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Integral transforms of the Minkowski question mark function

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Abstract

The Minkowski question mark function $F(x)$ arises as a real distribution function of rationals in the Farey (alias, Stern-Brocot or Calkin-Wilf) tree. In this thesis we introduce its three natural integral transforms: the dyadic period function $G(z)$, defined in the cut plane; the dyadic zeta function $\zeta_{\mathcal{M}}(s)$, which is an entire function; the characteristic function $\mathfrak{m}(t)$, which is an entire function as well. Each of them is a unique object, and is characterized by regularity properties and a functional equation, which reformulates in its own terms the functional equation for $F(x)$. We study the interrelations among these three objects and $F(x)$. It appears that the theory is completely parallel to the one for Maass wave forms for $\mathsf{PSL}_2(\mathbb{Z})$. One of the main purposes of this thesis is to clarify the nature of moments of the Minkowski question mark function.

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In 2006 I attended Palanga conference, were incidently J. Steuding (I am very obliged to him) posed a question about the mean value of the question mark function. This was a great impetus, and an outcome of this research is the current thesis.

Finally, without the inspiration from the music and literature, I hardly would have been able to accomplish anything. Amazing music of N.A. Rimsky-Korsakov, M.P. Mussorgsky, R. Wagner, A. Dvořak, B. Bartok, S.S. Prokofiev was always my companion.

Contents

One more legend, the final
one, and my chronicle is done.

Boris Godunov
A. S. Pushkin

Chapter 1

Introduction and summary

1.1 Introduction

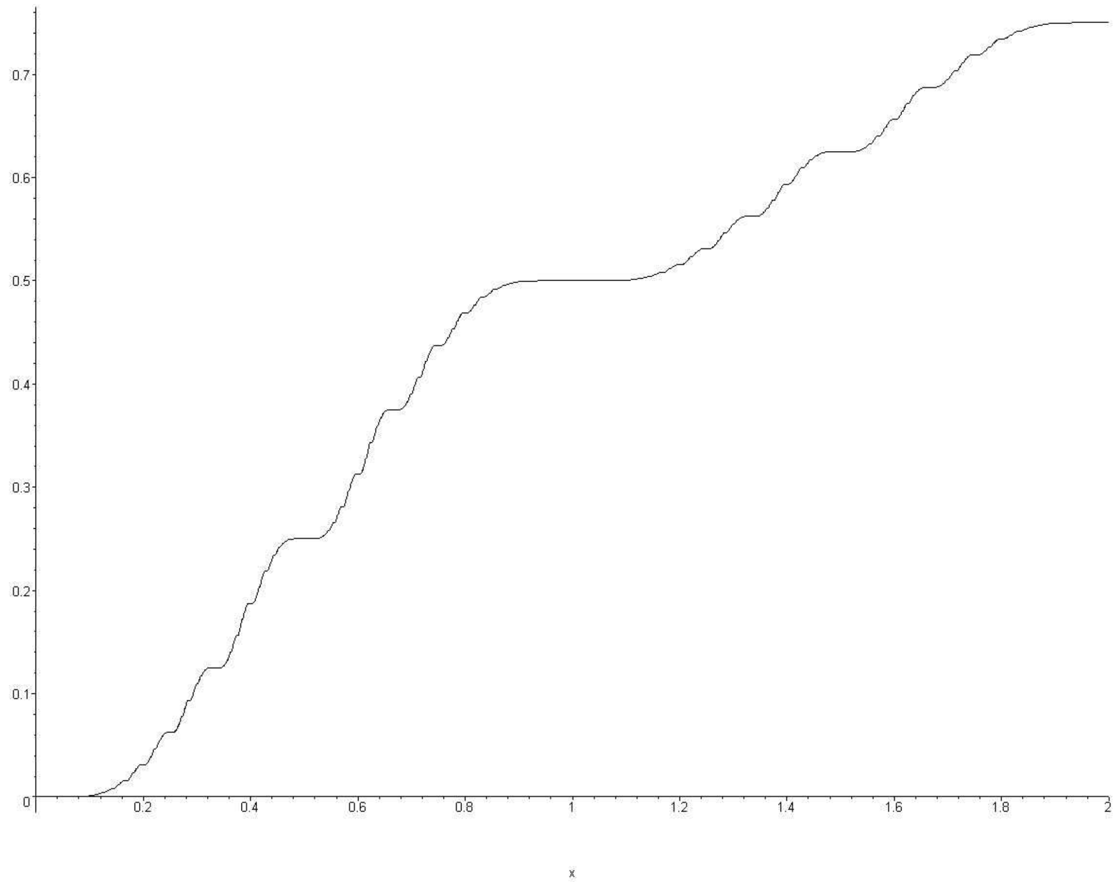


Figure 1.1: Minkowski's question mark $F(x)$, $x \in [0, 2]$

The main hero of this thesis is the function $F(x)$ (with the awkward name "the question mark function", which is now standard), which was introduced by Minkowski in 1904 [50] as an example of a continuous function $F : [0, \infty) \to [0, 1)$, which maps rationals to dyadic rationals, and quadratic irrationals to non-dyadic rationals. For non-negative real x it is defined by the expression

$$F([a_0, a_1, a_2, a_3, \ldots]) = 1 - 2^{-a_0} + 2^{-(a_0+a_1)} - 2^{-(a_0+a_1+a_2)} + \ldots, \tag{1.1}$$

where $x = [a_0, a_1, a_2, a_3, \ldots]$ stands for the representation of x by a (regular) continued fraction [36]. The latter explicit expression for the first time was given in [16]. Our definition slightly differs from the customary - usually one considers a function $?(x)$, defined only for $x \in [0, 1]$. Thus, we will make a convention that $?(x) := 2F(x)$ for $x \in [0, 1]$. For rational x the series terminates at the last nonzero element a_n of the continued fraction. Though being remembered rarely in the first half of the 20th century, this function received a substantial increase in interest in the past two decades; the number of publications is constantly growing. Next section gives a short overview of available literature. Nevertheless, the author of this thesis has a strong conviction that many hidden facts still need to be discovered, and many profound things are encoded in this simple definition. Why this object is so important in number theory, dynamic systems, complex dynamics, ergodic theory and the theory of automorphic forms? Recently, Calkin and Wilf [11] (re-)defined a binary tree which is generated by the iteration

$$\frac{a}{b} \quad \mapsto \quad \frac{a}{a+b}\,, \quad \frac{a+b}{b},$$

starting from the root $\frac{1}{1}$. Elementary considerations show that this tree contains any positive rational number exactly once, each being represented in lowest terms [11]. First four iterations lead to

$$\begin{tikzcd}[column sep=tiny]
 & & & & & & & \frac{1}{1} \arrow[dllll, dash] \arrow[drrrr, dash] & & & & & & & \\
 & & & \frac{1}{2} \arrow[dll, dash] \arrow[drr, dash] & & & & & & & & \frac{2}{1} \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & \frac{1}{3} \arrow[dl, dash] \arrow[dr, dash] & & & & \frac{3}{2} \arrow[dl, dash] \arrow[dr, dash] & & & & \frac{2}{3} \arrow[dl, dash] \arrow[dr, dash] & & & & \frac{3}{1} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{1}{4} & & \frac{4}{3} & & \frac{3}{5} & & \frac{5}{2} & & \frac{2}{5} & & \frac{5}{3} & & \frac{3}{4} & & \frac{4}{1}
\end{tikzcd} \tag{1.2}$$

Thus, the nth generation consists of 2^{n-1} positive rationals $x_n^{(i)}$, $1 \leq i \leq 2^{n-1}$. We denote this tree by $\mathcal{T}$, and the nth generation by $\mathcal{T}^{(n)}$. The limitation of this tree to the interval $[0, 1]$ is the well known Farey tree (albeit with a different order of rational numbers in the nth generation). Reading the tree line by line, this enumeration of $\mathbb{Q}_+$ starts with

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots$$

This sequence was already investigated by Stern [65] in 1858, where the definition of related the so called Stern-Brocot tree was presented. The sequence satisfies the remarkable iteration discovered by M. Newman [52]:

$$x_1 = 1, \quad x_{n+1} = 1/(2[x_n] + 1 - x_n),$$

thus giving an example of a simple recurrence which produces all positive rationals, and answering affirmatively to a question by D.E. Knuth. The nth generation of this binary tree consists of exactly those rational numbers, whose elements of the continued fraction sum up to n; this observation is due to Stern [65]. Indeed, this can be easily inherited directly from the definition. First, if rational number $\frac{a}{b}$ is represented as a continued fraction $[a_0, a_1, ..., a_r]$, then the map $\frac{a}{b} \to \frac{a+b}{b}$ maps $\frac{a}{b}$ to $[a_0+1, a_1..., a_r]$. Second, the map $\frac{a}{b} \to \frac{a}{a+b}$ maps $\frac{a}{b}$ to $[0, a_1+1, ..., a_r]$ in case $\frac{a}{b} < 1$, and to $[1, a_0, a_1, ..., a_r]$ in case $\frac{a}{b} > 1$. Hence, this fact is of utmost importance in our work: though it is not used in explicit form, this highly motivates the investigations of moments M_L and m_L, given by (2.4). The sequence of numerators of the Calkin-Wilf tree

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ...$$

is called the Stern diatomic sequence [65], and it satisfies the recurrence relations

$$s(0) = 0, \quad s(1) = 1, \quad s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1). \tag{1.3}$$

This sequence, and the pairs $(s(n), s(n+1))$, where also investigated by Reznick [59]; see also Lehmer [45]. The statistics of this sequence plays an important role in theories of phase transitions, spin-chains, and, naturally, in number theory and diophantine approximations. See, for example, [8], [13], [15], [21], [23], [32], [34], [51].

In the next section we will show that each generation of the Calkin-Wilf tree possesses a distribution function $F_n(x)$, and that $F_n(x)$ converges uniformly to $F(x)$. This is by far not a new fact. Nevertheless, we include the short proof of it for the sake of self-containedness. The function $F(x)$, as a distribution function, is uniquely determined by the functional equations (2.1). This implies the explicit expression (1.1) and the so called symmetry property $F(x)+F(1/x) = 1$. Surprisingly, the structure of the moments of $F(x)$, which is our principal concern, was not investigated before (apart from the mean value). On the other hand, the mean value of $F(x)$ was treated by several authors and was proved to be $3/2$ ([59], [66], [75]). We will obtain this result using quite a different method.

All papers mentioned in the next Section are concerned with the function $F(x)$ *per se*. On the other hand, the aim of this thesis is to give a different treatment of Minkowski's $?(x)$. It appears that there exist several natural integral transforms of $F(x)$, which are analytic functions and which encode certain (in fact, all) substantial information about the question mark function. Each of these transforms is characterized by a regularity property and a functional equation. Lastly, and most importantly, let us point out that there are striking similarities and analogies between the results proved in Chapters 2 and 4, with Lewis'-Zagier's [47] results on period functions for Maass wave forms. Let, for example, $u(z)$ be a Maass wave form for $\mathrm{PSL}_2(\mathbb{Z})$ with spectral parameter s. This similarity arises due to the fact that the limit value of $u(z)$ on the real line, given by $u(x+iy) \sim y^{1-s}U(x) + y^s U(x)$ as $y \to 0+$, satisfies (formal) functional equations $U(x+1) = U(x)$ and $|x|^{2s-2}U\big(-\frac{1}{x}\big) = U(x)$. Thus, these functional equations are completely analogous to those which $F(x)$ does satisfy (see (2.1)), save the fact that $U(x)$ is only a formal function - it is a distribution (e.g. a continuous functional in a properly defined space of functions). Thus, our objects $G(z)$, $\mathfrak{m}(t)$ and $M(t)$, defined in Chapter 2, are analogues of objects $\psi(z)$, $g(w)$ and $\phi(w)$ respectively, which are defined in [47]. In Section 4.1 we present more explanations on this topic. In Chapter 4 it is shown that in fact $L-$functions attached to Maass wave forms also do have an analogue in our setting - the dyadic zeta function $\zeta_{\mathcal{M}}(s)$.

1.2 Short literature overview

The Minkowski question mark function was investigated by many authors. In this section we give an overview of available literature.

Denjoy [16] gave an explicit expression of $F(x)$ in terms of continued fraction expansion (that is, the formula (1.1)). He also showed that $?(x)$ is purely singular: the derivative, in terms of the Lebesgue measure, vanishes almost everywhere (the short proof of this will be given in Section 2.1). Salem [62] proved that $?(x)$ satisfies the Lipschitz condition of order $\frac{\log 2}{2\log\gamma}$, where $\gamma = \frac{1+\sqrt{5}}{2}$, and this is in fact the best possible exponent for the Lipschitz condition. The Fourier-Stieltjes coefficients of $?(x)$, defined as $\int_0^1 e^{2\pi inx}\,\mathrm{d}?(x)$, where also investigated in the same paper (these coefficients also appeared in [9]; see also [58]). The author, as an application of Wiener's theorem about Fourier series, gives average results on these coefficients without giving an answer to yet unsolved problem whether these coefficients vanish, as $n \to \infty$. It is worth noting that in Section 4.7 we will encounter analogous coefficients (see Proposition 4.8). Kinney [37] proved that the Hausdorff dimension of growth points of $?(x)$ (denote this set by $\mathcal{A}$) is equal to $\alpha = \frac{1}{2}\big(\int_0^1 \log_2(1+x)\ ?(x)\big)^{-1}$ (see Section 5.8 for a numerical value of this constant). Also, if $x_0 \in \mathcal{A}$, $?(x)$ at a point x_0 satisfies the Lipschitz condition with an exponent α. The function $?(x)$ is mentioned in [14] in connection with a game called "box". In [42] Lagarias and Tresser introduced the so called $\mathbb{Q}-$tree: an extension of the Farey tree, which contains all (positive and negative) rationals. Tichy and Uitz [67] extended Kinney's approach (mainly, the calculation of a Hausdorff dimension) to a parametrized class of singular functions related to $?(x)$. Motivated by the investigation of Hermite problem - to represent real cubic irrationals as periodic sequences of integers - Beaver and Garrity [6] introduced a 2-dimensional analogue of $?(x)$. They showed that periodicity of Farey iterations corresponds to a class of cubic irrationals, and that $2-$dimensional analogue of $?(x)$ possesses similar singularity properties. Nevertheless, the Hermite problem remains open. Bower [10] considers the solution of the equation $?(x) = x$, different from $x = 0, \frac{1}{2}$ or 1. There are two of them (symmetric with respect to $x = \frac{1}{2}$), the first one is given by $x = 0.42037233_{+}$ [24]. Apparently, no closed form formula exists for it. In [20] Dilcher and Stolarsky introduced what they call Stern polynomials. The construction is analogous to the one given in Section 5.2. Nevertheless, in the work cited all polynomials have coefficients 0 and 1, and their structure is compatible with regular continued fraction algorithm, whereas in our case another algorithm is being introduced ($\mathsf{p}-$continued fractions). In [22] Dushistova and Moshchevitin find conditions in order $?'(x) = 0$ and $?'(x) = \infty$ to hold (for certain fixed positive real x) in terms of

$$\limsup_{t\to\infty} \frac{a_0 + a_1 + ... + a_t}{t} \text{ and } \liminf_{t\to\infty} \frac{a_0 + a_1 + ... + a_t}{t}$$

respectively, where $x = [a_0, a_1, a_2, ...]$ is represented by a continued fraction. The nature of singularity of $?(x)$ was clarified by Viader, Paradís and Bibiloni [55]. In particular, the existence of the derivative $?'(x)$ in $\mathbb{R}$ for fixed x forces it to vanish. Some other properties of $?(x)$ are demonstrated in [56]. In [35] Kesseböhmer and Stratmann study various fractal geometric aspects of the Minkowski question mark function $F(x)$. They show that the unit interval can be written as the union of three sets: $\Lambda_0 := \{x : F'(x) = 0\}$, $\Lambda_\infty := \{x : F'(x) = \infty\}$, and $\Lambda_\sim := \{x : F'(x) \text{ does not exist and } F'(x) \neq \infty\}$. Their main result is that the Hausdorff dimensions of these sets are related in the following way:

$$\dim_H(\nu_F) < \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H\left(\mathscr{L}(h_{\mathtt{top}})\right) < \dim_H(\Lambda_0) = 1.$$

Here $\mathscr{L}(h_{\mathtt{top}})$ refers to the level set of the Stern-Brocot multifractal decomposition at the topological entropy $h_{\mathtt{top}} = \log 2$ of the Farey map Q, and $\dim_H(\nu_F)$ denotes the Hausdorff dimension of the measure of maximal entropy of the dynamical system associated with Q. The notions and technique were developed earlier by authors in [34]. The paper [41] deals with the interrelations among the additive continued fraction algorithm, the Farey tree, the Farey shift and the Minkowski question mark function. The motivation for the work [54] is a fact that the function $?(x)$ can be characterized as the unique homeomorphism of the real unit interval that conjugates the Farey map with the tent map. In [54] Panti constructs an n-dimensional analogue of the Minkowski question mark function as the only homeomorphism of an n-simplex that conjugates the piecewise-fractional map associated to the Mönkemeyer continued fraction algorithm with an appropriate tent map. Marder [49] introduces two 2-dimensional analogues of $F(x)$, based on a map between Farey and barycentric subdivisions of the triangle. In [69], [70] and [71] Vepstas gives the treatment of various aspects of the question mark function from the perspective of computer science. In [9] Bonanno and Isola introduce a class of 1-dimensional maps which can be used to generate the binary trees in different ways, and study their ergodic properties. This leads to studying some random processes (Markov chains and martingales) arising in a natural way in this context. In the course of the paper the authors also introduce a function $\rho(x) = ?(\frac{x}{x+1})$, which is, of course, exactly $F(x)$. Okamoto and Wunsch [53] construct yet another generalization of $?(x)$, though their main concern is to introduce a new family of purely singular functions. Meanwhile, the paper by Grabner, Kirschenhofer and Tichy [29], out of all papers in the bibliography list, is the closest in spirit to the current thesis. In order to derive precise error bounds for the so called Garcia entropy of a certain measure, the authors consider the moments of the continuous and singular function

$$F_2([a_1, a_2, ...]) = \sum_{n=1}^{\infty}(-1)^{n-1}3^{-(a_1+...+a_n-1)}(q_n + q_{n-1}),$$

where $q_\star$ stand for a corresponding denominator of the convergent to $[a_1, a_2, ...]$. Lamberger [43] has shown that $F(x)$ and $F_2(x)$ are the first two members of a family (indexed by natural numbers) of mutually singular measures, derived from the subtractive Euclidean algorithm. The latter two papers are very interesting and promising, and the author of this thesis does intend to generalize the results about $F(x)$ to the whole family $F_j(x)$, $j \in \mathbb{N}$.

1.3 Summary of the thesis

1.3.1 Chapter 2

In Section 2.1 we demonstrate some elementary properties of the distribution function $F(x)$. Since the existence of all moments is guaranteed by the exponential decay of the tail, our main object is the generating function of moments, denoted by $G(z)$. In Section 2.2 we prove two functional equations for $G(z)$, which can be merged into a single one. In Section 2.3 this function is given the "Eisenstein" series expansion. Moreover, the uniqueness of solution of the functional equation is demonstrated, along with another representation of $G(z)$; the latter allows to calculate the moments numerically with much higher accuracy than directly from the Calkin-Wilf tree. Surprisingly, the Eisenstein series $G_1(z)$ appears on the stage. In Section 2.4 we prove the integral equation for the exponential generating power function, using standard techniques of Hankel transform. Further, a new class of functions emerging from eigen-functions of Hilbert-Schmidt operator (we call them "dyadic period functions") is introduced in Section 2.5. These are dyadic analogues of functions discovered by Wirsing [74] in connection with Gauss-Kuzmin-Lévy problem. In Section 2.6 we describe the $p-$adic distribution of rationals in the Calkin-Wilf tree. It appears that, differently from the real case, the distribution is uniform. In the final Section 2.7 some concluding remarks are presented. It took almost 8 months to accomplish this Chapter. Indeed, the starting point for these investigations was only the iteration $x \mapsto x+1, \frac{x}{x+1}$. It took quite a while for the wider picture to unfold.

1.3.2 Chapter 3

In Section 3.2 we establish the bounds for the moments m_L. In Section 3.2 this is improved to obtain the first asymptotic term for m_L (and the method allows to extract other terms in asymptotic expansion as well). The method is a variant of the Laplace method for asymptotic expansion of certain integrals which depend on one parameter. In our case we need some adjustments, though the main technique is a standard one.

1.3.3 Chapter 4

In Section 4.1 we present some explanations why it is natural to introduce three satellites of $F(x)$. In Section 4.2 we give a short proof of the three term functional equation (2.13), and prove the existence of certain distributions $F_{\lambda}(x)$, which can be thought as close relatives of $F(x)$. In Section 4.3 it is demonstrated that there are linear relations among moments M_L, and they are presented in an explicit manner. Moreover, we formulate a conjecture, based on the analogy with periods, that these are the only possible relations over $\mathbb{Q}$. In Section 4.4 we prove the exactness of a certain sequence of functional vector spaces and linear maps related to $F(x)$ in an essential way. This is an analogue of Kuzmin's result on the convergence of the iterates of the Gauss-Kuzmin-Lévy-Wirsing operator. Section 4.5 is devoted to calculation of a number of integrals, giving a rare example of Stieltjes integral, involving the question mark function, that "can" be calculated. Also, we exhibit that the "orthogonality" of dyadic eigen-functions $G_{\lambda}(z)$ can be stated in terms of a series, involving their Taylor coefficients. In Section 4.6 we compute the Fourier expansion of $F(x)$. The idea to expand a periodic function $\Psi(x) = 2^x(1 - F(x))$ into a Fourier series, though being a simple idea, was a breakthrough which led to other results in this Chapter. It is also shown that this establishes yet another relation among $\mathfrak{m}(t)$, $G(z)$ and $F(x)$ via Taylor coefficients and special values. In the penultimate Section 4.7, the associated Dirichlet series ("dyadic zeta function") $\zeta_{\mathcal{M}}(s)$ is introduced. Apparently, this zeta function has infinitely many zeros on the critical line $\Re s = 0$, though there are many other zeros apart from these. In the last Section 4.8 some concluding remarks are presented, regarding future research; relations between $F(x)$ and the Calkin-Wilf tree (and the Farey tree as well) to the known objects are established. Note also that we use the word "distribution" to describe a monotone function on $[0, \infty)$ with variation 1, and also for a continuous linear functional on some space of analytic functions. In each case the meaning should be clear from the context.

1.3.4 Chapter 5

In Section 5.1 we formulate the main results of the Chapter, which ought to be considered as a climax of this thesis. It states that in a half-plane $\Re z \leq \frac{1}{2}$ (apparently, $\Re z \leq 1$ is a precise region of convergence) the dyadic period function $G(z)$ can be expressed as a convergent sum of explicit rational function of the form $\mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}}$, where $\mathscr{B}_n(z)$ are certain polynomials of degree $n-1$. In Section 5.2, for each p, $1 \leq \mathsf{p} < \infty$, we introduce a generalization of the Farey (Calkin-Wilf) tree, denoted by $\mathcal{Q}_{\mathsf{p}}$. This leads to the notion of $\mathsf{p}-$continued fractions and $\mathsf{p}-$Minkowski question mark functions $F_{\mathsf{p}}(x)$. Though $\mathsf{p}-$continued fractions are of independent interest (one could define a transfer operator, to prove an analogue of the Gauss-Kuzmin-Lévy theorem, various metric results and introduce structural constants), we confine to the facts which are necessary for our purposes, and leave the deeper research for the future. In Section 5.3 we extend these results to the case of complex p, $\Re\,\mathsf{p} \geq 1$. The crucial proposition states that a function $\mathfrak{X}(\mathsf{p}, x)$ (which, if restricted to $\mathbb{Q}$, gives a bijection between the trees $\mathcal{Q}_1$ and $\mathcal{Q}_{\mathsf{p}}$) is a continuous function in x and is analytic function in p for $|\,\mathsf{p}-2| \leq 1$. Note that this section contains four unproved propositions: they are obvious heuristically, but very difficult to prove (see some remarks in this section). Hence, eventually our proof of Theorem 5.1 depends on one unproved statement. In Section 5.4 we introduce exactly the same integral transforms of $F_{\mathsf{p}}(x)$ as was done in a special (though most important) case of $F(x) = F_1(x)$. Also, in this section we prove certain relations among the moments. In Section 5.5 we give a proof of the three term functional equation for $G_{\mathsf{p}}(z)$ and the integral equation for $\mathfrak{m}_{\mathsf{p}}(t)$. The results of the latter Section were obtained almost automatically using precisely the same arguments as in Chapter 2. This was not very fascinating, up to the moment when we arrived at an idea that partial derivatives (with respect to p) of this three term functional equation does give a rich information about the case $\mathsf{p} = 1$! Thus, section 5.6 is devoted to demonstration how empirically one could arrive at the statement of Theorem 5.4. Finally, Theorem 5.1 is proved in Section 5.7. The hierarchy of sections is linear, and all results from previous ones is used in Section 5.7. There should be, of course, a straightforward way to verify Theorem 5.1 without introducing a variable p. Unfortunately, the recurrence for polynomials $\mathscr{B}_n(z)$ is very complicated, and the direct proof will certainly pose many technical difficulties. Supplementary Section 5.8 contains MAPLE codes to compute rational functions $\mathbf{H}_n(z)$ and $\mathbf{Q}_n(z)$, as well as some numerical calculations. Also, it contains a subsection where we speculate on yet another direction for investigations on the Calkin-Wilf tree. The Chapter also contains graphs of some $\mathsf{p}-$Minkowski question mark functions $F_{\mathsf{p}}(x)$ for real p, and also pictures of locus points of elements of trees $\mathcal{Q}_{\mathsf{p}}$ for certain characteristic values of p.

Chapter 2

Basic properties of integral transforms: the dyadic period function

2.1 Some properties of the distribution

The following proposition was proved by many authors in various forms, concerning (very related) Stern-Brocot, Farey or Calkin-Wilf trees. Though to our knowledge this was not written explicitly anywhere, this seems to be a well-known fact about a distribution of rationals in these trees. For the sake of completeness we present a short proof, since the functional equations for $G(z)$ and $\mathfrak{m}(t)$ (see Sections 2.2 and 2.4) heavily depend on the functional equation for $F(x)$ and are in fact reformulations of these in different terms.

Proposition 2.1. *Let $F_n(x)$ denote the distribution function of the nth generation, i.e.,*

$$F_n(x) = 2^{1-n}\#\{j : x_j^{(n)} \leq x\}.$$

Then uniformly $F_n(x) \to F(x)$. Thus, $F(0) = 0$, $F(\infty) = 1$. Moreover, $F(x)$ is continuous, monotone and singular, i.e., $F'(x) = 0$ almost everywhere.

Proof. Let $x \geq 1$. One half of the fractions in the $(n+1)$st generation do not exceed 1, and hence also do not exceed x. Further,

$$\frac{a+b}{b} \leq x \quad \Longleftrightarrow \quad \frac{a}{b} \leq x-1.$$

Hence,

$$2F_{n+1}(x) = F_n(x-1) + 1, \quad n \geq 1.$$

Now assume $0 < x < 1$. Then

$$\frac{a}{a+b} \leq x \quad \Longleftrightarrow \quad \frac{a}{b} \leq \frac{x}{1-x}.$$

Therefore,

$$2F_{n+1}(x) = F_n\Big(\frac{x}{1-x}\Big).$$

The distribution function F, defined by (1.1), satisfies the functional equation

$$2F(x) = \begin{cases} F(x-1)+1 & \text{if} \quad x \geq 1, \\ F(\frac{x}{1-x}) & \text{if} \quad 0 < x < 1. \end{cases} \tag{2.1}$$

For instance, the second identity is equivalent to $2F(\frac{t}{t+1}) = F(t)$ for all positive t. If $t = [b_0, b_1, ...]$, then $\frac{t}{t+1} = [0, 1, b_0, b_1, ..]$ for $t \geq 1$, and $\frac{t}{t+1} = [0, b_1+1, b_2, ...]$ for $t < 1$, and the statement follows immediately.

Now define $\delta_n(x) = F(x) - F_n(x)$. In order to prove the uniform convergence $F_n \to F$, it is sufficient to show that

$$\sup_{x \geq 0} |\delta_n(x)| \leq 2^{-n}. \tag{2.2}$$

It is easy to see that the assertion is true for $n = 1$. Now suppose the estimate is true for n. In view of the functional equation for both $F_n(x)$ and $F(x)$, we have

$$2\delta_{n+1}(x) = \delta_n\Big(\frac{x}{1-x}\Big)$$

for $0 < x < 1$, which gives $\sup_{0 \leq x < 1} |\delta_{n+1}(x)| \leq 2^{-n-1}$. Moreover, we have

$$2\delta_{n+1}(x) = \delta_n(x-1)$$

for $x \geq 1$, which yields the same bound for $\delta_n(x)$ in the range $x \geq 1$. This proves (2.2).

Clearly, F, as a distribution function, is monotonic; obviously, it is also continuous. It remains to prove that $F(x)$ is singular. Given an irrational number $\alpha = [a_0, a_1, a_2, ...]$, we consider the sequence

$$\alpha_n = [a_0, a_1, ..., a_{n-1}, a_n + 1, a_{n+1}, ...];$$

obviously, α_n is the real number which is defined by the continued fraction expansion of α, where the nth partial quotient a_n is replaced by $a_n + 1$. Depending on the parity of n, α_n is less than or greater than α. Thus, any real number y, which is sufficiently close to α, is contained between two terms of the sequence, α_L and α_{L+2} say. Then

$$\Big|\frac{F(y) - F(\alpha)}{y - \alpha}\Big| \leq \Big|\frac{F(\alpha_L) - F(\alpha)}{\alpha_{L+2} - \alpha}\Big|.$$

From the explicit form of F we deduce

$$|F(\alpha_L) - F(\alpha)| \leq \frac{1}{2} 2^{-(a_0 + a_1 + ... + a_L)}.$$

On the other hand,

$$\begin{aligned} |\alpha_{L+2} - \alpha| &\geq ([a_1, a_2, ..., a_{L+2}+1, ...] - [a_1, a_2, ..., a_{L+2}, ...])(a_0+1)^{-2} \\ &\geq \Big((a_0+1)(a_1+1)...(a_{L+2}+1)\Big)^{-2} \end{aligned}$$

by induction. Thus,

$$\Big|\frac{F(y)-F(\alpha)}{y-\alpha}\Big| \leq 2^{1-(a_0+a_1+\ldots+a_L)}\prod_{i=1}^{L+2}(a_i+1)^2.$$

The theorem of Khinchin ([36], p. 86) implies that $\prod_{i=1}^{n}(a_i+1)^{1/n}$ tends to a fixed constant limit almost everywhere. On the other hand, the same reasoning shows that $\frac{1}{n}\sum_{i=1}^{n}a_n$ tends to infinity for almost all x. Thus, almost everywhere the limit

$$\lim_{y\to\alpha}(F(y)-F(\alpha))(y-\alpha)^{-1}$$

exists and is equal 0. This finishes the proof. ■

As it was noted, the singularity of $F(x)$ was proved in [16]. By the same argument as for the singular behaviour of F we can show that $F'(\frac{\sqrt{5}+1}{2})=\infty$. Actually, the terms of $\mathcal{T}^{(n)}$ are densely concentrated around numbers with $F'(x)=\infty$ and scarcely around those where $F'(x)=0$. The value of $F(x)$ is rational iff x is either rational or quadratic irrationality, e.g.

$$F(1)=\frac{1}{2},\quad F(\sqrt{2})=\frac{3}{5},\quad F((\sqrt{5}+1)/2)=\frac{2}{3}.$$

This follows immediately from Lagrange's theorem which characterizes the quadratic irrationals by their eventually periodic continued fraction expansion. For Euler's number $e=[2,\overline{1,2n,1}]$ we find that $F(e)$ can be expressed in terms of special values of Jacobi theta functions.

Since $F(x)$ has a tail of exponential decay ($1-F(x)=O(2^{-x})$, as it is clear from (1.1)), all moments do exist. Let for $L\in\mathbb{N}_0$

$$M_L=\int\limits_0^\infty x^L\,\mathrm{d}F(x),$$

$$m_L=\int\limits_0^\infty\Big(\frac{x}{x+1}\Big)^L\mathrm{d}F(x)=2\int\limits_0^1 x^L\,\mathrm{d}F(x)=\int\limits_0^1 x^L\,\mathrm{d}?(x). \tag{2.3}$$

Therefore, M_L and m_L can also be defined as

$$\begin{aligned} M_L &= \lim_{n\to\infty}2^{1-n}\sum_{a_0+a_1+\ldots+a_s=n}[a_0,a_1,..,a_s]^L,\\ m_L &= \lim_{n\to\infty}2^{2-n}\sum_{a_1+\ldots+a_s=n}[0,a_1,..,a_s]^L, \end{aligned} \tag{2.4}$$

where the summation takes place over all rationals, whose elements of the continued fraction sum up to n. It is these expressions which highly motivate our research on moments. Numerically, one has

$$\begin{aligned} &M_1=1.5, && M_2=4.29092, && M_3=18.556, && M_4=107.03;\\ &m_1=0.5, && m_2=0.29092, && m_3=0.18638, && m_4=0.12699. \end{aligned}$$

We will see that the generating power function of m_L possesses some interesting properties. Let $\omega(x)$ be a continuous function of at most power growth: $\omega(x) = O(x^T)$, $x \to \infty$ for certain T. Then, as noted above, $F(x)$ has a tail of exponential decay; therefore, there exists next integral. The functional equation for $F(x)$ (2.1) gives $F(x+n) = 1 - 2^{-n} + 2^{-n}F(x)$, $x \geq 0$. Hence

$$\int\limits_0^\infty \omega(x)\,\mathrm{d}F(x) = \sum_{n=0}^{\infty}\int\limits_0^1 \omega(x+n)\,\mathrm{d}F(x+n) = \int\limits_0^1 \sum_{n=0}^{\infty}\frac{\omega(x+n)}{2^n}\,\mathrm{d}F(x).$$

Let $x = \frac{t}{t+1}$, $t \geq 0$. Since $F(\frac{t}{t+1}) = \frac{1}{2}F(t)$, this change of variables gives

$$\int\limits_0^\infty \omega(x)\,\mathrm{d}F(x) = \sum_{n=0}^{\infty}\int\limits_0^\infty \frac{\omega(\frac{t}{t+1}+n)}{2^{n+1}}\,\mathrm{d}F(t)$$

(All changes of order of summation and integration are easily justifiable minding the condition on $\omega(x)$). Let $\omega(x) = x^L$, $L \in \mathbb{N}_0$. Then, if we denote $B_s = \sum_{n=0}^{\infty}\frac{n^s}{2^{n+1}}$, we have

$$\int\limits_0^\infty x^L\,\mathrm{d}F(x) = \int\limits_0^\infty \sum_{i=0}^{L}\Big(\frac{x}{x+1}\Big)^i\binom{L}{i}B_{L-i}\,\mathrm{d}F(x).$$

Whence the relation

$$M_L = \sum_{i=0}^{L} m_i\binom{L}{i}B_{L-i}, \quad ,L \geq 0. \tag{2.5}$$

The generating exponential power function of B_i is

$$B(t) = \sum_{L=0}^{\infty}\frac{B_L}{L!}t^L = \sum_{L=0}^{\infty}\sum_{n=0}^{\infty}\frac{n^L t^L}{2^{N+1}L!} = \sum_{n=0}^{\infty}\frac{e^{nt}}{2^{n+1}} = \frac{1}{2-e^t}.$$

Denote by $M(t)$ and $\mathfrak{m}(t)$ the corresponding exponential generating power functions of the coefficients M_L and m_L respectively. Thus,

$$M(t) = \int\limits_0^\infty e^{xt}\,\mathrm{d}F(x), \quad \mathfrak{m}(t) = \int\limits_0^\infty \exp\Big(\frac{xt}{x+1}\Big)\,\mathrm{d}F(x) = 2\int\limits_0^1 e^{xt}\,\mathrm{d}F(x).$$

The relation (2.5) in terms of $M(t)$ and $\mathfrak{m}(t)$ reads as

$$M(t) = \sum_{L=0}^{\infty}\frac{M_L}{L!}t^L = \frac{1}{2-e^t}\sum_{L=0}^{\infty}\frac{m_L}{L!}t^L = \frac{1}{2-e^t}\mathfrak{m}(t). \tag{2.6}$$

We see that the function $\mathfrak{m}(t)$ is entire and the generating function $M(t)$ has a positive radius of convergence. The last identity implies the asymptotics for M_L.

Proposition 2.2. *For $L \in \mathbb{N}_0$,*

$$\begin{aligned} M_L &= \frac{\mathfrak{m}(\log 2)}{2\log 2}\Big(\frac{1}{\log 2}\Big)^L L! + O_\varepsilon\Big(((4\pi^2 + (\log 2)^{1/2} - \varepsilon)^{-L}\Big)L! \\ &= \Big(\frac{\mathfrak{m}(\log 2)}{2\log 2}\Big(\frac{1}{\log 2}\Big)^L + O(6.3^{-L})\Big)L! \end{aligned}$$

Proof. By Cauchy's formula, for any sufficiently small r,

$$M_L = \frac{L!}{2\pi i} \int_{|z|=r} \frac{M(z)}{z^{L+1}} \, \mathrm{d}z.$$

Changing the path of integration, we get by the calculus of residues

$$M_L = -\mathrm{Res}_{z=\log 2} \left(\frac{\mathfrak{m}(z)}{(2-e^z)z^{L+1}} \right) - \frac{L!}{2\pi i} \int_{|z|=R} \frac{\mathfrak{m}(z)}{2-e^z} \frac{\mathrm{d}z}{z^{L+1}},$$

where R satisfies $\log 2 < R < |\log 2 + 2\pi i|$ (which means that there is exactly one simple pole of the integrand located in the interior of the circle $|z| = R$). It is easily seen that the residue coincides with the main term in the formula of the lemma; the error term follows from estimating the integral. ■

Also, (2.6) gives the inverse to linear equations (2.5):

$$m_L = M_L - \sum_{s=0}^{L-1} M_s \binom{L}{s}, \quad L \geq 0. \tag{2.7}$$

Since $B(t)(2 - e^t) = 1$, the coefficients B_L can be calculated recursively: $B_L = \sum_{s=0}^{L-1} \binom{L}{s} B_s$. Thus, $B_0 = 1$, $B_1 = 1$, $B_2 = 3$, $B_3 = 13$, $B_4 = 75$, $B_5 = 541$. This sequence is labeled $A000670$ in [64].

In the future we will consider the integrals involving $\mathfrak{m}(t)$, and we need the evaluation of this function for negative t.

Proposition 2.3. *Let* $C = e^{-\sqrt{\log 2}} = 0.4349\ldots$. *Then* $C^{2\sqrt{t}} \ll \mathfrak{m}(-t) \ll C^{\sqrt{t}}$ *as* $t \to \infty$.

Proof. In fact, $\mathfrak{m}(-t) = \int_0^\infty \exp(-\frac{xt}{x+1}) \, \mathrm{d}F(x)$. Hence, $\mathfrak{m}(t)$ is positive for $t \in \mathbb{R}$. Let $0 < M < 1$. Since $1 - F(x) \asymp 2^{-x}$ as $x \to \infty$, and $F(x) + F(1/x) = 1$,

$$\mathfrak{m}(-t) = \Big(\int_0^M + \int_M^\infty \Big) \exp(-\frac{xt}{x+1}) \, \mathrm{d}F(x) \ll 2^{-1/M} + \exp\Big(-\frac{Mt}{M+1} \Big).$$

This is valid for every $M < 1$ and the implied constant is universal. Now choosing $M = \frac{\sqrt{\log 2}}{\sqrt{t}}$ gives the desired upper bound. To establish lower bound, note that

$$\mathfrak{m}(-t) > \int_0^M \exp(-\frac{xt}{x+1}) \, \mathrm{d}F(x) \gg 2^{-1/M} \cdot \exp\Big(-\frac{Mt}{M+1} \Big).$$

The same choice for M establishes the lower bound. Naturally, similar evaluation holds for the derivative, since $\mathfrak{m}'(-t) = \int_0^\infty \frac{x}{x+1} \exp(-\frac{xt}{x+1}) \, \mathrm{d}F(x)$. ■

We will prove one property of the function $\mathfrak{m}(t)$ which represents the symmetry of F given by $F(x) + F(1/x) = 1$.

Proposition 2.4. *We have* $\mathfrak{m}(t) = e^t \mathfrak{m}(-t)$.

