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Ratios of sums of two Fibonacci numbers equal to powers of 2

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Abstract. In this paper, we find all solutions to the Diophantine equation $F_n + F_m = 2^a(F_r + F_s)$, where $\{F_k\}_{k\geq 0}$ is the Fibonacci sequence. This paper continues and extends previous work, which investigated the powers of 2 that are sums of two Fibonacci numbers.

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1. Introduction

Let $\{F_k\}_{k\geq 0}$ be the *Fibonacci sequence* given by $F_{k+2} = F_{k+1} + F_k$, for all $k \geq 0$, where $F_0 = 0$ and $F_1 = 1$. The problem of determining all integer solutions to Diophantine equations with Fibonacci numbers has gained a considerable amount of interest among the mathematicians and there is a very broad literature on this subject. In addition, these numbers show up in many areas of mathematics and in nature. Also, there is the Lucas sequence, which is as important as the Fibonacci sequence. The Lucas sequence $\{L_k\}_{k\geq 0}$ follows the same recursive pattern as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book [4].

In the present paper we extend the work [2], which investigated the powers of 2 that are sums of two Fibonacci numbers. To be more precise, we find all solutions of the Diophantine equation

$$F_n + F_m = 2^a(F_r + F_s) \qquad \text{with} \qquad n, m, a, r, s \geq 0. \tag{1}$$

Let us first give some terminology. Given a positive integer N the Zeckendorf decomposition of N is a representation of the form

$$N = F_{n_1} + F_{n_2} + \cdots + F_{n_k},$$

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where $n_i - n_{i+1} \geq 2$. This always exists and up to identifying F_1 with F_2, it is unique. In (1), we ignore the solutions for which $n = m = r = s = 0$ (and any $a \geq 0$). If one or more of the Fibonacci numbers involved in (1) equals 1, we then assume that its index is 2. Finally, when $N = F_n + F_m$ and $M = F_r + F_s$, we assume that $n > m \geq 0$, $r > s \geq 0$ and that the above representations are the Zeckendorf decompositions of N and M, respectively. This rules out cases like $m = n - 1$, for which $N = F_n + F_{n-1} = F_{n+1}$, as well as $n = m$, for which $N = F_n + F_n = 2F_n = F_{n+1} + F_{n-2}$. Finally, we also ignore the trivial diagonal solutions $(n, m) = (r, s)$ and $a = 0$. The rest of solutions will be called *non-degenerate*.

The theorem is as follows.

Theorem 1. *Equation* (1) *has two parametric families of non-degenerate solutions* (n, m, a, r, s) *with* $n > m \geq 0$ *and* $r > s \geq 0$*, namely*

$$(n, n-3, 1, n-1, 0): \qquad F_n + F_{n-3} = 2F_{n-1}, \qquad n \geq 3;$$
$$(n, n-6, 1, n-2, n-4): \qquad F_n + F_{n-6} = 2(F_{n-2} + F_{n-4}), \qquad n \geq 6.$$

When $n = 4,\ 7$*, in the first and second families, we must take* $m = 2$ (*instead of* $m = 1$)*, respectively. In addition, putting* $N := F_n + F_m$*, there are exactly* 12 *values of* $N = F_n + F_m$ *yielding* 21 *more sporadic solutions, namely:*

$$\begin{aligned}
4 &= F_4 + F_2 = 2^2 F_2;\\
8 &= F_6 = 2^2 F_3 = 2^3 F_2;\\
16 &= F_7 + F_4 = 2^2(F_4 + F_2) = 2^3 F_3 = 2^4 F_2;\\
18 &= F_7 + F_5 = 2(F_6 + F_2);\\
24 &= F_8 + F_4 = 2^2(F_5 + F_2) = 2^3 F_4;\\
36 &= F_9 + F_3 = 2^2(F_6 + F_2);\\
56 &= F_{10} + F_2 = 2^2(F_7 + F_2) = 2^3(F_5 + F_3);\\
60 &= F_{10} + F_5 = 2^2(F_7 + F_3);\\
92 &= F_{11} + F_4 = 2^2(F_8 + F_3);\\
144 &= F_{12} = 2^2(F_9 + F_3) = 2^3(F_7 + F_5) = 2^4(F_6 + F_2);\\
288 &= F_{13} + F_{10} = 2^3(F_9 + F_3) = 2^4(F_7 + F_5) = 2^5(F_6 + F_2);\\
1008 &= F_{16} + F_8 = 2^4(F_{10} + F_6).
\end{aligned}$$

Our proof uses elementary considerations, linear forms in logarithms and reduction techniques.

