analysis we also use the partial differential equation in u. This enables us to obtain the desired results without making many assumptions. The transformations which we shall derive are not unique.

The sine-Gordon equation. The problem of ultrashort optical pulses, discussed by Lamb [55,56] has the equation

$$u_{xy} = \sin u. \tag{7.6}$$

Also a detailed discussion of equation (7.6) and more applications is given in Barone et al [7]. We shall use the sine-Gordon equation as a vehicle for demonstrating a classical method due to Bäcklund [5,6] and Clairin [23,24] for generating a BT which leaves a given differential equation invariant.

Consider the explicit special case of (7.4)

$$p=\psi(u,u',p',q'), \quad q=\phi(u,u',p',q'),$$

where u is a solution of (7.6) and u' also satisfies the same equation. It is easy to see from the integrability condition (7.5) that the coefficients of (t') and (r') give $\psi_{q'} = 0$ and $\phi_{p'} = 0$, respectively. Hence,

$$p=\psi(u,u',p'), \quad q=\phi(u,u',q'). \tag{7.7}$$

Since u satisfies (7.6) then we must have

$$E_1 = \frac{\partial p}{\partial y} - \sin u = 0 \text{ and} \tag{7.8}$$

$$E_2 = \frac{\partial q}{\partial x} - \sin u = 0. \tag{7.9}$$

Note that the integrability condition, E=0 (say), is given by $E = E_1 - E_2$. Therefore if (7.8) and (7.9) are satisfied then E=0 is also satisfied. Using (7.7), equations (7.8) and (7.9) become

$$E_1 = \phi\psi_u + q'\psi_{u'} + \sin u'\psi_{p'} - \sin u = 0, \tag{7.10}$$

$$E_2 = \psi\phi_u + p'\phi_{u'} + \sin u'\phi_{q'} - \sin u = 0. \tag{7.11}$$

Calculation of derivatives of E_1 and E_2 to the point where u and u' are no longer explicitly present, yields the following equations:

$$E_{1q'} = \phi_{q'}\psi_u + \psi_{u'} = 0, \tag{7.12}$$

$$E_{2p'} = \psi_{p'}\phi_u + \phi_{u'} = 0. \tag{7.13}$$

Equations (7.12) and (7.13) are free of explicit dependence upon u and u', although, of course, solutions of those equations will depend parametrically upon u and u'. Upon integration of equations (7.12) and (7.13), arbitrary functions will appear which are determined by the requirement that equations (7.10) and (7.11) must also satisfied. It is possible to separate the p' and q' dependence of (7.12) and (7.13) and write them as

$$\phi_{q'} = -\frac{\psi_{u'}}{\psi_u} = k(u,u'), \tag{7.14}$$

$$\psi_{p'} = -\frac{\phi_{u'}}{\phi_u} = m(u,u'), \tag{7.15}$$

where we have chosen $\psi_u \neq 0$ and $\phi_u \neq 0$, and we shall only present this case.

From the usual separation argument it follows that the two left hand sides terms of equations (7.14) and (7.15) do not depend on p' and q'. Hence, the introduction of $k(u,u')$ and $m(u,u')$ respectively. Recalling (7.7), integration of (7.14) and (7.15) give respectively

$$\psi = m(u,u')p' + n(u,u'), \tag{7.16}$$

$$\phi = k(u,u')q' + l(u,u'). \tag{7.17}$$

Now using (7.16) and the two right hand side terms of (7.14) we obtain

$$m_{u'} = -km_u, \quad n_{u'} = -kn_u. \tag{7.18a,b}$$

Similarly (7.17) and the two right hand side terms of (7.15) give

$$k_{u'} = -mk_u, \quad l_{u'} = -ml_u. \tag{7.19a,b}$$

Using (7.16) - (7.19) and equating coefficients of powers of p' in (7.10) and powers of q' in (7.11) lead to the following results

$$lm_u = 0, \quad nk_u = 0, \tag{7.20a,b}$$

$$ln_u + m\sin u' - \sin u = 0, \quad nl_u + k\sin u' - \sin u = 0. \tag{7.21a,b}$$

Assuming that $l \neq 0$ and $n \neq 0$ then from (7.20) $m_u = k_u = 0$, and furthermore (7.18a) and (7.19a) imply that $m=$ constant $= \lambda_1$ (say) and $k=$ constant $= \lambda_2$, respectively. Now equations (7.18) and (7.19) have the simple solutions

$$n=n(\xi), \quad l = l(\eta), \tag{7.22}$$

where $\xi = u - \lambda_2 u'$ and $\eta = u - \lambda_1 u'$, and we have assumed that $\lambda_1 \neq \lambda_2$.

Subtract (7.21b) from (7.21a) to give

$$ln_\xi - nl_\eta + (\lambda_1 - \lambda_2)\sin u' = 0. \tag{7.23}$$

Upon differentiation with respect to u equation (7.23) is expressible as

$$\frac{l_{\eta\eta}}{l} = \frac{n_{\xi\xi}}{n} = -\mu^2,$$

where μ is a constant. The classical solutions to this system are

$$l = \alpha_1\cos\mu\eta + \beta_1\sin\mu\eta, \tag{7.24}$$

$$n = \alpha_2\cos\mu\xi + \beta_2\sin\mu\xi. \tag{7.25}$$

In order to be able to satisfy equations (7.21) and (7.23) we need to take $\lambda_1 = 1$ and $\lambda_2 = -1$. When (7.24) and (7.25) are substituted into equation (7.23) and evaluated at $u = 0$ and odd and even functions are equated, one finds

$$\mu = \tfrac{1}{2}, \quad \alpha_1\beta_2 - \alpha_2\beta_1 = 0, \quad \alpha_1\alpha_2 + \beta_1\beta_2 = 4.$$

Furthermore equation (7.21a) [or (7.22b)] gives $\alpha_1 = \alpha_2 = 0$. Taking $\beta_1/2 = 2/\beta_2 = a$, equations (7.16) and (7.17) give the desired BTs

$$p = p' + 2a\sin\tfrac{1}{2}(u+u'), \tag{7.26a}$$

$$q = -q' + \frac{2}{a}\sin\tfrac{1}{2}(u-u'), \tag{7.26b}$$

where both u and u' satisfy the sine-Gordon equation. Hence, the transformation (7.26) is an auto-BT for (7.6).

Let us write (7.26) as

$$u_{1x} = u_{0x} + 2a_1\sin\tfrac{1}{2}(u_1 + u_0), \tag{7.27a}$$

$$u_{1y} = -u_{0y} + \frac{2}{a_1}\sin\tfrac{1}{2}(u_1 - u_0), \tag{7.27b}$$

where u_0 and u_1 are particular solutions of (7.6). Equations (7.27) provide a vehicle for relating pairs of solutions of the sine-Gordon equation. Thus if u_0 is a known solution of the sine-Gordon, another solution u_1 is obtained by solving the pair of first order differential equations (7.27).

By inspection, we see that $u_0 = 0$ is a particular solution of (7.6). Using equations (7.27) it is straightforward to show that

$$u_1 = 4\tan^{-1}[\exp(a_1y + x/a_1)], \tag{7.28}$$

is another particular solution. We can think of equation (7.28) as u_0 in equations (7.27) and generate an additional particular solution by integration. A sequence u_n of solutions is generated by continued repetition of the process.

In Bianchi [9,10] it is shown that a theorem of permutability (today called a nonlinear superposition) exists for the sine-Gordon equation. As we have seen, beginning with solution u_0 of (7.6), we can generate solutions u_1 and u_2 through a_1 and a_2 from (7.27), respectively. Then there is a solution u_3 which is generated from u_1 through a_2 and also from u_2 through a_1. Using the analytic expressions of these solutions, it can be shown that they are related by

$$\tan\left(\frac{u_3-u_0}{4}\right) = \frac{a_1+a_2}{a_1-a_2}\tan\left(\frac{u_1-u_2}{4}\right), \tag{7.29}$$

which is the classical theorem of permutability. The knowledge of three solutions enables one to recursively generate an infinite sequence of particular solutions to the sine-Gordon equation.

Perhaps the nonlinear superposition (7.29) should be expected, since each of the equations (7.26) is transformable into a Riccati equation

$$\frac{dy}{dx}+A(x)y+B(x)y^2 = C(x), \tag{7.30}$$

which is known to have a nonlinear superposition [31]. If y_1, y_2, y_3 and y_4 are particular solutions of (7.30), then these solutions are related by the expression

$$\frac{(y_1-y_3)(y_2-y_4)}{(y_1-y_4)(y_2-y_3)} = \lambda, \tag{7.31}$$

where λ is a constant. Letting $v = \tan\frac{1}{4}(u+u')$ into equation (7.26a) results in

$$v_x+av-\tfrac{1}{2}pv^2 = \tfrac{1}{2}p,$$

a Riccati equation.

The Korteweg-de Vries (KdV) equation. As a second example, but not in detail, an auto-BT of the KdV equation will be given here. A detailed discussion of the KdV equation,

$$v_y+6vv_x+v_{xxx} = 0, \tag{7.32}$$

is given in references [34] and [67]. Upon setting

$$u(x,y)=\int_{-\infty}^{x} v(t,y)dt,$$

equation (7.32) becomes

$$u_y+3u_x^2+u_{xxx} = 0, \tag{7.33}$$

after integration and the discard of an arbitrary function of integration.

Without presenting the detailed computations, Lamb [57] used the following form of BTs

$$p=\psi(u,u',p'), \quad q=\phi(u,u',p',q',r'),$$

to derive the auto-BT for the KdV equation,

$$p=-p'-\tfrac{1}{2}(u-u')^2+\lambda, \quad q=-q'+(u-u')(r-r')-2(p^2+up'+p'^2), \tag{7.34a,b}$$

where λ is an arbitrary constant.

Wahlquist and Estabrook [100] observed that there exists a theorem of permutability (nonlinear superposition) which permits the iterative construction of in infinite sequence of particular solutions for the KdV equation, provided one knows three solutions. More details about this can also be found in reference [3]. Note that setting $v=u-u'$ in (7.34a) results to

$$v_x-\tfrac{1}{2}v^2 = 2p-\lambda,$$

which is a Riccati equation.

3. Examples of Bäcklund Transformations Relating Different Differential Equations

The BTs of the previous section are of a special type, which are called auto (or self) BTs because they transform a given equation into itself. A more general use of the theory involves transformations between equations of different form. Some of these relate solutions of a nonlinear equation to those of a linear equation. Transformations of the Liouville equation $u_{xy} = e^u$ and the Burgers equation $u_y+uu_x = u_{xx}$ are of this nonlinear-linear type. Others relate two nonlinear equations wherein the solutions of the second are more easily obtained. Both processes are very useful. In what follows we shall present examples of both kind.

Liouville equation. The Liouville equation

$$u'_{xy} = e^{u'} \tag{7.35}$$

provides an excellent simple example of an equation for which BT exists relating u' satisfying (7.35) to u satisfying the linear equation

$$u_{xy} = 0. \tag{7.36}$$

As in the sine-Gordon equation, we consider the special case

$$p=\psi(u,u',p'), \quad q=\phi(u,u',q').$$

Since u satisfies (7.36) we require that

$$E_1 = \frac{\partial p}{\partial y} = 0, \quad E_2 = \frac{\partial q}{\partial x} = 0. \tag{7.37}$$

Therefore if $E_1 = 0$ and $E_2 = 0$ are satisfied, then the integrability condition is also satisfied. Recalling the form of the BT, equations (7.37) give

$$E_1 = \phi\psi_u + q'\psi_{u'} + e^{u'}\psi_{p'} = 0, \tag{7.38}$$

$$E_2 = \psi\phi_u + p'\phi_{u'} + e^{u'}\phi_{q'} = 0. \tag{7.39}$$

We note that these equations are the same as those for the sine-Gordon equation [(7.10) and (7.11)], except that $\sin u'$ is replaced by $e^{u'}$ and $\sin u$ by 0. For this reason the same analysis will apply.

Let us pick up the analysis from equation (7.23) with the same terminology and the understanding that $\sin u'$ is replaced by $e^{u'}$ and $\sin u$ by 0 in all previous equations. Upon differentiation with respect to u equation (7.23) is expressible as

$$\frac{l_{\eta\eta}}{l} = \frac{n_{\xi\xi}}{n} = \mu^2, \tag{7.40}$$

where real exponential are anticipated by the choice of sign in (7.40) and μ is a real constant. The classical solutions of (7.40) are

$$l=\alpha_1 e^{\mu\eta} + \beta_1 e^{-\mu\eta}, \tag{7.41}$$

$$n=\alpha_2 e^{\mu\xi} + \beta_2 e^{-\mu\xi}. \tag{7.42}$$

Upon substituting these solutions in (7.21) and (7.23) and requiring that these equations are identically satisfied, we find that

$$\lambda_1=1, \ \lambda_2=-1, \ k=\tfrac{1}{2}, \ \beta_1 = \alpha_2 = 0, \ \alpha_1\beta_2 = -2.$$

Choosing $\alpha_1 = a$ and $\beta_2 = -2/a$, then from (7.16) and (7.17) we obtain the desired BT

$$p = p' + a e^{\frac{1}{2}(u+u')}, \tag{7.43a}$$

$$q = -q' - \frac{2}{a} e^{-\frac{1}{2}(u-u')}, \tag{7.43b}$$

where u' satisfy equation (7.35) and u satisfy equation (7.36). When the general solution of $u_{xy} = 0$, namely $u = F(x) + G(y)$, F and G arbitrary, is inserted into (7.43), the solutions of the resulting simple first order equations readily yield the general solution of the Liouville equation (see chapter 1).

KdV and modified KdV equations. We have seen that the KdV equation

$$u'_y + 6u'u'_x + u'_{xxx} = 0 \tag{7.44}$$

admit an auto-BT. In this section a BT is stated which relates equation (7.44) and the modified KdV equation

$$u_y - 6u^2 u_x + u_{xxx} = 0. \tag{7.45}$$

The detailed computations of how to find this relation between these equations is given by Lamb [57]. He uses BTs of the form

$$p = \psi(u, u'),$$

$$q = \phi(u, u', p', r')$$

to derive the desired BT

$$p = \pm(u' + u^2), \tag{7.46a}$$

$$q = \mp r' - 2(up' + u'p). \tag{7.46b}$$

We note that (7.46a) is a BT by itself of the class $u = F(u', p')$. This class is examined in the next section where detailed examples are also given. BT (7.46a) is known as Miura transformation [66].

Burgers equation. The Burgers equation

$$u'_y + u'u'_x + u'_{xx} = 0 \tag{7.47}$$

arises in turbulence and wave propagation. The BT

$$p = \frac{1}{2}uu', \tag{7.48a}$$

$$q = -\frac{1}{2}(up' + u'p) \tag{7.48b}$$

relates (7.47) to the linear diffusion equation

$$u_y + u_{xx} = 0. \tag{7.49}$$

Equation (7.48a) is another BT of the class $u = F(u', p')$ which is known as Hopf-Cole transformation [29,45].

4. A Special Class of Bäcklund Transformations

In this section our attention will be on a class of BTs which are a special case to the standard BTs given by (7.4). These BTs have the form

$$u = F(u', p', q'), \tag{7.50}$$

where F might depend on r', s' and t' and even higher derivatives depending on the order of partial differential equations considered. We shall demonstrate how this special BT is used, by giving an example.

Consider the Burgers equation

$$u_y = 2uu_x + u_{xx} \tag{7.51}$$

and the heat equation

$$u'_y = u'_{xx} \tag{7.52}$$

Upon differentiating (7.50) with respect to x, it gives

$$u_x = p'F_{u'} + r'F_{p'} + s'F_{q'}$$

and since $r' = q'$ from (7.52) then

$$u_x = p'F_{u'} + q'F_{p'} + s'F_{q'}. \tag{7.53}$$

Similarly,

$$u_y = q'F_{u'} + s'F_{p'} + t'F_{q'}. \tag{7.54}$$

Now using (7.53), u_{xx} can be evaluated. Once u_x, u_y and u_{xx} are calculated using the

BT (7.50), we then substitute these into (7.51) to give

$$u_y - 2uu_x - u_{xx} = E(u', p', q', s') = 0, \tag{7.55}$$

for some function E which may be calculated explicitly in terms of F, its derivatives, u', p', q' and s'. In view of (7.51) E must be identically zero with u', p', q' and s' regarded as independent variables.

