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Graph Colouring and Frequency Assignment

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Abstract

In this thesis we study some graph colouring problems which arise from mathematical models of frequency assignment in radiocommunications networks, in particular from models formulated by Hale and by Tesman in the 1980s.

The main body of the thesis is divided into four chapters. Chapter 2 is the shortest, and is largely self-contained; it contains some early work on the frequency assignment problem, in which each edge of a graph is assigned a positive integer weight, and an assignment of integer colours to the vertices is sought in which the colours of adjacent vertices differ by at least the weight of the edge joining them.

The remaining three chapters focus on problems which combine frequency assignment with list colouring, in which each vertex has a list of integers from which its colour must be chosen. In Chapter 3 we study list colourings where the colours of adjacent vertices must differ by at least a fixed integer s, and in Chapter 4 we add the additional restriction that the lists must be sets of consecutive integers. In both cases we investigate the required size of the lists so that a colouring can always be found.

By considering the behaviour of these parameters as $s \to \infty$ we formulate

continuous analogues of the two problems, considering lists which are real intervals in Chapter 4, and arbitrary closed real sets in Chapter 5. This gives rise to two new graph invariants, the consecutive choosability ratio $\tau(G)$ and the choosability ratio $\sigma(G)$. We relate these to other known graph invariants, provide general bounds on their values, and determine specific values for various classes of graphs.

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'Can you do addition?' the White Queen asked. 'What's one and one and one and one and one and one and one and one and one?'
'I don't know,' said Alice. 'I lost count.'

Lewis Carroll, *Alice Through the Looking Glass*

Chapter 1

Introduction

1.1 Motivation

The study of graph colouring originally arose from the famous *four-colour problem* (see, for example, [1, 34] for comprehensive accounts), asking whether the regions of any map can be coloured with four colours so that countries with a common border have different colours. Whilst this can be considered to be a real-world application of graph theory, it is in some sense no more than a mathematical curiosity, since cartographers rarely concern themselves with minimising the number of colours they use.

However, graph colouring has benefited in recent decades from a surge in interest due to various real world applications (see [20] for a recent survey), with frequency assignment among the most prominent of these applications.

Consider a network of radio transmitters, each of which must be assigned one or more operating frequencies. If two nearby transmitters are operating on

the same frequency then they have the potential to interfere with each other. In the simplest model, the frequencies assigned to such pairs of transmitters are required to be distinct, and the objective is to minimise the total number of frequencies used.

Graph colouring is a natural model for this problem. The vertices of a graph represent the transmitters, and an edge is added between any pair of vertices (transmitters) which have the potential to interfere. Then the frequencies correspond to colours assigned to the vertices, where adjacent vertices must receive distinct colours.

In more elaborate models, the required separation between frequencies may be larger for pairs of transmitters which are closer together, or for multiple frequencies assigned to the same transmitter; the objective is usually then to minimise the difference between the minimum and maximum frequencies used (the *span* of the assignment). The assumption has generally been made that "frequencies should be assigned to discrete, evenly spaced points in a dedicated portion of the spectrum" [14], and consequently the colours are usually taken to be integers.

In a special issue of the Proceedings of the IEEE (Vol. 68, 1980) devoted entirely to radio spectrum management techniques, a paper by Hale [14] presents the first formulation in precise graph-theoretical language of various frequency assignment problems. Hale considers transmitters located in the plane, and derives restrictions on the frequencies assigned to each pair of nearby transmitters as a function of the Euclidean distance between them.

Two main models of frequency assignment have arisen from Hale's paper.

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In the first model, called *T-colouring*, a fixed set T of nonnegative integers is specified, and the difference between the colours of any pair of adjacent vertices must not lie in T. The second model, which has become known as 'the' frequency (or channel) assignment problem, assigns a positive integer weight to each edge of the graph, and the colours assigned to adjacent vertices must differ by at least the weight of the edge joining them.

Tesman [24, 25] combined the *T*-colouring model with *list colouring*, in which each vertex (transmitter) has a list from which its colour (frequency) must be chosen. This model has applications where there are pre-existing restrictions on the frequencies available for a given transmitter, for example, as a result of frequencies which are already in use.

As the demand placed upon the available radio spectrum has continued to grow, so has practical interest in frequency assignment problems, with the emphasis on finding approximate algorithms, and on generating upper and lower bounds to assess the efficiency of such algorithms. However, there is also a considerable volume of pure graph-theoretical research into various derivatives and generalisations of these models.

1.2 Graph definitions

For ease of reference we define here all the common graphs and graph properties which we will require later. For most of the standard notation we follow Bollobás [6].

A *graph G* consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*,

where $E(G) \subseteq V(G)^{(2)}$. By way of convention, when a single graph G is under consideration, we will write $V = V(G)$ for its vertex set and $E = E(G)$ for its edge set. Furthermore, n will always denote $|V|$, the *order* of G.

The *complete graph* K_n has n vertices and an edge between each pair of vertices. The *complement* of G, denoted $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = V(G)^{(2)} \setminus E(G)$. In particular, $\overline{K}_n$ is the *empty graph* on n vertices.

A *subgraph H* of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Note that for H to be a graph, we must have $E(H) \subseteq V(H)^{(2)}$. H is a *proper subgraph* of G if $H \neq G$. If $E(H) = V(H)^{(2)} \cap E(G)$ then H is an *induced* subgraph; writing $S = V(H)$, we call $H = G[S]$ the subgraph *induced by S*.

The *neighbourhood* of a vertex $v \in V$ is $N_G(v) = \{w \in V : vw \in E\}$, and the *degree* $d_G(v)$ of v equals $|N_G(v)|$. If a single graph G is under consideration, we write $N(v)$ for $N_G(v)$, and $d_G(v)$ for $d(v)$. The minimum and maximum degree over all vertices of G are denoted $\delta(G)$ and $\Delta(G)$ respectively.

If $S \subseteq V$ then $G - S = G[V \setminus S]$, the induced subgraph formed by removing the vertices in S. If $F \subseteq E$ then $G - E = (V, E \setminus F)$. For brevity, if $v \in V$ and $e \in E$ we write $G - v = G - \{v\}$ and $G - e = G - \{e\}$.

If G and H are two graphs with *disjoint* vertex sets, their *union* $G \cup H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G + H$ of G and H is $G \cup H$ together with all possible edges between $V(G)$ and $V(H)$.

A *graph isomorphism* from G to H is a bijective map $\phi : V(G) \to V(H)$ such that $\phi(u)\phi(v) \in E(H)$ iff $uv \in E(G)$. The automorphisms of G form a group $\mathrm{Aut}(G)$; G is (vertex-)transitive (respectively, edge-transitive) if there

is an automorphism $\phi \in \mathrm{Aut}(G)$ mapping any vertex (edge) of G to any other.

The *path P_n* $(n \geq 2)$ has vertices $v_1, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. The *cycle C_n* $(n \geq 3)$ consists of P_n plus an additional edge v_1v_n; cycles are *odd* or *even*, according as the number of vertices is odd or even.

A graph G is *connected* if given any two vertices $u, v \in V$, there is a path in G from u to v. A *component* of a graph G is a maximal connected subgraph.

An *independent set* in G is a subset $S \subseteq V$ such that $G[S]$ is empty. A graph G is *bipartite* if there is a partition of the vertices into two independent sets. The *complete bipartite graph $K_{r,s}$* has vertex set $V = V_1 \cup V_2$, where $|V_1| = r$ and $|V_2| = s$, and all possible edges between V_1 and V_2; it is isomorphic to $\overline{K_r} + \overline{K_s}$. A graph $K_{1,r}$ is called a *star graph*. By analogy we define *k-partite* graphs, and *complete k-partite graphs $K_{r_1,\ldots,r_k}$*.

If the vertices of G can be ordered $v_1, \ldots, v_n$ such that each vertex v_i is adjacent to at most k vertices v_j with $j < i$, then G is said to be *k-degenerate*. There are various equivalent definitions of this property; one is that every subgraph of G has minimum degree at most k.

1.3   Other notation

In Chapters 2 and 3 we work exclusively in the integers, and write $[a,b]$ for the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$.

In Chapters 4 and 5 we work in the reals as well as the integers, and reserve the notation $[\alpha, \beta]$ for the set $\{\lambda \in \mathbb{R} : \alpha \leq \lambda \leq \beta\}$. We write $\{a, \ldots, b\}$ for $\{x \in \mathbb{Z} : a \leq x \leq b\}$. We will also use Greek letters to denote real numbers

and sets and real-valued functions, to distinguish them from integers and sets of integers for which we use Roman letters. The only exceptions will normally be established terminology such as $\chi(G)$ for the chromatic number of G.

1.4 Graph colouring and list colouring

A *colouring* of a graph G is a function c which assigns a colour $c(v)$ to each vertex $v \in V$. A colouring c is *proper* if adjacent vertices have different colours, i.e. $c(v) \neq c(w)$ whenever $vw \in E$. Usually these colours are taken to be integers. Within this thesis we will also consider scenarios where the colours assigned are real numbers, and a different definition of 'proper' will apply.

A graph G is *k-colourable* if there exists a proper colouring $c : V \rightarrow \{1, \ldots, k\}$, and the *chromatic number* $\chi(G)$ of G is the smallest integer k such that G is k-colourable. The task of determining $\chi(G)$ is the 'classical' graph colouring problem. Note that G is k-colourable iff it is k-partite.

List colouring is a generalisation of graph colouring, formulated by Vizing [28] and independently by Erdős, Rubin and Taylor [9]. A *list assignment* for a graph G is a function L which assigns to each vertex $v \in V$ a list (set) of colours $L(v)$; in this thesis we will always have $L(v) \subseteq \mathbb{Z}$. If $|L(v)| = k$ for each $v \in V$, then L is a *k-list assignment*. An *L-colouring* of G is then a function $c : V \rightarrow \mathbb{Z}$ such that $c(v) \in L(v)$ for all $v \in V$. If a proper L-colouring of G exists for every k-list assignment L, then G is said to be *k-choosable*, and the *choosability* $\mathrm{ch}(G)$ of G is the smallest k such that G is k-choosable.

Clearly, if a graph is k-choosable then it is k-colourable, and so $\chi(G) \leq$

$\mathrm{ch}(G)$. However, these two parameters can be arbitrarily far apart; graphs are constructed in [9] for which $\chi(G) = 2$ but $\mathrm{ch}(G) \geq k$ for any $k \geq 2$.

If G is not connected, then $\chi(G)$ is simply the maximum of $\chi(C)$ over all components C of G. The same is true of $\mathrm{ch}(G)$ as well as of most colouring parameters. For this reason we will often restrict our attention to connected graphs, without loss of generality.

We will often construct a proper colouring of a graph by colouring the vertices in turn, or by starting from a pre-colouring of a subset of the vertices. Let $S \subset V$ and $v \in V \setminus S$, and let $c : S \to \mathbb{Z}$ be a proper colouring of $G[S]$. We will say that x is a *valid* colour for v if setting $c(v) = x$ gives a proper colouring of $G[S \cup \{v\}]$.

Chapter 2

Frequency assignment

2.1 Introduction

A *weighted graph* $(G, s) = (V, E, s)$ consists of a graph $G = (V, E)$ together with a *weight function* $s : E \to \mathbb{N}$ which assigns a positive integer weight to each edge. To simplify notation we will often extend the domain of s to all of $V^{(2)}$, setting $s(e) = 0$ if $e \in V^{(2)} \setminus E$.

A colouring $c : V \to [1, k]$ of (G, s) is *feasible* if $|c(v) - c(w)| \geq s(vw)$ for all $vw \in E$. The *span* $\mathrm{sp}(G, s)$ of (G, s) is the smallest k for which a feasible colouring $c : V \to [1, k]$ exists. (Note that some authors define the span to be the minimum difference between the smallest and largest colours used in a feasible colouring, giving a value one less than $\mathrm{sp}(G, s)$ as defined here.) If $s(e) = 1$ for all $e \in E$ then a feasible colouring of (G, s) is simply a proper colouring of G, and so $\mathrm{sp}(G, s) = \chi(G)$.

We extend some basic graph properties to weighted graphs. The *degree* of

a vertex v in (G, s) is defined to be

$$d_s(v) = d_{G,s}(v) = \sum_{w \in N(v)} s(vw),$$

and $\delta_s(G)$ and $\Delta_s(G)$ respectively denote the minimum and maximum of $d_s(v)$ over all $v \in V$.

We begin by giving the value of $\text{sp}(G, s)$ where s is constant on E, which implies an easy upper bound on $\text{sp}(G, s)$ for general (G, s). This is essentially a special case of a result of Cozzens and Roberts [8, Theorem 4], and also appears as Proposition 2.3.1 in [17].

Lemma 2.1 *If (G, s) is a weighted graph with $s(e) = s \in \mathbb{N}$ for all $e \in E$, then $\text{sp}(G, s) = s(\chi(G) - 1) + 1$.* $\qquad\square$

Corollary 2.2 *If (G, s) is a weighted graph and $M = max\{s(e) : e \in E\}$, then $\text{sp}(G, s) \leq M(\chi(G) - 1) + 1$.* $\qquad\square$

2.2   Brooks-type bounds

We look at analogues of the following well-known theorem of Brooks [7].

Theorem 2.3 (Brooks' Theorem) *For any graph G, $\chi(G) \leq \Delta(G) + 1$. Furthermore, if G is connected, then equality holds iff G is a complete graph or an odd cycle.* $\qquad\square$

This result can be generalised to weighted graphs in the natural way.

Theorem 2.4 *For a weighted graph* (G, s),

$$\text{sp}(G, s) \leq \Delta_s(G) + 1. \tag{2.1}$$

Proof. We construct a colouring $c : V \to [1, \Delta_s(G) + 1]$ as follows. Let $G_1 = G$, and choose a vertex v_1 of maximum weighted degree $\Delta_s(G_1)$. Let $c(v_1) = \Delta_s(G_1) + 1$, and let $G_2 = G_1 - v_1$. Now choose a vertex v_2 of maximum weighted degree $\Delta_s(G_2)$ in (G_2, s), and let $c(v_2) = \Delta_s(G_2) + 1$ and $G_3 = G_2 - v_2$. Continue in this way, assigning colour 1 to the final vertex v_n. Then c is a feasible colouring of (G, s). For suppose $v_i v_j \in E$, where $i < j$, and $s = s(v_i v_j)$. Then

$$c(v_j) = \Delta_s(G_j) + 1 = d_{G_j, s}(v_j) + 1 \leq d_{G_i, s}(v_j) + 1 - s \leq \Delta_s(G_i) + 1 - s = c(v_i) - s,$$

where the first inequality holds since the edge $v_i v_j$ is in G_i but not in G_j. $\square$

Theorem 2.4 is proved by McDiarmid in [16], using a different algorithm. However, the algorithm above can be augmented to prove a sharper bound based on the following result of Stacho [23]:

Theorem 2.5 (L. Stacho [23]) *The chromatic number* $\chi(G)$ *of a graph* G *satisfies*

$$\chi(G) \leq \Delta_2(G) + 1,$$

where

$$\Delta_2(G) = \max_{u \in V} \max_{\substack{v \in N(u) \\ d(v) \leq d(u)}} d(v).$$

Equivalently, $\Delta_2(G)$ is the largest degree that a vertex v can have subject to the condition that v is adjacent to a vertex whose degree is at least as large as its own. Hence Theorem 2.5 improves the bound $\chi(G) \leq \Delta(G) + 1$ if no two vertices of maximum degree are adjacent. Defining $\Delta_s^2(G)$ by substituting d_s for d throughout the definition of $\Delta_2(G)$, we can prove the following theorem.

Theorem 2.6 *For a weighted graph* (G, s),

$$\mathrm{sp}(G, s) \leq \Delta_s^2(G) + 1.$$

Proof. Let c be the colouring of (G, s) constructed in the proof of Theorem 2.4, and define $c'(v) = \min\{c(v), \Delta_s^2(G) + 1\}$. We will show that c' is also a feasible colouring of (G, s).

Let $I = \{v \in V : c(v) > \Delta_s^2(G) + 1\}$; then I is an independent set. Thus if $v_i v_j \in E$ then either $v_i \notin I$ and $v_j \notin I$, or w.l.o.g. $v_i \in I$ and $v_j \notin I$. In the first case, $c'(v_i) = c(v_i)$ and $c'(v_i) = c(v_j)$, and so $|c'(v_i) - c'(v_j)| \geq s(v_i v_j)$. In the second case, since $c(v_i) > c(v_j)$ we must have $i < j$, and so

$$c'(v_j) = d_{G_j,s}(v_j) + 1 \leq d_{G,s}(v_j) + 1 - s \leq \Delta_s^2(G) + 1 - s = c'(v_i) - s,$$

where the first inequality holds since $v_i v_j$ is in G but not in G_j, and the second inequality holds because v_j is adjacent to v_i in G, and $d_{G,s}(v_i) \geq d_{G_i,s}(v_i) = c(v_i) - 1 > \Delta_s^2(G)$. $\qquad\qquad\square$

Stacho shows that for fixed $\Delta_2(G) \geq 3$, determining whether $\chi(G) \leq \Delta_2(G)$ is an NP-complete problem. Hence the problem of determining whether the

bound in Theorem 2.6 is attained is also NP-complete.

Fiala et al. [10] prove a very general theorem regarding generalised list *T*-colourings, which we discuss further in Section 5.6. The full statement of the theorem is too long to include here, but we note that it can be used to characterise the weighted graphs for which equality holds in Theorem 2.4.

Theorem 2.7 *If (G, s) is a weighted graph and G is connected, then equality holds in Theorem 2.4 iff G is a complete graph or an odd cycle and s is constant on $E(G)$.* □

2.3　Counting feasible colourings

A classical result in graph theory, first proved by Birkhoff [5], states that the number of colourings of a simple graph G using t colours is a polynomial in t.

Theorem 2.8 *For any graph G, there exists a monic polynomial $P(G; t)$ of degree n, such that the number of proper colourings $c : V \to [1, t]$ of G is equal to $P(G; t)$ for all $t \in \mathbb{N} \cup \{0\}$.*

We can show that the corresponding statement holds for weighted graphs, provided that t is large enough. We will extend one of the many known proofs of Theorem 2.8, using the *Inclusion-Exclusion Principle*: for a finite collection of finite sets $A_1, \ldots, A_n$,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\emptyset \neq J \subseteq [1,n]} (-1)^{|J|-1} \left| \bigcap_{j \in J} A_j \right|.$$

Theorem 2.9 *Let (G, s) be a weighted graph. Then there exists a monic polynomial $P(G, s; t)$ of degree n, such that the number $f(G, s; t)$ of feasible colourings of (G, s) using t colours is equal to $P(G, s; t)$ for all integers $t \geq t_0$, where*

$$ t_0 = \max \left\{ \sum_{i=1}^{r-1} \left(s(v_i v_{i+1}) - 1 \right) : v_1 v_2 \cdots v_r \text{ a path in } G \right\}. $$

This result is proved by McDiarmid in [16] for $t \geq t_1 = (n-1)(M-1)$, where $M = \max\{s(e) : e \in E\}$, as well as in [32] with no explicit bound on t. Our proof, obtained independently, is simpler in nature, and refines McDiarmid's bound: clearly, any of the sums in the expression for t_0 has at most $n - 1$ terms, each of which is at most $M - 1$. Hence $t_0 \leq t_1$, and strict inequality will hold in many cases (for example, if G has no Hamiltonian path).

Proof of Theorem 2.9. By definition, $f(G, s; t)$ is the number of functions $c : V \to [1, t]$ for which $|c(v) - c(w)| \geq s(vw)$ for all $vw \in E$. Thus we have

$$ f(G, s; t) = \left| \bigcap_{vw \in E} \{c : V \to [1, t] : |c(v) - c(w)| \geq s(vw)\} \right| $$

$$ = t^n - \left| \bigcup_{vw \in E} \{c : V \to [1, t] : |c(v) - c(w)| < s(vw)\} \right|. $$

Applying the Inclusion-Exclusion Principle we obtain

$$ f(G, s; t) = \sum_{F \subseteq E} (-1)^{|F|} \left| \{c : V \to [1, t] : (\forall vw \in F) \ |c(v) - c(w)| < s(vw)\} \right|. $$

We now split the sets in this expression according to the specific values of $c(v) - c(w)$. To do this consistently we will need to choose (arbitrarily)

a reference orientation $\overrightarrow{vw}$ for each edge $vw \in E$. Let $\vec{E}$ be the set of these oriented edges. Then

$$f(G, s; t) = \sum_{F \subseteq E} (-1)^{|F|} \sum_{\substack{h: \vec{F} \to \mathbb{Z} \\ |h| < s}} \left| Q(F, h; t) \right|,$$

where

$$Q(F, h; t) = \{ c : V \to [1, t] : (\forall vw \in F) \, c(w) - c(v) = h(\overrightarrow{vw}) \},$$

and the shorthand $|h| < s$ means that $|h(\overrightarrow{vw})| < s(vw)$ for each $vw \in F$. Now we need to evaluate $|Q(F, h; t)|$ as a function of t for each $F \subseteq E$ and $h : \vec{F} \to \mathbb{Z}$ with $|h| < s$.

Suppose F contains a cycle $v_1 v_2 \cdots v_r v_1$. Then if $Q(F, h; t)$ is nonempty we must have

$$h(\overrightarrow{v_1 v_2}) + h(\overrightarrow{v_2 v_3}) + \cdots + h(\overrightarrow{v_{r-1} v_r}) + h(\overrightarrow{v_r v_1}) = 0, \tag{2.2}$$

where $h(\overrightarrow{wv}) = -h(\overrightarrow{vw})$. If (2.2) holds for all cycles in F, we say that h is *consistent*. Now assume that h is consistent. Let $\mathcal{C}(V, F)$ be the set of components of the graph (V, F), and for $C \in \mathcal{C}(V, F)$, define

$$\rho(C, h) = \max \left\{ \sum_{i=1}^{r-1} h(\overrightarrow{v_i v_{i+1}}) : v_1 v_2 \cdots v_r \text{ a path in } C \right\}.$$

Let $c \in Q(F, h; t)$, and let x be the smallest colour used by c on C. This fixes c on the rest of C, and the largest colour used must be $x + \rho(C, h)$. Thus the number of possible restrictions of $c \in Q(F, h; t)$ to C is $t - \rho(C, h)$ if $t \geq 1 + \rho(C, h)$, and zero otherwise. Since we can colour the components of (V, F) independently, we have

$$|Q(F, h; t)| = \begin{cases} 0 & \text{if } h \text{ is inconsistent,} \\ 0 & \text{if } t < \rho(C, h) \text{ for some } C, \text{ and} \\ \displaystyle\prod_{C \in \mathcal{C}(V, F)} \big(t - \rho(C, h)\big) & \text{otherwise.} \end{cases}$$

Note that this product is zero if $t = \rho(C, h)$ for any $C \in \mathcal{C}(V, F)$. Accordingly, we set

$$P(G, s; t) = \sum_{F \subseteq E} (-1)^{|F|} \sum_{\substack{h: \vec{F} \to \mathbb{Z} \\ |h| < s \\ h \text{ consistent}}} \prod_{C \in \mathcal{C}(V, F)} \big(t - \rho(C, h)\big). \tag{2.3}$$

Then $P(G, s; t)$ is a polynomial in t, and $P(G, s; t) = f(G, s; t)$ for all $t \in \mathbb{N}$ with

$$t \geq \max\{\rho(C, h) : C \subseteq G, \ |h| < s\} = \max\{\rho(G, h) : |h| < s\} = t_0.$$

To see that $P(G, s; t)$ is monic with degree n, note that the product in (2.3) has degree n iff F is empty. This completes the proof of Theorem 2.9. $\square$

Let G be a path $v_1 v_2 \cdots v_n$, and let s be any weight function on $E(G)$. If $t = t_0 - 1$ then the only terms on which $f(G, s; t)$ and $P(G, s; t)$ disagree correspond to $Q(E, \pm h; t)$, where $h(\overrightarrow{v_i v_{i+1}}) = s(v_i v_{i+1}) - 1$ for $i = 1, \ldots, n-1$.

Accordingly, $|f(G, s; t_0 - 1) - P(G, s; t_0 - 1)| = 2$, showing that our value of t_0 cannot be improved for these graphs. Note also that Theorem 2.9 reduces to Theorem 2.8 when $s(e) = 1$ for all $e \in E(G)$.

2.4 Vertex demands

Let (G, s) be a weighted graph with a *demand function* $t : V \to \mathbb{N}$, and a parameter $s_0 \in \mathbb{N}$ called the *vertex separation*. A *multicolouring* C of $(G, s; t, s_0)$ is a function which assigns to each vertex $v \in V$ a set $C(v) \subset \mathbb{N}$ of $t(v)$ distinct colours. We say that C is a feasible multicolouring of $(G, s; t, s_0)$ if

(i) for all $v \in V$ and $x, x' \in C(v)$ we have $|x - x'| \geq s_0$, and

(ii) for all $vw \in E$, $x \in C(v)$ and $y \in C(w)$, we have $|x - y| \geq s(vw)$.

The span of C is the largest colour appearing in any of the sets $C(v)$, and the span $\mathrm{sp}(G, s; t, s_0)$ of $(G, s; t, s_0)$ is the minimum span over all feasible multicolourings C.

The problem of computing the span of $(G, s; t, s_0)$ can be reduced to the problem of computing the span for an ordinary weighted graph as follows. Let $G^{(t)}$ be the graph formed by 'inflating' each $v \in V$ into a complete graph $K_{t(v)}$; more formally, $G^{(t)}$ has a set of t mutually adjacent vertices $v_1, \ldots, v_{t(v)}$ for each $v \in V(G)$, and an edge between v_i and w_j whenever $vw \in E(G)$. Let s^* be the weight function on $G^{(t)}$ with $s^*(v_i v_j) = s_0$ for all $v \in V(G)$ and

$1 \le i < j \le t(v)$, and $s^*(v_i w_j) = s(vw)$ whenever $vw \in E(G)$. Then

$$\mathrm{sp}(G, s; t, s_0) = \mathrm{sp}(G^{(t)}, s^*).$$

We end this chapter by computing an exact value of $\mathrm{sp}(G, s; t, s_0)$ in a special case. Let $s(e) = 1$ for all $e \in E(G)$, in which case we write the span as $\mathrm{sp}(G; t, s_0)$, and let G be a complete graph. Then there are two simple lower bounds for $\mathrm{sp}(G; t, s_0)$. The first is $\sum_{v \in V} t(v)$, since this is the total number of colours to be assigned, and they must all be distinct. The second is $s_0(t(v) - 1) + 1$, for any vertex $v \in V$, since the $t(v)$ colours assigned to v must all differ by at least s_0. We will show that these are essentially the only two factors which determine $\mathrm{sp}(K_n; t, s_0)$.

Theorem 2.10 *Let $n, s_0 \in \mathbb{N}$ and $G = K_n$, and let t be a demand function for G. Write*

$$T = \sum_{v \in V} t(v), \quad t_1 = \max_{v \in V} t(v), \quad \textit{and } M_1 = \{v \in V : t(v) = t_1\}.$$

Then

$$\mathrm{sp}(G; t, s_0) = \max \big\{ T, s_0(t_1 - 1) + |M_1| \big\}.$$

The following definition and lemma are required in the proof of Theorem 2.10. Let $\pi \in S_n$ be a permutation acting on $[1, n]$. We say that π is (p, k)-*separating* if for $i, j \in [1, n]$, $1 \le |i - j| < p$ implies $|\pi(i) - \pi(j)| \ge k$.

Lemma 2.11 *Let $n, p, k \in \mathbb{N}$ with $p, k \le n$. There exists a (p, k)-separating permutation on $[1, n]$ iff $n \ge pk$.*

Proof. To show that $n \geq pk$ is a necessary condition, let $\pi \in S_n$ be (p, k)-separating. Let $x = \pi^{-1}(k)$ and choose $S = [a, a + p - 1]$ such that $x \in S$. Then $|\pi(i) - \pi(j)| \geq k$ whenever $i, j \in S$ and $i \neq j$, and so $\min \pi(S) = k$ and $\max \pi(S) \geq k + (|S| - 1)k = pk$. But $\pi(S) \subseteq [1, n]$ and so $pk \leq n$.

Conversely, suppose $n \geq pk$. If n and k are coprime, we simply set $\pi(i) \equiv ki \pmod{n}$; then $1 \leq |i - j|_n < p$ implies $|\pi(i) - \pi(j)|_n \geq k$, where $|i - j|_n = \min\{|i - j|, n - |i - j|\}$. This implies that π is (p, k)-separating.