2. The proof

2.1. The cases $a = 0, 1$

Although mentioned in the title of the subsection, we do not have to deal with the case $a = 0$ because in this case $N = M$, and since we work with the Zeckendorf representations of N and M, we conclude that the only situations are the diagonal degenerate ones, namely $(n,m)=(r,s)$. Thus, $a\geq 1$. Assume next that $n>m$. Then

$$2F_n > F_n + F_m = 2^a(F_r+F_s) \geq 2(F_r+F_s) \geq 2F_r,$$

so $n>r$.

Next we deal with the case $a=1$. We have

$$F_n + F_m = 2(F_r+F_s) = 2F_r + 2F_s = F_{r+1} + F_{r-2} + 2F_s.$$

The case $s=0$ gives $r=n-1,\ m=r-2=n-3$, which is the first parametric family. If $s\geq 2$, then we get

$$F_n + F_m = F_{r+1} + F_{r-2} + F_{s+1} + F_{s-2}. \tag{2}$$

If $s\leq r-5$, then the right–hand side of (2) has a Zeckendorf decomposition of length 4 (if $s>2$) or 3 (if $s=2$), and the left–hand side has a Zeckendorf decomposition of length 2 if $m>0$ or 1 if $m=0$, a contradiction.

If $s=r-4$, then the right–hand side of (2) is

$$F_{r+1} + (F_{r-2}+F_{r-3}) + F_{r-6} = F_{r+1} + F_{r-1} + F_{r-6}.$$

This is a Zeckendorf decomposition of length 3 except if $r=6$, when it is a Zeckendorf decomposition with two terms, namely F_7+F_5. This gives $(n,m,r,s)=(7,5,6,2)$, which gives the only sporadic solution with $a=1$ for which $N=18$.

If $s=r-3$, then the right–hand side of (2) is

$$F_{r+1} + 2F_{r-2} + F_{r-5} = F_{r+1} + F_{r-1} + F_{r-4} + F_{r-5} = F_{r+1} + F_{r-1} + F_{r-3},$$

which is a Zeckendorf decomposition with 3 terms, which is not convenient for us.

Finally, if $s=r-2$, then we get that the right–hand side of (2) is

$$F_{r+1} + (F_{r-1}+F_{r-2}) + F_{r-4} = F_{r+1} + F_r + F_{r-4} = F_{r+2} + F_{r-4},$$

and this is a Zeckendorf decomposition of length 2 if $r>4$ and of length 1 if $r=4$. This gives $r=n-2,\ m=r-4=n-6$ and $s=r-2=n-4$, which is the second parametric family of solutions.

From now on, we may assume that $a\geq 2$.

2.2. Bounding a in terms of n and r

Let $(\alpha,\beta)=((1+\sqrt{5})/2,(1-\sqrt{5})/2)$ be the roots of the equation $x^2-x-1=0$. It is well-known that the Binet formula

$$F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}} \quad \text{holds for all} \quad n\geq 0.$$

We use

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \qquad \text{for all} \qquad k\geq 1,$$

to get that

$$\alpha^{n-2} \leq F_n + F_m = 2^a(F_r+F_s) \leq 2^a(2F_r) \leq 2^{a+1}\alpha^{r-1},$$

so

$$2^{a+1} \geq \alpha^{n-r-1}.$$

Also,

$$2\alpha^{n-1} \geq 2F_n \geq F_n + F_m = 2^a(F_r + F_s) \geq 2^a F_r \geq 2^a \alpha^{r-2},$$

which gives

$$2^{a-1} \leq \alpha^{n-r+1}.$$

We record these inequalities.