The analysis is continued by calculating various derivatives of E. Firstly,

$$E_{s's'} = F_{q'q'} = 0.$$

Consequently,

$$F = A(u', p')q' + B(u', p'), \tag{7.56}$$

for some functions $A(u', p')$ and $B(u', p')$. Using (7.56) and differentiating (7.55) with respect to s', we obtain

$$E_{s'} = 2(p'A_{u'} + q'A_{p'} + q'A^2 + AB) = 0. \tag{7.57}$$

Coefficients of (q') and (q'^0) in (7.57) give two equations in A and B from which one can easily deduce the following solutions

$$A = \frac{1}{p' + c(u')}, \quad B = \frac{p'c_{u'}}{p' + c(u')}, \tag{7.58}$$

where c is an arbitrary function of u'. Finally,

$$E_{q'} = \frac{(2c + p')p'c_{u'u'}}{(c + p')^2} = 0,$$

which implies that

$$c = \lambda_1 u' + \lambda_2, \tag{7.59}$$

where λ_1 and λ_2 are arbitrary constants. This final result satisfies the equation $E=0$. Choosing $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = 1$ and $\lambda_2 = 0$, from (7.56) we obtain the following two separate BTs:

$$u = F = \frac{u'_y}{u'_x}, \tag{7.60}$$

$$u = F = \frac{u'_x + u'_y}{u' + u'_x}, \tag{7.61}$$

which relate the equations (7.51) and (7.52).

If the function F does not depend on q', then the same analysis will lead to the known Hopf-Cole transformation [29,45], namely, $u = u'_x/u'$. We note that this transformation can also be obtained by eliminating u'_y from (7.60) and (7.61). A generalisation of the Hopf-Cole transformation is given by Sachdev [84]. He uses the BT

$$u = k(x,y)\frac{u'_x}{u'}$$

and enquires what nonlinear equations will be transformed to the linear parabolic equation

$$u'_y + bu'_x + cu' + f = au'_{xx},$$

where a, b, c and f are functions of x and y.

Two more examples of transformations of the form (7.50) are the Miura transformation [66] and the Nakamura transformation [70]. The Miura transformation, $u = u'^2 + \sqrt{-6}u'_x$, relates the KdV equation $u_y + u_{xxx} + uu_x = 0$ and the modified KdV equation $u'_y + u'_{xxx} + u'^2u'_x = 0$. Nakamura has shown that the sine-Gordon equation $u_{xy} = \sin u$ and the modified sine-Gordon $u'_{xy} = \sin u'\sqrt{1-(au'_x)^2}$ are related by $u = u' - \sin^{-1}(au'_x)$. Both of these results will be examined in more details in chapter 9.

An example of a BT outside the classes of transformations we have seen [(7.4) and (7.50)] is given by Kingston and Rogers [50], where the BT given by (7.4) is essentially amalgamated with a hodograph-type transformation. This example is beyond our scope.

The analysis which follows in the next two chapters is based on research work carried out by Kingston and Sophocleous [51,92,93]. We have looked for BTs for a family of partial differential equations instead for a single equation. In chapter 8 we explore the range of functions for which the generalised nonlinear Schrödinger Equations $iz_y + z_{xx} + f(z,\bar{z}) = 0$, admit BTs of the general class (7.4) [51]. These transformations all turn out to be auto-BTs. In chapter 9 we search for BTs of the special class (7.50) for equations of the type $u_{xy} = f(u, u_x)$ [92] and $u_y + u_{xxx} + f(u, u_x) = 0$ [93].

CHAPTER EIGHT

BÄCKLUND TRANSFORMATIONS FOR GENERALISED EQUATIONS

1. Introduction

The purpose of the present chapter is to construct BTs of the form (7.4) for a class of partial differential equations instead of a single equation. Nimmo and Crighton [72] considered the nonlinear parabolic equations of the form

$$u_y + u_{xx} + f(x,y,u,u_x) = 0 \tag{8.1}$$

and the BTs of the form (7.4). They investigated to see for what functions $f(x,y,u,u_x)$ equations (8.1) admit BTs. The result was that the only nonlinear equations (8.1) which admit BTs are the slight generalisations of the Burgers equation obtained by adding a forcing term,

$$u_y + u_{xx} + 2uu_x + a(x,y) = 0.$$

Similarly, the above authors [71] derived auto-BTs for generalised KdV equations of the form

$$u_y + u_{xxx} + 6uu_x + f(x,y,u) = 0.$$

McLaughlin and Scott [64] considered the class of equations

$$u_{xy} = f(u). \tag{8.2}$$

They proved the elegant result that the only equations of the form (8.2) admitting any members of a wide class of auto-BTs are those for which the function $f(u)$ satisfies the equation

$$f_{uu} = \lambda f, \tag{8.3}$$

where λ is a constant. We note that both the sine-Gordon and the Liouville equations are of this class of nonlinear partial differential equations

In the spirit of the above work, we shall explore in the next section in detail the range of functions $f(z,\bar{z})$ for which

$$iz_y + z_{xx} + f(z,\bar{z}) = 0 \tag{8.4}$$

admits BTs of a given general class. Here z is complex so that (8.4) may also be regarded as two coupled real partial differential equations. The complex equation (8.4) is a generalisation of the well-known nonlinear Schrödinger equation (NLSE)

$$iz_y + z_{xx} + z^2\bar{z} = 0.$$

Lamb [57] derived BTs for the NLSE using the method of Clairin [23,24]. In the process he made certain simplifying assumptions suggesting that it might be advantageous to look at other cases, although it is seen later that this does not turn out to be the case. Indeed the more general equation (8.4) is here exhaustively investigated with the conclusion that f must be one of the forms $z^2\bar{z}$, $z\ln\bar{z}$, $z\ln z$, $(z+\bar{z})^2$ or suitable combinations of these functions. The logarithmic nonlinearity $z\ln\bar{z}$ occurs in wave equations derived from quantum models (see e.g. ref [18]) and Steudl [95] has discussed invariant transformations of (8.4) when $f(z,\bar{z}) = z\ln z$. The BTs obtained for the case $f(z,\bar{z}) = (z+\bar{z})^2$ lead to auto-BTs for the Boussinesq equation in the form

$$u_{yy} + u_{xxxx} + (4u^2)_{xx} = 0,$$

$u(x,y)$ being the real part of $z(x,y)$. Fukushima et al [39] study a non-linear transmission line and model the circuit by a modified NLSE equivalent to $f(z,\bar{z}) = (z\bar{z})^{\frac{1}{2}}z$ in (8.4). They observe envelope solitons and obtain good agreement between experiment and theory. The results of this chapter show that BTs of a wide class do not exist for this equation.

2. A Class of Bäcklund Transformations for Generalised Nonlinear Schrödinger Equations

We consider BTs of the form

$$p = \psi(z,\bar{z},z',\bar{z}',p',\bar{p}',q',\bar{q}'), \tag{8.5}$$

$$q = \phi(z,\bar{z},z',\bar{z}',p',\bar{p}',q',\bar{q}'), \tag{8.6}$$

which relate the two equations

$$iq + r + f(z,\bar{z}) = 0 \tag{8.7}$$

and

$$iq' + r' + f'(z',\bar{z}') = 0, \tag{8.8}$$

where, with the usual notation, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, and similarly for p', q', r', s', and t' in terms of z'. The only difference of this form of BTs to the one used in chapter 7 (sections 2 and 3) is that ψ and ϕ also depend on the conjugates of their arguments. As long as p and q occur explicitly (8.5) and (8.6) represent the most general pair of complex functional relations connecting the dependent variables z and z' and their derivatives. This is a very wide class of BTs.

Equations (8.5) - (8.8) enable p, q, r, and r' (and their conjugates) to be expressed in terms of z, z', p', q' and their conjugates. We require this system of equations to be consistent in the sense that no new relations between these variables and s', t' and their conjugates are implied. Since p, q and r are connected by

$$r = \frac{\partial p}{\partial x}, \quad \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \tag{8.9}$$

these relations must therefore be identities in z, z', p', q', s', t' and their conjugates. It is these two identities which enable the BTs to be derived and which ultimately impose restrictions on the functional forms of f, f', ψ, and ϕ.

Since the main advantage of BTs is that they enable progress to be made in the study of non-linear equations we will assume that (8.7) is nonlinear and exclude cases in which $f(z,\bar{z})$ is linear in both z and $\bar{z}$.

Note that if (8.5) - (8.9) can be satisfied for suitable functions f, f', ψ and ϕ then under the linear transformation $z_1 = \alpha z + \beta$, $z_1' = \alpha' z' + \beta'$, where α, β, α' and β' are constants and α and α' are non-zero, the new transformed equations (8.5) - (8.9) take the same form and may be satisfied for suitable functions f_1, f_1', ψ_1 and ϕ_1. For example (8.7) becomes

$$iq_1 + r_1 + f_1(z_1, \bar{z}_1) = 0,$$

where $q_1 = \dfrac{\partial z_1}{\partial y}$, $r_1 = \dfrac{\partial^2 z_1}{\partial x^2}$, and where the function f_1 is defined by $f_1(z_1, \bar{z}_1) = \alpha f((z_1 - \beta)/\alpha, (\bar{z}_1 - \bar{\beta})/\bar{\alpha})$. This enables simplifications to be made in the subsequent analysis without loss of generality.

The two identities implied by (8.9) may be represented by

$$E_1 \equiv 0, \quad E_2 \equiv 0, \tag{8.10}$$

where

$$E_1 = \frac{\partial p}{\partial x} - r, \tag{8.11}$$

$$E_2 = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}, \tag{8.12}$$

the latter being the integrability condition. Using the forms of p, q, r, r' given by (8.5) - (8.8), we can successively calculate that

$$E_{2t'} = \psi_{q'}, \quad E_{2\bar{t}'} = \psi_{\bar{q}'}, \tag{8.13}$$

$$E_{2s'} = \psi_{p'} - \phi_{q'}, \quad E_{2\bar{s}'} = \psi_{\bar{p}'} - \phi_{\bar{q}'}, \tag{8.14}$$

$$E_{2q'q'} = 2\mathrm{i}\psi_{p'p'}, \quad E_{2\bar{q}'\bar{q}'} = -2\mathrm{i}\psi_{\bar{p}'\bar{p}'}. \tag{8.15}$$

All of these expressions must also be identically zero. Equations (8.13) show that ψ is independent of q' and $\bar{q}'$ while (8.15) gives that ψ is linear in p' and $\bar{p}'$. Together with (8.14) these deductions show that ψ and ϕ are of the form

$$\psi = kp'\bar{p}' + lp' + m\bar{p}' + n, \tag{8.16}$$

$$\phi = kq'\bar{q}' + lq' + k\bar{q}'p' + m\bar{q}' + H, \tag{8.17}$$

where k, l, m and n are functions of $z, z', \bar{z}$ and $\bar{z}'$ and H is a function of $z, z', \bar{z}, \bar{z}', p'$ and $\bar{p}'$.

Calculation of $E_{1\bar{q}'}$ $[=2\mathrm{i}(kp'+m)]$ immediately gives $k = m = 0$. It now follows that

$$E_{1\bar{p}'} - E_{2\bar{q}'} = 2\mathrm{i}H_{\bar{p}'}, \quad E_{2q'p'} = \mathrm{i}H_{p'p'},$$

so that

$$H = \tau p' + \chi, \tag{8.18}$$

where τ and χ are functions of $z, z', \bar{z}$ and $\bar{z}'$.

Now E_2 is a polynomial in $p', \bar{p}', q'$ and $\bar{q}'$ and E_1 is a polynomial in p' and $\bar{p}'$. All coefficients must vanish and this leads directly to the eleven conditions

$$\bar{l}l_{\bar{z}}+l_{\bar{z}'} = 0, \tag{8.19}$$

$$ll_z+l_{z'} = 0, \tag{8.20}$$

$$nl_z+\bar{n}l_{\bar{z}} = 0, \tag{8.21}$$

$$n_{\bar{z}}\bar{l}+n_{\bar{z}'} = 0, \tag{8.22}$$

$$n_z l+n_{z'} = -\mathrm{i}\tau, \tag{8.23}$$

$$-l_z\tau+\tau_z l+\tau_{z'} = 0, \tag{8.24}$$

$$-l_{\bar{z}}\bar{\tau}+\tau_{\bar{z}}\bar{l}+\tau_{\bar{z}'} = 0, \tag{8.25}$$

$$\mathrm{i}\chi+nn_z+\bar{n}n_{\bar{z}}+f-lf' = 0, \tag{8.26}$$

$$-n_{\bar{z}}\bar{\tau}+\chi_{\bar{z}}\bar{l}+\chi_{\bar{z}'} = 0, \tag{8.27}$$

$$-l_z\chi-l_{\bar{z}}\bar{\chi}-n_z\tau+\tau_z n+\tau_{\bar{z}}\bar{n}+\chi_z l+\chi_{z'} = 0, \tag{8.28}$$

$$-f'\tau-n_z\chi-n_{\bar{z}}\bar{\chi}+\chi_z n+\chi_{\bar{z}}\bar{n} = 0. \tag{8.29}$$

The BT now is of the form

$$p = lp'+n, \tag{8.30}$$

$$q = lq'+\tau p'+\chi. \tag{8.31}$$

The derivatives $n_{z'}$ and $n_{\bar{z}'}$ given by (8.23) and (8.22) are compatible providing [use is also made here of (8.19), its conjugate and (8.25)]

$$\bar{\tau}l_{\bar{z}} = 0. \tag{8.32}$$

Either $\tau = 0$ or $\tau \neq 0$. If $\tau \neq 0$ then $l_{\bar{z}} = 0$ and it is also clear from (8.23) that $n \neq 0$ so that (8.19) - (8.21) give all derivatives of l zero, showing that l is constant.

When l=constant we will only examine here the case where l is nonzero. If $l=0$ then it can be seen from (8.30) that $p=n(z,\bar{z},z',\bar{z}')$. Solving for z', (8.30) gives $z'=F(z,\bar{z},p,\bar{p})$. But this is the special BT given in chapter 7, section 4. In chapter 9 we investigate what equations of the form $u_{xy} = f(u,u_x)$ admit BTs of this special class. In a similar manner to the work carried out in the next chapter we have examined equations (8.7) but lead nowhere.

Note that in the constraints (8.19) - (8.29), when l is constant, the transformations $z \rightarrow lz, n \rightarrow ln, \tau \rightarrow l\tau, \chi \rightarrow l\chi, f \rightarrow lf, f' \rightarrow f'$ makes l "disappear".

That is, the same constraints are obtained as would have been obtained by setting $l = 1$. Hence we can take $l = 1$ without loss of generality in the l constant case with the proviso that any result may be generalised by reversing the above transformation.

We may now divide the analysis into
Case 1. $\tau = 0$, $l = 1$;
Case 2. $\tau = 0$, l not a constant;
Case 3. $\tau \neq 0$, $l = 1$.

Case 1: $\tau = 0, l = 1$

If the variables ξ and η are introduced:

$$\xi = z + z' , \quad \eta = z - z',$$

equations (8.22) and (8.23) show that n is independent of ξ and $\bar{\xi}$, that is $n = n(\eta, \bar{\eta})$. Also (8.27) and (8.28) give $\chi = \chi(\eta, \bar{\eta})$. Equation (8.26) now implies that $f(z,\bar{z}) - f'(z',\bar{z}')$ is independent of both ξ and $\bar{\xi}$. Taking partial derivatives with respect to ξ and $\bar{\xi}$ gives, respectively,

$$f_z - f'_{z'} = 0, \quad f_{\bar{z}} - f'_{\bar{z}'} = 0.$$

It follows, in particular, that $f(z,\bar{z})$ is linear in both z and $\bar{z}$ so that it is not possible for a nonlinear equation (8.7) to arise in this case.

Case 2: $\tau = 0$, l not a constant.

From equation (8.26)

$$\chi = \mathrm{i}(f - lf' + nn_z + \bar{n}n_{\bar{z}}), \tag{8.33}$$

and substituting for χ into (8.27) gives

$$\bar{l}f_{\bar{z}} - lf'_{\bar{z}'} = 0. \tag{8.34}$$

Also, (8.28) may be expressed (simplifying χ_z and $\chi_{z'}$) as

$$l_z\chi + l_{\bar{z}}\bar{\chi} = \mathrm{i}l(f_z - f'_{z'}). \tag{8.35}$$

Now define the operator T:

$$T \equiv l\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}. \tag{8.36}$$

Since $Tl = T\bar{l} = 0$ from (8.20) and the conjugate of (8.19), operating on (8.34) with T gives

$$\bar{l}g = g', \tag{8.37}$$

where $g(z,\bar{z}) = f_{z\bar{z}}$ and $g'(z',\bar{z}') = f'_{z'\bar{z}'}$. Further, applying T and its conjugate $\bar{T} = \bar{l}\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}'}$ to (8.37) gives, respectively,

$$l\bar{l}g_z = g'_{z'}, \tag{8.38}$$

$$\bar{l}^2 g_{\bar{z}} = g'_{\bar{z}'}. \tag{8.39}$$

Since $f_{\bar{z}}$ and $f'_{\bar{z}'}$ do or do not vanish together [see (8.34)] and the same is true for $f_{z\bar{z}}$ and $f_{z'\bar{z}'}$ [see (8.37)], the analysis may be divided into the three subcases :

(a) $f_{z\bar{z}} \neq 0, f'_{z'\bar{z}'} \neq 0$;

(b) $f_{z\bar{z}} = 0, f'_{z'\bar{z}'} = 0, f_{\bar{z}} \neq 0, f'_{\bar{z}'} \neq 0$;

(c) $f_{\bar{z}} = f'_{\bar{z}'} = 0$.