Proof. In fact,

$$\mathfrak{m}(t) = \int_0^\infty \exp\Big(\frac{xt}{x+1}\Big)\,\mathrm{d}F(x) = -\int_0^\infty \exp\Big(\frac{t/x}{1/x+1}\Big)\,\mathrm{d}F(1/x)$$

$$= \int_0^\infty \exp\Big(\frac{t}{x+1}\Big)\,\mathrm{d}F(x) = e^t \int_0^\infty \exp\Big(-\frac{xt}{x+1}\Big)\,\mathrm{d}F(x) = \mathfrak{m}(-t)e^t. \quad \blacksquare$$

Whence the relations

$$m_L = \sum_{s=0}^{L} \binom{L}{s}(-1)^s m_s, \quad , L \geq 0.$$

For example, $m_1 = m_0 - m_1$, which gives $m_1 = 1/2$, and thus we have $M_1 = 3/2$. For other coefficients we only have linear relations. Thus, $2m_3 = -1/2 + 3m_2$.

2.2 Generating function of moments: $G(z)$

We introduce the generating power function of moments

$$\mathcal{M}(z) = \sum_{L=0}^{\infty} m_L z^L.$$

A priori, this series converges in the unit circle. Since $\int_0^\infty x^n e^{-x}\,\mathrm{d}x = \Gamma(n+1) = n!$, we have, using the symmetry relation for $\mathfrak{m}(t)$, and for real $z < 1$:

$$\mathcal{M}(z) = \int_0^\infty \mathfrak{m}(zt)e^{-t}\,\mathrm{d}t = \int_0^\infty \mathfrak{m}(-zt)e^{-t(1-z)}\,\mathrm{d}t =$$

$$\int_0^\infty \mathfrak{m}\Big(t\frac{z}{z-1}\Big)\frac{1}{(1-z)}e^{-t}\,\mathrm{d}t = \mathcal{M}\Big(\frac{z}{z-1}\Big)\frac{1}{1-z}. \tag{2.8}$$

Both integrals converge for $z < 1$ (since $m_k \leq 1$, $|\mathfrak{m}(z)| \leq e^z$), hence for these values of z we have the above identity. $\mathcal{M}(z)$ is initially defined for $|z| < 1$; nevertheless, this identity gives us the holomorphic continuation of $\mathcal{M}(z)$ to the half plane $\Re z < 1/2$.

Proposition 2.5. *The function $\mathcal{M}(z)$ can be analytically continued to the domain $\mathbb{C}\backslash[1,\infty)$.*

Proof. In fact, $\mathfrak{m}(t) = \int_0^\infty \exp(\frac{x}{x+1}t)\,\mathrm{d}F(x)$. As noted above, $|\mathfrak{m}(t)| \leq e^t$ for positive t (in fact, Proposition 2.3 gives a slightly better estimate). Therefore, for real z, $z < 1$, we have:

$$\mathcal{M}(z) = \int_0^\infty \int_0^\infty \exp\Big(\frac{x}{x+1}zt\Big)e^{-t}\,\mathrm{d}F(x)\,\mathrm{d}t = \int_0^\infty \frac{1}{1-\frac{x}{x+1}z}\,\mathrm{d}F(x).$$

We already obtained the analytic continuation of $\mathcal{M}$ to the region $\{|z|<1\}\bigcup\{\Re z < 1/2\}$. Let $z=\sigma+iy$ with $y>0$ and $\sigma\geq 1/2$. In a small neighborhood of z the imaginary part is bounded: $y\geq y_0>0$, and also the real part is bounded: $\sigma\leq\sigma_0$. In this neighborhood the integral converges uniformly; in fact, there we have the estimate

$$\Big|(1-\frac{x}{x+1}z)\Big|\geq\max\{\Big|1-\frac{x}{x+1}\sigma\Big|,\Big|\frac{x}{x+1}t\Big|\}.$$

For $0\leq x\leq 1/\sigma$ we have the bound $\frac{1}{\sigma_0+1}$, and for $x>1/\sigma$ we have the bound $\frac{t_0}{\sigma_0+1}$. Hence, the function under integral in this neighborhood is uniformly bounded, which proves the uniform convergence of the integral and the statement of Proposition. ■

The system (2.5) gives us the expression of M_L in terms of m_s. In fact, there exists one more system which is independent of the distribution $F(x)$; it simply encodes the relation among functions $\left(\frac{x}{1-x}\right)^L$ and x^s given by

$$\left(\frac{x}{1-x}\right)^L=\sum_{s\geq L}\binom{s-1}{L-1}x^s,\quad L\geq 1,\quad 0\leq x<1.$$

Change $x=\frac{t}{t+1}$ gives

$$t^L=\sum_{s\geq L}\binom{s-1}{L-1}\left(\frac{t}{t+1}\right)^s\quad L\geq 1,\quad t\geq 0.$$

And ultimately,

$$M_L=\sum_{s\geq L}\binom{s-1}{L-1}m_s. \tag{2.9}$$

For the convenience, we introduce a function

$$G(z)=\frac{\mathcal{M}(z)-1}{z}=\sum_{L=1}^{\infty}m_Lz^{L-1}=\int_0^{\infty}\frac{\frac{x}{x+1}}{1-\frac{x}{x+1}z}\,\mathrm{d}F(x)=2\int_0^1\frac{x}{1-xz}\,\mathrm{d}F(x). \tag{2.10}$$

Our next purpose is to prove the main result about the function $G(z)$. The power series converges in the disc $|z|\leq 1$ (including the boundary, as can be inherited from (2.9); moreover, this implies that there exist all left derivatives of $G(z)$ at $z=1$). The integral converges in the cut plane $\mathbb{C}\setminus(1,\infty)$.

Theorem 2.6. *Let $m_L=\int_0^\infty(\frac{x}{x+1})^L\,\mathrm{d}F(x)$. Then the generating power function, defined as $G(z)=\sum_{L=1}^\infty m_Lz^{L-1}$, has an analytic continuation to the domain $\mathbb{C}\backslash\mathbb{R}_{x>1}$. It satisfies the functional equation*

$$-\frac{1}{1-z}-\frac{1}{(1-z)^2}G\Big(\frac{1}{1-z}\Big)+2G(z+1)=G(z), \tag{2.11}$$

and also the symmetry property

$$G(z+1)=-\frac{1}{z^2}G\Big(\frac{1}{z}+1\Big)-\frac{1}{z}.$$

Moreover, $G(z)=o(1)$ as $z\to\infty$ and the distance from z to $\mathbb{R}_+$ tends to infinity.

Proof. In analogy to $\mathcal{M}(z)$, for real $z<0$ define the following function: $\mathcal{M}_0(z) = \int_0^\infty M(zt)e^{-t}\,\mathrm{d}t$. In view of (2.6), this integral converges for real $z<0$. Thus,

$$\mathcal{M}_0(z) = \int\limits_0^\infty\int\limits_0^\infty \exp(xzt)e^{-t}\,\mathrm{d}F(x)\,\mathrm{d}t = \int\limits_0^\infty \frac{1}{1-xz}\,\mathrm{d}F(x).$$

In the same manner as with $\mathcal{M}(z)$, we deduce that $\mathcal{M}_0(z)$ extends as an analytic function to the region $\mathbb{C}\backslash\mathbb{R}_{>0}$. In this domain now we see that

$$\frac{\mathcal{M}_0(z)-1}{z} = \frac{\mathcal{M}(z+1)-1}{z+1}, \tag{2.12}$$

which is the consequence of an algebraic identity

$$\Big(\frac{1}{1-xz}-1\Big)\cdot\frac{1}{z} = \Big(\frac{1}{1-\frac{x}{x+1}(z+1)}-1\Big)\cdot\frac{1}{z+1}.$$

The relation (2.12) is independent of the specific distribution function, it simply encodes the information contained in (2.9) about the relation of x^L to $(x/(x+1))^s$. On the other hand, the specific information about $F(x)$ is encoded in (2.5), or in the form (2.6). The comparison of these two gives the desired functional equation for $G(z)$. In fact, for real $t<0$ the following estimate follows from (2.6) and Proposition 2.3: $|M(t)| = |\mathfrak{m}(t)(2-e^t)^{-1}| \leq |\mathfrak{m}(t)| \ll 1$; and thus for real $z<0$:

$$\mathcal{M}(z) = \int\limits_0^\infty \mathfrak{m}(zt)e^{-t}\,\mathrm{d}t = \int\limits_0^\infty (2-e^{zt})M(zt)e^{-t}\,\mathrm{d}t =$$

$$2\mathcal{M}_0(z) - \int\limits_0^\infty M(zt)e^{-t(1-z)}\,\mathrm{d}t = 2\mathcal{M}_0(z) - \mathcal{M}_0\Big(\frac{z}{1-z}\Big)\frac{1}{1-z}.$$

Finally, the substitution (2.12) gives us the functional equation

$$\frac{1-z}{1+z} - \frac{z}{1-z}\mathcal{M}\Big(\frac{1}{1-z}\Big) + 2\frac{z}{z+1}\mathcal{M}(z+1) = \mathcal{M}(z).$$

This equation from the principle of analytic continuation should be satisfied for all values of arguments in the region of holomorphicity of $\mathcal{M}(z)$. Direct inspection shows that for $G(z) = \frac{\mathcal{M}(z)-1}{z}$ this equation reads as (2.11). Also, the symmetry property is a reformulation of (2.8). This proves the first part of the theorem. Obviously, the last assertion follows from the integral representation of $G(z)$ given by (2.10). ■

We call the function $G(z)$ *the dyadic period function*, since its functional equation is completely parallel to a three term functional equations which are satisfied by rational period functions and period functions associated with Maass wave forms [47]. The word "dyadic" refers to the binary origin of the distribution function $F(x)$. Indeed, thorough inspection shows that the multiplier 2 in the equation (2.11) emerges exactly from the fact that every generation of $\mathcal{T}$ has twice as many members as a previous generation.

2.3 Uniqueness of $G(z)$

In this section we prove the uniqueness of the function having the properties, described in Theorem 2.6. Note that two functional equations for $G(z)$ can be merged into single one. It is easy to check that

$$\frac{1}{z}+\frac{1}{z^2}G\Big(\frac{1}{z}\Big)+2G(z+1)=G(z) \tag{2.13}$$

is equivalent to both together. In fact, the change $z\mapsto 1/z$ in the last equation gives the symmetry property and application of it to the term $G(1/z)$ of the above gives the functional equation (2.11).

Proposition 2.7. *The function $G(z)$, satisfying the conditions of Theorem 2.6, is unique.*

Proof. Suppose, there are two such functions. Then their difference $G_0(z)$ has the same behavior at ∞, and satisfies the homogenic form of the equation (2.13). Let $M=\sup_{-1\le x\le 0}|G_0(x)|$. We will show that $M=0$ and by the principle of analytic continuation this will imply that $G_0(z)\equiv 0$. Let z be real, $-1\le z\le 0$. Let us substitute $z\mapsto z-n$ in the equation (2.13), $n\in\mathbb{N}$, $n\ge 1$, and divide it by 2^n. Thus, we obtain:

$$\frac{G_0(z-n)}{2^n}-\frac{G_0(z-n+1)}{2^{n-1}}=\frac{1}{2^n(z-n)^2}G_0\Big(\frac{1}{z-n}\Big).$$

Note that for z in the interval $[-1,0]$, $\frac{1}{z-n}$ belongs to the same interval as well. Now sum this over $n\ge 1$. The series on both sides are absolutely convergent, minding the behavior of $G_0(z)$ at infinity. Therefore,

$$-G_0(z)=\sum_{n=1}^{\infty}\frac{1}{2^n(z-n)^2}G_0\Big(\frac{1}{z-n}\Big). \tag{2.14}$$

The evaluation of the right side gives

$$|G_0(z)|\le\sum_{n=1}^{\infty}\frac{1}{2^n n^2}M=\Big(\frac{\pi^2}{12}-\frac{1}{2}\log^2 2\Big)M\quad\text{for }-1\le z\le 0.$$

The constant is <1. Thus, unless $M=0$, this is contradictory for z_0 in the interval $[-1,0]$, such that $|G_0(z_0)|=M$. This proves the Proposition. ■

Note the similarity between (2.14) and the expression for the Gauss-Kuzmin-Wirsing operator $\mathbf{W}$. The latter is defined for bounded smooth functions $f:[0,1]\to\mathbb{R}$ by the formula

$$[\mathbf{W}f](x)=\sum_{k=1}^{\infty}\frac{1}{(k+x)^2}f\Big(\frac{1}{k+x}\Big).$$

The eigenvalue 1 corresponds to the function $\frac{1}{1+x}$ (see [36] chapter III, for Kuzmin's treatment). The second largest eigenvalue $-0.303663...$ (the Wirsing constant) leads to a function with no analytic expression known [74]; this eigenvalue determines the speed of convergence of iterates $[\mathbf{W}^{(n)}f](x)$ to $\frac{c}{1+x}$ (for certain $c\in\mathbb{R}$). The spectral analysis of our operator is presented in Section 2.5. Chapter 4 contains much more details and results in this direction.

Let $\Im z>0$. We remind that the Eisenstein series of weight 2 is defined as [63]

$$G_1(z)=\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}{}' \frac{1}{(m+nz)^2};$$

(mind the order of summation, since the series is not absolutely convergent). This series has the following Fourier expansion If $q=e^{2\pi iz}$, then

$$G_1(z)=\frac{\pi^2}{3}-8\pi^2\sum_{n=1}^{\infty}\sigma_1(n)q^n,\quad \sigma_1(n)=\sum_{d|n}d.$$

Then this function is not completely modular, but we have the following identities ([63], chapter VII):

$$G_1(z+1)=G_1(z),\quad G_1(-1/z)=z^2G_1(z)-2\pi iz.$$

Note that for $\Im z>0$, all arguments in (2.11) simultaneously belong to the upper half plane. It is surprising (but not coincidental) that the function $\frac{i}{2\pi}G_1(z)$ satisfies the functional equation (2.11) for $\Im z>0$ (see the remarks in Section 2.7 about possible connections in idelic setting). To check this statement, note that

$$\frac{i}{2\pi}G_1\Big(-\frac{1}{z-1}\Big)=\frac{i}{2\pi}\Big((z-1)^2G_1(z-1)-2\pi i(z-1)\Big)=\frac{i}{2\pi}(1-z)^2G_1(z)-(1-z).$$

Thus, plugging this into (2.11), we obtain an identity. If we define $G_1(z)=\overline{G_1(\overline{z})}$ for $\Im z<0$, one checks directly that the symmetry property is also satisfied. This is a surprising phenomena. See last section of Chapter 4 for more speculations on this topic, where the space of dyadic period functions in the upper half plane (denoted by `DPF`) is introduced.

We end this section with presenting a system of linear equations which the moments m_L do satisfy. This system is derived from the three term functional equation (2.13) and is a superior result in numerical calculations: whereas directly from the definition we can recover only a few digits of the moments, this method allows to calculate up to 60 digits and more.

Proposition 2.8. *Denote $c_L=\sum_{n=1}^{\infty}\frac{1}{2^n n^L}=Li_L(\frac{1}{2})$. The moments m_s satisfy the infinite system of linear equations*

$$m_s=\sum_{L=0}^{\infty}(-1)^L c_{L+s}\binom{L+s-1}{s-1}m_L,\quad s\geq 1.$$

Proof. Indeed, for $\Re z \leq 0$ we have (recall that $m_0 = 1$):

$$-G(z) = \sum_{n=1}^{\infty} \frac{1}{2^n(z-n)} + \sum_{n=1}^{\infty} \frac{1}{2^n(z-n)^2} G\Big(\frac{1}{z-n}\Big) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{L=0}^{\infty} m_L \Big(\frac{1}{z-n}\Big)^{L+1}.$$

This series is absolutely and uniformly convergent for $\Re z \leq 0$, as is implied by (2.9). We obtain the needed result after taking the sth left derivative at $z = 0$. ■

Numerical calculations are presented in Section 5.8. This method gives high precision values for other constants, including the Kinney's constant.

2.4 Exponential generating function $\mathfrak{m}(t)$

The aim of this section is to interpret (2.13) in terms of $\mathfrak{m}(t)$. The following Theorem uniquely determines the function $\mathfrak{m}(t)$, along with the boundary condition $\mathfrak{m}(0) = 1$ and regularity property as in Proposition 2.3.

Theorem 2.9. *The function $\mathfrak{m}(s)$ satisfies the integral equation*

$$\mathfrak{m}(-s) = (2e^s - 1) \int_0^{\infty} \mathfrak{m}'(-t) J_0(2\sqrt{st})\, \mathrm{d}t, \quad s \in \mathbb{R}_+, \tag{2.15}$$

where $J_0(\star)$ stands for the Bessel function:

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin x)\, \mathrm{d}x.$$

Proof. For $\Re z < 1$, we have that $G(z) = \int_0^{\infty} \mathfrak{m}'(zt) e^{-t}\, \mathrm{d}t$. Thus,

$$G(z) = -\frac{1}{z} \int_0^{\infty} \mathfrak{m}'(-t) e^{t/z}\, \mathrm{d}t \text{ for } \Re z < 0, \quad G(z) = \frac{1}{z} \int_0^{\infty} \mathfrak{m}'(t) e^{-t/z}\, \mathrm{d}t \text{ for } 0 < \Re z < 1.$$

Thus, the functional equation for $G(z)$ in the region $\Re z < -1$ in terms of $\mathfrak{m}'(t)$ reads as

$$\frac{1}{z} = \int_0^{\infty} \mathfrak{m}'(-t) \Big(\frac{2}{z+1} e^{\frac{t}{z+1}} + \frac{1}{z} e^{tz} - \frac{1}{z} e^{\frac{t}{z}}\Big)\, \mathrm{d}t. \tag{2.16}$$

Now, multiply this by e^{-sz} and integrate over $\Re z = -\sigma < -1$, where $s > 0$ is real. We have ([44], p. 465)

$$\int_{-\sigma - i\infty}^{-\sigma + i\infty} \frac{e^{-sz}}{z}\, \mathrm{d}z = -2\pi i;$$

$$2\int\limits_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{e^{\frac{t}{z+1}-sz}}{z+1}\,\mathrm{d}z=-2e^{s}\int\limits_{\sigma-1-i\infty}^{\sigma-1+i\infty}\frac{e^{sz-\frac{t}{z}}}{z}\,\mathrm{d}z=-2e^{s}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{e^{\sqrt{st}z-\frac{\sqrt{st}}{z}}}{z}\,\mathrm{d}z=-4\pi i e^{s}J_0(2\sqrt{st}),$$

where $\sigma_0=(\sigma-1)\sqrt{\frac{t}{s}}>0$, and $J_\lambda(*)$ stands for the Bessel function (see [44], p. 597 for the representation of the Bessel function by this integral). Further,

$$\int\limits_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{e^{(t-s)z}}{z}\,\mathrm{d}z=\begin{cases}-2\pi i & \text{if } s>t,\\ 0 & \text{if } s<t,\end{cases}\quad\int\limits_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{e^{\frac{t}{z}-sz}}{z}\,\mathrm{d}z=-2\pi i J_0(2\sqrt{st}).$$

Thus, eventually

$$-2\pi i=-2\pi i\int\limits_0^\infty \mathfrak{m}'(-t)(2e^s-1)J_0(2\sqrt{st})\,\mathrm{d}t-2\pi i\int\limits_0^s\mathfrak{m}'(-t)\,\mathrm{d}t;$$

since $\mathfrak{m}(0)=1$, this proves the theorem. ∎

Thus, we obtained an integral equation for $\mathfrak{m}(s)$, which corresponds to the functional equation (2.13) for $G(z)$. Since the Laplace transform of $J_0(2\sqrt{t})$ in variable z is $\frac{1}{z}e^{-1/z}$ ([44], p. 503), multiplying the integral equation by e^{-zs}, and integrating over $s\geq 0$, we obtain:

$$\int\limits_0^\infty\frac{\mathfrak{m}(-s)}{2e^s-1}e^{-zs}\,\mathrm{d}s=\int\limits_0^\infty\mathfrak{m}'(-t)\Big(\int\limits_0^\infty J_0(2\sqrt{st})e^{-zs}\,\mathrm{d}s\Big)\,\mathrm{d}t=\frac{1}{z}\int\limits_0^\infty\mathfrak{m}'(-t)e^{-\frac{t}{z}}\,\mathrm{d}t=G(-z).$$

Despite the fact that the first integral was calculated for $\Re z>0$, it does converge for $\Re z>-1$, and therefore we have an integral representation of $G(z)$ in a wider region $\Re z<1$ by a single integral

$$\int\limits_0^\infty\frac{\mathfrak{m}(-t)}{2e^t-1}e^{zt}\,\mathrm{d}t=G(z). \tag{2.17}$$

We conclude this chapter with yet another integral equation, which, unfortunately, is insufficient in determining the moments, since it represents only the symmetry property of $G(z)$. Consider the integral (2.17) in the half plane $\Re z<0$. Since $\mathfrak{m}(0)=1$, this reads as

$$\int\limits_0^\infty\mathfrak{m}(-t)\Big(\frac{e^{tz}}{2e^t-1}+\frac{1}{z^2}e^{\frac{t}{z}}\Big)\,\mathrm{d}t=-\frac{1}{z}. \tag{2.18}$$

Recall that, as usually, $K_n(*)$ denotes the Macdonald function, which for positive real t is defined as

$$K_n(t)=\frac{1}{2}\int\limits_0^\infty\frac{e^{-t/2(z+\frac{1}{z})}}{z^{n+1}}\,\mathrm{d}z.$$

([72], chapter VI, section 6.22). Then the following proposition holds.

Proposition 2.10. *We have an identity*

$$\int\limits_0^\infty M'(-t)K_0\Big(2\sqrt{(s+t)s}\Big)\,\mathrm{d}t = \frac{1}{2}K_0(2s),\ \textit{for } s>0.$$

Proof. If we substitute in (2.18) $z\to -z$, then this equality holds for $\Re z>0$. Multiply now both sides by $e^{-sz-\frac{s}{z}}$, $s>0$, and integrate over real $z\geq 0$. Thus,

$$\int\limits_0^\infty e^{-tz-sz-\frac{s}{z}}\,\mathrm{d}z = \int\limits_0^\infty \frac{e^{-\frac{t}{z}-\frac{s}{z}-sz}}{z^2}\,\mathrm{d}z = 2\sqrt{\frac{s}{s+t}}K_1(2\sqrt{(s+t)s}),\quad \int\limits_0^\infty \frac{e^{-sz-\frac{s}{z}}}{z}\,\mathrm{d}z = 2K_0(2s).$$

Therefore,

$$\int\limits_0^\infty \mathfrak{m}(-t)\Big(\frac{1}{2e^t-1}+1\Big)\sqrt{\frac{s}{s+t}}K_1(2\sqrt{(s+t)s})\,\mathrm{d}t = K_0(2s).$$

By (2.6), $\mathfrak{m}(-t)(\frac{1}{2e^t-1}+1)=2M(-t)$. Now integrate by parts. We have $K_0'(t)=-K_1(t)$ ([72], chapter III, section 3.71; this can be easily proved directly, given the integral representation of these functions). Thus,

$$\frac{\partial}{\partial t}K_0(2\sqrt{(s+t)s}) = -K_1(2\sqrt{(s+t)s})\sqrt{\frac{s}{s+t}},$$

and

$$-2K_0(\sqrt{(s+t)s})M(-t)\ |_0^\infty = 2K_0(2s),$$

whence the claim of the proposition follows. ■

Now we will make some formal calculations involving divergent series. They show that the last proposition is insufficient in determining $M(t)$ - it simply represents the symmetry property of $G(z)$ given in Theorem 2.6. One auxiliary lemma about integrals involving the Macdonald function is needed ([72], chapter XIII, section 13.47).

Lemma 2.11.

$$\int\limits_0^\infty \frac{K_\nu(a\sqrt{x^2+s^2})}{(x^2+s^2)^{\frac{1}{2}\nu}}x^{2\mu+1}\,\mathrm{d}x = \frac{2^\mu\Gamma(\mu+1)}{a^{\mu+1}s^{\nu-\mu-1}}K_{\nu-\mu-1}(as),\ \textit{for } a>0 \textit{ and } \Re\mu>-1.$$

Thus, after change of variables $(s+t)s=x^2+s^2$, (with $a=2$, $\nu=0$), we obtain

$$\int\limits_0^\infty t^nK_0\Big(2\sqrt{(s+t)s}\Big)\,\mathrm{d}t = n!K_{n+1}(2s),.$$

Since $K_{-n}(z) = K_n(z)$ ([72], chapter III, section 3.71), integration of the equation in Proposition 2.10 term by term leads to (a formal divergent sum)

$$\sum_{n=0}^{\infty}(-1)^n M_n K_n(2s) = \frac{1}{2}K_0(2s).$$

Formally (see (2.12), and the equation above), $\sum_{n=0}^{\infty}(-1)^n \frac{M_n}{z^{n+1}} = \frac{1}{z}\mathcal{M}_0(-\frac{1}{z})$, and therefore, after change of variables $z \to 1/z$, we get the following valid integral

$$\int_0^{\infty} e^{-s(z+\frac{1}{z})}\mathcal{M}_0(-z)\frac{\mathrm{d}z}{z} = K_0(2s).$$

This can be rewritten as

$$\int_1^{\infty} e^{-s(z+\frac{1}{z})}\Big(\mathcal{M}_0(-z) + \mathcal{M}_0(-1/z)\Big)\frac{\mathrm{d}z}{z} = K_0(2s).$$

The symmetry property implies $\mathcal{M}_0(z) + \mathcal{M}_0(1/z) \equiv 1$ (Theorem 2.6). Thus, indeed we have an identity, and this suggests that the last proposition only encodes the latter property of $G(z)$.

2.5 Dyadic eigenfunctions

We proceed with the definition of the sequence of functions $G_\lambda(z)$ which satisfy the functional equation analogous to (2.13).
Since $J_0'(\star) = -J_1(\star)$, $J_1(0) = 0$, integration by parts in (2.15) leads to

$$\int_0^{\infty} \frac{\mathfrak{m}(-t)}{\sqrt{t}} J_1(2\sqrt{st})\,\mathrm{d}t = \frac{1}{\sqrt{s}} - \frac{\mathfrak{m}(-s)}{\sqrt{s}(2e^s - 1)}.$$

Recall that the Hankel transform of degree $\nu > -1/2$ for the function $f(r)$ (provided that $\int_0^\infty f(r)\sqrt{r}\,\mathrm{d}r$ converges absolutely) is defined as

$$g(\rho) = \int_0^{\infty} f(r) J_\nu(r\rho) r\,\mathrm{d}r,$$

where $J_\nu(\star)$ is νth Bessel function. The inverse is given by the Hankel inversion formula with exactly the same kernel ([72], chapter XIV, section 14.4.). Thus, after proper change of variables,

$$g(\rho) = \int_0^{\infty} f(r) J_\nu(2\sqrt{r\rho})\,\mathrm{d}r \Leftrightarrow f(r) = \int_0^{\infty} g(\rho) J_\nu(2\sqrt{r\rho})\,\mathrm{d}\rho.$$

Thus, application of this inversion to our last identity yields

$$\frac{\mathfrak{m}(-s)}{\sqrt{s}} = \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{t}}\,\mathrm{d}t - \int_0^{\infty} \frac{\mathfrak{m}(-t)}{\sqrt{t}(2e^t - 1)} J_1(2\sqrt{st})\,\mathrm{d}t.$$

The first integral on the right hand side is equal to $-\frac{1}{\sqrt{s}}J_0(2\sqrt{st})\mid_{t=0}^{\infty}=\frac{1}{\sqrt{s}}$. Let $\psi(s)=(2e^s-1)^{1/2}$. Then this equation can be rewritten as

$$\frac{\mathfrak{m}(-s)}{\sqrt{s}\psi(s)}=\frac{1}{\sqrt{s}\psi(s)}-\int\limits_0^{\infty}\frac{\mathfrak{m}(-t)}{\sqrt{t}\psi(t)}\cdot\frac{J_1(2\sqrt{st})}{\psi(s)\psi(t)}\,\mathrm{d}t.$$

Hence, if we denote

$$\frac{J_1(2\sqrt{st})}{\psi(s)\psi(t)}=K(s,t),\quad \frac{\mathfrak{m}(-s)-1}{\sqrt{s}\psi(s)}=\mathsf{Y}(s),$$

we obtain a second type Fredholm integral equation with symmetric kernel ([39], chapter 9):

$$\mathsf{Y}(s)=\ell(s)-\int\limits_0^{\infty}\mathsf{Y}(t)K(s,t)\,\mathrm{d}t,$$

where

$$\ell(s)=-\frac{1}{\psi(s)}\int\limits_0^{\infty}\frac{J_1(2\sqrt{st})}{\sqrt{t}(2e^t-1)}\,\mathrm{d}t=\frac{1}{\sqrt{s}\psi(s)}\Big(\sum_{n=1}^{\infty}e^{-s/n}2^{-n}-1\Big).$$

The behavior of the Bessel function at infinity is given by the asymptotic formula

$$J_1(x)\sim\Big(\frac{2}{\pi x}\Big)^{1/2}\cos\Big(x-\frac{3}{4}\pi\Big)$$

([72], chapter VII, section 7.1). Therefore, obviously,

$$\int\limits_0^{\infty}\int\limits_0^{\infty}|K(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t<\infty,\quad \int\limits_0^{\infty}|\ell(s)|^2\,\mathrm{d}s<\infty.$$

Thus the operator associated with the kernel $K(s,t)$ is the Hilbert-Schmidt operator ([39], p. 532). The theorem of Hilbert-Schmidt ([39], p. 283) states that the solution of this type of integral equations reduces to finding the eigenvalues λ and the eigenfunctions $A_\lambda(s)$. We postpone the solution of this integral equation (or equation in the form (2.13)), for the future. Till the end of this section we deal with eigenfunctions. The integral operator, consequently, is compact self-conjugate operator in the Hilbert space, it possesses a complete orthogonal system of eigenfunctions, all λ are real and $\lambda_n\to 0$, as $n\to\infty$. If we denote $A_\lambda(s)\psi(s)=B_\lambda(s)$, then the equation for an eigenfunction reads as

$$\int\limits_0^{\infty}B_\lambda(t)\frac{J_1(2\sqrt{st})}{2e^t-1}\,\mathrm{d}t=\lambda B_\lambda(s).$$

This gives $B_\lambda(0)=0$. Since $A_\lambda(s)\in L_2(0,\infty)$, and $J_1(*)$ is bounded, this implies that $B_\lambda(s)$ is uniformly bounded for $s\geq 0$ as well. Moreover, since the Taylor expansion of $J_1(*)$ contains only odd powers of the variable, $B_\lambda(s)\sqrt{s}$ has a Taylor expansion with center 0. Now, multiply this by $\sqrt{s}e^{-s/z}$, $z > 0$, and integrate over $s \in \mathbb{R}_+$. The Laplace transform of $\sqrt{s}J_1(2\sqrt{s})$ is $\frac{1}{z^2}e^{-1/z}$ ([44], p. 503). Thus, we obtain

$$\frac{1}{\lambda}\int\limits_0^\infty \frac{B_\lambda(t)\sqrt{t}}{2e^t - 1}e^{-tz}\,\mathrm{d}t = \frac{1}{z^2}\int\limits_0^\infty B_\lambda(s)\sqrt{s}e^{-\frac{s}{z}}\,\mathrm{d}s. \tag{2.19}$$

Denote by $G_\lambda(-z)$ the function on both sides of the equality. Thus, $G_\lambda(z)$ is defined at least for $\Re z \leq 0$. Since $2e^{t(z+1)} - e^{tz} = (2e^t - 1)e^{tz}$, we have

$$\lambda\Big(2G_\lambda(z+1) - G_\lambda(z)\Big) = \int\limits_0^\infty B_\lambda(t)\sqrt{t}e^{tz}\,\mathrm{d}t = \frac{1}{z^2}G_\lambda(1/z).$$

Therefore, we have proved the first part of the following theorem.

Theorem 2.12. *For every eigenvalue λ of the integral operator, associated with the kernel $K(s,t)$, there exists at least one holomorphic function G_λ (defined for $z \in \mathbb{C} \setminus \mathbb{R}_{>1}$), such that the following holds:*

$$2G_\lambda(z+1) = G_\lambda(z) + \frac{1}{\lambda z^2}G_\lambda\Big(\frac{1}{z}\Big). \tag{2.20}$$

Moreover, $G_\lambda(z)$ for $\Re z < 0$ satisfies all regularity conditions imposed by it being an image under Laplace transform ([44], p. 469).
Conversely: for every λ, such that there exists a function, which satisfies (2.20) and these conditions, λ is the eigenvalue of this operator. The set of all possible λ's is countable, and $\lambda_n \to 0$, as $n \to \infty$.

Proof. The converse is straightforward, since, by the requirement, $G_\lambda(z)$ for $\Re z < 0$ is a Laplace image of a certain function, and all the above transformations are invertible. We leave the details. If the eigenvalue has multiplicity higher than 1, then these $\lambda-$forms span a finite dimensional $\mathbb{C}-$vector space. Note that the proof of Proposition 2.7 implies $|\lambda| < 0.342014...$ Finally, the functional equation (2.20) gives the analytic continuation of $G_\lambda(z)$ to the half-plane $\Re z \leq 1$. Further, if $z \in \mathcal{U}$, where $\mathcal{U} = \{0 \leq \Re z \leq 1\} \setminus \{|z| < 1\}$, we can continue $G_\lambda(z)$ to the region $\mathcal{U} + 1$, and, inductively, to $\mathcal{U} + n$, $n \in \mathbb{N}$. Let $\mathcal{U}_0$ be the union of these. We can, obviously, continue $G_\lambda(z)$ to the set $\mathcal{U}_0^{-1} + n$, $n \in \mathbb{N}$. Similar iterations cover the described domain. ■

Note that, in contrast to $G(z)$, we do not have a symmetry property for $G_\lambda(z)$.

Next calculations produce the first few eigenvalues. The Taylor expansion of $G_\lambda(z)$ is given by

$$G_\lambda(z) = \sum_{L=1}^{\infty} m_L^{(\lambda)} z^{L-1}.$$

It converges inside the unit circle, including its boundary (as is clear from (2.20), there exist all left derivatives at $z=1$). Thus, $m_L^{(\lambda)}$ have the same vanishing properties as m_L (which guarantees the convergence of the series in (2.9)). And therefore, as in the Proposition 2.8, we obtain:

$$\lambda m_s^{(\lambda)} = \sum_{L=1}^{\infty}(-1)^{L-1}c_{L+s}\binom{L+s-1}{s-1}m_L^{(\lambda)}, \quad s \geq 1.$$

Here $c_L = \sum_{n=1}^{\infty}\frac{1}{2^n n^L}$. If we denote $e_{s,L} = (-1)^{L-1}c_{L+s}\binom{L+s-1}{s-1}$, then λ is the eigenvalue of the infinite matrix $\mathcal{E}_{s,L=1}^{\infty} = (e_{s,L})_{s,L=1}^{\infty}$. The numerical calculations with the augmentation of this matrix at sufficiently high level give the following first eigenvalues in decreasing order, with all digits exact:

$$\lambda_1 = 0.25553210_+, \quad \lambda_2 = -0.08892666_+, \quad \lambda_3 = 0.03261586_+, \quad \lambda_4 = -0.01217621_+.$$

Figure 2.1 shows the functions $G_\lambda(z)$ (for the first six eigenvalues) for real z in the interval $[-1,-0.2]$. The choice of this interval is motivated by Theorem 2.12. Note also that functional equation implies $G_\lambda(0) = \left(\frac{1}{2}+\frac{1}{2\lambda}\right)G_\lambda(-1)$. Thus, one has $\frac{G_\lambda(0)}{G_\lambda(-1)} \to \infty$, as $\lambda \to 0$. This can also be seen empirically from Figure 2.1.

2.6 $p-$adic distribution

In the previous sections we were interested in the distribution of the nth generation of the tree $\mathcal{T}$ in the field of real numbers. Since the set of non-equivalent valuations of $\mathbb{Q}$ contains a valuation associated with any prime number p, it is natural to consider the distribution of the set of each generation in the field of $p-$adic numbers $\mathbb{Q}_p$. In this case we have an ultrametric inequality, which implies that two circles are either co-centric or do not intersect. We define

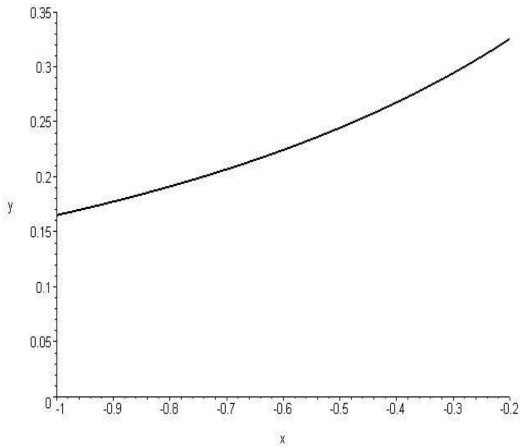
$$F_n(z,\nu) = 2^{-n+1}\#\{\frac{a}{b} \in \mathcal{T}^{(n)} : \mathrm{ord}_p(\frac{a}{b}-z) \geq \nu\}, \quad z \in \mathbb{Q}_p, \quad \nu \in \mathbb{Z}.$$

(When p is fixed, the subscript p in F_n is omitted). Note that in order to calculate $F_n(z,\nu)$ we can confine to the case $\mathrm{ord}_p(z) < \nu$; otherwise $\mathrm{ord}_p(\frac{a}{b}-z) \geq \nu \Leftrightarrow \mathrm{ord}_p(\frac{a}{b}) \geq \nu$. We shall calculate the limit distribution $\mu_p(z,\nu) = \lim_{n\to\infty} F_n(z,\nu)$, and also some characteristics of it, e.g. the zeta function

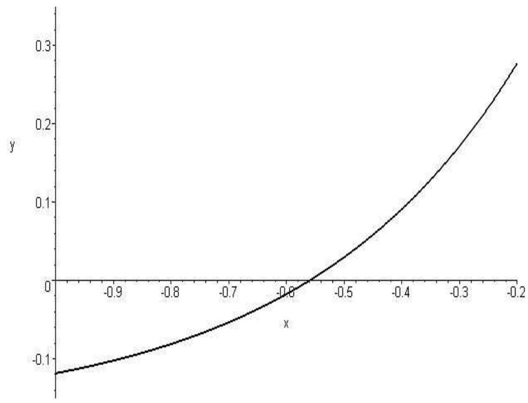
$$Z_p(s) = \int\limits_{u\in\mathbb{Q}_p} |u|^s d\mu_p, \quad s \in \mathbb{C}, \quad z \in \mathbb{Q}_p,$$

where $|*|$ stands for the $p-$adic valuation.