Lemma 1. *The inequalities*

$$2^{a-1} \leq \alpha^{n-r+1} \qquad \text{and} \qquad 2^{a+1} \geq \alpha^{n-r-1}$$

hold.

2.3. Matveev's theorem

We continue with some notations and terminologies from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the remaining of this paper without reference:

$$\begin{aligned} h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}), \\ h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2. \end{aligned}$$

In order to prove our main result Theorem 1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such bounds in the literature like that of Baker and Wüstholz from [1]. We use the one of Matveev from [5]. Matveev proved the following theorem, which is one of our main tools in this paper.

Theorem 2 (Matveev's theorem)**.** *Let $\alpha_1, \ldots, \alpha_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, let $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \alpha_1^{b_1} \cdots \alpha_t^{b_t} - 1,$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+\log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad \textit{for all} \qquad i = 1, \ldots, t.$$

2.4. Six linear forms in logarithms

We take $C_1 := 10^{10}$, $C_2 := 10^{12}$,

$$f_i(n) := C_1(2.2C_2)^{i-1}(1+\log n)^i, \qquad i = 1, 2, 3, 4,$$

and put

$$\mathcal{T} = \{n-m, r-s, r+s, n\} = \{t_1, t_2, t_3, t_4\},$$

where $t_1 \leq t_2 \leq t_3 \leq t_4$. We prove the following lemma.

Lemma 2. *We have*

$$t_i \leq f_i(n), \quad \textit{for} \quad i = 1, 2, 3, 4.$$

Notice that the lemma gives

$$n \leq t_4 \leq f_4(n), \qquad \text{which gives} \qquad n < 10^{56}.$$

In the next section, we will lower the upper bound for n.

Proof. We will apply Matveev's theorem to 6 linear forms in logarithms labelled as follows:

$$\Lambda_1, \Lambda_{2,1}, \Lambda_{2,2}, \Lambda_{3,1}, \Lambda_{3,2}, \Lambda_4,$$

where

$$\Lambda_U := \alpha^{-(n-\delta_U r)} 2^a \eta_U - 1, \quad U \in \{1, \{2,1\}, \{2,2\}, \{3,1\}, \{3,2\}, 4\},$$

where $\delta_U = 1$, except for $U \in \{\{3,1\}, 4\}$ when $\delta_U = 0$, and with

$$\eta_1 := 1, \quad \eta_{2,1} := (1+\alpha^{m-n})^{-1}, \quad \eta_{2,2} := 1+\alpha^{s-r},$$
$$\eta_{3,1} := \sqrt{5}(F_r+F_s), \quad \eta_{3,2} := \frac{1+\alpha^{r-s}}{1+\alpha^{m-n}}, \quad \eta_4 := \frac{\sqrt{5}(F_r+F_s)}{1+\alpha^{m-n}}. \tag{3}$$

In order to deduce the upper bounds on t_i, we show that

$$|\Lambda_{U_i}| < \frac{100}{\alpha^{t_{i+1}}}, \qquad \text{for} \qquad i = 0, 1, 2, 3, \tag{4}$$

where $U_0 = 1$, $U_1 \in \{\{2,1\}, \{2,2\}\}$, $U_2 \in \{\{3,1\}, \{3,2\}\}$, $U_3 = 4$. We also show that $\Lambda_{U_i} \neq 0$ for any $i = 0, 1, 2, 3$, and we show that (4) implies, via Matveev's theorem and recursively on i, that $t_{i+1} \leq f_{i+1}(n)$.

Since we have many things to prove, we will first explain how to deduce inequalities (4) for $i = 0, 1, 2, 3$. Then we will show how inequality (4) for $i = 0$ implies $t_1 \leq f_1(n)$. Then, for $i \geq 1$, we show how inequality (4) for i, Matveev's theorem, the assumption that $\Lambda_{U_i} \neq 0$, and the fact that $t_{j+1} \leq f_{j+1}(n)$ holds for $j = 0, 1, \ldots, i-1$, implies that $t_{i+1} \leq f_{i+1}(n)$.