If n and k have greatest common divisor $h > 1$, define $\rho(i) = \frac{k}{h}i \pmod{\frac{n}{h}}$, a $(p, \frac{k}{h})$-separating permutation on $[1, \frac{n}{h}]$. Construct $\pi \in S_n$ as follows:

$$
1 \leq i \leq \frac{n}{h} \quad \Longrightarrow \quad \pi(i) = h\rho(i),
$$

$$
\frac{n}{h} < i \leq \frac{2n}{h} \quad \Longrightarrow \quad \pi(i) = h\rho\left(i - \frac{n}{h}\right) - 1,
$$

$$
\frac{2n}{h} < i \leq \frac{3n}{h} \quad \Longrightarrow \quad \pi(i) = h\rho\left(i - \frac{2n}{h}\right) - 2,
$$

$$
\cdots
$$

$$
\frac{(h-1)n}{h} < i \leq n \quad \Longrightarrow \quad \pi(i) = h\rho\left(i - \frac{(h-1)n}{h}\right) - (h-1).
$$

Let $i, j \in [1, n]$ with $1 \leq j - i < p$; then $j - i < p \leq \frac{n}{k} \leq \frac{n}{h}$. Thus either $\frac{(t-1)n}{h} < i < j \leq \frac{tn}{h}$ for some $t \in [1, h]$, in which case

$$
|\pi(i) - \pi(j)| = h\left|\rho\left(i - \frac{(t-1)n}{h}\right) - \rho\left(j - \frac{(t-1)n}{h}\right)\right| \geq k,
$$

or $i \leq \frac{tn}{h} < j$. In this case $\rho\left(i - \frac{(t-1)n}{h}\right) = \frac{n}{h} - \left(\frac{tn}{h} - i\right)\frac{k}{h}$ and $\rho\left(j - \frac{tn}{h}\right) = \left(j - \frac{tn}{h}\right)\frac{k}{h}$, where these values are in $[1, \frac{n}{h}]$ because $\frac{tn}{h} - i < \frac{n}{k}$ and $j - \frac{tn}{h} < \frac{n}{k}$; and so

$$\pi(i) - \pi(j) = \left[n - \left(\frac{tn}{h} - i\right)k - (t-1)\right] - \left[\left(j - \frac{tn}{h}\right)k - t\right]$$
$$= n - (i - j)k + 1 \geq n - (p-1)k + 1 > k.$$

Thus $\pi \in S_n$ is (p, k)-separating, as required. $\qquad\square$

Note that in the definition of (p, k)-separating, if we were to replace $|i - j|$ by $|i - j|_n$, or $|\pi(i) - \pi(j)|$ by $|\pi(i) - \pi(j)|_n$, then Lemma 2.11 would no longer be true: for example, consider the case $n = 4$, $p = k = 2$.

Proof of Theorem 2.10. We have already observed that $\mathrm{sp}(G; t, s_0) \geq T$. Also, since the sets $C(v)$ in a feasible multicolouring are pairwise disjoint, we have $\min C(v) \geq |M_1|$ for some $v \in M_1$, and hence $\max C(v) \geq |M_1| + s_0(t_1 - 1)$. This establishes the lower bound; it remains to construct a feasible multicolouring C with the required span.

We proceed by induction on s_0. If $s_0 = 1$ then any choice of disjoint sets $C(v)$ of size $t(v)$ for each $v \in V$ is a feasible assignment for $(G; t, s_0)$, and so $\mathrm{sp}(G; t, s_0) = T$ and $T \geq t_1|M_1| \geq t_1 - 1 + |M_1|$, verifying the theorem in this case. So assume $s_0 \geq 2$. We consider two cases.

Case 1: $T \geq s_0(t_1 - 1) + |M_1|$. In this case we seek a feasible multicolouring C of $(G; t, s_0)$ with $\bigcup_{v \in V} C(v) = [1, T]$.

Note that if $T \geq s_0 t_1$ then Lemma 2.11 implies the result of the theorem, as follows. Let π be a (t_1, s_0)-separating permutation on $[1, T]$, and set $C(v_1) = \{\pi(1), \pi(2), \ldots, \pi(t(v_1))\}$, $C(v_2) = \{\pi(t(v_1) + 1), \ldots, \pi(t(v_1) + t(v_2))\}$, and

so on. Then C is a feasible multicolouring of $(G; t, s_0)$, since $|x - y| \geq s_0$ whenever $x, y \in C(v)$. So suppose $s_0 t_1 > T$; since we are assuming that $T \geq s_0(t_1 - 1) + |M_1|$, this implies that $s_0 > |M_1|$.

To set up the inductive step, let $G' = G - M_1$ and $V' = V(G')$, and define $T' = \sum_{v \in V'} t(v)$, $t_2 = \max\{t(v) : v \in V'\}$ and $M_2 = \{v \in V' : t(v) = t_2\}$; then $T' = T - t_1|M_1|$ and $t_2 \leq t_1 - 1$. Multicolour the vertices of $M_1 = \{v_1, \ldots, v_{|M_1|}\}$ by setting $C(v_i) = \{i, i + s_0, \ldots, i + s_0(t_1 - 1)\}$. Let

$$f : [1, T'] \to [1, T] \setminus \bigcup_{i=1}^{|M_1|} C(v_i)$$

be the unique order-preserving bijection between these two sets. Then for $x, y \in [1, T']$, $|x - y| \geq s_0 - |M_1|$ implies $|f(x) - f(y)| \geq s_0$. Thus, given a feasible multicolouring C' of $(G'; t, s_0 - |M_1|)$ with $C'(v) \subseteq [1, T']$ for each $v \in V'$, we can complete C to a feasible multicolouring of $(G; t, s_0)$ by setting $C(v) = f(C'(v))$ for each $v \in V'$. Since $T' = T - t_1|M_1| \geq s_0(t_1 - 1) + |M_1| - t_1|M_1| = (s_0 - |M_1|)(t_1 - 1) \geq (s_0 - |M_1|)t_2$, we can use Lemma 2.11 as before to show that such a C' exists.

Case 2: $s_0(t_1 - 1) + |M_1| > T$. In this case, since $T \geq t_1|M_1|$, we have $s_0(t_1 - 1) > (t_1 - 1)|M_1|$, and so $s_0 > |M_1|$. Let G', V', T', t_2 and M_2 be as in Case 1. Construct the multicolouring C of M_1 as before, and let

$$f : [1, (s_0 - |M_1|)(t_1 - 1)] \to [1, s_0(t_1 - 1) + |M_1|] \setminus \bigcup_{i=1}^{|M_1|} C(v_i)$$

be the order-preserving bijection between these sets. As before, in order to complete C we seek a feasible multicolouring C' of $(G'; t, s_0 - |M_1|)$, this time with $C'(v) \subseteq [1, (s_0 - |M_1|)(t_1 - 1)]$. We use the inductive hypothesis to show the existence of C'; thus we need to show that $T' \leq (s_0 - |M_1|)(t_1 - 1)$ and $(s_0 - |M_1|)(t_2 - 1) + |M_2| \leq (s_0 - |M_1|)(t_1 - 1)$.

The first of these two inequalities holds as $T' = T - t_1|M_1| < s_0(t_1 - 1) + |M_1| - t_1|M_1| = (s_0 - |M_1|)(t_1 - 1)$. Next, note that the number $t_2|M_2|$ of colours required by M_2 must be at most $T' < (s_0 - |M_1|)(t_1 - 1)$; and that since $1 \leq t_2 \leq t_1 - 1$, we have $(t_2 - (t_1 - 1))(t_2 - 1) \leq 0$, which rearranges to give $\frac{t_1 - 1}{t_2} \leq t_1 - t_2$. Thus $|M_2| < (s_0 - |M_1|)\frac{t_1 - 1}{t_2} \leq (s_0 - |M_1|)(t_1 - t_2)$, and so $(s_0 - |M_1|)(t_2 - 1) + |M_2| < (s_0 - |M_1|)(t_1 - 1)$.

This shows that C' exists, and as in Case 1 we complete C to a feasible multicolouring of $(G; t, s_0)$ by setting $C(v) = f(C'(v))$ for each $v \in V'$. Thus in either case we have constructed a feasible multicolouring of $(G; t, s_0)$ with the required span, and this completes the proof of Theorem 2.10. $\square$

Theorem 2.10 is proved independently by Gerke and McDiarmid [12], in a more general form: they consider a constant edge weight $s(e) = s_1 \in \mathbb{N}$, with $1 \leq s_1 \leq s_0$ (in our result, $s_1 = 1$). We include our proof since the method is different, and since the result of Lemma 2.11 may be of independent interest.

Chapter 3

List colouring with separation

3.1 Introduction

In this chapter we define and investigate *list colouring with separation*, also known as T_r-*list colouring*.

For a graph $G = (V, E)$ and $s \in \mathbb{N}$, a colouring $c : V \to \mathbb{Z}$ is said to have *separation s* if $|c(v) - c(w)| \geq s$ for all $vw \in E$. This is precisely a feasible colouring of the weighted graph (G, s) with $s(e) = s$ for all $e \in E$, as in Chapter 2, and a proper colouring is precisely a colouring with separation 1.

We combine list colouring and separation in the natural way. As defined in Chapter 1, a k-list assignment for G is a function L which assigns to each vertex $v \in V$ a list $L(v) \subseteq \mathbb{Z}$ with $|L(v)| = k$. The *choosability with separation s*, denoted $\text{ch}_s(G)$, is the smallest k such that for every k-list assignment L, there exists an L-colouring of G with separation s.

This topic was studied extensively by Tesman [24,25], and more recently by

Alon and Zaks [4] and others, under the name T_r-*choosability*, a development of T-*colouring* as initiated by Hale [14] as a model for radio frequency assignment. Given a graph $G = (V, E)$ and a set T of non-negative integers, a T-*colouring* of G is a function $c : V \to \mathbb{Z}$ such that $|c(v) - c(w)| \notin T$ for all $vw \in E$. Usually, T is assumed to contain 0; and in particular, $T_r = \{0, 1, \ldots, r\}$, so that a T_r-colouring of G has $|c(v) - c(w)| \geq r + 1$ for all $vw \in E$.

Given G, T as above and a list assignment L for G, in Tesman's notation, an L-T-*colouring* of G is an L-colouring which is also a T-colouring (using a slight abuse of terminology). If an L-T-colouring exists for every k-list assignment L then G is T-k-*choosable*, and the T-*choosability* T-ch(G) of G is the smallest k for which G is T-k-choosable.

Thus a T_r-colouring is the same as a colouring with separation $r + 1$, and T_r-ch$(G) = \text{ch}_{r+1}(G)$. Our notation is based on the use of weights in the frequency assignment problem and has the advantage of being less cumbersome, and the results we obtain are expressed more neatly in terms of s than $r = s - 1$. However, it does not admit the full generality of T-colouring.

Exact values of $\text{ch}_s(G)$ are computed by Tesman in [24] for G a complete graph, a tree, or an odd cycle, and are summarised in the following theorem.

Theorem 3.1 (Tesman [24]) *Let $s \in \mathbb{N}$.*

(i) *If $G = K_n$ then $\text{ch}_s(G) = s(n - 1) + 1$.*

(ii) *If G is an odd cycle C_{2r+1}, then $\text{ch}_s(G) = 2s + 1$.*

(iii) *If G is a tree on n vertices then $\text{ch}_s(G) = \left\lfloor 2s\left(1 - \frac{1}{n}\right) \right\rfloor + 1$.* $\square$

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Tesman described the value of $\mathrm{ch}_s(G)$ for even cycles as 'the most glaring open problem' in his thesis [24]. The correct value was conjectured by Alon and Zaks [4], and proved by Sitters [21].

Theorem 3.2 (Sitters [21]) *If* $s \in \mathbb{N}$ *and* $r \geq 2$, *then*

$$\mathrm{ch}_s(C_{2r}) = \left\lfloor 2s\left(1 - \tfrac{1}{4r-1}\right)\right\rfloor + 1. \qquad \square$$

In Section 3.2, we consider the problem of finding an upper bound for $\mathrm{ch}_s(G)$ given only the values of s and $\mathrm{ch}(G)$. In Section 3.3 we investigate how large $\mathrm{ch}_s(G)$ can be for planar and outerplanar graphs G.

3.2 Bounding $\mathrm{ch}_s(G)$ in terms of $\mathrm{ch}(G)$

3.2.1 Preliminaries

A lower bound on $\mathrm{ch}_s(G)$ in terms of s and $\mathrm{ch}(G)$ is presented by Alon and Zaks in [4].

Theorem 3.3 (Alon, Zaks [4]) *For any graph* G *and any* $s \in \mathbb{N}$,

$$\mathrm{ch}_s(G) \geq s(\mathrm{ch}(G) - 1) + 1. \qquad \square$$

Taking G to be a complete graph, and comparing Theorem 3.1(i), we see that Theorem 3.3 is tight for all values of s and $\mathrm{ch}(G)$.

The work in this section is centred around the following conjecture.

Conjecture 3.4 *For any graph G and any $s \in \mathbb{N}$,*

$$\mathrm{ch}_s(G) \leq (2s - 1)(\mathrm{ch}(G) - 1) + 1.$$

Clearly, if Conjecture 3.4 holds as well as Theorem 3.3, then given any G and s as above, knowing $\mathrm{ch}(G)$ would determine $\mathrm{ch}_s(G)$ to within a factor of 2.

D. R. Woodall has observed that for $s \geq 2$,

$$\mathrm{ch}_s(G) \leq (s - 1)(\mathrm{ch}_2(G) - 1) + 1. \tag{3.1}$$

It follows that a linear bound on $\mathrm{ch}_2(G)$ in terms of $\mathrm{ch}(G)$ would imply a linear bound on $\mathrm{ch}_s(G)$ in terms of $\mathrm{ch}(G)$ for any $s \geq 2$. In Section 5.2 we generalise (3.1) to give bounds on $\mathrm{ch}_s(G)$ in terms of $\mathrm{ch}_r(G)$ whenever $r, s \geq 2$.

A graph G is *chordal* if every cycle in G has a chord, that is, if G has no induced subgraph isomorphic to C_m for $m \geq 4$. Tesman [24] proved the following theorem for such graphs, which implies that Conjecture 3.4 holds (and that $\mathrm{ch}(G) = \chi(G)$) if G is chordal.

Theorem 3.5 (Tesman [24]) *If G is a chordal graph and $s \in \mathbb{N}$, then* $\mathrm{ch}_s(G) \leq (2s - 1)(\chi(G) - 1) + 1.$

3.2.2   Upper bounds using degeneracy

Conjecture 3.4 is motivated by the next theorem, for which we need to define the *colouring number* col(G) of a graph G:

$$\text{col}(G) = 1 + \max_{H \subseteq G} \delta(H). \tag{3.2}$$

Equivalently, col(G) is the smallest $k \in \mathbb{N}$ such that G is $(k-1)$-degenerate. The colouring number is an upper bound for ch(G) (and hence for $\chi(G)$) as follows: let the vertices of G be ordered $v_1, \ldots, v_n$ so that each v_i is adjacent to at most col$(G) - 1$ vertices v_j with $j < i$. Then if each vertex has a list of size col(G), we can construct a proper colouring by giving each vertex v_i in turn the smallest colour $c(v_i)$ not already used on any of its neighbours.

Theorem 3.6 *For any graph G and any $s \in \mathbb{N}$,*

$$\text{ch}_s(G) \leq (2s - 1)(\text{col}(G) - 1) + 1,$$

and this bound is tight for all values of s and col(G).

Proof. As in the proof that ch$(G) \leq$ col(G), let the vertices of G be ordered $v_1, \ldots, v_n$ so that each v_i is adjacent to at most col$(G) - 1$ vertices v_j with $j < i$. Colouring each vertex v_i in turn, for each already coloured vertex v_j there are $2s - 1$ colours $[c(w) - (s - 1), c(w) + s - 1]$ which are not valid for v_i. Thus if the size of the lists assigned to the vertices exceeds $(2s-1)(\text{col}(G)-1)$, we can complete the colouring with separation s, and the bound follows.

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To show that the bound can be attained, we need a k-degenerate graph whose vertices are assigned lists of size $(2s-1)k$, for which no colouring with separation s exists. The graph K_1 trivially suffices if $k=0$, so assume $k \geq 1$. Let $t = (2s-1)k$ and take $G = K_{k,t^k}$, with vertex set $V = \{u_0, \ldots, u_{k-1}\} \cup \{v_{i_0,\ldots,i_{k-1}} : i_0, \ldots, i_{k-1} \in [0, t-1]\}$. Assign the following lists:

$$L(u_p) = [2pt, (2p+1)t - 1],$$

$$L(v_{i_0,\ldots,i_{k-1}}) = \bigcup_{0 \leq p \leq k-1} [2pt + i_p - (s-1), 2pt + i_p + s - 1].$$

The intervals in the above union are disjoint, as $(2(p+1)t + i_{p+1} - (s-1)) - (2pt + i_p + s - 1) \geq 2t - (t-1) - (2s-2) = (2s-1)k - (2s-3) \geq 2 > 0$, and hence $|L(v)| = t$ for all $v \in V$. But whichever choice of colours $c(u_p) = 2pt + i_p$ we make for the vertices $u_0, u_1, \ldots, u_{k-1}$, we find that $L(v_{i_0,\ldots,i_{k-1}})$ contains precisely those colours which are not valid for the vertex $v_{i_0,\ldots,i_{k-1}}$, and so there is no colouring of G with separation s. $\hspace{1cm}\square$

Observe that with $t = (2s-1)k$ and $G = K_{k,t^k}$ as in the above proof, $t^k \geq k^k$ and so G is not k-choosable, as noted in [9]. Hence $\mathrm{ch}(G) = \mathrm{col}(G) = k+1$, which implies that if Conjecture 3.4 is true, it is also tight for all values of s and $\mathrm{ch}(G)$.

Theorem 3.6 can be used in conjunction with the following result due to Alon [3], which links $\mathrm{ch}(G)$ to the minimum degree $\delta(G)$, to prove an exponential bound on $\mathrm{ch}_s(G)$ in terms of s and $\mathrm{ch}(G)$.

Theorem 3.7 (Alon [3]) *If* $s \in \mathbb{N}$ *and* G *is a graph whose minimum degree*

$\delta(G)$ *satisfies*

$$\delta(G) > \frac{4(s^2+1)^2}{(\log_2 \mathrm{e})^2} 2^{2s},$$

then $\text{ch}(G) > s$.   □

Theorem 3.8 *For any graph* G *with* $\text{ch}(G) = k$ *and any* $s \in \mathbb{N}$,

$$\text{ch}_s(G) \leq (2s-1)\frac{(k^2+1)^2}{(\log_2 \mathrm{e})^2} 4^{k+1} + 1.$$

Proof. Writing $\text{ch}(G) = k$, and noting that $\text{ch}(H) \leq k$ for all subgraphs $H \subseteq G$, we can use Theorem 3.7 to bound $\text{col}(G)$:

$$\text{col}(G) = 1 + \max_{H \subseteq G} \delta(H) \leq \frac{4(k^2+1)^2}{(\log_2 \mathrm{e})^2} 2^{2k} + 1.$$

Hence, applying Theorem 3.6, we obtain the required result.   □

Though this is a very long way from the bound of Conjecture 3.4, it does at least establish that some bound of the required form does exist. Estimating the bound in Theorem 3.8 as $\text{ch}(G) = k \to \infty$, we obtain

$$\text{ch}_s(G) = O(sk^4 4^k).$$

Even in the case of 2-choosable graphs G, the bound given by Theorem 3.8 is approximately $\text{ch}_s(G) \leq 1538s - 768$, which is quite some way from the correct bound which we are about to prove.

3.2.3   Bound for 2-choosable graphs

In this section we prove the following theorem, verifying Conjecture 3.4 in the case $\mathrm{ch}(G) = 2$.

Theorem 3.9 *For any graph* G *with* $\mathrm{ch}(G) = 2$ *and any* $s \in \mathbb{N}$,

$$\mathrm{ch}_s(G) \leq 2s.$$

The proof of Theorem 3.9 relies on A. L. Rubin's characterisation of 2-choosable graphs [9], for which we need the following two definitions.

The *core* of a graph G is obtained by successively removing vertices of degree 1 until none remain. If G is assigned lists of size 2, and we remove a vertex v of degree 1 and colour $G - v$ from its lists, we can always choose a colour for v which differs from that of its neighbour. By iterating this 'pruning' process, we see that G is 2-choosable iff its core is 2-choosable.

The Θ-*graph* $\Theta_{a,b,c}$ consists of two distinguished vertices u, w connected by three paths P_1, P_2 and P_3 of lengths a, b and c respectively. Two examples are shown in Figure 3.1.
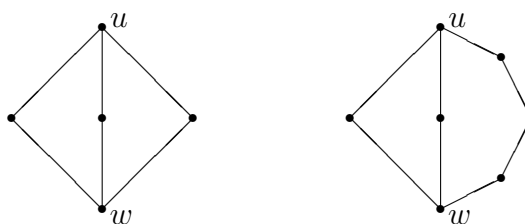
Figure 3.1: $\Theta_{2,2,2}$ $(\cong K_{2,3})$, and $\Theta_{2,2,4}$.

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Now we can state the theorem characterising 2-choosable graphs:

Theorem 3.10 (Rubin [9]) *A connected graph G is 2-choosable iff its core is K_1, or C_{2m+2} or $\Theta_{2,2,2m}$ for some $m \geq 1$.*

For the two lemmas to follow we will need some additional terminology. Let $P = uv_1v_2 \cdots v_{l-1}w$ be a path of length l, with a list assignment L such that $|L(v_p)| = 2s$ for $1 \leq p \leq l-1$. Construct the set $F = F(P) \subseteq L(u) \times L(w)$ as follows: $(x, y) \in F$ iff there is *no L*-colouring of P with separation s such that $c(u) = x$ and $c(w) = y$. Clearly there is an L-colouring of P with separation s iff $F \neq L(u) \times L(w)$.

The set F can be characterised as follows: for $x \in L(u)$ and $y \in L(w)$, let

$$
\begin{aligned}
I_1(x) &= [x - (s-1), x + s - 1], \\
I_2(x) &= \bigcap_{c \in L(v_1) \setminus I_1(x)} [c - (s-1), c + s - 1], \\
I_3(x) &= \bigcap_{c \in L(v_2) \setminus I_2(x)} [c - (s-1), c + s - 1], \\
&\quad \cdots \\
I_l(x) &= \bigcap_{c \in L(v_{l-1}) \setminus I_{l-1}(x)} [c - (s-1), c + s - 1].
\end{aligned}
$$

Evidently, $|I_p(x)| \leq 2s - 1 < 2s = |L(v_p)|$ for each $p = 1, \ldots, l-1$, and so $L(v_p) \setminus I_p(x) \neq \emptyset$. If we want to colour P from its lists, starting with $c(u) = x$, the sets $I_p(x)$ are constructed precisely so that the colour we choose for v_1 must not be in $I_1(x)$, the colour we choose for v_2 must not be in $I_2(x)$, and so

on. It follows that

$$(x, y) \in F \iff y \in I_l(x). \tag{3.3}$$

Alternatively, we can start at w, and define

$$J_{l-1}(y) = [y - (s-1), y + s - 1],$$

$$J_{l-2}(y) = \bigcap_{c \in L(v_{l-1}) \setminus J_{l-1}(y)} [c - (s-1), c + s - 1],$$

$$\dots$$

$$J_0(y) = \bigcap_{c \in L(v_1) \setminus J_1(y)} [c - (s-1), c + s - 1].$$

Then

$$(x, y) \in F \iff x \in J_0(y), \tag{3.4}$$

and furthermore, for each $p = 1, \dots, l - 1$,

$$(x, y) \in F \iff L(v_p) \subseteq I_p(x) \cup J_p(y), \tag{3.5}$$

since there is an L-colouring of P with $c(u) = x$, $c(v_p) = c$ and $c(w) = y$ if and only if $c \in L(v_p) \setminus (I_p(x) \cup J_p(y))$.

Since $I_l(x)$ and $J_0(y)$ are intervals, (3.3) and (3.4) respectively imply the following properties (which we call *convexity*):

$$y_1 < y_2 < y_3 \text{ and } (x, y_1), (x, y_3) \in F \implies (x, y_2) \in F, \tag{3.6}$$

$$x_1 < x_2 < x_3 \text{ and } (x_1, y), (x_3, y) \in F \implies (x_2, y) \in F. \tag{3.7}$$

The sets I_p are nonincreasing in size with increasing p. To see this, fix $p \in [1, l-1]$ and let a and b be the minimum and maximum elements of $L(v_p) \setminus I_p(x)$. Then

$$
\begin{aligned}
|I_{p+1}(x)| &= |[a-(s-1), a+s-1] \cap [b-(s-1), b+s-1]| \\
&= \max\{2s - |[a,b]|, 0\} \\
&\le |L(v_p)| - |L(v_p) \setminus I_p(x)|,
\end{aligned}
$$

which gives

$$
|I_{p+1}(x)| \le |L(v_p) \cap I_p(x)| \le |I_p(x)|. \tag{3.8}
$$

The following lemma is used later as an auxiliary result, but is stated separately as it may be of interest in itself.

Lemma 3.11 *Let $P = uv_1 v_2 \cdots v_{l-1} w$ be a path of length l, whose vertices are assigned lists $L(v)$ such that $|L(v_p)| = 2s$ for $1 \le p \le l-1$, and $|L(u)| + |L(w)| > 2s$. Then there is an L-colouring of P with separation s, such that at least one of u and w is assigned the minimum colour in its list.*

Proof. We use induction on the path length l. Write $L(u) = \{x_1, \ldots, x_i\}$ and $L(w) = \{y_1, \ldots, y_j\}$, with the elements arranged in ascending order in each case. First we considering the case $l = 1$, i.e. $P = uw$. Since $i + j > 2s$, we have $(x_i - x_1) + (y_j - y_1) \ge 2s - 1$, and hence either $x_i - y_1 \ge s$ or $y_j - x_1 \ge s$. In the former case we have an L-colouring of P by setting $c(u) = x_i$ and $c(w) = y_1$, and in the latter case, by setting $c(u) = x_1$ and $c(w) = y_j$.

Now assume that $l > 1$, and that the lemma is true for a path of length $l-1$. Suppose no colouring of P of the required type exists, so that $(x_s, y_1) \in F$ for each $s = 1, \ldots, i$ and $(x_1, y_t) \in F$ for each $t = 1, \ldots, j$. We will obtain a contradiction.

By (3.3), $\{y_1, \ldots, y_j\} \in I_l(x_1)$ and so, by repeated application of (3.8),

$$|L(v_1) \cap I_1(x_1)| \geq |I_2(x_1)| \geq |I_l(x_1)| \geq j. \tag{3.9}$$

We consider two separate cases.

Case 1: $\max L(v_1) \notin I_1(x_1)$. Then, since $(x_1, y_1) \in F$ and $(x_i, y_1) \in F$,

$$\max L(v_1) \in J_1(y_1) \qquad \text{by (3.5)}$$
$$\implies \min L(v_1) \notin J_1(y_1) \qquad \text{since } |J_1(y_1)| \leq 2s - 1 < |L(v_1)|$$
$$\implies \min L(v_1) \in I_1(x_i) \qquad \text{by (3.5)}$$
$$\implies \min L(v_1) \geq x_i - (s - 1)$$
$$\implies |L(v_1) \cap I_1(x_1)| \leq (2s - 1) - (x_i - x_1) \leq 2s - i.$$

Together with (3.9), this shows that $i + j \leq 2s$, contrary to the hypothesis of the lemma. In this case, the above 'exchange' trick using (3.5) allows us to establish the result without using the inductive hypothesis.

Case 2: $\max L(v_1) \in I_1(x_1)$. This means that $\min L(v_1) \notin I_1(x_1)$ and hence, by (3.5),

$$\min L(v_1) \in J_1(y_t) \text{ for } 1 \leq t \leq j. \tag{3.10}$$

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Also, since $\max L(v_1) \le x_1 + s - 1$, we have $|L(v_1) \cap I_1(x_i)| \le (2s-1) - (x_i - x_1) \le 2s - i$, and so $L(v_1)$ contains elements $\min L(v_1) = x'_1 < x'_2 < \cdots < x'_i$ such that $x'_s \notin I_1(x_i)$ and hence

$$x'_s \in J_1(y_1) \text{ for } 1 \le s \le i. \tag{3.11}$$

Thus, writing $P' = v_1 v_2 \cdots v_{l-1} w$ and $L'(v_1) = \{x'_1, \ldots, x'_i\}$, $L'(w) = L(w) = \{y_1, \ldots, y_j\}$ and $L'(v_p) = L(v_p)$ for $2 \le p \le l - 1$, by (3.10) and (3.11) there is no L-colouring of P' with separation s such that at least one of v_1 and w is assigned the minimum colour in its list. This contradicts the inductive hypothesis, and so completes the proof of Lemma 3.11. $\qquad \square$

The key to the proof of Theorem 3.9 is the following lemma.

Lemma 3.12 *If the vertices of* $G = \Theta_{2a,2b,2c}$ *are assigned lists* $L(v)$ *of size* $2s$ *($a, b, c, s \in \mathbb{N}$), and* $L(u) \ne L(w)$, *then there is an* L-*colouring of* G *with separation* s.

Proof. For $h \in \{1, 2, 3\}$ construct the set $F_h = F(P_h)$ as above. Then there is an L-colouring of G with separation s iff $F_1 \cup F_2 \cup F_3 \ne L(u) \times L(w)$.

So suppose there is no such colouring of G, i.e.

$$F_1 \cup F_2 \cup F_3 = L(u) \times L(w). \tag{3.12}$$

By examining the sets F_1, F_2 and F_3 we will eventually deduce that $L(u) = L(w)$, contradicting the hypothesis of the lemma.