In what follows it will be seen that only the first and third of these cases lead to non-linear forms of $f(z,\bar{z})$ and $f'(z',\bar{z}')$.

(a) $f_{z\bar{z}} \neq 0$, $f'_{z'\bar{z}'} \neq 0$. In this case $g \neq 0$ and $g' \neq 0$, and if $\bar{l}$ is eliminated between (8.37) and (8.39) and the variables separated, it is found that

$$g = 1/(\bar{\alpha}\bar{z} + h(z)), \quad g' = 1/(\bar{\alpha}\bar{z}' + h'(z')), \tag{8.40}$$

where α is constant and h, h' are arbitrary functions. Equation (8.38), using (8.37) and (8.40), may now be separated to give

$$h_z(\alpha z + \bar{h})/(\bar{\alpha}\bar{z} + h) = h'_{z'}(\alpha z' + \bar{h}')/(\bar{\alpha}\bar{z}' + h') = \text{constant}. \tag{8.41}$$

It is now straightforward to deduce that h is linear in z and that h' is linear in z', so that $1/g$ is linear in z and $\bar{z}$ and $1/g'$ is linear in z' and $\bar{z}'$. Neither g nor g' can be constant unless both are constants [see (8.38)] and then (8.37) is contradicted since l is not constant. The existence of a BT and the form of the differential equations (8.7) and (8.8) are unaffected by linear transformations of z and z' (see beginning of the section). Thus without loss of generality we may consider separately $g = 1/\bar{z}$ and $g = 1/(z + \mu\bar{z} + \nu)$, μ and ν being constants. However, the latter means that $f_{\bar{z}}$ will involve the "ln" function and remembering that $\bar{l} = g'/g$ it is clear that (8.34) cannot be satisfied. Hence $g = 1/\bar{z}$ and f must take the form

$$f(z,\bar{z}) = z\ln\bar{z} + j(\bar{z}) + k(z). \tag{8.42}$$

A similar argument gives

$$f'(z',\bar{z}') = z'\ln\bar{z}' + j'(\bar{z}') + k'(z') \tag{8.43}$$

and hence

$$l = z/z'. \tag{8.44}$$

Equation (8.34) now gives that j, j' are constants, which may be absorbed in k and k' (i.e., taken as zero). Also, from (8.21),

$$n = 0. \tag{8.45}$$

Finally, (8.22) gives

$$k(z) = \alpha z\ln z + \beta z, \quad k'(z') = \alpha z'\ln z' + \beta' z', \tag{8.46}$$

where β and β' are complex constants, and from (8.26)

$$\chi = \mathrm{i}[z\ln(\bar{z}/\bar{z}') + \alpha z\ln(z/z') + z(\beta-\beta')]. \tag{8.47}$$

Equations (8.19) - (8.29) are now all satisfied. For this case, therefore, the partial differential equation (8.7) becomes

$$\mathrm{i}z_y + z_{xx} + z\ln\bar{z} + \alpha z\ln z + \beta z = 0, \tag{8.48}$$

which transforms under the BT

$$z_x = \left(\frac{z}{z'}\right)z'_x, \tag{8.49a}$$

$$z_y = \left(\frac{z}{z'}\right)z'_y + \mathrm{i}\left[z\ln\left(\frac{\bar{z}}{\bar{z}'}\right) + \alpha z\ln\left(\frac{z}{z'}\right) + z(\beta-\beta')\right] \tag{8.49b}$$

to

$$\mathrm{i}z'_y + z'_{xx} + z'\ln\bar{z}' + \alpha z'\ln z' + \beta' z' = 0. \tag{8.50}$$

Equations (8.49) can, in fact, be integrated to give

$$z = z'F(y), \tag{8.51}$$

where $F(y)$ satisfies the first-order ordinary differential equation

$$\frac{\mathrm{d}F}{\mathrm{d}y} = \mathrm{i}[F\ln\bar{F} + \alpha F\ln F + (\beta-\beta')F]. \tag{8.52}$$

(b) $f_{z\bar{z}} = f'_{z'\bar{z}'} = 0$, $f_{\bar{z}} \neq 0$, $f'_{\bar{z}'} \neq 0$. Here f and f' now take the forms

$$f = j(z) + \overline{k(z)}, \tag{8.53}$$

$$f' = j'(z') + \overline{k'(z')} \tag{8.54}$$

and the conjugate of (8.34) becomes

$$lk_z = \bar{l}k'_{z'}. \tag{8.55}$$

Eliminating l it follows that when $f_{\bar{z}} \neq 0$ and $f'_{\bar{z}'} \neq 0$, k_z, $k'_{z'}$ and $\bar{l}/l$ are all constant. Equation (8.35) and its conjugate now show that j_z and $j'_{z'}$ are also constant so that this case leads only to linear forms of f and f'.

(c) $f_{\bar{z}} = f'_{\bar{z}'} = 0$. In addition to $Tl = T\bar{l} = 0$ equation (8.23) and the conjugate of (8.22) also give $Tn = T\bar{n} = 0$. It then follows that $Tl_z = (Tl)_z - l_z^2 = -l_z^2$ and similarly $Tl_{\bar{z}} = -l_z l_{\bar{z}}$, $Tn_z = -n_z l_z$, $Tn_{\bar{z}} = -n_z l_{\bar{z}}$, $T\bar{n}_z = -l_z \bar{n}_z$ and $T\bar{n}_{\bar{z}} = -l_{\bar{z}}\bar{n}_z$. Equation (8.26) now gives, using (8.21), $T\chi = il(f_z - f'_{z'})$ and $T\bar{\chi} = 0$. Applying T to (8.35) and making use of these results shows that

$$lf_{zz} - f'_{z'z'} = 0. \tag{8.56}$$

Note that $f_{zz} = 0$ can be excluded since (8.56) again gives both f and f' linear, and substituting for l into (8.20) gives

$$f_{zzz}\, f'^2_{z'z'} = f'_{z'z'z'}\, f^2_{zz}.$$

Separating the variables, solving for $f(z)$ and $f'(z')$ and making linear transformations to z and z', without loss of generality, we find that $f(z) = \alpha z \ln z + \beta z$ and $f'(z') = \alpha z' \ln z' + \beta' z'$, so that (8.56) gives $l = z/z'$. This is essentially a subset of the case 2(a) since in (8.48) it is possible to make scale changes in x and y to introduce a parameter coefficient of the $z \ln \bar{z}$ term, without otherwise changing the form of the equation. Setting this parameter to be zero gives the present case. The form of $F(y)$ in (8.52) may however now be obtained explicitly. Summing up,

$$iz_y + z_{xx} + \alpha z \ln z + \beta z = 0 \tag{8.57}$$

transforms to

$$iz'_y + z'_{xx} + \alpha z' \ln z' + \beta' z' = 0 \tag{8.58}$$

under the transformation (c.f. ref [95])

$$z = z' \exp\left[\gamma e^{i\alpha y} - (\beta - \beta')/\alpha\right], \tag{8.59}$$

with γ being a constant.

Case 3: $\tau \neq 0, l = 1$

It is now convenient to introduce new independent variables ξ, η, such that

$$z = (\xi+\eta)/2, \quad z' = (\xi-\eta)/2. \tag{8.60}$$

Equations (8.19)-(8.21) are now satisfied. Equations (8.24) and (8.25) reduce to $\tau_{\xi} = \tau_{\bar{\xi}} = 0$, so that

$$\tau = \tau(\eta,\bar{\eta}). \tag{8.61}$$

Equations (8.22) and (8.23) reduce to

$$n_{\bar{\xi}} = 0, \quad n_{\xi} = -i\tau/2,$$

so that n is of the form

$$n = -i\tau\xi/2+im(\eta,\bar{\eta}). \tag{8.62}$$

Equations (8.27) and (8.28) give expressions for $\chi_{\bar{\xi}}$ and χ_{ξ} respectively, which may be integrated to give

$$\chi = \tfrac{1}{4}i[-\bar{\tau}\tau_{\bar{\eta}}\xi\bar{\xi}+2\bar{\tau}m_{\bar{\eta}}\bar{\xi}+(-\tau^2+2\tau m_{\eta}$$

$$-2m\tau_{\eta}+2\bar{m}\tau_{\bar{\eta}})\xi-4A(\eta,\bar{\eta})], \tag{8.63}$$

$A(\eta,\bar{\eta})$ being an arbitrary function of η and $\bar{\eta}$.

It only remains now to satisfy equations (8.26) and (8.29). That is, suitable functions $m(\eta,\bar{\eta})$, $A(\eta,\bar{\eta})$, $\tau(\eta,\bar{\eta})$, $f(z,\bar{z})$, $f'(z',\bar{z}')$ must be found so that these last two equations become identities in ξ, $\bar{\xi}$, η, $\bar{\eta}$.

The first of these two equations, (8.26), gives

$$f(z,\bar{z})-f'(z',\bar{z}') = \xi^2P_1+\xi\bar{\xi}P_2+\xi P_3+\bar{\xi}P_4+P_5 = F(\xi,\bar{\xi},\eta,\bar{\eta}), \tag{8.64}$$

where

$$P_1 = \tfrac{1}{4}\tau\tau_{\eta}, \quad P_2 = -\tfrac{1}{2}\bar{\tau}\tau_{\bar{\eta}}, \quad P_3 = \bar{m}\tau_{\bar{\eta}}-m\tau_{\eta},$$

$$P_4 = \bar{\tau}m_{\bar{\eta}}, \quad P_5 = G(\eta,\bar{\eta}). \tag{8.65}$$

and where

$$G(\eta,\bar{\eta}) = mm_{\eta}-\bar{m}m_{\bar{\eta}}-m\tau/2-A(\eta,\bar{\eta}). \tag{8.66}$$

The nature of the left hand side of (8.64) imposes conditions on P_1, P_2, P_3, P_4 and P_5. Specifically, $F_{zz'} = F_{z\bar{z}'} = F_{\bar{z}z'} = F_{\bar{z}\bar{z}'} = 0$, or

$$F_{\xi\xi} - F_{\eta\eta} = F_{\bar{\xi}\bar{\xi}} - F_{\bar{\eta}\bar{\eta}} = F_{\eta\bar{\xi}} - F_{\bar{\eta}\xi} = F_{\xi\bar{\xi}} - F_{\eta\bar{\eta}} = 0. \tag{8.67}$$

Hence,

$$P_1 = \tfrac{1}{4}\tau\tau_\eta = \alpha_1\eta + \beta_1\bar{\eta} + \gamma_1, \tag{8.68}$$

$$P_2 = -\tfrac{1}{2}\bar{\tau}\tau_{\bar{\eta}} = 2\beta_1\eta + \gamma_2, \tag{8.69}$$

$$P_3 = \bar{m}\tau_{\bar{\eta}} - m\tau_\eta = \alpha_3\eta + \beta_3\bar{\eta} + \gamma_3, \tag{8.70}$$

$$P_4 = \bar{\tau}m_{\bar{\eta}} = \beta_3\eta + \beta_4\bar{\eta} + \gamma_4, \tag{8.71}$$

$$P_5 = G = \bar{\eta}(\beta_1\eta^2 + \gamma_2\eta + \varepsilon_1) + \tfrac{1}{3}\alpha_1\eta^3 + \gamma_1\eta^2 + \delta_2\eta + \varepsilon_2, \tag{8.72}$$

where $\alpha_1, \alpha_3, \beta_1, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_2, \varepsilon_1, \varepsilon_2$ are constants. Note that once the functional form of $F(\xi, \bar{\xi}, \eta, \bar{\eta})$ has been established, f and f' may be obtained directly from F, in terms of ξ, $\bar{\xi}$, η, and $\bar{\eta}$, by the substitutions

$$f = F((\xi+\eta)/2, (\bar{\xi}+\bar{\eta})/2, (\xi+\eta)/2, (\bar{\xi}+\bar{\eta})/2) + \varepsilon_3 - \varepsilon_2, \tag{8.73}$$

$$f' = -F((\xi-\eta)/2, (\bar{\xi}-\bar{\eta})/2, (\eta-\xi)/2, (\bar{\eta}-\bar{\xi})/2) + \varepsilon_3, \tag{8.74}$$

where $\varepsilon_3[=f(0,0)]$ is an arbitrary constant and where ε_2 in (8.73) is the value of $F(0,0,0,0)$, obtained from (8.64) and (8.72).

The remaining equation (8.29) may be written in terms of ξ and η as

$$E_3 = 0, \tag{8.75}$$

where

$$E_3 = -\tau f' + n(\chi_\xi + \chi_\eta) + \bar{n}(\chi_{\bar{\xi}} + \chi_{\bar{\eta}}) - \chi(n_\xi + n_\eta) - \bar{\chi}(n_{\bar{\xi}} + n_{\bar{\eta}}) \tag{8.76}$$

and where τ, n, χ and f' are given by (8.61), (8.62), (8.63) and (8.74) respectively. Direct calculation gives

$$E_{3\xi\xi\xi} = -\tau\alpha_1 \tag{8.77}$$

and

$$E_{3\bar{\xi}\bar{\xi}} = -\beta_4(\tau+\bar{\tau})/2 \tag{8.78}$$

so that

$$\alpha_1 = 0 \tag{8.79}$$

and

$$\beta_4 = 0 \text{ or } \bar{\tau} = -\tau. \tag{8.80}$$

With $\alpha_1 = 0$ (8.68) integrates to give

$$\tau^2 = 8(\beta_1\eta\bar{\eta} + \gamma_1\eta + \overline{p(\eta)}), \tag{8.81}$$

where $p(\eta)$ is an arbitrary function of η. Hence

$$\tau\tau_{\bar{\eta}} = 4(\beta_1\eta + \bar{p}_{\bar{\eta}}). \tag{8.82}$$

Comparison of (8.82) and (8.69) and using $|\bar{\tau}/\tau| = 1$ shows that p must be linear. Taking $p(\eta) = \delta_1\eta + \delta_3$, (8.69) and (8.82) give

$$\tau(\beta_1\eta + \gamma_2/2) = -\bar{\tau}(\beta_1\eta + \bar{\delta}_1), \tag{8.83}$$

where

$$\tau^2 = 8(\beta_1\eta\bar{\eta} + \gamma_1\eta + \bar{\delta}_1\bar{\eta} + \bar{\delta}_3). \tag{8.84}$$

Squaring (8.83), substituting for τ^2 from (8.84) and equating coefficients of different powers of η and $\bar{\eta}$ gives that β_1 must be real. In addition, conditions arise which may be conveniently separated into 5 discrete cases. The corresponding forms of $\tau(\neq 0)$ are also given:

(a) $\beta_1 = 0,\ \gamma_2 = 0,\ \delta_1 = 0,\ \gamma_1 = 0,\ \tau$ constant;

(b) $\beta_1 = 0,\ \gamma_2 = 0,\ \delta_1 = 0,\ \gamma_1 \neq 0,\ \beta_4 = 0,$
$\tau^2 = 8(\gamma_1\eta + \bar{\delta}_3)$;

(c) $\beta_1 = 0,\ \gamma_1 \neq 0,\ \gamma_2 = 2\bar{\gamma}_1,\ \delta_1 = \gamma_1,$
$\bar{\delta}_3 = \delta_3,\ \bar{\tau} = -\tau,\ \tau^2 = 8(\gamma_1\eta + \bar{\gamma}_1\bar{\eta} + \bar{\delta}_3)$;

(d) $\beta_1 = 0,\ \gamma_1 \neq 0,\ \gamma_2 \neq 2\bar{\gamma}_1,\ \gamma_2\bar{\gamma}_2 = 4\gamma_1\bar{\gamma}_1,$
$\delta_1 = \frac{1}{4}\frac{\bar{\gamma}_2^2}{\gamma_1},\ \bar{\delta}_3 = \frac{1}{4}\frac{\gamma_2^2\delta_3}{\bar{\gamma}_1^2},\ \bar{\tau} = -2\frac{\tau\bar{\gamma}_1}{\gamma_2},$
$\tau^2 = 8(\gamma_1\eta + \frac{1}{4}\frac{\gamma_2^2\bar{\eta}}{\bar{\gamma}_1} + \bar{\delta}_3)$;

(e) $\beta_1 = \bar{\beta}_1 \neq 0,\ \gamma_2 = 2\bar{\gamma}_1,\ \delta_1 = \gamma_1,\ \bar{\delta}_3 = \delta_3,$
$\bar{\tau} = -\tau,\ \tau^2 = 8(\beta_1\eta\bar{\eta} + \gamma_1\eta + \bar{\gamma}_1\bar{\eta} + \delta_3)$.