So, let us first see how they work. Let $i = 0$. We rewrite our equation (1) using the Binet formula for the Fibonacci numbers as

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} + \frac{\alpha^m - \beta^m}{\sqrt{5}} = 2^a \left(\frac{\alpha^r - \beta^r}{\sqrt{5}} + \frac{\alpha^s - \beta^s}{\sqrt{5}} \right),$$

giving

$$\begin{aligned} |\alpha^n - 2^a\alpha^r| &= |\beta^n - \alpha^m + \beta^m + 2^a\alpha^s - 2^a\beta^r - 2^a\beta^s| \\ &\leq |\beta|^n + |\beta|^m + \alpha^m + 2^a\alpha^s + 2^a|\beta|^r + 2^a|\beta|^s \\ &\leq 2 + \alpha^m + 2^a\alpha^s + 2^{a+1} \leq 3(\alpha^m + 2^a\alpha^s) \\ &\leq 3(\alpha^m + 2\alpha^{n-r+s+1}) \leq 3(2\alpha+1)\alpha^{\max\{m, n-r+s\}}, \end{aligned}$$

where in the above we used that $|\beta| < 1$ and Lemma 1. Dividing across by α^n, we get

$$|\Lambda_1| = |\alpha^{-(n-r)}2^a - 1| < \frac{3(2\alpha+1)}{\alpha^{\min\{n-m, r-s\}}} < \frac{100}{\alpha^{t_1}}, \tag{5}$$

and we recognise as (4) for $i = 0$. Note that we also get that $t_1 = \min\{n-m, r-s\}$. In the same way, we prove that (4) holds for $i = 1, 2, 3$. Let's see the details.

For $i = 1$, if $t_1 = n - m$, then we rewrite (1) as

$$\begin{aligned} |\alpha^n(1 + \alpha^{m-n}) - 2^a\alpha^r| &= |\beta^n + \beta^m + 2^a\alpha^s - 2^a\beta^r - 2^a\beta^s| \\ &\leq 2 + 2^a\alpha^s + 2^{a+1} < 3(1 + 2^a\alpha^s) \\ &< 3(1 + 2\alpha^{n-r+s+1}) < 3(2\alpha+1)\alpha^{n-r+s}, \end{aligned}$$

so, dividing across by $\alpha^n(1 + \alpha^{m-n})$, we get

$$|\Lambda_{2,1}| = |\alpha^{-(n-r)}2^a(1 + \alpha^{m-n})^{-1} - 1| < \frac{3(2\alpha+1)}{\alpha^{r-s}(1 + \alpha^{m-n})} < \frac{100}{\alpha^{t_2}}, \tag{6}$$

which is (4) at $i = 2$. We also note that in this case $t_1 = n - m,\ t_2 = r - s$. On the other hand, if $t_1 = r - s$, then we rewrite (1) as

$$\begin{aligned} |\alpha^n - 2^a\alpha^r(1 + \alpha^{s-r})| &= | - \alpha^m + \beta^n + \beta^m - 2^a\beta^r - 2^a\beta^s| \\ &\leq \alpha^m + 2 + 2^{a+1}|\beta|^s = \alpha^m + 2 + 2^{a+1}\alpha^{-s} \\ &< 3(\alpha^m + 2\alpha^{n-r-s+1}) < 3(2\alpha+1)\alpha^{\max\{m, n-r-s\}}, \end{aligned}$$

and dividing across by α^n, we get

$$|\Lambda_{2,2}| = |\alpha^{-(n-r)}2^a(1 + \alpha^{s-r}) - 1| < \frac{3(2\alpha+1)}{\alpha^{\min\{n-m, r+s\}}} \leq \frac{100}{\alpha^{t_2}}\cdot \tag{7}$$