We will write $L(u) = \{x_1, x_2, \ldots, x_{2s}\}$ and $L(w) = \{y_1, y_2, \ldots, y_{2s}\}$, with the elements arranged in ascending order. The following pair of properties of the sets F_h is crucial to the proof of the lemma:

$$i \leq j \text{ and } (x_1, y_i) \in F_h \implies (x_{2s}, y_j) \notin F_h, \tag{3.13}$$

$$i \leq j \text{ and } (x_i, y_1) \in F_h \implies (x_j, y_{2s}) \notin F_h. \tag{3.14}$$

Proof of (3.13) and (3.14): For $h = 1, 2, 3$ let $m = a, b, c$ respectively.

Suppose $(x_1, y_i) \in F_h$ and $(x_{2s}, y_j) \in F_h$. So by (3.3), $I_{2m}(x_1) \neq \emptyset \neq I_{2m}(x_{2s})$ and by (3.8), for $1 \leq p \leq 2m - 1$,

$$|L(v_p) \cap I_p(x_1)| \geq |I_{p+1}(x_1)| \geq |I_{2m}(x_1)| > 0,$$

and similarly, $L(v_p) \cap I_p(x_{2s}) \neq \emptyset$.

Now $x_{2s} - x_1 \geq 2s - 1$ since $|L(u)| = 2s$, and so

$$\max I_1(x_1) = x_1 + s - 1 < x_{2s} - (s - 1) = \min I_1(x_{2s}). \tag{3.15}$$

Since $L(v_1) \cap I_1(x_1) \neq \emptyset \neq L(v_1) \cap I_1(x_{2s})$,

$$\min L(v_1) \leq \max I_1(x_1) < \min I_1(x_{2s}) \leq \max L(v_1),$$

and so $\min L(v_1) \notin I_1(x_{2s})$ and $\max L(v_1) \notin I_1(x_1)$. Again, $\max L(v_1) - \min L(v_1) \geq 2s - 1$ since $|L(v_1)| = 2s$, and so

$$\max I_2(x_{2s}) = \min L(v_1) + s - 1 < \max L(v_1) - (s - 1) = \min I_2(x_1).$$

Since $L(v_2) \cap I_2(x_1) \neq \emptyset \neq L(v_2) \cap I_2(x_{2s})$,

$$\min L(v_2) \leq \max I_2(x_{2s}) < \min I_2(x_1) \leq \max L(v_2),$$

and so $\min L(v_2) \notin I_2(x_1)$ and $\max L(v_2) \notin I_2(x_{2s})$. Repeating the above process, we see that for $1 \leq q \leq m$,

$$\max I_{2q-1}(x_1) < \min I_{2q-1}(x_{2s})$$

$$\text{and} \quad \max I_{2q}(x_{2s}) < \min I_{2q}(x_1).$$

Since $y_i \in I_{2m}(x_1)$ and $y_j \in I_{2m}(x_{2s})$, it follows that $y_i > y_j$ and thus $i > j$, which establishes (3.13). The proof of (3.14) is very similar. $\square$

It follows from (3.13) and (3.14) that no two of (x_1, y_1), (x_1, y_{2s}) and (x_{2s}, y_{2s}) can belong to the same F_h, and similarly if (x_1, y_{2s}) is replaced by (x_{2s}, y_1). So we may assume that

$$(x_1, y_1) \in F_1, \ (x_{2s}, y_{2s}) \in F_2, \ (x_1, y_{2s}) \in F_3, \ \text{and} \ (x_{2s}, y_1) \in F_3.$$

We need a final definition: the *border* of $L(u) \times L(w)$ is defined as

$$B = \big\{(x, y) \in L(u) \times L(w) : x \in \{x_1, x_{2s}\} \text{ or } y \in \{y_1, y_{2s}\}\big\};$$

it consists of the starred cells in Figure 3.2.

	x_1	x_2	$\cdots$	x_{2s}
y_1	*	*	*	*
y_2	*			*
$\vdots$	*			*
y_{2s}	*	*	*	*

Figure 3.2: The border B of $L(u) \times L(w)$.

It follows from (3.13), (3.14) and convexity ((3.6) and (3.7)) that

$$F_1 \cap B = \{(x_t, y_1) : 1 \le t \le i\} \cup \{(x_1, y_t) : 1 \le t \le j\} \qquad (3.16)$$

for some values $1 \le i, j \le 2s - 1$. Now we use Lemma 3.11 to bound the size of the set $F_1 \cap B$. It can be seen that if $i + j > 2s$, (3.16) precisely contradicts the statement of Lemma 3.11 for the path P_1. Hence $i + j \le 2s$, and $|F_1 \cap B| \le 2s - 1$.

By symmetry (interchanging the roles of u and w, I_p and J_p, etc.) we can deduce the corresponding properties of F_2:

$$F_2 \cap B = \{(x_t, y_{2s}) : 2s - k + 1 \le t \le 2s\}$$

$$\cup \{(x_{2s}, y_t) : 2s - l + 1 \le t \le 2s\}. \qquad (3.17)$$

Since Lemma 3.11 is equally valid with the word 'maximum' substituted for 'minimum', we deduce that $k + l \leq 2s$, and hence $|F_2 \cap B| \leq 2s - 1$.

Finally we consider $F_3 \cap B$. Since we are assuming (3.12) holds, F_3 must contain those elements of B not in F_1 or F_2, that is,

$$
\begin{aligned}
F_3 \cap B \supseteq\ & \{(x_t, y_1) : i + 1 \leq t \leq 2s\} \\
& \cup \{(x_{2s}, y_t) : 1 \leq t \leq 2s - l\} \\
& \cup \{(x_1, y_t) : j + 1 \leq t \leq 2s\} \\
& \cup \{(x_t, y_{2s}) : 1 \leq t \leq 2s - k\}.
\end{aligned}
\tag{3.18}
$$

Note that (3.13) and (3.14) respectively imply the following:

$$
(x_1, y_t) \in F_3 \implies (x_{2s}, y_t) \notin F_3
$$

$$
\text{and } (x_t, y_1) \in F_3 \implies (x_t, y_{2s}) \notin F_3,
$$

which together with (3.18) show that $i + 1 > 2s - k$ and $j + 1 > 2s - l$, so that $i + k \geq 2s$ and $j + l \geq 2s$. But since $i + j \leq 2s$ and $k + l \leq 2s$, we must in fact have equality throughout (as well as in (3.18)):

$$
i + j = k + l = i + k = j + l = 2s.
$$

The third and second terms in (3.18) respectively give $\{y_{j+1}, \ldots, y_{2s}\} \subseteq I_{2c}(x_1)$ and $\{y_1, \ldots, y_{2s-l}\} \subseteq I_{2c}(x_{2s})$ (recalling that the path P_3 has length $2c$), so that $|I_{2c}(x_1)| \geq 2s - j = i$ and $|I_{2c}(x_{2s})| \geq 2s - l = j$. Repeated

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application of (3.8) then shows that, for $p = 1, \ldots, 2c - 1$,

$$|L(v_p) \cap I_p(x_1)| \geq i \text{ and } |L(v_p) \cap I_p(x_{2s})| \geq j. \tag{3.19}$$

However, since $I_1(x_1) \cap I_1(x_{2s}) = \emptyset$, the sum of the left-hand sides in (3.19) when $p = 1$ is at most $|L(v_1)| = 2s = i + j$, and so equality must hold (and $L(v_1) \subseteq I_1(x_1) \cup I_1(x_{2s})$). In particular, by (3.15), this tells us that $\min L(v_1) \in I_1(x_1)$ and $\max L(v_1) \in I_1(x_{2s})$. We now use the 'exchange' trick as in the proof of Lemma 3.11, using the facts that $(x_1, y_{2s}) \in F_3$ and $(x_i, y_{2s}) \in F_3$, by (3.18):

$$
\begin{aligned}
\min L(v_1) \in I_1(x_1) &\implies \max L(v_1) \notin I_1(x_1) \\
&\implies \max L(v_1) \in J_1(y_{2s}) \qquad \text{by (3.5)} \\
&\implies \min L(v_1) \notin J_1(y_{2s}) \\
&\implies \min L(v_1) \in I_1(x_i) \qquad \text{by (3.5)} \\
&\implies \min L(v_1) \geq x_i - (s - 1) \tag{3.20} \\
&\implies |L(v_1) \cap I_1(x_1)| \leq (2s - 1) - (x_i - x_1) \\
&\qquad\qquad\qquad\qquad \leq 2s - i = j. \tag{3.21}
\end{aligned}
$$

Combined with (3.19) this gives $i \leq j$. But similar reasoning using the facts that $\max L(v_1) \in I_1(x_{2s})$, $(x_{2s}, y_1) \in F_3$ and $(x_{i+1}, y_1) \in F_3$ shows that also $j \leq |L(v_1) \cap I_1(x_{2s})| \leq i$. Hence we must have equality throughout:

$$i = j = k = l = s. \tag{3.22}$$

Figure 3.3 summarises what we know about each $F_h \cap B$ at this point.

	x_1	$\cdots$	x_s	x_{s+1}	$\cdots$	x_{2s}
y_1	F_1	F_1	F_1	F_3	F_3	F_3
$\vdots$	F_1					F_3
y_s	F_1					F_3
y_{s+1}	F_3					F_2
$\vdots$	F_3					F_2
y_{2s}	F_3	F_3	F_3	F_2	F_2	F_2

Figure 3.3: The intersection of the border B of $L(u) \times L(w)$ with each F_h.

The conclusion (3.22) shows that we must also have equality in (3.20) and (3.21), as well as in (3.19) for each p. Hence $\min L(v_1) = x_s - (s-1) = x_1$ and $|L(v_1) \cap I_1(x_1)| = s$, and symmetrically, $\max L(v_1) = x_{s+1} + s - 1 = x_{2s}$ and $|L(v_1) \cap I_1(x_{2s})| = s$. Thus we deduce that

$$L(u) = L(v_1) = [x_1, x_1 + s - 1] \cup [x_{2s} - (s-1), x_{2s}].$$

We can now use the definitions of $I_p(x)$ directly:

$$I_2(x_1) = \bigcap_{x_{2s}-(s-1) \leq c \leq x_{2s}} [c - (s-1), c + s - 1] = [x_{2s} - (s-1), x_{2s}]$$

$$\text{and } I_2(x_{2s}) = \bigcap_{x_1 \leq c \leq x_1 + s - 1} [c - (s-1), c + s - 1] = [x_1, x_1 + s - 1].$$

Since we have equality in (3.19), $|L(v_2) \cap I_2(x_1)| = |L(v_2) \cap I_2(x_{2s})| = s$, and it follows that $L(v_2) = L(v_1)$ as above. Continuing this process we see that $L(u) = L(v_1) = L(v_2) = \cdots = L(v_{2c-1}) = L(w)$, which completes the proof of Lemma 3.12. $\qquad\square$

It follows from the above proof that if there is no colouring of $G = \Theta_{2a,2b,2c}$ with separation s from lists of size $2s$, the list assignment must in fact be constant on one of the paths P_1, P_2 or P_3. Note that we can indeed have $\mathrm{ch}_s(G) > 2s$ if $b > 1$ and $c > 1$. A list assignment illustrating this for $s = 1$ and $G = \Theta_{2,4,4}$ is shown in Figure 3.4, which generalises to any $s \in \mathbb{N}$ if each colour i in each list is replaced with the interval $[is, is + s - 1]$.

$$
\begin{array}{ccc}
 & 01 & \\
02 & & 12 \\
23 & 01 & 23 \\
03 & & 13 \\
 & 01 &
\end{array}
$$
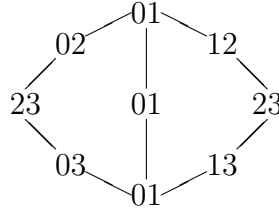
Figure 3.4: A list assignment showing that $\mathrm{ch}(\Theta_{2,4,4}) > 2$.

Proof of Theorem 3.9. First, we observe that the technique of pruning vertices of degree 1 also works for separation s. Suppose G is assigned lists of size $2s$, and we remove a vertex v of degree 1 and colour $G - v$ from its lists. We must choose a colour for v not contained in $[c(w) - (s - 1), c(w) + s - 1]$, where w is the sole neighbour of v in G, which we can do because the size of this interval is $2s - 1$. Thus if H is the core of G,

$$
\mathrm{ch}_s(G) \leq 2s \iff \mathrm{ch}_s(H) \leq 2s.
$$

Applying Theorem 3.10, it suffices to consider the cases $H = K_1$, $H = C_{2m+2}$ ($m \geq 1$), and $H = \Theta_{2,2,2m}$ ($m \geq 1$). If $H = K_1$ then G is 1-degenerate (i.e. a tree), and the result follows from Theorem 3.6. We now establish the

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claims $\mathrm{ch}_s(C_{2m+2}) \leq 2s$ and $\mathrm{ch}_s(\Theta_{2,2,2m}) \leq 2s$ simultaneously by induction on m.

Note that $C_{2m+2} \subset \Theta_{2,2,2m}$, and so at each step we need only show that $\mathrm{ch}_s(H) \leq 2s$ for $H = \Theta_{2,2,2m}$. Let the vertices of H be assigned lists $L(v)$ of size $2s$. Then by Lemma 3.12, we know H has an L-colouring with separation s if $L(u) \neq L(w)$. So assume $L(u) = L(w)$, and form the graph H' from H by identifying the vertices u and w (to give a new vertex u' with $N(u') = N(u) \cup N(w)$), and set $L(u') = L(u)$. Then any colouring of H' with separation s will yield a colouring of H by setting $c(u) = c(w) = c(u')$.

If $m = 1$, then $H' \cong K_{1,3}$ (see Figure 3.5, top) and H' has an L-colouring with separation s by Theorem 3.6. If $m > 1$, then $\mathrm{core}(H') \cong C_{2m}$ (Figure 3.5, bottom), and H' has a colouring by the inductive hypothesis. $\square$

We remark that the case $\mathrm{ch}_s(C_{2m+2}) \leq 2s$ was first established by Tesman [24, Theorem 3.10], and the inductive argument in the above proof can be avoided by using this result.

3.3   Planar and outerplanar graphs

We investigate the maximum value which $\mathrm{ch}_s(G)$ can take where $s \in \mathbb{N}$ is fixed and G ranges over all planar or outerplanar graphs. Let $m_{\mathrm{pl}}(s)$ denote the maximum of $\mathrm{ch}_s(G)$ over all planar graphs G, and $m_{\mathrm{op}}(s)$ the maximum over all outerplanar graphs.

Every planar graph is 5-degenerate, and every outerplanar graph is 2-degenerate [33, pp.240,243]. Together with Theorem 3.6, this tells us that

$m_{\mathrm{pl}}(s) \leq 10s - 4$ and $m_{\mathrm{op}}(s) \leq 4s - 1$. For outerplanar graphs, we will use the next lemma to show that this is the correct value.

Lemma 3.13 *Let H be a graph with $\mathrm{ch}_s(H) = k$, and let $G = K_1 + tH$. Then $\mathrm{ch}_s(G) \leq k + 2s - 1$, and equality holds if $t \geq k + 2s - 2$.*

Proof. Let L be a $(k+2s-1)$-list assignment for G. Let u be the 'hub' vertex of G, and choose any $x \in L(u)$. Define $L'(v) = L(v) \setminus [x - (s-1), x + s - 1]$ for $v \in V \setminus \{u\}$. Then $|L'(v)| \geq k$ for each $v \in V \setminus \{u\}$, and so there is an L'-colouring with separation s of each copy of H. This gives an L-colouring of $G - u$ and leaves x as a valid colour for u, thus proving the inequality.

It now suffices to prove that equality holds when $t = k + 2s - 2$. Let $H_1, \ldots, H_t$ be the disjoint copies of H in $G - u$, and let L_i be a $(k-1)$-list assignment for H_i such that there is no L_i-colouring of H_i with separation s; we may assume that none of these lists contains any element smaller than
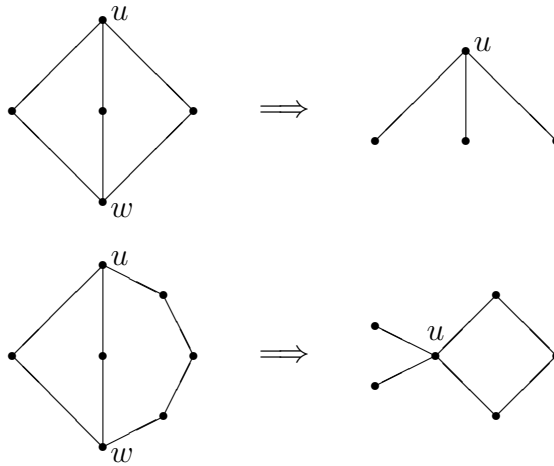
Figure 3.5: Forming H' from $H = \Theta_{2,2,2m}$ (for $m = 1, 2$).

$t + s$. Let $L(u) = [1, t]$, and for each $i = 1, \ldots, t$ and $v \in V(H_i)$, let $L(v) = [i - (s - 1), i + s - 1] \cup L_i(v)$. Then L is a t-list assignment for G.

Suppose c is an L-colouring of G with separation s; then $c(u) = i$ for some $i \in [1, t]$. But then $c(v) \notin [i - (s - 1), (i + s - 1)]$ for $v \in H_i$, and so $c(v) \in L_i(v)$. This gives an L_i-colouring of H_i with separation s, contradicting the definition of L_i. Thus no such colouring c exists of G, and so $\mathrm{ch}_s(G) > t = k + 2s - 2$, as required. $\qquad\qquad\square$

Theorem 3.14 *For every $s \in \mathbb{N}$, $m_{\mathrm{op}}(s) = 4s - 1$.*

Proof. We observed above that $m_{\mathrm{op}}(s) \leq 4s - 1$, and so we need an outerplanar graph G for which $\mathrm{ch}_s(G) = 4s - 1$. By Theorem 3.1(iii), $\mathrm{ch}_s(P_{2s}) = 2s$; so we apply Lemma 3.13 with $H = P_{2s}$ and $t = 4s - 2$, to give $G = K_1 + (4s - 1)P_{2s}$ and $\mathrm{ch}(G) = 4s - 1$ as required. $\qquad\qquad\square$

Series-parallel (K_4-minor-free) graphs are a larger class than outerplanar, and so $m_{\mathrm{sp}}(s) = \max\{\mathrm{ch}_s(G) : G$ is series-parallel$\} \geq m_{\mathrm{op}}(s) = 4s - 1$. On the other hand, series-parallel graphs are also 2-degenerate, and so $m_{\mathrm{sp}}(s) \leq 4s - 1$.

The result for planar graphs, as is often the case, is not so easily obtained. Thomassen [26] provided an elegant proof that $\mathrm{ch}(G) \leq 5$ for every planar graph G, and various authors [13, 18, 29] have given examples of planar graphs which are not 4-choosable; thus $m_{\mathrm{pl}}(1) = 5$. By generalising these non-4-choosable graphs we can show that $m_{\mathrm{pl}}(s) \geq 5s$, but for $s > 2$ we can improve upon this as in the following theorem.

Theorem 3.15 *For every $s \geq 2$, $6s - 2 \leq m_{\mathrm{pl}}(s) \leq 8s - 3$.*

Proof. The lower bound follows by applying Lemma 3.13 a second time to the graph G in Theorem 3.14. Alternatively, we can show that $\mathrm{ch}(G) > 6s - 3$ if $G = \overline{K}_2 + tP_{2s}$ and $t \geq (6s - 3)^2$, by adapting the proof of Lemma 3.13 to allow for two hub vertices.

Thomassen's proof [26] that $\mathrm{ch}(G) \leq 5$ for any planar graph G is easily adapted to show that $\mathrm{ch}_s(G) \leq 8s - 3$ for every $s \in \mathbb{N}$. Our proof, although obtained independently, is essentially identical to that of the same result given by Wallace in his Ph.D. thesis [30], and is therefore omitted. $\qquad\square$

Chapter 4

Consecutive list colouring

4.1 Introduction

In this introduction we define the concept of consecutive list colouring, and describe its relationship to some other graph colouring problems.

Consecutive list colouring can be regarded either as a generalisation of classical graph colouring, or as a restriction of list colouring. A *consecutive list assignment* is a function I which assigns to each vertex $v \in V$ a set $I(v) \subseteq \mathbb{Z}$ of consecutive integer colours. It is a *k-consecutive list assignment* if $|I(v)| = k$ for each $v \in V$. We can identify any k-consecutive list assignment with a function $a : V \to \mathbb{Z}$, by setting $I(v) = \{a(v), \ldots, a(v) + k - 1\}$. If a proper I-colouring of G exists for every k-consecutive list assignment I, then G is said to be *k-consecutive-choosable*, and the *consecutive choosability* $\mathrm{cch}(G)$ of G is the smallest k such that G is k-consecutive-choosable.

Given the relationships between the three graph colouring problems de-

scribed above, it is clear that

$$\chi(G) \leq \mathrm{cch}(G) \leq \mathrm{ch}(G) \tag{4.1}$$

for every graph G, since if G can be coloured from any lists of size k, then it can certainly be coloured from any consecutive lists of size k; and if G can be coloured from any consecutive lists of size k, then it can certainly be coloured from the list $\{1, \ldots, k\}$ assigned to each vertex.

As noted in Section 1.4, $\chi(G)$ and $\mathrm{ch}(G)$ can be arbitrarily far apart. Given this fact, we may be surprised by the following result.

Theorem 4.1 *For any graph G, $\mathrm{cch}(G) = \chi(G)$.*

Proof. Write $k = \chi(G)$; we need to show that $\mathrm{cch}(G) \leq k$. Let $h : V \to \{1, \ldots, k\}$ be a proper k-colouring of G. Now given any k-consecutive list assignment I, we wish to find a proper I-colouring of G. But for each $v \in V$ there must exist an integer $c(v) \in I(v)$ such that $c(v) \equiv h(v) \pmod{k}$. The function $c : V \to \mathbb{Z}$ thus defined is a proper I-colouring of G. $\qquad\square$

Hence $\mathrm{cch}(G)$ is not a new parameter. However, consecutive choosability becomes interesting when we introduce separation, as defined in Section 3.1. The *consecutive choosability with separation s*, denoted $\mathrm{cch}_s(G)$, is the smallest k such that for every k-consecutive list assignment I, there exists an I-colouring c with separation s, which we recall means that $|c(v) - c(w)| \geq s$ for all $vw \in E$.

Analogously to (4.1), we have the following pair of inequalities for any

graph G and any $s \in \mathbb{N}$:

$$\mathrm{sp}(G, s) \leq \mathrm{cch}_s(G) \leq \mathrm{ch}_s(G). \tag{4.2}$$

Consecutive choosability with separation was briefly studied by Tesman, in Section 3.3 of his Ph.D. thesis [24]. In his notation, as described in Section 3.1, our $\mathrm{cch}_s(G)$ corresponds to Tesman's $T_{s-1}\text{-cch}(G)$. However, his investigation of the topic only extends to a comparison of $\mathrm{ch}_s(G)$ and $\mathrm{cch}_s(G)$, showing that they are equal for trees and odd cycles, but not for all graphs (specifically, he shows that $\mathrm{ch}_s(C_4) > \mathrm{cch}_s(C_4)$ for $s = 7, 9, 11, 12, 13, \ldots$). No further results on consecutive choosability were published either in Tesman's thesis or elsewhere, although some of the known results in the general (non-consecutive) case can be applied here.

As we shall see, $\mathrm{cch}_s(G)$ can take only a small range of values for fixed values of s and $\chi(G)$; specifically,

$$s(\chi(G) - 1) + 1 \leq \mathrm{cch}_s(G) \leq s\chi(G).$$

This in turn motivates the definition of a new graph invariant, which we call the *consecutive choosability ratio*:

$$\tau(G) = \lim_{s \to \infty} \frac{\mathrm{cch}_s(G)}{s}.$$

In Section 4.2 we show that this limit exists, and find general relationships between $\tau(G)$, $\mathrm{cch}_s(G)$ and $\chi(G)$. In Section 4.3 we give an alternative char-

acterisation of $\tau(G)$ in terms of assignments of real intervals, and show that $\tau(G)$ is rational for all graphs G. Values of $\tau(G)$ for various classes of graphs G are determined in Section 4.4. The relationship between $\tau(G)$ and the circular chromatic number $\chi_{\mathrm{c}}(G)$, and the spectrum of τ, are studied in Section 4.5.

4.2 The consecutive choosability ratio $\tau(G)$

Upper and lower bounds on $\mathrm{cch}_s(G)$, in terms of s and $\chi(G)$, are relatively easy to obtain. Lemma 2.1 and equation (4.2) combine to give a lower bound on $\mathrm{cch}_s(G)$ as in the following theorem. We give a direct proof of the result, since it illustrates a method of proof which will be useful later.

Theorem 4.2 *For any graph G and any $s \in \mathbb{N}$,*

$$\mathrm{cch}_s(G) \geq s(\chi(G) - 1) + 1.$$

Proof. The result is trivial if G is empty (i.e. has no edges). Assuming G is not empty, assign to each vertex v the list $I(v) = \{0, \ldots, s(\chi(G) - 1) - 1\}$. We show that no I-colouring of G has separation s, and so $\mathrm{cch}_s(G) > s(\chi(G) - 1)$.

Let $c : V \to \mathbb{Z}$ be any I-colouring of G, and define $h : V \to \mathbb{Z}$ by setting $h(v) = \left\lfloor \frac{c(v)}{s} \right\rfloor$. Then $0 \leq h(v) \leq \chi(G) - 2$ for each $v \in V$. Now h cannot be a proper colouring of G, since it only uses $\chi(G) - 1$ colours. Thus there exists some edge $vw \in E$ such that $h(v) = h(w)$. Then $c(v), c(w) \in \{sh(v), \ldots, s(h(v) + 1) - 1\}$, and so $|c(v) - c(w)| \leq s - 1$. Thus c is not a colouring with separation s. $\qquad\square$

An upper bound for $\mathrm{cch}_s(G)$ can be obtained by emulating the proof of Theorem 4.1 above.

Theorem 4.3 *For any graph G and any $s \in \mathbb{N}$,*

$$\mathrm{cch}_s(G) \le s\chi(G).$$

Proof. Write $k = \chi(G)$, and let $h : V \to \{1, \ldots, k\}$ be a proper k-colouring of G. Now given any consecutive lists $I(v)$ of size sk, we wish to find a proper I-colouring of G with separation s. But for each $v \in V$ there must exist an integer $c(v) \in I(v)$ such that $c(v) \equiv sh(v) \pmod{sk}$. The function $c : V \to \mathbb{Z}$ thus defined is an I-colouring of G with separation s. $\qquad\square$

Thus we have the following bounds on $\mathrm{cch}_s(G)$:

$$s(\chi(G) - 1) + 1 \le \mathrm{cch}_s(G) \le s\chi(G), \tag{4.3}$$

and dividing through by s, we see that

$$\chi(G) - 1 < \frac{\mathrm{cch}_s(G)}{s} \le \chi(G). \tag{4.4}$$

A natural question now arises: does the central term $\frac{\mathrm{cch}_s(G)}{s}$ in (4.4) tend to a limit as $s \to \infty$? To show that it does, we need the following lemma which relates the values of $\mathrm{cch}_s(G)$ and $\mathrm{cch}_r(G)$ for any pair of natural numbers $s, r \in \mathbb{N}$. Note that this result is a generalisation of (4.3), since $\mathrm{cch}_1(G) = \mathrm{cch}(G) = \chi(G)$ by Theorem 4.1.

Lemma 4.4 *For any graph G and any $s, r \in \mathbb{N}$,*

$$\left\lfloor \frac{r}{s}(\text{cch}_s(G) - 1) \right\rfloor + 1 \leq \text{cch}_r(G) \leq \left\lceil \frac{r}{s}\text{cch}_s(G) \right\rceil .$$

Proof. Write $c_s = \text{cch}_s(G)$. Firstly, note that the upper and lower bounds are in fact equivalent:

$$c_r \leq \left\lceil \frac{r}{s} c_s \right\rceil \iff c_r \leq \frac{r}{s} c_s + \frac{s-1}{s} \iff \frac{s}{r} c_r \leq c_s + \frac{s-1}{r}$$

$$\iff \frac{s}{r}(c_r - 1) - \frac{r-1}{r} + 1 \leq c_s \iff \left\lfloor \frac{s}{r}(c_r - 1) \right\rfloor + 1 \leq c_s ,$$

where the first and last implications hold because c_r and c_s are integers. So it suffices to prove the upper bound $c_r \leq \left\lceil \frac{r}{s} c_s \right\rceil$. This means that, given any function $a : V \to \mathbb{Z}$, we need to show that there is an I-colouring with separation r from the lists $I(v) = \{a(v), \ldots, a(v) + \left\lceil \frac{r}{s} c_s \right\rceil - 1\}$.