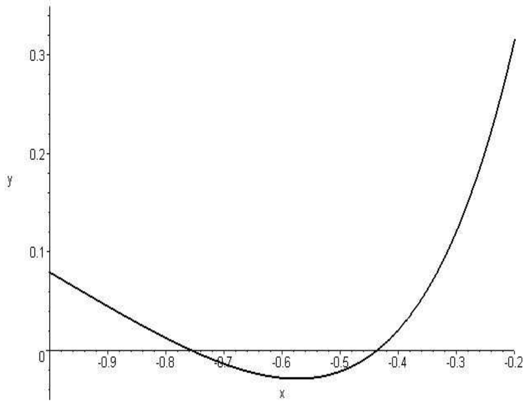
To illustrate how the method works, we will calculate the value of F_n in two special cases. Let $p=2$ and let $E(n)$ be the number of rational numbers in the nth generation with one of a or b being even, and let $O(n)$ be the corresponding



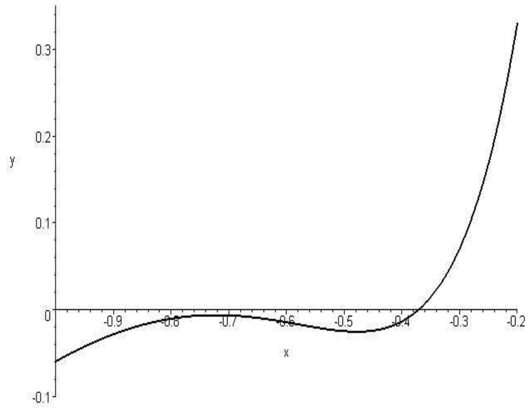
$\lambda_1 = 0.25553210_+$



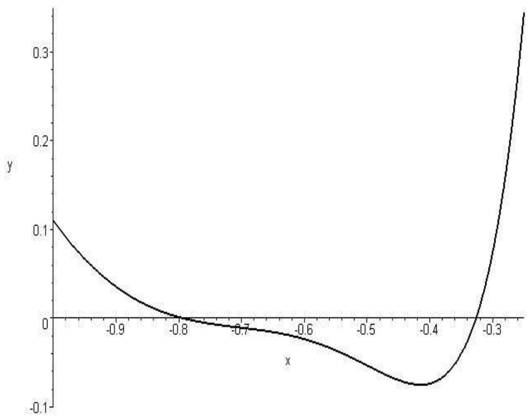
$\lambda_2 = -0.08892666_+$



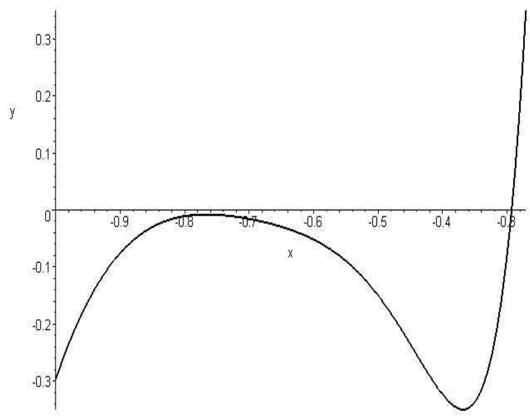
$\lambda_3 = 0.03261586_+$



$\lambda_4 = -0.01217621_+$



$\lambda_5 = 0.00458154_+$



$\lambda_6 = -0.00173113_+$

Figure 2.1: Eigenfunctions $G_\lambda(z)$ for $z \in [-1, -0.2]$

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number of fractions with both a and b odd. Then $E(n)+O(n)=2^{n-1}$. Since $\frac{a}{b}$ in the nth generation generates $\frac{a}{a+b}$ and $\frac{a+b}{b}$ in the $(n+1)$st generation, each fraction $\frac{a}{b}$ with one of the a, b even will generate one fraction with both numerator and denominator odd. If both a, b are odd, then their two offsprings will not be of this kind. Therefore, $O(n+1)=E(n)$. Similarly, $E(n+1)=E(n)+2O(n)$. This gives the recurrence $E(n+1)=E(n)+2E(n-1)$, $n\geq 2$, and this implies

$$E(n)=\frac{2^n+2(-1)^n}{3},\quad O(n)=\frac{2^{n-1}+2(-1)^{n-1}}{3},\quad \mu_2(0,0)=\frac{2}{3}.$$

(For the last equality note that $\frac{a}{b}$ and $\frac{b}{a}$ simultaneously belong to $\mathcal{T}^{(n)}$, and so the number of fractions with $\mathrm{ord}_2(*)>0$ is $E(n)/2$). We will generalize this example to odd prime $p\geq 3$. Let $L_i(n)$ be the part of the fractions in the nth generations such that $ab^{-1}\equiv i \bmod p$ for $0\leq i\leq p-1$ or $i=\infty$ (that is, $b\equiv 0 \bmod p$). Thus,

$$\sum_{i\in\mathbb{F}_p\cup\infty} L_i(n)=1;$$

in other words, $L_i(n)=F_n(i,1)$. For our later investigations we need a result from the theory of finite Markov chains.

Lemma 2.13. *Let* $\mathbf{A}$ *be a matrix of a finite Markov chain with* s *stages. That is,* $a_{i,j}\geq 0$*, and* $\sum_{j=1}^{s}a_{i,j}=1$ *for all* i*. Suppose that* $\mathbf{A}$ *is irreducible (for all pairs* (i,j)*, and some* m*, the entry* $a_{i,j}^{(m)}$ *of the matrix* $\mathbf{A}^m$ *is strictly positive), acyclic and recurrent (this is satisfied, if all entries of* $\mathbf{A}^m$ *are strictly positive for some* m*). Then the eigenvalue* 1 *is simple and if* λ *is another eigenvalue, then* $|\lambda|<1$*, and* $\mathbf{A}^m$*, as* $m\to\infty$*, tends to the matrix* $\mathbf{B}$*, with entries* $b_{i,j}=\pi_j$*, where* $(\pi_1,...,\pi_s)$ *is a unique left eigenvector with eigenvalue* 1*, such that* $\sum_{j=1}^{s}\pi_j=1$*.*

A proof of this lemma can be found in [33], Section 3.1., Theorem 1.3.

Proposition 2.14. $\mu_p(z,1)=\frac{1}{p+1}$ *for* $z\in\mathbb{Z}_p$*.*

Proof. Similarly as in the above example, a fraction $\frac{a}{b}$ from the nth generation generates $\frac{a}{a+b}$ and $\frac{a+b}{b}$ in the $(n+1)$st generation, and it is routine to check that

$$L_i(n+1)=\frac{1}{2}L_{\frac{i}{1-i}}(n)+\frac{1}{2}L_{i-1}(n)\quad\text{for}\quad i\in\mathbb{F}_p\cup\{\infty\},\tag{2.21}$$

(Here we make a natural convention for $\frac{i}{1-i}$ and $i-1$, if $i=1$ or ∞). In this equation, it can happen that $i-1\equiv\frac{i}{1-i} \bmod p$; thus, $(2i-1)^2\equiv -3 \bmod p$. The recurrence for this particular i is to be understood in the obvious way, $L_i(n+1)=L_{i-1}(n)$. Therefore, if we denote the vector-column $(L_\infty(n),L_0(n),...,L_{p-1}(n))^T$ by $\mathbf{v}_n$, and if $\mathcal{A}$ is a matrix of the system (2.21), then $\mathbf{v}_{n+1}=\mathcal{A}\mathbf{v}_n$, and hence

$$\mathbf{v}_n=\mathcal{A}^{n-1}\mathbf{v}_1,$$

where $\mathbf{v}_1 = (0,0,1,0,...,0)^T$. In any particular case, this allows us two find the values of L_i explicitly. For example, if $p = 7$, the characteristic polynomial is

$$f(x) = \frac{1}{16}(x-1)(2x-1)(2x^2+1)(4x^4+2x^3+2x+1).$$

The list of roots is

$$\alpha_1 = 1, \quad \alpha = \frac{1}{2}, \quad \alpha_{3,4} = \pm\frac{i}{\sqrt{2}}, \quad \alpha_{5,6,7,8} = \frac{-1-\sqrt{17}}{8} \pm \frac{\sqrt{1+\sqrt{17}}}{2\sqrt{2}},$$

(with respect to the two values for the root $\sqrt{17}$), the matrix is diagonalisible, and the Jordan normal form gives the expression

$$L_i(n) = \sum_{s=1}^{8} C_{i,s}\alpha_s^n.$$

Note that the elements in each row of the $(p+1)\times(p+1)$ matrix $\mathcal{A}$ are non-negative and sum up to 1, and thus, we have a matrix of a finite Markov chain. We need to check that it is acyclic. Let $\tau(i) = i-1$, and $\sigma(i) = \frac{i}{1-i}$ for $i \in \mathbb{F}_p \cup \{\infty\}$. The entry $a_{i,j}^{(m)}$ of $\mathcal{A}^m$ is

$$a_{i,j}^{(m)} = \sum_{i_1,\dots,i_{m-1}} a_{i,i_1}\cdot a_{i_1,i_2}\cdot\ldots\cdot a_{i_{m-1},j}.$$

Therefore, we need to check that for some fixed m, the composition of m $\sigma's$ or $\tau's$ leads from any i to any j. One checks directly that for any positive k, and $i,j \in \mathbb{F}_p$,

$$\begin{aligned} \tau^{p-1-j}\circ\sigma\circ\tau^k\circ\sigma\circ\tau^{i-1}(i) &= j, \\ \tau^{p-1-j}\circ\sigma\circ\tau^k(\infty) &= j, \\ \tau^k\circ\sigma\circ\tau^{i-1}(i) &= \infty; \end{aligned}$$

(for $i = 0$, we write τ^{-1} for τ^{p-1}). For each pair (i,j), choose k in order the amount of compositions used to be equal (say, to m). Then obviously all entries of $\mathcal{A}^m$ are positive, ant this matrix satisfies the conditions of lemma. Since all columns also sum up to 1, $(\pi_1,...,\pi_{p+1})$, $\pi_j = \frac{1}{p+1}$, $1 \le j \le p+1$, is the needed eigenvector. This proves the Proposition. ■

Next theorem describes $\mu(z,\nu)$ in all cases.

Theorem 2.15. *Let $\nu \in \mathbb{Z}$ and $z \in \mathbb{Q}_p$, and $\mathrm{ord}_p(z) < \nu$ (or $z = 0$). Then, if z is $p-$adic integer,*

$$\mu(z,\nu) = \frac{1}{p^\nu + p^{\nu-1}}.$$

If z is not integer, $\mathrm{ord}_p(z) = -\lambda < 0$,

$$\mu(z,\nu) = \frac{1}{p^{\nu+2\lambda} + p^{\nu+2\lambda-1}}.$$

For $z=0$, $-\nu\leq 0$, we have

$$\mu(0,-\nu)=1-\frac{1}{p^{\nu+1}+p^{\nu}}.$$

This theorem allows the computation of the associated zeta-function:

Corollary 2.16. *For s in the strip $-1<\Re s<1$,*

$$Z_p(s)=\int_{u\in\mathbb{Q}_p}|u|^s d\mu_p=\frac{(p-1)^2}{(p-p^{-s})(p-p^s)},$$

and $Z_p(s)=Z_p(-s)$.

The proof is straightforward. It should be noted that this expression encodes all the values of $\mu(0,\nu)$ for $\nu\in\mathbb{Z}$.

Proof of Theorem 2.15. For shortness, when p is fixed, denote $\mathrm{ord}_p(*)$ by $v(*)$. As before, we want a recurrence relation among the numbers $F_n(i,\kappa)$, $i\in\mathbb{Q}_+$. For each integral κ, we can confine to the case $i<p^{\kappa}$. If $i=0$, we only consider $\kappa>0$ and call these pairs (i,κ) "admissible". We also include $G_n(0,-\kappa)$ for $\kappa\geq 1$, where these values are defined in the same manner as F_n, only inverting the inequality, considering $\frac{a}{b}\in\mathcal{T}^{(n)}$, such that $v(\frac{a}{b})\leq-\kappa$; this is the ratio of fractions in the nth generation outside this circle. As before, a fraction $\frac{a}{b}$ in the nth generation generates the fractions $\frac{a}{a+b}$ and $\frac{a+b}{b}$ in the $(n+1)$st generation. Let $\tau(i,\kappa)=((i-1)\bmod p^{\kappa},\kappa)$. Then for all admissible pairs (i,κ), $i\neq 0$, the pair $\tau(i,\kappa)$ is also admissible, and

$$v(\frac{a+b}{b}-i)=\kappa\Leftrightarrow v(\frac{a}{b}-(i-1))=\kappa.$$

Second, if $\frac{a}{a+b}=i+p^{\kappa}u$, $i\neq 1$, $u\in\mathbb{Z}_p$, and (i,κ) is admissible, then

$$\frac{a}{b}-\frac{i}{1-i}=\frac{p^{\kappa}u}{(1-i)(1-i-p^{\kappa}u)}.$$

Since $v(\frac{i}{1-i})=v(i)-v(1-i)$, this is 0 unless i is an integer, it equals $v(i)$ if the latter is >0 and equals $-v(1-i)$ if $v(1-i)>0$. Further, this difference has valuation $\geq\kappa_0=\kappa$, if $i\in\mathbb{Z}, i\not\equiv 1 \bmod p$, valuation $\geq\kappa_0=\kappa-2v(1-i)$, if $i\in\mathbb{Z}, i\equiv 1\bmod p$, and valuation $\geq\kappa_0=\kappa-2v(i)$ if i is not integer. In all three cases, easy to check, that, if we define $i_0=\frac{i}{1-i}\bmod p^{\kappa_0}$, the pair $\sigma(i,\kappa)=^{\mathrm{def}}(i_0,\kappa_0)$ is admissible. For the converse, let $\frac{a}{b}=i_0+p^{\kappa_0}u$, $u\in\mathbb{Z}_p$. Then

$$\frac{a}{a+b}-\frac{i_0}{1+i_0}=\frac{p^{\kappa_0}}{(1+i_0+p^{\kappa_0}u)(1+i_0)}.$$

If $i=\frac{i_0}{1+i_0}$ is a $p-$adic integer, $i\not\equiv 1\bmod p$, this has a valuation $\geq\kappa=\kappa_0$; if i is a $p-$adic integer, $i\equiv 1(p)$, this has valuation $\geq\kappa=\kappa_0-2v(i_0)=\kappa_0+2v(1-i)$; if i is not a $p-$adic integer, this has valuation $\geq\kappa=\kappa_0-2v(1+i_0)=\kappa_0+2v(i)$. Thus,

$$v(\frac{a}{a+b}-i)\geq\kappa\Leftrightarrow v(\frac{a}{b}-i_0)\geq\kappa_0.$$

Let $i=1$. If $\frac{a}{a+b}=1+p^{\kappa}u$, then $\kappa>0$, $u\in\mathbb{Z}_p$, and we obtain $\frac{a}{b}=-1-\frac{1}{p^{\kappa}u}$, $v(\frac{a}{b})\leq-\kappa$. Converse is also true. Finally, for $\kappa\geq 1$,

$$v(\frac{a+b}{b})\leq-\kappa\Leftrightarrow v(\frac{a}{b})\leq-\kappa,$$

and

$$v(\frac{a}{a+b})\leq-\kappa\Leftrightarrow v(\frac{a}{b}+1)\geq\kappa.$$

Therefore, we have the recurrence relations:

$$\begin{cases} F_{n+1}(i,\kappa) &= \frac{1}{2}F_n(\tau(i,\kappa))+\frac{1}{2}F_n(\sigma(i,\kappa)),\ \text{if } (i,\kappa) \text{ is admissible},\\ F_{n+1}(1,\kappa) &= \frac{1}{2}F_n(0,\kappa)+\frac{1}{2}G_n(0,-\kappa), \kappa\geq 1,\\ G_{n+1}(0,-\kappa) &= \frac{1}{2}G_n(0,-\kappa)+\frac{1}{2}F_n(-1,\kappa), \kappa\geq 1.\end{cases} \tag{2.22}$$

Thus, we have an infinite matrix $\mathcal{A}$, which is a change matrix for the Markov chain. If $\mathbf{v}_n$ is an infinite vector-column of F_n's and G_n's, then $\mathbf{v}_{n+1}=\mathcal{A}\mathbf{v}_n$, and, as before, $\mathbf{v}_n=\mathcal{A}^{n-1}\mathbf{v}_1$. It is direct to check that each column also contains exactly two nonzero entries $\frac{1}{2}$, or one entry, equal to 1. In terms of Markov chains, we need to determine the classes of orbits. Then in proper rearranging, the matrix $\mathcal{A}$ looks like

$$\begin{pmatrix} \mathbf{P}_1 & 0 & \dots & 0 & \dots\\ 0 & \mathbf{P}_2 & \dots & 0 & \dots\\ \vdots & & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & \mathbf{P}_s & 0\\ \vdots & \vdots & \dots & 0 & \ddots \end{pmatrix},$$

where $\mathbf{P}_s$ are finite Markov matrices. Thus, we claim that the length of each orbit is finite, every orbit has a representative $G_*(0,-\kappa)$, $\kappa\geq 1$, the length of it is $p^{\kappa}+p^{\kappa-1}$, and the matrix is recurrent (that is, every two positions communicate). In fact, from the system above and form the expression of the maps $\tau(i,\kappa)$ and $\sigma(i,\kappa)$, the direct check shows that the complete list of the orbit of $G_*(0,-\kappa)$ consists of (and each pair of states are communicating):

$$G_*(0,-\kappa),$$
$$F_*(i,\kappa)\quad (i=0,1,2,...,p^{\kappa}-1),$$
$$F_*(p^{-\lambda}u,\kappa-2\lambda)\quad (\lambda=1,2,...,\kappa-1, u\in\mathbb{N}, u\not\equiv 0 \bmod p, u\leq p^{\kappa-\lambda}).$$

In total, we have

$$1+p^{\kappa}+\sum_{\lambda=1}^{\kappa-1}(p^{\kappa-\lambda}-p^{\kappa-\lambda-1})=p^{\kappa}+p^{\kappa-1}$$

members in the orbit. Thus, each $\mathbf{P}_{\kappa}$ in the matrix above is a finite dimensional $\ell_{\kappa}\times\ell_{\kappa}$ matrix, where $\ell_{\kappa}=p^{\kappa}+p^{\kappa-1}$. For $\kappa=1$, the matrix $\mathbf{P}_1$ is exactly the matrix of the system (2.21). As noted above, the vector column $(1,1,...,1)^T$ is the left eigenvector. As in the previous theorem, it is straightforward to check that this matrix is irreducible and acyclic (that is, the entries of $\mathbf{P}_{\kappa}^{n}$ are strictly positive for sufficiently large n). In fact, since by our observation, each two members in the orbit communicate, and since we have a move $G_{*}(0,-\kappa)\rightarrow G_{*}(0,-\kappa)$, the proof of the last statement is immediate: there exists n such that any position is reachable from another in exactly n moves, and this can be achieved at the expense of the move just described. Therefore, all entries of $\mathbf{P}_{\kappa}^{n}$ are strictly positive. Thus, the claim of the theorem follows from the Lemma 2.13. ■

2.7 Conclusion

We end the Chapter with the following remarks. As is implied by Theorem 2.15, the measure μ_p of those rationals in the Calkin-Wilf tree which are invertible elements of $\mathbb{Z}_p$ is equal to $\frac{p-1}{p+1}$. We follow the line of the Tate thesis [12], and modify this measure in order $\mathbb{Z}_p^{*}$ to have measure 1; accordingly, let us define $\mu'_p=\frac{p+1}{p-1}\mu_p$. Thus, we are lead to the formal definition of the zeta function

$$\zeta_{\mathcal{T}}(s)=\prod_{p}\int_{u\in\mathbb{Q}_p}|u|^{s}\,\mathrm{d}\mu'_p=\prod_{p}\Big(1-\frac{1}{p^2}\Big)\prod_{p}\frac{1}{1-p^{-s-1}}\cdot\frac{1}{1-p^{s-1}}=\frac{6}{\pi^2}\zeta(s+1)\zeta(-s+1).$$

This product diverges everywhere; nevertheless, if we apply the functional equation of the Riemann ζ function for the second multiplier, we obtain

$$\zeta_{\mathcal{T}}(s)=\frac{12}{\pi^2}(2\pi)^{-s}\cos\Big(\frac{\pi s}{2}\Big)\Gamma(s)\zeta(s)\zeta(s+1).$$

From the above definition it is clear that, formally, this zeta function is the sum of the form $\sum_{r\in\mathbb{Q}^{+}}\mu_r r^{-s}$, where, if $r\in\mathbb{Q}_{+}$, and μ_r stands for the limit measure of those rationals in the nth generation of $\mathcal{T}$, which have precisely the same valuation as r at every prime which appears in the decomposition of r, times the factor $\prod_{\,\mathrm{ord}_p(r)\neq 0}\frac{p+1}{p-1}$. Surprisingly, the product $\zeta(s)\zeta(s+1)$ is the zeta function of the Eisenstein series $G_1(z)$, which is related to the distribution of rationals in $\mathcal{T}$ at the infinite prime $\mathbb{Q}_{\infty}=\mathbb{R}$. In fact,

$$\int\limits_{0}^{\infty}\Big(G_1(iz)-G_1(i\infty)\Big)z^{s-1}\,\mathrm{d}z=-8\pi^2(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+1).$$

This is a strong motivation to investigate the tree $\mathcal{T}$ and the Minkowski question mark function in a more general - idelic - setting, thus revealing the true connection between $p-$adic and real distribution, and clarifying the nature of continued fractions in this direction. We hope to implement this in the subsequent papers.

Unfortunately, currently we left the most interesting question, the explicit description of the moments of $F(x)$, unanswered. It is desirable to give the function $G(z)$, and, more generally, dyadic forms $G_{\lambda}(z)$, certain modified Fourier series expansion. This is in part accomplished in Chapter 4, which is a direct continuation of thw current one. Among other results, the dyadic zeta function $\zeta_{\mathcal{M}}(s)$ is introduced, given by $\zeta_{\mathcal{M}}(s)\Gamma(s+1) = \int\limits_0^\infty x^s \, \mathrm{d}F(x)$, the nature of dyadic eigenfunctions $G_\lambda(s)$ is clarified, and certain integrals involving $F(x)$ are computed.

Chapter 3

Asymptotic formula for the moments

3.1 Formulation

In this chapter we establish an asymptotic formula for the moments m_L, which are defined by (2.3). As said before, this sequence is of definite number-theoretic significance because of (2.4). It is not clear whether there exists a closed form formula for the moments m_L. This would be greatly desirable minding the expression (2.4). As a matter of fact, we cannot present a precise definition of the expression "closed form". The above limit clearly gives the sequence of explicitly constructed rational numbers whose limit is m_L. Moreover, Chapter 5 is devoted to finding another such representation, in a much more fundamental way. On the other hand, one has the following asymptotic result. Let $\mathsf{C} = e^{-2\sqrt{\log 2}} = 0.189169995269...$

Proposition 3.1. *The following estimate holds, as $L \to \infty$:*

$$\mathsf{C}^{\sqrt{L}} \ll m_L \ll L^{1/4}\mathsf{C}^{\sqrt{L}}.$$

Both implied constants are absolute.

Though the result of Proposition 3.1 is contained in the next Theorem 3.2, we indulge in presenting a separate proof for it, since it is considerably shorter and is less technical.

Thus, the final aim of this Chapter is to prove the first exact asymptotic term for the sequence of moments m_L. Though from our point of view this is of inferior significance than the closed form formula, nevertheless, it is of definite interest too.

Theorem 3.2. *Let m_L be defined by (2.3). Then the following holds as $L \to \infty$:*

$$m_L = \sqrt[4]{4\pi^2 \log 2} \cdot c_0 \cdot L^{1/4}\mathsf{C}^{\sqrt{L}} + O(\mathsf{C}^{\sqrt{L}} L^{-1/4}).$$

Here the constant c_0 is given by $c_0 = \int_0^1 \Psi(x)\,\mathrm{d}x = 1.030199563382+$, where $\Psi(x)$ is $1-$periodic function given by $\Psi(x) = 2^x(1 - F(x))$.

Remark 1. The periodicity of $\Psi(x)$ is obvious from (2.1). Numerically, $\sqrt[4]{4\pi^2\log 2}\cdot c_0 = 2.356229889908+$. The sequence of moments M_L (see definition (2.4)) is equally important. These moments represent special values of higher left derivatives of $G(z)$ at $z=1$; more precisely, formal Taylor expansion at $z=1$ is given by (see (2.6))

$$G(z+1)=\sum_{L=1}^{\infty}M_L z^{L-1}.$$

As is implied by Proposition 2.2, one has the asymptotic formula $M_L \sim L!\frac{c_0}{(\log 2)^L}$. Moreover, in this case there exists exact convergent asymptotic series (see Proposition 4.8). Though sequences M_L and m_L are linearly dependent via relations (2.7), it is nevertheless significant that the same structural constant c_0 does manifest in asymptotic formulae for both of these sequences.

Remark 2. As it is clear from the proof, other terms of asymptotic expansion can be calculated as well. We confine to the first term since calculations are standard (though tedious) with no new ideas being introduced. Note that structural constants $c_s=\int_0^1 x^s\Psi(x)\,\mathrm{d}x$ do appear in the asymptotic expansion, and the method used shares some similarities with the Euler-Maclaurin summation.

Remark 3. If we start from the representation of m_L via the last integral of (2.3) (which is the most natural), then to obtain the representation by the first integral we use the second equation in (2.1). Moreover, one of the core stages of the proof relies on the periodicity of $\Psi(x)$ and so uses the first equation in (2.1). Since these two equations are characteristic only to the Minkowski question mark function, generally speaking, our asymptotic formula for the moments of $F(x)$ is unique among similar results for other probabilistic distributions with proper support on the interval $[0,1]$.

3.2 Estimate for the moments m_L

This section gives the proof of Proposition 3.1. *A priori*, as it is implied by the fact that the radius of convergence of $G(z)$ at $z=0$ is 1, and by the fact (2.9), for every $\varepsilon>0$ and $p>1$, one has $\frac{1}{L^p}\gg m_L\gg(1-\varepsilon)^L$, as $L\to\infty$.

Proof of Proposition 3.1. Fix $J\in\mathbb{N}$, and choose an increasing sequence of positive real numbers $\mu_j<1$, $1\leq j\leq J$. We will soon specify μ_j in such a way that $\mu_j\to 0$ uniformly as $L\to\infty$. An estimate for m_L is obtained via the defining integral (recall that $F(x)+F(1/x)=1$):

$$m_L=\Big(\int\limits_0^{\mu_1}+\sum_{j=1}^{J-1}\int\limits_{\mu_j}^{\mu_{j+1}}+\int\limits_{\mu_J}^{\infty}\Big)\Big(\frac{1}{x+1}\Big)^L\,\mathrm{d}F(x)<$$
$$F(\mu_1)+\sum_{j=1}^{J-1}\Big(\frac{1}{\mu_j+1}\Big)^L F(\mu_{j+1})+\Big(\frac{1}{\mu_J+1}\Big)^L.$$

This indeed holds: the integrand is bounded by 1 in the first integral. In the middle integrals, we choose the largest value of integrand, and change bounds of integration to $[0,\mu_{j+1}]$. The same is done with the last integral, with bounds being changed into $[0,\infty)$. Now choose $\mu_j = \frac{1}{c_j\sqrt{L}}$ for some decreasing sequence of constants c_j. Functional equation for $F(x)$ implies $F(x+n) = 1-2^{-n}+2^{-n}F(x)$, $x \geq 0$. Thus, $1-F(x) \asymp 2^{-x}$, as $x \to \infty$ (the implied constants being $\min$ and $\max$ of the function $\Psi(x)$; see Figure 4.1, Section 4.6). Using the identity $F(x)+F(1/x) = 1$, we therefore obtain

$$m_L \ll 2^{-c_1\sqrt{L}} + \sum_{j=1}^{J-1}\Big(\frac{1}{\frac{1}{c_j\sqrt{L}}+1}\Big)^L 2^{-c_{j+1}\sqrt{L}} + \Big(\frac{1}{\frac{1}{c_J\sqrt{L}}+1}\Big)^L \ll$$
$$e^{-\sqrt{L}c_1\log 2} + \sum_{j=1}^{J-1} e^{-\sqrt{L}(\frac{1}{c_j}+c_{j+1}\log 2)} + e^{-\sqrt{L}\frac{1}{c_J}}. \qquad (3.1)$$

At this point we need an elementary Lemma.

Lemma 3.3. *For given $J\in\mathbb{N}$, there exists a unique sequence of positive real numbers $c_1^*,...,c_J^*$, such that*

$$c_1^* = \frac{1}{c_1^*}+c_2^* = \frac{1}{c_2^*}+c_3^* = ... = \frac{1}{c_{J-1}^*}+c_J^* = \frac{1}{c_J^*}.$$

Moreover, this sequence $\{c_j^*, 1\leq j\leq J\}$ is decreasing, and it is given by

$$c_j^* = \frac{\sin\frac{(j+1)\pi}{J+2}}{\sin\frac{j\pi}{J+2}}, \quad j=1,2,...,J \Rightarrow c_1^* = 2\cos\frac{\pi}{J+2}.$$

Proof. Indeed, we see that $c_1^* = x$ determines the sequence c_j^* uniquely. First, $c_2 = x - \frac{1}{x} = \frac{x^2-1}{x}$. Let $F_1(x) = x$, $F_2(x) = x^2-1$. Suppose we have shown that $c_j = \frac{F_j(x)}{F_{j-1}(x)}$ for a certain sequence of polynomials. Then from the above equations one obtains

$$c_{j+1} = c_1 - \frac{F_{j-1}(x)}{F_j(x)} = \frac{xF_j(x)-F_{j-1}(x)}{F_j(x)}.$$

Thus, using induction we see that $c_j = \frac{F_j(x)}{F_{j-1}(x)}$, where polynomials $F_j(x)$ are given by the initial values $F_0(x) = 1$, $F_1(x) = x$, and then for $j\geq 1$ recurrently by $F_{j+1}(x) = xF_j(x)-F_{j-1}(x)$. This shows that $F_j(2x) = U_j(x)$, where $U(x)$ stand for the classical Chebyshev $U-$polynomials, given by

$$U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}.$$

The last equation $c_1^* = \frac{1}{c_J^*}$ implies $F_{J+1}(x) = 0$. Thus, $U_{J+1}(x/2) = 0$, and all possible values for c_1^* are given by $c_1^* = x = 2\cos\frac{k\pi}{J+2}$, $k = 1,2,...,J+1$. Thus,

$$c_j^* = \frac{F_j(x)}{F_{j-1}(x)} = \frac{U_j(x/2)}{U_{j-1}(x/2)} = \frac{\sin\frac{k(j+1)\pi}{J+2}}{\sin\frac{kj\pi}{J+2}}.$$

Since we are concerned with only positive solutions, this gives the last statement of the Lemma. Finally, monotonicity is easily verifiable. Indeed, the system of equations implies $c_2^* < c_1^*$, and then we act by induction. ■

Thus, $c_1^* > 2 - \frac{b}{J^2}$ for some constant $b > 0$. Let us return to the proof of the Proposition. For a given J, let c_j^* be the sequence in Lemma, and let $c_i^* = c_i\sqrt{\log 2}$. Thus,

$$c_1 \log 2 = \frac{1}{c_1} + c_2 \log 2 = \frac{1}{c_2} + c_3 \log 2 = ... = \frac{1}{c_{J-1}} + c_J \log 2 = \frac{1}{c_J}.$$

Choosing exactly this sequence for the estimate (3.1), and using the bound for c_1^*, we get:

$$m_L \ll (J+1)e^{-\sqrt{L}c_1 \log 2} < (J+1)\mathsf{C}^{\sqrt{L}}e^{\frac{b\sqrt{\log 2}}{J^2}\sqrt{L}}.$$

Finally, the choice $J = [L^{1/4}]$ establishes the upper bound.

The lower estimate is immediate. In fact, let $\mu = \frac{1}{c\sqrt{L}}$. Then

$$m_L > \int\limits_0^\mu \Big(\frac{1}{x+1}\Big)^L \mathrm{d}F(x) > \Big(\frac{1}{\mu+1}\Big)^L F(\mu) \gg 2^{-c\sqrt{L}} \cdot e^{-\sqrt{L}\frac{1}{c}}$$

The choice $c = \log^{-1/2} 2$ establishes the desired bound. ■

The constants in Proposition can also be calculated without great effort. As was said before as a Remark 3, it should be noted that, if we start directly from the second definition (2.3) of m_L, then in the course of the proof of Proposition 3.1 we use both equalities $F(x) + F(1/x) = 1$ and $2F(\frac{x}{x+1}) = F(x)$. Since these two determine $F(x)$ uniquely, generally speaking, our estimate for m_L is characteristic only to $F(x)$. A direct inspection of the proof also reveals that the true asymptotic "action" in the second definition (2.3) of m_L takes place in the neighborhood of 1. This, obviously, is a general fact for probabilistic distributions with proper support on the interval $[0, 1]$. Additionally, calculations show that the sequence $m_L/(L^{1/4}\mathsf{C}^{\sqrt{L}})$ is monotonically decreasing. This is indeed the case, and there exists a limit $A = \lim_{L\to\infty} \frac{m_L}{L^{1/4}\mathsf{C}^{\sqrt{L}}}$, as stated in Theorem 3.2.

3.3 Asymptotic formula

This section gives a proof of Theorem 3.2.

3.3.1 Preliminary calculations

We use the notation of Landau to denote by $\mathcal{B}$ some absolutely bounded function in certain neighborhood of a variable; in our case $\mathcal{B}$ depends mostly on L and we consider the case $L \to \infty$. As a convention, $\mathcal{B}$ stands for different function if considered in another or even the same formula. Occasionally, this notation is used to denote dependence on other variables and in each case it should be clear which variable and which neighborhood is implied in notation $\mathcal{B}$. The main tool of the proof is a variant of saddle-point method. The latter is used in finding asymptotic expansion of certain integrals depending on parameter and it traces its history back from the works of Laplace. Though in our case certain corrections, amendments and variations are necessary, the main technique is standard and can be found, for example, in [44].

Since $1 - F(x) = 2^{-x}\Psi(x)$, identity (2.3) implies

$$\begin{aligned} m_L = \int\limits_0^\infty \Big(\frac{x}{x+1}\Big)^L \mathrm{d}(F(x)-1) &= (F(x)-1)\Big(\frac{x}{x+1}\Big)^L\Big|_0^\infty + \\ L\int\limits_0^\infty 2^{-x}\frac{x^{L-1}}{(x+1)^{L+1}}\Psi(x)\,\mathrm{d}x &= L\int\limits_0^\infty 2^{-x}\frac{x^{L-1}}{(x+1)^{L+1}}\Psi(x)\,\mathrm{d}x. \end{aligned}$$

Put $c = \log 2$. The function $\mathsf{f}(x) = L\cdot\log\frac{x}{x+1} - cx$ achieves its maximum at $x = x_0$, where

$$x_0(x_0+1) = \frac{L}{c} \Rightarrow x_0 = \frac{-1+\sqrt{1+\frac{4L}{c}}}{2} \Rightarrow x_0 = \sqrt{\frac{L}{c}} - \frac{1}{2} + \frac{\mathcal{B}}{\sqrt{L}}.$$

Let $c_0 = \int_0^1 \Psi(x)\,\mathrm{d}x$, and let us rewrite the expression for m_L as

$$m_L = Lc_0\int\limits_0^\infty \frac{1}{x(x+1)}e^{\mathsf{f}(x)}\,\mathrm{d}x + L\int\limits_0^\infty(\Psi(x)-c_0)\frac{1}{x(x+1)}e^{\mathsf{f}(x)}\,\mathrm{d}x = c_0Lg_L + Lr_L.$$

Our specific choice in extracting c_0 out of $\Psi(x)$ as a dominant ingredient can be motivated for the following reason. Since $0.9 < \Psi(x) < 1.2$ (see Figure 4.1), the main weight of the integral defining m_L (as far as $\Psi(x)$ is concerned) befalls on a certain constant in the range $(0.9, 1.2)$. Moreover, it is easy to verify that for any continuous periodic function $\Upsilon(x)$ one has

$$\begin{aligned} \int\limits_{-\infty}^{\infty} \Upsilon(x)e^{-Ax^2}\,\mathrm{d}x &= \frac{\sqrt{\pi}\Upsilon(0)}{\sqrt{A}} + O(A^{-1}) \text{ as } A\to\infty \text{ (if } \Upsilon(x) \text{ is smooth)}, \\ \int\limits_{-\infty}^{\infty} \Upsilon(x)e^{-Ax^2}\,\mathrm{d}x &= \frac{\sqrt{\pi}\int_0^1\Upsilon(x)\,\mathrm{d}x}{\sqrt{A}} + O(1) \text{ as } A\to 0+. \end{aligned}$$

This can be seen empirically from the fact that in the first case the dominant weight of the integral is supported only in the neighborhood of 0, while in the second case this interval has length tending to infinity. As it is seen from the next subsection, we have the second case (*a posteriori*, the weight of m_L is properly supported on the interval $(\sqrt{L/c} - L^{2/7}, \sqrt{L/c} + L^{2/7})$, and hence the correct constant is c_0 rather then $\Psi(x_0)$.