Here, $t_1 = r - s$ and $t_2 = \min\{n-m, r+s\}$. A similar argument works for $i = 2$ distinguishing the various possibilities for t_1, t_2. In the most asymmetric case $t_1 = r - s, t_2 = r + s$, we rewrite equation (1) as

$$|\alpha^n - 2^a\sqrt{5}(F_r + F_s)| = |-\alpha^m + \beta^m + \beta^n| \le 3\alpha^m,$$

so, dividing across by α^n we get

$$|\Lambda_{3,1}| = |\alpha^{-n}2^a\sqrt{5}(F_r + F_s) - 1| < \frac{3}{\alpha^{n-m}} < \frac{100}{\alpha^{t_3}}, \tag{8}$$

which is what we wanted. In the remaining cases, we have $\{t_1, t_2\} = \{n-m, r-s\}$, and then we rewrite (1) as

$$\begin{aligned}|\alpha^n(1+\alpha^{m-n}) - 2^a\alpha^r(1+\alpha^{s-r})| &= |\beta^m + \beta^n + 2^a\beta^r + 2^a\beta^s| \\ &\le 2|\beta|^m + 2^{a+1}|\beta|^s < 2\alpha^{-m} + 4\alpha^{n-r-s+1} \\ &\le (2+4\alpha)\alpha^{\max\{-m, n-r-s\}},\end{aligned}$$

which after dividing it by $\alpha^n(1+\alpha^{m-n})$ we recognise that it leads to

$$|\Lambda_{3,2}| = \left|\alpha^{-(n-r)}2^a\left(\frac{1+\alpha^{s-r}}{1+\alpha^{m-n}}\right) - 1\right| < \frac{2+4\alpha}{\alpha^{\min\{n+m, r+s\}}(1+\alpha^{m-n})} < \frac{100}{\alpha^{t_3}}. \tag{9}$$

Here, we take $t_3 = \min\{r+s, n\}$. Cleary, $n > \max\{n-m, r-s\}$ (because $n > r$), so we cannot have $n \in \{t_1, t_2\}$. If $n \ne t_4$, then we get that $n = t_3$, which leads to $t_4 = r + s < 2n$. Thus, by the $i = 2$ step, we would get that $n < f_3(n)$, and later on $t_4 = r + s < 2n < 2f_3(n) < f_4(n)$. So, the inequality for $i = 3$ follows right away from the inequality of $i = 2$. It remains to study the case when $n = t_4$. In this case, we rewrite equation (1) as

$$|\alpha^n(1+\alpha^{m-n}) - 2^a(\sqrt{5}(F_r + F_s))| = |\beta^n + \beta^m| \le 2,$$

and dividing across by $\alpha^n(1+\alpha^{m-n})$ we get

$$|\Lambda_4| = \left|\alpha^{-n}2^a\left(\frac{\sqrt{5}(F_r+F_s)}{1+\alpha^{m-n}}\right) - 1\right| < \frac{2}{\alpha^n(1+\alpha^{m-n})} < \frac{2}{\alpha^n} < \frac{100}{\alpha^{t_4}}, \tag{10}$$

which is inequality (4) at $i = 3$.

Having justified inequalities (4), let us see how to deduce the upper bounds on t_i. We will prove that $\Lambda_U \ne 0$ later. So far, to get a lower bound on Λ_U, note that

$$\Lambda_U = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1,$$

where

$$\alpha_1 = \alpha, \quad \alpha_2 = 2, \quad \alpha_3 = \eta_U, \quad b_1 = -(n - \delta_U r), \quad b_2 = a, \quad b_3 = 1.$$

Notice that $\mathbb{K} := \mathbb{Q}(\alpha)$ has degree $D = 2$ and contains $\alpha_1, \alpha_2, \alpha_3$. Next, $b_1 \le n$. As for b_2, Lemma 1 tells us that $2^{a-1} \le \alpha^{n-r+1}$. If $r \ge 2$, then we get $a < n$