To do this we define a scaling function $p : \mathbb{Z} \to \mathbb{Z}$ on the integers, by setting $p(i) = \left\lfloor \frac{r}{s} i \right\rfloor$. We will make use of the easily proved inequalities

$$\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a + b \rfloor \leq \lfloor a \rfloor + \lceil b \rceil. \tag{4.5}$$

Let $J(v)$ be the consecutive list $\{i \in \mathbb{Z} : p(i) \in I(v)\}$. Each $J(v)$ has size at least c_s. To see this, let $x = \min J(v)$. Then $p(x - 1) \leq a(v) - 1$, and so

$$p(x - 1 + c_s) \leq a(v) - 1 + \left\lceil \frac{r}{s} c_s \right\rceil = \max I(v),$$

using the second inequality in (4.5). Thus $\{x, \ldots, x + c_s - 1\} \subseteq J(v)$, and so $|J(v)| \geq c_s$.

By the definition of $c_s = \text{cch}_s(G)$, it follows that there exists a J-colouring $h : V \to \mathbb{Z}$ with separation s. So set $c(v) = p(h(v))$. This gives an I-colouring since $c(v) \in p(J(v)) \subseteq I(v)$. To show that it has separation r, let vw be any edge of G. Then w.l.o.g. $h(w) \geq h(v) + s$, and so

$$c(w) \geq p(h(v) + s) \geq p(h(v)) + p(s) = c(v) + r,$$

using the first inequality in (4.5). Thus c is an I-colouring with separation r, and this completes the proof of Lemma 4.4. $\qquad\square$

Using Lemma 4.4 we can now assert that $\frac{\text{cch}_s(G)}{s}$ does indeed tend to a limit as $s \to \infty$, and define the *consecutive choosability ratio* $\tau(G)$ to equal this limit.

Theorem 4.5 *For any graph G, the limit*

$$\tau(G) = \lim_{s \to \infty} \frac{\text{cch}_s(G)}{s}$$

exists. Furthermore, $\chi(G) - 1 \leq \tau(G) \leq \chi(G)$.

Proof. Writing $c_s = \text{cch}_s(G)$ as before, the result of Lemma 4.4 implies that $\frac{r}{s}(c_s - 1) < c_r < \frac{r}{s}c_s + 1$, so that

$$\frac{c_s}{s} - \frac{1}{s} < \frac{c_r}{r} < \frac{c_s}{s} + \frac{1}{r}$$

and therefore

$$\left| \frac{c_r}{r} - \frac{c_s}{s} \right| < \frac{1}{\min\{r, s\}} \ .$$

Hence $\left(\frac{\mathrm{cch}_s(G)}{s} \right)_{s \in \mathbb{N}}$ is a Cauchy sequence, and so converges in $\mathbb{R}$ to a limit $\tau(G)$. Moreover, since by (4.4) each of the terms in the sequence lies in the real interval $(\chi(G) - 1, \chi(G)]$, it follows that $\chi(G) - 1 \leq \tau(G) \leq \chi(G)$. $\square$

We can obtain a small but significant improvement to our upper bound on $\tau(G)$. In the proof of Theorem 4.3, the colouring c used only multiples of s. We could equally have chosen to use members of any congruence class modulo s, and we use this choice to our advantage in the following lemma.

Lemma 4.6 *If G is a graph on n vertices and $s \in \mathbb{N}$, then*

$$\mathrm{cch}_s(G) \leq s\chi(G) - \left\lceil \frac{s\chi(G)}{n} \right\rceil + 1.$$

Proof. Write $k = \chi(G)$ and $t = \lceil sk/n \rceil - 1 < sk/n$, and let $h : V \to \{1, \ldots, k\}$ be a proper k-colouring of G. Given any consecutive lists $I(v)$ of size $sk - t$, we want to produce an I-colouring of G with separation s. Choose $b \in \{0, \ldots, sk - 1\}$. For each vertex v there exists $c(v) \in I(v)$ such that

$$c(v) \equiv b + sh(v) \pmod{sk}, \tag{4.6}$$

for all but t possible choices of b. But since $nt < sk$, there is some choice of b for which a suitable $c(v)$ can be chosen for every vertex v. With this b, (4.6) gives an I-colouring of G with separation s as required. $\square$

Dividing the result of Lemma 4.6 through by s, and taking the limit as $s \to \infty$, we obtain the upper bound in the following theorem.

Theorem 4.7 *For any graph G on n vertices,*

$$\chi(G) - 1 \leq \tau(G) \leq \chi(G)\left(1 - \tfrac{1}{n}\right). \qquad \square$$

Observe that the lower and upper bounds on $\tau(G)$ in Theorem 4.7 are equal if $\chi(G) = n$, i.e. if G is a complete graph:

Corollary 4.8 $\tau(K_n) = n - 1$. $\qquad \square$

As another consequence of Theorem 4.7, we can now state that $\tau(G)$ is a *refinement* of $\chi(G)$. By this we mean that if we know $\tau(G)$ for a finite graph G, we can obtain $\chi(G)$ immediately, since $\tau(G) < \chi(G) \leq \tau(G) + 1$ and so

$$\chi(G) = \lfloor \tau(G) \rfloor + 1.$$

(This need not be true if G is infinite, and $\tau(G)$ is not strictly a refinement of $\chi(G)$ for infinite graphs. We return to this observation in Section 4.5.) On the other hand, two graphs with the same chromatic number can have different values of $\tau(G)$. Thus, like the circular chromatic number $\chi_{\mathrm{c}}(G)$, $\tau(G)$ contains more information about the structure of G than $\chi(G)$. The relationship between $\tau(G)$ and $\chi_{\mathrm{c}}(G)$ is discussed in Section 4.5.

In fact, as we shall see in Theorem 4.10, $\tau(G)$ contains precisely enough information to tell us the value of $\mathrm{cch}_s(G)$ for *every* $s \in \mathbb{N}$ (recall that

$\mathrm{cch}_1(G) = \mathrm{cch}(G) = \chi(G))$. In the result of Lemma 4.4, note that the expressions inside the floor $\lfloor \cdots \rfloor$ and ceiling $\lceil \cdots \rceil$ are integers when r is a multiple of s. Hence if $r = ps$, we can write

$$p(\mathrm{cch}_s(G) - 1) + 1 \leq \mathrm{cch}_{ps}(G) \leq p\,\mathrm{cch}_s(G). \tag{4.7}$$

As in Lemma 4.6, we can obtain a small improvement to this upper bound.

Lemma 4.9 *If G is a graph on n vertices and $p, s \in \mathbb{N}$, then*

$$\mathrm{cch}_{ps}(G) \leq p\,\mathrm{cch}_s(G) - \lceil p/n \rceil + 1.$$

Proof. Set $t = \lceil p/n \rceil - 1 < p/n$ and write $c_s = \mathrm{cch}_s(G)$ as before. Given any consecutive lists $I(v)$ of size $pc_s - t$, we want to produce an I-colouring of G with separation ps. Choose $b \in \{0, \ldots, p-1\}$ and for each vertex v, let $J(v)$ be the consecutive list $\{i \in \mathbb{Z} : b + ip \in I(v)\}$.

Then $|J(v)| = c_s$, except for t of the possible values of b, for which $|J(v)| = c_s - 1$. But since $nt < p$, there is some choice of b for which all the lists $J(v)$ have size c_s. Hence for this b, there exists a J-colouring $h : V \to \mathbb{Z}$ with separation s. Now set $c(v) = b + ph(v)$ to obtain an I-colouring of G with separation ps. $\qquad\qquad\square$

Note that if we specialise Lemma 4.9 to the case $s = 1$, we only obtain

$$\mathrm{cch}_p(G) \leq p\,\mathrm{cch}_1(G) - \lceil p/n \rceil + 1 = p\chi(G) - \lceil p/n \rceil + 1,$$

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which is not as strong as Lemma 4.6: our improvement to the upper bound is missing a factor of $\chi(G)$. Nevertheless, Lemma 4.9 is enough to imply the following theorem, which determines any $\mathrm{cch}_s(G)$ in terms of $\tau(G)$.

Theorem 4.10 *For any graph G and any $s \in \mathbb{N}$,*

$$\mathrm{cch}_s(G) = \lfloor s\,\tau(G) \rfloor + 1.$$

Proof. Combining equation (4.7) and Lemma 4.9, we see that

$$p(\mathrm{cch}_s(G) - 1) + 1 \le \mathrm{cch}_{ps}(G) \le p\,\mathrm{cch}_s(G) - \lceil p/n \rceil + 1.$$

Dividing through by p, and taking the limit as $p \to \infty$, we obtain

$$\mathrm{cch}_s(G) - 1 \le s\tau(G) \le \mathrm{cch}_s(G) - 1/n.$$

Since $\mathrm{cch}_s(G)$ is an integer, this implies the result of the theorem. $\qquad\square$

4.3 Real intervals and the rationality of $\tau(G)$

So far the lists we have considered have been sets of consecutive integers. Instead of increasing the separation s, we can equivalently fix the separation to be 1, and choose the elements of $I(v)$ at intervals of $1/s$ along the real line. Now in the limit as $s \to \infty$, we can regard $I(v)$ as approximating a closed interval of real numbers.

For $\kappa \in \mathbb{R}^+$, a *κ-interval assignment* is a function Γ which assigns to

each vertex $v \in V$ a closed real interval $\Gamma(v)$ of length κ. We can identify any κ-interval assignment with a function $\alpha : V \to \mathbb{R}$ by setting $\Gamma(v) = [\alpha(v), \alpha(v) + \kappa]$. A Γ-colouring $\gamma : V \to \mathbb{R}$ of G is *proper* if $|\gamma(v) - \gamma(w)| \geq 1$ for all $vw \in E$.

We define the *interval choosability* $\tau^*(G)$ to be the infimum of all $\kappa \in \mathbb{R}^+$ such that for every κ-interval assignment Γ, there exists a proper Γ-colouring. As the following lemma shows, the infimum in the definition of $\tau^*(G)$ is always attained, and so can be replaced by a minimum.

Lemma 4.11 *For any graph G, let $\kappa = \tau^*(G)$. Then for every κ-interval assignment Γ, there exists a proper Γ-colouring.*

Proof. Given any function $\alpha : V \to \mathbb{R}$ we need to construct a proper Γ-colouring from the intervals $\Gamma(v) = [\alpha(v), \alpha(v) + \kappa]$. For each $r \in \mathbb{N}$, $\tau^*(G) < \kappa + \frac{1}{r}$ and so we know that there exists a proper colouring $\gamma_r : V \to \mathbb{R}$ from the intervals $\Gamma_r(v) = \left[\alpha(v), \alpha(v) + \kappa + \frac{1}{r}\right]$.

Write $V = \{v_1, \ldots, v_n\}$. The vectors $\left(\gamma_r(v_1), \ldots, \gamma_r(v_n)\right)$ $(r = 1, 2, \ldots)$ form a bounded sequence in $\mathbb{R}^n$, and so by the Bolzano-Weierstrass Theorem, there is a subsequence converging to a limit $\left(\gamma(v_1), \ldots, \gamma(v_n)\right)$. This defines a colouring $\gamma : V \to \mathbb{R}$ of G. If $v_i v_j \in E$ then $|\gamma_r(v_i) - \gamma_r(v_j)| \geq 1$ for each $r \in \mathbb{N}$, and so $|\gamma(v_i) - \gamma(v_j)| \geq 1$; thus γ is a proper colouring of G. Furthermore, for each $v_i \in V$,

$$\gamma(v_i) \in \bigcap_{r \in \mathbb{N}} \left[\alpha(v_i), \, \alpha(v_i) + \kappa + \tfrac{1}{r}\right] = [\alpha(v_i), \, \alpha(v_i) + \kappa] = \Gamma(v_i),$$

and so $\gamma : V \to \mathbb{R}$ is a Γ-colouring of G. $\qquad\square$

Our next task is to show that the consecutive choosability ratio $\tau(G)$ and the interval choosability $\tau^*(G)$ are in fact the same for all graphs G.

Theorem 4.12 *For any graph G, $\tau(G) = \tau^*(G)$.*

Proof. First, we show that $\tau^*(G) \leq \tau(G)$. For $s \in \mathbb{N}$, let $\kappa = \frac{\mathrm{cch}_s(G)}{s}$. Then given any κ-interval assignment $\Gamma(v) = [\alpha(v), \alpha(v) + \kappa]$ for G, define consecutive lists $I(v) = [s\alpha(v), s(\alpha(v) + \kappa)] \cap \mathbb{Z}$. Each of the lists $I(v)$ contains at least $\mathrm{cch}_s(G)$ elements, and so there exists an I-colouring $c : V \to \mathbb{Z}$ of G with separation s. Now $\gamma(v) = c(v)/s$ defines a proper Γ-colouring of G and so, using Theorem 4.10,

$$\tau^*(G) \leq \kappa = \frac{\mathrm{cch}_s(G)}{s} = \frac{\lfloor s\,\tau(G) \rfloor + 1}{s} \leq \tau(G) + \frac{1}{s}. \tag{4.8}$$

Since (4.8) holds for any $s \in \mathbb{N}$, we deduce that $\tau^*(G) \leq \tau(G)$.

For the reverse inequality, take $s \in \mathbb{N}$ and let $k = \lfloor s\tau^*(G) \rfloor + 1$. Given any k-consecutive list assignment $I(v) = \{a(v), \ldots, a(v) + k - 1\}$, we will show that there exists an I-colouring of G with separation s. Let $\Gamma(v) = [a(v)/s,\ a(v)/s + \tau^*(G)]$. By Lemma 4.11 we know that there exists a proper Γ-colouring $\gamma : V \to \mathbb{R}$. Now let $c(v) = \lfloor s\gamma(v) \rfloor$ for $v \in V$. Then $a(v) \leq c(v) \leq a(v) + \lfloor s\tau^*(G) \rfloor$, since $a(v)$ is an integer, and so $c : V \to \mathbb{Z}$ defines an I-colouring of G. To show that it has separation s, let vw be any edge of G. Then w.l.o.g. $\gamma(w) \geq \gamma(v) + 1$, and so $c(w) = \lfloor s\gamma(w) \rfloor \geq \lfloor s\gamma(v) + s \rfloor = \lfloor s\gamma(v) \rfloor + s = c(v) + s$. Thus c has the required properties.

Hence $\mathrm{cch}_s(G) \leq \lfloor s\tau^*(G) \rfloor + 1$ for any $s \in \mathbb{N}$. Dividing through by s and taking the limit as $s \to \infty$, we see that $\tau(G) \leq \tau^*(G)$ as required. $\qquad\square$

Theorem 4.12 tells us that the above definition of interval choosability is simply an alternative definition of $\tau(G)$. When determining values of $\tau(G)$ for various graphs G in later sections, sometimes it will be easier to phrase the proof in terms of consecutive lists of integers, and in other cases it will be easier to use real intervals.

We conclude this section by showing that $\tau(G)$ is rational, and that its lowest denominator is at most the number of vertices of G. To do this we introduce yet another definition of $\tau(G)$, which can be seen to be equivalent to that given above for $\tau^*(G)$. Given a graph $G = (V, E)$ and a function $\alpha : V \to \mathbb{R}$, let $\kappa_G(\alpha)$ be the infimum of all $\kappa \in \mathbb{R}^+$ such that there exists a proper Γ-colouring from the intervals $\Gamma(v) = [\alpha(v), \alpha(v) + \kappa]$. Then it is easy to see that

$$\tau(G) = \tau^*(G) = \sup_{\alpha:V\to\mathbb{R}} \kappa_G(\alpha). \tag{4.9}$$

We can express $\kappa_G(\alpha)$ as the infimum of

$$\max\{\gamma(v) - \alpha(v) : v \in V\} \tag{4.10}$$

over all proper colourings $\gamma : V \to \mathbb{R}$ such that $\gamma(v) \geq \alpha(v)$ for all $v \in V$. For example, if $\alpha(v) = 0$ for all $v \in V$ then (4.10) is minimised by setting $\gamma(v) = c(v)$, where $c : V \to \{0, \ldots, \chi(G) - 1\}$ is any proper $\chi(G)$-colouring of G, and hence $\kappa_G(\alpha) = \chi(G) - 1$.

To simplify notation, we identify functions $\alpha : V \to \mathbb{R}$ with vectors $\mathbf{x} \in \mathbb{R}^n$ by setting $x_i = \alpha(v_i)$, where $v_1, \ldots, v_n$ is a fixed ordering of V. This allows us

to regard κ_G as a function from $\mathbb{R}^n$ to $\mathbb{R}$.

Theorem 4.13 *For any graph G on n vertices, $\tau(G)$ is rational, and can be expressed as $\tau(G) = p/q$ with $q \leq n$.*

Proof. Write $\tau = \tau(G)$. We assume that G is not empty, so that $\tau > 0$ and κ_G is not identically zero. Fix $\mathbf{x} \in \mathbb{R}^n$ and an ordering $v_1, \ldots, v_n$ of the vertices, and let $e_{ij} = 1$ if $v_i v_j \in E$ and $e_{ij} = 0$ otherwise.

We seek a colouring $\gamma : V \to \mathbb{R}$ such that $\gamma(v_i) \geq x_i$ for $1 \leq i \leq n$, and which minimises (4.10). To do this, for each permutation $\sigma \in S_n$ we find such a colouring which minimises (4.10) subject to the additional constraint

$$\gamma(v_{\sigma(1)}) \leq \gamma(v_{\sigma(2)}) \leq \ldots \leq \gamma(v_{\sigma(n)}).$$

This is achieved by assigning, for each $i = 1, \ldots, n$ in turn, the smallest colour $\gamma(v_{\sigma(i)}) \geq x_{\sigma(i)}$ such that $\gamma(v_{\sigma(i)}) - \gamma(v_{\sigma(j)}) \geq e_{\sigma(i)\,\sigma(j)}$ for every $j < i$. Taking $\sigma = \mathrm{id}(S_n)$ for clarity, this algorithm proceeds as follows:

$$\gamma(v_1) = x_1,$$
$$\gamma(v_2) = \max\{x_2, \gamma(v_1) + e_{12}\} = \max\{x_2, x_1 + e_{12}\},$$
$$\gamma(v_3) = \max\{x_3, \gamma(v_1) + e_{13}, \gamma(v_2) + e_{23}\}$$
$$= \max\{x_3, x_1 + e_{13}, x_2 + e_{23}, x_1 + e_{12} + e_{23}\}, \ldots$$

and hence, for this colouring, we obtain the following expression for (4.10):

$$\max\{\gamma(v_i) - x_i : 1 \leq i \leq n\} = \max_{1 \leq j \leq i \leq n}\{x_j + t(i,j) - x_i\},$$

where each $t(i,j)$ is an integer between 0 and $i-j$ (so that $t(i,i)=0$). Finally, we minimise over all $\sigma \in S_n$ to obtain an expression for $\kappa_G(\mathbf{x})$:

$$\kappa_G(\mathbf{x}) = \min_{\sigma \in S_n} \max_{1 \leq j \leq i \leq n} \left\{ x_{\sigma(j)} + t_\sigma(i,j) - x_{\sigma(i)} \right\}. \tag{4.11}$$

Let F be the set of all the nonzero functionals $x_{\sigma(j)} + t_\sigma(i,j) - x_{\sigma(i)}$ in (4.11). (We will not need to consider the zero functionals, as we assumed that $\tau > 0$.)

From (4.11) we deduce that $\kappa_G : \mathbb{R}^n \to \mathbb{R}$ is continuous and piecewise linear. It follows that the supremum in (4.9) is attained, and we can choose $\mathbf{z} \in \mathbb{R}^n$ such that $\kappa_G(\mathbf{z}) = \tau$. Now let $S = \{f \in F : f(\mathbf{z}) = \tau\}$, and

$$A = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = g(\mathbf{x}) \ \forall f, g \in S\}.$$

Clearly, A is an affine subspace of $\mathbb{R}^n$, and $\mathbf{z} \in A$. Choose a representative $f \in S$, and let $\varepsilon = \min\{|g(\mathbf{z}) - \tau| : g \in F \setminus S\}$, and $N = \{\mathbf{x} \in A : \|\mathbf{x} - \mathbf{z}\| < \varepsilon/4\}$. Then for all $g \in F$ and $\mathbf{x} \in N$, $|g(\mathbf{x}) - g(\mathbf{z})| < \varepsilon/2$, and it follows that

$$g(\mathbf{x}) < f(\mathbf{x}) \ \text{ if } \ g(\mathbf{z}) < f(\mathbf{z}),$$
$$g(\mathbf{x}) > f(\mathbf{x}) \ \text{ if } \ g(\mathbf{z}) > f(\mathbf{z}),$$
$$\text{and } g(\mathbf{x}) = f(\mathbf{x}) \ \text{ if } \ g(\mathbf{z}) = f(\mathbf{z}) \ \text{(i.e. if } g \in S).$$

Together with (4.11), this implies that $\kappa_G(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in N$. But since $f(\mathbf{z}) = \tau = \sup\{\kappa_G(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ and f is linear, this implies that f has constant value τ on N and hence on A, and so do all the other functionals in

S since they are equal to f on A.

Let $T \subseteq S$ be minimal such that $A = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = g(\mathbf{x}) \; \forall f, g \in T\}$. Form a (multi)graph H with vertex set V, and an edge $e_r = v_{i_r} v_{j_r}$ for each of the functionals $f_r(\mathbf{x}) = x_{j_r} - x_{i_r} + t_r$ in T. We will now show that H contains a cycle (possibly in the form of a pair of parallel edges). So suppose this is not the case; then H is simple and 1-degenerate, that is, the vertices can be reordered as $v_1, \ldots, v_n$ so that each vertex v_i is adjacent to at most one vertex v_j with $j < i$. Choose $\lambda \in \mathbb{R}$ such that $\lambda \neq \tau$, and construct $\mathbf{x} \in \mathbb{R}^n$ as follows: set $x_1 = 0$; then for each $i \in \{2, \ldots, n\}$ in turn, set $x_i = x_{j_r} - \lambda + t_r$ if there is an edge $e_r = v_{i_r} v_{j_r}$ with $j_r < i_r = i$, and $x_i = 0$ otherwise. This ensures that $f(\mathbf{x}) = \lambda$ for each $f \in T$, and hence $\mathbf{x} \in A$. However, this is a contradiction since we showed above that $f(\mathbf{x}) = \tau$ for all $f \in S$ and $\mathbf{x} \in A$. This shows that H must contain a cycle C.

By relabelling the edges e_r of H (and the corresponding functionals f_r) we may assume that $C = e_1 \cdots e_m$, where $2 \leq m \leq n$. Now orient the edges of C so that it is a directed cycle; then let $s_m = +1$ if $\overrightarrow{e_r} = \overrightarrow{v_{i_r} v_{j_r}}$, and $s_m = -1$ if e_r has the opposite orientation. Then

$$\sum_{r=1}^{m} s_r f_r(\mathbf{z}) = \sum_{r=1}^{m} s_r t_r = p \in \mathbb{Z},$$

since we have oriented C so that the x_{i_r} and x_{j_r} terms cancel. But

$$\sum_{r=1}^{m} s_r f_r(\mathbf{z}) = \sum_{r=1}^{m} s_r \tau = q\tau,$$

where $q \in \mathbb{Z}$ and $|q| \leq m \leq n$. Note also that q cannot be zero, since this would imply a dependence among the functionals $f_1, \ldots, f_m$ and contradict the minimality of T. Hence $q \neq 0$, and $\tau(G) = \tau = p/q$ as required. $\qquad \square$

4.4 Determining $\tau(G)$

In this section we introduce various techniques for finding upper and lower bounds on $\tau(G)$, and use them to determine the value of $\tau(G)$ for various classes of graphs G.

4.4.1 Bipartite graphs

We begin by using results of Tesman [24] to determine $\tau(G)$ for all bipartite graphs G.

Theorem 4.14 *If G is a connected bipartite graph on n vertices, then*

$$\tau(G) = 2\left(1 - \tfrac{1}{n}\right).$$

Proof. For any $s \in \mathbb{N}$ and any tree B_n on n vertices, Tesman computes the value of $\mathrm{ch}_s(B_n)$, as given in Theorem 3.1(iii), and later observes that $\mathrm{cch}_s(B_n) = \mathrm{ch}_s(B_n)$ [24, Proposition 3.3.3]; thus $\mathrm{cch}_s(B_n) = \left\lfloor 2s\left(1 - \tfrac{1}{n}\right) \right\rfloor + 1$. Taking the limit as $s \to \infty$ we see that $\tau(B_n) = 2\left(1 - \tfrac{1}{n}\right)$.

Now let G be a connected bipartite graph on n vertices; since it contains a spanning tree B_n, we deduce that

$$2\left(1 - \tfrac{1}{n}\right) = \tau(B_n) \leq \tau(G) \leq 2\left(1 - \tfrac{1}{n}\right),$$

where the final inequality is the upper bound from Theorem 4.7. □

Since Theorem 4.14 is integral to much of what follows, we will present a complete proof for the case when G is a star graph. Tesman's proof of Theorem 3.1(iii) uses a similar method for star graphs, and proves the general result by a complicated induction on the tree-height of a rooted tree.

Proof of Theorem 4.14 for star graphs. Let $G = K_{1,r}$ with vertices $u, v_1, \ldots, v_r$. We want to show that $\tau(G) = 2\left(1 - \tfrac{1}{r+1}\right)$. By Theorem 4.10, this is equivalent to showing that $\mathrm{cch}_s(G) = \left\lfloor 2s(1 - \tfrac{1}{r+1}) \right\rfloor + 1$ for each $s \in \mathbb{N}$.

Let $s, t \in \mathbb{N}$ with $t < 2s$, and let I be an assignment of consecutive lists of size $2s - t$. For each $i \leq r$, suppose $I(v_i) = \{a_i, \ldots, a_i + 2s - t - 1\}$. Then there are precisely t integers $a_i + s - t, \ldots, a_i + s - 1$ whose distance from every element of $I(v_i)$ is less than s, and which therefore cannot be used to colour u. Let $F_i = \{a_i + s - t, \ldots, a_i + s - 1\}$ and $F = \bigcup_{i \leq r} F_i$; now the colour $c(u)$ we choose for u extends to a colouring of G iff $c(u) \notin F$.

Now suppose $I(u) = \{0, \ldots, 2s - t - 1\}$ and $a_i = it - s$ for $i \leq r$, so that $F = \{0, \ldots, rt - 1\}$. We see that $I(u) \subseteq F$ (and hence there is no I-colouring of G) if $2s - t \leq rt$, that is, if $t \geq \tfrac{2s}{r+1}$. Hence $\mathrm{cch}_s(G) > 2s\left(1 - \tfrac{1}{r+1}\right)$, and thus $\mathrm{cch}_s(G) \geq \left\lfloor 2s(1 - \tfrac{1}{r+1}) \right\rfloor + 1$ since $\mathrm{cch}_s(G)$ is an integer.

On the other hand, if $2s - t > 2s(1 - \frac{1}{r+1})$ then $|I(u)| = 2s - t > rt \geq |F|$, and hence we can always find a suitable $c(u) \in I(u) \setminus F$ which extends to a colouring of all of G. The upper bound $\mathrm{cch}_s(G) \leq \left\lfloor 2s(1 - \frac{1}{r+1}) \right\rfloor + 1$ follows. (Note that we can also deduce this from Lemma 4.6). $\qquad\square$

For a disconnected graph G, $\tau(G)$ is the maximum of $\tau(H)$ over all the connected components H of G. Hence the result of Theorem 4.14 holds for all bipartite graphs G by setting n equal to the number of vertices in the largest component of G.

4.4.2 Some graphs for which $\tau(G) = \chi(G) - 1$

We know that for any graph G, $\tau(G) \geq \chi(G) - 1$. One class of graphs for which this bound is attained is the class of colour-critical graphs. A graph G is *k-colour-critical* if $\chi(G) = k$, but $\chi(G - v) < k$ for every $v \in V$.

Theorem 4.15 *If G is a k-colour-critical graph then $\tau(G) = k - 1$.*

Proof. We use the characterisation of $\tau(G)$ based on real intervals. Let Γ be any $(k{-}1)$-interval assignment for G. Suppose w.l.o.g. that the smallest element in any of the intervals $\Gamma(v)$ is 0, and choose $u \in V$ such that $\Gamma(u) = [0, k - 1]$. For $v \in V \setminus \{u\}$, define $I(v) = \Gamma(v) \cap \mathbb{N}$, so that $|I(v)| \geq k - 1$. Then $G - u$ has a proper I-colouring c by Theorem 4.1, since $\mathrm{cch}_1(G - u) = \chi(G - u) = k - 1$. Finally, c extends to a Γ-colouring of G by setting $c(u) = 0$. $\qquad\square$

This result allows us to determine $\tau(G)$ for odd cycles and (generalised) odd wheels.

Corollary 4.16 *For $r \in \mathbb{N}$, $\tau(C_{2r+1}) = 2$ and $\tau(K_k + C_{2r+1}) = k + 2$.* $\qquad\square$

Another class of graphs for which $\tau(G) = \chi(G) - 1$ consists of graphs of the form $G = K_k + \overline{K}_r$, where $k \geq 2$ and $r \geq 1$. To prove this result we use the following lemma, which is quite similar to Theorem 4.15 above, although it does not seem to be possible to deduce one from the other.