In cases (b) and (d) $\bar{\tau} \neq -\tau$ so that, from (8.80), $\beta_4 = 0$.

For each of these five cases (8.71) may be integrated directly to give an expression for $m(\eta, \bar{\eta})$ which will, of course, involve an arbitrary function of η. Substituting

τ and m into (8.70) then leads to further constraints on the constants and also specifies, in most cases, a more precise form of m. In summary equations (8.68) - (8.72) imply the following forms of τ, m and G corresponding to cases (a) - (e) above.

(a) τ constant,

$$m = (\tfrac{1}{2}\beta_4\bar{\eta}^2+\gamma_4\bar{\eta})/\bar{\tau}+V(\eta),\ \ V \text{ arbitrary},$$

$$G = \bar{\eta}\varepsilon_1+\eta\delta_2+\varepsilon_2;$$

(b) $$\tau^2 = 8(\gamma_1\eta+\bar{\delta}_3),\ \ \gamma_1 \neq 0,$$

$$m = -\tfrac{1}{4}\tau(\alpha_3\eta+\gamma_3)/\gamma_1,$$

$$G = \gamma_1\eta^2+\delta_2\eta+\varepsilon_1\bar{\eta}+\varepsilon_2;$$

(c) $$\tau^2 = 8(\gamma_1\eta+\bar{\gamma}_1\bar{\eta}+\delta_3),\ \ \delta_3 \text{ real},\ \ \bar{\tau} = -\tau,$$

$$m = -\frac{\tau}{12\gamma_1^2\bar{\gamma}_1}[\gamma_1(\gamma_1\bar{\alpha}_3+2\bar{\gamma}_1\alpha_3)\eta+\bar{\gamma}_1(\gamma_1\bar{\alpha}_3-\bar{\gamma}_1\alpha_3)\bar{\eta}$$

$$-2\delta_3(\gamma_1\bar{\alpha}_3-\bar{\gamma}_1\alpha_3)+3\gamma_1^2\gamma_4],$$

$$G = 2\bar{\gamma}_1\eta\bar{\eta}+\gamma_1\eta^2+\delta_2\eta+\varepsilon_1\bar{\eta}+\varepsilon_2;$$

(d) $$\tau^2 = 8(\gamma_1\eta+\tfrac{1}{4}\gamma_2^2\bar{\eta}/\bar{\gamma}_1+\bar{\delta}_3),\ \ \bar{\tau} = -2\tau\bar{\gamma}_1/\gamma_2,$$

$$m = -\tfrac{1}{4}\tau(\alpha_3\eta/\gamma_1+2\gamma_4/\gamma_2)+\varepsilon_6,$$

$$G = \gamma_2\eta\bar{\eta}+\gamma_1\eta^2+\delta_2\eta+\varepsilon_1\bar{\eta}+\varepsilon_2;$$

where

$$\bar{\varepsilon}_6 = \frac{\varepsilon_6\bar{\gamma}_2}{\gamma_2},\ \ \bar{\alpha}_3 = \frac{\alpha_3\bar{\gamma}_1}{\gamma_1},\ \ \gamma_2\bar{\gamma}_2 = 4\gamma_1\bar{\gamma}_1,$$

$$\bar{\delta}_3 = \tfrac{1}{4}\frac{\delta_3\gamma_2^2}{\bar{\gamma}_1^2},\ \ \bar{\gamma}_3 = 2\frac{\gamma_3\bar{\gamma}_1}{\gamma_2},\ \ \bar{\gamma}_4 = -4\frac{\gamma_1\bar{\gamma}_1}{\gamma_2^3}(2\gamma_1\gamma_4-\gamma_2\gamma_3),\ \ \gamma_1 \neq 0,\ \ \gamma_2 \neq 2\bar{\gamma}_1;$$

(e) $$\tau^2 = 8(\beta_1\eta\bar{\eta}+\gamma_1\eta+\bar{\gamma}_1\bar{\eta}+\delta_3),\ \ \bar{\tau} = -\tau,$$

$$m = -\tfrac{1}{4}\tau\bar{\alpha}_3/\beta_1+\varepsilon_6(\beta_1\eta+\bar{\gamma}_1),$$

$$G = \beta_1\eta^2\bar{\eta}+\gamma_1\eta^2+2\bar{\gamma}_1\eta\bar{\eta}+\delta_2\eta+\varepsilon_1\bar{\eta}+\varepsilon_2;$$

where β_1, δ_3 and ε_6 are all real and $\beta_1 \neq 0$.

In each case equation (8.64) enables F to be found in a form which is consistent with its representation $F = f(z,\bar{z}) - f'(z',\bar{z}')$. Here f and f' may be calculated using the substitutions given in (8.73) and (8.74). Of the original equations (8.19) - (8.29) only (8.29), which was expressed earlier [see (8.75) and (8.76)] as $E_3 = 0$, remains to be satisfied.

Using the above results E_3 may be expressed in terms of ξ, $\bar{\xi}$, η and $\bar{\eta}$ and must be identically zero. Of the five cases above, three [(a), (b) and (d)] lead to no non-linear forms of $f(z,\bar{z})$. For case (b) direct calculation leads to

$$(E_3/\tau)_{\xi\xi} = -\alpha_3,$$

giving $\alpha_3 = 0$, and then to

$$(\tau E_3)_{\xi\eta\eta} = -4\gamma_1^2,$$

leading to the contradiction $\gamma_1 = 0$.

For case (d),

$$(\tau E_3)_{\xi\xi\eta} = -8\gamma_1\alpha_3,$$

so that, since $\gamma_1 \neq 0$, $\alpha_3 = 0$. Hence

$$(\tau E_3)_{\xi\eta\eta} = -4\gamma_1^2,$$

giving, again, the contradiction $\gamma_1 = 0$.

In case (a)

$$E_{3\xi\xi} = \tfrac{1}{2}\tau^2 V_{\eta\eta},$$

so that $V(\eta)$ is linear. Then

$$E_{3\eta\eta\eta\eta} = 6\beta_4\bar{\beta}_4^2/(\tau^2\bar{\tau}),$$

giving $\beta_4 = 0$ and hence

$$F = \gamma_4\bar{\xi} + \delta_2\eta + \varepsilon_1\bar{\eta} + \varepsilon_2,$$

which leads only to linear forms for both $f(z,\bar{z})$ and $f'(z',\bar{z}')$.

Case (e) leads to the cubic Schrödinger equation. The identities $(\tau E_3)_{\xi\eta\bar{\eta}} = 0$, $(\tau E_3)_{\xi\eta\eta} = 0$, $(\tau E_3)_{\xi\eta} = 0$ and $(\tau E_3)_{\eta\bar{\eta}} = 0$ give in turn

$$\bar{\delta}_2 = \delta_2,$$

$$\varepsilon_1 = \tfrac{1}{4}\frac{\bar{\alpha}_3^2+4\bar{\gamma}_1^2}{\beta_1},$$

$$\varepsilon_2 = \tfrac{1}{4}\frac{4\beta_1\bar{\gamma}_1\delta_2-4\gamma_1\bar{\gamma}_1^2+\gamma_1\bar{\alpha}_3^2}{\beta_1^2},$$

$$\varepsilon_3 = \frac{1}{16\beta_1^2}(8\beta_1\bar{\gamma}_1\delta_2+4\beta_1\bar{\alpha}_3\delta_2-8\gamma_1\bar{\gamma}_1^2$$

$$+2\gamma_1\bar{\alpha}_3^2+4\bar{\gamma}_1^2\alpha_3-\bar{\alpha}_3^2\alpha_3).$$

E_3 is now identically zero.

The forms of $f(z,\bar{z})$ and $f'(z',\bar{z}')$ may be calculated from (8.73) and (8.74) and these may be simplified by replacing z by $z-\frac{1}{4}(2\bar{\gamma}_1+\bar{\alpha}_3)/\beta_1$ and z' by $z'-\frac{1}{4}(-2\bar{\gamma}_1+\bar{\alpha}_3)/\beta_1$. In addition the constants involved may be simplified by setting

$$\delta_2 = \mu+\tfrac{1}{2}(4\gamma_1\bar{\gamma}_1+\alpha_3\bar{\alpha}_3)/\beta_1,$$

$$\delta_3 = -\tfrac{1}{2}\beta_1\lambda_1+\gamma_1\bar{\gamma}_1/\beta_1,$$

$$\varepsilon_6 = \lambda_2/\beta_1,$$

where μ, λ_1 and λ_2 are all real constants. With these substitutions

$$f(z,\bar{z}) = 4\beta_1 z^2\bar{z}+\mu z,$$

$$f'(z',\bar{z}') = 4\beta_1 z'^2\bar{z}'+\mu z'$$

and auto-BTs for the equation

$$\mathrm{i}z_y+z_{xx}+4\beta_1 z^2\bar{z}+\mu z = 0 \qquad (8.85)$$

are given by

$$p = p'+n, \quad q = q'+\tau p'+\chi, \qquad (8.86a,b)$$

where

$$\tau^2 = 4\beta_1(2(z-z')(\bar{z}-\bar{z}')-\lambda_1),$$

$$n = -\tfrac{1}{2}\mathrm{i}\tau(z+z')+\mathrm{i}\lambda_2(z-z'),$$

$$\chi = \mathrm{i}\lambda_2\tau z+\mathrm{i}\mu(z-z')+\mathrm{i}\lambda_1\beta_1(z+z')-\mathrm{i}\lambda_2^2(z-z')$$

$$+2\mathrm{i}\beta_1(z-z')(z\bar{z}'-z'\bar{z}+2z'\bar{z}').$$

This agrees with Lamb [57].

The only remaining case is case (c). Here, the identity $(\tau E_3)_{\xi\eta\eta} = 0$ gives

$$\alpha_3 = \pm\frac{2i\gamma_1}{\sqrt{3}}.$$

Considering $\alpha_3 = +\frac{2i\gamma_1}{\sqrt{3}}$, the identities $(\tau E_3)_{\xi\eta} = 0$, $(\tau E_3)_{\xi} = 0$, $(\tau E_3)_{\eta\eta} = 0$, $(\tau E_3)_{\eta\bar{\eta}} = 0$ and $(\tau E_3)_{\eta} = 0$ give in turn

$$\varepsilon_1 = [\sqrt{3}\gamma_1\bar{\gamma}_1\bar{\delta}_2 - i(2\gamma_1^2\gamma_4+\bar{\gamma}_1^2\bar{\gamma}_4)]/(\sqrt{3}\gamma_1^2),$$

$$\varepsilon_2 = \tfrac{1}{4}(\gamma_1^4\gamma_4^2 - r_1)/(\gamma_1^3\bar{\gamma}_1^2),\ r_1 \text{ real},$$

$$\bar{\delta}_2 = \delta_2,$$

$$\delta_2 = i(\gamma_1^2\gamma_4 - \bar{\gamma}_1^2\bar{\gamma}_4)/(\sqrt{3}\gamma_1\bar{\gamma}_1),$$

$$\varepsilon_3 = \tfrac{1}{8}[r_1(-\sqrt{3}+i) + \gamma_1^4\gamma_4^2(\sqrt{3}+i) - 2i\bar{\gamma}_1^4\bar{\gamma}_4^2]/(\sqrt{3}\gamma_1^3\bar{\gamma}_1^2).$$

E_3 is now identically zero.

Here $f(z,\bar{z})$, $f'(z',\bar{z}')$, n, and χ may now be calculated from (8.73) and (8.74). Their forms are considerably simplified by first writing

$$\gamma_4 = \bar{\gamma}_1\gamma_5(\sqrt{3}-i)/\gamma_1,$$

$$r_1 = -\gamma_1^2\bar{\gamma}_1^2(8\gamma_5\bar{\gamma}_5 + 3r_2),\ r_2 \text{ real},$$

$$\delta_3 = \tfrac{1}{16}[2\sqrt{3}(\gamma_5+\bar{\gamma}_5) + 6i(\gamma_5-\bar{\gamma}_5) - r_5],\ r_5 \text{ real}.$$

i.e. replace constants γ_4, r_1, δ_3 by constants γ_5, r_2, r_5 and then applying linear transformations to z and z' as follows

$$z \rightarrow \tfrac{1}{8}(\sqrt{3}-i)(\sqrt{3}z - (\gamma_5+\bar{\gamma}_5))/\gamma_1,$$

$$z' \rightarrow -\tfrac{1}{8}[z'(3+i\sqrt{3}) + \gamma_5(\sqrt{3}-i) + 2\bar{\gamma}_5 i]/\gamma_1.$$

The two partial differential equations then simplify to

$$iz_y + z_{xx} + (z+\bar{z})^2 + r_2 = 0, \qquad (8.87a)$$

$$iz'_y + z'_{xx} + (z'+\bar{z}')^2 + r_2 = 0 \qquad (8.87b)$$

and the BTs relating them are

$$p = -\tfrac{1}{2}(1+\mathrm{i}\sqrt{3})p' + n, \tag{8.88a}$$

$$q = -\tfrac{1}{2}(1+\mathrm{i}\sqrt{3})q' + \tau p' + \chi, \tag{8.88b}$$

where

$$\tau^2 = \tfrac{1}{4}[8\sqrt{3}\mathrm{i}(z+z') - (12-\mathrm{i}4\sqrt{3})(\bar{z}+z') + r_5(1-\mathrm{i}\sqrt{3})],$$

$$n = \frac{\tau}{36}[z(2\sqrt{3}-18\mathrm{i}) - \bar{z}(2\sqrt{3}+6\mathrm{i}) + z'(-8\sqrt{3}+12\mathrm{i}) - \bar{z}'4\sqrt{3} - r_5(\sqrt{3}+\mathrm{i})],$$

$$\chi = [-12z^2(1+\mathrm{i}3\sqrt{3}) + 24z\bar{z}(1+\mathrm{i}\sqrt{3}) - 12\bar{z}^2(1-\mathrm{i}\sqrt{3}) - 12z'^2(13-\mathrm{i}\sqrt{3})$$
$$-96z'\bar{z}'(2-\mathrm{i}\sqrt{3}) - 12\bar{z}'^2(7-\mathrm{i}3\sqrt{3}) - 12zz'(1-\mathrm{i}5\sqrt{3}) - 12z\bar{z}'(5+\mathrm{i}3\sqrt{3})$$
$$-12\bar{z}\bar{z}'(7+\mathrm{i}\sqrt{3}) - 12\bar{z}z'(11+\mathrm{i}\sqrt{3}) + r_5(-4z(6-\mathrm{i}\sqrt{3}) - 4\mathrm{i}\sqrt{3}\bar{z}$$
$$+2z'(9-\mathrm{i}5\sqrt{3}) - 2\bar{z}'(3+\mathrm{i}\sqrt{3})) + r_5^2(1+\mathrm{i}\sqrt{3}) - 108r_2(1-\mathrm{i}\sqrt{3})] / (72\sqrt{3}).$$

r_5 is the Bäcklund parameter.

For the case $\alpha_3 = -2\mathrm{i}\gamma_1/\sqrt{3}$ the above may be followed through in exactly same way arriving at the same partial differential equations. For the BTs, however i should be replaced by -i where it occurs explicitly in the expressions for p, q, τ, n and χ above. We therefore have two different auto-BTs, each with an arbitrary parameter for the partial differential equation (8.87a).

These BTs incorporate auto-BTs for the Boussinesq equation since if $z = u + \mathrm{i}v$, equation (8.87a) becomes

$$\mathrm{i}u_y - v_y + u_{xx} + \mathrm{i}v_{xx} + 4u^2 + r_2 = 0.$$

If this is separated into real and imaginary parts and v eliminated, u satisfies

$$u_{yy} + u_{xxxx} + (4u^2)_{xx} = 0,$$

which is a form of the well-known Boussinesq equation.

We do believe that a nonlinear superposition may be obtained for equation (8.87a) from the BTs (8.88), but as Lamb [57] points out for the nonlinear Schrödinger equation, the result appears to be too complicated to be useful for computational purposes.

CHAPTER NINE

ON BÄCKLUND TRANSFORMATIONS OF THE CLASS $u=F(u', u_x', u_y')$

1. Introduction

Chapter 9 is also in the spirit of the previous chapter, where here we use BTs of the class

$$u=F(u', u_x', u_y'). \tag{9.1}$$

We explore in detail the range of functions $f(u, u_x)$ and $f'(u', u_x')$ for which the partial differential equations

$$u_{xy} = f(u, u_x), \tag{9.2}$$

$$u_{xy}' = f'(u', u_x') \tag{9.3}$$

are related by the BTs (9.1). Such a case is the BT $u=u'-\sin^{-1}(au_x')$ which relates the sine-Gordon equation $u_{xy} = \sin u$ and the modified sine-Gordon equation $u_{xy}' = \sin u'\sqrt{1-(au_x')^2}$ [70]. This result will be generalised here.