since $\alpha < 2$. On the other hand, if $r = 1$, then $F_n + F_m = 2^a$, which implies that $n \le 7$ by the main result in [2], and then the inequalities $t_i < f_i(n)$ hold anyway for all $i = 1, 2, 3, 4$. Thus, we may take $B := n > \max\{|b_1|, |b_2|, |b_3|\}$. We take $A_1 := \log \alpha$, $A_2 := 2\log 2$. At $i = 0$, we take $U_0 := 1$, $\eta_{U_0} = \eta_1 = 1$, so we have a linear form in two logarithms only. By Matveev's Theorem 2, we get

$$|\Lambda_{U_0}| > \exp(-C_0(1 + \log n)),$$

where

$$C_0 = 1.4 \times 30^5 \times 2^{4.5} 2^2 (1 + \log 2)(\log \alpha)(2\log 2) < 4 \times 10^9.$$

Applying inequality (4) at $i = 0$, we get

$$t_1 \log \alpha < \log 100 + C_0(1 + \log n) < 4.1 \times 10^9 (1 + \log n),$$

so

$$t_1 < \frac{4.1}{\log \alpha} \times 10^9 (1 + \log n) < 10^{10}(1 + \log n) < f_1(n).$$

This is the start. Assume now that $i \ge 2$ and that $t_j \le f_j(n)$ has been established for $j = 1, \ldots, i-1$. We apply Matveev's Theorem 2 to $|\Lambda_{U_{i-1}}|$. We then get that

$$|\Lambda_{U_{i-1}}| > \exp(-C_3(1 + \log n)(2h(\eta_{U_{i-1}}))),$$

where

$$C_3 = 1.4 \times 30^6 \times 3^{4.5} 2^2 (1 + \log 2)(\log \alpha)(2\log 2) < 7 \times 10^{11}.$$

It remains to bound $h(\eta_{U_{i-1}})$. Note that

$$h(\eta_{U_{i-1}}) \le \begin{cases} t_1(\log \alpha)/2 + \log 2 & i = 2, \\ (r+s)\log \alpha + \log 2 + (\log 5)/2, & i = 3, \text{ or} \\ (r-s)(\log \alpha)/2 + (n-m)(\log \alpha)/2 + 2\log 2, & i = 3, \\ (n-m)(\log \alpha)/2 + (r+s)\log \alpha + 2\log 2 + (\log 5)/2, & i = 4. \end{cases} \quad (11)$$

Since $2\log 2 + (\log 5)/2 < 3$, it follows from the above that

$$h(\eta_{U_{i-1}}) < \frac{3t_{i-1}\log \alpha}{2} + 3 < \frac{3}{2}\left(f_{i-1}(n)\log \alpha + 2\right).$$

We thus get that

$$t_i \log \alpha < \log 100 + 7 \times 10^{11} \times 3(f_{i-1}(n)\log \alpha + 2)(1 + \log n),$$

which gives

$$\begin{aligned} t_i &< \frac{\log 100}{\log \alpha} + 7 \times 10^{11} \times 3\left(f_{i-1}(n) + \frac{2}{\log \alpha}\right)(1 + \log n) \\ &< 2.2 \times 10^{12} f_{i-1}(n)(1 + \log n) = f_i(n), \end{aligned}$$

which is what we wanted. In the above, we used the fact that

$$\begin{aligned}
&\frac{\log 100}{\log\alpha} + 7\times 10^{11}\times 3\left(f_{i-1}(n)+\frac{2}{\log\alpha}\right)(1+\log n)\\
&\quad < (0.71\times 10^{12}\times 3)\left(f_{i-1}(n)+\frac{2}{\log\alpha}\right)(1+\log n)\\
&\quad < (2.13\times 10^{12})(1.01 f_{i-1}(n))(1+\log n)\\
&\quad < 2.2\times 10^{12} f_{i-1}(n)(1+\log n) = f_i(n),
\end{aligned}$$

for any $i \geq 2$ and any $n \geq 2$. □

2.5. Justifying that $\Lambda_U \neq 0$

For $i=0$, the form Λ_1 appears on the left–hand side of (5). This is zero if and only if $\alpha^{n-r}=2^a$. This implies $n=r$ and $a=0$, which is not allowed. For $i=1$, the form is the one appearing on the left–hand sides of one of (6) or (7). This gives