Lemma 4.17 *Let $G = (V, E)$ be a graph with $u \in V$. If a proper I-colouring exists for every assignment I of consecutive lists such that $|I(u)| \geq \chi(G)$ and $|I(v)| \geq \chi(G) - 1$ for $v \in V \setminus \{u\}$, then $\tau(G) = \chi(G) - 1$.*

Proof. Again, we use the characterisation of $\tau(G)$ based on real intervals. Let $k = \chi(G) - 1$ and let Γ be any k-interval assignment for G. Suppose w.l.o.g. that $\Gamma(u) = [0, k]$, and let $I(v) = \Gamma(v) \cap \mathbb{Z}$ for $v \in V$. Then $|I(v)| \geq k$ for $v \in V$, and $|I(u)| = k + 1 = \chi(G)$. Thus I is an assignment of consecutive lists satisfying the conditions stated in the Lemma, and hence there exists a proper I-colouring, which in turn is also a Γ-colouring. $\square$

Theorem 4.18 *For $k, r \in \mathbb{N}$,*

$$\tau(K_k + \overline{K}_r) = \begin{cases} \frac{2r}{r+1} & k = 1 \\ k & k \geq 2. \end{cases}$$

Proof. The result for $k = 1$ follows from Theorem 4.14 for bipartite graphs. So assume that $k \geq 2$, and let $H = K_k$ and $G = H + \overline{K}_r$.

Let $v_1, \ldots, v_k$ be the vertices of H; we will apply Lemma 4.17 to G, with $u = v_k$. Given a consecutive list assignment I for G such that $|I(v_k)| = k + 1$ and $|I(v)| = k$ for $v \in V(G) \setminus \{v_k\}$, let $a_i = \min I(v_i)$ for $i = 1, \ldots, k$. If $a_i = a$ for all $i \leq k$, then colour H by setting $c(v_i) = a + i - 1$ for $i = 1, \ldots, k - 1$ and

$c(v_k) = a + k$. On the other hand, if the a_i are not all the same, we do not need the fact that $|I(v_k)| > k$, and so we rearrange the vertices of H so that $a_1 \leq a_2 \leq \cdots \leq a_k$ and colour the vertices of H by setting $c(v_i) = a_i + i - 1$ for $i = 1, \ldots, k$.

This gives a proper I-colouring of H, and ensures that the set S of k colours assigned to its vertices is not a list of consecutive integers. Now for each remaining vertex $v \in G \setminus H$, the interval $I(v)$ contains k integers which *are* consecutive. Hence $I(v) \neq S$, and there exists some colour in $I(v) \setminus S$ which we can assign to v. This completes the I-colouring of G. Hence, by Lemma 4.17, $\tau(G) = \chi(G) - 1 = k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.4.3 More techniques for bounding $\tau(G)$ from above

Lemma 4.17 uses consecutive list assignments for G where the lists are not all the same size to prove that $\tau(G) \leq \chi(G) - 1$. In the following two generalisations we use the same technique to establish more general upper bounds.

Lemma 4.19 *Let $G = (V, E)$ be a graph with $u \in V$. If an I-colouring with separation s exists for every assignment I of consecutive lists such that $|I(u)| \geq k + 1$ and $|I(v)| \geq k$ for $v \in V \setminus \{u\}$, then $\tau(G) \leq k/s$.*

Proof. Let Γ be any (k/s)-interval assignment for G, and suppose w.l.o.g. that $\Gamma(u) = [0, k/s]$. For $v \in V$, write $\Gamma(v) = [\alpha(v), \alpha(v) + k/s]$, and let $I(v) = [s\alpha(v), s\alpha(v) + k] \cap \mathbb{Z}$. Then $|I(v)| \geq k$ for $v \in V$, and $|I(u)| = k + 1$. Thus I satisfies the conditions stated in the Lemma, and hence there exists an

I-colouring c with separation s. Finally, we obtain a proper Γ-colouring γ by setting $\gamma(v) = c(v)/s$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.20 *Let $G = (V, E)$ be a graph, and let $U \subseteq V$ and $1 \leq q \leq |U|$. Suppose that whenever $Q \subseteq U$ and $|Q| = q$, a proper I-colouring exists for every assignment I of consecutive lists such that $|I(v)| \geq \chi(G)$ for $v \in Q$ and $|I(v)| \geq \chi(G) - 1$ for $v \in V \setminus Q$. Then*

$$\tau(G) \leq \chi(G) - 1 + \frac{q-1}{|U|}.$$

Proof. Write $m = |U|$ and let $\kappa = \chi(G) - 1 + (q-1)/m + \delta$, where $\delta > 0$. Let Γ be any κ-interval assignment for G. We will choose $\beta \in [0, 1)$ and let $I_\beta(v) = \Gamma(v) \cap (\mathbb{Z} + \beta)$ for $v \in V$, where $\mathbb{Z} + \beta = \{n + \beta : n \in \mathbb{Z}\}$. Note that whatever β we choose, $|I_\beta(v)| \geq \chi(G) - 1$ for all $v \in V$.

For $v \in U$, let $f_v(\beta) = 1$ if $|I_\beta(v)| = \chi(G)$ and $f_v(\beta) = 0$ otherwise, and let $f(\beta) = \sum_{v \in U} f_v(\beta)$. Then $f_v(\beta) = 1$ iff $\beta \in [\alpha(v), \alpha(v) + (q-1)/m + \delta]$ (mod 1), where $\Gamma(v) = [\alpha(v), \alpha(v) + \kappa]$, and so

$$\int_{[0,1)} f_v(\beta)\mathrm{d}\beta = (q-1)/m + \delta;$$

and summing over $v \in U$, we obtain

$$\int_{[0,1)} f(\beta)\mathrm{d}\beta = q - 1 + m\delta.$$

Thus there exists $\beta \in [0, 1)$ such that $f(\beta) \geq q - 1 + m\delta$, and hence $f(\beta) \geq q$ since $f(\beta) \in \mathbb{Z}$. It follows that for this choice of β, there is a set $Q \subseteq U$ of

size q such that $|I(v)| \geq \chi(G)$ for all $v \in Q$. Thus I_β is an assignment of lists of consecutive integers (shifted by β) satisfying the conditions stated in the lemma, and so there exists a proper I_β-colouring, which in turn is also a Γ-colouring. Hence $\tau(G) \leq \chi(G) - 1 + (q-1)/m + \delta$, and since $\delta > 0$ was arbitrary, the result of the lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As an application of Lemma 4.20, we now determine $\tau(G)$ for graphs of the form $G = K_1 + rK_k$, formed by taking r disjoint copies of K_k and joining them to a single additional vertex.

Theorem 4.21 *For $k, r \in \mathbb{N}$, let $G = K_1 + rK_k$. Then $\tau(G) = k + 1 - \frac{2}{r+1}$.*

Proof. If $r = 1$, then $G = K_{k+1}$ and the result follows from Corollary 4.8; if $k = 1$, then G is a star graph and the result follows from Theorem 4.14. So assume $k \geq 2$ and $r \geq 2$. Let u be the universal vertex in G, and let $\{v_{ij} : 1 \leq i \leq r, 1 \leq j \leq k\}$ be the remaining vertices, such that $v_{i1}, \ldots, v_{ik}$ induce a complete graph for each $i \leq r$. We will first establish that $\tau(G) \geq k + 1 - \frac{2}{r+1}$.

Let H be the subgraph of G induced by the vertices $U = \{u, v_{11}, \ldots, v_{r1}\}$; then $H \cong K_{1,r}$ and $\tau(H) = 2 - \frac{2}{r+1}$. Let $\kappa = 2 - \frac{2}{r+1} - \delta$, where $\delta > 0$. Then there exists a function $\alpha : U \to \mathbb{R}$ such that there is no proper Γ-colouring of H from the intervals $\Gamma(v) = [\alpha(v), \alpha(v) + \kappa]$. Extend the function α to all of V by setting $\alpha(v_{ij}) = \alpha(v_{i1})$ for all i, j.

We will now show by contradiction that there is no proper Γ'-colouring of G from the intervals $\Gamma'(v) = [\alpha(v), \alpha(v) + k - 1 + \kappa]$. Suppose such a colouring

$\gamma' : V \to \mathbb{R}$ exists. For each $i \leq r$ we may assume w.l.o.g. that

$$\gamma'(v_{i1}) \leq \gamma'(v_{i2}) - 1 \leq \gamma'(v_{i3}) - 2 \leq \cdots \leq \gamma'(v_{ik}) - k + 1,$$

using the fact that these vertices are mutually adjacent. Thus for $1 \leq j \leq k$, $\gamma'(v_{i1}) - j + 1 \leq \gamma'(v_{ij}) \leq \gamma'(v_{ik}) - k + j$, and so

$$\alpha(v_{i1}) + j - 1 \leq \gamma'(v_{ij}) \leq \alpha(v_{i1}) + j - 1 + \kappa.$$

Let $t = \min\{\lfloor \gamma'(u) - \alpha(u) \rfloor, k - 1\}$. Then $\gamma'(u) \in [\alpha(u) + t, \alpha(u) + t + \kappa]$, and for $1 \leq i \leq r$, $\gamma'(v_{i\,t+1}) \in [\alpha(v_{i1}) + t, \alpha(v_{i1}) + t + \kappa]$. But this allows us to construct a proper Γ-colouring γ of H by setting $\gamma(u) = \gamma'(u) - t$ and $\gamma(v_{i1}) = \gamma'(v_{i\,t+1}) - t$ for $1 \leq i \leq r$. This contradiction implies that $\tau(G) > k - 1 + \kappa = k + 1 - \frac{2}{r+1} - \delta$, and hence $\tau(G) \geq k + 1 - \frac{2}{r+1}$, since $\delta > 0$ was arbitrary.

To show that $\tau(G) \leq k + 1 - \frac{2}{r+1}$ we use Lemma 4.20, with U as above and $q = r$. There are two cases to consider for the set Q: either $Q = U \setminus u$, or $Q = U \setminus \{v_{i1}\}$, where w.l.o.g. $i = 1$. Now given an assignment I of consecutive lists as in Lemma 4.20, noting that $\chi(G) = k + 1$, we can properly I-colour the vertices of G in the following order if $Q = U \setminus u$:

$$u; \; v_{12}, \ldots, v_{1k}, v_{11}; \; v_{22}, \ldots, v_{2k}, v_{21}; \; \ldots \; v_{r2}, \ldots, v_{rk}, v_{r1};$$

and in the following order if $Q = U \setminus \{v_{11}\}$:

$$v_{11}, \ldots, v_{1k}; \;\; u; \;\; v_{22}, \ldots, v_{2k}, v_{21}; \;\; v_{32}, \ldots, v_{3k}, v_{31}; \;\; \ldots \;\; v_{r2}, \ldots, v_{rk}, v_{r1}.$$

As we come to colour each vertex, the size of its list is strictly greater than the number of its neighbours which are already coloured, and so we can complete the colouring. Thus, by Lemma 4.20, $\tau(G) \le \chi(G) - 1 + \frac{r-1}{r+1} = k + 1 - \frac{2}{r+1}$ as required. $\qquad\square$

4.4.4   Complete multipartite graphs

We begin this section with the result that the upper bound $\tau(G) \le \chi(G)\big(1 - \frac{1}{n}\big)$ is obtained for balanced complete multipartite graphs. The proof uses the following important lemma, which follows easily from Theorem 4.10.

Lemma 4.22 *For any graph G and any $s \in \mathbb{N}$,*

$$\tau(G) \ge \frac{\mathrm{cch}_s(G) - 1}{s}.$$

Proof. By Theorem 4.10, $\mathrm{cch}_s(G) \le s\,\tau(G) + 1$. Rearranging this expression gives the above result. $\qquad\square$

Theorem 4.23 *For $k, r \ge 2$, if $G = K_{r,\ldots,r}$ is the complete k-partite graph with r vertices in each class, then $\tau(G) = k - \frac{1}{r}$.*

Proof. Since G has kr vertices, Theorem 4.7 gives an upper bound of $\tau(G) \le k(1 - \frac{1}{kr}) = k - \frac{1}{r}$. We use Lemma 4.22 to obtain the lower bound. Let

$\{v_{ij} : 1 \leq i \leq r, 1 \leq j \leq k\}$ be the vertices of G, such that $v_{1j}, \ldots, v_{rj}$ is a colour class for each j. Assign consecutive lists $I(v_{ij}) = \{i, \ldots, i+kr-2\}$; this is illustrated for $G = K_{2,2,2}$ in Figure 4.1. We will suppose that c is an I-colouring of G with separation r, and derive a contradiction.

$$I(v_{1j}) = \{1, 2, 3, 4, 5\}$$

$$I(v_{2j}) = \{2, 3, 4, 5, 6\}$$
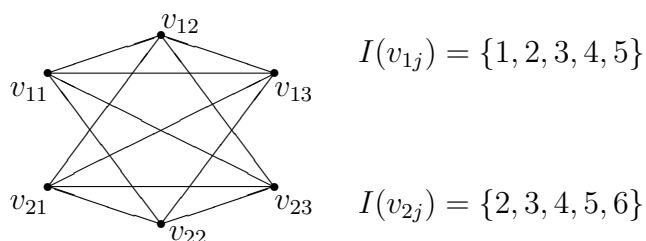
Figure 4.1: $K_{2,2,2}$ with lists assigned as in Theorem 4.23.

The vertices $v_{r1}, \ldots, v_{rk}$ are pairwise adjacent and are all assigned the same list $\{r, \ldots, r+kr-2\}$. By our assumption that c is an I-colouring of G with separation r, any two of the k colours received by these vertices must differ by at least r, and so the smallest of these colours must be at most $r + kr - 2 - (k-1)r = 2r - 2$. So w.l.o.g. we may suppose that $r \leq c(v_{r1}) \leq 2r - 2$. Now let $i = c(v_{r1}) - r + 1$ and observe that $1 \leq i \leq r - 1$.

The vertices v_{r1} and $v_{i2}, \ldots, v_{ik}$ are pairwise adjacent. Since c is an I-colouring of G, we have $c(v_{ij}) \geq i$ for $2 \leq j \leq k$; but since $i > c(v_{r1}) - r$ and c has separation r, we must in fact have $c(v_{ij}) \geq c(v_{r1}) + r = i + 2r - 1$ for $2 \leq j \leq k$. Any two of the $k-1$ colours received by the vertices $v_{i2}, \ldots, v_{ik}$ must differ by at least r, and so the largest of these colours must be at least $i + 2r - 1 + (k-2)r = i + kr - 1$. This exceeds the largest colour $i + kr - 2$ in the any of these vertices, contradicting our assumptions about c.

The above argument shows that $\mathrm{cch}_r(G) > kr - 1$, that is, $\mathrm{cch}_r(G) \geq kr$. Thus by Lemma 4.22, $\tau(G) \geq \frac{kr-1}{r} = k - \frac{1}{r}$ as required. $\qquad\square$

Note that not every complete multipartite graph attains the upper bound $\tau(G) = \chi(G)\left(1 - \frac{1}{n}\right)$, as evidenced by Theorem 4.18. In our next theorem we show that if the size of each colour class of G is smaller than the number of classes, then the bound is only attained when G is balanced.

Theorem 4.24 *Let $k \geq 2$, and let $r_1, \ldots, r_k \in \mathbb{N}$ be such that $r_i < k$ for each $i \in \{1, \ldots, k\}$. Let $G = K_{r_1, \ldots, r_k}$ and $n = |V(G)| = \sum_{i=1}^{k} r_i$. Then $\tau(G) \leq k\left(1 - \frac{1}{n}\right)$, and equality holds iff $r_1 = r_2 = \cdots = r_k$.*

Proof. The inequality follows from Theorem 4.7 since $\chi(G) = k$, and if $r_1 = r_2 = \cdots = r_k$ then equality follows from Theorem 4.23. Conversely, suppose $\tau(G) = k\left(1 - \frac{1}{n}\right)$; then Theorem 4.10 implies that $\mathrm{cch}_n(G) = nk - k + 1$. We will prove the theorem by showing that this implies that $r_1 = \ldots = r_k$.

Let $V_1, \ldots, V_k$ be the colour classes, so that $|V_i| = r_i$. Draw n $(k \times k)$-grids, where the k rows in each grid correspond to the colour classes, and the nk columns correspond to the elements of $\{0, \ldots, nk - 1\}$ in such a way that the columns of each grid correspond to a residue class modulo n. This is illustrated in Figure 4.2 for $G = K_{1,1,2}$ ($k = 3$, $r_1 = r_2 = 1$, $r_3 = 2$, $n = 4$).

By our assumption, there exists an $(nk - k)$-consecutive list assignment I for G for which there is no I-colouring of G with separation n. For each $v \in V$, $v \in V_i$ for some $i \in \{1, \ldots, k\}$; for each of the k elements $x \in \{0, \ldots, nk - 1\}$ such that $I(v)$ contains no colour congruent to x modulo nk, draw a cross in the cell (V_i, x). Since $k \leq n$ and the lists are consecutive, we draw a cross in

at most one cell of each grid for each vertex. Also, the total number of crosses (counting multiple crosses in the same cell) is exactly nk.

We call a set of k ticks in one of the grids a *solution set* if each row and each column of that grid contains exactly one tick, and no cell contains both a tick and a cross. An example of a solution set is given in Figure 4.2. Given a solution set, we can construct an I-colouring of G with separation n as follows: for each class V_i, there is a single $t_i \in \{0, \ldots, nk - 1\}$ such that the cell (V_i, t_i) contains a tick. Then for each V_i and each $v \in V_i$, we choose $c(v) \in I(v)$ such that $c(v) \equiv t_i \pmod{nk}$, which gives the required colouring.

Since we assumed there is no I-colouring of G with separation n, there is also no solution set. Looking at each grid, we can deduce that there must be at least k crosses in that grid; since there are nk crosses overall, we deduce that there are exactly k crosses in each grid, and hence in each grid, the k crosses must form either a complete row or a complete column.

However, each vertex contributes at most one cross to each grid, and so there are at most $|V_i| = r_i < k$ crosses in each row of each grid. This means that the k crosses in each grid must form a column, and hence that there are

Figure 4.2: The grids defined in Theorem 4.24, for $G = K_{1,1,2}$.

n crosses in each row. Thus $r_1 = r_2 = \cdots = r_k = \frac{n}{k}$, as required. $\square$

We now turn our attention to the complete tripartite graphs $G = K_{1,r,r}$. Determining $\tau(G)$ for these graphs will enable us to do the same for even wheels $G = K_1 + C_{2r}$ as an immediate corollary.

Theorem 4.25 *For $r \geq 2$, $\tau(K_{1,r,r}) = 3 - \frac{2}{r+1}$.*

Proof. Let $G = K_{1,r,r}$. The lower bound $\tau(G) \geq 3 - \frac{2}{r+1}$ follows from Theorem 4.21, since $K_1 + rK_2 \subset G$.

Let the vertex set be $V = \{u\} \cup \{v_1, \ldots, v_r\} \cup \{v_{r+1}, \ldots, v_{2r}\}$, and let I be an assignment of consecutive lists such that $|I(u)| = 3r + 2$ and $|I(v_i)| = 3r + 1$ for $1 \leq i \leq 2r$. We will show that there must exist an I-colouring with separation $s = r + 1$, and hence $\tau(G) \leq \frac{3r+1}{r+1} = 3 - \frac{2}{r+1}$ by Lemma 4.19. Suppose w.l.o.g. that $I(u) = \{0, \ldots, 3r + 1\}$, and let $J = \{0, \ldots, r\} \cup \{2r + 2, \ldots, 3r + 1\} \subset I(u)$. Note that $|J| = 2r + 1$, and that $x \in J$ implies $x + r + 1 \notin J$. We will choose $b \in J$, and construct an I-colouring such that $c(u) = b$ and $c(v_i) \equiv b$ (mod $r + 1$) for $i = 1, \ldots, 2r$. Let

$$D_i = \{d \in \mathbb{Z} \setminus \{0\} : b + d(r+1) \in I(v_i)\};$$

we exclude zero from D_i, since we cannot have $c(v_i) = b$. The list $I(v_i)$ must contain at least two multiples of $r + 1$, and so $D_i \neq \emptyset$; furthermore, $|D_i| \geq 2$ unless $I(v_i) \subseteq \{b - r, \ldots, b + 2r + 1\}$ or $I(v_i) \subseteq \{b - (2r + 1), \ldots, b + r\}$. Equivalently, writing $I(v_i) = \{a(v_i), \ldots, a(v_i) + 3r\}$ for each $i = 1, \ldots, 2r$, we

have $|D_i| \geq 2$ unless $b \in F_i$, where

$$F_i = \{a(v_i) + r - 1,\ a(v_i) + r,\ a(v_i) + 2r,\ a(v_i) + 2r + 1\}.$$

Since $x \in J$ implies $x + r + 1 \notin J$, we have $|J \cap F_i| \leq 2$ for $i = 1, \ldots, 2r$. Summing over i, we have

$$\sum_{i=1}^{2r} |J \cap F_i| \leq 4r < 4r + 2 = 2|J|.$$

We can re-express this as

$$\sum_{b \in J} |\{i : b \in F_i\}| < 2|J|,$$

and hence, for some $b \in J$, there is at most one $i \in \{1, \ldots, 2r\}$ for which $b \in F_i$, and hence $|D_i| = 1$. With this choice of b, we may suppose that w.l.o.g. $|D_1| \geq 1$, and $|D_i| \geq 2$ for $i = 2, \ldots, 2r$. Observe that if $|D_i| \geq 2$ then either D_i contains two consecutive (non-zero) integers, or D_i contains both -1 and $+1$. Partitioning $\mathbb{Z} \setminus \{0\}$ into sets $S = \{\ldots, -6, -4, -2, 1, 3, 5, \ldots\}$ and $T = \{\ldots, -5, -3, -1, 2, 4, 6, \ldots\}$, it follows that $D_i \cap S \neq \emptyset$ and $D_i \cap T \neq \emptyset$ for $i = 2, \ldots, 2r$.

Let $d_1 \in D_1$; w.l.o.g. $d_1 \in S$. Then choose $d_i \in D_i$ such that $d_i \in S$ for $2 \leq i \leq r$ and $d_i \in T$ for $r + 1 \leq i \leq 2r$. Finally, we have an I-colouring of G with separation $r + 1$ as required, by setting $c(u) = b$ and $c(v_i) = b + d_i(r + 1)$ for $i = 1, \ldots, 2r$. By Lemma 4.19, this completes the proof of the theorem. $\square$

It follows that $\tau(G) = 3 - \frac{2}{r+1}$ for any graph G such that $K_1 + rK_2 \subseteq G \subseteq K_{1,r,r}$, which in particular is the case for even wheels $K_1 + C_{2r}$.

Corollary 4.26 *For $r \geq 2$, $\tau(K_1 + C_{2r}) = 3 - \frac{2}{r+1}$.* $\square$

4.4.5   The Petersen graph

As an application of two of the above lemmas, we determine the value of the consecutive choosability ratio for the Petersen graph P. It has ten vertices and chromatic number 3, which by Theorem 4.7 tells us that $2 \leq \tau(P) \leq 27/10$. We will use various properties of the Petersen graph and its automorphism group $\text{Aut}(P)$ without proof; proofs can be found, for example, in the book [15].
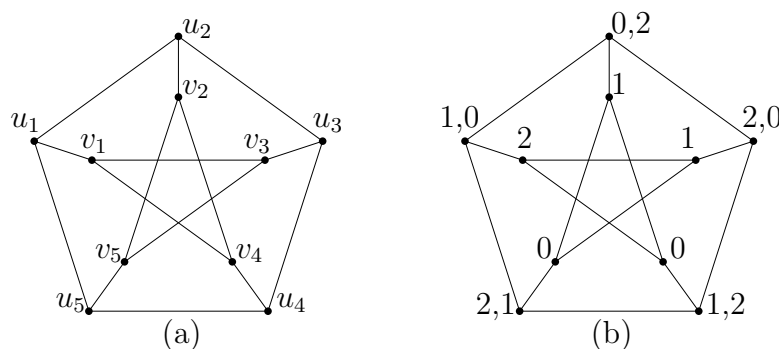

Figure 4.3: The Petersen graph P.

Theorem 4.27 $\tau(P) = 5/2$.

Proof. First we show that $\text{cch}_2(P) > 5$. With the vertices labelled as in Figure 4.3(a), assign lists $I(u_i) = \{0, \ldots, 4\}$ and $I(v_i) = \{1, \ldots, 5\}$ $(i = 1, \ldots, 5)$. Since adjacent colours must differ by at least 2, the sequence of colours 0,2,4

must occur, either clockwise or anticlockwise, around the outer pentagon; we may assume w.l.o.g. that $c(u_1) = 0$, $c(u_2) = 2$, and $c(u_3) = 4$. This implies that $c(v_2) \in \{4, 5\}$ and $c(v_3) \in \{1, 2\}$, and hence $c(v_5) = 3$; however, this leaves no valid colour for the vertex u_5. Hence $\mathrm{cch}_2(P) \geq 6$, and so by Lemma 4.22, $\tau(P) \geq 5/2$.

To prove that $\tau(P) \leq 5/2$ we use Lemma 4.20, with $U = V(P)$ and $q = 6$. Thus suppose we have six vertices with consecutive lists $I(v)$ of size 3, and four vertices, which we call *deficient* vertices, with lists of size 2; if we show that a proper colouring from these lists always exists, the required result will follow. For each $v \in V(P)$ we form a list $L(v) \subseteq \{0, 1, 2\}$ from $I(v)$ by replacing each element by its residue modulo 3, and seek an L-colouring of P.

We consider first the case where there are two adjacent deficient vertices; since P is edge-transitive, we may assume they are v_1 and v_3. Choose a third deficient vertex w; there is a 5-cycle $C \subset P$ containing v_1, v_3 and w, and since $\mathrm{Aut}(P)$ is transitive on 5-cycles, we may assume that $V(C) = \{v_1, \ldots, v_5\}$. Now C contains at least one vertex with a list of size 3, and so we can colour C from its lists; and since a 5-cycle has a unique 3-colouring up to rotation and permutation of the colours, we may assume that C is coloured as in Figure 4.3(b). Let c_1 and c_2 be the two colourings of P given in Figure 4.3(b), using the first and the second colour respectively from each pair of colours for the vertices u_i. Since at most one of the vertices u_i is deficient, one of c_1 and c_2 must give a proper L-colouring of P.

It remains to consider the case where the four deficient vertices form an independent set. There are five independent sets of size 4 in P, obtained from

each other by rotation of Figure 4.3(a), and so we may assume the deficient vertices are u_1, u_3, v_4 and v_5. One of the colours 0,1,2 must occur at least three times in the lists of these four vertices; we may assume it is 0. If 0 appears in all four of these lists, then c_2 as defined above is a proper L-colouring of P. Otherwise, 0 appears in exactly three of the lists; w.l.o.g. $0 \notin L(u_1)$, and in this case, swapping the colours of u_2 and u_3 under c_1 gives a proper L-colouring.

Hence an L-colouring c of P always exists, and we can form an I-colouring by choosing $h(v) \in I(v)$ such that $h(v) \equiv c(v) \pmod 3$. Hence the conditions of Lemma 4.20 are satisfied, and so $\tau(P) \leq 3 - 1 + \frac{6-1}{10} = 5/2$ as required. $\square$

4.4.6   Analogues of some well-known results

We first observe that the analogue for $\tau(G)$ of Brooks' Theorem (Theorem 2.3) is not difficult to derive.

Theorem 4.28 *For any graph G, $\tau(G) \leq \Delta(G)$. Furthermore, if G is connected, then equality holds iff G is a complete graph or an odd cycle.*

Proof. We may assume G is connected. If G is a complete graph or an odd cycle then $\tau(G) = \chi(G) - 1 = \Delta(G)$, as shown in the previous section. Otherwise, by Theorems 2.3 and 4.7, we have $\tau(G) < \chi(G) \leq \Delta(G)$. $\square$

The Four Colour Theorem tells us that $\chi(G) \leq 4$ if G is planar, and if G is outerplanar then $\chi(G) \leq 3$ since it is 2-degenerate, as observed in Section 3.3. It follows that $\tau(G) < 4$ for planar graphs, and $\tau(G) < 3$ for outerplanar graphs. Moreover, we can use Theorem 4.21 to get arbitrarily close to these bounds, since $K_1 + rK_3$ is planar and $K_1 + rK_2$ is outerplanar. By Theorem

4.10, this last observation also implies that the bounds $\text{cch}_s(G) \leq 4s$ for planar graphs and $\text{cch}_s(G) \leq 3s$ for outerplanar graphs can be attained for all $s \in \mathbb{N}$.

4.5   Relationship to circular colouring

We have alluded in previous sections to similarities between the consecutive choosability ratio $\tau(G)$ and the circular chromatic number $\chi_c(G)$. In this section we discuss these similarities, and use them to derive additional information about $\tau(G)$.

4.5.1   Circular colouring

The circular chromatic number was introduced by Vince [27], under the name 'star-chromatic number' and with a slightly different definition from that given below. We will only quote some known results regarding $\chi_c(G)$, and refer the reader to the survey by Zhu [36] for proofs and further information.