3.3.2 Evaluation of g_L.

Choose $\delta = \delta(L)$ such that $\frac{\delta}{L^{1/4}} \to \infty$ and $\frac{\delta}{L^{1/3}} \to 0$. For example, henceforth we fix $\delta = \delta(L) = L^{2/7}$. Separate the integral defining g_L into four parts:

$$g_L = \int_0^1 + \int_1^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^{\infty} \frac{1}{x(x+1)} e^{\mathbf{f}(x)}\,\mathrm{d}x = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4.$$

Evaluation of $\mathscr{I}_3$

First, $\mathbf{f}'(x_0) = 0$ and $\mathbf{f}''(x_0) < 0$. Suppose $|x - x_0| \le \delta$. Then the Taylor formula implies that for certain $\theta_x \in [x_0, x]$ one has

$$\mathbf{f}(x) = \mathbf{f}(x_0) - \alpha(x-x_0)^2 + \beta(x-x_0)^3 + \frac{f^{(4)}(\theta_x)}{24}(x-x_0)^4.$$

Direct calculations show that

$$\begin{aligned}
\mathbf{f}''(x) &= -\frac{L(2x+1)}{x^2(x+1)^2} \Rightarrow \mathbf{f}''(x_0) = -\frac{c^2}{L}(2x_0+1) \Rightarrow \alpha = \frac{c^{3/2}}{\sqrt{L}} + \frac{\mathcal{B}}{L^{3/2}},\\
\beta &= \frac{\mathbf{f}'''(x_0)}{6} = \frac{L(3x_0^2+3x_0+1)}{3x_0^3(x_0+1)^3} = \frac{c^2}{L} + \frac{\mathcal{B}}{L^{3/2}},\\
\mathbf{f}^{(4)}(x) &= 6L\Big(\frac{1}{(x+1)^4} - \frac{1}{x^4}\Big) \Rightarrow \mathbf{f}^{(4)}(\theta_x) = \frac{\mathcal{B}}{L^{3/2}}.
\end{aligned}$$

In the same fashion,

$$\frac{1}{x(x+1)} = \frac{1}{x_0(x_0+1)} + \gamma(x-x_0) + \sigma_x(x-x_0)^2,$$

where

$$\gamma = -\frac{2x_0+1}{x_0^2(x_0+1)^2} = -\frac{2c^{3/2}}{L^{3/2}} + \frac{\mathcal{B}}{L^{5/2}}, \quad \sigma_x = \frac{\mathcal{B}}{L^2}.$$

Now let us evaluate the value of $e^{\mathbf{f}(x_0)}$. For $x \to \infty$, we have

$$\begin{aligned}
\log\frac{x}{x+1} &= -\frac{1}{x} + \frac{1}{2x^2} + \frac{\mathcal{B}}{x^3} \Rightarrow\\
L\log\frac{x_0}{x_0+1} = -\frac{L}{\sqrt{\frac{L}{c} - \frac{1}{2} + \frac{\mathcal{B}}{\sqrt{L}}}} + \frac{c}{2} + \frac{\mathcal{B}}{\sqrt{L}} &= -\sqrt{cL} + \frac{\mathcal{B}}{\sqrt{L}}.
\end{aligned}$$

Further,

$$\begin{aligned}
-cx_0 = -\sqrt{cL} + \frac{c}{2} + \frac{\mathcal{B}}{\sqrt{L}} \Rightarrow\\
e^{\mathbf{f}(x_0)} = \exp\Big(-2\sqrt{cL} + \frac{c}{2} + \frac{\mathcal{B}}{\sqrt{L}}\Big) = e^{-2\sqrt{cL}}\sqrt{2}\Big(1 + \frac{\mathcal{B}}{\sqrt{L}}\Big).
\end{aligned}$$

We already have obtained all necessary components to evaluate $\mathscr{I}_3$. Note that

$$\beta(x-x_0)^3 = \frac{\mathcal{B}}{L} \cdot L^{6/7} = o(1), \quad \frac{\mathcal{B}}{L^{3/2}}(x-x_0)^4 = o(1).$$

Since $e^y = 1 + y + \mathcal{B}y^2$ for $y = o(1)$, in the range $(x_0 - \delta, x_0 + \delta)$ the function under the integral can be written as

$$\begin{aligned}\frac{1}{x(x+1)}e^{\mathbf{f}(x)} &= \Big(\frac{1}{x_0(x_0+1)} + \gamma(x-x_0) + \frac{\mathcal{B}}{L^2}(x-x_0)^2\Big)\\ \cdot \exp\Big(\mathbf{f}(x_0) - \alpha(x-x_0)^2 + \beta(x-x_0)^3 + \frac{\mathcal{B}}{L^{3/2}}(x-x_0)^4\Big) &= e^{\mathbf{f}(x_0)}e^{-\alpha(x-x_0)^2}\cdot\\ \Big(1 + \beta(x-x_0)^3 + \frac{\mathcal{B}}{L^{3/2}}(x-x_0)^4 + \frac{\mathcal{B}}{L^2}(x-x_0)^6\Big)&\cdot\\ \Big(\frac{1}{x_0(x_0+1)} + \gamma(x-x_0) + \frac{\mathcal{B}}{L^2}(x-x_0)^2\Big)&\end{aligned}$$

(The bound $|x - x_0| = \mathcal{B}L^{1/2}$ was used in merging $\frac{\mathcal{B}\beta}{L^{3/2}}(x-x_0)^7$ and $\frac{\mathcal{B}}{L^3}(x-x_0)^8$ into $\frac{\mathcal{B}}{L^2}(x-x_0)^6$). Thus, we have decomposed the function under integral into the sum of twelve functions. It is important to note that two of these functions

$$e^{\mathbf{f}(x_0)}e^{-\alpha(x-x_0)^2}\gamma(x-x_0) \text{ and } e^{\mathbf{f}(x_0)}e^{-\alpha(x-x_0)^2}\frac{1}{x_0(x_0+1)}\beta(x-x_0)^3,$$

thought being comparatively large, are odd function in $(x - x_0)$ and consequently contribute 0 after integration. We are left with evaluating all the rest summands. This is a routine job. For example,

$$\begin{aligned} e^{\mathbf{f}(x_0)}\gamma\beta \int_{x_0-\delta}^{x_0+\delta} e^{-\alpha(x-x_0)^2}(x-x_0)^4\,\mathrm{d}x &= \\ e^{\mathbf{f}(x_0)}\gamma\beta\frac{1}{\alpha^{5/2}} \int_{-\sqrt{\alpha}\delta}^{\sqrt{\alpha}\delta} y^4 e^{-y^2}\,\mathrm{d}y = \mathcal{B}\frac{L^{5/4}}{L^{5/2}}\mathsf{C}^{\sqrt{L}} &= \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.\end{aligned}$$

Second example:

$$\begin{aligned} e^{\mathbf{f}(x_0)}\frac{\mathcal{B}\gamma}{L^{3/2}} \int_{x_0-\delta}^{x_0+\delta} e^{-\alpha(x-x_0)^2}|x-x_0|^5\,\mathrm{d}x &= \\ e^{\mathbf{f}(x_0)}\frac{\mathcal{B}\gamma}{L^{3/2}\alpha^3} \int_{-\sqrt{\alpha}\delta}^{\sqrt{\alpha}\delta} |y|^5 e^{-y^2}\,\mathrm{d}y = \mathcal{B}L^{-3/2-3/2+3/2}\mathsf{C}^{\sqrt{L}} &= \mathcal{B}L^{-3/2}\mathsf{C}^{\sqrt{L}}.\end{aligned}$$

Another example:

$$e^{\mathbf{f}(x_0)}\frac{\mathcal{B}}{x_0(x_0+1)L^{3/2}} \int_{x_0-\delta}^{x_0+\delta} e^{-\alpha(x-x_0)^2}(x-x_0)^4\,\mathrm{d}x = \mathcal{B}\frac{L^{5/4}}{L^{5/2}}\mathsf{C}^{\sqrt{L}} = \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.$$

Therefore, direct inspection shows that all functions apart from the main term contribute at most $\mathcal{B}\mathsf{C}^{\sqrt{L}}L^{-5/4}$ into the value of g_L. We are left with determining the magnitude of the main term. Since $\sqrt{\alpha}\delta \sim c^{3/4}L^{1/28}$, this yields

$$\int_{-\sqrt{\alpha}\delta}^{\sqrt{\alpha}\delta} e^{-y^2}\,\mathrm{d}y = \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y + \mathcal{B}\int_{\sqrt{\alpha}\delta}^{\infty} e^{-y^2}\,\mathrm{d}y = \sqrt{\pi} + \mathcal{B}\exp(-c^{3/2}L^{1/14}).$$

And so, the main term is given by

$$\frac{1}{x_0(x_0+1)}e^{\mathbf{f}(x_0)}\int\limits_{-\delta}^{\delta} e^{-\alpha x^2}\,\mathrm{d}x = \Big(\sqrt{\pi}+\mathcal{B}\exp(-c^{3/2}L^{1/14})\Big)\frac{1}{\sqrt{\alpha}}\frac{1}{x_0(x_0+1)}e^{\mathbf{f}(x_0)} =$$
$$\frac{c}{L}\sqrt{2\pi}e^{-2\sqrt{cL}}\sqrt[4]{L}c^{-3/4}\Big(1+\frac{\mathcal{B}}{\sqrt{L}}\Big).$$

Thus we obtain the main asymptotic term

$$\mathscr{I}_3 = \sqrt[4]{4\pi^2\log 2}\cdot L^{-3/4}\mathsf{C}^{\sqrt{L}} + \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.$$

Evaluation of $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_4$

Trivially,

$$\mathscr{I}_1 = \int\limits_0^1 \frac{x^{L-1}}{(x+1)^{L+1}}2^{-x}\,\mathrm{d}x = \mathcal{B}2^{-L}.$$

To evaluate $\mathscr{I}_2$, we use standard inequality

$$\log\Big(\frac{x}{x+1}\Big) < -\frac{1}{x}+\frac{1}{2x^2} \text{ for } x \geq 1. \tag{3.2}$$

Thus, since $\mathbf{f}(x)$ is increasing function in the interval $[1, x_0]$, we have

$$\mathscr{I}_2 = \int\limits_1^{x_0-\delta}\frac{1}{x(x+1)}e^{\mathbf{f}(x)}\,\mathrm{d}x < \log 2\cdot e^{\mathbf{f}(x_0-\delta)} = \mathcal{B}e^{\mathbf{f}(x_0-\delta)}.$$

Further, using (3.2), we have

$$\mathbf{f}(x_0-\delta) = L\log\Big(\frac{x_0-\delta}{x_0-\delta+1}\Big) - c(x_0-\delta) < -\frac{L}{x_0-\delta}+\frac{L}{2(x_0-\delta)^2} - cx_0 + c\delta =$$
$$-\frac{L}{x_0} - cx_0 + \mathcal{B} + c\delta + \Big(\frac{L}{x_0} - \frac{L}{x_0-\delta}\Big) =$$
$$-\frac{L}{x_0} - cx_0 + \mathcal{B} + c\delta - \frac{L\delta}{x_0(x_0+1)} - L\delta\Big(\frac{1}{x_0(x_0-\delta)} - \frac{1}{x_0(x_0+1)}\Big) =$$
$$-\frac{L}{x_0} - cx_0 + \mathcal{B} - \frac{L\delta^2}{x_0^3} = -2\sqrt{cL} + \mathcal{B} - c^{3/2}L^{1/14}$$

(recall that $\delta = L^{2/7}$). Therefore,

$$\mathscr{I}_2 = \mathcal{B}\mathsf{C}^{\sqrt{L}}\exp(-c^{3/2}L^{1/14}) = \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.$$

In the same vein,

$$\mathbf{f}(x_0+\delta) = -2\sqrt{cL} + \mathcal{B} - c^{3/2}L^{1/14}.$$

Since $\mathbf{f}(x)$ is decreasing in the interval $[x_0, \infty)$, we obtain

$$\mathscr{I}_4 = \int\limits_{x_0+\delta}^{\infty} \frac{1}{x(x+1)} e^{\mathbf{f}(x)}\,\mathrm{d}x < e^{\mathbf{f}(x_0+\delta)} \int\limits_{1}^{\infty} \frac{\mathrm{d}x}{x(x+1)} = \mathcal{B}\mathsf{C}^{\sqrt{L}} \exp(-c^{3/2}L^{1/14}) = \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.$$

Combining all estimates for $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$ and $\mathscr{I}_4$, we eventually obtain

$$g_L = \sqrt[4]{4\pi^2 \log 2} \cdot L^{-3/4}\mathsf{C}^{\sqrt{L}} + \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.$$

3.3.3 Evaluation of r_L.

In this subsection we can be more concise, since the method is the same as we used to evaluate g_L. Recall that

$$r_L = \int\limits_{0}^{\infty} (\Psi(x) - c_0) \frac{1}{x(x+1)} e^{\mathbf{f}(x)}\,\mathrm{d}x.$$

Let

$$\hat{\Psi}(x) = \int\limits_{0}^{x} (\Psi(t) - c_0)\,\mathrm{d}t.$$

Definition of c_0 yields that $\hat{\Psi}(x)$ is bounded (and periodic) function. Using integration by parts, we can rewrite r_L as

$$r_L = \int\limits_{0}^{\infty} \hat{\Psi}(x) h(x) e^{\mathbf{f}(x)}\,\mathrm{d}x \Rightarrow r_L = \mathcal{B} \int\limits_{0}^{\infty} h(x) e^{\mathbf{f}(x)}\,\mathrm{d}x,$$

where

$$h(x) = \frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{1}{x(x+1)}\mathbf{f}'(x) = -\frac{L}{x^2(x+1)^2} + \frac{c}{x(x+1)} + \frac{2x+1}{x^2(x+1)^2}.$$

Note that

$$h(x_0) = \frac{2x_0+1}{x_0^2(x_0+1)^2} = \frac{2c^{3/2}}{L^{3/2}} + \frac{\mathcal{B}}{L^{5/2}}.$$

As a matter of fact, we have completely analogous integral to the one defining g_L, with the major difference in the estimate $h(x_0) = \mathcal{B}L^{-3/2}$, whereas $\frac{1}{x_0(x_0+1)} = \mathcal{B}L^{-1}$. Thus, using the same method to evaluate r_L as we did with g_L, one gets

$$r_L = \mathcal{B}L^{1/4-3/2}\mathsf{C}^{\sqrt{L}} = \mathcal{B}L^{-5/4}\mathsf{C}^{\sqrt{L}}.$$

Since $m_L = c_0 L g_L + L r_L$, this finishes the proof of Theorem 3.2. ■

3.3.4 Final remarks

As can be inherited from the proof, the share of integrals $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_4$ into the exact value of m_L is of order $\mathsf{C}^{\sqrt{L}}\exp(-c^{3/2}L^{1/14})$. Thus, standard machinery of asymptotic expansion shows that all terms for asymptotic series of m_L are supported on the neighborhood $(x_0-\delta, x_0+\delta)$. More thorough inspection reveals that

$$m_L \sim L^{1/4}\mathsf{C}^{\sqrt{L}} \cdot \sum_{i=0}^{\infty} \frac{\mathsf{A}_i}{L^{i/2}}.$$

As was mentioned, every term of this expansion can be obtained by (increasingly tedious) computations. For example, A_1 is a sum of c_0 and $c_1 = \int_0^1 x\Psi(x)\,\mathrm{d}x$, each multiplied by some explicit constant (constant c_1 occurs while extracting the main term out of the integral r_L). The main idea is a classical fact that for any continuous periodic function $\Upsilon(x)$, a function

$$\Upsilon_n(x) \overset{\mathsf{def}}{=} \int_0^x \int_0^{x_1} \dots \int_0^{x_{n-1}} \Upsilon(x_n)\,\mathrm{d}x_n\,\mathrm{d}x_{n-1}\dots\mathrm{d}x_1 = \frac{1}{(n-1)!}\int\limits_0^x (x-t)^{n-1}\Upsilon(t)\,\mathrm{d}t$$

is a sum of certain polynomial and some periodic function (of course, this idea is the core of the Euler-Maclaurin summation). Nevertheless, from number-theoretic point of view our main concern is the structure of values m_L rather than their magnitude (though dominant) in some small neighborhood of x_0, and hence we have confined only in exhibiting such a possibility of asymptotic expansion.

We finish with providing a table for some values of constants m_L. Here $m_L^\star = \frac{m_L}{\sqrt[4]{L}\mathsf{C}^{\sqrt{L}}}$. The main result of this Chapter implies that sequence $m_L^\star$ tends to the limit $c_0\sqrt[4]{4\pi^2\log 2} = 2.3562298899_+$. Unfortunately, we do not have yet any evidence that the closed form expression exists for c_0 (as well as for m_L with $L \geq 2$). Finally, we remark that the result of Theorem 3.2 should be considered in conjunction with linear relations $m_L = \sum_{s=0}^{L}\binom{L}{s}(-1)^s m_s$, $L \geq 0$ (Proposition 2.4), which the sequence m_L satisfies, and with the natural inequalities, imposed by the fact that m_L is a sequence of moments of probabilistic distribution with support on the interval $[0,1]$. We thus have Hausdorff conditions, which state that for all non-negative integers m and n, one has

$$2\int\limits_0^1 x^n(1-x)^m\,\mathrm{d}F(x) = \sum_{i=0}^{m}\binom{m}{i}(-1)^i m_{i+n} > 0.$$

This is, of course, the consequence of monotonicity of $F(x)$.

Sequence m_L		
L	m_L	$m_L^\star$
1	0.5000000000	2.643125297
2	0.2909264764	2.577573745
3	0.1863897146	2.533204605
4	0.1269922584	2.509329792
5	0.09016445494	2.496320715
6	0.06592816257	2.488147649
7	0.04929431046	2.481940613
8	0.03751871185	2.476544438
9	0.02897962203	2.471583746
10	0.02266585817	2.466982861
11	0.01792085923	2.462750421
12	0.01430468951	2.458897371
20	0.003008686707	2.438565967
30	0.0006211064464	2.425096683
40	0.0001622371309	2.416702495
50	0.00004937221843	2.410831724
100	0.0000004445933003	2.395743861

Chapter 4

Further properties of integral transforms: the zeta function

4.1 Four objects

It was shown in Chapter 2 that each generation of the Calkin-Wilf tree possesses a distribution function $F_n(x)$, and that $F_n(x)$ converges uniformly to $F(x)$. This is, of course, a well known fact about the Farey tree. The function $F(x)$, as a distribution function is uniquely determined by the functional equation (2.1)

On the other hand (as was mentioned in the introduction), almost all the results of other authors reveal the properties of the Minkowski question mark function as a function itself. Our goal and main motivation of this thesis is to show that in fact there exist several unique and very interesting analytic objects associated with $F(x)$ which encode a great deal of essential information about it. Two of these objects were introduced in Chapter 2. As was already noticed, let us point out that, surprisingly, there are striking similarities and analogies between the results proved here as well as in Chapter 2, with Lewis'-Zagier's [47] results on period functions for Maass wave forms. That work is an expanded and clarified exposition of an earlier paper by Lewis [46]. The concise exposition of these objects, their properties and relations to Selberg zeta function can be found in [76]. The reader who is not indifferent to the beauty of the Minkowski question mark function is strongly urged to compare results in this thesis with those in [47]. Thus, instead of making quite numerous references to [47] at various stages of the work (mainly in Sections 4.2, 4.6 and 4.7), it is more useful to give a table of most important functions encountered there, juxtaposed with analogous object in this work. Here is the summary (some notations were already introduced, others will be explained in Sections 4.6 and 4.7).

Maass wave form	$u(z)$	$\Psi(x)$	*Periodic function on the real line*
Period function	$\psi(z)$	$G(z)$	*Dyadic period function*
Distribution	$U(x)\,\mathrm{d}x$	$\mathrm{d}F(x)$	*Minkowski's "question mark"*
$L-$functions	$L_0(\rho), L_1(\rho)$	$\zeta_{\mathcal{M}}(s)$	*Dyadic zeta function*
Entire function	$g(w)$	$\mathfrak{m}(t)$	*Generating function of moments*
Entire function	$\phi(w)$	$M(t)$	*Generating function of moments*
Spectral parameter	s	$\frac{1}{2}$; 1	*Analogue of spectral parameter*

As a matter of fact, the first entry is the only one where the analogy is not precise. Indeed, the distribution $U(x)$ is the limit value of the Maass wave form $u(x+iy)$ on the real line (as $y \to +0$), in the sense that $u(x+iy) \sim y^{1-s}U(x) + y^{s}U(x)$, whereas $\Psi(x)$ is the same $F(x)$ made periodic. As far as the last entry of the table in concerned, the "analogue" of spectral parameter, sometimes this role is played by 1, sometimes by $\frac{1}{2}$. This occurs, obviously, because the relation between Maass forms and $F(x)$ is only the analogy which is not strictly defined.

Summarizing, these are the three objects associated with the Minkowski question mark function.

- *Distribution $F(x)$ = Functional equations (2.1) + Continuity.*

- *Period function $G(z)$ = Three term functional equation (2.13) + Mild growth condition (as in Theorem 2.6).*

- *Exponential generating function $\mathfrak{m}(t)$ = Integral equation (2.15) + boundary value and diminishing condition on the negative real line (Proposition 2.4).*

Each of these objects is characterized by the functional equation, and subject to some regularity conditions, it is unique, and thus arises exactly from $F(x)$. The objects are described via the "equality" *Function = Equation + Condition*. This means that the object on the left possesses both features; conversely - any object with these properties is necessarily the function on the left.

As expected, here we encounter the phenomena of "bootstrapping": in all cases, regularity conditions can be significantly relaxed, and they are sufficient for the uniqueness, which automatically imply stronger regularity conditions. Here we show the rough picture of this phenomena. In each case, we suppose that the object satisfies the corresponding functional equation. See Chapter 2 for the details.

(i) $F(x)$ is continuous at one point $\Rightarrow F(x)$ is continuous.

(ii) There exists $\varepsilon < 1$ such that for every z with $\Re z < 0,$ we have
$G(z-x) = O(2^{\varepsilon x})$ as $x \to \infty \Rightarrow G(z) = O(|z|^{-1})$ as $dist(z, \mathbb{R}_{+}) \to \infty.$

(iii) $\mathfrak{m}'(-t) = O(t^{-1})$ as $t \to \infty \Rightarrow |\mathfrak{m}(-t)| \ll e^{-\sqrt{t \log 2}}$ as $t \to \infty.$

Corresponding converse results were proved in Chapter 2. For $F(x)$, this was in fact the starting point of these investigations, since the distribution of rationals in the Calkin-Wilf tree is a certain continuous function satisfying (2.1); thus, it is exactly $F(x)$. The converse result for $\mathfrak{m}(t)$ follows from Fredholm alternative, since all eigenvalues of the operator of the Section 2.6 are strictly less than 1 in absolute value. Finally, the converse theorem for $G(z)$ follows from a technical detail in the proof, which is the numerical estimate $0 < \frac{\pi^2}{12} - \frac{\log^2 2}{2} < 1$; as a matter of fact, it appears that this is essentially the same argument as in the case of $\mathfrak{m}(t)$, since this constant gives the upper bound for the moduli of eigenvalues.

One of the aims of this Chapter is to clarify the connections among these three objects, and to add the final fourth satellite, associated with $F(x)$. Henceforth, we have the complete list:

- *Zeta function $\zeta_{\mathcal{M}}(s)$ (see definition (4.11) below) = Functional equation with symmetry $s \to -s$ (4.12) + Regularity behavior in vertical strips.*

In this case, we do not present a proof of a converse result. Indeed, the converse result for $G(z)$ is strongly motivated by its relation to Eisenstein series $G_1(z)$ (see Chapter 2 and its last section). In the case of $\zeta_{\mathcal{M}}(s)$, this question is of small importance, and we rather concentrate on the direct result and its consequences.

4.2 Three term functional equation, distributions $F_{\lambda}(x)$

In this section, we give a proof of (2.13) different from the one presented in Chapter 2, since it is considerably shorter. For our purposes, it is convenient to work in a slightly greater generality. Suppose that $\lambda \in \mathbb{R}$ has the property that there exists a function $F_{\lambda}(x)$, $x \in [0, \infty)$, such that

$$\mathrm{d}F_{\lambda}(x+1) = \frac{1}{2}\,\mathrm{d}F_{\lambda}(x),\ \mathrm{d}F_{\lambda}\Big(\frac{1}{x}\Big) = \frac{1}{\lambda}\,\mathrm{d}F_{\lambda}(x). \tag{4.1}$$

We omitted the word "continuous" in the description of the function intentionally. For a moment, consider $F_{\lambda}(x) = F(x)$ with $\lambda = -1$. Then $F_{-1}(x)$ is certainly continuous. The reason for introducing λ will be apparent later. Let

$$G_{\lambda}(z) = \int\limits_{0}^{\infty} \frac{1}{x+1-z}\,\mathrm{d}F_{\lambda}(x).$$

Since $F(x)+F(1/x)=1$, we see that for $\lambda=-1$ this agrees with the definition (2.10). This integral converges to an analytic function in the cut plane $\mathbb{C}\setminus(1,\infty)$. We have

$$2G_\lambda(z+1)=2\int\limits_0^1\frac{1}{x-z}\,\mathrm{d}F_\lambda(x)+2\int\limits_1^\infty\frac{1}{x-z}\,\mathrm{d}F(x)=$$

$$2\int\limits_0^\infty\frac{1}{\frac{x}{x+1}-z}\,\mathrm{d}F_\lambda\Big(\frac{x}{x+1}\Big)+2\int\limits_0^\infty\frac{1}{x+1-z}\,\mathrm{d}F_\lambda(x+1)=$$

$$\frac{2}{z}\int\limits_0^\infty\Big(\frac{x+1}{x+1-\frac{1}{z}}-1+1\Big)\,\mathrm{d}F_\lambda\Big(\frac{1}{x+1}\Big)+G_\lambda(z)=$$

$$\frac{\alpha}{\lambda z}+\frac{1}{\lambda z^2}G_\lambda\Big(\frac{1}{z}\Big)+G_\lambda(z),\ \text{where } \alpha=\int\limits_0^\infty\mathrm{d}F_\lambda(x).$$

For $\lambda=-1$ and $F_{-1}(x)=F(x)$, this gives Theorem 2.6. Further, suppose $\lambda\neq-1$. Then

$$\alpha=\int\limits_0^\infty\mathrm{d}F_\lambda(x)=\int\limits_1^\infty\mathrm{d}F_\lambda(x)+\int\limits_0^1\mathrm{d}F_\lambda(x)=\frac{\alpha}{2}-\frac{\alpha}{2\lambda}\Rightarrow\alpha=0.$$

Therefore, the last functional equation reads as

$$2G_\lambda(z+1)=\frac{1}{\lambda z^2}G_\lambda\Big(\frac{1}{z}\Big)+G_\lambda(z).$$

As a matter of fact, there cannot be any reasonable function $F_\lambda(x)$ which satisfies (4.1). Nevertheless, the last functional equation is identical to (2.20). Thus, Theorem 2.12 gives a description of all such possible λ. This suggests that we can still find certain distributions $F_\lambda(x)$. Further, as it was mentioned, -1 is not an-eigen value of operator from the Section 2.6. Due to the minus sign in front of the operator, this is exactly the exceptional eigenvalue, which is essential in the Fredholm alternative. The above proof (rigid at least in case $\lambda=-1$), surprisingly, proves that the next tautological sentence has a certain point: "-1 is not an eigenvalue because it is -1". Indeed, we obtain a non-homogeneous part of the three term functional equation only because $\lambda=-1$, since otherwise $\alpha=0$ and the equation is homogenic.

Distributions $F_\lambda(x)$ can indeed be strictly defined, at least in the space of functions, which are analytic in the disk $\mathbf{D}=\{z:|z-\frac{1}{2}|\leq\frac{1}{2}\}$, including its boundary. This space is equipped with a topology of uniform convergence, and a distribution on this space is any continuous linear functional. Denote this space by C^ω. Now, since

$$\int\limits_0^1\frac{x}{1-xz}\,\mathrm{d}F_\lambda(x)=-\frac{\lambda}{2}G_\lambda(z):=\sum_{L=1}^\infty m_L^{(\lambda)}z^{L-1},$$

define a distribution F_λ on the space C^ω by $\langle z^L, F_\lambda\rangle = m_L^\lambda$, $L \geq 1$, $\langle 1, F_\lambda\rangle = 0$, and for any analytic function $B(z) \in \mathsf{C}^\omega$, $B(z) = \sum_{L=0}^{\infty} b_L z^L$, by

$$\langle B, F_\lambda\rangle = \sum_{L=0}^{\infty} b_L \langle z^L, F_\lambda\rangle.$$

First, $\langle *, F_\lambda\rangle$ is certainly a linear functional and is properly defined, since the functional equation (2.20) implies that $G_\lambda(z)$ possesses all left derivatives at $z = 1$; as a consequence, the series $\sum_{L=1}^{\infty} L^p |m_L^{(\lambda)}|$ converges for any $p \in \mathbb{N}$ (see Proposition 3.1 for the estimates on moments m_L). Second, let $B_n(z) = \sum_{L=0}^{\infty} b_L^{(n)} z^L$, $n \geq 1$, converge uniformly to $B(z)$ in the circle $|z| \leq 1$. Thus, $\sup_{|z|\leq 1} |B_n(z) - B(z)| = r_n \to 0$. Then by Cauchy formula,

$$b_L^{(n)} = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{B_n(z)}{z^{L+1}}\,\mathrm{d}z.$$

This obviously implies that $|b_L^{(n)} - b_L| \leq r_n$, $L \geq 0$, and therefore $\langle *, F_\lambda\rangle$ is continuous, and hence it is a distribution. Using the condition $\mathrm{d}F_\lambda(x+1) = \frac{1}{2}\,\mathrm{d}F_\lambda(x)$, these distributions can be extended to other spaces. Summarizing, we have shown that Minkowski question mark function has an infinite sequence of "peers" $F_\lambda(x)$ which are also related to continued fraction expansion, in somewhat similar manner. $F(x)$ is the only one among them being "non-homogeneous".

4.3 Linear relations among moments M_L

In this section we clarify the nature of linear relations among the moments M_L. This was mentioned in Chapter 2, but not done in explicit form. Note that Proposition 2.4 gives linear relations among moments m_L: $m_L = \sum_{s=0}^{L} \binom{L}{s}(-1)^s m_s$, $L \geq 0$. These linear relations can be written in terms of M_L. Despite the fact that these relations form a general phenomena for symmetric distributions, in conjunction with (2.6) they give an essential information about $F(x)$. Let us denote

$$\mathsf{q}(x,t) = (2 - e^t)e^{xt} - (2e^t - 1)e^{-xt} = \sum_{n=1}^{\infty} \mathsf{Q}_n(x)\frac{t^n}{n!}.$$

We see that $\mathsf{Q}_n(x)$ are polynomials with integer coefficients and they are given by

$$\mathsf{Q}_n(x) = 2x^n - (x+1)^n - 2(1-x)^n + (-x)^n. \tag{4.2}$$

The following table gives the first few polynomials.

n	$\mathsf{Q}_n(x)$	n	$\mathsf{Q}_n(x)$
1	$2x - 3$	5	$2x^5 - 15x^4 + 10x^3 - 30x^2 + 5x - 3$
2	$2x - 3$	6	$6x^5 - 45x^4 + 20x^3 - 45x^2 + 6x - 3$
3	$2x^3 - 9x^2 + 3x - 3$	7	$2x^7 - 21x^6 + 21x^5 - 105x^4 + 35x^3 - 63x^2 + 7x - 3$
4	$4x^3 - 18x^2 + 4x - 3$	8	$8x^7 - 84x^6 + 56x^5 - 210x^4 + 56x^3 - 84x^2 + 8x - 3$

Moreover, the following statement holds.

Proposition 4.1. *Polynomials* $\mathsf{Q}_n(x)$ *have the following properties:*

$$
\begin{array}{ll}
(i) & \mathsf{Q}_{2n}(x) \in L_{\mathbb{Q}}\big(\mathsf{Q}_1(x), \mathsf{Q}_3(x), ..., \mathsf{Q}_{2n-1}(x)\big), \quad n \geq 1; \\
(ii) & \deg \mathsf{Q}_{2n} = 2n-1, \quad \deg \mathsf{Q}_{2n-1} = 2n-1, \quad n \geq 1; \\
(iii) & \widehat{\mathsf{Q}}_{2n}(x) := \dfrac{\mathsf{Q}_{2n}(x)+3}{x} \textit{ is reciprocal} : \widehat{\mathsf{Q}}_{2n}(x) = x^{2n-2}\widehat{\mathsf{Q}}_{2n}\Big(\dfrac{1}{x}\Big); \\
(iv) & \displaystyle\int_0^\infty \mathsf{Q}_n(x)\,\mathrm{d}F(x) = 0.
\end{array}
$$

Naturally, it is property (iv) which makes these polynomials very important in the study of the Minkowski question mark function. Here $L_{\mathbb{Q}}(*)$ denotes the $\mathbb{Q}-$linear space spanned by the specified polynomials.

Proof. (i) Let $\mathsf{q}_e(x,t) = \frac{1}{2}\big(\mathsf{q}(x,t) + \mathsf{q}(x,-t)\big)$, and $\mathsf{q}_o(x,t) = \frac{1}{2}\big(\mathsf{q}(x,t) - \mathsf{q}(x,-t)\big)$. Direct calculation shows that, if $e^t = T$, then

$$
2\mathsf{q}_e = e^{xt}(3 - T - \frac{2}{T}) + e^{-xt}(3 - \frac{1}{T} - 2T), \quad 2\mathsf{q}_o = e^{xt}(1 - T + \frac{2}{T}) - e^{-xt}(1 - \frac{1}{T} + 2T).
$$

This yields

$$
\sum_{n=1}^{\infty} \mathsf{Q}_{2n}(x)\frac{t^{2n}}{(2n)!} = \mathsf{q}_e(x,t) = \frac{T-1}{T+1}\mathsf{q}_o(x,t) = \frac{e^t-1}{e^t+1}\sum_{n=0}^{\infty} \mathsf{Q}_{2n+1}(x)\frac{t^{2n+1}}{(2n+1)!}.
$$

The multiplier on the right, $\frac{e^t-1}{e^t+1} = \tanh(t/2)$, is independent of x, and this obviously proves the part (i). Also, part (ii) follows easily from (4.2).

(iii) Since $\widehat{\mathsf{Q}}_{2n}(x) = \frac{1}{x}(3x^{2n} - (x+1)^{2n} - 2(x-1)^{2n} + 3)$, the proof is immediate.

(iv) In fact, Proposition 2.4 gives $(2-e^t)M(t) = (2e^t-1)M(-t)$. For real $|t| < \log 2$, we have $M(t) = \int_0^\infty e^{xt}\,\mathrm{d}F(x)$. This implies

$$
\int_0^\infty \mathsf{q}(x,t)\,\mathrm{d}F(x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^\infty \mathsf{Q}_n(x)\,\mathrm{d}F(x) \equiv 0, \quad \text{for } |t| < \log 2,
$$

and this completes the proof. ■

Consequently, there exist linear relations among the moments M_L. Thus, for example, part (iv) (in case $n=1$ and $n=3$) implies $2M_1 - 3 = 0$ and $2M_3 - 9M_2 + 3M_1 = 3$ respectively. The exact values of M_L belong to the class of constants, which can be thought as emerging from arithmetic-geometric chaos. This resembles the situation concerning polynomial relations among various periods. We will not present the definition of a period (it can be found in [40]). In particular, the authors conjecture (and there is no support for possibility that it can be proved wrong) that *"if a period has two integral representations, then one can pass from one formula to another using only additivity, change of variables, and Newton-Leibniz formula, in which all functions and domains of integration are algebraic with coefficients in* $\overline{\mathbb{Q}}$*"*. Thus, for example, the conjecture predicts the possibility to prove directly that $\iint_{\frac{x^2}{4}+3y^2\leq 1} \mathrm{d}x\,\mathrm{d}y = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt[3]{(1-x)(1+x)^2}}$, without knowing that they both are equal to $\frac{2\pi}{\sqrt{3}}$, and this indeed can be done. Similarly, returning to the topic of this paper, we believe that any finite $\mathbb{Q}-$linear relation among the constants M_L can be proved simply by applying the functional equation of $F(x)$, by means of integration by parts and change of variables. The last proposition supports this claim. In other words, we believe that there cannot be any other miraculous coincidences regarding the values of M_L. More precisely, we formulate

Conjecture 4.2. *Suppose, $r_k \in \mathbb{Q}$, $0 \leq k \leq L$, are rational numbers such that $\sum_{k=0}^{L} r_k M_k = 0$. Let $\ell = \left[\frac{L-1}{2}\right]$. Then*

$$\sum_{k=0}^{L} r_k x^k \in L_{\mathbb{Q}}\big(\mathsf{Q}_1(x), \mathsf{Q}_3(x), ..., \mathsf{Q}_{2\ell+1}(x)\big).$$

This conjecture, if true, should be difficult to prove. It would imply, for example, that M_L for $L \geq 2$ are irrational. On the other hand, this conjecture seems to be much more natural and approachable, compared to similar conjectures regarding arithmetic nature of constants emerging from geometric chaos, e.g. spectral values s for Maass wave forms (say, for $\mathsf{PSL}_2(\mathbb{Z})$), or those coming from arithmetic chaos, like non-trivial zeros of Riemann's $\zeta(s)$. We cannot give any other evidence, save the last proposition, to support this conjecture.

4.4 Exact sequence

In this section we prove the exactness of a sequence of continuous linear maps, intricately related to the Minkowski question mark function $F(x)$. Let $\mathsf{C}[0,1]$ denote the space of continuous, complex-valued functions on the interval $[0,1]$ with supremum norm. For $f \in \mathsf{C}[0,1]$, one has the identity

$$\int_0^1 f(x)\,\mathrm{d}F(x) = \sum_{n=1}^{\infty} \int_0^1 f\Big(\frac{1}{x+n}\Big) 2^{-n}\,\mathrm{d}F(x), \tag{4.3}$$

Indeed, using functional the equation (2.1), we have

$$\int_0^1 f(x)\,\mathrm{d}F(x) = \int_1^{\infty} f\Big(\frac{1}{x}\Big)\,\mathrm{d}F(x) = \sum_{n=1}^{\infty} \int_0^1 f\Big(\frac{1}{x+n}\Big)\,\mathrm{d}F(x+n),$$

which is exactly (4.3). Let C^{ω} denote, as before, the space of analytic functions in the disk $\mathbf{D} = |z-\frac{1}{2}| \leq \frac{1}{2}$, including its boundary. We equip this space with the topology of uniform convergence (as a matter of fact, we have a wider choice of spaces; this one is chosen as an important example). Now, consider a continuous functional on C^{ω} given by $T(f) = \int_0^1 f(x)\,\mathrm{d}F(x)$, and a continuous non-compact linear operator $[\mathcal{L}f](x) = f(x) - \sum_{n=1}^{\infty} f\big(\frac{1}{x+n}\big)2^{-n}$. Finally, let i stand for the natural inclusion $i: \mathbb{C} \to \mathsf{C}^{\omega}$.

Theorem 4.3. *The following sequence of maps is exact:*

$$0 \to \mathbb{C} \xrightarrow{i} \underset{(*)}{\mathsf{C}^{\omega}} \xrightarrow{\mathcal{L}} \underset{(**)}{\mathsf{C}^{\omega}} \xrightarrow{T} \mathbb{C} \to 0. \tag{4.4}$$

Proof. First, i is obviously a monomorphism. Let $f \in \mathrm{Ker}(\mathcal{L})$. This means that $f(x) = \sum_{n=1}^{\infty} f\big(\frac{1}{x+n}\big)2^{-n}$. Let $x_0 \in [0,1]$ be such that $|f(x_0)| = \sup\limits_{x\in[0,1]} |f(x)|$. Since $\sum_{n=1}^{\infty} 2^{-n} = 1$, this yields $f(\frac{1}{x_0+n}) = f(x_0)$ for $n \in \mathbb{N}$. By induction, $f([0, n_1, n_2, ..., n_I + x_0]) = f(x_0)$ for all $I \in \mathbb{N}$, and all $n_i \in \mathbb{N}$, $1 \leq i \leq I$; here $[\star]$ stands for the (regular) continued fraction. Since this set is everywhere dense in $[0,1]$ and f is continuous, this forces $f(x) \equiv \mathrm{const}$ for $x \in [0,1]$. Due to the analytic continuation, this is valid for $x \in \mathbf{D}$ as well. Hence, we have the exactness at the term $(*)$.

Next, T is obviously an epimorphism. Further, identity (4.3) implies that $\mathrm{Im}(\mathcal{L}) \subset \mathrm{Ker}(T)$. The task is to show that indeed we have an equality. At this stage, we need the following lemma. Denote $[\mathcal{S}f](x) = \sum_{n=1}^{\infty} f\big(\frac{1}{x+n}\big)2^{-n}$.

Lemma 4.4. *Let $f \in \mathsf{C}^{\omega}$. Then $[\mathcal{S}^n f](x) = 2T(f) + O(\gamma^{-2n})$ for $x \in \mathbf{D}$; here $T(f)$ stands for the constant function, $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden section, and the bound implied by O is uniform for $x \in \mathbf{D}$.*

Proof. In fact, lemma is true for any function with continuous derivative. Let $x \in \mathbf{D}$. We have

$$[\mathcal{S}^r f](x) = \sum_{n_1, n_2, \ldots, n_r = 1}^{\infty} 2^{-(n_1+n_2+\ldots+n_r)} f([0, n_1, n_2, ..., n_r + x]).$$

The direct inspection of this expression and (1.1) shows that this is exactly twice the Riemann sum for the integral $\int\limits_0^1 f(x)\,\mathrm{d}F(x)$, corresponding to the division of unit interval into intervals with endpoints being $[0, n_1, n_2, ..., n_r]$, $n_i \in \mathbb{N}$. From the basic properties of Möbius transformations we inherit that the set $[0, n_1, n_2, ..., n_r + x]$ for $x \in \mathbf{D}$ is a circle $\mathbf{D}_r$ whose diagonal is one of these intervals, say I_r. For fixed r, the largest of these intervals has endpoints $\frac{F_{r-1}}{F_r}$ and $\frac{F_r}{F_{r+1}}$, where F_r stands for the usual Fibonacci sequence. Thus, its length is $\frac{1}{F_r F_{r+1}} \sim c\gamma^{-2r}$. Let $x_0, x_1 \in \mathbf{D}_r$, and $\sup_{x\in\mathbf{D}} |f'(x)| = A$. We have

$$\sup_{x_0, x_1 \in \mathbf{D}_r} |f(x_0) - f(x_1)| \leq Ac\gamma^{-2r}.$$

Thus, the Riemann sum deviates from the Riemann integral no more than

$$|[\mathcal{S}^r f](x) - 2T(f)| \leq Ac\gamma^{-2r} \sum_{n_1,n_2,\ldots,n_r=1}^{\infty} 2^{-(n_1+n_2+\ldots+n_r)} = Ac\gamma^{-2r}.$$

This proves the Lemma. ■

Thus, let $f \in \mathrm{Ker}(T)$. All we need is to show that the equation $f = g - \mathcal{S}g$ has a solution $g \in \mathsf{C}^{\omega}$. Indeed, let $g = f + \sum_{n=1}^{\infty} \mathcal{S}^n f$. By the above lemma, $\|\mathcal{S}^n f\| = O(\gamma^{-2n})$. Thus, the series defining g converges uniformly and hence g is an analytic function. Finally, $g - \mathcal{S}g = f$; this shows that $\mathrm{Ker}(T) \subset \mathrm{Im}(\mathcal{L})$ and the exactness at the term $(**)$ is proved. ■

A generalization of Theorem 4.3 is the following

Proposition 4.5. *Let $\ell \geq 2$ be a fixed positive integer. Then the following sequence of maps is exact:*

$$0 \to \mathbb{C} \xrightarrow{i} \underset{(*)}{\mathsf{C}^{\omega}} \xrightarrow{\mathcal{L}^{\ell}} \underset{(**)}{\mathsf{C}^{\omega}} \xrightarrow{T} \mathbb{C} \to 0.$$

Proof. Let $f \in \mathrm{Ker}(\mathcal{L}^{\ell})$. This means $\mathcal{L}^{\ell}(f) = 0$. By Theorem 4.3, this implies that $\mathcal{L}^{\ell-1}(f) \equiv c$, where c is a constant function. Thus, on the one hand, $T \circ \mathcal{L}^{\ell-1}(f) = T(c) = c/2$. On the other, using exactness of the sequence (4.4), we obtain $T \circ \mathcal{L}^{\ell-1}(f) = T \circ \mathcal{L} \circ \mathcal{L}^{\ell-2}(f) = 0$. Whence $c = 0$. Therefore, if $\ell \geq 2$, $\mathcal{L}^{\ell}(f) = 0$ implies $\mathcal{L}^{\ell-1}(f) = 0$. By induction, this yields $\mathcal{L}(f) = 0 \Rightarrow f \equiv$ const., and this proves the exactness at the term $(*)$.

As before, $\mathrm{Im}(\mathcal{L}^{\ell}) \subset \mathrm{Ker}(T)$. Let $f \in \mathrm{Ker}(T)$. We need to show that $\mathcal{L}^{\ell}(g) = f$ has a solution $g \in \mathsf{C}^{\omega}$. Indeed, let

$$g = \sum_{n=0}^{\infty} \binom{n+\ell-1}{n} \mathcal{S}^n f.$$

Lemma 4.4 states that this series converges uniformly to an analytic function. Further, using Pascal's identity, we obtain

$$\mathcal{L}g = g - \mathcal{S}g = \sum_{n=0}^{\infty} \binom{n+\ell-2}{n} \mathcal{S}^n f.$$

We act by induction, and this gives exactly $\mathcal{L}^{\ell}(g) = f$. This proves the exactness at the term $(**)$. ■

These results imply that, for example, $\mathsf{Q} := \mathrm{Im}(\mathcal{L})$ is a linear subspace of C^{ω} of codimension 1, and that $\mathcal{L}|_{\mathsf{Q}}$ is an isomorphism.