$$\alpha^{-(n-r)}2^a(1+\alpha^{-t_1})^{\pm 1}=1.$$

Taking norms and absolute values in $\mathbb{K}$, we get that

$$2^{2a}=|N(1+\alpha^{-t_1})|^{\pm 1}=|N(\alpha^{t_1}+1)|^{\pm 1}.$$

The one with a negative exponent cannot hold since $1+\alpha^{t_1}$ is an algebraic integer. The one with a positive exponent gives

$$2^{2a}=(\alpha^{t_1}+1)(\beta^{t_1}+1)=(\alpha\beta)^{t_1}+1+(\alpha^{t_1}+\beta^{t_1})=L_{t_1}+1+(-1)^{t_1}.$$

If t_1 is odd, we get $L_{t_1}=2^{2a}$. Since 8 never divides L_k for any k, we get $a=1$ and $t_1=3$. If t_1 is even, we get

$$2^{2a}=L_{t_1}+2=\begin{cases}5F_{t_1/2}^2 & \text{if } 2\|t_1,\\ L_{t_1/2}^2 & \text{if } 4\mid t_1.\end{cases}$$

The first case is impossible since 5 does not divide 2^{2a}. The second case leads to $L_{t_1/2}=2^a$ with $t_1/2$ being even, which gives again that $a=1$. However, the case $a=1$ was treated by elementary arguments using Zeckendorf decompositions in the first section of the proof and we are in the case $a\geq 2$. Thus, Λ_{U_1} is nonzero.

For $i=3$, if $\eta_{U_2}=\sqrt{5}(F_r+F_s)$, then the form appears on the left–hand side of (8). If it is zero, then $-\alpha^m+\beta^m+\beta^n=0$. If $m=0$, we get $\beta^n=0$, which is impossible, while if $m\neq 0$, then $m\geq 2$, so $\alpha^2\leq\alpha^m=|\beta^m+\beta^n|<2$, which is a contradiction. If $i=3$ and $\eta_{U_2}=(1+\alpha^{s-r})/(1+\alpha^{m-n})$, then the form appears on the left–hand side of (9). If it zero, we get

$$2^a\alpha^{-(n-r)}=\frac{1+\alpha^{m-n}}{1+\alpha^{r-s}}.$$

Taking norms we get

$$2^{2a} = \left| N\left(\frac{1+\alpha^{m-n}}{1+\alpha^{s-r}}\right)\right| = \frac{|N(\alpha^{n-m}+1)|}{|N(\alpha^{r-s}+1)|} = \frac{L_{n-m}+1+(-1)^{n-m}}{L_{r-s}+1+(-1)^{r-s}}\cdot$$

If $n-m$ is odd, we get $2^{2a} = L_{n-m}/(L_{r-s}+1+(-1)^{r-s})$. Since $8 \nmid L_k$ for any $k \geq 1$, we get that $a = 1$, which is not convenient for us. Thus, $n-m$ is even. If $2\|n-m$, we get

$$2^{2a} = \frac{5F_{(n-m)/2}^2}{L_{r-s}+1+(-1)^{r-s}}\cdot$$

If $r-s$ is odd, the denominator on the right–hand side above is L_{r-s}, a number coprime to 5, so the above equation is impossible since 5 does not divide 2^{2a}. If $4 \mid r-s$, then the denominator on the right–hand side above is $L_{(r-s)/2}^2$, a number coprime to 5, and we get the same contradiction. Finally, it follows that $2\|r-s$, so the equation is

$$2^a = \frac{F_{(n-m)/2}}{F_{(r-s)/2}}\cdot$$

Since $(n-m)/2$ is odd, it follows that $F_{(n-m)/2}$ is even but not a multiple of 4, so $a = 1$, again a contradiction. Finally, if $4 \mid n-m$, we get

$$2^{2a} = \frac{L_{(n-m)/2}^2}{L_{r-s}+1+(-1)^{r-s}}\cdot$$

Note that $L_{(n-m)/2}$ can be even but not a multiple of 4 since $(n-m)/2$ is even. This shows again that $a = 1$, a contradiction. Thus, $\Lambda_{U_2} \neq 0$ in all cases. When $i = 4$, the form Λ_{U_3} appears on the left–hand side of (10). The condition $\Lambda_{U_3} = 0$ then implies $\beta^n + \beta^m = 0$, so $\beta^{n-m} = -1$, which is impossible. Thus, $\Lambda_{U_3} \neq 0$.