We begin by defining circular colourings and $\chi_c(G)$. For $\rho \in \mathbb{R}^+$, let $C = [0, \rho)$. A ρ-*circular colouring* of a graph G is a function $\phi : V \to C$ such that $1 \leq |\phi(v) - \phi(w)| \leq \rho - 1$ for all $vw \in E$. This is equivalent to considering C as a circle of length ρ, and ensuring that whenever $vw \in E$, the distance around the circle between $\phi(v)$ and $\phi(w)$ is at least 1. The *circular chromatic number* $\chi_c(G)$ is then defined as the infimum of all $\rho \in \mathbb{R}^+$ such that there exists a ρ-circular colouring of G.

The infimum in the above definition is always attained, and $\chi_c(G)$ is rational for all finite graphs G. Furthermore, $\chi_c(G)$ can be expressed as p/q with

$q \leq \alpha(G)$, where $\alpha(G)$ is the largest size of an independent set in G.

It can be shown that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and so $\chi(G) = \lceil \chi_c(G) \rceil$. Recalling that $\chi(G) - 1 \leq \tau(G) < \chi(G)$ and $\chi(G) = 1 + \lfloor \tau(G) \rfloor$, we see that $\chi_c(G)$ and $\tau(G)$ are both refinements of $\chi(G)$. (The applicability of these statements to infinite graphs is discussed below.)

The following result establishes a direct relationship between $\tau(G)$ and $\chi_c(G)$. It is a refinement of the upper bound in Theorem 4.7, since $\chi_c(G) \leq \chi(G)$ for all graphs G.

Theorem 4.29 *For any graph G on n vertices,*

$$\tau(G) \leq \chi_c(G)\left(1 - \tfrac{1}{n}\right).$$

Proof. Let $\rho = \chi_c(G)$ and $C = [0, \rho)$, and let $\phi : V \to C$ be a ρ-circular colouring of G. Given an interval assignment $\Gamma(v) = [\alpha(v), \alpha(v) + \rho(1 - \tfrac{1}{n})]$, we need to show that there exists a proper Γ-colouring γ of G. To do this we seek $\beta \in C$ such that for each $v \in V$, there exists $\gamma(v) \in \Gamma(v)$ such that

$$\gamma(v) \equiv \beta + \phi(v) \pmod{\rho}. \tag{4.12}$$

Regarding C as a circle of length ρ, for each $v \in V$ there is an open interval of length ρ/n of values of $\beta \in C$ for which no colour $\gamma(v)$ satisfies (4.12). But n open intervals of length ρ/n cannot cover a circle of length ρ, and so some $\beta \in C$ exists such that (4.12) can be satisfied for all $v \in V$. Furthermore, for each $vw \in E$, we have $\gamma(v) - \gamma(w) \equiv \phi(v) - \phi(w) \in [1, \rho - 1] \pmod{\rho}$ and so

$|\gamma(v) - \gamma(w)| \geq 1$. Thus γ is a proper Γ-colouring of G. $\square$

4.5.2   Infinite graphs

The inequalities $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ can be shown to hold for infinite as well as finite graphs; hence $\chi(G) = \lceil \chi_c(G) \rceil$, and $\chi_c(G)$ is a refinement of $\chi(G)$, for all graphs G.

In contrast, the statement $\chi(G) = 1 + \lfloor \tau(G) \rfloor$ need not hold if G is infinite, and hence $\tau(G)$ cannot be said to be a refinement of $\chi(G)$ for infinite graphs, as the following example illustrates. Let G_1 be an infinite collection of independent edges, and G_2 be an infinite path. Both graphs have chromatic number 2. However, $\tau(G_1) = 1$, since $\tau(H) = 1$ for each component $H \cong K_2$ of G_1; whereas $\tau(G_2) = 2$, since for each $n \in \mathbb{N}$, G_2 contains the finite path P_n on n vertices, and hence $\tau(G_2) \geq \tau(P_n) = 2\left(1 - \frac{1}{n}\right)$.

The equation $\chi(G) = 1 + \lfloor \tau(G) \rfloor$ fails to hold for G_2 because $\tau(G_2) = \chi(G_2) = 2$. More generally, for $k \in \mathbb{N}$ we can obtain an infinite graph G for which $\tau(G) = \chi(G) = k+1$, by taking infinitely many copies of K_k and joining them to a single additional vertex. The proof that $\tau(G) = k + 1$ follows from Theorem 4.21, since $K_1 + rK_k \subseteq G$ for all $r \in \mathbb{N}$.

4.5.3   The spectrum of τ

We conclude by considering the spectrum of τ, that is, the set of possible values of $\tau(G)$ over all finite graphs G. The corresponding question for $\chi_c(G)$ is answered by Vince [27]: for any rational $p/q \geq 2$, let G be the graph

with vertex set $\{0, \ldots, p-1\}$ and an edge xy iff $q \leq |x - y| \leq p - q$; then $\chi_c(G) = p/q$. On the other hand, $\chi_c(G)$ can take no non-integer value less than 2. Thus the set of values taken by $\chi_c(G)$ over all finite graphs G is

$$\{0, 1\} \cup \left(\mathbb{Q} \cap [2, \infty)\right).$$

We have already seen that $\tau(G)$ is rational, and that if G is bipartite then $\tau(G) = 2\left(1 - \frac{1}{n}\right)$, where n is the size of the largest component of G. On the other hand, if G is not bipartite then $\tau(G) \geq 2$, and we conjecture that $\tau(G)$, like $\chi_c(G)$, can take any rational value in this range.

Conjecture 4.30 *The set of values taken by $\tau(G)$ over all finite graphs G is*

$$\left\{2\left(1 - \tfrac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \left(\mathbb{Q} \cap [2, \infty)\right).$$

As a partial solution to this problem, we show that the spectrum of τ is *dense* in the interval $[2, \infty)$. Given two graphs H and K, the *lexicographic product* $H[K]$ is defined to be the graph with vertex set $V(H) \times V(K)$ and an edge between (u_1, v_1) and (u_2, v_2) if $u_1 u_2 \in E(H)$, or if $u_1 = u_2$ and $v_1 v_2 \in E(K)$. Informally, we think of each vertex of H as being 'inflated' into a copy of K.

Theorem 4.31 *For any natural numbers $p, q \in \mathbb{N}$ with $p > 2q$, let H be any graph with $\chi_c(H) = p/q$, and let $G = H[\overline{K}_q]$. Then*

$$\frac{p-1}{q} \leq \tau(G) < \frac{p}{q}.$$

Proof. Write $V(H) = \{v_1, \ldots, v_n\}$, and $V(G) = \{v_i^r : 1 \le i \le n, 1 \le r \le q\}$ where $v_i^r v_j^s \in E(G)$ iff $v_i v_j \in E(H)$.

Let $\gamma : V(H) \to [0, p/q)$ be a p/q-circular colouring of H. Then $\gamma' : V(G) \to [0, p/q)$ defined by $\gamma'(v_i^r) = \gamma(v_i)$ is a p/q-circular colouring of G. Thus $\chi_c(G) = \chi_c(H) = p/q$, and so $\tau(G) < p/q$ by Theorem 4.29.

To obtain the lower bound, we will show that $\mathrm{cch}_q(G) \ge p$; by Lemma 4.22, this implies that $\tau(G) \ge \frac{p-1}{q}$ as required. We define consecutive lists

$$I(v_i^r) = \{r - 1, \ldots, p + r - 3\},$$

and will show by contradiction that no I-colouring of G with separation q exists. So suppose c is such a colouring. Define $r_i = \max\{r : c(v_i^r) \le p - 2\}$ (which is well defined since $c(v_i^1) \le p - 2$) and a function $\gamma : V(H) \to \left[0, \frac{p-1}{q}\right)$ where $\gamma(v_i) = c(v_i^{r_i})/q$. We now show that γ is a $\frac{p-1}{q}$-circular colouring of H, which is a contradiction since $\chi_c(H) > \frac{p-1}{q}$.

Let $v_i v_j$ be any edge of H, and suppose that $c(v_j^{r_j}) \ge c(v_i^{r_i})$. We know that $c(v_j^{r_j}) - c(v_i^{r_i}) \ge q$, and thus $\gamma(v_j) - \gamma(v_i) \ge 1$; we also need to show that $\gamma(v_j) - \gamma(v_i) \le \frac{p-1}{q} - 1$, that is, $c(v_j^{r_j}) - c(v_i^{r_i}) \le p - 1 - q$. This condition must be satisfied if $c(v_i^{r_i}) \ge q - 1$, since $c(v_j^{r_j}) \le p - 2$; so suppose that $c(v_i^{r_i}) \le q - 2$. Then $r_i \le q - 1$, since $c(v_i^{r_i}) \ge r_i - 1$ by the definition of $I(v_i^{r_i})$, and thus $c(v_i^{r_i+1}) > p - 2 \ge c(v_j^{r_j})$ by the definition of r_i. Finally, since $v_i^{r_i+1} v_j^{r_j} \in E(G)$ and $c(v_i^{r_i+1}) \le p + r_i - 2$, we deduce that

$$c(v_j^{r_j}) \le c(v_i^{r_i+1}) - q \le p + r_i - 2 - q \le p + c(v_i^{r_i}) - 1 - q$$

as required, and so γ is a $\frac{p-1}{q}$-circular colouring of H. This contradiction completes the proof of the theorem. $\qquad\square$

Note that for any real $\kappa > 2$, we can use the Theorem 4.31 to generate sequences of graphs $(G_q)_{q\in\mathbb{N}}$ and $(H_q)_{q\in\mathbb{N}}$ such that $\tau(G_q) \to \kappa$ from below, by setting $p = \lfloor q\kappa \rfloor$ so that $\frac{p-1}{q} \leq \tau(G_q) < \frac{p}{q} \leq \kappa < \frac{p+1}{q}$, and $\tau(H_q) \to \kappa$ from above, by setting $p = \lceil q\kappa \rceil + 1$ so that $\frac{p-2}{q} < \kappa \leq \frac{p-1}{q} \leq \tau(H_q) < \frac{p}{q}$. Furthermore, by taking the disjoint union of the graphs G_q, we can generate an infinite graph $G = \bigcup_{q\in\mathbb{N}} G_q$ such that $\tau(G) = \kappa$.

4.6 Future research

An extensive literature has appeared regarding the circular chromatic number since its introduction [27], and it is hoped that this research will stimulate a similar level of interest in this new graph invariant.

A possible question is whether, for fixed $k \in \mathbb{N}$ with $k \geq 3$, there is a characterisation of those k-chromatic graphs G for which $\tau(G) = k - 1$. (For $k = 2$ it is simply those graphs with no two incident edges.) It would also be of interest to say precisely which complete k-partite graphs, and perhaps which graphs in general, attain the bound $\tau(G) = \chi(G)\left(1 - \frac{1}{n}\right)$.

It would be of interest to see Conjecture 4.30 settled, and particularly to obtain a construction of a graph G with any given rational value of $\tau(G) \geq 2$. Other possible questions related to the spectrum include the following: what rational values $2 \leq \tau(G) < 4$ can be obtained with G a planar graph, or $2 \leq \tau(G) < 3$ with G an outerplanar graph?

Chapter 5

The choosability ratio

5.1 Introduction

In Chapter 4 we defined the consecutive choosability ratio of a graph G to be the limit of $\frac{\mathrm{cch}_s(G)}{s}$ as $s \to \infty$. In this chapter we show that $\frac{\mathrm{ch}_s(G)}{s}$ also tends to a limit as $s \to \infty$, which we call the *choosability ratio* $\sigma(G)$ of G, and investigate this limit.

The generalisation from consecutive sets to arbitrary sets of integers makes the problem much more difficult; compare, for example, the relatively easily derived bounds on $\mathrm{cch}_s(G)$ in terms of $\chi(G)$ (equation 4.1) with our efforts to bound $\mathrm{ch}_s(G)$ in terms of $\mathrm{ch}(G)$ (Section 3.2). This difficulty is further manifested in that are unable to show directly, as we did in the consecutive case (Lemma 4.4 and Theorem 4.5), that $\left(\frac{\mathrm{ch}_s(G)}{s}\right)_{s \in \mathbb{N}}$ is a Cauchy sequence. The best direct analogue of Lemma 4.4 we can obtain is the following.

Lemma 5.1 *Let G be a graph and $s \geq r \geq 2$. Then*

$$\mathrm{ch}_s(G) \leq \left\lceil \frac{s-1}{r-1} \right\rceil (\mathrm{ch}_r(G) - 1) + 1.$$

Proof. Let $t = \left\lceil \frac{s-1}{r-1} \right\rceil$ and $k = \mathrm{ch}_r(G)$, and let L be a an assignment of lists for G of size $t(k-1) + 1$. For each vertex $v \in V$, define a new list

$$L'(v) = \big\{ x \in \mathbb{Z} : L(v) \cap \{xt, \ldots, (x+1)t - 1\} \neq \emptyset \big\}.$$

Then $|L'(v)| > k - 1$ since $|L(v)| > t(k-1)$, and so there is an L'-colouring h of G with separation r. For each $v \in V$ choose $c(v) \in L(v)$ such that $h(v)t \leq c(v) \leq (h(v)+1)t - 1$. Then for each edge $vw \in E$, w.l.o.g. $h(w) - h(v) \geq r$, and so $c(w) - c(v) \geq h(w)t - ((h(v)+1)t - 1) \geq (r-1)t + 1 \geq s$. Thus c is an L-colouring of G with separation s, as required. $\qquad\square$

If $r = 2$ then this result reduces to equation (3.1). In Section 5.2 we will improve on the bound of Lemma 5.1 when $s - 1$ is not a multiple of $r - 1$.

Due to the difficulties mentioned above, we introduce in Section 5.2 the continuous definition of $\sigma(G)$ first, with arbitrary closed sets as lists, and then show that it is equal to the limit of $\frac{\mathrm{ch}_s(G)}{s}$ as $s \to \infty$. We will need to make use of the *Lebesgue measure* $\mu(\Lambda)$ of a set $\Lambda \subseteq \mathbb{R}$. For the most part, the only property of Lebesgue measure we will need is that if Λ is a disjoint union of intervals (closed, open or half-open) then $\mu(\Lambda)$ is equal to the sum of the lengths of those intervals. (Some formal definitions are given in the Appendix to this chapter, where they are needed to prove a technical lemma.)

In Section 5.3 we show that we do not need to consider all possible closed lists when defining $\sigma(G)$, but only finite unions of closed intervals. An analogue of Brooks' Theorem appears in Section 5.4, and general bounds and some specific values for $\sigma(G)$ are investigated in Section 5.5.

5.2   Choosability and measure

For $\kappa \in \mathbb{R}^+$, a *real κ-list assignment* is a function Λ which assigns to each vertex $v \in V$ a nonempty closed set $\Lambda(v) \subset \mathbb{R}$ of Lebesgue measure κ. As before, a Λ-colouring $\gamma : V \to \mathbb{R}$ of G is *proper* if $|\gamma(v) - \gamma(w)| \geq 1$ for all $vw \in E$. We define the *measure choosability* $\sigma^*(G)$ to be the infimum of all $\kappa \in \mathbb{R}^+$ such that for every real κ-list assignment Λ, there exists a proper Λ-colouring.

We have specified that the lists must be nonempty to ensure that the infimum in the definition of $\sigma^*(G)$ is attained for empty graphs. If $G = \overline{K}_n$ then we can choose any colours from any assignment of nonempty lists, and so $\sigma^*(G) = 0$; but if we allowed empty lists, we would have a real 0-list assignment admitting no proper colouring. We will show later in Lemma 5.6, as we did for $\tau^*(G)$ in Lemma 4.11, that the infimum in the definition of $\sigma^*(G)$ is attained for all graphs G, and so can be replaced by a minimum.

Firstly, for any $s \in \mathbb{N}$, we relate the value of $\mathrm{ch}_s(G)$ to $\sigma^*(G)$.

Lemma 5.2 *For any graph G and any $s \in \mathbb{N}$,*

$$(s-1)\sigma^*(G) + 1 \leq \mathrm{ch}_s(G) \leq s\sigma^*(G) + 1.$$

Proof. Let $k = \mathrm{ch}_s(G) - 1$ and $\varepsilon \in (0, 1)$, and let L be a k-list assignment for G such that G has no L-colouring with separation s. Now for $v \in V$, define

$$\Lambda(v) = \bigcup_{x \in L(v)} \left[\tfrac{x}{s}, \tfrac{x+1-\varepsilon}{s}\right].$$

Suppose $\gamma : V \to \mathbb{R}$ is a Λ-colouring of G, and set $c(v) = \lfloor s\gamma(v) \rfloor$ for each $v \in V$; then c is an L-colouring of G, and so it cannot have separation s. Thus there is an edge $vw \in E$ such that $|c(v) - c(w)| \leq s - 1$. But then $|\gamma(v) - \gamma(w)| \leq \frac{s-\varepsilon}{s} < 1$, and so γ is not a proper colouring of G. This shows that $\sigma^*(G) > \mu(\Lambda(v)) = \frac{1-\varepsilon}{s}(\mathrm{ch}_s(G) - 1)$. By rearranging this expression we obtain $\mathrm{ch}_s(G) < \frac{s}{1-\varepsilon}\sigma^*(G) + 1$, and the upper bound of the lemma follows since $\varepsilon \in (0, 1)$ was arbitrary.

The lower bound is immediate if $s = 1$, so assume $s \geq 2$. Let $\kappa = \frac{\mathrm{ch}_s(G)-1}{s-1}$, and let Λ be any real κ-list assignment for G. For $v \in V$, define

$$L(v) = \left\{x \in \mathbb{Z} : \Lambda(v) \cap \left[\tfrac{x}{s-1}, \tfrac{x+1}{s-1}\right) \neq \emptyset\right\}.$$

No closed set of measure κ can lie within a union of $\mathrm{ch}_s(G) - 1$ of these half-open intervals, and so $|L(v)| \geq \mathrm{ch}_s(G)$ for each $v \in V$. Thus there exists an L-colouring $c : V \to \mathbb{Z}$ with separation s. For each $v \in V$ choose $\gamma(v) \in \Lambda(v)$ such that $\frac{c(v)}{s-1} \leq \gamma(v) < \frac{c(v)+1}{s-1}$. Then for each edge $vw \in E$, w.l.o.g. $c(w) - c(v) \geq s$, and so $\gamma(w) - \gamma(v) > \frac{c(w)}{s-1} - \frac{c(v)+1}{s-1} \geq 1$. Hence γ is a proper Λ-colouring of G, and so $\sigma^*(G) \leq \kappa$. Rearranging gives the lower bound of the lemma. $\qquad\square$

Dividing through the result of Lemma 5.2 by s, and taking the limit as

$s \to \infty$, we see that the sequence $\left(\frac{\mathrm{ch}_s(G)}{s}\right)_{s \in \mathbb{N}}$ does indeed converge to $\sigma^*(G)$.
Now we can formally define the *choosability ratio*

$$\sigma(G) = \lim_{s \to \infty} \frac{\mathrm{ch}_s(G)}{s},$$

and state the following theorem.

Theorem 5.3 *For any graph G, $\sigma(G) = \sigma^*(G)$.* $\qquad\qquad\qquad$ □

Lemma 5.2 also implies the following relation between $\mathrm{ch}_s(G)$ and $\mathrm{ch}_r(G)$ for any pair of natural numbers $s, r \in \mathbb{N}$.

Theorem 5.4 *For any graph G and any $s, r \in \mathbb{N}$,*

$$\left\lceil \frac{s-1}{r}(\mathrm{ch}_r(G) - 1) \right\rceil + 1 \le \mathrm{ch}_s(G) \le \left\lfloor \frac{s}{r-1}(\mathrm{ch}_r(G) - 1) \right\rfloor + 1.$$

If $r = 1$ then we interpret the last quantity in Theorem 5.4 as being infinite. Note also that if $r = 1$, the lower bound is weaker than Theorem 3.3, and if $r = 2$, the upper bound is weaker than equation 3.1: we lose some precision by appealing to Lemma 5.2. On the other hand, if $s - 1$ is *not* a multiple of $r - 1$, the upper bound of Theorem 5.4 improves on that of Lemma 5.1.

Proof of Theorem 5.4. We use the result of Lemma 5.2 twice to obtain each inequality. Rearranging that result and substituting r for s, we have

$$\frac{\mathrm{ch}_r(G) - 1}{r} \le \sigma^*(G) \le \frac{\mathrm{ch}_r(G) - 1}{r - 1}. \qquad\qquad (5.1)$$

Consequently, $(s-1)\frac{\mathrm{ch}_r(G)-1}{r} + 1 \le (s-1)\sigma^*(G) + 1 \le \mathrm{ch}_s(G) \le s\sigma^*(G) + 1 \le$

$s\frac{\mathrm{ch}_r(G)-1}{r-1} + 1$. The theorem now follows since $\mathrm{ch}_s(G)$ is an integer. $\square$

We can deduce immediately from Theorems 3.1 and 3.2 the value of $\sigma(G)$ when G is a complete graph, a cycle or a tree.

Theorem 5.5   *(i) If $G = K_n$ then $\sigma(G) = n - 1$.*

   (ii) If G is an odd cycle C_{2r+1}, then $\sigma(G) = 2$.

   (iii) If G is a tree on n vertices then $\sigma(G) = 2\left(1 - \frac{1}{n}\right)$.

   (iv) If G is an even cycle C_{2r}, then $\sigma(G) = 2\left(1 - \frac{1}{4r-1}\right)$. $\square$

Note that $\sigma(G) = \tau(G)$ for complete graphs, trees and odd cycles, but not for even cycles.

Having proved that $\sigma^*(G) = \sigma(G)$ we use exclusively the latter notation for the rest of this chapter, although we refer mostly to its formulation in terms of closed real sets.

5.3   Simplifying $\sigma(G)$

Arbitrary closed sets can be cumbersome to work with as lists; this section contains some results that simplify the situation, by showing that the only lists we need to consider have much simpler forms. We may begin by observing that if any list $\Lambda(v)$ is unbounded, we can easily find a valid colour for v after colouring the other vertices; thus we may assume that $\Lambda(v)$ is bounded, and hence compact, for each $v \in V$. This observation allows us to deduce that the infimum in the definition of $\sigma^*(G)$ $(= \sigma(G))$ is attained.

Lemma 5.6 *For any nonempty graph G, let $\kappa = \sigma(G)$. Then for every real κ-list assignment Λ, there exists a proper Λ-colouring.*

Proof. Let Λ be any real κ-list assignment for G. For each vertex $v \in V$, we may choose $\beta_v \in \mathbb{R}$ such that $\beta_v \notin \Lambda(v)$ and $\beta_v > \inf \Lambda(v)$, and let $\alpha_v = \sup\{\lambda < \beta_v : \lambda \in \Lambda(v)\}$. Then $\alpha_v \in \Lambda(v)$ since $\Lambda(v)$ is closed and nonempty, and $\Lambda(v) \cap (\alpha_v, \beta_v] = \emptyset$.

For each $r \in \mathbb{N}$, define lists $\Lambda_r(v) = \Lambda(v) \cup (\alpha_v, \alpha_v + \frac{1}{r}]$. If $r \geq \frac{1}{\beta_v - \alpha_v}$ then $\Lambda(v)$ and $(\alpha_v, \alpha_v + \frac{1}{r}]$ are disjoint, and so $\mu(\Lambda_r(v)) = \kappa + \frac{1}{r}$; thus if $r \geq \max\{\frac{1}{\beta_v - \alpha_v} : v \in V\}$, Λ_r is a $(\kappa + \frac{1}{r})$-list assignment for G, and so there exists a proper Λ_r-colouring $\gamma_r : V \to \mathbb{R}$. As noted above, we may assume that $\Lambda(v)$ is bounded for each $v \in V$, and we observe that any convergent sequence of colours $\gamma_r(v) \in \Lambda_r(v)$ must have its limit in $\Lambda(v)$. Now we complete the proof exactly as for Lemma 4.11. $\qquad\square$

The next important lemma allows us to deduce that for any graph G, there is a constant $M \in \mathbb{R}^+$ such that in the definition of $\sigma(G)$ we need only consider lists which are closed subsets of $[0, M]$.

Lemma 5.7 *Let G be a graph on n vertices. Suppose there exists a Λ-colouring of G for every real κ-list assignment Λ for which $\Lambda(v) \subseteq [0, \kappa n]$ for each $v \in V$. Then $\sigma(G) \leq \kappa$.*

The following lemma is a crucial step in the proof of Lemma 5.7. Its proof is technical and relies on results from measure theory, and thus is given in the Appendix to this chapter.

Lemma 5.8 *Let $\Lambda \subseteq \mathbb{R}$ be a Borel set with $\mu(\Lambda) = \lambda$, and define $f : \mathbb{R} \to [0, \lambda]$ where $f(x) = \mu(\Lambda \cap (-\infty, x])$. Then for any Borel set $B \subseteq \mathbb{R}$,*

$$\mu(f(B)) = \mu(\Lambda \cap B).$$

Proof of Lemma 5.7. Let Λ be *any* real κ-list assignment for G; we will construct a proper Λ-colouring of G. As noted above, we may assume that $\Lambda(v)$ is compact for each $v \in V$. Write $\Lambda = \bigcup_{v \in V} \Lambda(v)$ and $\lambda = \mu(\Lambda)$; clearly $\lambda \leq \sum_{v \in V} \mu(\Lambda(v)) = \kappa n$. We define a function $f : \mathbb{R} \to [0, \lambda]$, where

$$f(x) = \mu\big(\Lambda \cap (-\infty, x]\big).$$

Note that given $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|$, and so f is continuous.

Define new lists $\Lambda'(v) = f(\Lambda(v))$. By Lemma 5.8, $\mu(\Lambda'(v)) = \mu(\Lambda(v)) = \kappa$, and since f is continuous and $\Lambda(v)$ is compact, $\Lambda'(v)$ is closed; thus Λ' is a κ-list assignment for G. Also, we clearly have $\Lambda'(v) \subseteq [0, \lambda] \subseteq [0, \kappa n]$. Thus the conditions in the hypothesis of the lemma are satisfied, and so there exists a proper Λ'-colouring $\gamma' : V \to \mathbb{R}$. Finally, we can find $\gamma(v) \in \Lambda(v)$ such that $f(\gamma(v)) = \gamma'(v)$ for each $v \in V$; and as above, if $vw \in E$ then

$$|\gamma(v) - \gamma(w)| \geq |f(\gamma(v)) - f(\gamma(w))| = |\gamma'(v) - \gamma'(w)| \geq 1,$$

and so γ is a proper Λ-colouring of G, as required. $\qquad\qquad\square$

If G has n vertices then $\sigma(G) \leq \sigma(K_n) = n - 1$, and so Lemma 5.7 has the following immediate corollary.

Corollary 5.9 *Let G be a graph on n vertices, and $M = n(n-1)$. Then $\sigma(G)$ is the infimum of all $\kappa \in \mathbb{R}^+$ such that for every real κ-list assignment Λ with $\Lambda(v) \subseteq [0, M]$, there exists a proper Λ-colouring.* □

Next, we will show that the only lists we need to consider are finite unions of closed intervals. For $r \in \mathbb{N}$, we call Λ a *κ-r-interval assignment* if each set $\Lambda(v)$ is the union of at most r closed intervals of total length κ. We define $\tau_r(G)$ to be the infimum of all $\kappa \in \mathbb{R}^+$ such that for every κ-r-interval assignment Λ, there exists a proper Λ-colouring of G.

It is clear from these definitions that $\tau(G) = \tau_1(G) \leq \tau_2(G) \leq \cdots$, and that $\tau_r(G) \leq \sigma(G)$ for all $r \in \mathbb{N}$; hence the sequence $(\tau_r(G))_{r \in \mathbb{N}}$ converges to a limit. In fact, we can deduce from the proof of Lemma 5.2 that this limit is $\sigma(G)$, and so $\sigma(G)$ is the infimum of all $\kappa \in \mathbb{R}^+$ such that for every $r \in \mathbb{N}$ and every κ-r-interval assignment Λ, there exists a proper Λ-colouring of G.

Lemma 5.10 *For any graph G, $\sigma(G) = \lim_{r \to \infty} \tau_r(G)$.*

Proof. In the proof of the upper bound on $\text{ch}_s(G)$ in Lemma 5.2, the lists $\Lambda(v)$ we used each consisted of $k = \text{ch}_s(G) - 1$ closed intervals, and thus we proved that $\text{ch}_s(G) \leq s\tau_k(G) + 1$ for any $s \in \mathbb{N}$. Assuming G is nonempty, we have $k \to \infty$ as $s \to \infty$; thus dividing by s, and taking the limit as $s \to \infty$, we obtain $\sigma(G) \leq \lim_{k \to \infty} \tau_k(G)$ as required. □

Note that Lemma 5.8 can be proved for finite unions of closed intervals without resorting to measure-theoretic techniques. Lemma 5.10 can then be used to deduce Lemma 5.7 and Corollary 5.9.

We can go further than Lemma 5.10, and show that some $\tau_r(G)$ is equal to $\sigma(G)$ (and hence the sequence of values of $\tau_r(G)$ is eventually constant.) We need the following lemma to establish this result. Given a real list assignment Λ for $G - u$, define $\Xi(u)$ to be the set of values $\xi \in \mathbb{R}$ for which there is a proper colouring γ of G with $\gamma(u) = \xi$ and $\gamma(v) \in \Lambda(v)$ for $v \in V \setminus \{u\}$.