In the present chapter, without presenting the detailed computations, we also explore the range of functions $f(u, u_x)$ and $f'(u', u_x')$ for which the KdV type equations

$$u_y + u_{xxx} + f(u, u_x) = 0, \tag{9.4}$$

$$u_y' + u_{xxx}' + f'(u', u_x') = 0 \tag{9.5}$$

are connected by BT (9.1), where, in this example, F depends also on the second derivatives of u' (u_{xx}', $u_{xy}'$and u_{yy}'). Such an example is the well-known Miura transformation $u = u'^2 + u_x'$ which relates the KdV equation $u_y + u_{xxx} + 6uu_x = 0$ and the modified KdV equation $u_y' + u_{xxx}' - 6u'^2 u_x' = 0$ [66].

In section 2 the class of BTs for the first example [equations (9.2) and (9.3)] is considered and the analysis partitions naturally into four exclusive cases. At the end of this section a list of all the results obtained is given. In section 3 we search for BTs of the class $u=F(u', u_x', u_y', u_{xx}', u_{xy}', u_{yy}')$ which link equations of the form (9.4). In the final section we explain how to generate BTs of the general class (7.4) for equations which are known to admit BTs of the special class (9.1).

2. A Class of Bäcklund Transformations for Equations of the Type $u_{xy} = f(u, u_x)$

We consider BTs of the form

$$u = F(u',p',q') \tag{9.6}$$

which relate the two partial differential equations

$$s = f(u,p) \tag{9.7}$$

and

$$s' = f'(u',p') \tag{9.8}$$

where, with the usual notation, $p=u_x$, $q=u_y$, $r=u_{xx}$, $s = u_{xy}$, $t=u_{yy}$, and similarly for p', q', r', s', t' in terms of u'. In ref [8] it is shown that a number of partial differential equations of the form (9.2) admit auto-BTs of the class (7.4). It is shown here that these equations also admit BTs of the class (9.1). In ref [80], all the possible functions $f(u,u_x)$ for which equation (9.2) describes pseudospherical surfaces, are given. These functions, excluding those linear in u and u_x, are also shown in this example to admit transformations of the class (9.1).

Since $p=\dfrac{\partial u}{\partial x}$ and $s=\dfrac{\partial p}{\partial y}$, using (9.6) - (9.8) we obtain respectively,

$$p=p'F_{u'}+r'F_{p'}+f'F_{q'}, \tag{9.9}$$

$$\begin{aligned} f(F, p'F_{u'}+r'F_{p'}+f'F_{q'}) &= p'q'F_{u'u'}+p'f'F_{u'p'}+p't'F_{u'q'}+f'F_{u'}+ \\ &r'q'F_{u'p'}+r'f'F_{p'p'}+r't'F_{p'q'}+p'f'_{u'}F_{p'}+r'f'_{p'}F_{p'}+q'f'F_{u'q'}+ \\ &f'^2F_{p'q'}+t'f'F_{q'q'}+q'f'_{u'}F_{q'}+f'f'_{p'}F_{q'}. \end{aligned} \tag{9.10}$$

In the following analysis we search for functions $F(u',p',q')$, $f(F,p)$ and $f'(u',p')$ which satisfy identically (9.10). If f and f' are identical functions the BT is an auto-BT. We will exclude cases in which each of f and f' is one of the following three forms :

(Fa) Linear in both u and p (or linear in both u' and p').

(Fb) Form for which equation (9.7) (or equation (9.8)) can be reduced to first order by direct integration.

(Fc) Form which makes equation (9.7) [or (9.8)] a known equation which has well known analytic solutions.

In the work which follows whenever f and f' is one of the above forms we will refer to it as (Fa), (Fb) and (Fc). We will also exclude point transformations.

The coefficients of $(r't')$, (t'), and (t'^0) in (9.10) give respectively

$$F = A+B, \tag{9.11}$$

$$p'B_{u'q'} + f'B_{q'q'} = 0, \tag{9.12}$$

$$f = p'q'A_{u'u'} + p'q'B_{u'u'} + p'f'A_{u'p'} + f'A_{u'} + f'B_{u'} + r'q'A_{u'p'} + r'f'A_{p'p'} +$$

$$p'A_{p'}f'_{u'} + r'A_{p'}f'_{p'} + q'f'B_{u'q'} + q'B_{q'}f'_{u'} + B_{q'}f'f'_{p'}, \tag{9.13}$$

where A is a function of u' and p' and B is a function of u' and q'. From equation (9.12) it follows that either f' linear in p' or $B=b_1q'+c$, b_1 being a constant and c a function of u'. The right hand side of (9.13) is linear in r' and therefore $f_{r'r'} = 0$ which gives $f_{pp}A^2_{p'} = 0$. Hence, either f is linear in p or $A_{p'} = 0$. We can therefore conclude that the analysis may be split into four cases:

1. f linear in p, f' linear in p' and $F=A+B$;
2. f' linear in p' and $F=B$;
3. f linear in p and $F=b_1q'+A$;
4. $F = b_1q'+c$.

Note in cases 2 and 4 that A, since it is a function of u' only, has been absorbed in B and c respectively.

The above cases may be made exclusive by specifying that, where the forms of f and f' are not stated, they must not be linear in p and p', respectively. In cases 3 and 4, where f' is not linear p', equation (9.12) is satisfied and one only needs to consider equation (9.13). In the cases 1 and 2 both equations (9.12) and (9.13) need to be considered.

Case 1: f linear in p and f' linear in p'.

In this case we have

$$u = F = A+B, \tag{9.14}$$

$$f = Lp+M, \tag{9.15}$$

$$f'=kp'+l, \tag{9.16}$$

where L and M are functions of u and k and l are functions of u'. Using (9.16) it follows from equation (9.12) that either (a) $l=0$ and $B_{q'q'}\neq 0$ or (b) $B_{q'q'}=0$.

(a) Since $l=0$, f' takes the excluded form (Fb). Considering the coefficients of (r') and (r'^0) in equation (9.13) and differentiating twice with respect to q', it can be deduced that L and M are given by

$$L = \lambda_2 e^{\lambda_1 u}+\lambda_3, \quad M = \lambda_4 e^{\lambda_1 u}+\lambda_5,$$

where $\lambda_{1,\ldots,}\ \lambda_5$ are all constants. The constants λ_1 and λ_4 must not be zero because otherwise f also becomes the excluded form (Fb). Using the forms of L and M equation (9.13) becomes a polynomial in r' and q'. Equating coefficients of r' and q' we obtain four equations which force f to be of the form $f = \lambda_4 e^{\lambda_1 u}+\lambda_5$, which makes equation (9.7) the well-known Liouville equation. Therefore f is also of the excluded form (Fc). We note that if the analysis is continued a form of F may be obtained which relates the equations $u_{xy} = e^u$ and $u'_{xy} = 0$ and gives the general solution of Liouville's equation (see chapter 1).

(b) Since $B_{q'q'} = 0$, using (9.12), equation (9.14) becomes

$$F = b_1 q'+A, \tag{9.17}$$

where b_1 is a constant. Using (9.15) and (9.16), the coefficients of (r') and (r'^0) in (9.13) give respectively

$$A_{p'}L = q'A_{u'p'}+p'kA_{p'p'}+lA_{p'p'}+kA_{p'}, \tag{9.18}$$

$$(p'A_{u'}+b_1p'k+b_1l)L+M = p'q'A_{u'u'}+p'^2kA_{u'p'}+p'lA_{u'p'}+p'kA_{u'}+lA_{u'}+$$

$$p'^2k_{u'}A_{p'}+p'l_{u'}A_{p'}+b_1p'q'k_{u'}+b_1q'l_{u'}+b_1p'k^2+b_1kl. \tag{9.19}$$

Since the right hand side of (9.18) is linear in q', one must have $A_{p'}L_{q'q'} = 0$ and hence, $b_1^2A_{p'}L_{uu} = 0$. Therefore case 1 may be conveniently separated into three disjoint subcases:

(i) L linear in u, $b_1 \neq 0$, $A_{p'} \neq 0$;

(ii) $A_{p'} = 0$, $b_1 \neq 0$;

(iii) $b_1 = 0$.

In all three subcases functions $L(u)$, $M(u)$, $k(u')$, $l(u')$, and $A(u',p')$ need to be found, which satisfy equations (9.18) and (9.19).

(i) Since L is linear in u, from (9.19) it is easy to deduce that M is also linear in u. Hence,

$$f = (\mu_1 u + \mu_2)p + \mu_3 u + \mu_4, \tag{9.20}$$

where μ_1, μ_2, μ_3 and μ_4 are all constants. Using (9.20) the coefficients of (q') and (q'^0) in (9.18) give respectively,

$$A = Je^{b_1\mu_1 u'} + d, \tag{9.21}$$

$$p'J_{p'p'}k + J_{p'}k - \mu_1 JJ_{p'}e^{b_1\mu_1 u'} + J_{p'p'}l - \mu_1 J_{p'}d - \mu_2 J_{p'} = 0, \tag{9.22}$$

where J is a function of p' and d is a function of u'. Since $A_{p'} \neq 0$ then $J_{p'} \neq 0$. Identity (9.22) implies that there exists a linear relation between $p'J_{p'p'}$, $J_{p'}$, $JJ_{p'}$ and $J_{p'p'}$, unless $k=l=\mu_1 = \mu_2 = 0$. This linear relation may be integrated with respect to p' to give

$$\lambda_1 p'J_{p'} + \lambda_2 J_{p'} + \lambda_3 J^2 + \lambda_4 J + \lambda_5 = 0, \tag{9.23}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and λ_5 are constants. Taking $\mu_1 \neq 0$ and solving (9.23), it is seen that J must take one of the following forms:

$$J = \lambda_4 \tan\left[(\lambda_5 \ln(\lambda_1 p' + \lambda_2)\right] + \lambda_3, \tag{9.24}$$

$$J = \lambda_4 \tan(\lambda_1 p' + \lambda_2) + \lambda_3, \tag{9.25}$$

$$J = \frac{\lambda_4}{\ln(\lambda_1 p' + \lambda_2)} + \lambda_3. \tag{9.26}$$

If $\mu_1 = 0$, then J is given by one of the following forms:

$$J = \lambda_4 \ln(p' + \lambda_2) + \lambda_3, \tag{9.27}$$

$$J = \lambda_4 (p' + \lambda_2)^{\lambda_1} + \lambda_3, \tag{9.28}$$

$$J = \lambda_4 e^{\lambda_1 p'} + \lambda_3, \tag{9.29}$$

where the constants λ_i have been redefined. Using these forms of J, each separately, equations (9.22) and (9.19) will give the forms of k, l and d. The result is that, all six forms of J produce both f and f' of the excluded forms. For example, substituting (9.25) into equation (9.22) gives

$$k=0, \quad l=\frac{\mu_1\lambda_4}{2\lambda_1}e^{b_1\mu_1u'}, \quad d=-\lambda_3e^{b_1\mu_1u'}, \quad \mu_2=0.$$

Now using the above results, the coefficient of q' in (9.19) gives $\mu_3 = 0$. Therefore f is of the form (Fc) (Liouville's equation) and f' is of the form (Fb). The coefficient of (q'^0) in (9.19) will give more restrictions on the forms of f and f', but there is no point in continuing the analysis, since both f and f' are already of the excluded forms.

A similar conclusion is reached when the other forms of J are used. Care must be taken when (9.28) is used, because in order to be able to equate coefficients of powers of p' in (9.22) and (9.19), one needs to consider separately (a)$\lambda_1 = 1$, (b) $\lambda_1=-1$ and (c) $\lambda_1 \neq -1, 1$. Finally if $k=l=\mu_1 = \mu_2=0$ the coefficient of q' in (9.19) gives $\mu_3 = 0$, which implies $f=$ constant and $f'=0$.

(ii) Since $A_{p'} = 0$, equation (9.17) becomes

$$u=F=b_1q'+c, \tag{9.30}$$

where c is a function of u'. Equation (9.18) vanishes and differentiating (9.19) twice with respect to q' and then with respect to p' gives

$$(c_{u'}+b_1k)L_{uu} = 0.$$

Hence, either $c_{u'} \neq -b_1k$ and L linear in u or $c_{u'} = -b_1k$.

When $L=\mu_1u+\mu_2$, μ_1 and μ_2 are constants, it is easy to deduce from (9.19) that M is also linear in u. Using the forms of L and M, (9.19) becomes a polynomial in p' and q'. Equating coefficients we obtain four identities which involve k, l and c. One needs to take $\mu_1 = 0$ and $\mu_1 \neq 0$ separately, because these lead to different forms of k, l and c. In both cases we obtain excluded forms of f and f'.

When $c_{u'}=-b_1k$, equation (9.19) reduces to

$$b_1lL+M = b_1q'l_{u'}. \tag{9.31}$$

If the operator $D_1 \equiv c_{u'}\dfrac{\partial}{\partial q'} - b_1\dfrac{\partial}{\partial u'}$ is applied to (9.31), we obtain

$$L = \frac{q' l_{u'u'}}{l_{u'}} - \frac{c_{u'}}{b_1}. \tag{9.32}$$

Note that $l_{u'} \neq 0$ since otherwise from (9.9) p=constant. Applying D_1 to (9.32) now gives

$$b_1 q' \left[\frac{l_{u'u'}}{l_{u'}} \right]_{u'} - c_{u'u'} - \frac{l_{u'u'} c_{u'}}{l_{u'}} = 0. \tag{9.33}$$

The coefficients of (q') and (q'^0) in (9.33) will give the forms of l and c, which are either both linear in u' or $l = \lambda_2 e^{\lambda_1 u'} + \lambda_3$ and $c = \lambda_4 e^{-\lambda_1 u'} + \lambda_5$. Using the forms of l and c, the functions L and M can be found from equations (9.32) and (9.31) respectively. When both l and c are linear the result is that both f and f' are of the form (Fa). In the other case we obtain the result that the pair of partial differential equations

$$u_{xy} = \frac{\lambda_1}{b_1}(u - \lambda_5) u_x - \lambda_1 \lambda_3 (u - \lambda_5) - \lambda_1 \lambda_2 \lambda_4, \tag{9.34a}$$

$$u'_{xy} = \frac{\lambda_1 \lambda_4}{b_1} e^{-\lambda_1 u'} u'_x + \lambda_2 e^{\lambda_1 u'} + \lambda_3 \tag{9.34b}$$

are related by

$$u = b_1 u'_y + \lambda_4 e^{-\lambda_1 u'} + \lambda_5. \tag{9.34c}$$

iii) In this subcase we have $u = A(u', p')$ and equations (9.18) and (9.19) become linear in q'. Equating coefficients of powers of q' in both of these equations give

$$u = A = J + \lambda_1 u', \tag{9.35}$$

$$J_{p'} L = p' J_{p'p'} k + J_{p'p'} l + J_{p'} k, \tag{9.36}$$

$$\lambda_1 p' L + M = \lambda_1 p' k + \lambda_1 l + p'^2 J_{p'} k_{u'} + p' J_{p'} l_{u'}, \tag{9.37}$$

where λ_1 is a constant and J is a function of p'. Note that $J_{p'} \neq 0$ because otherwise we have a point transformation. Differentiating (9.36) with respect to u' and then integrating with respect to p' gives

$$\lambda_1 L = k_{u'} p' J_{p'} + l_{u'} J_{p'} + g, \tag{9.38}$$

g being a function of u'. Using (9.37) and (9.38), L can be eliminated and then applying the operator $D_2 = J_{p'} \frac{\partial}{\partial u'} - \lambda_1 \frac{\partial}{\partial p'}$ to the new equation gives

$$(\lambda_1 k_{u'} - g_{u'}) p' J_{p'} + \lambda_1 J_{p'} l_{u'} - \lambda_1^2 k + \lambda_1 g = 0. \tag{9.39}$$

Also eliminating L from (9.36) and (9.38) we obtain

$$\lambda_1 p' J_{p'p'} k + \lambda_1 J_{p'p'} l + \lambda_1 J_{p'} k - p' J_{p'}^2 k_{u'} - J_{p'}^2 l_{u'} - J_{p'} g = 0. \tag{9.40}$$

Equations (9.39) and (9.40) will give the functions J, k, l and g and then from (9.36) [or (9.38)] and (9.37) one can find L and M, respectively. If the coefficients of $(p'J_{p'})$, $(J_{p'})$ and the term independent of p' in (9.39) do not all vanish, then from (9.39) either $J = \lambda_4 \ln(p'+\lambda_2)+\lambda_3$ or $J = \lambda_4 p' + \lambda_3$, where $\lambda_2, \lambda_3, \lambda_4$ are constants and λ_4 is non-zero. For the first form of J, equation (9.7) becomes Liouville equation and (9.8) can be reduced to first order by direct integration. The second form implies that both f and f' are of the form (Fa).