The eigenfunctions of $\mathcal{S}$ acting on the space C^{ω} are given by $G^{\star}(-x) = \int_0^{-x} G_{\lambda}(z)\,\mathrm{d}z + \int_{-1}^{0} G_{\lambda}(z)\,\mathrm{d}z$ (see equations (4.6) and (4.7) in the next section). Thus, the problem of convergence of $\mathcal{S}^n f$ is completely analogous to the problem of convergence for the iterates of Gauss-Kuzmin-Wirsing operator. Let us remind that if $f \in \mathsf{C}[0,1]$, it is given by $[\mathbf{W}f](x)=\sum_{n=1}^{\infty}\frac{1}{(x+n)^2}f\big(\frac{1}{x+n}\big)$. Dominant eigenvalue 1 correspond to an eigenfunction $\frac{1}{1+x}$. As it was proved by Kuzmin, provided that $f(x)$ has a continuous derivative, there exists $c>0$, such that

$$[\mathbf{W}^n f](x)=\frac{A}{1+x}+O(e^{-c\sqrt{n}}),\text{ as } n\to\infty;\quad A=\frac{1}{\log 2}\int\limits_0^1 f(x)\,\mathrm{d}x.$$

The proof can be found in [36]. Note that this was already conjectured by Gauss, but he did not give the proof nor for the main neither for the error term. For the most important case, when $f(x)=1$, Lévy established the error term of the form $O(C^n)$ for $C=0.7$. Finally, Wirsing [74] gave the exact result in terms of eigenfunctions of $\mathbf{W}$, establishing the error term of the form $c^n\Psi(x)+O(x(1-x)\mu^n)$, where $c=-0.303663...$ is the sub-dominant eigenvalue (Gauss-Kuzmin-Wirsing constant), $\Psi(x)$ is a corresponding eigenfunction, and $\mu<|c|$. Returning to our case, we have completely analogous situation: operator $\mathbf{W}$ is replaced by $\mathcal{S}$, and the measure $\mathrm{d}x$ is replaced by $\mathrm{d}F(x)$. The leading eigenvalue 1 corresponds to the constant function. However strange, Wirsing did not notice that eigenvalues of $\mathbf{W}$ are in fact eigenvalues of certain Hilbert-Schmidt operator. This was later clarified by Bobenko [7]. Recently, Gauss-Kuzmin-Lévy theorem was generalized by Manin and Marcolli in [48]. The paper is very rich in ideas and results; in particular, in sheds a new light on the theorem just mentioned.

Concerning spaces for which Theorem 4.3 holds, we can investigate the space $\mathrm{C}[0,1]$ as well. However, if $f\in\mathrm{C}[0,1]$ and $f\in\mathrm{Ker}(T)$, the significant difficulty arises in proving uniform convergence of the series $\sum_{n=0}^{\infty}\mathcal{S}^n f$. Moreover, operator $\mathcal{S}$, acting on the space $\mathrm{C}[0,1]$, has additional point spectra apart from λ. Indeed, let $P_n(y)=y^n+\sum\limits_{i=0}^{n-1}a_i y^i$ be a polynomial of degree n which satisfies yet another variation of three term functional equation

$$2P_n(1-2y)-P_n(1-y)=\frac{1}{\delta_n}P(y)$$

for certain δ_n. The comparison of leading terms shows that $\delta_n=\frac{(-1)^n}{2^{n+1}-1}$, and that indeed for this δ_n there exists a unique polynomial, since each coefficient a_j can be uniquely determined with the knowledge of coefficients a_i for $i>j$. Thus,

$$\begin{aligned}
&P_1(y)=y-\tfrac{1}{4}, && P_2(y)=y^2-\tfrac{3}{5}y+\tfrac{1}{15},\\
&P_3(y)=y^3-\tfrac{21}{22}y^2+\tfrac{3}{11}y-\tfrac{7}{352}, && P_4(y)=y^4-\tfrac{30}{23}y^3+\tfrac{14}{23}y^2-\tfrac{45}{391}y+\tfrac{37}{5865}.
\end{aligned}$$

The equation for P_n implies that

$$\delta_n P_n(y)=\sum_{n=1}^{\infty}\frac{1}{2^n}P_n\Big(\frac{1-y}{2^n}\Big). \tag{4.5}$$

Then we have

Proposition 4.6. *The function $P_n(F(x))$ is the eigenfunction of $\mathcal{S}$, acting on the space* $\mathsf{C}[0,1]$, *and eigenvalue $\frac{(-1)^n}{2^{n+1}-1}$ belongs to the point spectra of $\mathcal{S}$.*

Proof. Indeed,

$$[\mathcal{S}(P_n \circ F)](x) = \sum_{n=1}^{\infty} \frac{1}{2^n} P_n \circ F\Big(\frac{1}{x+n}\Big) \overset{(2.1)}{=} \frac{1}{2^n} P_n\Big(1 - F(x+n)\Big) \overset{(2.1)}{=}$$
$$\sum_{n=1}^{\infty} \frac{1}{2^n} P_n\Big(2^{-n} - 2^{-n}F(x)\Big) \overset{(4.5)}{=} \delta_n P_n(F(x)). \quad \blacksquare$$

Thus, operator $\mathcal{S}$ behaves differently in spaces $\mathsf{C}[0,1]$ and C^{ω}. We postpone the analysis of this operator in various spaces for the future.

4.5 Integrals which involve $F(x)$

In this section we calculate certain integrals. Only rarely it is possible to express an integral involving $F(x)$ in closed form. In fact, all results we possess come from the identity $M_1 = \frac{3}{2}$, and any iteration of identities similar to (4.3). The following theorem adds identities of quite a different sort.

Theorem 4.7. *Let $G_{\lambda}(z)$ be any function, which satisfies the hypotheses of Theorem 2.12. Then*

$$(i) \quad \frac{\lambda}{\lambda+1}\int_0^1 G_{\lambda}(-x)\,\mathrm{d}x = \int_0^1 G_{\lambda}(-x)F(x)\,\mathrm{d}x;$$

$$(ii) \quad -\int_0^1 \log x\,\mathrm{d}F(x) = 2\int_0^1 \log(1+x)\,\mathrm{d}F(x) = \int_0^1 G(-x)\,\mathrm{d}x;$$

$$(iii) \quad \int_0^1 G(-x)(1+x^2)\,\mathrm{d}F(x) = \frac{1}{4};$$

$$(iv) \quad \int_0^1 G_{\lambda}(-x)\Big(1 - \frac{x^2}{\lambda}\Big)\,\mathrm{d}F(x) = 0.$$

Proof. We first prove identity (i). By (2.20), for every integer $n \geq 1$, we have

$$2G_{\lambda}(-z-n+1) - G_{\lambda}(-z-n) = \frac{1}{\lambda(z+n)^2} G_{\lambda}\Big(-\frac{1}{z+n}\Big).$$

Divide this by 2^n and sum over $n \geq 1$. By Theorem 2.6, the sum on the left is absolutely convergent. Thus,

$$G_{\lambda}(-z) = \sum_{n=1}^{\infty} \frac{1}{\lambda 2^n (z+n)^2} G_{\lambda}\Big(-\frac{1}{z+n}\Big)$$

Let $G^{\star}_{\lambda}(x)=\int_0^x G_{\lambda}(z)\,\mathrm{d}z$. In terms of $G^{\star}_{\lambda}(x)$, the last identity reads as

$$-G^{\star}_{\lambda}(-x)=\sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^{\star}_{\lambda}\Big(-\frac{1}{x+n}\Big)-\sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^{\star}_{\lambda}\Big(-\frac{1}{n}\Big). \tag{4.6}$$

In particular, setting $x=1$, one obtains

$$\sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^{\star}_{\lambda}\Big(-\frac{1}{n}\Big)=(\frac{1}{\lambda}-1)G^{\star}_{\lambda}(-1). \tag{4.7}$$

Now we are able to calculate the following integral (we use integration by parts in Stieltjes integral twice).

$$\int_0^1 G_{\lambda}(-x)F(x)\,\mathrm{d}x=-\int_0^1\frac{\mathrm{d}}{\mathrm{d}x}G^{\star}_{\lambda}(-x)F(x)\,\mathrm{d}x=-\frac{1}{2}G^{\star}_{\lambda}(-1)+\int_0^1 G^{\star}_{\lambda}(-x)\,\mathrm{d}F(x)\overset{(4.6)}{=}$$

$$-\frac{1}{2}G^{\star}_{\lambda}(-1)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\lambda 2^n}G^{\star}_{\lambda}\Big(-\frac{1}{n}\Big)-\frac{1}{\lambda}\sum_{n=1}^{\infty}\int_0^1 G^{\star}_{\lambda}\Big(-\frac{1}{x+n}\Big)2^{-n}\,\mathrm{d}F(x)\overset{(4.3),(4.7)}{=}$$

$$-\frac{1}{2}G^{\star}(-1)+\frac{1}{2}(\frac{1}{\lambda}-1)G^{\star}_{\lambda}(-1)-\frac{1}{\lambda}\int_0^1 G^{\star}_{\lambda}(-x)\,\mathrm{d}F(x)=-G^{\star}(-1)-\frac{1}{\lambda}\int_0^1 G_{\lambda}(-x)F(x)\,\mathrm{d}x.$$

Thus, the same integral is on the both sides, and this gives

$$\int_0^1 G_{\lambda}(-x)F(x)\,\mathrm{d}x=-\frac{\lambda}{\lambda+1}G^{\star}_{\lambda}(-1).$$

This establishes the statement (i).

Now we proceed with second identity. Integral (2.10) and Fubini theorem imply

$$\int_0^1 G(-z)\,\mathrm{d}z=2\int_0^1\int_0^1\frac{x}{1+xz}\,\mathrm{d}z\,\mathrm{d}F(x)=2\int_0^1\log(1+x)\,\mathrm{d}F(x).$$

Lastly, we apply (4.3) twice to obtain the needed equality. Indeed,

$$I=\int_0^1\log(1+x)\,\mathrm{d}F(x)\overset{(4.3)}{=}\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1\log\Big(1+\frac{1}{x+n}\Big)\,\mathrm{d}F(x)=$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1\log(x+n)\,\mathrm{d}F(x)-I\overset{(4.3)}{=}-\int_0^1\log x\,\mathrm{d}F(x)-I.$$

This finishes the proof of (ii).

In proving (iii), we can be more concise, since the pattern of the proof goes along the same line. One has

$$G(-z)=-\sum_{n=1}^{\infty}\frac{1}{2^n(z+n)^2}G\Big(-\frac{1}{z+n}\Big)+\sum_{n=1}^{\infty}\frac{1}{2^n(z+n)}.$$

Thus,

$$\int_0^1 G(-x)\,\mathrm{d}F(x) = -\sum_{n=1}^{\infty}\int_0^1 \frac{1}{2^n(x+n)^2}G\Big(-\frac{1}{x+n}\Big)\,\mathrm{d}F(x) +$$
$$\sum_{n=1}^{\infty}\int_0^1 \frac{1}{2^n(x+n)}\,\mathrm{d}F(x) \stackrel{(4.3)}{=} -\int_0^1 x^2 G(-x)\,\mathrm{d}F(x) + \int_0^1 x\,\mathrm{d}F(x).$$

Since $\int_0^1 x\,\mathrm{d}F(x) = \frac{m_1}{2} = \frac{1}{4}$, this finishes the proof of (iii). Part (iv) is completely analogous. ■

Part (iii), unfortunately, gives no new information about the sequence m_L. Indeed, the identity can be rewritten as

$$\sum_{L=1}^{\infty} m_L(-1)^{L-1}(m_{L-1}+m_{L+1}) = \frac{1}{2},$$

which, after regrouping, turns into the identity $m_0 m_1 = \frac{1}{2}$.

Concerning part (iv), and taking into account Theorem 4.3, one could expect that in fact $\mathrm{Ker}(T)$ is equal to the closure of vector space spanned by functions $G_\lambda(-x)(1-\frac{x^2}{\lambda})$. If this is the case, then these functions, along with $G(z)(1+x^2)$, produce a Schauder basis for C^ω. Thus, if $x^L = \sum_\lambda a_L^{(\lambda)} G_\lambda(-x)(1-\frac{x^2}{\lambda})$, then $a_L^{(-1)} = 2m_L$. We hope to return to this point in the future.

Concerning (i), note that the values of both integrals depend on the normalization of G_λ, since it is an eigenfunction. Replacing $G_\lambda(z)$by $cG_\lambda(z)$ for some $c\in\mathbb{R}$, we deduce that the left integral is equal to 1 or 0. Then (i) states that $\int_0^1 F(x)G_\lambda(-x)\,\mathrm{d}x = \frac{\lambda}{\lambda+1}$ or 0 (apparently, it is never equal to 0). The presence of $\lambda+1$ in the denominator should come as no surprise, minding that λ is the eigenvalue of the Hilbert-Schimdt operator. The Fredholm alternative gives us a way of solving the integral equation in terms of eigenfunctions. Since $|\lambda| \leq \lambda_1 = 0.25553210... < 1$, the integral equation is *a posteriori* solvable, and $\lambda+1$ appears in the denominators. Curiously, it is possible to approach this identity numerically. One of the motivations is to check its validity, since the result heavily depends on the validity of almost all the preceding results in Chapter 2. The left integral causes no problems, since Taylor coefficients of $G_\lambda(z)$ can be obtained at high precision as an eigenvector of a finite matrix, which is the truncation of an infinite one. On the other hand, the right integral can be evaluated with less precision, since it involves $F(x)$, and thus requires more time and space consuming continued fractions algorithm. Nevertheless, the author of this thesis have checked it with completely satisfactory outcome, confirming the validity.

Just as interestingly, results (i) and (iv) can be though as a reflection of a "pair-correlation" between eigenvalues λ and eigenvalue -1 (see Section 4.2 for some remarks on this topic). Moreover, minding properties of distributions $F_{\mu}(x)$ (here μ simply means another eigenvalue), the following result can be obtained. Given the conditions enforced on F_{μ} by (4.1), identity (4.3) is replaced by (rigid for $f \in \mathrm{C}^{\omega}$)

$$\int\limits_{0}^{1} f(x)\,\mathrm{d}F_{\mu}(x) = -\frac{1}{\mu}\sum_{n=1}^{\infty}\int\limits_{0}^{1} f\Big(\frac{1}{x+n}\Big)2^{-n}\,\mathrm{d}F_{\mu}(x).$$

Then our trick works smoothly again, and this yields an identity

$$\int\limits_{0}^{1} G_{\lambda}(-x)(\lambda+\mu x^{2})\,\mathrm{d}F_{\mu}(x) = 0.$$

This fact, consequently, is an interesting example of "pair correlation" between eigenvalues of the Hilbert-Schmidt operator in Section 2.6. Using definition of distribution F_{μ}, the last identity is equivalent to

$$\sum_{L=1}^{\infty}(-1)^{L}\big(m_{L}^{(\mu)}m_{L+1}^{(\lambda)}\lambda - m_{L}^{(\lambda)}m_{L+1}^{(\mu)}\mu\big) = 0,$$

and thus is a property of "orthogonality" of $G_{\lambda}(z)$. This expression is symmetric regarding μ and λ. As could be expected, it is void in case $\mu = \lambda$. As a matter of fact, the proof of the above identity is fallacious, since the definition of distributions F_{λ} does not imply properties (4.1) (these simply have no meaning). Nevertheless, numerical calculations show that the last identity truly holds. We also hope to return to this topic in the future.

4.6 Fourier series

Minkowski question mark function $F(x)$, originally defined for $x \geq 0$ by (1.1), can be extended naturally to $\mathbb{R}$ simply by the functional equation $F(x+1) = \frac{1}{2} + \frac{1}{2}F(x)$. Such an extension is still given by the expression (1.1), with the difference that a_0 can be negative integer. Naturally, the second functional equation is not preserved for negative x. Thus, we have

$$2^{x+1}(F(x+1)-1) = 2^{x}(F(x)-1) \text{ for } x \in \mathbb{R}.$$

So, $2^{x}(F(x)-1)$ is a periodic function, which we will denote by $-\Psi(x)$. Figure 4.1 gives the graph of $\Psi(x)$ for $x \in [0,1]$. Thus, $F(x) = -2^{-x}\Psi(x) + 1$. Since $F(x)$ is singular, the same is true for $\Psi(x)$: it is differentiable almost everywhere, and for these regular points one has $\Psi'(x) = \log 2 \cdot \Psi(x)$. As a periodic function, it has an associated Fourier series expansion $\Psi(x) \sim \sum\limits_{n=-\infty}^{\infty} c_{n}e^{2\pi i n x}$. Since $F(x)$ is real function, this gives $c_{-n} = \overline{c_{n}}$, $n \in \mathbb{Z}$. Let for $n \geq 1$, $c_{n} = a_{n} + ib_{n}$, and $a_{0} = \frac{c_{0}}{2}$. Here we list initial numerical values for $c_{n}^{\star} = c_{n}(2\log 2 - 4\pi i n)$ (see the next Proposition for the

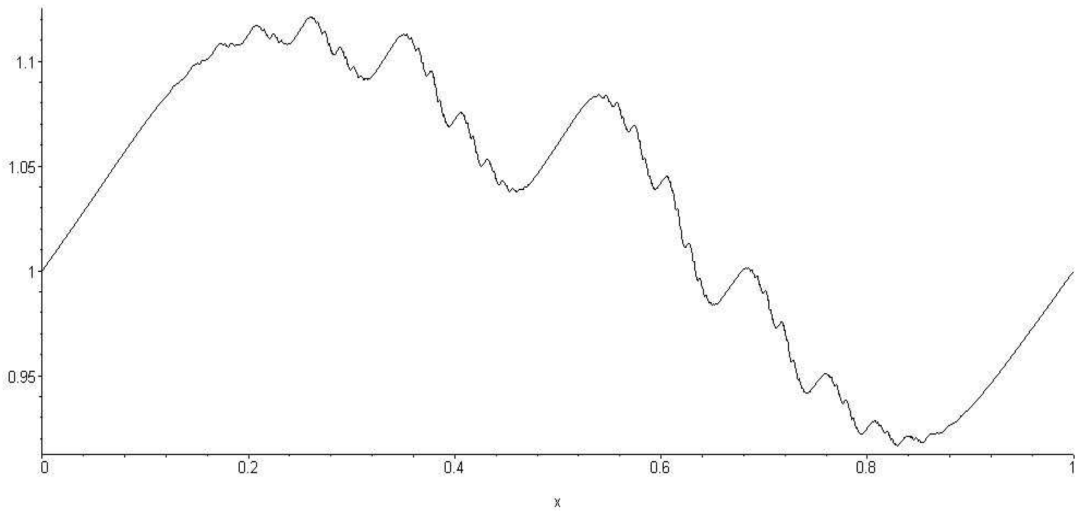


Figure 4.1: Periodic function $\Psi(x)$

reason of this normalization).

$$c_0^\star = 1.428159, \quad c_3^\star = +0.128533 - 0.026840i, \quad c_6^\star = -0.262601 + 0.004128i,$$
$$c_1^\star = -0.521907 + 0.148754i, \quad c_4^\star = -0.140524 - 0.021886i, \quad c_7^\star = +0.198742 - 0.013703i,$$
$$c_2^\star = -0.334910 - 0.017869i, \quad c_5^\star = +0.285790 + 0.003744i, \quad c_8^\star = -0.008479 + 0.024012i.$$

It is important to note that we do not pose the question about the convergence of this Fourier series. For instance, the authors of [62] and [58] give examples of singular monotone increasing functions $f(x)$, whose Fourier-Stieltjes coefficients $\int_0^1 e^{2\pi inx}\,\mathrm{d}f(x)$ do not vanish, as $n \to \infty$. In [62], the author even investigated $f(x) = ?(x)$. In our case, the convergence problem is far from clear. Nevertheless, in all cases we substitute $-2^{-x}\Psi(x)$ instead of $(F(x)-1)$ under an integral. Let, for example, $W(x)$ be a continuous function of at most polynomial growth, as $x \to \infty$, and let $\Psi_N(x) = \sum\limits_{n=-N}^{N} c_n e^{2\pi inx}$. Then

$$\Big|\int\limits_0^\infty W(x)\Big((F(x)-1) + 2^{-x}\Psi_N(x)\Big)\,\mathrm{d}x\Big| \ll \sum_{r=0}^{\infty} |W(r)|2^{-r} \cdot \int\limits_0^1 |2^x(F(x)-1) + \Psi_N(x)|\,\mathrm{d}x.$$

Since $2^x(F(x)-1) \in \mathcal{L}_2[0,1]$, the last integral tends to 0, as $N \to \infty$. As it was said, this makes the change of $(F(x)-1)$ into $-2^{-x}\Psi(x)$ under integral legitimate, and this also justifies term-by-term integration. Henceforth, we will omit a step of changing $\Psi(x)$ into $\Psi_N(x)$, and taking a limit $N \to \infty$.

A general formula for the Fourier coefficients is given by

Proposition 4.8. *Fourier coefficients c_n are related to special values of exponential* *generating function* $\mathfrak{m}(t)$ *through the equality*

$$c_n = \frac{\mathfrak{m}(\log 2 - 2\pi i n)}{2\log 2 - 4\pi i n}, \text{ and } c_n = O(n^{-1}).$$

Proof. We have (note that $F(1) = \frac{1}{2}$):

$$c_n = -\int_0^1 2^x (F(x) - 1) e^{-2\pi i n x}\,\mathrm{d}x = -\frac{1}{\log 2 - 2\pi i n}\int_0^1 (F(x) - 1)\,\mathrm{d}e^{x(\log 2 - 2\pi i n)} =$$

$$\frac{1}{\log 2 - 2\pi i n}\int_0^1 e^{x(\log 2 - 2\pi i n)}\,\mathrm{d}F(x) = \frac{\mathfrak{m}(\log 2 - 2\pi i n)}{2\log 2 - 4\pi i n}.$$

The last assertion of proposition is obvious. ■

This proposition is a good example of intrinsic relations among the three functions $F(x)$, $G(z)$ and $\mathfrak{m}(t)$. Indeed, the moments m_L of $F(x)$ give Taylor coefficients of $G(z)$, which are proportional (up to the factorial multiplier) to Taylor coefficients of $\mathfrak{m}(t)$. Finally, special values of $\mathfrak{m}(t)$ on a discrete set of vertical line produce "Fourier coefficients" of $F(x)$.

Next proposition describes explicit relations among Fourier coefficients and the moments. Additionally, in the course of the proof we obtain the expansion of $G(z)$ for negative real z in terms of incomplete gamma integrals.

Proposition 4.9. *For* $L \geq 1$*, one has*

$$M_L = L! \sum_{n\in\mathbb{Z}} \frac{c_n}{(\log 2 - 2\pi i n)^L}. \tag{4.8}$$

Proof. Let $z < 0$ be fixed negative real. Then integration by parts gives

$$G(z+1) = \int_0^\infty \frac{x}{1 - xz}\,\mathrm{d}(F(x) - 1) = \int_0^\infty \frac{1}{(1 - xz)^2} 2^{-x}\Psi(x)\,\mathrm{d}x =$$

$$\sum_{n=-\infty}^{\infty} c_n \int_0^\infty \frac{1}{(1 - xz)^2} 2^{-x} e^{2\pi i n x}\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} c_n V_n(z),$$

where

$$V_n(z) = \int_0^\infty \frac{1}{(1 - xz)^2} e^{-x(\log 2 - 2\pi i n)}\,\mathrm{d}x = \frac{1}{\log 2 - 2\pi i n}\int_0^{\infty(\log 2 - 2\pi i n)} \frac{1}{(1 - \frac{yz}{\log 2 - 2\pi i n})^2} e^{-y}\,\mathrm{d}y.$$

Since by our convention $z<0$, the function under integral does not have poles for $\Re y>0$, and Jordan's lemma gives

$$V_n(z)=\frac{1}{\log 2-2\pi in}\int\limits_0^\infty \frac{1}{(1-\frac{yz}{\log 2-2\pi in})}e^{-y}\,\mathrm{d}y=$$

$$\frac{1}{\log 2-2\pi in}\cdot V\Big(\frac{z}{\log 2-2\pi in}\Big),\text{ where } V(z)=\int\limits_0^\infty \frac{1}{(1-yz)^2}e^{-y}\,\mathrm{d}y.$$

The function $V(z)$ is defined for the same values of z as $G(z+1)$ and therefore is defined in the cut plane $\mathbb{C}\setminus(0,\infty)$. Consequently, this implies

$$G(z+1)=\sum_{n\in\mathbb{Z}}\frac{c_n}{\log 2-2\pi in}\cdot V\Big(\frac{z}{\log 2-2\pi in}\Big). \tag{4.9}$$

The formula is only valid for real $z<0$. The obtained series converges uniformly, since $|1-y\frac{z}{\log 2-2\pi in}|\geq 1$ for $n\in\mathbb{Z}$ and $z<0$. Since

$$V\Big(\frac{1}{z}\Big)=-ze^{-z}\int\limits_1^\infty \frac{1}{y^2}e^{yz}\,\mathrm{d}y,$$

this gives us the expansion of $G(z+1)$ on negative real line in terms of incomplete gamma integrals. As noted before, and this can be seen from (2.9), the function $G(z)$ has all left derivatives at $z=1$. Further, $(L-1)-$fold differentiation of $V(z)$ gives

$$V^{(L-1)}(z)=L!\int\limits_0^\infty \frac{y^{L-1}}{(1-yz)^{L+1}}e^{-y}\,\mathrm{d}y\Rightarrow V^{(L-1)}(0)=L!(L-1)!.$$

Comparing with (4.9) and (2.9), this gives the desired relation among moments M_L and Fourier coefficients, as stated in the Proposition. ■

It is important to compare this expression with the first equality of (2.6). Indeed, since $\mathfrak{m}(t)$ is entire, that equality via Cauchy residue formula implies the result obtained as Proposition 2.2, i.e.

$$M_L\sim\frac{\mathfrak{m}(\log 2)}{2\log 2}\Big(\frac{1}{\log 2}\Big)^L L! \tag{4.10}$$

It is exactly the leading term in (4.8), corresponding to $n=0$.

4.7 Associated zeta function

Recall that for complex c and s, c^s is multi-valued complex function, defined as $e^{s\log c}=e^{s(\log|c|+i\arg(c))}$. Henceforth, we fix the branch of the logarithm by requiring that the value of $\arg c$ for c in the right half plane $\Re c>0$ is in the range $(-\pi/2,\pi/2)$.

Thus, if $s=\sigma+it$, and if we denote $r_n=\log 2+2\pi in$, then $|r_n^{-s}|=|r_n|^{-\sigma}e^{t\arg r_n}\sim|r_n|^{-\sigma}e^{\pm\pi t/2}$ as $n\to\pm\infty$. Minding this convention and the identity (4.8), we introduce the zeta function, associated with Minkowski question mark function.

Definition 4.10. *The dyadic zeta function $\zeta_{\mathcal{M}}(s)$ is defined in the half plane $\Re s>0$ by the series*

$$\zeta_{\mathcal{M}}(s)=\sum_{n\in\mathbb{Z}}\frac{c_n}{(\log 2-2\pi in)^s}, \tag{4.11}$$

where c_n are Fourier coefficients of $\Psi(x)$, and for each n, $(\log 2-2\pi in)^s$ is understood in the meaning just described.

Then we have

Theorem 4.11. *$\zeta_{\mathcal{M}}(s)$ has an analytic continuation as an entire function to the whole plane $\mathbb{C}$, and satisfies the functional equation*

$$\zeta_{\mathcal{M}}(s)\Gamma(s)=-\,\zeta_{\mathcal{M}}(-s)\Gamma(-s). \tag{4.12}$$

Further, $\zeta_{\mathcal{M}}(L)=\frac{M_L}{L!}$ for $L\geq 1$. $\zeta_{\mathcal{M}}(s)$ has trivial zeros for negative integers: $\zeta_{\mathcal{M}}(-L)=0$ for $L\geq 1$, $\zeta_{\mathcal{M}}(0)=1$, and $\zeta_{\mathcal{M}}'(-L)=(L-1)!(-1)^L M_L$. Additionally, $\zeta_{\mathcal{M}}(s)$ is real on the real line, and thus $\zeta_{\mathcal{M}}(\overline{s})=\overline{\zeta_{\mathcal{M}}(s)}$. The behavior of $\zeta_{\mathcal{M}}(s)$ in vertical strips is given by estimate

$$|\,\zeta_{\mathcal{M}}(\sigma+it)|\ll t^{-\sigma-1/2}\cdot e^{\pi|t|/2}$$

uniformly for $a\leq\sigma\leq b$, $|t|\to\infty$.

As we will see, these properties are immediate (subject to certain regularity conditions) for any distribution $f(x)$ with a symmetry property $f(x)+f(1/x)=1$. Nevertheless, it is a unique characteristic of $F(x)$ that the corresponding zeta function can be given a Dirichlet series expansion, like (4.11). We do not give the proof of the converse result, since there is no motivation for this. But empirically, we see that this functional equation is equivalent exactly to the symmetry property. Additionally, the presence of a Dirichlet series expansion yields a functional equation of the kind $f(x+1)=\frac{1}{2}f(x)+\frac{1}{2}$. Generally speaking, these two together are unique for $F(x)$. Note also that the functional equation implies that $\zeta_{\mathcal{M}}(it)\Gamma(1+it)=\int\limits_0^\infty x^{it}\,\mathrm{d}F(x)$ is real for real t. Figures 4.2-4.4 shows its graph for $1.5\leq t\leq 270$. Further calculations support the claim that this function has infinitely many zeros on the critical line $\Re s=0$. On the other hand, numerical calculations of contour integrals reveal that there exist much more zeros apart from these.

We need one classical integral.

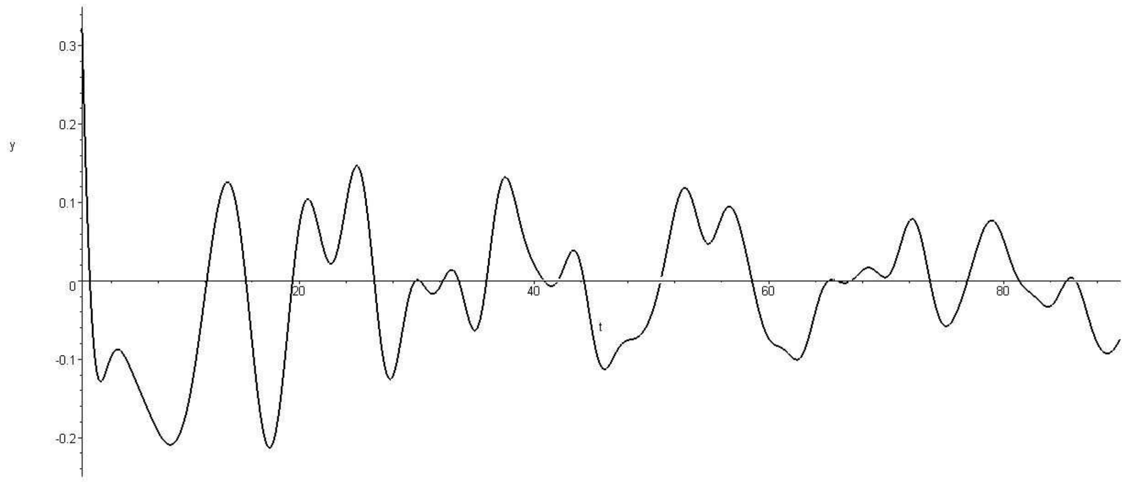


Figure 4.2: $\zeta_{\mathcal{M}}(it)\Gamma(1+it)$, $1.5 \leq t \leq 90$.

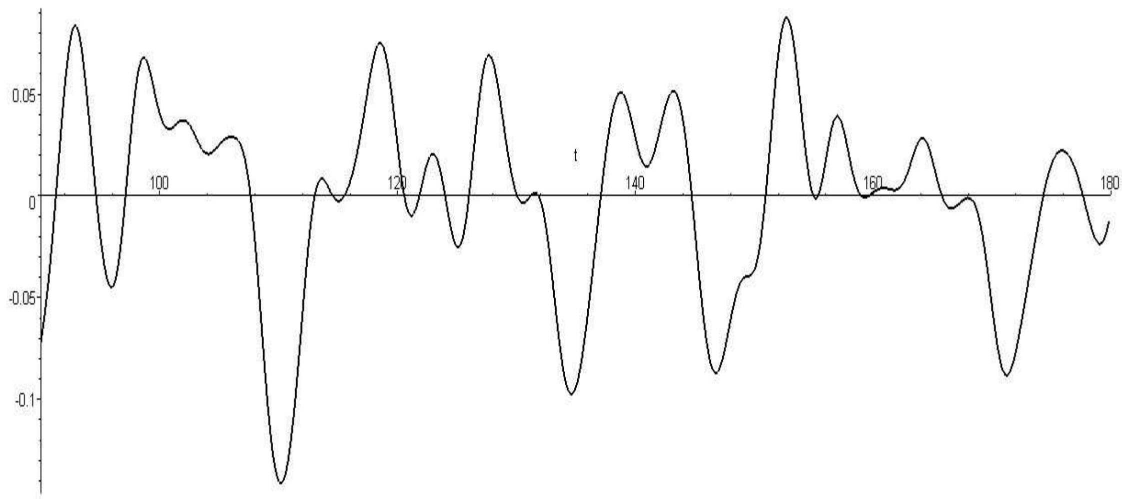


Figure 4.3: $\zeta_{\mathcal{M}}(it)\Gamma(1+it)$, $90 \leq t \leq 180$.

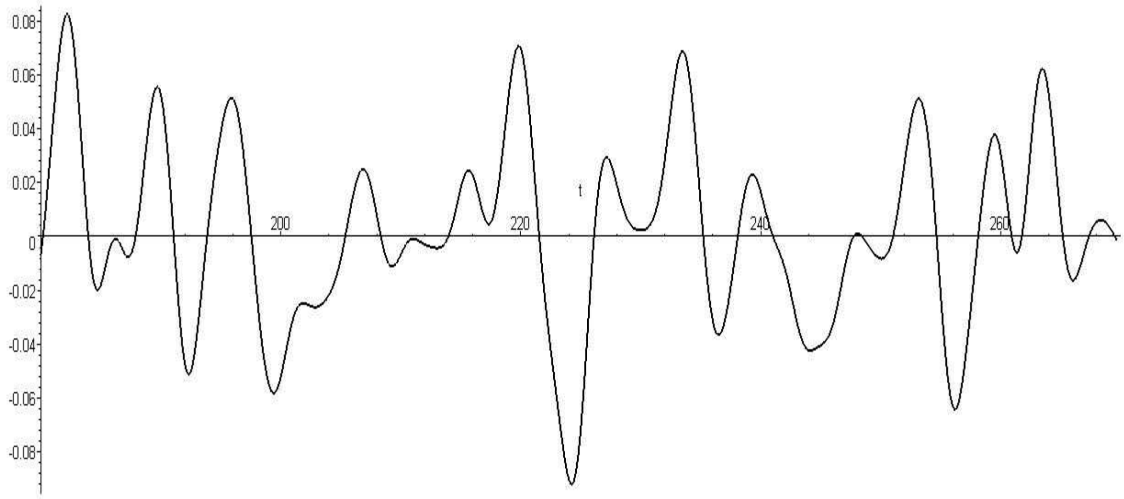


Figure 4.4: $\zeta_{\mathcal{M}}(it)\Gamma(1+it)$, $180 \leq t \leq 270$.

Lemma 4.12. *Let* A *be real number,* $\arctan(A) = \phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *and* $\Re s > 0$. *Then*

$$\int\limits_0^\infty x^{s-1} e^{-x} \cos(Ax)\, \mathrm{d}x = \frac{1}{(1+A^2)^{s/2}} \cos(\phi s)\Gamma(s).$$

The same is valid with $\cos$ *replaced by* $\sin$ *on both sides.*

This can be found in any extensive table of gamma integrals or tables of Mellin transforms.

Proof of Theorem 4.11. Let for $n \geq 0$, $\arctan \frac{2\pi n}{\log 2} = \phi_n$. We will calculate the following integral. Let $\Re s > 0$. Then integrating by parts and using Lemma 4.12, one obtains

$$\int\limits_0^\infty x^s\, \mathrm{d}(F(x)-1) = s\int\limits_0^\infty x^{s-1} 2^{-x} \Psi(x)\, \mathrm{d}x = s\sum_{n\in\mathbb{Z}} c_n \int\limits_0^\infty x^{s-1} 2^{-x} e^{2\pi i n x}\, \mathrm{d}x$$

$$= s\sum_{n=0}^{\infty} \int\limits_0^\infty x^{s-1}\Big(2a_n \cos(2\pi n x) - 2b_n \sin(2\pi n x)\Big) 2^{-x}\, \mathrm{d}x =$$

$$2s\Gamma(s)\sum_{n=0}^{\infty} |\log 2 + 2\pi n i|^{-s}\Big(a_n \cos(\phi_n s) - b_n \sin(\phi_n s)\Big) = s\Gamma(s)\sum_{n\in\mathbb{Z}} \frac{c_n}{(\log 2 - 2\pi i n)^s}.$$

Note that the function $\int_0^\infty x^s\, \mathrm{d}F(x)$ is clearly analytic and entire. Thus, $s\Gamma(s)\,\zeta_{\mathcal{M}}(s)$ is an entire function, and this proves the first statement of the theorem. Since $F(x) + F(1/x) = 1$, this gives $\int_0^\infty x^s\, \mathrm{d}F(x) = \int_0^\infty x^{-s}\, \mathrm{d}F(x)$, and this, in turn, implies the functional equation. All other statements follow easily from this, our previous results, and known properties of the $\Gamma-$function. In particular, if $s = \sigma + it$,

$$|\zeta_{\mathcal{M}}(s)\Gamma(s+1)| \leq \int_0^\infty |x^s| \,\mathrm{d}F(x) = \zeta_{\mathcal{M}}(\sigma)\Gamma(\sigma+1),$$

and the last statement of the theorem follows from the Stirling's formula for $\Gamma-$function: $|\Gamma(\sigma+it)| \sim \sqrt{2\pi} t^{\sigma-1/2} e^{-\pi|t|/2}$ uniformly for $a \leq \sigma \leq b$, as $|t| \to \infty$. ■

At this stage, we will make some remarks, concerning the analogy and differences with the classical results known for the Riemann zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$. Let $\theta(x)$ denote the usual theta function: $\theta(x) = \sum\limits_{n\in\mathbb{N}} e^{\pi i n^2 x}$, $\Im x > 0$. The following table summarizes all ingredients, which eventually produce the functional equation both for $\zeta(s)$ and $\zeta_{\mathcal{M}}(s)$.

Function	$\zeta(s)$	$\zeta_{\mathcal{M}}(s)$
Dirichlet series exp.	*Periodicity:* $\theta(x+2) = \theta$	*Periodicity:* $F'(x+1) = \frac{1}{2}F'(x)$
Functional equation	$\theta(ix) = \frac{1}{\sqrt{x}}\theta(\frac{i}{x})$	$F'(x) = -F'(\frac{1}{x})$

Since $F(x)$ is a singular function, its derivative should be considered as a distribution on the real line. For this purpose, it is sufficient to consider a distribution $U(x)$ as a derivative of a continuous function $V(x)$, for which the scalar product $\langle U, f\rangle$, defined for functions $f \in C^\infty(\mathbb{R})$ with compact support, equals to $-\langle V, f'\rangle = -\int\limits_{\mathbb{R}} f'(x)V(x)\,\mathrm{d}x$. Thus, both $\theta(x)$ and $2^x F'(x)$ are periodic distributions. This guarantees that the appropriate Mellin transform can be factored into the product of Dirichlet series and gamma factors. Finally, the functional equation for the distribution produces the functional equation for the Mellin transform. The difference arises from the fact that for $\theta(x)$ the functional equation is symmetry property on the imaginary line, whereas for $F'(x)$ we have the symmetry on the real line instead. This explains the unusual fact that in (4.11) we have the summation over the discrete set of the vertical line, instead of the summation over integers.

We will finish by proving another result, which links $\zeta_{\mathcal{M}}(s)$ to the Mellin transform of $G(-z+1)$. This can be done using expansion (4.9), but we rather chose a direct way. Let $\int\limits_0^\infty G(-z+1)z^{s-1}\,\mathrm{d}z = G^*(s)$. Symmetry property of Theorem 2.6 implies that $G(-z+1)$ has a simple zero, as $z \to \infty$ along the positive real line. Thus, basic properties of Mellin transform imply that $G^*(s)$ is defined for $0 < \Re s < 1$. For these values of s, we have the following classical integral:

$$\int_0^\infty \frac{z^{s-1}}{1+z}\,\mathrm{d}z \overset{\frac{z}{1+z}\to x}{=} \int_0^1 x^{s-1}(1-x)^{-s}\,\mathrm{d}x = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Thus, using (2.10), we get

$$G^*(s) = \int\limits_0^\infty \int\limits_0^\infty \frac{xz^{s-1}}{1+xz}\,\mathrm{d}F(x)\,\mathrm{d}z = \int\limits_0^\infty \int\limits_0^\infty \frac{z^{s-1}}{1+z} x^{1-s}\,\mathrm{d}z\,\mathrm{d}F(x) = \frac{\pi}{\sin \pi s} \int\limits_0^\infty x^{1-s}\,\mathrm{d}F(x).$$

This holds for $0 < \Re s < 1$. Due to the analytic continuation, this gives

Proposition 4.13. *For $s \in \mathbb{C} \setminus \mathbb{Z}$, we have an identity*

$$G^*(s) = \zeta_{\mathcal{M}}(s-1)\Gamma(s) \cdot \frac{\pi}{\sin \pi s}.$$

Therefore, $G^*(s)$ is a meromorphic function, $G^*(s+1) = -G^*(-s+1)$, and $\mathrm{res}_{s=L} G^*(s) = (-1)^L M_{L-1}$. This is, of course, the general property of the Mellin transform, since formally $G(z+1) = \sum_{L=0}^{\infty} M_L z^{L-1}$. Thus, $G(z+1) \sim \sum\limits_{L=0}^{M} M_L z^{L-1}$ in the left neighborhood of $z = 0$.