Lemma 5.11 *For G, u and Λ as above, $\Xi(u)$ is either empty, or is of the form*

$$(-\infty, \alpha_1] \cup [\beta_1, \alpha_2] \cup \cdots \cup [\beta_{r-1}, \alpha_r] \cup [\beta_r, \infty),$$

for some $r \in \mathbb{N}$. Furthermore, $r \leq r_0$, where r_0 depends only on G.

Proof. Clearly, $\Xi(u)$ is empty iff there is no proper Λ-colouring of $G - u$. So suppose this is not the case. Fix an ordering $v_1, \ldots, v_n$ of the vertices, and let $e_{ij} = 1$ if $v_i v_j \in E$ and $e_{ij} = 0$ otherwise. For each permutation $\pi \in S_n$, let Γ_π be the set of colourings $\gamma : V \to \mathbb{R}$ satisfying $\gamma(v) \in \Lambda(v)$ for $v \in V \setminus \{u\}$ and $\gamma(v_{\pi(1)}) \leq \gamma(v_{\pi(2)}) \leq \ldots \leq \gamma(v_{\pi(n)})$. We define $\Xi_\pi(u) = \{\gamma(u) : \gamma \in \Gamma_\pi\}$, and observe that $\Xi(u) = \bigcup \{\Xi_\pi(u) : \pi \in S_n\}$.

Set $k \in \{1, \ldots, n\}$ such that $v_{\pi(k)} = u$. For $i = 1, 2, \ldots, k-1$ in turn, assign the smallest colour $\gamma(v_{\pi(i)}) \in \Lambda(v_{\pi(i)})$ such that $\gamma(v_{\pi(i)}) - \gamma(v_{\pi(j)}) \geq e_{\pi(i)\pi(j)}$ for $j < i$. Then for $i = n, n-1, \ldots, k+1$ in turn, assign the largest colour $\gamma(v_{\pi(i)}) \in \Lambda(v_{\pi(i)})$ such that $\gamma(v_{\pi(j)}) - \gamma(v_{\pi(i)}) \geq e_{\pi(i)\pi(j)}$ for $j > i$. In any colouring in Γ_π, the colour ξ of $u = v_{\pi(i)}$ must satisfy

$$\max_{1 \leq i < k} \left(\gamma(v_{\pi(i)}) + e_{\pi(i)\pi(k)} \right) \leq \xi \leq \min_{k < i \leq n} \left(\gamma(v_{\pi(i)}) - e_{\pi(i)\pi(k)} \right); \qquad (5.2)$$

and for any ξ in this range, setting $\gamma(u) = \xi$ gives a colouring $\gamma \in \Xi_\pi(u)$. If $k = 1$, the lower bound in (5.2) is vacuous, and $\Xi_\pi(u)$ is of the form $(-\infty, \alpha]$; if $k = n$, the upper bound is vacuous, and $\Xi_\pi(u)$ is of the form $[\beta, \infty)$. Otherwise, $\Xi_\pi(u)$ is either a closed interval, or is empty (if the lower bound in (5.2) exceeds the upper bound). Since $\Xi(u) = \bigcup \{\Xi_\pi(u) : \pi \in S_n\}$, this shows that $\Xi(u)$ is of the required form. The maximum number of bounded closed intervals $\Xi_\pi(u)$ that can arise in this way is $(n-2)(n-1)!$; hence the total number of intervals in $\Xi(u)$ is at most $(n-2)(n-1)! + 2$, and so $r \leq r_0 = (n-2)(n-1)! + 1$. $\square$

Theorem 5.12 *For any graph G there is $r \in \mathbb{N}$ such that $\tau_r(G) = \sigma(G)$.*

Proof. Let $n = |V|$, and r_0 be as in Lemma 5.11. Given any $\kappa < \sigma(G)$, there exists a real κ-list assignment Λ such that there is no proper Λ-colouring of G. We will show that there is an assignment Λ^* such that $\Lambda^*(v)$ is a union of at most r_0 closed intervals and $\mu(\Lambda^*(v)) \geq \kappa$ for each $v \in V$, and which admits no proper Λ^*-colouring; thus $\kappa < \tau_{r_0}(G)$. This will imply that $\tau_{r_0}(G) \geq \sigma(G)$ and hence prove the theorem.

We construct Λ^* from Λ as follows. For each vertex $u \in V$ in turn, form $\Xi(u)$ as in Lemma 5.11. Since there is no Λ-colouring of G, $\Lambda(u)$ must be disjoint from $\Xi(u)$; we will replace $\Lambda(u)$ by a list $\Lambda^*(u)$ which is a union of at most r closed intervals, and satisfies $\Lambda(u) \subseteq \Lambda^*(u) \subseteq \mathbb{R} \setminus \Xi(u)$. This is easy if $\Xi(u)$ is empty, so assume it is not; then $\mathbb{R} \setminus \Xi(u) = \bigcup_{i=1}^{r}(\alpha_i, \beta_i)$. Now let $\Lambda_i = \Lambda(u) \cap (\alpha_i, \beta_i)$, which is closed since $\alpha_i, \beta_i \notin \Lambda(u)$, and

$$\Lambda^*(u) = \bigcup_{i=1}^{r} [\min \Lambda_i, \max \Lambda_i],$$

disregarding any i for which Λ_i is empty. By repeating this process for each $u \in V$, we transform Λ into Λ^* as required, where each $\Lambda^*(u)$ is a union of at most $r \leq r_0$ intervals. This completes the proof. $\square$

Let $r^*(G)$ denote the smallest $r \in \mathbb{N}$ such that $\tau_r(G) = \sigma(G)$. The above theorem establishes that $r^*(G)$ exists; however, the upper bound, of order $n!$, is not very satisfying. It should surely be the case that $\tau_r(G) \leq n$; and we offer the following stronger conjecture.

Conjecture 5.13 *If $k \geq 1$ and G is a k-degenerate graph, then $r^*(G) \leq k$.*

Equivalently, we conjecture that $r^*(G) \leq \max\{1, \mathrm{col}(G) - 1\}$. We know from Theorems 4.14 and 5.5 that $\tau(G) = \tau_1(G) = \sigma(G)$ for trees, and so Conjecture 5.13 holds when $k = 1$.

Theorem 5.12 greatly simplifies some of the theory surrounding the choosability ratio: it tells us that we only need to consider lists which are finite unions of closed intervals, and it is the key to proving that those graphs for which $\tau(G) < \Delta(G)$ (cf. Theorem 4.28) also satisfy $\sigma(G) < \Delta(G)$.

5.4   A Brooks-type bound for $\sigma(G)$

The investigation of the relationship between $\sigma(G)$ and $\Delta(G)$ is motivated by the following two theorems.

Theorem 5.14 (Vizing [28]) *For any graph G, $\mathrm{ch}(G) \leq \Delta(G) + 1$, and if G is connected, equality holds iff G is a complete graph or an odd cycle.* $\square$

Theorem 5.15 (Waller [31]) *For any graph* G, $\mathrm{ch}_s(G) \leq s\Delta(G) + 1$*, and if* G *is connected, equality holds iff* G *is a complete graph or an odd cycle.* $\square$

Dividing the result of Theorem 5.15 through by s and taking the limit as $s \to \infty$, we see that $\sigma(G) \leq \Delta(G)$ for all graphs G. In this section we will show that strict inequality holds for the same set of graphs as in the above theorems. To do this we will need to consider lists which are finite unions of half-open intervals $[a, b)$. When referring to half-open intervals we will always mean bounded intervals of the form $[a, b)$, and never $(a, b]$.

Lemma 5.16 *Let* G *be a graph and* $\kappa \in \mathbb{R}$*. Suppose that a proper* Λ*-colouring exists for every assignment* Λ *of lists such that*

    (i) each $\Lambda(v)$ *is a union of* r *half-open intervals,*

    (ii) $\mu(\Lambda(v)) \geq \kappa$ *for each* $v \in V(G)$*.*

Then $\tau_r(G) < \kappa$*.*

Proof. The conditions of the lemma clearly imply that $\tau_r(G) \leq \kappa$, since any union of closed intervals contains a union of half-open intervals of the same measure. Suppose the result is false; then we have a graph G and $\kappa = \tau_r(G)$, such that a proper Λ-colouring exists for every Λ satisfying (i) and (ii). We will prove the theorem by deriving a contradiction.

For each $s \in \mathbb{N}$, $\tau_r(G) > \kappa - \frac{1}{s}$ and so there exists a list assignment Λ_s such that each $\Lambda_s(v)$ is a union of at most r closed intervals of total length $\kappa - \frac{1}{s}$, and such that no proper Λ_s-colouring of G exists. Write $V = \{v_1, \ldots, v_n\}$, and

parameterise the lists as follows:

$$\Lambda_s(v_i) = \bigcup_{p=1}^{r} \left[\alpha_p^s(v_i),\, \alpha_p^s(v_i) + \beta_p^s(v_i)\right],$$

where $\sum_{p=1}^{r} \beta_p^s(v_i) = \kappa - \frac{1}{s}$ for $i = 1, \ldots, n$. By Corollary 5.9, we may also assume that $\Lambda_s(v_i) \subseteq [0, n(n-1)]$ for each $s \in \mathbb{N}$ and $i = 1, \ldots, n$.

Let $\mathbf{x}_s \in \mathbb{R}^{2rn}$ be the vector whose coordinates are the values of $\alpha_p^s(v_i)$ and $\beta_p^s(v_i)$, where $1 \le i \le n$ and $1 \le p \le r$. These vectors form a bounded sequence in $\mathbb{R}^{2rn}$, and so by the Bolzano-Weierstrass theorem, there is a subsequence converging to a limit $\mathbf{x}$, with coordinates $\alpha_p(v_i)$ and $\beta_p(v_i)$, where $\sum_{p=1}^{r} \beta_p(v_i) = \kappa$ for $i = 1, \ldots, n$. Let these coordinates define lists

$$\Lambda(v_i) = \bigcup_{p=1}^{r} \left[\alpha_p(v_i),\, \alpha_p(v_i) + \beta_p(v_i)\right).$$

By our assumption there is a proper Λ-colouring $\gamma : V \to \mathbb{R}$. Now let $\varepsilon > 0$ be such that, for each $i = 1, \ldots, n$,

$$\gamma(v_i) \in \bigcup_{p=1}^{r} \left[\alpha_p(v_i),\, \alpha_p(v_i) + \beta_p(v_i) - \varepsilon\right].$$

Find $s \in \mathbb{N}$ such that $|\alpha_p^s(v_i) - \alpha_p(v_i)| < \varepsilon/3$ and $|\beta_p^s(v_i) - \beta_p(v_i)| < \varepsilon/3$ whenever $1 \le i \le n$ and $1 \le p \le r$. Now we can construct a proper Λ_s-colouring $\gamma' : V \to \mathbb{R}$ by setting $\gamma'(v_i) = \gamma(v_i) + \varepsilon/3$ for $i = 1, \ldots, n$. This is a contradiction, since we asserted that no such colouring existed. $\qquad\square$

The next four lemmas are based on lemmas in Waller's paper [31] leading up to the proof that $\mathrm{ch}_s(G) \leq s\Delta(G)$.

Lemma 5.17 *Let $G = (V, E)$ be a connected graph, and let $u \in V$. Let Λ be an assignment of lists such that*

(i) each $\Lambda(v)$ is a finite union of half-open intervals,

(ii) $\mu(\Lambda(u)) > d(u)$,

(iii) $\mu(\Lambda(v)) \geq d(v)$ for $v \in V \setminus \{u\}$.

Then G has a proper Λ-colouring.

Proof. We use induction on $n = |V|$. If $n = 1$ then the sole vertex u has a non-empty list and the result is trivial. So assume $n \geq 2$. Let $\Lambda = \bigcup_{v \in V} \Lambda(v)$, and $\beta = \min \Lambda$; this minimum is attained in Λ since Λ is a finite union of half-open intervals. Let $X = \{v \in V : \beta \in \Lambda(v)\}$. We consider two cases.

Case 1: $X = \{u\}$. Set $\gamma(u) = \beta$. Now we use induction to colour each component C of $G - u$. Let $u' \in C$ be adjacent to u; then $\min \Lambda(u') > \beta$, and so the set of colours $\Lambda'(u') = \Lambda(u') \cap [\beta + 1, \infty)$ available for u' satisfies $\mu(\Lambda'(u')) > \mu(\Lambda(u')) - 1 \geq d_G(u) - 1 \geq d_C(u')$. Hence the conditions of the lemma are satisfied for each C, and so we can complete the colouring of G.

Case 2: $X \setminus \{u\} \neq \emptyset$. Choose $x \in X \setminus \{u\}$ at maximum distance from u, and set $\gamma(x) = \beta$. Let C be the component of $G - x$ that contains u. If $v \in C$ was adjacent to x in G, then the set of colours $\Lambda'(v)$ available for v satisfies $\mu(\Lambda'(v)) \geq \mu(\Lambda(v)) - 1 \geq d_G(v) - 1 \geq d_C(v)$; and the middle inequality is strict if $v = u$. Hence we can colour C by the induction hypothesis.

Every other component of $G - x$ contains a vertex $u' \in N(x)$, which is at greater distance from u than x, and so $\beta \notin \Lambda(u')$. Hence we can complete the colouring on these components in the same way as in Case 1. $\qquad \square$

Note that condition (i) in Lemma 5.17 can be replaced by the weaker statement 'every bounded decreasing sequence in $\Lambda(v)$ has its limit in $\Lambda(v)$'.

Lemma 5.18 *Let G be an even circuit, and let Λ be an assignment of lists such that*

   (i) each $\Lambda(v)$ is a finite union of half-open intervals,

   (ii) $\mu(\Lambda(v)) \geq d(v)$ for $v \in V(G)$.

Then G has a proper Λ-colouring.

Proof. By Theorem 5.5(iv) we know that $\sigma(G) < 2$. Clearly, then, we can find $\Lambda'(v) \subset \Lambda(v)$ such that $\Lambda'(v)$ is closed and $\mu(\Lambda'(v)) \geq \sigma(G)$, for each $v \in V(G)$. By definition of $\sigma(G)$ we then have a Λ'-colouring of G, which is also a Λ-colouring. $\qquad \square$

Lemma 5.19 *Let G be an even circuit with one chord, and let Λ be an assignment of lists as in Lemma 5.18. Then G has a proper Λ-colouring.*

Proof. Let Λ, β and X be as in the proof of Lemma 5.17. Again we consider two cases.

   Case 1: $X = V(G)$. Let x and y be the endvertices of the chord, and let $u \neq y$ be a vertex adjacent to x. Let $\gamma(u) = \beta$, and $\gamma(v) = \beta$ for every vertex v at even distance from u in the circuit $G - xy$. Any uncoloured vertex

$v \in V \setminus \{x, y\}$ can now be coloured with any colour $\gamma(v) \geq \beta + 1$ from its list. If y has already been assigned colour β then x can also be coloured with any colour $\gamma(x) \geq \beta + 1$ in $\Lambda(x)$. Otherwise, let $\Lambda' = (\Lambda(x) \cup \Lambda(y)) \cap [\beta + 1, \infty)$, and $\beta' = \min \Lambda'$; w.l.o.g. $\beta' \in \Lambda(x)$. We can now complete the colouring of G by setting $\gamma(x) = \beta'$ and choosing $\gamma(y) \geq \beta' + 1$ from $\Lambda(y)$.

Case 2: $X \neq V(G)$. In this case there exists a vertex $x \in X$ which is adjacent to a vertex $u \in V(G) \setminus X$. Set $\gamma(x) = \beta$. As in the proof of Lemma 5.17, with $\Lambda'(u) = \Lambda(u) \cap [\beta + 1, \infty)$, we have $\mu(\Lambda'(u)) > \mu(\Lambda(u)) - 1 \geq d_G(u) - 1 \geq d_{G-x}(u)$. Hence u satisfies condition (ii) of Lemma 5.17 in $G - x$; and every other vertex satisfies condition (iii) in $G - x$. Since $G - x$ is connected, we can complete the colouring of G by Lemma 5.17. $\qquad\square$

Lemma 5.20 *Let G be a connected graph, and let G contain as an induced subgraph an even circuit, or an even circuit with one chord. Let Λ be an assignment of lists as in Lemma 5.18. Then G has a proper Λ-colouring.*

Proof. Let $S \subseteq V$ be the vertex set of the induced subgraph. Let Λ, β and X be as in the proof of Lemma 5.17. We use induction on $|V \setminus S|$; if $|V \setminus S| = 0$ then the result follows directly from Lemma 5.18 or 5.19. So assume $|V \setminus S| \geq 1$. We distinguish three cases.

Case 1: $X \setminus S \neq \emptyset$. Choose $x \in X \setminus S$ at maximum distance from any vertex of S, and set $\gamma(x) = \beta$. Let C be the component of $G - x$ that contains S. If $v \in C$ was adjacent to x in G, then the set of colours $\Lambda'(v)$ available for v satisfies $\mu(\Lambda'(v)) \geq \mu(\Lambda(v)) - 1 \geq d_G(v) - 1 \geq d_C(v)$. Hence we can colour C by the induction hypothesis. Every other component of $G - x$ contains a vertex

$u' \in N(x)$, which is at greater distance from S than x, so we can complete the colouring on these components as in Lemma 5.17.

Case 2: $X \subsetneq S$. In this case there exists a vertex $x \in X$ which is adjacent to a vertex $u \in S \setminus X$. Set $\gamma(x) = \beta$. All the vertices in $S \setminus \{x\}$ are contained in the same component C of $G - x$. Proceeding as in the proof of Lemma 5.17, we have $\mu(\Lambda'(u)) > \mu(\Lambda(u)) - 1 \geq d_G(u) - 1 \geq d_{G-x}(u)$. Hence u satisfies condition (ii) of Lemma 5.17 in C; and every other vertex satisfies condition (iii) in C. By Lemma 5.17, we can complete the colouring on this component. Every other component of $G - x$ contains only vertices not in S, and hence not in X, and so we can complete the colouring on these components as above.

Case 3: $X = S$. Choose vertices of S to be given colour β as in Case 1 of Lemma 5.19 (if the subgraph induced by S has no chord, then u can be any vertex of S). Now remove the vertices of G coloured β to form G', and let C be any component of G'. If C contains a vertex $v \in S$ then, since v has two neighbours in G with colour β, we have $\mu(\Lambda'(v)) \geq \mu(\Lambda(v)) - 1 > d_G(v) - 2 \geq d_C(v)$; hence v satisfies condition (ii) of Lemma 5.17 in C, and we can use Lemma 5.17 to colour C. Otherwise C consists of vertices not in S, and hence not in X, and so we can complete the colouring on C as above. $\qquad\square$

The final step we need is the following theorem due to A.L. Rubin [9]. Together with the above lemmas, it is then a short step to the theorem we are aiming for.

Theorem 5.21 (Rubin [9]) *If G is a 2-connected graph, and G is not a complete graph or an odd circuit, then G contains as an induced subgraph an*

even circuit, or an even circuit with one chord. □

Lemma 5.22 *Let G be a connected graph which is not a complete graph or an odd circuit, and let Λ be an assignment of lists such that*

 (i) each $\Lambda(v)$ is a finite union of half-open intervals,

 (ii) $\mu(\Lambda(v)) \geq \Delta(G)$ for $v \in V(G)$.

Then G has a proper Λ-colouring.

Proof. If G is 2-connected, the result is implied by Theorem 5.21 and Lemma 5.20. So let G be connected but not 2-connected. If any block B of G is not a complete graph or an odd circuit, the result follows by applying Theorem 5.21 to B and then Lemma 5.20 to G. So assume each block of G is a complete graph or an odd circuit.

Let B be an endblock of G; it contains a unique vertex v adjacent to vertices in $G - B$. Taking $u \in B - v$ we have $d(u) < d(v)$, and so G is non-regular. The result now follows immediately from Lemma 5.17, since $\mu(\Lambda(v)) \geq \Delta(G) > d(u)$. □

Theorem 5.23 *For any graph G, $\sigma(G) \leq \Delta(G)$, and if G is connected, then equality holds iff G is a complete graph or an odd cycle.*

Proof. We may restrict our attention to connected graphs G. By Theorem 5.5 we know that if G is a complete graph or an odd circuit then $\sigma(G) = \chi(G) - 1 = \Delta(G)$. Otherwise, by Lemma 5.16 and Theorem 5.22, we know that $\tau_r(G) < \Delta(G)$ for each $r \in \mathbb{N}$. By Theorem 5.12, there is $r \in \mathbb{N}$ such that $\tau_r(G) = \sigma(G)$, and hence $\sigma(G) < \Delta(G)$ as required. □

5.5 Bounds and values for $\sigma(G)$

In this section we briefly discuss general bounds on $\sigma(G)$, and investigate the value of $\sigma(G)$ for two classes of graphs: wheels and complete bipartite graphs. Along the way we will show that the natural analogues for $\sigma(G)$ of various results from Chapter 4 regarding $\tau(G)$ do not hold.

5.5.1 General bounds and inequalities

Equation (5.1) gives a lower bound on $\sigma(G)$ in terms of $\mathrm{ch}_s(G)$ for all $s \in \mathbb{N}$, and an upper bound when $s \geq 2$. For $s = 1$ we have the following conjecture, which is implied by Conjecture 3.4.

Conjecture 5.24 *For any graph G and any $s \in \mathbb{N}$,*

$$\sigma(G) \leq 2(\mathrm{ch}(G) - 1).$$

Recall Theorem 4.10, which stated that for any graph G and $s \in \mathbb{N}$, $\mathrm{cch}_s(G) = \lfloor s\,\tau(G) \rfloor + 1$. Lemma 5.2 tells us that $\mathrm{ch}_s(G) \leq \lfloor s\,\sigma(G) \rfloor + 1$, but equality does not hold in general. When equality does hold for all $s \in \mathbb{N}$, we will say that G is σ-*exact*.

All the graphs G for which we have stated the value of $\sigma(G)$ so far are σ-exact, but we already have enough information to prove that $K_{2,36}$ is not σ-exact. In the proof of Theorem 3.6 we showed that $\mathrm{ch}_2(K_{2,36}) = 7$, and so $\sigma(K_{2,36}) \geq \frac{7-1}{2} = 3$ by Lemma 5.2. However, $\mathrm{ch}(K_{2,36}) = 3$, and so we have $\mathrm{ch}_1(K_{2,36}) = 3 < \lfloor \sigma(K_{2,36}) \rfloor + 1$.

5.5.2   Wheels

Theorem 4.21 gives the value of $\tau(G)$ for graphs of the form $G = K_1 + rK_k$. We begin by determining $\mathrm{ch}_s(G)$, and hence $\sigma(G)$, for these graphs.

Theorem 5.25 *For $k, r \in \mathbb{N}$, let $G = K_1 + rK_k$. Then for each $s \in \mathbb{N}$, $\mathrm{ch}_s(G) = \left\lfloor (k+1)s\left(1 - \frac{1}{kr+1}\right)\right\rfloor + 1$, and hence $\sigma(G) = (k+1)\left(1 - \frac{1}{kr+1}\right)$.*

Proof. If $r = 1$ then $G = K_{k+1}$, and if $k = 1$ then $G = K_{1,r}$; in either case the required result follows from Theorem 3.1. So assume $k \geq 2$ and $r \geq 2$. As in the proof of Theorem 4.21, let u be the universal vertex in G, and let $\{v_{ij} : 1 \leq i \leq r, 1 \leq j \leq k\}$ be the remaining vertices.

Fix $s \in \mathbb{N}$ and $t \in \{1, \ldots, s-1\}$, and for $j = 1, \ldots, k$, let $L(v_{1j}) = \{1, \ldots, (k+1)s - t\}$. Suppose c is a colouring of $G[u, v_{11}, \ldots, v_{1k}] \cong K_{k+1}$ with separation s such that $c(v_{1j}) \in L(v_{1j})$ for $j = 1, \ldots, k$; we consider the possible values of $c(u)$. We may assume w.l.o.g. that

$$c(v_{11}) \leq c(v_{12}) - s \leq c(v_{13}) - 2s \leq \cdots \leq c(v_{1k}) - (k-1)s,$$

and one of the following must hold: $c(u) \leq c(v_{11}) - s$, in which case $c(u) \leq s - t$; $c(u) \geq c(v_{1k}) + s$, in which case $c(u) \geq ks + 1$; or $c(v_{1\,j-1}) + s \leq c(u) \leq c(v_{1j}) - s$ for some $j \in \{2, \ldots, k\}$, in which case $(j-1)s + 1 \leq c(u) \leq js - t$. Let F_1 be the set of colours $x \in \mathbb{Z}$ such that there is *no* colouring c of $G[u, v_{11}, \ldots, v_{1k}]$ with separation s such that $c(u) = x$ and $c(v_{1j}) \in L(v_{1j})$ for $j = 1, \ldots, k$; it follows from the above results that $F_1 = \bigcup_{j=1}^{k} \{js - t + 1, \ldots, js\}$, and so $|F_1| = kt$.

Similarly, for $i = 2, \ldots, r$, let $L(v_{ij}) = \{(i-1)ks + 1, \ldots, (ik+1)s - t\}$ for each $j \in \{1, \ldots, k\}$, and define F_i by analogy with F_1 above. Then the sets $F_1, \ldots, F_r$ are pairwise disjoint, and $|F_i| = kt$. Now if $(k+1)s - t \leq krt$, we can complete L to a $((k+1)s - t)$-list assignment for G with $L(u) \subseteq F_1 \cup \cdots \cup F_r$, and hence so that there is no L-colouring of G with separation s. Thus $\mathrm{ch}_s(G) > (k+1)s - t$ whenever $(k+1)s - t \leq krt$, that is, when $t \geq \frac{(k+1)s}{kr+1}$; and so, since $\mathrm{ch}_s(G)$ is an integer, $\mathrm{ch}_s(G) \geq \left\lfloor (k+1)s\left(1 - \frac{1}{kr+1}\right) \right\rfloor + 1$.

Now let L be any $((k+1)s - t)$-list assignment for G. The upper bound $\mathrm{ch}_s(G) \leq \left\lfloor (k+1)s\left(1 - \frac{1}{kr+1}\right) \right\rfloor + 1$ will follow once we have shown that in general, with F_i as defined above, $|F_i| \leq kt$ for each $i \in \{1, \ldots, r\}$. It will suffice to show this just for $i = 1$.

Let $W = \{v_{11}, \ldots, v_{1k}\}$. For $q \in \{0, \ldots, k\}$, let c_q be the L-colouring of $G[W]$ with separation s which uses the smallest possible colours for some q vertices in W, and the largest possible colours for the remaining vertices. Formally, we construct c_q as follows:

- If $q \geq 1$, let $x_1 = \min \bigcup_{v \in W} L(v)$. Choose $v_1 \in W$ so that $x_1 \in L(v_1)$ (these subscripts are independent of the double subscripts above), and set $c_q(v_1) = x_1$.

- For $2 \leq p \leq q$, let $x_p = \min\{x \in \bigcup_{v \in W \setminus \{v_1, \ldots, v_{p-1}\}} L(v) : x \geq x_{p-1} + s\}$. Choose $v_p \in W \setminus \{v_1, \ldots, v_{p-1}\}$ so that $x_p \in L(v_p)$, and set $c_q(v_p) = x_p$.

- Now let $W_q = W \setminus \{v_1, \ldots, v_q\}$. If $q < k$, let $y_k^q = \max \bigcup_{v \in W_q} L(v)$. Choose $w_k^q \in W_q$ so that $y_k^q \in L(w_k^q)$, and set $c_q(w_k^q) = y_k^q$.

- For $k > p > q$, let $y_p^q = \max\{y \in \bigcup_{v \in W_q \setminus \{w_{p+1}^q, \ldots, w_k^q\}} L(v) : y \leq y_{p+1}^q - s\}$.

  Choose $w_p^q \in W_q \setminus \{w_{p+1}^q, \ldots, v_k^q\}$ so that $y_p^q \in L(w_p^q)$, and set $c_q(w_p^q) = y_p^q$.

Note that the values of x_p and y_p are independent of q, whereas the superscripts on y_p^q and w_p^q indicate that their values may depend on q.

Under c_k, any colour $x \geq c_k(v_k) + s$ is valid for u, and under c_0, any colour $x \leq c_0(w_0^1) - s$ is valid for u; furthermore, for $q \in \{1, \ldots, k-1\}$, any colour x such that $c_q(v_q) + s \leq x \leq c_q(w_{q+1}^q) - s$ is valid for $c_q(u)$. Hence

$$F_1 \subseteq \bigcup_{q=1}^{k} \{c_{q-1}(w_q^{q-1}) - s + 1, \ldots, c_q(v_q) + s - 1\}.$$

Since we only need to show that $|F_1| \leq kt$, we are done if we show that $|\{c_{q-1}(w_q^{q-1}) - s + 1, \ldots, c_q(v_q) + s - 1\}| \leq t$ for each $q \in \{1, \ldots, k\}$, that is, $c_{q-1}(w_q^{q-1}) - c_q(v_q) \geq 2s - t - 1$.