Now if the coefficients of $(p'J_{p'})$, $(J_{p'})$ and the term independent of p' in (9.39) all vanish, that is

$$\lambda_1 k_{u'} - g_{u'} = \lambda_1 l_{u'} = \lambda_1(\lambda_1 k - g) = 0, \tag{9.41}$$

only equation (9.40) needs to be considered to give J, k, l and g. If $\lambda_1 \neq 0$ then $l=$ constant and $g=\lambda_1 k$. These results imply that $J=$ constant. When $\lambda_1 = 0$ (9.41) gives $g = \mu_1$, μ_1 constant and equation (9.40) becomes

$$p' J_{p'} k_{u'} + J_{p'} l_{u'} + \mu_1 = 0. \tag{9.42}$$

Note that if $k_{u'} = l_{u'} = \mu_1 = 0$ then from (9.37), $M=0$ and therefore both partial differential equations, (9.7) and (9.8), can be reduced to first order by direct integration. Thus, equation (9.42) implies, as before, that $J=\lambda_4 \ln(p'+\lambda_2)+\lambda_3$ or $J=\lambda_4 p'+\lambda_2$. It is straightforward to show that the second form leads to excluded forms of f and f'. Finally the first form of J gives k and l linear in u', from (9.42). Then from equations (9.36) and (9.37) the forms of L and M can be found. The pair of partial differential equations which are obtained are the same as in (9.34a,b), with u and u' interchanged. These two equations are related by

$$u = \lambda_4 \ln(\lambda_2 + u_x') + \lambda_3. \tag{9.43}$$

Note that (9.43) can also be obtained from (9.34c).

Case 2: f non-linear in p, f' linear in p'.

In this case we have

$$u = F = B(u', q'), \tag{9.44}$$

$$f' = kp' + l, \tag{9.45}$$

where k and l are functions of u'. Equations (9.12) and (9.13) give respectively,

$$p'B_{u'q'}+kp'B_{q'q'}+lB_{q'q'} = 0, \tag{9.46}$$

$$f(B, p'B_{u'}+kp'B_{q'}+lB_{q'}) = p'q'B_{u'u'}+kp'B_{u'}+lB_{u'}+p'q'B_{q'}k_{u'}+$$

$$q'B_{q'}l_{u'}+p'B_{q'}k^2+B_{q'}kl. \tag{9.47}$$

Since the right hand side of (9.47) is linear in p' then $f_{p'p'} = 0$ and therefore $(B_{u'}+kB_{q'})f_{pp} = 0$. The function f is not linear in p and hence,

$$B_{u'}+kB_{q'} = 0. \tag{9.48}$$

Since $l \neq 0$, [otherwise $p=0$ from (9.9)], equations (9.48) and (9.46) give $B=b_1q'+c$, where b_1 is a non-zero constant and c a function of u'. Therefore (9.48) and (9.47) become

$$c_{u'}+b_1k = 0,$$

$$f(b_1q'+c, b_1l) = b_1q'l_{u'}.$$

Since $p=b_1l(u')$, then $u' = l^{-1}(p/b_1)$, where l^{-1} is the inverse function of l, and from (9.44) $q' = \frac{1}{b_1}[u-c(l^{-1}(p/b_1))]$. Hence, the pair of partial differential equations

$$u_{xy} = [u-c(l^{-1}(u_x/b_1))]l'(l^{-1}(u_x/b_1)), \tag{9.49a}$$

$$u'_{xy} = -\frac{1}{b_1}c'(u')u'_x+l(u') \tag{9.49b}$$

are related by the BT

$$u = b_1u'_y+c(u'), \tag{9.49c}$$

where l and c are any arbitrary functions and l' and c' are the derived functions of l and c, respectively.

Since $u' = l^{-1}(u_x/b_1)$, in case 3 subcases in which F is a function of p' only, can be excluded because by interchanging primed and non-primed variables in (9.49) one gets the same results as in these subcases. One example of (9.49) is that the pair of equations $u_{xy} = ue^{-u_x}+1$ and $u'_{xy} = u'_x+\ln u'$ are related by $u = u'_y-u'$.

Case 3: f linear in p, f' non-linear in p'.

In this case we have

$$F=b_1q'+A, \tag{9.50}$$

$$f = Lp+M. \tag{9.51}$$

Using (9.50) and (9.51) the coefficients of powers of r' in (9.13) give

$$A_{p'}L = q'A_{u'p'}+f'A_{p'p'}+A_{p'}f'_{p'}, \tag{9.52}$$

$$(p'A_{u'}+b_1f')L+M = p'q'A_{u'u'}+p'f'A_{u'p'}+f'A_{u'}+$$

$$p'A_{p'}f'_{u'}+b_1q'f'_{u'}+b_1f'f'_{p'}. \tag{9.53}$$

As in case 1(b), it can easily be shown that there are three disjoint subcases:

(a) $L=\mu_1u+\mu_2$, $M=\mu_3u+\mu_4$, $A_{p'} \neq 0$ and $b_1\neq 0$, where μ_1, μ_2, μ_3 and μ_4 are constants,

(b) $A_{p'} = 0$ and $b_1 \neq 0$,

(c) $b_1=0$.

(a) Equating coefficients of powers of q' in (9.52) one obtains

$$A=Je^{b_1\mu_1u'}+d, \tag{9.54}$$

$$f' = \frac{\frac{1}{2}\mu_1A^2+\mu_2A+g}{A_{p'}}, \tag{9.55}$$

where d and g are functions of u' and J is a function of p'. The coefficient of q' in (9.53) gives

$$A_1p'J_{p'}+A_2J_{p'}+A_3J+A_4 = 0, \tag{9.56}$$

where

$$A_1 = (d_{u'u'}-b_1\mu_1d_{u'})e^{b_1\mu_1u'}, \quad A_2 = -b_1\mu_3e^{b_1\mu_1u'},$$

$$A_3 = (b_1\mu_1d_{u'}-b_1^2\mu_1^2d-b_1^2\mu_1\mu_2)e^{b_1\mu_1u'},$$

$$A_4 = b_1\mu_1dd_{u'}+b_1\mu_2d_{u'}+b_1g_{u'}-b_1^2\mu_1^2d^2-2b_1^2\mu_1\mu_2d-2b_1^2\mu_1g.$$

If $A_1 = A_2 = A_3 = A_4 = 0$ then

$$d = \lambda_1 e^{b_1\mu_1 u'} + \lambda_2,\ \ \mu_3 = 0,$$

$$g = \tfrac{1}{2}\lambda_3^2 e^{2b_1\mu_1 u'} + \tfrac{1}{2}\mu_1\lambda_2^2,\ \ \mu_2 = -\lambda_2\mu_1, \tag{9.57}$$

where λ_1, λ_2 and λ_3 are constants. The constant μ_1 must not be zero, because otherwise using (9.53) it can be shown that $f'=f'(p')$ and since $\mu_3 = 0$, both f and f' are of the excluded form (Fb). Using (9.57) equation (9.53) gives

$$\mu_4 = 0,$$

$$8\mu_1\lambda_3^2 p' J_{p'}^3 + J_{p'}[\mu_1(J+\lambda_1)^2+\lambda_3^2]_{p'}^2 - 2J_{p'p'}[\mu_1(J+\lambda_1)^2+\lambda_3^2]^2 = 0. \tag{9.58}$$

Integrating (9.58) with respect to p' gives

$$\frac{J_{p'}}{\mu_1(J+\lambda_1)^2+\lambda_3^2} = \frac{1}{2}\frac{1}{\sqrt{\tau^2-\mu_1\lambda_3^2 p'^2}}, \tag{9.59}$$

where τ^2 is a constant of integration . Using (9.57), (9.55) gives

$$f' = \tfrac{1}{2}e^{b_1\mu_1 u'}[\mu_1(J+\lambda_1)^2+\lambda_3^2]/J_{p'},$$

and therefore from (9.59)

$$f' = e^{b_1\mu_1 u'}\sqrt{\tau^2-\mu_1\lambda_3^2 p'^2}, \tag{9.60}$$

where $\tau \neq 0$ because otherwise f' becomes linear in p'. Hence, the transformation

$$u = (J+\lambda_1)e^{b_1\mu_1 u'} + b_1 u_y' + \lambda_2 \tag{9.61a}$$

relates the two partial differential equations

$$u_{xy} = (\mu_1 u - \lambda_2\mu_1)u_x, \tag{9.61b}$$

$$u_{xy}' = e^{b_1\mu_1 u'}\sqrt{\tau^2-\mu_1\lambda_3^2 u_x'^2}, \tag{9.61c}$$

where J [from (9.59)] is given by

$$J = \frac{\lambda_3^2 p'}{\tau+\sqrt{\tau^2-\mu_1\lambda_3^2 p'^2}} - \lambda_1, \tag{9.61d}$$

the constant of integration having been absorbed in λ_1.

Now if not all A_i vanish in (9.56), then there exists a non-trivial linear relationship connecting $p'J_{p'}$, $J_{p'}$, J and the term independent of p'. From this relationship it can be deduced that J must be one of the following forms

$$J=\lambda_4 e^{\lambda_1 p'}+\lambda_3, \quad \lambda_4 \neq 0, \lambda_1 \neq 0, \tag{9.62}$$

$$J=\lambda_4 \ln(p'+\lambda_2)+\lambda_3, \quad \lambda_4 \neq 0, \tag{9.63}$$

$$J=\lambda_4 (p'+\lambda_2)^{\lambda_1}+\lambda_3, \quad \lambda_4 \neq 0, \lambda_1 \neq 0, \tag{9.64}$$

where λ_1, λ_2, λ_3 and λ_4 are constants. Note that these constants are not the same as in (9.57). Using equation (9.53) and each form of J separately, the functions d and g can be found. The result is that all three forms of J will give f' linear in p'. Using (9.63), coefficients of $(\ln p')^4$ and $(\ln p')^2$ in (9.53) give $\mu_1 = 0$ and $\mu_2 = 0$ respectively. Hence, from (9.55) f' is linear in p'. A similar contradiction arises from (9.62). Finally using (9.64), equation (9.53) has the form

$$q' \sum_{j=0}^{1} (A_j p' + B_j)(p'+\lambda_2)^{j\lambda_1} + \sum_{j=0}^{4} (C_j p' + D_j)(p'+\lambda_2)^{j\lambda_1} = 0,$$

where A_j, B_j, C_j, D_j are determined functions of d, g and u'. Because one needs to equate coefficients of powers of p', attention must be paid to when $\lambda_1 = 1$ or -1. In fact if $\lambda_1 > 0$ the leading coefficient, C_4, gives $\lambda_1 = -1$ which is a contradiction. Similarly if $\lambda_1 < -1$, the coefficient of (p') gives again $\lambda_1 = -1$. Now taking $\lambda_1 = -1$, the coefficients of $(q'p'^2)$ and (p'^5) give respectively

$$g = \lambda_5 e^{2b_1\mu_1 u'} - \lambda_3 e^{b_1\mu_1 u'}(\mu_1 d + \mu_2) - \tfrac{1}{2}\mu_1 d^2 - \mu_2 d,$$

$$\lambda_5 = -\tfrac{1}{2}\mu_1 \lambda_3^2,$$

where λ_5 is another constant. Using the above results, (9.19) gives the contradiction that f' is linear in p'.

(b) In this subcase

$$u=b_1 q' + c, \tag{9.65}$$

where c is a function of u'. Equation (9.52) vanishes and using (9.53), since f' is not linear in p', it can be shown, in the same way as in case 1(b)(ii) that $L=\mu_1 u+\mu_2$ and $M=\mu_3 u+\mu_4$, where μ_1, μ_2, μ_3 and μ_4 are constants. Taking $\mu_1 \neq 0$ the coefficients of (q') and (q'^0) in (9.53) give respectively

$$f' = J e^{b_1\mu_1 u'} - c_{u'} p'/b_1 - \mu_3/b_1\mu_1, \tag{9.66}$$

$$p' J_{p'} c_{u'} + b_1\mu_1 J c - b_1 J J_{p'} e^{b_1\mu_1 u'} + \mu_3 J_{p'}/\mu_1 +$$

$$b_1\mu_2 J + \mu_4 e^{-b_1\mu_1 u'} - \mu_2\mu_3 e^{-b_1\mu_1 u'}/\mu_1 = 0, \tag{9.67}$$

where J is a non-linear function of p'. Note that the coefficients of $c_{u'}$ and c in (9.67) do not vanish. Therefore there exists a linear relationship connecting $c_{u'}$, u, 1, $e^{b_1\mu_1 u'}$ and $e^{-b_1\mu_1 u'}$. From this relationship one can deduce that c must be one of the following forms:

$$c=\lambda_2 e^{\mu_1 b_1 u'}+\lambda_3 e^{-\mu_1 b_1 u'}+\lambda_4 e^{\lambda_1 u'}+\lambda_5,\ \lambda_1 \neq 0, \mu_1 b_1,\ -\mu_1 b_1, \tag{9.68}$$

$$c=\lambda_2 e^{\mu_1 b_1 u'}+\lambda_3 e^{-\mu_1 b_1 u'}+\lambda_4 u' e^{\mu_1 b_1 u'}+\lambda_5,\ \lambda_4 \neq 0, \tag{9.69}$$

$$c=\lambda_2 e^{\mu_1 b_1 u'}+\lambda_3 e^{-\mu_1 b_1 u'}+\lambda_4 u' e^{-\mu_1 b_1 u'}+\lambda_5,\ \lambda_4 \neq 0, \tag{9.70}$$

where λ_1, λ_2, λ_3, λ_4, and λ_5 are constants. Using each form of c separately and equation (9.67), one can find J. It is easy to check that (9.69) and (9.70) give J=0 and J linear in p' respectively. Finally using (9.68), the coefficient of $(e^{b_1\mu_1 u'}e^{\lambda_1 u'})$ in (9.67) gives either $\lambda_4 = 0$ or $J = p'^{(-b_1\mu_1/\lambda_1)}$. If $J=p'^{(-b_1\mu_1/\lambda_1)}$ then from (9.67) $b_1 = 0$ which is a contradiction. Taking $\lambda_4 = 0$, coefficients of $(e^{\lambda_1 u'})$, (u'^0), and $(e^{b_1\mu_1 u'})$ give $\mu_3 = 0, \mu_2 = -\mu_1\lambda_5, \lambda_3 = 0, \mu_4 = 0$ and

$$\mu_1\lambda_2 p' J_{p'} - J J_{p'} + \mu_1\lambda_2 J = 0. \tag{9.71}$$

Equation (9.71) can be solved to give

$$J = \lambda_2\mu_1 p' + \sqrt{\lambda_2^2\mu_1^2 p'^2 - \tau^2}, \tag{9.72}$$

where τ^2 is a non-zero constant of integration. Therefore we have the result that the pair of partial differential equations

$$u_{xy} = (\mu_1 u - \mu_1\lambda_5)u_x, \tag{9.73a}$$

$$u'_{xy} = e^{b_1\mu_1 u'}\sqrt{\lambda_2^2\mu_1^2 u_x'^2 - \tau^2} \tag{9.73b}$$

are related by the BT

$$u = b_1 u'_y + \lambda_2 e^{b_1\mu_1 u'} + \lambda_5. \tag{9.73c}$$

Note that the partial differential equations in (9.73) are the same as in (9.61) but the BTs which connect them are different.

Now let $\mu_1 = 0$. From (9.53) $f' = J - p'c_{u'}/b_1 + \mu_3 u'$ and either $c = \lambda_2 e^{\lambda_1 u'} + \lambda_3 u' + \lambda_4$ or $c = \lambda_2 u'^2 + \lambda_3 u' + \lambda_4$. Both of these forms of c lead to f' being a function of p' only and $\mu_3 = 0$ which means that both f and f' are of the excluded form (Fb).

(c) Equating coefficients of powers of q' in (9.62) and (9.63) we obtain

$$u=A = J+\lambda_1 u', \tag{9.74}$$

$$J_{p'}L = J_{p'p'}f'+J_{p'}f'_{p'}, \tag{9.75}$$

$$\lambda_1 p'L+M = \lambda_1 f'+p'J_{p'}f'_{u'}, \tag{9.76}$$

where J is a function of p' and λ_1 is non-zero constant. If $\lambda_1=0$ then A is a function of p' only and as was noted in case 2, we are simply led to the partial differential equations (9.49a,b) with the primed and non-primed variables interchanged. The BT which relates them is given by $u = l^{-1}(u'_x/b_1)$.