4.8 Concluding remarks

4.8.1 Dyadic period functions in $\mathbb{H}$

As noted in Chapter 2, one encounters the surprising fact that in the upper half plane $\mathbb{H}$, the equation (2.13) is also satisfied by $\frac{i}{2\pi} G_1(z)$, where $G_1(z)$ stands for the Eisenstein series (see page 18). Let $f_0(z) = G(z) - \frac{i}{2\pi} G_1(z)$, where $G(z)$ is the function in Theorem 2.6. Then for $z \in \mathbb{H}$, $f_0(z)$ satisfies the homogeneous form of the three term functional equation (2.13); moreover $f_0(z)$ is bounded, when $\Im z \to \infty$. Thus, if $f(z) = f_0(z)$,

$$-\frac{1}{(1-z)^2} f\Big(\frac{1}{1-z}\Big) + 2f(z+1) = f(z).$$

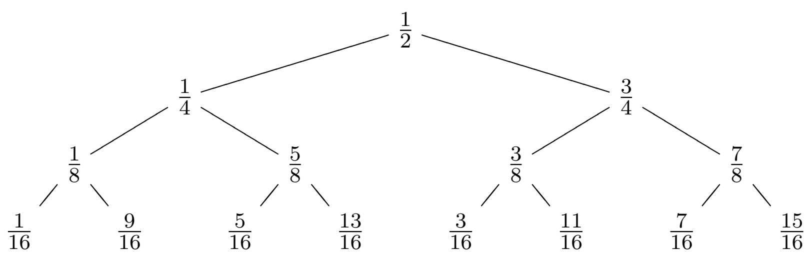
Therefore, denote by $\mathtt{DPF}^0$ the $\mathbb{C}-$linear vector space of solutions of this three term functional equation, which are holomorphic in $\mathbb{H}$ and are bounded at infinity, and call it *the space of dyadic period functions in the upper half-plane*. Consequently, this space is at least one-dimensional. If we abandon the growth condition, then the corresponding space $\mathtt{DPF}$ is infinite-dimensional. This is already true for periodic solutions. Indeed, if $f(z)$ is a periodic solution, then $f(z) = \frac{1}{z^2} f(-1/z)$. Let $P(z) \in \mathbb{C}[z]$, and suppose that $j(z)$ stands, as usually, for the $j-$invariant. Then any modular function of the form $j'(z)P(j(z))$ satisfies this equation. Additionally, there are non-periodic solutions, given by $f_0(z)P(j(z))$. Therefore, $G(z)$ surprisingly enters the profound domain of classical modular forms and functions for $\mathsf{PSL}_2(\mathbb{Z})$. Moreover, in the space $\mathtt{DPF}$, one establishes the relation between real quadratic irrationals (via $G(z)$, Minkowski question mark function $F(x)$ and continued fraction algorithm), and imaginary quadratic irrationals (via $j-$invariant and its special values). Hence, it is greatly desirable to give the full description and structure of spaces $\mathtt{DPF}^0$ and $\mathtt{DPF}$.

4.8.2 Where should the true arithmetic zeta function come from?

Here we present some remarks, concerning the zeta function $\zeta_{\mathcal{M}}(s)$. This object is natural for the question mark function - its Dirichlet coefficients are the Fourier coefficients of $F(x)$, and its special values at integers are proportional to the moments M_L. Moreover, its relation to $G(z)$, $\mathfrak{m}(t)$ and $F(x)$ is the same as the role of $L-$series of Maass wave forms against analogous objects [76]. Nevertheless, one expects richer arithmetic object associated with Calkin-Wilf tree, since the latter consists or rational numbers, and therefore can be canonically embedded into the group of ídeles $\mathbb{A}_{\mathbb{Q}}$. The $p-$adic distribution of rationals in the $n-$th generation of Calkin-Wilf tree was investigated in Section 2.6. Surprisingly, Eisenstein series $G_1(z)$ yet again manifest, as in case of $\mathbb{R}$ (see previous subsection). Nevertheless, there is no direct way of normalizing moments of the $n-$th generation in order for them to converge in the $p-$adic norm. There is an exception. As can easily be seen,

$$\sum_{a_0+a_1+\ldots+a_s=n}[a_0,a_1,..,a_s]=3\cdot 2^{n-2}-\frac{1}{2},$$

and thus we have a convergence only in the $2-$adic topology, namely to the value $-\frac{1}{2}$. The investigation of $p-$adic values of moments is relevant for the following reason. Let us apply $F(x)$ to each rational number in the Calkin-Wilf tree. What we obtain is the following:



Using (2.1), we deduce that this tree starts from the root $\frac{1}{2}$, and then inductively each rational r produces two offsprings: $\frac{r}{2}$ and $\frac{r}{2}+\frac{1}{2}$. One is therefore led to the following

Task. *Produce a natural algorithm, which takes into account $p-$adic and real properties of the above tree, and generates Riemann zeta function $\zeta(s)$.*

We emphasize that the choice of $\zeta(s)$ is not accidental. In fact, $\mathbb{R}-$distribution of the above tree is a uniform one with support $[0,1]$. Further, there is a natural algorithm to produce "characteristic function of ring of integers of $\mathbb{R}$" (that is, $e^{-\pi x^2}$)

from the uniform distribution via the central limit theorem through the expression

$$\int\limits_{\mathbb{R}} f(x)e^{-\pi x^2}\,\mathrm{d}x = \lim_{N\to\infty}\frac{1}{2^N}\int\limits_{-1}^{1}\mathrm{d}x_1 ... \int\limits_{-1}^{1}\mathrm{d}x_N f\Big(\frac{x_1+...+x_N}{\sqrt{\frac{2}{3}\pi N}}\Big).$$

(For clarity, here we take the uniform distribution in the interval $[-1,1]$). This formula and this explanation and treatment of $e^{-\pi x^2}$ as "characteristic function of the ring of integer of $\mathbb{R}$" is borrowed from [30], p. 7. Further, the operator which is invariant under uniform measure has the form $[\mathcal{U}f](x) = \frac{1}{2}f\big(\frac{x}{2}\big) + \frac{1}{2}f\big(\frac{x}{2}+\frac{1}{2}\big)$. Indeed, for every $f \in \mathsf{C}[0,1]$, one has $\int_0^1 [\mathcal{U}f](x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$. The spectral analysis of $\mathcal{U}$ shows that its eigenvalues are 2^{-n}, $n \geq 0$, with corresponding eigenfunctions being Bernoulli polynomials $B_n(x)$ [27]. These, as is well known from the time of Euler, are intricately related with $\zeta(s)$. Moreover, the partial moments of the above tree can be defined as $\sum_{i=1}^{2^N}\big(\frac{2i-1}{2^N}\big)^L$. These values are also expressed in term of Bernoulli polynomials. As we know, there are famous Kummer congruences among Bernoulli numbers, which later led to the introduction of the $p-$adic zeta function $\zeta_p(s)$. Thus, the real distribution of the above tree and its spectral decomposition is deeply related to the p-adic properties. This justifies the choice in the task of $\zeta(s)$.
Therefore, returning to Calkin-Wilf tree, one expects that moments can be $p-$adically interpolated, and some natural arithmetic zeta function can be introduced, as a "preimage" of $\zeta(s)$ under map F.

Chapter 5

Explicit series for the dyadic period function

5.1 Introduction and main results

We wish to emphasize that the main motivation for previous research was clarification of the nature and structure of the moments m_L. It was greatly desirable to give these constants (emerging as if from geometric chaos) some other expression than the one obtained directly from the Farey (or Calkin-Wilf) tree, which could reveal their structure to greater extent. This is accomplished in the current Chapter. Thus, the main result can be formulated as follows.

Theorem 5.1. *There exist canonical and explicit sequence of rational functions* $\mathbf{H}_n(z)$*, such that for* $\{\Re z \leq \frac{1}{2}\}\bigcup\{|z| \leq 1\}$*, one has an absolutely convergent series*

$$G(z) = \int\limits_0^\infty \frac{1}{x+1-z}\,\mathrm{d}F(x) = \sum_{n=0}^{\infty}(-1)^n \mathbf{H}_n(z), \quad \mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}},$$

where $\mathscr{B}_n(z)$ *is the polynomial with rational coefficients of degree* $n-1$*. For* $n \geq 1$ *it has the following reciprocity property:*

$$\mathscr{B}_n(z+1) = (-1)^n z^{n-1} \mathscr{B}_n\Big(\frac{1}{z}+1\Big), \quad \mathscr{B}_n(0) = 0.$$

The following table gives initial polynomials $\mathscr{B}_n(z)$.

n	$\mathscr{B}_n(z)$	n	$\mathscr{B}_n(z)$
0	-1	4	$-\frac{2}{27}z^3+\frac{53}{270}z^2-\frac{53}{270}z$
1	0	5	$\frac{4}{81}z^4-\frac{104}{675}z^3+\frac{112}{675}z^2-\frac{224}{2025}z$
2	$-\frac{1}{6}z$	6	$-\frac{8}{243}z^5+\frac{47029}{425250}z^4-\frac{1384}{14175}z^3-\frac{787}{30375}z^2+\frac{787}{60750}z$
3	$\frac{1}{9}z^2-\frac{2}{9}z$	7	$\frac{16}{729}z^6-\frac{1628392}{22325625}z^5+\frac{272869}{22325625}z^4+\frac{5392444}{22325625}z^3-\frac{238901}{637875}z^2+\frac{477802}{3189375}z$

Example. In fact, apparently the true region of convergence of the series in question is the half plane $\Re z \leq 1$. Take, for example, $z_0 = \frac{2}{3} + 4i$. Then by (2.13) and symmetry property one has

$$G(z_0) = \frac{1}{2}G(z_0 - 1) - \frac{1}{2(z_0-1)^2}G\Big(\frac{1}{z_0-1}\Big) - \frac{1}{2(z_0-1)} =$$
$$-\frac{1}{2(z_0-2)^2}G\Big(\frac{z_0-1}{z_0-2}\Big) - \frac{1}{2(z_0-1)^2}G\Big(\frac{1}{z_0-1}\Big) - \frac{1}{2(z_0-2)} - \frac{1}{2(z_0-1)}.$$

Both arguments under G on the right belong to the unit circle, and thus we can use Taylor series for $G(z)$. Using numerical values of m_L, obtained via the method described on page 18, we obtain: $G(z_0) = 0.078083_+ {+}0.205424_+ i$, with all digits exact. On the other hand, the series in Theorem 5.1 for $n = 60$ gives

$$\sum_{n=0}^{60}(-1)^n\mathbf{H}_n(z_0) = 0.078090_+ + 0.205427_+ i.$$

Finally, based on the last integral in (2.10), we can calculate $G(z)$ as a Stieltjes integral. If we divide the unit interval into $N = 3560$ equal subintervals, and use Riemann-Stieltjes sum, we get an approximate value $G(z_0) \approx 0.078082_+ + 0.205424i$. All evaluations match very well.

With a slight abuse of notation, we will henceforth write $f^{(L-1)}(z_0)$ instead of $\frac{\partial^{L-1}}{\partial z^{L-1}}f(z)\big|_{z=z_0}$.

Corollary 5.2. *The moments m_L can be expressed in the closed form by the convergent series of rational numbers:*

$$\begin{aligned} m_L &= \lim_{n\to\infty} 2^{2-n} \sum_{a_1+a_2+\ldots+a_s=n} [0, a_1, a_2, \ldots, a_s]^L = \\ &= \frac{1}{(L-1)!}\sum_{n=0}^{\infty}(-1)^n\mathbf{H}_n^{(L-1)}(0). \end{aligned}$$

The speed of convergence is given by the following estimate: $\Big|\mathbf{H}_n^{(L-1)}(0)\Big| \ll \frac{1}{n^M}$, for every $M \in \mathbb{N}$. The implied constant depends only on L and M.

Thus, $m_2 = \sum_{n=0}^{\infty}(-1)^n \mathbf{H}'_n(0) = 0.290926476_+$. Regarding the speed, numerical calculations show that in fact the convergence is geometric. For example, Theorem 5.1 in case $z = 1$ gives

$$M_1 = G(1) = 1 + 0 + \sum_{n=0}^{\infty} \frac{1}{6}\Big(\frac{2}{3}\Big)^n = \frac{3}{2},$$

which we already know. Geometric convergence would be the consequence of the fact that analytic functions $m_L(\mathsf{p})$ extend beyond $\mathsf{p} = 1$ (see below). This is supported by the phenomena represented in Theorem 5.4. Meanwhile, we are able to prove only the given rate. Theorem 5.1 gives a convergent series for the moments M_L as well. This is exactly the same as the series in the Corollary 5.2, only one needs to use a point $z = 1$ instead of $z = 0$. To this account, Proposition 5.14 suggests the following prediction, which is highly supported by numerical calculations, and which holds for $L = 1$.

Prediction 5.3. *For $L \geq 1$, the series*

$$M_L(\mathsf{p}) = \frac{1}{(L-1)!} \cdot \sum_{n=0}^{\infty} (\mathsf{p} - 2)^n \mathbf{H}_n^{(L-1)}(1)$$

has exactly $2 - \frac{1}{\sqrt[L]{2}}$ as radius of convergence.

The following two tables give starting values for the sequence $\mathbf{H}'_n(0)$.

n	$\mathbf{H}'_n(0)$	n	$\mathbf{H}'_n(0)$	n	$\mathbf{H}'_n(0)$
0	$\frac{1}{4}$	5	$-\frac{7}{2 \cdot 3^4 \cdot 5^2}$	10	$-\frac{8026531718888633}{2^{12} \cdot 3^9 \cdot 5^7 \cdot 7^4 \cdot 11 \cdot 17^2}$
1	0	6	$-\frac{787}{2^8 \cdot 3^5 \cdot 5^3}$	11	$\frac{7972095369765557079423}{2^{11} \cdot 3^{10} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 17^3 \cdot 31}$
2	$\frac{1}{48}$	7	$\frac{238901}{2^7 \cdot 3^6 \cdot 5^4 \cdot 7}$	12	$\frac{4198988799919158293319845971}{2^{14} \cdot 3^{11} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13 \cdot 17^4 \cdot 31^2}$
3	$-\frac{1}{72}$	8	$-\frac{181993843}{2^{10} \cdot 3^7 \cdot 5^5 \cdot 7^2}$	13	$-\frac{12702956822417247965298252330349561}{2^{10} \cdot 3^{12} \cdot 5^{10} \cdot 7^7 \cdot 11^4 \cdot 13^2 \cdot 17^5 \cdot 31^3}$
4	$\frac{53}{8640}$	9	$\frac{12965510861}{2^6 \cdot 3^8 \cdot 5^6 \cdot 7^3 \cdot 17}$	14	$\frac{72261916360136752928335145486035163954998 99}{2^{16} \cdot 3^{13} \cdot 5^{11} \cdot 7^8 \cdot 11^5 \cdot 13^3 \cdot 17^6 \cdot 31^4}$

n	$\mathbf{H}'_n(0)$
15	$-\frac{1293371830090421418537484507305813697332268574439156 17}{2^{15}\cdot 3^{14}\cdot 5^{12}\cdot 7^{9}\cdot 11^{6}\cdot 13^{4}\cdot 17^{7}\cdot 31^{5}\cdot 43\cdot 127}$
16	$\frac{31258186275777197041073243752715109842753785598306812028984213251}{2^{18}\cdot 3^{15}\cdot 5^{13}\cdot 7^{10}\cdot 11^{7}\cdot 13^{5}\cdot 17^{8}\cdot 31^{6}\cdot 43^{2}\cdot 127^{2}}$
17	$-\frac{328252050122963975599776202270732170439777688894846986095983045977441444448 3}{2^{12}\cdot 3^{16}\cdot 5^{14}\cdot 7^{11}\cdot 11^{8}\cdot 13^{6}\cdot 17^{9}\cdot 31^{7}\cdot 43^{3}\cdot 127^{3}\cdot 257}$

The float values of the last three rational numbers are

$$-0.000025804822076, \quad 0.000018040274062, \text{ and } -0.000010917558446$$

respectively. The alternating sum of the elements in the table is $\sum_{n=0}^{N}(-1)^n\mathbf{H}'_n(0) = 0.2909255862_+$ (where $N = 17$), whereas $N = 40$ gives 0.2909264880_+, and $N = 50$ gives 0.2909264784_+. Note that the manifestation of Fermat and Mersenne primes in the denominators at an early stage is not accidental, minding the exact value of the determinant in Lemma 5.21, Section 5.7 (see below).

As will be apparent later, the result in Theorem 5.1 is derived from the knowledge of $\mathsf{p}-$derivatives of $G(\mathsf{p}, z)$ at $\mathsf{p} = 2$ (see below). On the other hand, since there are two points ($\mathsf{p} = 2$ and $\mathsf{p} = 0$) such that all higher $\mathsf{p}-$derivatives of $G(\mathsf{p}, z)$ are rational functions in z, it is not completely surprising that the approach through $\mathsf{p} = 0$ also gives convergent series for the moments, though in this case we are forced to use Borel summation. At this point, the author does not have a strict mathematical proof of this result (since the function $G(\mathsf{p}, z)$ is meanwhile defined only for $\Re\,\mathsf{p} \geq 1$), though numerical calculations provide overwhelming evidence for its validity.

Theorem 5.4. *(Heuristic result). Define the rational functions (with rational coefficients)* $\mathbf{Q}_n(z)$, $n \geq 0$, *by*

$$\mathbf{Q}_0(z) = -\frac{1}{2z}, \text{ and recurrently by } \mathbf{Q}_n(z) = \frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{j!}\cdot\frac{\partial^j}{\partial z^j}\mathbf{Q}_{n-j-1}(-1)\cdot\Big(z^j - \frac{1}{z^{j+2}}\Big).$$

Then

$$\begin{aligned} m_L &= \lim_{n\to\infty} 2^{2-n}\sum_{a_1+a_2+\ldots+a_s=n}[0, a_1, a_2, \ldots, a_s]^L = \\ &= \frac{1}{(L-1)!}\sum_{r=0}^{\infty}\Big(\sum_{n=0}^{\infty}\frac{\mathbf{Q}_n^{(L-1)}(-1)}{n!}\cdot\int_r^{r+1} t^n e^{-t}\,\mathrm{d}t\Big). \end{aligned}$$

Moreover,

$$\mathbf{Q}_n(z) = \frac{(z+1)(z-1)\mathscr{D}_n(z)}{z^{n+1}}, \quad n \geq 1,$$

where $\mathscr{D}_n(z)$ are polynomials with rational coefficients ($\mathbb{Q}_p$ integers for $p \neq 2$) of degree $2n-2$ with the reciprocity property

$$\mathscr{D}_n(z) = z^{2n-2}\mathscr{D}_n\Big(\frac{1}{z}\Big).$$

Note the order of summation in the series for m_L, since the reason for introducing exponential function is because we use Borel summation. For example,

$$\text{“}1-2+4-8+16-32+\ldots\text{”} \overset{\texttt{Borel}}{=} \sum_{r=0}^{\infty}\Big(\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}\cdot\int_r^{r+1} t^n e^{-t}\,\mathrm{d}t\Big) = \frac{1}{3}.$$

The following table gives initial results.

n	$\mathscr{D}_n(z)$	n	$\mathscr{D}_n(z)$
1	$\frac{1}{4}$	4	$\frac{1}{8}(2z^6-3z^5+6z^4-3z^3+6z^2-3z+2)$
2	$\frac{1}{4}(z^2+1)$	5	$\frac{1}{4}(z^8-2z^7+4z^6-7z^5+4z^4-7z^3+4z^2-2z+1)$
3	$\frac{1}{4}(z^4-z^3+z^2-z+1)$	6	$\frac{1}{8}(2z^{10}-5z^9+12z^8-20z^7+37z^6-$
			$-20z^5+37z^4-20z^3+12z^2-5z+2)$

The next table gives $\mathbf{Q}'_n(-1) = 2(-1)^n\mathscr{D}_n(-1)$ explicitly, which appear in the series defining the first non-trivial moment m_2. Also, since these numbers are p–adic integers for $p \neq 2$, there is a hope for the successful implementation of the idea from the Section 4.8; that is, possibly one can define moments m_L as p–adic integers as well.

n	$\mathbf{Q}'_n(-1)$	n	$\mathbf{Q}'_n(-1)$	n	$\mathbf{Q}'_n(-1)$	n	$\mathbf{Q}'_n(-1)$
0	$\frac{1}{2}$	8	$\frac{1417}{4}$	16	$\frac{206836175}{64}$	24	$\frac{1685121707817}{32}$
1	$-\frac{1}{2}$	9	$-\frac{8431}{8}$	17	$-\frac{339942899}{32}$	25	$-\frac{92779913448103}{512}$
2	1	10	$\frac{50899}{16}$	18	$\frac{1125752909}{32}$	26	$\frac{80142274019997}{128}$
3	$-\frac{5}{2}$	11	-9751	19	$-\frac{15014220659}{128}$	27	$-\frac{1111839248032133}{512}$
4	$\frac{25}{4}$	12	30365	20	$\frac{25188552721}{64}$	28	$\frac{7740056893342455}{1024}$
5	-16	13	$-\frac{3069719}{32}$	21	$-\frac{170016460947}{128}$	29	$-\frac{13515970598654393}{512}$
6	43	14	$\frac{1227099}{4}$	22	$\frac{1153784184807}{256}$	30	$\frac{47354245650630005}{512}$
7	$-\frac{971}{8}$	15	$-\frac{31719165}{32}$	23	$-\frac{983668214037}{64}$	31	$-\frac{665632101181145115}{2048}$

The final table in this section lists float values of the constants

$$\vartheta_r = \sum_{n=0}^{\infty}\frac{\mathbf{Q}'_n(-1)}{n!}\cdot\int_r^{r+1} t^n e^{-t}\,\mathrm{d}t, \quad r \in \mathbb{N}_0, \quad \sum_{r=0}^{\infty}\vartheta_r = m_2,$$

appearing in Borel summation.

r	ϑ_r	r	ϑ_r
0	0.2327797875	6	0.0004701146
1	0.0471561089	7	0.0004980015
2	0.0085133626	8	0.0004005270
3	0.0005892453	9	0.0002722002
4	−0.0001872357	10	0.0001607897
5	0.0002058729	11	0.0000812407

Thus, $\sum_{r=0}^{11} \vartheta_r = 0.2909400155_{+} = m_2 + 0.000013539_{+}$.

5.2 p−question mark functions and p−continued fractions

In this section we introduce a family of natural generalizations of the Minkowski question mark function $F(x)$. Let $1 \leq \mathsf{p} < 2$. Consider the following binary tree, which we denote by $\mathcal{Q}_{\mathsf{p}}$. We start from the root $x = 1$. Further, each element ("root") x of this tree generates two "offsprings" by the following rule:

$$x \mapsto \frac{\mathsf{p}x}{x+1}, \quad \frac{x+1}{\mathsf{p}}.$$

We will use the notation $\mathcal{T}_{\mathsf{p}}(x) = \frac{x+1}{\mathsf{p}}$, $\mathcal{U}_{\mathsf{p}}(x) = \frac{\mathsf{p}x}{x+1}$. When p is fixed, we will sometimes discard the subscript. Thus, the first four generations look like

$$\begin{array}{cccccccc}
 & & & \frac{1}{1} & & & & \\
 & \frac{\mathsf{p}}{2} & & & & \frac{2}{\mathsf{p}} & & \\
\frac{\mathsf{p}^2}{\mathsf{p}+2} & & \frac{\mathsf{p}+2}{2\,\mathsf{p}} & & \frac{2\,\mathsf{p}}{\mathsf{p}+2} & & \frac{\mathsf{p}+2}{\mathsf{p}^2} & \\
\frac{\mathsf{p}^3}{\mathsf{p}^2+\mathsf{p}+2} & \frac{\mathsf{p}^2+\mathsf{p}+2}{\mathsf{p}^2+2\,\mathsf{p}} & \frac{\mathsf{p}^2+2\,\mathsf{p}}{3\,\mathsf{p}+2} & \frac{3\,\mathsf{p}+2}{2\,\mathsf{p}^2} & \frac{2\,\mathsf{p}^2}{3\,\mathsf{p}+2} & \frac{3\,\mathsf{p}+2}{\mathsf{p}^2+2\,\mathsf{p}} & \frac{\mathsf{p}^2+2\,\mathsf{p}}{\mathsf{p}^2+\mathsf{p}+2} & \frac{\mathsf{p}^2+\mathsf{p}+2}{\mathsf{p}^3}
\end{array} \tag{5.1}$$

We refer the reader to the paper [23], where authors consider a rather similar construction, though having a different purpose in mind (see also [9]). Denote by $T_n(\mathsf{p})$ the sequence of polynomials, appearing as numerators of fractions of this tree. Thus, $T_1(\mathsf{p}) = 1$, $T_2(\mathsf{p}) = \mathsf{p}$, $T_3(\mathsf{p}) = 2$. Directly from the definition of this tree we inherit that

$$\begin{aligned}
T_{2n}(\mathsf{p}) &= \mathsf{p}T_n(\mathsf{p}) \text{ for } n \geq 1, \\
T_{2n-1}(\mathsf{p}) &= T_{n-1}(\mathsf{p}) + \mathsf{p}^{-\epsilon}T_n(\mathsf{p}) \text{ for } n \geq 2,
\end{aligned}$$

where $\epsilon = \epsilon(n) = 1$ if $n = 2^k$ and $\epsilon = 0$ otherwise. Thus, the definition of these polynomials is almost the same as it appeared in [38] (these polynomials were named

Stern polynomials by the authors), with the distinction that in [38] everywhere one has $\epsilon = 0$. Naturally, this difference produces different sequence of polynomials.

There are 2^{n-1} positive real numbers in each generation of the tree $\mathcal{Q}_{\mathsf{p}}$, say $a_k^{(n)}$, $1 \leq k \leq 2^{n-1}$. Moreover, they are all contained in the interval $[\mathsf{p}-1, \frac{1}{\mathsf{p}-1}]$. Indeed, this holds for the initial root $x = 1$, and

$$\mathsf{p}-1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow \mathsf{p}-1 \leq \frac{\mathsf{p}x}{x+1} \leq 1,$$
$$\mathsf{p}-1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow 1 \leq \frac{x+1}{\mathsf{p}} \leq \frac{1}{\mathsf{p}-1}.$$

This also shows that the left offspring is contained in the interval $[\mathsf{p}-1, 1]$, while the right one - in the interval $[1, \frac{1}{\mathsf{p}-1}]$. The real numbers appearing in this tree have intrinsic relation with p−*continuous fractions algorithm*. The definition of the latter is as follows. Let $x \in (\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$. Consider the following procedure:

$$R_{\mathsf{p}}(x) = \begin{cases} \mathcal{T}^{-1}(x) = \mathsf{p}x - 1, & \text{if} \quad 1 \leq x < \frac{1}{\mathsf{p}-1}, \\ \mathcal{I}(x) = \frac{1}{x}, & \text{if} \quad \mathsf{p}-1 < x < 1, \\ \text{STOP}, & \text{if} \quad x = \mathsf{p}-1. \end{cases}$$

Then each such x can be uniquely represented as p−continuous fraction

$$x = [a_0, a_1, a_2, a_3, ...]_{\mathsf{p}},$$

where $a_i \in \mathbb{N}$ for $i \geq 1$, and $a_0 \in \mathbb{N} \cup \{0\}$. This notation means that in the course of iterations $R_{\mathsf{p}}^{\infty}(x)$ we apply $\mathcal{T}^{-1}(x)$ exactly a_0 times, then once $\mathcal{I}$, then we apply $\mathcal{T}^{-1}$ exactly a_1 times, then $\mathcal{I}$, and so on. The procedure terminates exactly for those $x \in (\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$, which are the members of the tree $\mathcal{Q}_{\mathsf{p}}$ (" p-rationals"). Also, direct inspection shows that if procedure does terminate, the last entry $a_s \geq 2$. Thus, we have the same ambiguity for the last entry exactly as is the case with ordinary continued fractions. At this point it is straightforward to show that the nth generation of $\mathcal{Q}_{\mathsf{p}}$ consists of $x = [a_0, a_1, ..., a_s]_{\mathsf{p}}$ such that $\sum_{j=0}^{s} a_j = n$, exactly as in the case $\mathsf{p} = 1$ and tree (1.2).

Now, consider a function $\mathfrak{X}_{\mathsf{p}}(x)$ with the following property: $\mathfrak{X}_{\mathsf{p}}(x) = \overline{x}$, where x is a rational number in the Calkin-Wilf tree (1.2), and $\overline{x}$ is a corresponding number in the tree (5.1). In other words, $\mathfrak{X}_{\mathsf{p}}(x)$ is simply a bijection between these two trees. First, if $x < y$, then $\overline{x} < \overline{y}$. Also, all positive rationals appear in the tree (1.2) and they are everywhere dense in $\mathbb{R}_+$. Moreover, $\mathcal{T}$ and $\mathcal{U}$ both preserve order, and $[\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$ is a disjoint union of $\mathcal{T}[\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$ and $\mathcal{U}[\mathsf{p}-1, \frac{1}{\mathsf{p}-1})$. Now it is obvious that the function $\mathfrak{X}_{\mathsf{p}}(x)$ can be extended to a continuous monotone increasing function

$$\mathfrak{X}_{\mathsf{p}}(\star) : \mathbb{R}_+ \to \Big[\mathsf{p}-1, \frac{1}{\mathsf{p}-1}\Big), \quad \mathfrak{X}_{\mathsf{p}}(\infty) = \frac{1}{\mathsf{p}-1}.$$

Thus,

$$\mathfrak{X}_{\mathsf{p}}\big([a_0, a_1, a_2, a_3...]\big) = [a_0, a_1, a_2, a_3...]_{\mathsf{p}}.$$

As can be seen from the definitions of both trees (1.2) and (5.1), this function satisfies functional equations

$$\begin{aligned} \mathfrak{X}_{\mathsf{p}}(x+1) &= \frac{\mathfrak{X}_{\mathsf{p}}(x)+1}{\mathsf{p}}, \\ \mathfrak{X}_{\mathsf{p}}\Big(\frac{x}{x+1}\Big) &= \frac{\mathsf{p}\mathfrak{X}_{\mathsf{p}}(x)}{\mathfrak{X}_{\mathsf{p}}(x)+1}, \\ \mathfrak{X}_{\mathsf{p}}\Big(\frac{1}{x}\Big) &= \frac{1}{\mathfrak{X}_{\mathsf{p}}(x)}. \end{aligned} \tag{5.2}$$

The last one (symmetry property) is a consequence of the first two. We are not aware whether this notion of $\mathsf{p}-$continuous fractions is new or not. For example,

$$\begin{aligned} \frac{1+\sqrt{1+4\,\mathsf{p}}}{2\,\mathsf{p}} &= [1,1,1,1,1,1,...]_{\mathsf{p}} = \mathfrak{X}_{\mathsf{p}}\Big(\frac{1+\sqrt{5}}{2}\Big), \\ \sqrt{3} &= [4,2,1,10,1,1,2,1,5,1,1,2,1,2,1,1,2,1,3,7,4,...]_{\frac{3}{2}}, \\ 2 &= [4,1,1,\overline{2,1,1}]_{\sqrt{2}}. \end{aligned}$$

Now fix p, $1 \leq \mathsf{p} < 2$. The following proposition follows immediately from the properties of $F(x)$.

Proposition 5.5. *There exists a limit distribution of the nth generation of the tree $\mathcal{Q}_{\mathsf{p}}$ as $n \to \infty$, defined as*

$$F_{\mathsf{p}}(x) = \lim_{n\to\infty} 2^{-n+1}\#\{k : a_k^{(n)} < x\}.$$

This function is continuous, $F_{\mathsf{p}}(x) = 0$ for $x \leq \mathsf{p}-1$, $F_{\mathsf{p}}(x) = 1$ for $x \geq \frac{1}{\mathsf{p}-1}$, and it satisfies two functional equations:

$$2\,F_{\mathsf{p}}(x) = \begin{cases} F_{\mathsf{p}}(\,\mathsf{p}x-1)+1, & \textit{if} \quad 1 \leq x \leq \frac{1}{\mathsf{p}-1}, \\ F_{\mathsf{p}}(\frac{x}{\mathsf{p}-x}), & \textit{if} \quad \mathsf{p}-1 \leq x \leq 1. \end{cases} \tag{5.3}$$

Additionally,

$$F_{\mathsf{p}}(x) + F_{\mathsf{p}}\Big(\frac{1}{x}\Big) = 1 \text{ for } x > 0.$$

The explicit expression for $F_{\mathsf{p}}(x)$ is given by

$$F_{\mathsf{p}}([a_0,a_1,a_2,a_3,...]_{\mathsf{p}}) = 1 - 2^{-a_0} + 2^{-(a_0+a_1)} - 2^{-(a_0+a_1+a_2)} + ...$$

We will refer to the last functional equation as *the symmetry property*. As was said, it is a consequence of the other two, though it is convenient to separate it.

Proof. Indeed, as it is obvious from the observations above, we simply have

$$F_{\mathsf{p}}\big(\mathfrak{X}_{\mathsf{p}}(x)\big) = F(x), \quad x \in [0,\infty).$$

Therefore, two functional equations follow from (5.2) and (2.1). All the other statements are immediate and follow from the properties of $F(x)$. ∎

Equally important, consider the binary tree (5.1) for $\mathsf{p} > 2$. In this case analogous proposition holds.

Proposition 5.6. *Let $\mathsf{p} > 2$. Then there exists a limit distribution of the nth generation as $n \to \infty$. Denote it by $f_{\mathsf{p}}(x)$ This function is continuous, $f_{\mathsf{p}}(x) = 0$ for $x \leq \frac{1}{\mathsf{p}-1}$, $f_{\mathsf{p}}(x) = 1$ for $x \geq \mathsf{p} - 1$, and it satisfies two functional equations:*

$$2f_{\mathsf{p}}(x) = \begin{cases} f_{\mathsf{p}}(\mathsf{p}x - 1) & \text{if } \ 1 \leq x \leq \mathsf{p} - 1, \\ f_{\mathsf{p}}(\frac{x}{\mathsf{p}-x}) + 1 & \text{if } \ \frac{1}{\mathsf{p}-1} \leq x \leq 1, \end{cases}$$

and

$$f_{\mathsf{p}}(x) + f_{\mathsf{p}}\Big(\frac{1}{x}\Big) = 1 \text{ for } x > 0.$$

Proof. The proof is analogous to the one of Proposition 5.5, only this time we use equivalences

$$\mathsf{p} - 1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow 1 \leq \frac{\mathsf{p}x}{x+1} \leq \mathsf{p} - 1,$$
$$\mathsf{p} - 1 \leq x \leq \frac{1}{\mathsf{p}-1} \Leftrightarrow \frac{1}{\mathsf{p}-1} \leq \frac{x+1}{\mathsf{p}} \leq \mathsf{p} - 1. \quad \blacksquare$$

For the sake of uniformity, we introduce $F_{\mathsf{p}}(x) = 1 - f_{\mathsf{p}}(x)$ for $\mathsf{p} > 2$. Then $F_{\mathsf{p}}(x)$ satisfies exactly the same functional equations (5.3), with a slight difference that $F_{\mathsf{p}}(x) = 1$ for $x \leq \frac{1}{\mathsf{p}-1}$ and $F_{\mathsf{p}}(x) = 0$ for $x \geq \mathsf{p} - 1$. Consequently, we will not separate these two cases and all our subsequent results hold uniformly. To this account it should be noted that, for example, in case $\mathsf{p} > 2$ the integral $\int_{\mathsf{p}-1}^{1} \star \,\mathrm{d}\star$ should be understood as $-\int_{1}^{\mathsf{p}-1} \star \,\mathrm{d}\star$. Figure 5.1 gives graphic images of typical cases for $F_{\mathsf{p}}(x)$.

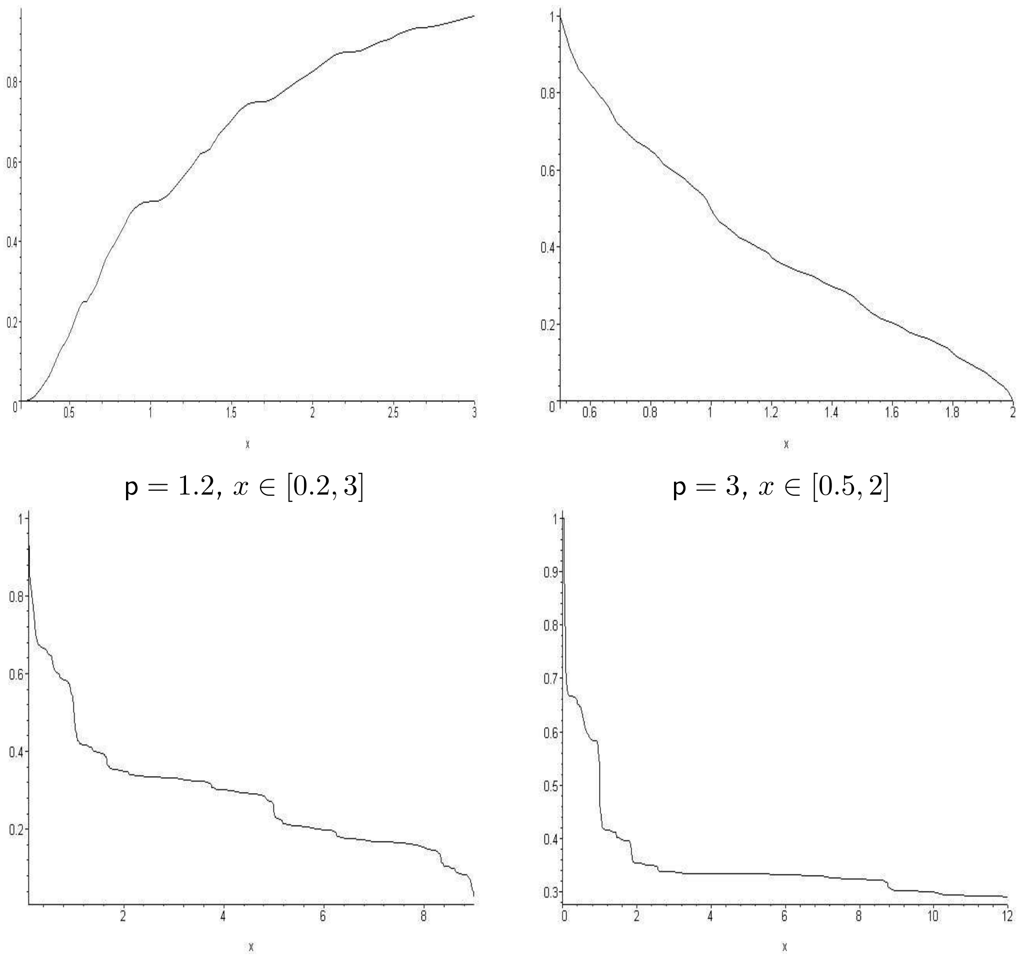
5.3 Complex case of the tree $\mathcal{T}_{\mathsf{p}}$

After dealing the case of real p, $1 \leq \mathsf{p} < \infty$, let us consider a tree (5.1), when $\mathsf{p} \in \mathbb{C}$.

Fact 5.7. *Fix p, $\Re\mathsf{p} \geq 1$, $\mathsf{p} \neq 1$. Then all members of $\mathcal{Q}_{\mathsf{p}}$ belong to a compact set. Moreover, for $|\mathsf{p} - 2| \leq 1$ this set is contained in the half plane $\Re z > -\frac{1}{2}$.*

As a consequence, since this set is invariant under $z \to \frac{1}{z}$, for $|\mathsf{p} - 2| \leq 1$ this set is contained outside the circle $|z + 1| \leq 1$.