For all $v \in W$ and $q \geq 1$, $\{x \in L(v) : x < c_q(v_1)\} = \emptyset$ by our choice of x_1. Then for all $v \in W \setminus \{v_1\}$ and $q \geq 2$, $\{x \in L(v) : c_q(v_1) + s \leq x < c_q(v_2)\} = \emptyset$ by our choice of x_2, and so $|\{x \in L(v) : x < c_q(v_2)\}| \leq s$. Continuing in this way, for each $p \in \{1, \ldots, q\}$, if $v \in W \setminus \{v_1, \ldots, v_{p-1}\}$, we obtain $|\{x \in L(v) : x < c_q(v_p)\}| \leq (p-1)s$; and in particular, since $w_q^{q-1} \in W \setminus \{v_1, \ldots, v_q - 1\}$,

$$\left|\{x \in L(w_q^{q-1}) : x < c_q(v_q)\}\right| \leq (q-1)s. \tag{5.3}$$

By an analogous argument, if $q \leq p \leq k$ and $v \in W_{q-1} \setminus \{w_{p+1}^{q-1}, \ldots, w_k^{q-1}\}$ then $|\{x \in L(v) : x > c_{q-1}(w_p^{q-1})\}| \leq (k-p)s$; and in particular, since

$$w_q^{q-1} \in W_{q-1} \setminus \{w_{q+1}^{q-1}, \dots, w_k^{q-1}\},$$

$$\left|\{x \in L(w_q^{q-1}) : x > c_{q-1}(w_q^{q-1})\}\right| \leq (k-q)s. \tag{5.4}$$

Finally, we have

$$
\begin{aligned}
c_{q-1}(w_q^{q-1}) - c_q(v_q) &= |\{c_q(v_q), \dots, c_{q-1}(w_q^{q-1})\}| - 1 \\
&\geq |L(w_q^{q-1}) \cap \{c_q(v_q), \dots, c_{q-1}(w_q^{q-1})\}| - 1 \\
&\geq |L(w_q^{q-1})| - (q-1)s - (k-q)s - 1 \\
&= ((k+1) - (q-1) - (k-q))s - t - 1 = 2s - t - 1,
\end{aligned}
$$

where the second inequality is implied by (5.3) and (5.4). Thus $|F_1| \leq kt$, and similarly $|F_i| \leq kt$ for $i = 2, \dots, r$. This allows us to deduce the required upper bound on $\mathrm{ch}_s(G)$, and completes the proof of the theorem. $\qquad \square$

Comparing Theorems 4.21 and 5.25, we see that $\sigma(G) > \tau(G)$ for all graphs $G = K_1 + rK_k$ with $k \geq 2$ and $r \geq 2$.

For $r \geq 2$, since $K_1 + rK_2 \subset K_1 + C_{2r}$, Theorem 5.25 implies that

$$\sigma(K_1 + C_{2r}) \geq \sigma(K_1 + rK_2) \geq 3\left(1 - \tfrac{1}{2r+1}\right),$$

and for odd wheels $K_1 + C_{2r+1}$ $(r \geq 1)$, from Corollary 4.16 we have

$$\sigma(K_1 + C_{2r+1}) \geq \tau(K_1 + C_{2r+1}) \geq 3.$$

At first glance it is tempting to conjecture that equality holds in both cases; but we will use the next lemma to show that in fact $\sigma(K_1 + C_m) \to 4$ as $m \to \infty$ (irrespective of the parity of m). Note that since $\mathrm{ch}_s(C_m) \le 2s + 1$, we have $\mathrm{ch}_s(K_1 + C_m) \le 4s$ by Lemma 3.13, and so $\sigma(K_1 + C_m) \le 4$.

Lemma 5.26 *Let $r \in \mathbb{N}$ and $n_1, \ldots, n_r \ge 2$. For $i = 1, \ldots, r$ let B_i be a tree on n_i vertices, and let $G = K_1 + (B_1 \cup \cdots \cup B_r)$. Then*

$$\sigma(G) \ge \min\left\{ 2\left(2 - \frac{1}{n_1}\right), \ldots, 2\left(2 - \frac{1}{n_r}\right), \frac{2}{r+1}\left(2r - \sum_{i=1}^{r} \frac{1}{n_i}\right) \right\}.$$

Proof. Fix $s \in \mathbb{N}$, and let $k_i = \mathrm{ch}_s(B_i) - 1$ for $i = 1, \ldots, r$; by Theorem 3.1(iii), $k_i = \left\lfloor 2s\left(1 - \frac{1}{n_i}\right)\right\rfloor$. Let L_i be a k_i-list assignment for B_i such that there is no L_i-colouring of B_i with separation s. Now let $k \in \mathbb{N}$ be such that $k \ge 2s > \max\{k_i : i = 1, \ldots, r\}$ and $k < \min\{k_i + 2s : i = 1, \ldots, r\}$, and set $b_i = k - k_i$ for $i = 1, \ldots, r$. Define a list assignment L for G as follows:

$$L(v) = L_i(v) \cup \{a_i, \ldots, a_i + b_i - 1\} \quad \text{if } v \in B_i, \tag{5.5}$$

$$L(u) = \bigcup_{i=1}^{r}\{a_i + b_i - s, \ldots, a_i + s - 1\}, \tag{5.6}$$

where the values $a_1, \ldots, a_r \in \mathbb{Z}$ are chosen so that the unions in (5.5) and (5.6) are disjoint, and the restrictions on k ensure that $a_i \le a_i + b_i - 1$ and $a_i + b_i - s \le a_i + s - 1$. There is no L-colouring of G with separation s; for suppose c is such a colouring. Then $c(u) \in \{a_i + b_i - s, \ldots, a_i + s - 1\}$ for some $i \in \{1, \ldots, r\}$, and so $c(v) \notin \{a_i, \ldots, a_i + b_i - 1\}$ for all $v \in B_i$. But this implies that $c(v) \in L_i(v)$, and so c restricted to B_i is an L_i-colouring with

separation s, contradicting the definition of L_i.

Clearly $|L(v)| = k$ for each $v \in V \setminus \{u\}$, and it follows that $\mathrm{ch}_s(G) > k$ provided that $|L(u)| \geq k$, that is, if

$$k \leq \sum_{i=1}^{r}(2s - b_i) = 2rs - rk + \sum_{i=1}^{r} k_i$$

$$\iff (r+1)k \leq 2rs + \sum_{i=1}^{r} k_i = 4rs - \sum_{i=1}^{r} \left\lceil \frac{2s}{n_i} \right\rceil.$$

Recalling that the definitions (5.5) and (5.6) are only valid provided that $k < \min\{k_i + 2s : i = 1, \ldots, r\} = \min\left\{ \left\lfloor 2s\left(2 - \frac{1}{n_i}\right) \right\rfloor : i = 1, \ldots, r \right\}$, we have

$$\mathrm{ch}_s(G) \geq \min\left\{ \left\lfloor 2s\left(2 - \frac{1}{n_1}\right) \right\rfloor, \ldots, \left\lfloor 2s\left(2 - \frac{1}{n_r}\right) \right\rfloor, \frac{4rs - \sum_{i=1}^{r} \left\lceil \frac{2s}{n_i} \right\rceil}{r+1} \right\}.$$

Dividing through this inequality by s and taking the limit as $s \to \infty$, we obtain the result of the lemma. $\qquad \square$

Corollary 5.27 *If $r \geq 2$ and $m \geq r^2$, then $\sigma(K_1 + C_m) \geq 4 - \frac{6}{r+1}$.*

Proof. Apply Lemma 5.26 with $n_i = r$ and $B_i \cong P_r$ for each $i = 1, \ldots, r$. Then $G = K_1 + rP_r \subset K_1 + C_m$, and so $\sigma(K_1 + C_m) \geq \sigma(K_1 + rP_r) \geq \min\left\{4 - \frac{2}{r}, \frac{2}{r+1}(2r - 1)\right\} = \min\left\{4 - \frac{2}{r}, 4 - \frac{6}{r+1}\right\} = 4 - \frac{6}{r+1}$. $\qquad \square$

Corollary 5.28 *If $m \geq 25$ then $\sigma(K_1 + C_m) > 3$.*

Proof. Apply Lemma 5.26 with $r = 6$, $n_1 = \cdots = n_5 = 4$ and $n_6 = 5$, and each B_i a path. Then $\sum_{i=1}^{6} n_i = 25$, and so $G \subset K_1 + C_m$ and $\sigma(K_1 + C_m) \geq \sigma(G) \geq \min\left\{4 - \frac{2}{4}, \frac{2}{7}\left(12 - \frac{5}{4} - \frac{1}{5}\right)\right\} = \min\left\{\frac{7}{2}, \frac{240 - 25 - 4}{70}\right\} = 3 + \frac{1}{70}$. $\qquad \square$

A little trial and error shows that $m \geq 25$ is the best bound we can derive in Corollary 5.28 from Lemma 5.26. It follows from this last result that there is no analogue of Lemma 4.15 for the choosability ratio of critical graphs: $G = K_1 + C_{25}$ is *choice-critical* in the sense that $\mathrm{ch}(G - v) < \mathrm{ch}(G)$ for all $v \in V$, and we even have $\mathrm{ch}(G) = \chi(G) = 4$ and $\mathrm{ch}(H) = \chi(H) < 4$ for all proper subgraphs $H \subset G$; but $\sigma(G) > 3 = \mathrm{ch}(G) - 1$.

5.5.3   Complete bipartite graphs

Let $G = K_{k,m}$. If $m \geq k^k$ then $\mathrm{ch}(G) = k + 1$, as noted in [9], and if $m \geq (2s - 1)^k k^k$ then $\mathrm{ch}_s(G) = (2s - 1)k + 1$ as in the proof of Theorem 3.6. We also note that for fixed $k \in \mathbb{N}$, Theorem 3.6 and equation (5.1) imply that $\sigma(K_{k,m}) \to 2k$ as $m \to \infty$. Asymptotic results for general graphs of this form are known: Alon and Zaks [4] use probabilistic methods to show that there exist constants $c_1, c_2 > 0$ such that for all $s \geq 1$ and $n \geq 2$, $c_1 s \log n \leq \mathrm{ch}_s(K_{n,n}) \leq c_2 s \log n$, and it follows that $c_1 \log n \leq \sigma(K_{n,n}) \leq c_2 \log n$.

The next theorem gives a lower bound on $\sigma(K_{k,m})$ when m is a kth power.

Theorem 5.29 *For any $k, r \in \mathbb{N}$, let $G = K_{k,r^k}$. Then $\sigma(G) \geq \frac{2rk}{r+k}$.*

Proof. By Lemma 5.16 and Theorem 5.12, the result will follow if we exhibit an assignment Λ of lists for G such that each $\Lambda(v)$ is a union of half-open intervals, and $\mu(\Lambda(v)) \geq \frac{2rk}{r+k}$ for each $v \in V$.

As in the proof of Theorem 3.6, let the vertex set $V = \{u_0, \ldots, u_{k-1}\} \cup$

$\{v_{i_0,\ldots,i_{k-1}} : i_0,\ldots,i_{k-1} \in \{0,\ldots,r-1\}\}$. Assign the following lists:

$$\Lambda(u_p) = [4pk, 4pk + 2k\delta),$$

$$\Lambda(v_{i_0,\ldots,i_{k-1}}) = \bigcup_{p=0}^{k-1}[4pk + 2k\delta\tfrac{i_p+1}{r} - 1, 4pk + 2k\delta\tfrac{i_p}{r} + 1),$$

where we set $\delta \in (0, \frac{r}{k})$ to ensure that the upper end of each of these intervals is larger than the lower.

For each $p = 0,\ldots, k-1$, whatever colour $\gamma(u_p)$ we choose for u_p, there is some $i_p \in \{0,\ldots, r-1\}$ such that $\gamma(u_p) \in [4pk + 2k\delta\frac{i_p}{r}, 4pk + 2k\delta\frac{i_p+1}{r})$. But then for every $\lambda \in \Lambda(v_{i_0,\ldots,i_{k-1}})$, $|\lambda - \gamma(u_p)| < 1$ for some $p \in \{0,\ldots, k-1\}$, and so there is no valid colour for $v_{i_0,\ldots,i_{k-1}}$ in its list. Thus there is no proper Λ-colouring of G.

The lists all have the same measure when $2k\delta = k(2-\frac{2k\delta}{r})$, which rearranges to give $\delta = \frac{r}{k+r}$ and $\mu(\Lambda(v)) = \frac{2kr}{k+r}$ for each $v \in V$. Hence in this case, Λ satisfies all the conditions we required, and we deduce that $\sigma(G) \geq \frac{2kr}{k+r}$. $\quad\square$

It is somewhat surprising that the bound we obtain in Theorem 5.29 is symmetric in k and r, since the graph G is not symmetric in these parameters. We remark that equality cannot hold for all $k, r \in \mathbb{N}$, which can be seen from the following two cases of Theorem 5.29:

$$k = 2,\ r = 8 \implies \sigma(K_{2,64}) \geq \tfrac{32}{10} = 3.2,$$

$$k = 6,\ r = 2 \implies \sigma(K_{6,64}) \geq \tfrac{24}{8} = 3.$$

It is clear that equality cannot hold in the second case since $K_{2,64} \subset K_{6,64}$,

and thus $\sigma(K_{6,64}) \geq \sigma(K_{2,64}) \geq 3.2$. However, if $k = 1$ or $r = 1$ then G is a star graph, and Theorem 5.5 implies that equality holds. Furthermore, we conjecture that equality holds at least for $k = 2$:

Conjecture 5.30 *For $r \in \mathbb{N}$, let $G = K_{2,r^2}$. Then $\sigma(G) = \frac{4r}{r+2} = 4 - \frac{8}{r+2}$.*

The technique of 'forbidden sets' used in the proof of Theorem 3.9 can be used to make some progress towards Conjecture 5.30. Let $G = K_{2,m}$ with vertex set $V = \{u, w\} \cup \{v_1, \ldots, v_m\}$, and let Λ be a real list assignment for G. For $i = 1, \ldots, m$ we define

$$F_i = \{\langle \alpha, \beta \rangle \in \mathbb{R}^2 : \Lambda(v_i) \subseteq (\alpha - 1, \alpha + 1) \cup (\beta - 1, \beta + 1)\}, \tag{5.7}$$

the set of pairs of colours $\langle \alpha, \beta \rangle$ such that if $\gamma(u) = \alpha$ and $\gamma(w) = \beta$, there is no valid colour for v_i in $\Lambda(v_i)$. (Here we use angle brackets $\langle \alpha, \beta \rangle$ for ordered pairs, to distinguish them from open intervals.) Then there is no proper Λ-colouring of G iff $\Lambda(u) \times \Lambda(w) \subseteq \bigcup_{i=1}^{m} F_i$.

We will now derive a general bound on the measure of F_i in terms of the measure of $\Lambda(v_i)$. Let $\mu(\Lambda(v_i)) = \kappa \in [2, 4)$, and define

$$\lambda_i^1 = \inf \Lambda(v_i),$$

$$\lambda_i^2 = \inf \{\lambda \in \Lambda(v_i) : \mu(\Lambda(v_i) \cap [\lambda, \infty)) \leq 2\},$$

$$\lambda_i^3 = \sup \{\lambda \in \Lambda(v_i) : \mu(\Lambda(v_i) \cap (-\infty, \lambda]) \leq 2\},$$

$$\lambda_i^4 = \sup \Lambda(v_i).$$

Note that $\lambda_i^1, \lambda_i^2, \lambda_i^3, \lambda_i^4 \in \Lambda(v_i)$ since $\Lambda(v_i)$ is closed, and $\lambda_i^1 \leq \lambda_i^2 < \lambda_i^3 \leq \lambda_i^4$

since $2 \le \kappa < 4$.

Now suppose $\langle \alpha, \beta \rangle \in F_i$, so that $\Lambda(v_i) \subseteq (\alpha - 1, \alpha + 1) \cup (\beta - 1, \beta + 1)$. Note that $\alpha \ne \beta$, since $\mu(\Lambda(v_i)) \ge 2$ and so $\Lambda(v_i) \not\subseteq (\alpha - 1, \alpha + 1)$, and that $\langle \alpha, \beta \rangle \in F_i$ iff $\langle \beta, \alpha \rangle \in F_i$. Now assume $\alpha < \beta$. We must have $\lambda_i^1 \in (\alpha - 1, \alpha + 1)$ and $\lambda_i^4 \in (\beta - 1, \beta + 1)$; since $\lambda_i^3 - \lambda_i^1 \ge 2$ and $\lambda_i^4 - \lambda_i^2 \ge 2$, these imply respectively that $\lambda_i^3 \in (\beta - 1, \beta + 1)$ and $\lambda_i^2 \in (\alpha - 1, \alpha + 1)$. Thus

$$\alpha \in (\lambda_i^2 - 1, \lambda_i^1 + 1) \quad \text{and} \quad \beta \in (\lambda_i^4 - 1, \lambda_i^3 + 1). \tag{5.8}$$

Let $F_i^- = \{ \langle \alpha, \beta \rangle \in F_i : \alpha < \beta \}$; since $\lambda_i^2 - \lambda_i^1 \ge \kappa - 2$ and $\lambda_i^4 - \lambda_i^3 \ge \kappa - 2$, equation (5.8) tells us that $\mu(F_i^-) \le (4 - \kappa)^2$. We can reduce this bound: suppose $\alpha = \lambda_i^2 - 1 + \delta$ and $\beta = \lambda_i^3 + 1 - \varepsilon$. Then since $\mu\big(\Lambda(v_i) \cap (-\infty, \lambda_i^2]\big) = \mu\big(\Lambda(v_i) \cap [\lambda_i^3, \infty)\big) = \kappa - 2$, we have $\mu(\Lambda(v_i) \cap (\alpha - 1, \alpha + 1)) \le \kappa - 2 + \delta$ and $\mu(\Lambda(v_i) \cap (\beta - 1, \beta + 1)) \le \kappa - 2 + \varepsilon$. But $\langle \alpha, \beta \rangle \in F_i$ then implies that $\kappa - 2 + \delta + \kappa - 2 + \varepsilon \ge \mu(\Lambda(v_i)) = \kappa$, and so $\delta + \varepsilon \ge 4 - \kappa$. Combined with the previous observations, we deduce that $\mu(F_i^-) \le \frac{1}{2}(4 - \kappa)^2$.

Writing $F_i^+ = F_i \setminus F_i^-$, clearly $\mu(F_i^+) = \mu(F_i^-)$, and so

$$\mu(F_i) \le (4 - \kappa)^2. \tag{5.9}$$

If $\Lambda(v_i) = [0, \kappa]$ we can show that $\mu(F_1) = (4 - \kappa)^2$, so this general bound is sharp. Equation (5.9) allows us to derive the following result.

Lemma 5.31 *Let $\kappa \in [2, 4)$ and $m \le \frac{2\kappa^2}{(4 - \kappa)^2}$, and $G = K_{2,m}$. Then $\sigma(G) \le \kappa$.*

Proof. Let Λ be a real κ-list assignment for G. Using the above notation, we

know there is a proper Λ-colouring of G if $\Lambda(u) \times \Lambda(w) \not\subseteq \bigcup_{i=1}^{m} F_i$, which must hold if $\sum_{i=1}^{m} \mu(F_i \cap (\Lambda(u) \times \Lambda(w))) < \mu(\Lambda(u) \times \Lambda(w)) = \kappa^2$.

As we observed above, $\langle \alpha, \alpha \rangle \notin F_i$ for any $\alpha \in \mathbb{R}$; so if $\alpha \in \Lambda(u) \cap \Lambda(w)$, we can find a suitable colouring γ with $\gamma(u) = \gamma(w) = \alpha$. So assume that $\Lambda(u) \cap \Lambda(w) = \emptyset$; this implies that $(\Lambda(u) \times \Lambda(w)) \cap (\Lambda(w) \times \Lambda(u)) = \emptyset$. By the symmetry of F_i, $\mu(F_i \cap (\Lambda(u) \times \Lambda(w))) = \mu(F_i \cap (\Lambda(w) \times \Lambda(u)))$, and so

$$\mu(F_i \cap (\Lambda(u) \times \Lambda(w))) \leq \tfrac{1}{2}(4 - \kappa)^2.$$

Hence we can find a proper Λ-colouring of G if $\tfrac{1}{2}(4 - \kappa)^2 m < \kappa^2$, that is, if $m < \frac{2\kappa^2}{(4-\kappa)^2}$, and so $\sigma(G) \leq \kappa$ in this case. Finally, if $m = \frac{2\kappa^2}{(4-\kappa)^2}$ and $\kappa < \kappa' < 4$ then $m < \frac{2\kappa'^2}{(4-\kappa')^2}$, and so $\sigma(G) \leq \kappa'$. Since κ' is arbitrary, this implies that $\sigma(G) \leq \kappa$, and this completes the proof of the lemma. $\qquad \square$

Writing $\kappa = 4 - \delta$, we transform the bound in Lemma 5.31 as follows:

$$m \leq \frac{2\kappa^2}{(4 - \kappa)^2} \iff m \leq \frac{2(4 - \delta)^2}{\delta^2}$$

$$\iff 0 \leq \frac{32}{\delta^2} - \frac{16}{\delta} + (2 - m)$$

$$\iff \frac{1}{\delta} \geq \frac{16 + \sqrt{16^2 - 4 \cdot 32(2 - m)}}{2 \cdot 32} = \frac{2 + \sqrt{2m}}{8}$$

$$\iff \kappa \geq 4 - \frac{8}{2 + \sqrt{2m}} \,.$$

Setting $m = r^2$, and combining the above result with Theorem 5.29, we obtain the following bounds on $\sigma(K_{2,r^2})$.

Theorem 5.32 *For $r \geq 2$, let $G = K_{2,r^2}$. Then*

$$4 - \frac{8}{2 + r} \leq \sigma(G) \leq 4 - \frac{8}{2 + r\sqrt{2}} \, . \qquad \square$$

5.6 Future research

Our efforts to determine various exact values of $\sigma(G)$, even for such small graphs as $G = K_{2,4}$ or $G = K_1 + C_5$, have been frustratingly unsuccessful. As further evidence for the difficulty of computing such parameters, we quote Alon and Zaks, who state that their proof that $\mathrm{ch}_s(C_4) \leq \left\lfloor \frac{12s}{7} \right\rfloor + 1$ "is a rather lengthy case-by-case analysis, and is therefore omitted" (from their paper [4]).

Conjectures 3.4 and 5.24 are perhaps the most compelling for further study. As with the $(a : b)$-choosability conjectures (see, for example, Woodall's survey [35]), it is likely that some greater insight into structural or other properties of k-choosable graphs will be needed before significant progress can be made. Attempts to adapt the techniques of Section 4.3 to prove that $\sigma(G)$ is rational for finite graphs G have also proved unsuccessful, though we conjecture that this is the case.

We could ask questions about the spectrum of $\sigma(G)$, by analogy with results in Section 4.5, but with our limited ability to produce upper bounds on $\sigma(G)$, answers are unlikely to be forthcoming. We may at least look at the possible values of $\sigma(G) < 2$, and ask whether they form a discrete set as do the values of $\tau(G) < 2$.

Fiala, Král and Škrekovski [10, 11] have recently introduced a common

generalisation of the channel assignment problem and T-colouring, in which each vertex has a list $L(v) \subseteq \mathbb{Z}$ and each edge has a set $t(e) \subseteq \mathbb{N}_0$. The goal is to find a colouring c with $c(v) \in L(v)$ for $v \in V$, and $|c(v) - c(w)| \notin t(vw)$ for $uv \in E$. They prove a Brooks-type theorem characterising exactly those list assignments L with $|L(v)| = \sum_{w \in N(v)} |t(vw)|$ for all $v \in V$ which admit a colouring as described. As ever more generalised colouring problems are introduced, it becomes correspondingly harder to derive interesting results about them, but this is still an area worthy of further study.

5.7   Appendix: Lebesgue measure

We include here the formal definitions from measure theory which are needed to define Lebesgue measure and to prove Lemma 5.8.

Let X be any set. A family $\Sigma \subseteq \mathcal{P}(X)$ is a *σ-algebra* if it satisfies

(i) $\emptyset \in \Sigma$ and $X \in \Sigma$,

(ii) if $A \in \Sigma$ then $X \setminus A \in \Sigma$, and

(iii) if $A_1, A_2, \ldots \in \Sigma$ then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

The smallest σ-algebra containing a family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called the σ-algebra *generated* by $\mathcal{F}$. The *Borel σ-algebra* $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by the open subsets of $\mathbb{R}$; its elements are the *Borel sets.* Naturally, there are many other families which generate $\mathcal{B}(\mathbb{R})$, and one such family is $\{(-\infty, x] : x \in \mathbb{R}\}$ which we will use below.

Given a σ-algebra $\Sigma \subseteq \mathcal{P}(X)$, a function $\mu : \Sigma \to [0, \infty]$ is a *measure* on Σ if it satisfies

(i) $\mu(\emptyset) = 0$, and

(ii) if $A_1, A_2, \ldots \in \Sigma$ are disjoint then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Theorem 5.33 *There is a unique measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$, called* Lebesgue measure, *which satisfies $\mu([a, b]) = b - a$ whenever $-\infty < a \leq b < \infty$.* $\square$

The proof of Lemma 5.8 requires the following additional definition and lemma. A family $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *Dynkin system* if it satisfies

(i) $X \in \mathcal{D}$,

(ii) if $A, B \in \mathcal{D}$ and $A \subseteq B$ then $B \setminus A \in \mathcal{D}$, and

(iii) if $A_1, A_2, \ldots \in \mathcal{D}$ and $A_1 \subseteq A_2 \subseteq \cdots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{D}$.

Lemma 5.34 (Dynkin's Lemma) *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be closed under intersection. Then any Dynkin system containing $\mathcal{F}$ also contains the σ-algebra generated by $\mathcal{F}$.* $\square$

Corollary 5.35 *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be closed under intersection, and let Σ be the σ-algebra generated by $\mathcal{F}$. Suppose that μ_1 and μ_2 are measures on Σ with $\mu_1(X) = \mu_2(X) < \infty$. If μ_1 and μ_2 agree on $\mathcal{F}$, then $\mu_1 = \mu_2$.* $\square$

For the proof of Dynkin's Lemma see, for example, [2, p.153]. Corollary 5.35 follows by showing that $\mathcal{D} = \{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$ is a Dynkin system.

Proof of Lemma 5.8. Let $\mathcal{F} = \{(-\infty, x] : x \in \mathbb{R}\}$, and for a Borel set $B \subseteq \mathbb{R}$, define $\mu_1(B) = \mu(f(B))$ and $\mu_2(B) = \mu(\Lambda \cap B)$. The Borel sets are generated as a σ-algebra by $\mathcal{F}$, and $\mathcal{F}$ is closed under intersection. Since f is continuous and nondecreasing, if $B_x = (-\infty, x]$ then either $f(B_x) = (0, f(x)]$ or $f(B_x) = [0, f(x)]$, and so

$$\mu_1(B_x) = \mu(f(B_x)) = f(x) = \mu(\Lambda \cap B_x) = \mu_2(B_x).$$

Thus μ_1 and μ_2 agree on $\mathcal{F}$, and so Lemma 5.8 follows from Corollary 5.35 once we have shown that μ_1 and μ_2 are measures.

Let $B_1, B_2, \ldots$ be a sequence of disjoint Borel sets and $B = \bigcup_{i=1}^{\infty} B_i$; we need to show that $\mu_1(B) = \sum_{i=1}^{\infty} \mu_1(B_i)$ and $\mu_2(B) = \sum_{i=1}^{\infty} \mu_2(B_i)$. The second statement is straightforward, since

$$\mu_2(B) = \mu(\Lambda \cap B) = \mu\left(\bigcup_{i=1}^{\infty}(\Lambda \cap B_i)\right) = \sum_{i=1}^{\infty} \mu(\Lambda \cap B_i) = \sum_{i=1}^{\infty} \mu_2(B_i).$$

Turning now to μ_1, we will need the fact that $\mu(f(B_i) \cap f(B_j)) = 0$ when $i \neq j$. For each $y \in f(B_i) \cap f(B_j)$, there exists $x \in B_i$ such that $f(x) = y$, and $x' \in B_j$ such that $f(x') = y$. Let $\Gamma_y \subseteq \mathbb{R}$ be the interval $[x, x']$ or $[x', x]$ (according as $x < x'$ or $x' < x$). The intervals Γ_y all have nonzero length and are pairwise disjoint, and so there can only be countably many of them. Hence $f(B_i) \cap f(B_j)$ is countable, and so $\mu(f(B_i) \cap f(B_j)) = 0$.

Define $C_1 = f(B_1)$, and $C_i = f(B_i) \setminus \bigcup_{j<i} C_j$ for $i \geq 2$; then the above argument shows that $\mu(C_i) = \mu(f(B_i))$. Furthermore, the sets $C_1, C_2, \ldots$ are disjoint, and $\bigcup_{i=1}^{\infty} C_i = f(B)$; hence

$$\mu_1(B) = \mu(f(B)) = \sum_{i=1}^{\infty} \mu(C_i) = \sum_{i=1}^{\infty} \mu(f(B_i)) = \sum_{i=1}^{\infty} \mu_1(B_i).$$

Hence μ_1 and μ_2 are measures, as required. $\qquad\square$

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