Now differentiating (9.75) with respect to u' and integrating with respect to p' one obtains

$$\lambda_1 L = J_{p'}f'_{u'}+g, \tag{9.77}$$

where g is a function of u'. Equations (9.76) and (9.77) now yield

$$M=\lambda_1 f'-p'g. \tag{9.78}$$

Eliminating f' from (9.77) and (9.78) one obtains

$$\lambda_1^2 L = \lambda_1 J_{p'}M_u+p'J_{p'}g_{u'}+\lambda_1 g. \tag{9.79}$$

Applying the operator $D_2 \equiv J_{p'}\dfrac{\partial}{\partial u'}-\lambda_1\dfrac{\partial}{\partial p'}$ onto (9.79) now gives

$$\lambda_1^2 M_u = Ig_{u'u'}-\lambda_1 p'g_{u'}, \tag{9.80}$$

where $I=p'J_{p'}^2/J_{p'p'}$. Note that it has been assumed that $J_{p'p'} \neq 0$. The case $J_{p'p'}=0$ is examined later. Finally application of the operator D_2 onto (9.80) gives

$$J_{p'}Ig_{u'u'u'}-\lambda_1(I_{p'}+p'J_{p'})g_{u'u'}+\lambda_1^2 g_{u'} = 0. \tag{9.81}$$

From (9.81) it may be deduced that there exists a non-trivial linear relation connecting $g_{u'}$, $g_{u'u'}$ and $g_{u'u'u'}$. This differential equation for g can be solved to give the following four possible solutions:

$$g=\lambda_3 e^{\lambda_2 u'}+\lambda_5 e^{\lambda_4 u'}+\lambda_6,\ \ \lambda_2 \neq \lambda_4,\ \lambda_2\neq 0,\ \lambda_4\neq 0, \tag{9.82}$$

$$g = \lambda_3 e^{\lambda_2 u'}+\lambda_5 u'e^{\lambda_2 u'}+\lambda_6,\ \ \lambda_2\neq 0,\ \lambda_5\neq 0, \tag{9.83}$$

$$g = \lambda_3 e^{\lambda_2 u'}+\lambda_5 u'+\lambda_6,\ \ \lambda_2\neq 0,\ \lambda_3\neq 0,\ \lambda_5\neq 0, \tag{9.84}$$

$$g = \lambda_3 u'^2+\lambda_5 u'+\lambda_6. \tag{9.85}$$

Note that $g \neq$constant because otherwise f' is linear in p' from (9.80) and (9.78).

Using each form of g separately, the functions J, M, f' and L can be found from the equations (9.81), (9.80), (9.78) and (9.77), respectively, and then equation (9.75) will give more restrictions on the forms of these four functions. It is straightforward, using these equations, to show that the forms of g given by (9.83) to (9.85) lead to no results. Substituting the final form of g, given by (9.82), into (9.81) it can be seen, by considering the coefficients of $(e^{\lambda_2 u'})$ and $(e^{\lambda_4 u'})$, that this analysis can be split into two cases:(i) $\lambda_4 \neq 0$ and (ii) $\lambda_4 = 0$.

(i) The coefficients of $(e^{\lambda_2 u'})$ and $(e^{\lambda_4 u'})$ in (9.81) give

$$J_{p'} = \frac{\lambda_1}{\lambda_2}(\mu^2 + p'^2)^{-\frac{1}{2}}, \tag{9.86}$$

$$\lambda_4 = -\lambda_2, \tag{9.87}$$

where μ^2 is a constant of integration. If this constant is zero, equations (9.80) and (9.78), imply that f' is linear in p'. The constant μ^2 has been taken as positive but the negative case may easily be obtained by replacing μ by $i\mu$ and suitably modifying other constants to give real results . Solving (9.86) gives

$$J = \frac{\lambda_1}{\lambda_2}\sinh^{-1}\frac{p'}{\mu} + \nu, \tag{9.88}$$

where ν is a constant of integration. Using (9.86) - (9.88), equations (9.80), (9.78) and (9.77) imply that

$$M = -\mu(\lambda_3 e^{\lambda_2(u-\nu)/\lambda_1} - \lambda_5 e^{-\lambda_2(u-\nu)/\lambda_1}) + \tau,$$

$$f' = \frac{1}{\lambda_1}[\lambda_6 p' - (\lambda_3 e^{\lambda_2 u'} - \lambda_5 e^{-\lambda_2 u'})\sqrt{\mu^2 + p'^2}] + \tau,$$

$$L = \lambda_6 / \lambda_1,$$

respectively, where τ is a constant of integration. Equation (9.75) now gives $\lambda_6 = 0$ and $\tau = 0$. Thus, finally,

$$u = \frac{\lambda_1}{\lambda_2}\sinh^{-1}\frac{u'_x}{\mu} + \lambda_1 u' + \nu \tag{9.89a}$$

relates the pair of partial differential equations

$$u_{xy} = -\mu(\lambda_3 e^{\lambda_2(u-v)/\lambda_1} - \lambda_5 e^{-\lambda_2(u-v)/\lambda_1}), \tag{9.89b}$$

$$u'_{xy} = -\frac{1}{\lambda_1}(\lambda_3 e^{\lambda_2 u'} - \lambda_5 e^{-\lambda_2 u'})\sqrt{\mu^2 + p'^2}. \tag{9.89c}$$

The Bäcklund transformation (9.89) is a generalisation of Nakamura's result [70], where the sine-Gordon and sinh-Gordon equations are related to the modified sine-Gordon, $u_{xy} = \sin u\sqrt{1-(\mu u_x)^2}$, and modified sinh-Gordon, $u_{xy} = \sinh u\sqrt{1-(\mu u_x)^2}$, respectively. Choosing suitable constants (9.89a) also provides a BT relating Liouville's equation to the modified Liouville's equation $u_{xy} = e^u\sqrt{1-(\mu u_x)^2}$.

(ii) Since $\lambda_4 = 0$, $g = \lambda_3 e^{\lambda_2 u'} + \lambda_6$, where λ_5 has been absorbed in λ_6. Integrating (9.81) twice with respect to p', gives

$$\mu J_{p'} + \lambda_1 p' J_{p'} e^{-\lambda_2 J/\lambda_1} + \frac{\lambda_1^2}{\lambda_2} e^{-\lambda_2 J/\lambda_1} = v, \tag{9.90}$$

where μ and v are constants. From equations (9.80), (9.78) and (9.77) it is found that

$$M = \frac{\lambda_3}{\lambda_1}\mu e^{\lambda_2 u/\lambda_1} + \tau,$$

$$f' = \frac{\mu\lambda_3}{\lambda_1^2} e^{\lambda_2 J/\lambda_1} e^{\lambda_2 u'} + \frac{1}{\lambda_1}(\lambda_3 e^{\lambda_2 u'} + \lambda_6)p' + \frac{\tau}{\lambda_1},$$

$$L = \frac{\lambda_2\lambda_3 v}{\lambda_1^3} e^{\lambda_2 u/\lambda_1} + \frac{\lambda_6}{\lambda_1},$$

respectively, where τ is a constant of integration. Using equation (9.75) we obtain $\tau = 0$ and $\lambda_6 = 0$. Therefore we have the result that the pair of partial differential equations

$$u_{xy} = \frac{\lambda_2\lambda_3 v}{\lambda_1^3} e^{\lambda_2 u/\lambda_1} u_x + \frac{\mu\lambda_3}{\lambda_1} e^{\lambda_2 u/\lambda_1}, \tag{9.91a}$$

$$u'_{xy} = \frac{\mu\lambda_3}{\lambda_1^2} e^{\lambda_2 J/\lambda_1} e^{\lambda_2 u'} + \frac{\lambda_3}{\lambda_1} e^{\lambda_2 u'} u'_x \tag{9.91b}$$

are related by the Bäcklund transformation

$$u = J + \lambda_1 u', \tag{9.91c}$$

where J is given by the ordinary differential equation (9.90). If $v = 0$ and $\mu = -\mu$ in (9.91) and (9.90), then from (9.90), $J = \frac{\lambda_1}{\lambda_2}\sinh^{-1}(\lambda_1 p'/\mu)$. We note that if μ is replaced by μ/λ_1 and let $\lambda_5 = 0$ in (9.89), then the two BTs, (9.89) and (9.91), become identical.

Case 3 will be concluded by taking $J=\lambda_2 p'+\lambda_3$. Application of the operator $D_2 = \lambda_2\frac{\partial}{\partial u'} - \lambda_1\frac{\partial}{\partial p'}$ onto (9.79) gives $g = \lambda_4 u' + \lambda_5$. Applying D_2 to (9.78) gives

$$\lambda_1^2 f'_{p'} - \lambda_1\lambda_2 f'_{u'} = \lambda_1\lambda_4 u' - \lambda_2\lambda_4 p' + \lambda_1\lambda_5. \tag{9.92}$$

Equation (9.92) is a first order partial differential equation which can be solved by the method of characteristics. Hence,

$$f' = \Phi(\lambda_1 u' + \lambda_2 p') - \frac{\lambda_4}{2\lambda_2}u'^2 - \frac{\lambda_4\lambda_2}{2\lambda_1^2}p'^2 - \frac{\lambda_5}{\lambda_2}u', \tag{9.93}$$

where Φ is an arbitrary function. From equations (9.78) and (9.77) one can find M and L respectively, and from (9.75) $\lambda_4 = 0$. This leads to the result that the BT

$$u = \lambda_2 u'_x + \lambda_1 u' + \lambda_3 \tag{9.94a}$$

relates the pair of partial differential equations

$$u_{xy} = \lambda_2 \Phi'(u-\lambda_3)u_x + \lambda_1\Phi(u-\lambda_3) - \frac{\lambda_5}{\lambda_2}[u-\lambda_3], \tag{9.94b}$$

$$u'_{xy} = \Phi(\lambda_1 u' + \lambda_2 u'_x) - \frac{\lambda_5}{\lambda_2}u', \tag{9.94c}$$

where Φ' is the derived function of Φ. Since it is required that equation (9.94c) must not be linear in p', the function Φ must not be linear.

Case 4: f non-linear in p and f' non-linear in p'

In this case F is given by

$$u=F=b_1 q' + c, \quad b_1 \neq 0. \tag{9.95}$$

Equation (9.13) reduces to

$$f(u,p)=p'q'c_{u'u'} + f'c_{u'} + b_1 q' f'_{u'} + b_1 f' f'_{p'}. \tag{9.96}$$

The right hand side of (9.96) is linear in q', so that $f_{q'q'}=0$ and therefore $b_1 f_{uu} = 0$, since p is independent of q' from (9.9). Hence,

$$f = uR + S, \tag{9.97}$$

where R and S are functions of p. Equating coefficients of powers of q' in (9.96) we obtain

$$b_1R = p'c_{u'u'} + b_1f'_{u'}, \tag{9.98}$$

$$cR + S = f'c_{u'} + b_1f'f'_{p'}. \tag{9.99}$$

Equation (9.98) is of the form $b_1R(p) = p_{u'}$ which has solution of the form

$$p = Q(u'+J), \tag{9.100}$$

for some function Q, where J is a function of p'. Letting $\xi = u'+J$, equations (9.98) and (9.99) may be written

$$b_1f' = -p'c_{u'} + Q(\xi), \tag{9.101}$$

$$S(Q) = \frac{1}{b_1}Q_\xi[-p'J_{p'}c_{u'} + QJ_{p'} - c], \tag{9.102}$$

respectively. Note from (9.101) that J must not be a constant because otherwise f' is linear in p'. Let $u' = \xi - J$ and then differentiate (9.102) with respect to p' keeping ξ constant. This gives

$$c_{u'u'}p'J_{p'}^2 - c_{u'}p'J_{p'p'} + QJ_{p'p'} = 0. \tag{9.103}$$

Now take $J_{p'p'} \neq 0$, (the case $J_{p'p'} = 0$ is examined at the end of this section). If equation (9.103) is divided through by $J_{p'p'}$ and then differentiated with respect to p', keeping ξ constant, we obtain

$$c_{u'} - c_{u'u'}(p'J_{p'} + I_{p'}) + c_{u'u'u'}IJ_{p'} = 0, \tag{9.104}$$

where $I = p'J_{p'}^2/J_{p'p'}$. Identity (9.104) is the same as (9.81), with g replaced by c and $\lambda_1 = 1$. Therefore, as in case 3(c), c must take one of four different forms. As before, only the form $c = \lambda_3e^{\lambda_3u'} + \lambda_5e^{\lambda_4u'} + \lambda_6$ will be considered. It is easy to check that the other three forms lead nowhere. As in case 3(c) there are two subcases:(i) $c = \lambda_3e^{\lambda_2u'} + \lambda_5e^{-\lambda_2u'} + \lambda_6$ and (ii) $c = \lambda_3e^{\lambda_2u'} + \lambda_6$.

(i) J is given by (9.88), where $\lambda_1 = 1$. The function Q can be found from (9.103) and then the functions f', R and S can be found from (9.101), (9.98) and (9.102) respectively. Therefore it is seen that the pair of partial differential equations

$$u_{xy} = \frac{\lambda_2}{b_1}(u-\lambda_6)\sqrt{u_x^2 - 4\lambda_2^2\lambda_3\lambda_5\mu^2}, \tag{9.105a}$$

$$u'_{xy} = \frac{\lambda_2}{b_1}(\lambda_3e^{\lambda_2u'} + \lambda_5e^{-\lambda_2u'})\sqrt{u_x'^2 + \mu^2} \tag{9.105b}$$

are related by the BT

$$u = b_1 u'_y + \lambda_3 e^{\lambda_2 u'} + \lambda_5 e^{-\lambda_2 u'} + \lambda_6. \tag{9.105c}$$

A different BT to that obtained earlier in (9.89) for the modified sine-Gordon and modified sinh-Gordon equations can be obtained from (9.105).

(ii) Comparing with case 3(c)(ii) and using the same equations as in case 4(i) it can be shown that the two equations

$$u_{xy} = \frac{\lambda_2}{b_1} u u_x + \frac{\lambda_2 \nu}{\mu b_1} u_x^2 - \frac{\lambda_2 \lambda_6}{b_1} u_x, \tag{9.106a}$$

$$u'_{xy} = -\frac{\lambda_2 \lambda_3}{b_1} e^{\lambda_2 u'} (p' + \mu e^{\lambda_2 J}) \tag{9.106b}$$

are related by the BT

$$u = b_1 u'_y + \lambda_3 e^{\lambda_2 u'} + \lambda_6, \tag{9.106c}$$

where J is given by the ordinary differential equation

$$\mu J_{p'} + p' J_{p'} e^{-\lambda_2 J} + e^{-\lambda_2 J} / \lambda_2 = \nu, \tag{9.107}$$

μ and ν being constants. It is noted that $\nu = 0, \mu = -\mu, \lambda_2 = b_1 \mu, \lambda_6 = \lambda_5$ and $\lambda_3 = \lambda_2$ in (9.106), makes this BT identical to (9.73).

Finally if $J = \lambda_1 p' + \lambda_2$ it can be shown that the transformation

$$u = b_1 u'_y + \lambda_3 u' + \lambda_4 \tag{9.108a}$$

relates the pair of partial differential equations

$$u_{xy} = \frac{1}{b_1} u Q'(Q^{-1}(u_x)) + \frac{1}{b_1} Q'(Q^{-1}(u_x))[\lambda_1 u_x - \lambda_3 Q^{-1}(u_x) + \lambda_2 \lambda_3 - \lambda_4] \tag{9.108b}$$

$$u'_{xy} = -\frac{\lambda_3}{b_1} u'_x + \frac{1}{b_1} Q(\lambda_1 u'_x + u' + \lambda_2), \tag{9.108c}$$

where λ_1, λ_2, λ_3 and λ_4 are constants, λ_1 and λ_3 are non-zero, Q is an arbitrary function and Q' the derived function of Q.

Summary: We now list the main results of this section. Note that if (9.6)-(9.8) can be satisfied for suitable functions $F, f,$ and f' then under linear transformations $u_1 = \alpha u + \beta$, $u'_1 = \alpha' u' + \beta'$, where α, β, α' and β' are constants and α and α' are non-zero, the new transformed equations (9.6)-(9.8) take the same form and may be satisfied for suitable functions F_1, f_1 and f_1'. This enables some constants to be removed which clarifies the nature of some of the following transformations. We note that these transformations can then easily be generalised by making linear transformations. In what follows, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu, v, \tau$ and b_1 are all constants. We also note that the BTs (9.34) and (9.73) are special cases of the BT (9.106) and (9.89) is a special case of (9.91).

1.
$$u_{xy} = uu_x - \lambda_2\lambda_4 \text{ and}$$
$$u'_{xy} = \lambda_4 e^{-u'} u'_x + \lambda_2 e^{u'} \tag{9.34}$$
are related by the BT $u = u'_y + \lambda_4 e^{-u'}$.

2.
$$u_{xy} = [u - c(l^{-1}(u_x))]\, l'(l^{-1}(u_x)) \text{ and}$$
$$u'_{xy} = -c'(u')u'_x + l(u') \tag{9.49}$$
are related by the BT $u = u'_y + c(u')$,

where l and c are arbitrary functions and l' and c' are the derived functions of l and c respectively.