We want to extend the definition of $\mathfrak{X}(\mathsf{p}, x)$, given for a positive p in the previous Section, to complex values of p. Thus, as before, let us define $\mathfrak{X}(\mathsf{p}, x) = \overline{x}$ for $x \in$



$\mathsf{p} = 1.2,\ x \in [0.2, 3]$ $\mathsf{p} = 3,\ x \in [0.5, 2]$

$\mathsf{p} = 10,\ x \in [0.1, 9]$ $\mathsf{p} = 25,\ x \in [0, 10]$

Figure 5.1: Functions $F_\mathsf{p}(x)$

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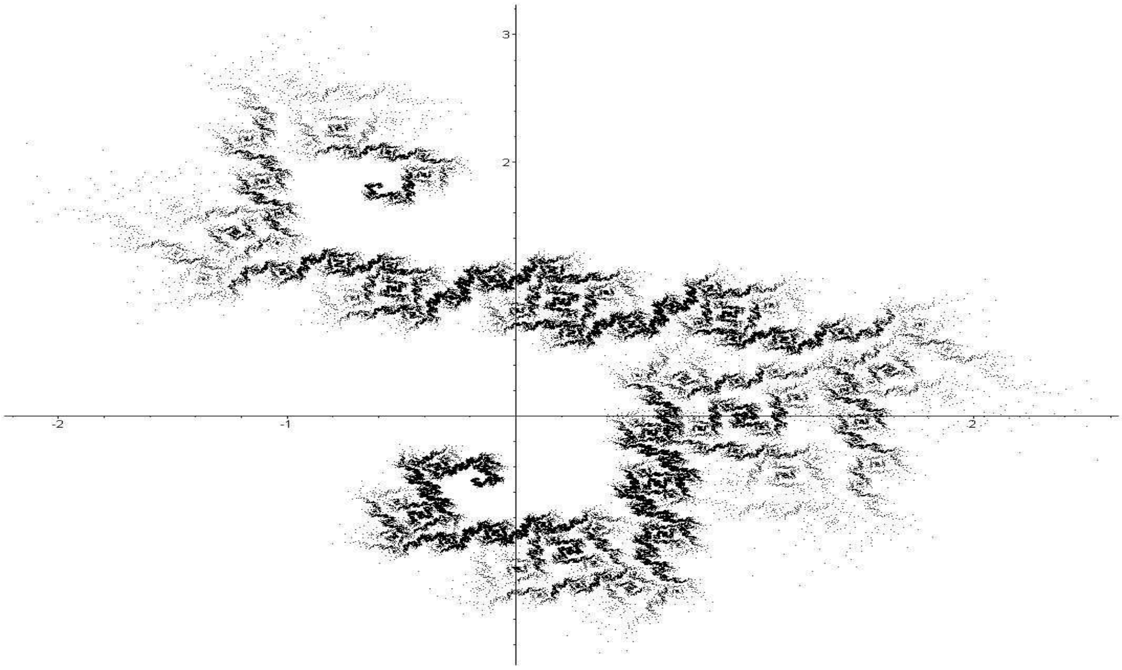

Figure 5.2: $\mathscr{I}_{\mathsf{p}}$, $\mathsf{p} = 0.4 + 1.8i$

$\mathcal{Q}_1$, where $\overline{x}$ is a corresponding element of the tree $\mathcal{Q}_{\mathsf{p}}$. Then Fact 5.7 after some preliminary calculations implies

Fact 5.8. *Fix* $\Re\,\mathsf{p} \geq 1$. *Then the function* $\mathfrak{X}(\mathsf{p}, x) : \mathbb{Q}_+ \to \mathbb{C}$ *is uniformly continuous function, and consequently it can be extended by continuity to* $\mathfrak{X}(\mathsf{p}, x) : [0, \infty) \cup \{\infty\} \to \mathbb{C}$. *Therefore, the curve* $\mathfrak{X}(\mathsf{p}, [0, \infty])$ *(denote it by* $\mathscr{I}_{\mathsf{p}}$*) is a closed set in* $\mathbb{C}$. *As a consequence,* $0 \notin \mathscr{I}_{\mathsf{p}}$ *for* $\mathsf{p} \neq 1$.

This curve $\mathscr{I}_{\mathsf{p}}$ has a natural fractal structure: it decomposes into two parts, namely $\frac{\mathscr{I}_{\mathsf{p}}+1}{\mathsf{p}}$ and $\frac{\mathsf{p}\mathscr{I}_{\mathsf{p}}}{\mathscr{I}_{\mathsf{p}}+1}$, with a single common point $z = 1$. Additionally, $\mathscr{I}_{\mathsf{p}} = \frac{1}{\mathscr{I}_{\mathsf{p}}}$. Thus, each point z on this curve has a unique representation of $\mathsf{p}-$continued fraction of the form $z = [a_0, a_1, a_2, ...]_{\mathsf{p}}$, where $a_0 \in \mathbb{N}_0$, and $a_i \in \mathbb{N}$ for $i \geq 1$. For this reason, the curve is not self intersecting (except for $\mathsf{p} = 2$, since in this case $\mathscr{I}_2$ is a single point). Figures 5.2-5.6 show the images of $\mathscr{I}_{\mathsf{p}}$ (we take sixteen generations of $\mathcal{Q}_{\mathsf{p}}$) for certain characteristic values of p. They are indeed all continuous curves, at least for $\Re\,\mathsf{p} \geq 1$!

Now we will pass to the next level. Namely, it appears that the function $\mathfrak{X}(\mathsf{p}, x) : [0, \infty] \to \mathbb{C}$ has a derivative in p, $\Re\,\mathsf{p} \geq 1$, and it is a continuous and bounded function for $\mathsf{p} \neq 1$. On the other hand, the point $\mathsf{p} = 1$ must be treated separately. It appears that there exist all derivatives at $\mathsf{p} = 1$ as well, though this time they are continuous functions only for irrational x. This is a generic situation: higher derivatives $\frac{\mathrm{d}^T}{\mathrm{d}\,\mathsf{p}^T}\mathfrak{X}(\mathsf{p}, x)$ for $T \geq 2$, $\Re\,\mathsf{p} \geq 1$, are also continuous functions only for $x \in \mathbb{R}_+ \setminus \mathbb{Q}_+$.

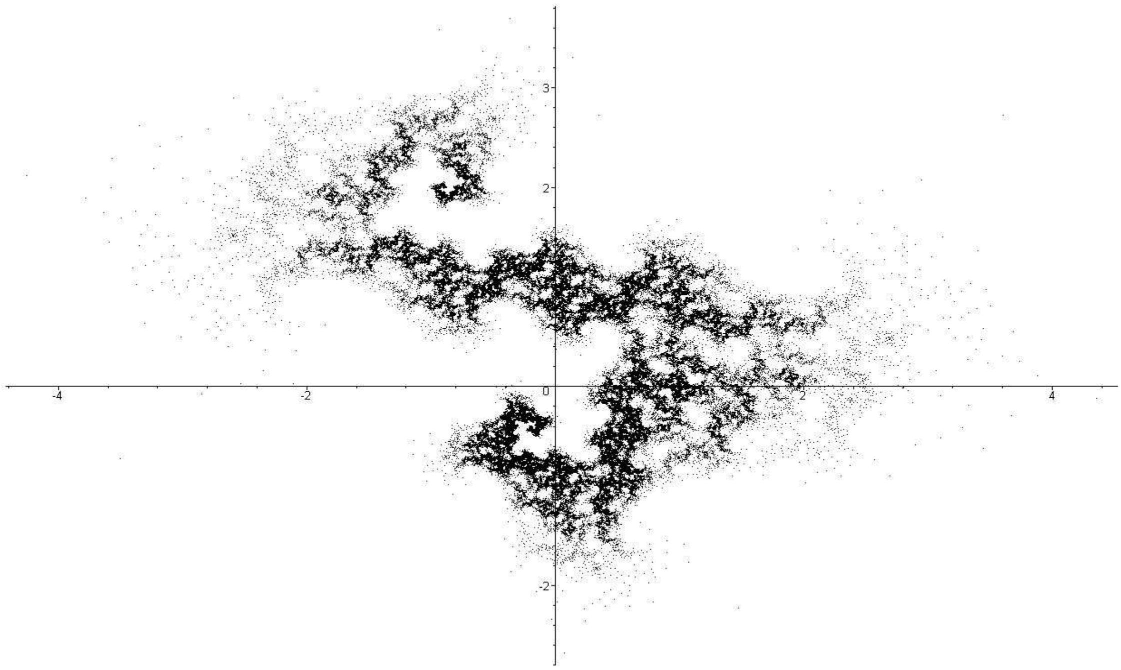

Figure 5.3: $\mathscr{I}_{\mathsf{p}}$, $\mathsf{p} = 0.1 + 2i$.

Luckily, this will have a small impact on the analyticity of $m_L(\mathsf{p})$ in the disc $|\mathsf{p}-2| \leq 1$ (Proposition 5.11).

Fact 5.9. *Let $x, y \in \mathbb{Q}_+$ be elements in (1.2), and $\overline{x}$ and $\overline{y}$ be the corresponding rational functions in (5.1). Suppose $\Re\,\mathsf{p}_0 \geq 1$, $\mathsf{p}_0 \neq 1$. Then, as x varies over $[0, \infty]$, complex numbers $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}}\overline{x}|_{\mathsf{p}=\mathsf{p}_0}$ belong to compact set. Moreover, if $x, y \to \alpha$, $x, y \in \mathbb{Q}_+$, $\alpha \in \mathbb{R}_+$, then $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}}\overline{x}|_{\mathsf{p}=\mathsf{p}_0}$ and $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}}\overline{y}|_{\mathsf{p}=\mathsf{p}_0}$ tend to the same finite limit.*

For example, Figure 5.7 shows the image of the curve $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}}\mathfrak{X}(\mathsf{p}, x)|_{\mathsf{p}=1.5+0.5i}$, $x \in [0, \infty]$.

We are left to tackle the case $\mathsf{p} = 1$.

Fact 5.10. *There exists $\mathcal{S}_N(x) = \frac{\mathrm{d}^N}{\mathrm{d}\,\mathsf{p}^N}\mathfrak{X}(\mathsf{p}, x)|_{\mathsf{p}=1}$. This function is continuous for irrational x. Moreover, $\mathcal{S}_N(x) \ll_N x^{N+1}$ for $x \geq 1$, and $\mathcal{S}_N(x) \ll_N 1$ for $x \in (0, 1)$.*

Surprisingly, all straightforward attempts to prove the Fact 5.7 fail. Facts 5.8, 5.9 and 5.10 are almost direct corollaries of the latter. As a matter of fact, the investigations of the tree $\mathcal{Q}_{\mathsf{p}}$ deserves a separate paper. I am very grateful to my colleagues Jeffrey Lagarias and Stefano Isola, who sent me various references, also informing about the intrinsic relations of this problem with: Julia sets of rational maps of the Riemann sphere; iterated function systems; forward limit sets of semigroups; various topics from complex dynamics and geometry of discrete groups. Thus, the problem is much more subtle and involved than it appears to be. This poses the

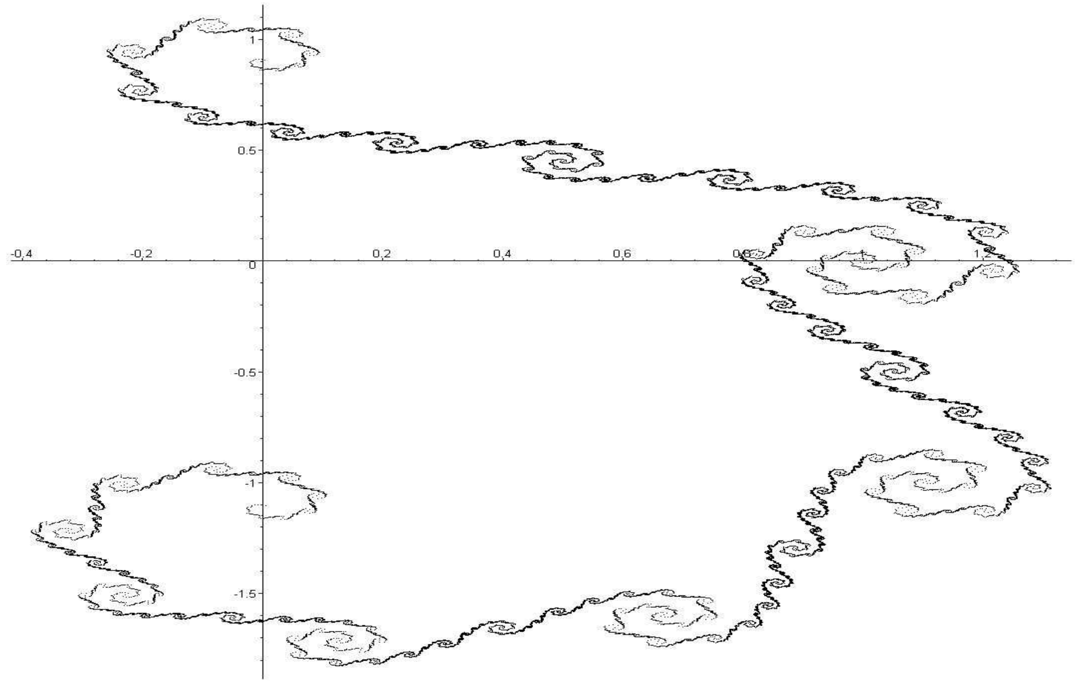

Figure 5.4: $\mathscr{I}_{\mathsf{p}}$, $\mathsf{p} = 1 + 0.9i$. This is a continuous curve!

deep question on the limit set of the semigroup, generated by transformations $\mathcal{U}_{\mathsf{p}}$ and $\mathcal{T}_{\mathsf{p}}$, or any other two "conjugate" analytic maps of the Riemann sphere (say, two analytic maps $\mathcal{A}$ and $\mathcal{B}$ are "conjugate", if $\mathcal{A}(\alpha) = \alpha$, $\mathcal{B}(\beta) = \beta$, $\mathcal{A}(\beta) = \mathcal{B}(\alpha) = \gamma$ for some three points α, β and γ on the Riemann sphere. We construct the same tree, starting from the root γ. The limit set should be some curve with endpoints α, β). The case of one rational map is rather well understood, and it is treated in [5]. On the other hand, the main Theorem 5.1 of this chapter is not directly related to these topics. Therefore, we believe that graphic images of the curves $\mathscr{I}_{\mathsf{p}}$ (and their "derivatives") should certainly convince the reader that the last four propositions do certainly hold. Hence we do not present the strict proofs of the last four propositions, with an intention to investigate this problem in a separate paper.

With all these preliminary results, we formulate the main proposition of this section, which is crucial in the final stage in the proof of Theorem 5.1. Let us define

$$m_L(\mathsf{p}) = 2\int\limits_0^1 \mathfrak{X}^L(\mathsf{p}, x)\,\mathrm{d}F(x) = \lim_{n\to\infty} 2^{2-n} \sum_{a_1+a_2+\ldots+a_s=n} [0, a_1, a_2, .., a_s]^L_{\mathsf{p}}.$$

Proposition 5.11. *The function $m_L(\mathsf{p})$ is analytic in the disc $|\mathsf{p}-2| \leq 1$, including its boundary. In particular, if in this disc*

$$\widehat{m}_L(\mathsf{p}) := \frac{m_L(\mathsf{p})}{\mathsf{p}^L} = \sum_{v=0}^{\infty} \eta_{v,L}(\mathsf{p}-2)^v,$$

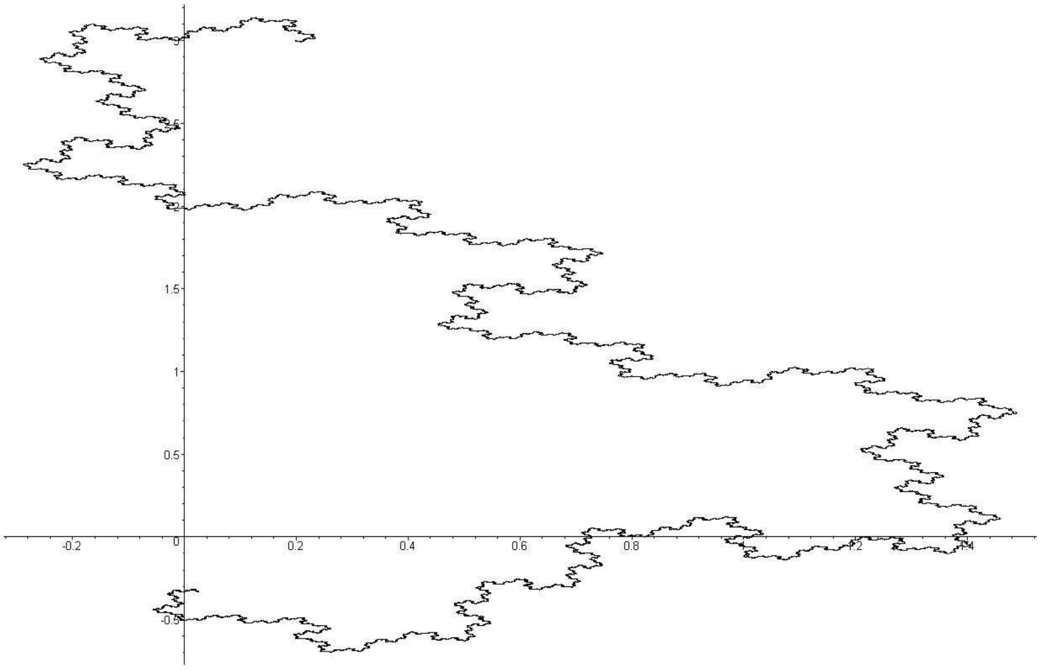

Figure 5.5: $\mathscr{I}_{\mathsf{p}},\ \mathsf{p} = 1.2 + 3i$

then for any $M \in \mathbb{N}$, one has an estimate $\eta_{v,L} \ll v^{-M}$ as $v \to \infty$.

Proof. The function $\mathfrak{X}(\mathsf{p}, x)$ possesses a derivative in p for $\Re\,\mathsf{p} \geq 1,\ \mathsf{p} \neq 1$, and these are bounded and continuous functions for $x \in \mathbb{R}_+$. Therefore $m_L(\mathsf{p})$ has a derivative. For $\mathsf{p} = 1$, there exists $\frac{\mathrm{d}^M}{\mathrm{d}\,\mathsf{p}^M}\mathfrak{X}(\mathsf{p}, x) \ll x^{M+1}$, and it is a continuous function for irrational x. Additionally, $F'(x) = 0$ for $x \in \mathbb{Q}_+$. This proves the analyticity of $m_L(\mathsf{p})$ in the disc $|\,\mathsf{p} - 2| \leq 1$. Then an estimate for the Taylor coefficients is the standard fact from Fourier analysis. In fact,

$$\eta_{v,L} = \int_0^1 \widehat{m}_L(2 + e^{2\pi i \vartheta}) e^{-2\pi i v \vartheta}\,\mathrm{d}\vartheta.$$

The function $\widehat{m}_L(2 + e^{2\pi i \vartheta}) \in \mathsf{C}^\infty(\mathbb{R})$, hence the iteration of integration by parts implies the needed estimate. ∎

Definition 5.12. *We define Minkowski $\mathsf{p}-$question mark function $F_{\mathsf{p}}(x) : \mathscr{I}_{\mathsf{p}} \to [0, 1]$, by*

$$F_{\mathsf{p}}(\mathfrak{X}(\mathsf{p}, x)) = F(x), \quad x \in [0, \infty].$$

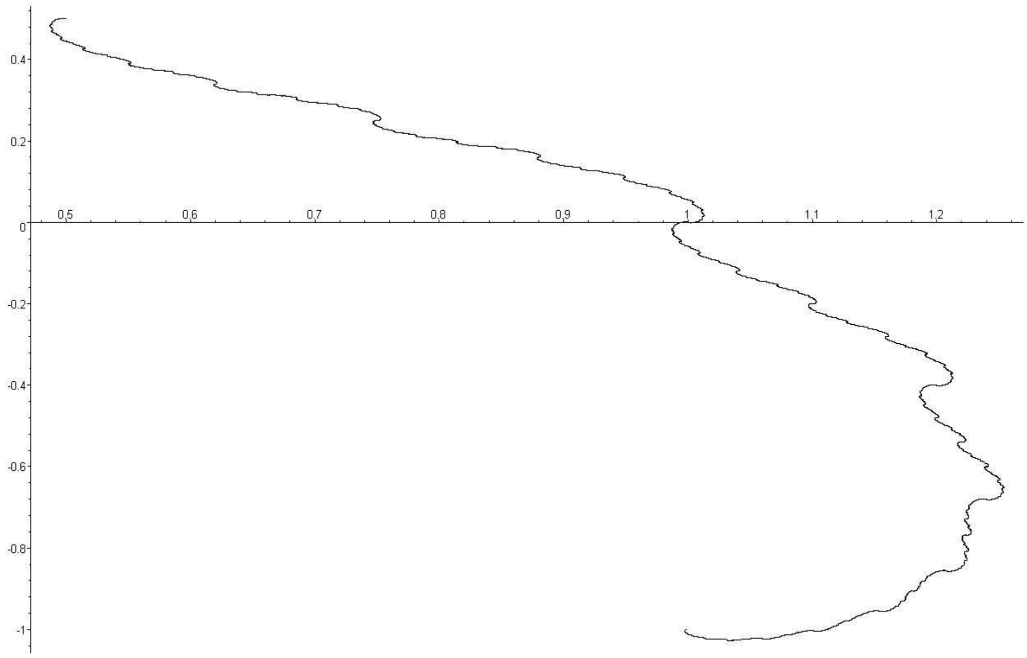

Figure 5.6: $\mathscr{I}_{\mathsf{p}},\ \mathsf{p}=1.5+0.5i$

5.4 Properties of integral transforms of $F_{\mathsf{p}}(x)$

For given p, $\Re\,\mathsf{p}\geq 1$, we define

$$\chi_n=\frac{\mathsf{p}+\mathsf{p}^{n-1}-2}{\mathsf{p}^{n-1}(\mathsf{p}-1)},\qquad \mathscr{I}_n=[\chi_n,\chi_{n+1}]=\mathfrak{X}(\mathsf{p},[n,n+1])\text{ for }n\in\mathbb{N}_0.$$

Complex numbers χ_n stand for the analogue of non-negative integers on the curve $\mathscr{I}_{\mathsf{p}}$. In other words, $\chi_n=\mathcal{U}^n(\mathsf{p}-1)$. We consider $\mathscr{I}_n$ as part of the curve $\mathscr{I}_{\mathsf{p}}$ contained between the points χ_n and χ_{n+1}. Thus, $\chi_0=\mathsf{p}-1$, $\chi_1=1$, and the sequence χ_n is "increasing", in the sense that χ_j as a point on a curve $\mathscr{I}_{\mathsf{p}}$ is between χ_i and χ_k if $i<j<k$. Moreover, $\bigcup\limits_{n=0}^{\infty}\mathscr{I}_n\bigcup\{\frac{1}{\mathsf{p}-1}\}=\mathscr{I}_{\mathsf{p}}$.

Proposition 5.13. *Let* $\omega(x):\mathscr{I}_{\mathsf{p}}\to\mathbb{C}$ *be a continuous function. Then*

$$\int\limits_{\mathscr{I}_{\mathsf{p}}}\omega(x)\,\mathrm{d}\,F_{\mathsf{p}}(x)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\int\limits_{\mathscr{I}_{\mathsf{p}}}\omega\Big(\frac{x}{\mathsf{p}^{n-1}(x+1)}+\frac{\mathsf{p}^n-1}{\mathsf{p}^{n+1}-\mathsf{p}^n}\Big)\,\mathrm{d}\,F_{\mathsf{p}}(x).$$

Proof. Indeed, using (5.3) we obtain:

$$\int\limits_{\mathscr{I}_{\mathsf{p}}}\omega(x)\,\mathrm{d}\,F_{\mathsf{p}}(x)=\sum_{n=0}^{\infty}\int\limits_{\mathscr{I}_n}\omega(x)\,\mathrm{d}\,F_{\mathsf{p}}(x)=\sum_{n=0}^{\infty}\int\limits_{\mathcal{T}^n(\mathscr{I}_0)}\omega(x)\,\mathrm{d}\,F_{\mathsf{p}}(x)\overset{x\to\mathcal{T}^n x}{=}$$

$$\sum_{n=0}^{\infty}\frac{1}{2^n}\int\limits_{\mathscr{I}_0}\omega(\mathcal{T}^n x)\,\mathrm{d}\,F_{\mathsf{p}}(x)\overset{x\to\mathcal{U}x}{=}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\int\limits_{\mathscr{I}_{\mathsf{p}}}\omega(\mathcal{T}^n\mathcal{U}x)\,\mathrm{d}\,F_{\mathsf{p}}(x),$$

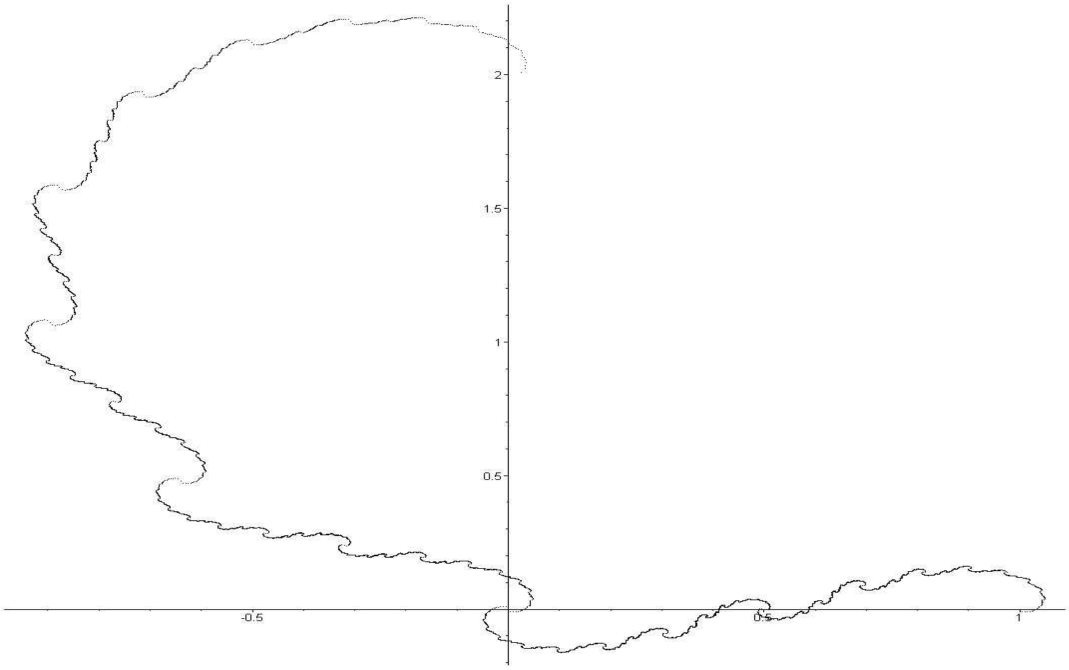

Figure 5.7: $\frac{\mathrm{d}}{\mathrm{d}\,\mathsf{p}}\mathfrak{X}(\mathsf{p},[0,\infty])|_{\mathsf{p}=\mathsf{p}_0}$, $\mathsf{p}_0 = 1.5 + 0.5i$

and this is exactly the statement of the proposition. ■

For $L, T \in \mathbb{N}_0$ let us introduce

$$B_{L,T}(\mathsf{p}) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}\,\mathsf{p}^{Tn}} \Big(\frac{\mathsf{p}^n - 1}{\mathsf{p}^{n+1} - \mathsf{p}^n} \Big)^L.$$

For example,

$$\begin{aligned}
B_{0,T}(\mathsf{p}) &= \frac{\mathsf{p}^T}{2\,\mathsf{p}^T - 1}, \quad B_{1,T}(\mathsf{p}) = \frac{\mathsf{p}^T}{(2\,\mathsf{p}^T - 1)(2\,\mathsf{p}^{T+1} - 1)},\\
B_{2,T}(\mathsf{p}) &= \frac{\mathsf{p}^T(2\,\mathsf{p}^{T+1} + 1)}{(2\,\mathsf{p}^{T+2} - 1)(2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)},\\
B_{3,T}(\mathsf{p}) &= \frac{\mathsf{p}^T(4\,\mathsf{p}^{2T+3} + 4\,\mathsf{p}^{T+2} + 4\,\mathsf{p}^{T+1} + 1)}{(2\,\mathsf{p}^{T+3} - 1)(2\,\mathsf{p}^{T+2} - 1)(2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)},\\
B_{4,T}(\mathsf{p}) &= \frac{p^T(2\,\mathsf{p}^{T+2} + 1)(4\,\mathsf{p}^{2T+4} + 6\,\mathsf{p}^{T+3} + 8\,\mathsf{p}^{T+2} + 6\,\mathsf{p}^{T+1} + 1)}{(2\,\mathsf{p}^{T+4} - 1)(2\,\mathsf{p}^{T+3} - 1)(2\,\mathsf{p}^{T+2} - 1)(2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)}.
\end{aligned}$$

As it is easy to see, $B_{L,T}(\mathsf{p})$ are rational functions in p for $L, T \in \mathbb{N}_0$. Indeed,

$$B_{L,T}(\mathsf{p}) = \frac{1}{(\mathsf{p}-1)^L} \cdot \sum_{n=0}^{\infty} \frac{1}{\mathsf{p}^{Tn} 2^{n+1}} \Big(1 - \frac{1}{\mathsf{p}^n}\Big)^L = \frac{1}{2(\mathsf{p}-1)^L} \cdot \sum_{s=0}^{L} (-1)^s \binom{L}{s} \sum_{n=0}^{\infty} \frac{1}{2^n\,\mathsf{p}^{n(s+T)}} =$$
$$\frac{\mathsf{p}^T}{(\mathsf{p}-1)^L} \cdot \sum_{s=0}^{L} (-1)^s \binom{L}{s} \frac{\mathsf{p}^s}{2\,\mathsf{p}^{s+T} - 1} = \frac{\mathsf{p}^T \mathscr{R}_{L,T}(\mathsf{p})}{(2\,\mathsf{p}^{T+L} - 1)(2\,\mathsf{p}^{T+L-1} - 1) \cdot \ldots \cdot (2\,\mathsf{p}^{T+1} - 1)(2\,\mathsf{p}^T - 1)},$$

where $\mathscr{R}_{L,T}(\mathsf{p})$ are polynomials. This follows from the observation that $\mathsf{p}=1$ is a root of numerator of multiplicity not less than L.

As in case $\mathsf{p}=1$, our main concern are the moments of distributions $F_{\mathsf{p}}(x)$, which are defined by

$$\begin{aligned} m_L(\mathsf{p}) &= 2\int\limits_{\mathscr{I}_0} x^L \,\mathrm{d}\,F_{\mathsf{p}}(x) = \int\limits_{\mathscr{I}_{\mathsf{p}}} \Big(\frac{\mathsf{p}x}{x+1}\Big)^L \mathrm{d}\,F_{\mathsf{p}}(x) = 2\int\limits_0^1 \mathfrak{X}^L(\mathsf{p},x)\,\mathrm{d}F(x), \\ M_L(\mathsf{p}) &= \int\limits_{\mathscr{I}_{\mathsf{p}}} x^L \,\mathrm{d}\,F_{\mathsf{p}}(x). \end{aligned}$$

Thus, if $\sup_{z\in\mathscr{I}_{\mathsf{p}}}|z| = \rho_{\mathsf{p}} > 1$, which is finite for $\Re\,\mathsf{p}\geq 1,\ \mathsf{p}\neq 1$ (see Section 5.3), then $|M_L(\mathsf{p})|\leq \rho_{\mathsf{p}}^L$.

Proposition 5.14. *Functions $M_L(\mathsf{p})$ and $m_L(\mathsf{p})$ are related via rational functions $B_{L,T}(\mathsf{p})$ in the following way:*

$$M_L(\mathsf{p}) = \sum_{s=0}^{L} m_s(\mathsf{p}) B_{L-s,s}(\mathsf{p}) \binom{L}{s}.$$

Proof. Indeed, this follows from the definitions and Proposition 5.13 in case $\omega(x)=x^L$. ■

Let us introduce, following Chapter 2 in case $\mathsf{p}=1$, the following generating functions:

$$\begin{aligned} \mathfrak{m}_{\mathsf{p}}(t) &= \sum_{L=0}^{\infty} m_L(\mathsf{p})\frac{t^L}{L!} = 2\int\limits_{\mathscr{I}_0} e^{xt}\,\mathrm{d}\,F_{\mathsf{p}}(x) = \int\limits_{\mathscr{I}_{\mathsf{p}}} \exp\Big(\frac{\mathsf{p}xt}{x+1}\Big)\,\mathrm{d}\,F_{\mathsf{p}}(x); \\ G_{\mathsf{p}}(z) &= \sum_{L=1}^{\infty} \frac{m_L(\mathsf{p})}{\mathsf{p}^L} z^{L-1} = \int\limits_{\mathscr{I}_{\mathsf{p}}} \frac{1}{x+1-z}\,\mathrm{d}F_{\mathsf{p}}(x) = \int\limits_0^{\infty} \frac{1}{\mathfrak{X}(\mathsf{p},x)+1-z}\,\mathrm{d}F(x). \end{aligned} \tag{5.4}$$

The limit situation $\mathsf{p}=2$ is particularly important, since all these functions can be explicitly calculated, and it provides the case where all the subsequent results can be checked directly and the starting point in proving Theorem 5.1. Thus,

$$\mathfrak{m}_2(t) = e^t, \quad G_2(z) = \frac{1}{2-z}.$$

By the definition, expressions $m_L(\mathsf{p})/\,\mathsf{p}^L$ are Taylor coefficients of $G_{\mathsf{p}}(z)$ at $z=0$. Differentiation of $L-1$ times under the integral defining $G_{\mathsf{p}}(z)$, and substitution $z=1$ gives

$$G_{\mathsf{p}}^{(L-1)}(1) = (L-1)!\int\limits_{\mathscr{I}_{\mathsf{p}}} \frac{1}{x^L}\,\mathrm{d}\,F_{\mathsf{p}}(x) = M_L(\mathsf{p}) \Rightarrow G_{\mathsf{p}}(z+1) = \sum_{L=0}^{\infty} M_L(\mathsf{p}) z^{L-1}, \tag{5.5}$$

with a radius of convergence equal to ρ_{p}^{-1}. As was proved in Chapter 2 and mentioned before, in case $\mathsf{p} = 1$ ($\rho_1 = \infty$) this must be interpreted that there exist all left derivatives at $z = 1$. The next proposition shows how symmetry property reflects in $\mathfrak{m}_{\mathsf{p}}(t)$.

Proposition 5.15. *One has*

$$\mathfrak{m}_{\mathsf{p}}(t) = e^{\mathsf{p}t}\mathfrak{m}_{\mathsf{p}}(-t).$$

Proof. Indeed,

$$\mathfrak{m}_{\mathsf{p}}(t) = \int\limits_{\mathscr{I}_{\mathsf{p}}} \exp\Big(\frac{\mathsf{p}xt}{x+1}\Big)\,\mathrm{d}\,F_{\mathsf{p}}(x) = \int\limits_{\mathscr{I}_{\mathsf{p}}} \exp\Big(\mathsf{p}t - \frac{\mathsf{p}t}{x+1}\Big)\,\mathrm{d}\,F_{\mathsf{p}}(x) =$$

$$e^{\mathsf{p}t}\int\limits_{\mathscr{I}_{\mathsf{p}}} \exp\Big(-\frac{\mathsf{p}t}{x+1}\Big)\,\mathrm{d}\,F_{\mathsf{p}}(x) \overset{x\to\frac{1}{x}}{=} e^{\mathsf{p}t}\mathfrak{m}_{\mathsf{p}}(-t). \quad \blacksquare$$

This result allows to obtain linear relations among moments $m_L(\mathsf{p})$ and the exact value of the first (trivial) moment $m_1(\mathsf{p})$.

Corollary 5.16. *One has*

$$m_1(\mathsf{p}) = \frac{\mathsf{p}}{2}, \quad M_1(\mathsf{p}) = \frac{\mathsf{p}^2+2}{4\,\mathsf{p}-2}.$$

Proof. Indeed, the last propositions implies

$$m_L(\mathsf{p}) = \sum_{s=0}^{L}\binom{L}{s}(-1)^s m_s(\mathsf{p})\,\mathsf{p}^{L-s}, \quad L \geq 0.$$

For $L = 1$ this gives the first statement of the Corollary. Additionally, Proposition 5.14 for $L = 1$ reads as

$$M_1(\mathsf{p}) = \frac{\mathsf{p}}{2\,\mathsf{p}-1}\cdot m_1(\mathsf{p}) + \frac{1}{2\,\mathsf{p}-1},$$

and we are done. $\blacksquare$

5.5 Three term functional equation

Theorem 5.17. *The function $G_{\mathsf{p}}(z)$ can be extended to analytic function in the domain $\mathbb{C}\setminus(\mathscr{I}_{\mathsf{p}}+1)$. It satisfies the functional equation*

$$\frac{1}{z} + \frac{\mathsf{p}}{z^2}G_{\mathsf{p}}\Big(\frac{\mathsf{p}}{z}\Big) + 2G_{\mathsf{p}}(z+1) = \mathsf{p}G_{\mathsf{p}}(\mathsf{p}z), \ \textit{for } z \notin \frac{\mathscr{I}_{\mathsf{p}}+1}{\mathsf{p}}. \tag{5.6}$$

Its consequence is the symmetry property

$$G_{\mathsf{p}}(z+1) = -\frac{1}{z^2}G_{\mathsf{p}}\Big(\frac{1}{z}+1\Big) - \frac{1}{z}.$$

Moreover, $G_{\mathsf{p}}(z) \to 0$ *if dist*$(z, \mathscr{I}_{\mathsf{p}}) \to \infty$.
Conversely - the function satisfying this functional equation and regularity property is unique.

Proof. Let $w(x,z) = \frac{1}{x+1-z}$. Then it is straightforward to check that

$$\begin{aligned} w(\frac{x+1}{\mathsf{p}}, z+1) &= \mathsf{p}\cdot w(x, \mathsf{p}z), \\ w(\frac{\mathsf{p}}{x+1}, z+1) &= -\frac{\mathsf{p}}{z^2}w(x, \frac{\mathsf{p}}{z}) - \frac{1}{z}. \end{aligned}$$

Thus, for $\Re\mathsf{p} \geq 1,\ \mathsf{p}\neq 2$,

$$2G_{\mathsf{p}}(z+1) = 2\int\limits_{\mathscr{I}_0} w(x,z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x) + 2\int\limits_{\mathscr{I}_{\mathsf{p}}\setminus\mathscr{I}_0} w(x,z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x) =$$

$$2\int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{\mathsf{p}x}{x+1}, z+1)\,\mathrm{d}\,F_{\mathsf{p}}\Big(\frac{\mathsf{p}x}{x+1}\Big) + 2\int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{x+1}{\mathsf{p}}, z+1)\,\mathrm{d}\,F_{\mathsf{p}}\Big(\frac{x+1}{\mathsf{p}}\Big) =$$

$$\int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{\mathsf{p}}{x+1}, z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x) + \int\limits_{\mathscr{I}_{\mathsf{p}}} w(\frac{x+1}{\mathsf{p}}, z+1)\,\mathrm{d}\,F_{\mathsf{p}}(x) = -\frac{1}{z} - \frac{\mathsf{p}}{z^2}G_{\mathsf{p}}\Big(\frac{\mathsf{p}}{z}\Big) + \mathsf{p}G_{\mathsf{p}}(\mathsf{p}z).$$

(In the first integral we used a substitution $x \to \frac{1}{x}$). The functional equation holds in case $\mathsf{p}=2$ as well, which can be checked directly. The holomorphicity of $G_{\mathsf{p}}(z)$ follows exactly as in case $\mathsf{p}=1$, see Chapter 2, Proposition 2.5. All we need is the first integral in (5.4) and the fact that $\mathscr{I}_{\mathsf{p}}$ is a closed set.

As was mentioned, the uniqueness of function satisfying (5.6) for $\mathsf{p}=1$ was proved in Chapter 2, Proposition 2.7. Thus, the converse implication follows from analytic continuation principle for the function in two complex variables (p, z) (see Lemma 5.22 below, where the proof in case $\mathsf{p}=2$ is presented. Similar argument works for general p). ■

Corollary 5.18. *Let* $\mathsf{p}\neq 1$*, and* $\mathscr{C}$ *be any closed smooth contour which circumvents the curve* $\mathscr{I}_{\mathsf{p}}+1$ *once in the positive direction. Then*

$$\frac{1}{2\pi i}\oint\limits_{\mathscr{C}} G_{\mathsf{p}}(z)\,\mathrm{d}z = -1.$$

Proof. Indeed, this follows from functional equation as well as from symmetry property. It is enough to take a sufficiently large circle $\mathscr{C}=\{|z|=R\}$ such that $\mathscr{C}^{-1}+1$ is contained in a small neighborhood of $z=1$, for which $(\mathscr{C}^{-1}+1)\cap(\mathscr{I}_{\mathsf{p}}+1)=\emptyset$. This is possible since $0\notin\mathscr{I}_{\mathsf{p}}$ (see Fact 5.7). ■

We finish with providing an integral equation for $\mathfrak{m}_{\mathsf{p}}(t)$. We indulge in being concise since the argument directly generalizes the one used in Chapter 2, Theorem 2.9, to prove the integral functional equation for $\mathfrak{m}(t)$ (in our notation, this is $\mathfrak{m}_1(t)$).