3.
$$u_{xy} = uu_x \text{ and}$$
$$u'_{xy} = e^{b_1 u'}\sqrt{\tau^2 - (\lambda_3 u'_x)^2} \tag{9.61}$$
are related by the BT $u = Je^{b_1 u'} + b_1 u'_y$,

where
$$J = \frac{\lambda_3^2 p'}{\tau + \sqrt{\tau^2 - \lambda_3^2 p'^2}}.$$

4.
$$u_{xy} = uu_x \text{ and}$$
$$u'_{xy} = \lambda_2 e^{u'}\sqrt{u_x'^2 - \tau^2} \tag{9.73}$$
are related by the BT $u = u'_y + \lambda_2 e^{u'}$.

5.
$$u_{xy} = \mu(\lambda_3 e^{\lambda_2 u} - \lambda_5 e^{-\lambda_2 u}) \text{ and}$$
$$u'_{xy} = (\lambda_3 e^{\lambda_2 u'} - \lambda_5 e^{-\lambda_2 u'})\sqrt{\mu^2 + u_x'^2} \tag{9.89}$$
are related by the BT $u = \frac{1}{\lambda_2}\sinh^{-1}\frac{u'_x}{\mu} + u'$.

6. $$u_{xy} = \lambda_2 v e^{\lambda_2 u} u_x + \mu e^{\lambda_2 u} \text{ and}$$

$$u'_{xy} = \mu e^{\lambda_2 J} e^{\lambda_2 u'} + e^{\lambda_2 u'} u'_x \tag{9.91}$$

are related by the BT $u = J + u'$,

where J is given by the equation

$$\lambda_2 \mu J_{p'} + \lambda_2 p' J_{p'} e^{-\lambda_2 J} + e^{-\lambda_2 J} = \lambda_2 v. \tag{9.109}$$

7. $$u_{xy} = \Phi'(u) u_x + \lambda_1 \Phi(u) - \lambda_5 u \text{ and}$$

$$u'_{xy} = \Phi(\lambda_1 u' + u'_x) - \lambda_5 u' \tag{9.94}$$

are related by the BT $u = u'_x + \lambda_1 u'$,

where Φ is an arbitrary function and Φ' its derived function.

8. $$u_{xy} = \lambda_2 u \sqrt{u_x^2 - 4\lambda_2^2 \lambda_3 \lambda_5 \mu^2} \text{ and}$$

$$u'_{xy} = \lambda_2 (\lambda_3 e^{\lambda_2 u'} + \lambda_5 e^{-\lambda_2 u'}) \sqrt{u_x'^2 + \mu^2} \tag{9.105}$$

are related by the BT $u = u'_y + \lambda_3 e^{\lambda_2 u'} + \lambda_5 e^{-\lambda_2 u'}$.

9. $$u_{xy} = \lambda_2 u u_x + \frac{\lambda_2 v}{\mu} u_x^2 \text{ and}$$

$$u'_{xy} = -\lambda_2 \lambda_3 e^{\lambda_2 u'} (u'_x + \mu e^{\lambda_2 J}) \tag{9.106}$$

are related by the BT $u = u'_y + \lambda_3 e^{\lambda_2 u'}$,

where J is given by the ordinary differential equation (9.109).

10. $$u_{xy} = Q'(Q^{-1}(u_x))[\lambda_1 u_x - \lambda_3 Q^{-1}(u_x) + u + \lambda_2 \lambda_3] \text{ and}$$

$$u'_{xy} = -\lambda_3 u'_x + Q(\lambda_1 u'_x + u' + \lambda_2) \tag{9.108}$$

are related by the BT $u = u'_y + \lambda_3 u'$,

where Q is an arbitrary function and Q' is its derived function.

In ref [8] is shown that the modified Liouville equation, the modified sine-Gordon equation and the modified sinh-Gordon equation admit auto-BTs. Here, in addition to other results, we have shown that the modified Liouville equation can also be related to equation $u_{xy} = u u_x$ by two different BTs [(9.61) and (9.73)) and to the standard Liouville equation by Bäcklund transformation (9.89), the latter being a generalisation of Nakamura's result [70]. Also in this paper we have found two different BTs for each of the modified sine-Gordon and the modified sinh-Gordon equations. One of these is the Nakamura transformation and the other is given by the BT (9.105).

3. Bäcklund Transformations for KdV Type Equations

Let us now consider the KdV type equations

$$u_y + u_{xxx} + f(u, u_x) = 0 \qquad (9.110)$$

and

$$u'_y + u'_{xxx} + f'(u', u'_x) = 0. \qquad (9.111)$$

We briefly explain how one can explore the range of functions f and f' for which the equations (9.110) and (9.111) admit BTs of the class

$$u = F(u', p', q', r', s', t'). \qquad (9.112)$$

As in the previous example we will exclude cases where both f and f' are linear in both of their arguments, and also point transformations. The results of this section could be seen as a generalisation of Miura transformation (see section 1) [66].

Upon differentiation of (9.112) with respect to x and y we obtain respectively

$$u_x = p'F_{u'} + r'F_{p'} + s'F_{q'} + \alpha'F_{r'} + \beta'F_{s'} + \gamma'F_{t'}, \qquad (9.113)$$

$$u_y = q'F_{u'} + s'F_{p'} + t'F_{q'} + \beta'F_{r'} + \gamma'F_{s'} + \delta'F_{t'}, \qquad (9.114)$$

where $\alpha' = u'_{xxx}$, $\beta' = u'_{xxy}$, $\gamma' = u'_{xyy}$ and $\delta' = u'_{yyy}$. Remembering that $\alpha' = -q' - f'$, from equation (9.111), differentiating (9.113) twice with respect to x, a similar expression for u_{xxx} can be obtained. Upon substitution of this expression for u_{xxx} and (9.114) into (9.110) results to

$$u_y + u_{xxx} + f = E(u', p', q', r', s', t', \beta', \gamma', \varepsilon') = 0, \qquad (9.115)$$

where $\varepsilon' = u'_{xxyy}$, for some function E. This function may be calculated explicitly in terms of F, its derivatives, f, its derivatives, u', p', q', r', s', t', β', γ' and ε'. In view of (9.110) E must be identically zero with u', p', q', r', s', t', β', γ' and ε' regarded as independent variables.

A sequence of calculations of various derivatives of E, namely, $E_{\varepsilon'}$, $E_{\gamma'\gamma'}$, $E_{t'}$, $E_{\beta'\beta'}$, $E_{\beta'}$ and $E_{s's'}$, imply that F does not depend on t', s' and q'. Now $E_{s'}$ imply that

$$F = \lambda r' + A(u', p'),$$

where λ is a constant. Differentiating E with respect to q' twice it gives

$$E_{q'q'} = \lambda^2 f_{pp} = 0.$$

From this last result it is seen that the analysis may be split into two disjoint cases:(1) $\lambda \neq 0$ and f linear in p and (2) $\lambda = 0$.

Case 1: Calculations of $E_{q'}$, $E_{r'r'}$ and $E_{r'}$ lead to the following results

$$F = \lambda r' - \tfrac{1}{6}\lambda^2\mu_1 p'^2 + a(u')p' + b(u'), \tag{9.116}$$

$$f = (\mu_1 u + \mu_2)p, \tag{9.117}$$

$$f' = -\tfrac{1}{18}\lambda^2\mu_1^2 p'^3 + \tfrac{1}{2}\mu_1 a p'^2 + \mu_1 b p' + \mu_2 p' + \frac{3}{2\lambda} a_{u'} p'^2, \tag{9.118}$$

where,

$$a = \nu_1 \exp(-\tfrac{1}{3}\lambda\mu_1 u') + \nu_2,$$

$$b = \nu_5 \exp(\tfrac{2}{3}\lambda\mu_1 u') - \frac{3\nu_1}{8\mu_1\lambda^2}\exp(-\tfrac{2}{3}\lambda\mu_1 u') - \frac{\nu_1\nu_2}{\mu_1\lambda^2}\exp(-\tfrac{1}{3}\lambda\mu_1 u') - \frac{3(\nu_2^2 + \nu_4)}{4\mu_1\lambda^2},$$

where μ_1, μ_2, ν_1, ν_2, ν_3, ν_4 and ν_5 are all constants. The constant μ_1 has been taken as nonzero because otherwise we obtain f and f' both of the excluded (linear) forms.

Finally equating coefficients of powers of p' in $E=0$, we must either have (a) $\nu_2 = \nu_3 = 0$ or (b) $\nu_2 \neq 0$, $\nu_1 = \nu_5 = 0$. These two subcases give all possible forms of F, f and f' (equations (9.116) - (9.118)).

We conclude this case by giving an example. Letting $\mu_1 = 6$ and $\mu_2 = \nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu_5 = 0$ we deduce that the pair of partial differential equations

$$u_y + u_{xxx} + 6uu_x = 0,$$

$$u'_y + u'_{xxx} - 2\lambda^2 u_x'^3 = 0$$

are related by the BT

$$u = \lambda u'_{xx} - \lambda^2 u_x'^2.$$

Case 2: Upon differentiating E firstly, with respect to q' and secondly, twice with respect to r' yield

$$F = \lambda p' + c(u'),$$

$$f = L(u)p + M(u),$$

respectively. Let us now equate coefficients of powers of r' in $E=0$ to give

$$\lambda L = \lambda f'_{p'} - 3p' c_{u'u'}, \tag{9.119}$$

$$p' c_{u'} L + M = f' c_{u'} - p'^3 c_{u'u'u'} + \lambda p' f'_{u'}. \tag{9.120}$$

These two equations impose restrictions on the functional forms of L, M, f' and c and that ultimately enable the BTs to be derived.

Without presenting any more detailed computations, we state the transformations which are obtained in this case. If $c_{u'u'} \neq 0$, then equations (9.119) and (9.120) lead to the following theorem:

Theorem 9.1: The pair of partial differential equations

$$u_y + u_{xxx} + \frac{1}{\lambda^2}[\tfrac{3}{2}\mu(u-\mu\tau)^2 - 3\tau(1-\mu^2)(u-\theta) -$$

$$\tfrac{3}{2}(\phi^2+\mu\theta^2) + \nu]u_x = 0, \tag{9.121a}$$

$$u'_y + u'_{xxx} + \tfrac{1}{2}\mu u_x'^3 - \frac{3}{2\lambda^2}c_{u'}^2 u'_x + \frac{\nu}{\lambda^2}u'_x = 0 \tag{9.121b}$$

are related by the BT

$$u = \lambda u'_x + c, \tag{9.121c}$$

where c is a function of u' and is given by the ordinary differential equation

$$c_{u'u'} + \mu c = \tau, \quad c_{u'u'} \neq 0, \tag{9.121d}$$

where $\lambda, \nu, \mu, \tau, \phi, \theta$ are constants, $\mu = 0, -1, 1$ and ϕ and θ are constants of integration obtained by solving (9.121d). Some of these constants may be fixed without loss of generality. In particular by appropriate scaling of u and u' and writing equations (9.121a) and (9.121b) in a moving frame we may take $\lambda = 1$ and $\nu = 0$. Nevertheless we have not fixed these constants, so one can easily deduce special cases of this theorem. For example, taking $\lambda=1, \tau = -2, \mu = \nu = \phi = \theta = 0$ leads to the Miura's result. For $\mu = -1, 1$ we obtain transformations between modified KdV equation and some equations introduced by Calogero and Degasperis [20].

Finally, if $c = \nu u + \tau$, one finds the following two transformations:

(a) $$u = \lambda u'_x + \nu u' + \tau$$

relates the two partial differential equations

$$u_y + u_{xxx} + [\lambda M'(u) + \mu] u_x + M(u) = 0,$$

$$u'_y + u'_{xxx} + \frac{1}{\nu}[M(\lambda u'_x + \nu u' + \tau) + \mu u'_x] = 0,$$

where M is an arbitrary function, M' is its derived function and μ is a constant.

(b) $$u = \lambda u'_x + \tau$$

relates the equations

$$u_y + u_{xxx} + J'\left(\frac{u-\tau}{\lambda}\right) u_x - \frac{\mu}{\lambda} u + \frac{\tau\mu}{\lambda} = 0$$

$$u'_y + u'_{xxx} - \frac{\mu}{\lambda} u' + J(u'_x) = 0,$$

where J is an arbitrary function, J' is its derived function and μ is a constant.

4. Generation of the General Class of BTs Using the Special Class

In chapter 7 section 3 we saw the BTs which relate the KdV and the modified KdV equations derived by Lamb [57]. We noted that one of the BTs is in fact the Miura transformation. Now the following question arises: how does one generate BTs of the general class (7.4) using the special class (7.50)? This can be done if F in (7.50) does not depend on both p' and q'. Let, say, F is independent of q', then solving (7.50) for p' we obtain

$$p' = \Psi(u, u'), \tag{9.122}$$

where Ψ is determined function. Also consider

$$q' = \Phi(u, u', p). \tag{9.123}$$

Hence, (9.122) and (9.123) are a complete pair of BTs of the general class. The usual procedure, which was described in chapter 7 section 2, will also determine the function Φ.

Let us proceed by giving a detailed example. Consider the pair of partial differential equations

$$u_{xy} = \cosh u\sqrt{u_x^2+\mu^2}, \tag{9.124a}$$

$$u'_{xy} = u'\sqrt{u_x'^2-\mu^2} \tag{9.124b}$$

which are related by the BT

$$u' = u_y+\cosh u. \tag{9.124c}$$

The BT (9.124) is a special case of (9.105), but with primed and unprimed variables interchanged. This was done in order to have the same notation as in chapters 7 and 8.

Solving (9.124c) for u_y one finds

$$q=u'-\cosh u. \tag{9.125a}$$

Now consider

$$p=\Psi(u,u',p',q'), \tag{9.125b}$$

which completes the pair of BTs. Since $s=\dfrac{\partial p}{\partial y}$ and $s=\dfrac{\partial q}{\partial x}$, equations (9.125a,b) and equations (9.124a,b) yield the following identities

$$(u'-\cosh u)\Psi_u+q'\Psi_{u'}+u'\sqrt{p'^2-\mu^2}\Psi_{p'}+t'\Psi_{q'} = \cosh u\sqrt{\Psi^2+\mu^2} \tag{9.126}$$

$$p'-\Psi\sinh u = \cosh u\sqrt{\Psi^2+\mu^2} \tag{9.127}$$

Consequently, equation (9.127) can be solved for Ψ to give

$$\Psi = -p'\sinh u\pm\cosh u\sqrt{p'^2-\mu^2}.$$

Identity (9.126) is satisfied only if

$$\Psi = -p'\sinh u+\cosh u\sqrt{p'^2-\mu^2}. \tag{9.128}$$

We note that the integrability condition ($E = \dfrac{\partial p}{\partial y}-\dfrac{\partial q}{\partial x} = 0$) is also satisfied. Hence, (9.128) and (9.125a) are the BTs of the equations (9.124a,b).

Let us now complete this section by generalising Lamb's result [57], by giving the corresponding results of the Theorem 9.1 (previous section).

Theorem 9.2: The BTs

$$p=\frac{1}{\lambda}[u'-c(u)],$$

$$q=-\frac{1}{\lambda}r'+\frac{1}{\lambda^2}p'c_u-\tfrac{1}{2}\mu p^3+\frac{1}{2\lambda^2}pc_u^2+\frac{1}{\lambda}p^2c_{uu}-\frac{1}{\lambda^2}\nu p$$

relate the pair of partial differential equations

$$u_y+u_{xxx}+\tfrac{1}{2}\mu u_x^3-\frac{3}{2\lambda^2}c_u^2u_x+\frac{\nu}{\lambda^2}u_x=0,$$

$$u'_y+u'_{xxx}+\frac{1}{\lambda^2}[\tfrac{3}{2}\mu(u'-\mu\tau)^2-3\tau(1-\mu^2)(u'-\theta)-\tfrac{3}{2}(\phi^2+\mu\theta^2)+\nu]u'_x=0,$$

where c is a function of u and is given by the equation $c_{uu}+\mu c=\tau$ and λ, ν, μ, τ, ϕ and θ as defined in Theorem 9.1.

For a number of partial differential equation, it is an impossible task to derive BTs of the class (7.4) even with use of today's computer algebra systems. A possible alternative is to attempt to obtain BTs of the class (7.50). These transformations then, if they do not depend on both p' and q', can be used to generate BTs of the general class (7.4). Therefore for equations of the type (9.2), BTs of the class (7.4) can be generated using the transformations derived in section 2, with the exception of (9.61). Thus, for the problem of determining BTs of the class (7.4), which is investigated at the moment by the author, cases for which either Φ or Ψ is a function of u and u' only must be disregarded.

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