Proposition 5.19. *Let $1 \leq \mathsf{p} < \infty$ be real. Then the function $\mathfrak{m}_{\mathsf{p}}(t)$ satisfies the boundary condition $\mathfrak{m}_{\mathsf{p}}(0)=1$, regularity property $\mathfrak{m}_{\mathsf{p}}(-t) \ll e^{-\sqrt{t\log 2}}$, and the integral equation*

$$\mathfrak{m}_{\mathsf{p}}(-s) = \int\limits_0^\infty \mathfrak{m}'_{\mathsf{p}}(-t)\Big(2e^s J_0(2\sqrt{\mathsf{p}st}) - J_0(2\sqrt{st})\Big)\,\mathrm{d}t, \quad s \in \mathbb{R}_+.$$

For instance, in the case $\mathsf{p}=2$ this reads as

$$2e^s\int\limits_0^\infty e^{-t}J_0(2\sqrt{2st})\,\mathrm{d}t = 2e^se^{-2s} = e^{-s}+e^{-s} = e^{-s} + \int\limits_0^\infty e^{-t}J_0(2\sqrt{st})\,\mathrm{d}t,$$

which is an identity (see [72]).

Proof. Indeed, the functional equation for $G_{\mathsf{p}}(z)$ in the region $\Re z < -1$ in terms of $\mathfrak{m}'_{\mathsf{p}}(t)$ reads as

$$\frac{1}{z} = \int\limits_0^\infty \mathfrak{m}'_{\mathsf{p}}(-t)\Big(\frac{2}{z+1}e^{\frac{\mathsf{p}t}{z+1}} + \frac{1}{z}e^{tz} - \frac{1}{z}e^{\frac{t}{z}}\Big)\,\mathrm{d}t.$$

Now, multiply this by e^{-sz} and integrate over $\Re z = -\sigma < -1$, where $s>0$ is real. All the remaining steps are exactly the same as on the page 19. ∎

Remark. If $\mathsf{p}\neq 1$, the regularity bound is easier than in case $\mathsf{p}=1$. Take, for example, $1 < \mathsf{p} < 2$. Then

$$|\mathfrak{m}_{\mathsf{p}}(t)| \leq \int\limits_{\mathsf{p}-1}^{\frac{1}{\mathsf{p}-1}} \Big|\exp\Big(\frac{\mathsf{p}xt}{x+1}\Big)\Big|\,\mathrm{d}\,F_{\mathsf{p}}(x) < \int\limits_{\mathsf{p}-1}^{\frac{1}{\mathsf{p}-1}} e^t\,\mathrm{d}\,F_{\mathsf{p}}(x) = e^t.$$

Thus, Proposition 5.15 gives $|\mathfrak{m}_{\mathsf{p}}(-t)| < e^{(1-\mathsf{p})t}$. The same argument shows that for $\mathsf{p}>2$ we have $|\mathfrak{m}_{\mathsf{p}}(-t)| < e^{-t}$.

5.6 Approach through $\mathsf{p}=0$: into the realm of unknown

Let us rewrite the functional equation for $G_{\mathsf{p}}(z) = G(\mathsf{p},z)$ as

$$\frac{1}{z} + \frac{\mathsf{p}}{z^2}G\Big(\mathsf{p},\frac{\mathsf{p}}{z}\Big) + 2G(\mathsf{p},z+1) = \mathsf{p}G(\mathsf{p},\mathsf{p}z). \tag{5.7}$$

With a slight abuse of notation, we will use the expression $\frac{\partial^s}{\partial\mathsf{p}^s}G(0,z)$ to denote $\frac{\partial^s}{\partial\mathsf{p}^s}G(\mathsf{p},z)\big|_{\mathsf{p}=0}$ for $s\in\mathbb{N}_0$. Though the function $G(\mathsf{p},z)$ is defined only for $\Re\mathsf{p}\geq 1$,

$z \notin (\mathscr{I}_{\mathsf{p}}+1)$, assume that we are able to prove that it is analytic in p in a certain wider domain containing a disc $|\mathsf{p}| < \varpi$, $\varpi > 0$. These are only formal calculations, but they unexpectedly yield Theorem 5.4 (see Section 5.1), and numerical calculations do strongly confirm the validity of it.

Thus, substitution $\mathsf{p}=0$ into (5.7) gives $G(0,z)=\frac{1}{2(1-z)}$. Partial differentiation of (5.7) with respect to p yields

$$\frac{1}{z^2}G(\mathsf{p},\frac{\mathsf{p}}{z})+\frac{\mathsf{p}}{z^2}\frac{\partial}{\partial\,\mathsf{p}}G(\mathsf{p},\frac{\mathsf{p}}{z})+\frac{\mathsf{p}}{z^3}\frac{\partial}{\partial z}G(\mathsf{p},\frac{\mathsf{p}}{z})+2\frac{\partial}{\partial\,\mathsf{p}}G(\mathsf{p},z+1)=$$
$$G(\mathsf{p},\mathsf{p}z)+\mathsf{p}\frac{\partial}{\partial\,\mathsf{p}}G(\mathsf{p},\mathsf{p}z)+\mathsf{p}z\frac{\partial}{\partial z}G(\mathsf{p},\mathsf{p}z).$$

Consequently, after substitution $\mathsf{p}=0$, we get

$$\frac{1}{z^2}G(0,0)+2\frac{\partial}{\partial\,\mathsf{p}}G(0,z+1)=G(0,0)\Rightarrow\frac{\partial}{\partial\,\mathsf{p}}G(0,z)=\frac{(z-1)^2-1}{4(z-1)^2}.$$

In the same fashion, differentiating the second time, we obtain $\frac{\partial^2}{\partial\,\mathsf{p}^2}G(0,z)=\frac{(z-1)^4-1}{2(z-1)^3}$. In general, direct induction shows that the following "chain-rule" holds:

$$\frac{\partial^n}{\partial\,\mathsf{p}^n}\Big(\mathsf{p}G(\mathsf{p},\mathsf{p}z)\Big)=\sum_{i+j=n}\binom{n}{j}\mathsf{p}\frac{\partial^i\,\partial^j}{\partial\,\mathsf{p}^i\,\partial z^j}G(\mathsf{p},\mathsf{p}z)z^j+$$
$$\sum_{i+j=n-1}n\binom{n-1}{j}\frac{\partial^i\,\partial^j}{\partial\,\mathsf{p}^i\,\partial z^j}G(\mathsf{p},\mathsf{p}z)z^j, \qquad (5.8)$$

where in the summation it is assumed that $i,j\geq 0$. Thus, differentiating (5.7) $n\geq 1$ times with respect to p, and substituting $\mathsf{p}=0$, we obtain:

$$2\frac{\partial^n}{\partial\,\mathsf{p}^n}G(0,z+1)=\sum_{i+j=n-1}n\binom{n-1}{j}\frac{\partial^i\,\partial^j}{\partial\,\mathsf{p}^i\,\partial z^j}G(0,0)\Big(z^j-\frac{1}{z^{j+2}}\Big).$$

Let

$$\frac{1}{n!}\cdot\frac{\partial^n}{\partial\,\mathsf{p}^n}G(0,z)=\overline{\mathbf{Q}}_n(z).$$

Then

$$2\overline{\mathbf{Q}}_n(z+1)=\sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\overline{\mathbf{Q}}_{n-j-1}(0)\Big(z^j-\frac{1}{z^{j+2}}\Big).$$

Consequently, we have a recurrent formula to compute rational functions $\overline{\mathbf{Q}}(z)$. Let $\mathbf{Q}_n(z)=\overline{\mathbf{Q}}_n(z+1)$. Thus,

$$\mathbf{Q}_n(z)=\frac{(z+1)(z-1)\mathscr{D}_n(z)}{z^{n+1}},\quad n\geq 1,$$

where $\mathscr{D}_n$ are polynomials of degree $2n-2$ with the reciprocity property $\mathscr{D}_n(z)=z^{2n-2}\mathscr{D}_n\Big(\frac{1}{z}\Big)$ (this is obvious from the recurrence relation which defines $\mathbf{Q}_n(z)$). Moreover, the coefficients of $\mathscr{D}_n$ are $\mathbb{Q}_p$ integers for any prime $p\neq 2$. These calculations yield the following formal result.

Proposition 5.20. *(Heuristic result). One has*

$$G(\mathsf{p}, z) \text{``}=\text{''} \sum_{n=0}^{\infty} \mathsf{p}^n \cdot \mathbf{Q}_n(z-1) = \sum_{n=0}^{\infty} \mathsf{p}^n \frac{z(z-2)\mathscr{D}_n(z-1)}{(z-1)^{n+1}}.$$

This produces the "series" for the second and higher moments of the form

$$m_2(\mathsf{p}) = \mathsf{p}^2 \cdot \sum_{n=0}^{\infty} \mathsf{p}^n \mathbf{Q}'_n(-1).$$

In particular, inspection of the table in Section 5.1 (where the initial values for $\mathbf{Q}'_n(-1)$ are listed) shows that this series for $\mathsf{p} = 1$ does not converge. However, the Borel sum is properly defined and it converges exactly to the value m_2. This gives empirical evidence for the validity of Theorem 5.4. The principles of Borel summation also suggest the mysterious fact that indeed $G(\mathsf{p}, z)$ analytically extends to the interval $\mathsf{p} \in [0, 1]$.

Additionally, numerical calculations reveal the following fact: the sequence $\sqrt[n]{|\mathbf{Q}'_n(-1)|}$ is monotonically increasing (apparently, tends to ∞), while $\frac{1}{n} \log |\mathbf{Q}'_n(-1)| - \log n$ monotonically decreases (possibly, tends to $-\infty$). Thus,

$$A^n < |\mathbf{Q}'_n(-1)| < (cn)^n,$$

for $c = 0.02372$ and $A = 3.527$, $n \geq 150$. We do not have enough evidence to conjecture the real growth of this sequence. If $c = c(n) \to 0$, as $n \to \infty$, then the function

$$\Lambda(t) = \sum_{n=0}^{\infty} \frac{\mathbf{Q}'_n(-1)}{n!} t^n$$

is entire, and Theorem 5.4 is equivalent to the fact that

$$\int_0^{\infty} \Lambda(t) e^{-t} \, \mathrm{d}t = m_2.$$

5.7 Closed form formula: approach through $\mathsf{p} = 2$

In this section we provide rigid calculations which yield explicit series for $G(\mathsf{p}, z)$ in terms of powers of $(\mathsf{p} - 2)$ and certain rational functions. The function $G(\mathsf{p}, z)$ is analytic in $\{|\mathsf{p} - 2| \leq 1\} \times \{|z| \leq 1\}$. This follows from results is Section 5.3, Fact 5.7, integral representation (5.4), and also from (5.5) and explanation afterwards. Thus, for $\{|\mathsf{p} - 2| < 1\} \times \{|z| < 1\}$ it has a Taylor expansion

$$G(\mathsf{p}, z) = \sum_{L=1}^{\infty} \sum_{v=0}^{\infty} \eta_{v,L} \cdot z^{L-1} (\mathsf{p} - 2)^v. \tag{5.9}$$

Moreover, the function $G(2+e^{2\pi i\vartheta},e^{2\pi i\varphi})\in\mathsf{C}^{\infty}(\mathbb{R}\times\mathbb{R})$, and it is double-periodic. Thus,

$$\eta_{v,L}=\int\limits_0^1\int\limits_0^1 G(2+e^{2\pi i\vartheta},e^{2\pi i\varphi})e^{-2\pi iv\vartheta-2\pi i(L-1)\varphi}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi,\quad v\geq 0,\quad L\geq 1.$$

A standard trick from Fourier analysis (using iteration of integration by parts) shows that $\eta_{v,L}\ll_M (Lv)^{-M}$ for any $M\in\mathbb{N}$. Thus, (5.9) holds for $(\mathsf{p},z)\in\{|\mathsf{p}-2|\leq 1\}\times\{|z|\leq 1\}$.

Our idea is a simple one. Indeed, let us look at (5.4). This implies the Taylor series for $m_L(\mathsf{p})/\mathsf{p}^L=\sum_{v=0}^{\infty}\eta_{v,L}(\mathsf{p}-2)^v$, convergent in the disc $|\mathsf{p}-2|\leq 1$. Due to the absolute convergence, the order of summation in (5.9) is not essential. This yields

$$G(\mathsf{p},z)=\sum_{v=0}^{\infty}(\mathsf{p}-2)^v\Big(\sum_{L=1}^{\infty}\eta_{v,L}\cdot z^{L-1}\Big).$$

Therefore, let

$$\frac{1}{n!}\frac{\partial^n}{\partial\,\mathsf{p}^n}G(\mathsf{p},z)\Big|_{\mathsf{p}=2}=\mathbf{H}_n(z)=\sum_{L=1}^{\infty}\eta_{n,L}\cdot z^{L-1}.$$

We already know that $\mathbf{H}_0(z)=\frac{1}{2-z}$. Though $m_L(\mathsf{p})$ are obviously highly transcendental functions, the series for $\mathbf{H}_n(z)$ is in fact a rational function in z, and this is the main point of our approach. Moreover, we will show that

$$\mathbf{H}_n(z)=\frac{\mathscr{B}_n(z)}{(z-2)^{n+1}},$$

where $\mathscr{B}_n(z)$ is a polynomial with rational coefficients of degree $n-1$ with the reciprocity property $\mathscr{B}_n(z+1)=(-1)^n z^{n-1}\mathscr{B}_n(\frac{1}{z}+1)$, $\mathscr{B}_n(0)=0$. We argue by induction on n. First we need an auxiliary lemma.

Let $\mathbb{Q}[z]_{n-1}$ denote the linear space of dimension n of polynomials of degree $\leq n-1$ with rational coefficients. Consider the following linear map $\mathcal{L}_{n-1}:\mathbb{Q}[z]_{n-1}\to\mathbb{Q}[z]_{n-1}$, defined by

$$\mathcal{L}_{n-1}(P)(z)=P(z+1)-\frac{1}{2^{n+1}}P(2z)+\frac{(-1)^{n+1}}{2^{n+1}}P\Big(\frac{2}{z}\Big)z^{n-1}.$$

Lemma 5.21. *For $n\in\mathbb{N}$, det$(\mathcal{L}_{n-1})\neq 0$. Accordingly, $\mathcal{L}_{n-1}$ is the isomorphism.*

Remark. Let $m=\big[\frac{n}{2}\big]$. Then it can be proved that indeed $\det(\mathcal{L}_{n-1})=\frac{\prod_{i=1}^{m}(4^i-1)}{2^{m^2+m}}$.

Proof. Suppose $P\in\ker(\mathcal{L}_{n-1})$. Then the rational function $\mathbf{H}(z)=\frac{P(z)}{(z-2)^{n+1}}$ satisfies the three term functional equation

$$\mathbf{H}(z+1)-\mathbf{H}(2z)+\mathbf{H}\Big(\frac{2}{z}\Big)\frac{1}{z^2}=0,\quad z\neq 1. \tag{5.10}$$

Also, $\mathbf{H}(z)=o(1)$, as $z\to\infty$. Now the result follows from the following

Lemma 5.22. *Let $\Upsilon(z)$ be any analytic function in the domain $\mathbb{C}\setminus\{1\}$. Then if $\mathbf{H}(z)$ is a solution of the equation*

$$\mathbf{H}(z+1)-\mathbf{H}(2z)+\mathbf{H}\Big(\frac{2}{z}\Big)\frac{1}{z^2}=\Upsilon(z),$$

such that $\mathbf{H}(z)\to 0$ as $z\to\infty$, $\mathbf{H}(z)$ is analytic in $\mathbb{C}\setminus\{2\}$, then such $\mathbf{H}(z)$ is unique.

Proof. All we need is to show that with the imposed diminishing condition, homogeneous equation (5.10) admits only the solution $\mathbf{H}(z)\equiv 0$. Indeed, let $\mathbf{H}(z)$ be such a solution. Put $z\to 2^n z+1$. Thus,

$$\mathbf{H}(2^n z+2)-\mathbf{H}(2^{n+1}z+2)+\frac{1}{(2^n z+1)^2}\mathbf{H}\Big(\frac{2}{2^n z+1}\Big)=0.$$

This is valid for $z\neq 0$ (since $\mathbf{H}(z)$ is allowed to have a singularity at $z=2$). Now sum this over $n\geq 0$. Due to the diminishing assumption, one gets (after additional substitution $z\to z-2$)

$$\mathbf{H}(z)=-\sum_{n=0}^{\infty}\frac{1}{(2^n z-2^{n+1}+1)^2}\mathbf{H}\Big(\frac{2}{2^n z-2^{n+1}+1}\Big).$$

For clarity, put $z\to -z$ and consider a function $\widehat{\mathbf{H}}(z)=\mathbf{H}(-z)$. Thus,

$$\widehat{\mathbf{H}}(z)=-\sum_{n=0}^{\infty}\frac{1}{(2^n z+2^{n+1}-1)^2}\widehat{\mathbf{H}}\Big(\frac{2}{2^n z+2^{n+1}-1}\Big).$$

Consider this for $z\in[0,2]$. As can be easily seen, then all arguments on the right also belong to this interval. We want to prove the needed result simply by applying the maximum argument. The last identity is still insufficient. For this reason consider its second iteration. This produces a series

$$\widehat{\mathbf{H}}(z)=\sum_{n,m=0}^{\infty}\frac{1}{(2^{n+m+1}z+2^{n+m+2}-2^n z-2^{n+1}+1)^2}\widehat{\mathbf{H}}\Big(\omega_m\circ\omega_n(z)\Big),$$

where $\omega_n(z)=\frac{2}{2^n z+2^{n+1}-1}$. As said, $\omega_m\circ\omega_n(z)\in[0,2]$ for $z\in[0,2]$. Since a function $\mathbf{H}(z)$ is continuous in the interval $[0,2]$, let $z_0\in[0,2]$ be such that $M=|\widehat{\mathbf{H}}(z_0)|=\sup_{z\in[0,2]}|\widehat{\mathbf{H}}(z)|$. Consider the above expression for $z=z_0$. Thus,

$$M=|\mathbf{H}(z_0)|\leq\sum_{n,m=0}^{\infty}\Big|\frac{1}{(2^{n+m+1}z_0+2^{n+m+2}-2^n z_0-2^{n+1}+1)^2}\widehat{\mathbf{H}}\Big(\omega_m\circ\omega_n(z_0)\Big)\Big|\leq$$

$$M\sum_{n,m=0}^{\infty}\frac{1}{(2^{n+m+2}-2^{n+1}+1)^2}=0.20453_{+}M.$$

This is contradictory unless $M=0$. By the principle of analytic continuation, $\mathbf{H}(z)\equiv 0$, and this proves the Lemma. ■

Remark. Direct inspection of the proof reveals that the statement of Lemma still holds with a weaker assumption that $\mathbf{H}(z)$ is real-analytic on $(-\infty, 0]$.

Now, let us differentiate (5.7) n times with respect to p, use (5.8) and afterwards substitute $\mathsf{p} = 2$. This gives

$$\sum_{j=1}^{n} \frac{2}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j}(2z) z^j + \sum_{j=0}^{n-1} \frac{1}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j-1}(2z) z^j -$$

$$\sum_{j=1}^{n} \frac{2}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j}\Big(\frac{2}{z}\Big) \frac{1}{z^{j+2}} - \sum_{j=0}^{n-1} \frac{1}{j!} \frac{\partial^j}{\partial z^j} \mathbf{H}_{n-j-1}\Big(\frac{2}{z}\Big) \frac{1}{z^{j+2}} =$$

$$2\mathbf{H}_n(z+1) - 2\mathbf{H}_n(2z) + 2\mathbf{H}_n\Big(\frac{2}{z}\Big)\frac{1}{z^2}. \tag{5.11}$$

We note that this implies the reciprocity property

$$\mathbf{H}_n(z+1) = -\frac{1}{z^2}\mathbf{H}_n\Big(\frac{1}{z} + 1\Big), \quad n \geq 1.$$

A posteriori, this clarifies how the identity $F(x) + F(1/x) = 1$ reflects in the series for $G(z)$, as stated in Theorem 5.1: reciprocity property (non-homogeneous for $n = 0$ and homogeneous for $n \geq 1$) is reflected in each of the summands separately, whereas the three term functional equation heavily depends on inter-relations among $\mathbf{H}_n(z)$.

Now, suppose we know all $\mathbf{H}_j(z)$ for $j < n$.

Lemma 5.23. *The left hand side of the equation (5.11) is of the form*

$$\textit{l.h.s.} = \frac{\mathscr{I}_n(z)}{(z-1)^{n+1}},$$

where $\mathscr{I}_n(z) \in \mathbb{Q}[z]_{n-1}$.

Proof. First, as it is clear from the appearance of l.h.s., we need to verify that z does not divide a denominator, if l.h.s. is represented as a quotient of two co-prime polynomials. Indeed, using the symmetry property in (5.7) for the term $G(\mathsf{p}, \frac{\mathsf{p}}{z})$, we obtain the three term functional equation of the form

$$-\frac{1}{\mathsf{p}-z} - \frac{\mathsf{p}}{(\mathsf{p}-z)^2} G\Big(\mathsf{p}, \frac{\mathsf{p}}{\mathsf{p}-z}\Big) + 2G(\mathsf{p}, z+1) = \mathsf{p}G(\mathsf{p}, \mathsf{p}z).$$

Let us perform the same procedure which we followed to arrive at equation (5.11). Thus, differentiation n times with respect to p and substitution $\mathsf{p} = 2$ gives the expression of the form

$$\text{l.h.s.}_2 = 2\mathbf{H}_n(z+1) - 2\mathbf{H}_n(2z) - 2\mathbf{H}_n\Big(\frac{2}{2-z}\Big)\frac{1}{(2-z)^2},$$

where lh.s.$_2$ is expressed in terms of $\mathbf{H}_j(z)$ for $j < n$. Nevertheless, this time the common denominator of l.h.s.$_2$ is of the form $(z-1)^{n+1}(z-2)^{n+2}$. As a corollary, z

does not divide it. Finally, due to the reciprocity property, for $n \geq 1$ one has

$$\mathbf{H}_n\Big(\frac{2}{2-z}\Big)\frac{1}{(2-z)^2} = -\mathbf{H}_n\Big(\frac{2}{z}\Big)\frac{1}{z^2}.$$

This shows that actually $\mathrm{l.h.s.} = \mathrm{l.h.s.}_2$, and therefore if this is expressed as a quotient of two polynomials in lowest terms, the denominator is a power of $(z-1)$. Finally, it is obvious that this exponent is exactly $n+1$, and one easily verifies that $\deg \mathscr{I}_n(z) \leq n-1$. (Possibly, $\mathscr{I}_n(z)$ can be divisible by $(z-1)$, but this does not have an impact on the result). ■

Proof of Theorem 5.1. Now, using Lemma 5.21, we inherit that there exists a unique polynomial $\mathscr{B}_n(z)$ of degree $\leq n-1$ such that $\mathscr{B}_n(z) = \frac{1}{2}\mathcal{L}_{n-1}^{-1}(\mathscr{I}_n)(z)$. Summarizing, $\mathbf{H}_n(z) = \frac{\mathscr{B}_n(z)}{(z-2)^{n+1}}$ solves the equation (5.11). On the other hand, Lemma 5.22 implies that the solution of (5.11) we obtained is indeed the unique one. This reasoning proves that for $|\mathsf{p}-2| \leq 1$, $|z| \leq 1$ we have the series

$$G(\mathsf{p}, z) = \sum_{n=0}^{\infty}(\mathsf{p}-2)^n \cdot \mathbf{H}_n(z).$$

This finally establishes the validity of Theorem 5.1. Note also that each summand satisfies the symmetry property. The series converges absolutely for any z, $|z| \leq 1$, and if this holds for z, the same does hold for $\frac{z}{z-1}$, which gives a half-plane $\Re z \leq \frac{1}{2}$. ■

Curiously, one could formally verify that the function defined by this series does indeed satisfy (5.6). Indeed, using (5.11), we get:

$$2G(\mathsf{p}, z+1) = 2\mathbf{H}_0(z+1) + 2\sum_{n=1}^{\infty}(\mathsf{p}-2)^n\mathbf{H}_n(z+1) =$$

$$2\mathbf{H}_0(z+1) + \sum_{n=1}^{\infty}(\mathsf{p}-2)^n\Bigg(\sum_{j=0}^{n}\frac{2}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j}(2z)z^j + \sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j-1}(2z)z^j -$$
$$\sum_{j=0}^{n}\frac{2}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j}\Big(\frac{2}{z}\Big)\frac{1}{z^{j+2}} - \sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j-1}\Big(\frac{2}{z}\Big)\frac{1}{z^{j+2}}\Bigg)$$

Denote $n-j=s$. Then interchanging the order of summation for the first term of the sum in the brackets, we obtain:

$$2\sum_{n=1}^{\infty}(\mathsf{p}-2)^n\sum_{j=0}^{n}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j}(2z)z^j = 2\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}(\mathsf{p}-2)^{j+s}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_s(2z)z^j - 2\mathbf{H}_0(2z) =$$
$$2\sum_{s=0}^{\infty}(\mathsf{p}-2)^s\mathbf{H}_s(2z+(\mathsf{p}-2)z) - 2\mathbf{H}_0(2z) = 2G(\mathsf{p}, \mathsf{p}z) - 2\mathbf{H}_0(2z).$$

The same works for the second sum:

$$\sum_{n=1}^{\infty}(\mathsf{p}-2)^n\sum_{j=0}^{n-1}\frac{1}{j!}\frac{\partial^j}{\partial z^j}\mathbf{H}_{n-j-1}(2z)z^j = (\mathsf{p}-2)G(\mathsf{p}, \mathsf{p}z).$$

We perform the same interchange of summation for the second and the third summand respectively. Thus, this yields

$$2G(\mathsf{p}, z+1) = \mathsf{p}G(\mathsf{p}, \mathsf{p}z) - \frac{\mathsf{p}}{z^2}G\Big(\mathsf{p}, \frac{\mathsf{p}}{z}\Big) + 2\mathbf{H}_0(z+1) - 2\mathbf{H}_0(2z) + \frac{2}{z^2}\mathbf{H}_0\Big(\frac{2}{z}\Big) = \\ \mathsf{p}G(\mathsf{p}, \mathsf{p}z) - \frac{\mathsf{p}}{z^2}G\Big(\mathsf{p}, \frac{\mathsf{p}}{z}\Big) - \frac{1}{z}.$$

On the other hand, it is unclear how one can make this argument to work. This would require rather detailed investigation of the linear map $\mathcal{L}_{n-1}$ and recurrence (5.11), and this seems to be very technical.

5.8 Appendices

5.8.1 Numerical calculations

Unfortunately, the Corollary 5.2 is not very useful in finding exact decimal digits of m_2. In fact, the vector $(m_1, m_2, m_3 ...)$ is the solution of an (infinite) system of linear equations, which encodes the functional equation (2.13). Namely, if we denote $c_L = \sum_{n=1}^{\infty}\frac{1}{2^n n^L} = \mathrm{Li}_L(\frac{1}{2})$, we have a linear system for m_s which describes the coefficients m_s uniquely (see page 18):

$$m_s = \sum_{L=0}^{\infty}(-1)^L c_{L+s}\binom{L+s-1}{s-1}m_L, \quad s \geq 1.$$

Note that this system is not homogeneous ($m_0 = 1$). We truncate this matrix at sufficiently high order to obtain float values. By a lucky chance, the accuracy of this calculation can be checked on the test value $m_1 = 0.5$. This approach yields (for the matrix of order 325):

$$\begin{aligned} m_2 &= 0.29092647642930873638069776273912029008043710219559436654 92... \\ m_3 &= 0.18638971464396310457104664410868043512065565329339154982 38... \\ m_4 &= 0.12699225840744313520289222788021163884118514576172571810 16... \end{aligned}$$

with all 58 digits exact (note that $3m_2 - 2m_3 = 0.5$). In fact, the truncation of matrix at an order 325 gives rather accurate values for m_L for $1 \leq L \leq 125$, well in correspondence with Theorem 3.2. Higher numerical moments tend to deviate from this expression rather quickly.

Kinney [37] has proved that the Hausdorff dimension of growth points of $?(x)$ is equal to

$$\alpha = \frac{1}{2}\Big(\int_0^1 \log_2(1+x)\,\mathrm{d}?(x)\Big)^{-1}.$$

Based on the calculations of Lagarias, the author in [24] reproduces the following estimates: $0.8746 < \alpha < 0.8749$. We have (note that $?(1-x)+?(x)=1$):

$$A := \int\limits_0^1 \log(1+x)\,\mathrm{d}?(x) = \int\limits_0^1 \log\Big(1-\frac{1-x}{2}\Big)\,\mathrm{d}?(x) + \int\limits_0^1 \log 2\,\mathrm{d}?(x) =$$

$$-\sum_{L=1}^{\infty}\frac{1}{L\cdot 2^L}\int\limits_0^1 (1-x)^L\,\mathrm{d}?(x) + \log 2 = -\sum_{L=1}^{\infty}\frac{m_L}{L\cdot 2^L} + \log 2.$$

Thus, we are able to present much more precise result:

$$\alpha = \frac{\log 2}{2A} = 0.87471630510821114221515290421915975 7...$$

with all 35 digits exact. Additionally, the constant c_0 in Theorem 3.2 (and Proposition 2.2) is given by

$$c_0 = \int\limits_0^1 2^x(1-F(x))\,\mathrm{d}x = \frac{\mathfrak{m}(\log 2)}{2\log 2} = \frac{1}{2}\sum_{L=0}^{\infty}\frac{m_L}{L!}(\log 2)^{L-1}.$$

This series is fast convergent, and we obtain

$$c_0 = 1.030199563382694623156004112564478676694158859182 40...$$

5.8.2 Rational functions $\mathbf{H}_n(z)$

The following is MAPLE code to compute rational functions $\mathbf{H}_n(z)$=`h[n]` and coefficients $\mathbf{H}'_n(0)$=`alpha[n]` for $0 \le n \le 50$.

```
> restart;
> with(LinearAlgebra):
> U:=50:
> h[0]:=1/(2-z):
> for n from 1 to U do
>    j[n]:=1/2*simplify(
>        add( unapply(diff(h[n-j],z$j),z)(2*z)*2/j!*(z^(j)),j=1..n)+
>        add( unapply(diff(h[n-j-1],z$j),z)(2*z)*1/j!*(z^(j)),j=1..n-1)+
>            unapply(h[n-1],z)(2*z)    ):
>    k[n]:=simplify((z-1)^(n+1)*(unapply(j[n],z)(z)-
>                             unapply(j[n],z)(1/z)/z^2)):
>    M[n,1]:=Matrix(n,n):M[n,2]:=Matrix(n,n): M[n,3]:=Matrix(n,n):
>          for tx from 1 to n do for ty from tx to n do
>                 M[n,1][ty,tx]:=binomial(n-tx,n-ty)
>                     end do: end do:
```

```
>          for tx from 1 to n do M[n,2][tx,tx]:=2^(n-tx) end do:
>          for tx from 1 to n do M[n,3][tx,n+1-tx]:=2^(tx-1) end do:
>    Y[n]:=M[n,1]-1/2^(n+1)*M[n,2]+(-1)^(n+1)/2^(n+1)*M[n,3]:
>    A[n]:=Matrix(n,1):
>          for tt from 1 to n do A[n][tt,1]:=coeff(k[n],z,n-tt) end do:
>    B[n]:=MatrixMatrixMultiply(MatrixInverse(Y[n]),A[n]):
>    h[n]:=add(z^(n-s)*B[n][s,1](s,1),s=1..n)/(z-2)^(n+1):
>            end do:
>
> for n from 0 to U do alpha[n]:=unapply(diff(h[n],z$1),z)(0) end do;
```

It causes no complications to compute h[n] on a standard home computer for $0 \leq n \leq 60$, though the computations heavily increase in difficulty for $n > 60$.

5.8.3 Rational functions $\mathbf{Q}_n(z)$

This program computes $\mathbf{Q}_n(z)$ =q[n] and the values
$\mathbf{Q}'_n(-1)$ =beta[n] for $0 \leq n \leq 50$.

```
> restart;
>q[0]:=-1/(2*z);
>N:=50:
>q[1]:=simplify(1/2*unapply(q[0],z)(-1)*(1-1/z^2)):
> for n from 1 to N do
>   q[n]:=1/2*simplify(
>       add(unapply(diff(q[n-j-1],z$j),z)(-1)/j!*(z^(j)-1/z^(j+2)),j=1..n-1)+
>       unapply(q[n-1],z)(-1)*(1-1/z^2)
>                      ):
                end do:
> for w from 0 to N do beta[w]:=unapply(diff(q[w],z$1),z)(-1) end do;
```

5.8.4 Summatory function of the Calkin-Wilf tree

In this thesis we were interested in the properties of the nth generation of the Calkin-Wilf tree (1.2) as a whole, without taking into account the order of rationals in this generation. Let x_i, $i \geq 1$, be the sequence of rational numbers in this tree, read line-by-line. In this appendix our main interest is a summatory function

$$\mathcal{S}(N)=\sum_{n=1}^{N}x_n.$$

Since $M_1=\frac{3}{2}$, this implies $\mathcal{S}(2^N-1)\sim\frac{3}{2}\cdot 2^N$. Moreover, as was proved by Reznick [59], one has a stronger result:

$$\mathcal{S}(N)=\frac{3}{2}\cdot N+O((\log N)^2).$$

As a matter of fact, since $x_{2^N-1}=N$, there does not exist a continuous approximation to $\mathcal{S}(N)-\frac{3}{2}N$. Figure 5.8 shows the point-plot of the set $\{N,\mathcal{S}(N)-\frac{3}{2}N\}$ for $1\leq N\leq 2^{16}$. Due to this reason, our main object of future investigation will be the

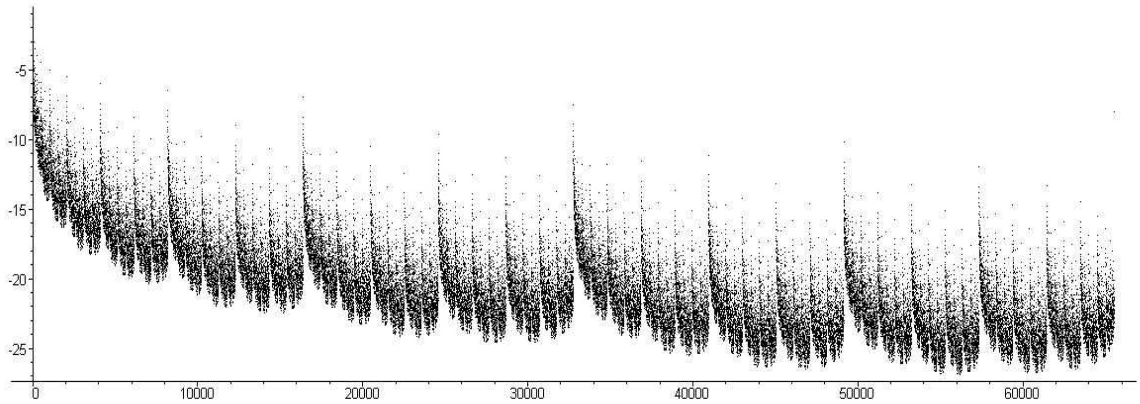

Figure 5.8: $\mathcal{S}(N)-\frac{3}{2}N, 1\leq N\leq 2^{16}$.

"smoothed" version of $\mathcal{S}(N)$. Therefore, let us introduce

$$\mathscr{S}(N)=\sum_{n=1}^{N}(N-n+1)\cdot x_n.$$

Using Abel summation formula we immediately obtain the first asymptotic term: $\mathscr{S}(N)\sim\frac{3}{4}\cdot N^2$. To extract other terms, we use numerical calculations. We take $N=2^{24}$. Denote $\Big(\mathscr{S}(2^N)-\frac{3}{4}N^2\Big)=\mathscr{R}(N)$. In fact, these calculations reveal that there exists constants A and B such that there exists a finite positive limit

$$\lim_{N\to\infty}2^{-AN}N^{-B}\mathscr{R}(N).$$

Let us try to interpolate. For example, our aim to find c, α and β, such that $\mathscr{R}(2^N)=c\cdot\alpha^N\cdot N^{\beta}$ is an equality for $N=22,23$ and 24. This gives $\alpha=2.000541765...$, $\beta=1.518618751...$, $c=-0.2932455601...$ Based on these calculations, it ought to be deduced that indeed $\alpha=2$ (this is one would anticipate). The following Figure 5.9 gives the point-plot of the set

$$\{\log_2(N),\Big(\mathscr{R}(N)+0.27639833\cdot N\cdot(\log_2(N))^{1.5409112498}\Big)N^{-1}\},\quad 2^6\leq N\leq 2^{16}$$

This lead to the following

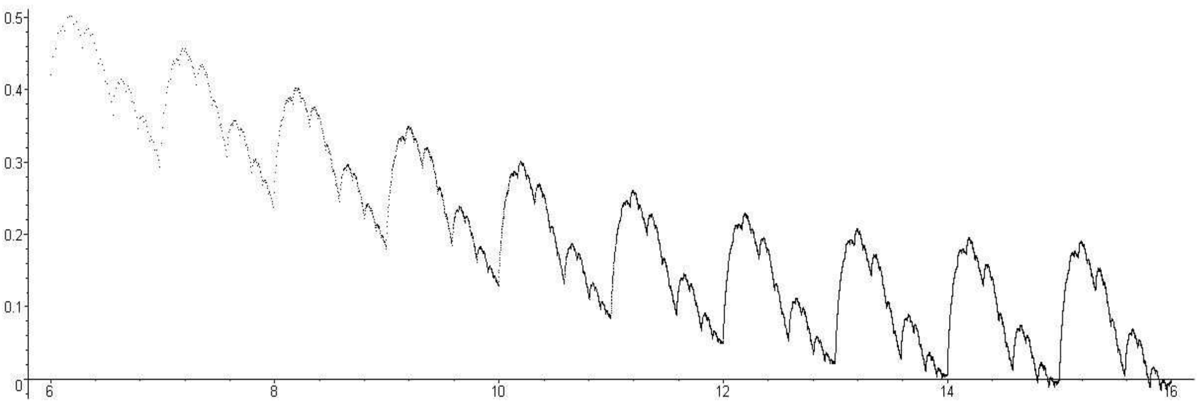

Figure 5.9: $\Big(\mathcal{R}(N) - cN(\log_2 N)^B\Big)N^{-1}, 2^6 \leq N \leq 2^{16}$.

Prediction 5.24. *There exists constants $c < 0$, $B > 0$ and a continuous $1-$periodic function $\Psi(x)$, such that*

$$\mathscr{S}(N) = \frac{3}{4}N^2 + cN(\log_2 N)^B + N \cdot \Psi(\log_2 N) + o(N).$$

Moreover, the function $\Psi(x)$ is nowhere differentiable, it possesses fractal structure (the point $x = \log_2 \frac{3}{2}$ divides it in "quasi-similar" parts), and the points $\log_2 \frac{\ell}{2^T}$, ℓ is odd integer, $2^T < \ell < 2^{T+1}$, serve as local extrema.

As could be expected, this prediction is void for the function $\mathcal{S}(N)$. Though it certainly gives the following

$$\mathcal{S}(N) = \mathscr{S}(N) - \mathscr{S}(N-1) =$$
$$= \frac{3}{2}N + c(\log_2 N)^B + d(\log_2 N)^{B-1} + N\Psi(\log_2 x)\Big|_{N-1}^{N} + o(N) = \frac{3}{2}N + o(N).$$

Thus, all "interesting" terms are devoured by a mysterious fluctuation $o(N)$.

If this prediction is true, the picture we would obtain is completely analogous to the results known for $q-$ary expansion digit summatory function (Delange), and arbitrary $q-$multiplicative function (Grabner). We hope to prove this prediction in the future.

The list of publications

Four publications, which are the basis of the current thesis, are given in the bibliography. Here we list the rest of them.

- A generalization of the Rödseth-Gupta theorem on binary partitions, *Lith. Math. J.* **43** (2) (2003), 103-110.

- Dirichlet series associated with strongly $q-$multiplicative functions, *Ramanujan J.* **8** (1) (2004), 13-21.

- Prime and composite numbers as integer parts of powers (with A. Dubickas), *Acta Math. Hungar.* **105** (3) (2004), 249-256.

- Functional equation related to quadratic and norm forms, *Lith. Math. J.* **45** (2) (2005), 123-141.

- A curious proof of Fermat's little theorem, *Amer. Math. Monthly* (to appear).

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