2.6. Reduction tools

During the course of our calculations, we got $n < 10^{56}$. This is too large, thus we need to reduce it. To do so, we use some results from the theory of continued fractions. Specifically, for a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [3], Lemma 5a).

For a real number X, we write $||X|| := \min\{|X-n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3. *Let M be a positive integer, p/q a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ some real numbers with $A > 0$ and $B > 1$. If $\varepsilon := ||\mu q|| - M||\tau q|| > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}\cdot$$

The above lemma cannot be applied when μ is a linear combination of 1 and τ since then $\varepsilon < 0$. In this case, we use the following criterion of Legendre (see Theorem 8.2.4 and the top of page 287 in [6]).

Lemma 4 (Legendre)**.** *Let $\tau = [a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of a real number τ, and let x, y be integers such that*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Then $x/y = p_k/q_k$ is a convergent of τ. Furthermore,

$$\left|\tau - \frac{x}{y}\right| \geq \frac{1}{(a_{k+1}+2)y^2}.$$

2.7. Lowering the bounds

We need to find better bounds on t_i for $i = 1, 2, 3, 4$ than the ones implied by Lemma 2 for $n < 10^{56}$.

2.7.1. Bounding t_1

Assume $t_1 \geq 12$. Then the right–hand side of inequality (5) is $< 1/2$. It thus follows that

$$|a \log 2 - (n-r)\log\alpha| < \frac{200}{\alpha^{t_1}}.$$

Dividing across by $(n-r)\log 2$, we get

$$\left|\frac{a}{n-r} - \frac{\log\alpha}{\log 2}\right| < \frac{200}{(n-r)(\log 2)\alpha^{t_1}}. \tag{12}$$

Suppose that $t_1 \geq 290$. Then the right–hand side above is less than $1/(2(n-r)^2)$. Indeed, the inequality

$$\frac{200}{(n-r)(\log 2)\alpha^{t_1}} < \frac{1}{2(n-r)^2}$$

is equivalent to

$$\left(\frac{400}{\log 2}\right)(n-r) < \alpha^{t_1}$$

and this last inequality is fulfilled for $t_1 > 290$ since $n - r \leq n < 10^{56}$. By Legendre's Lemma 4, $a/(n-r) = p_k/q_k$ for some convergent p_k/q_k of $(\log\alpha)/(\log 2)$. Since $q_{113} \leq 10^{56} < q_{114}$, it follows that $k \leq 113$. Since $\max\{a_j : 0 \leq j \leq 114\} = 134$, we get, again by Lemma 4, that the left–hand side of (12) is bounded below by $1/(136(n-r)^2)$. We thus get that

$$\frac{1}{136(n-r)^2} < \frac{200}{(n-r)(\log 2)\alpha^{t_1}},$$

so

$$\alpha^{t_1} < 136\left(\frac{200}{\log 2}\right)(n-r) < 4 \times 10^{60}, \qquad \text{so} \qquad t_1 < 290,$$

a contradiction. This shows that $t_1 \leq 290$.

2.7.2. Bounding t_2

We assume that $t_2 \geq 300$. We work with inequality (6) or (7), according to whether $t_1 = n - m$ or $t_1 = r - s$, respectively. In either case, since $100/\alpha^{t_2} < 1/2$, we get that

$$|a \log 2 - (n-r) \log \alpha \pm \log L| < \frac{200}{\alpha^{t_2}}, \qquad \text{where} \qquad L := 1 + \alpha^{-t_1}.$$

Dividing both sides by $\log \alpha$, we get

$$|a\tau - (n-r) \pm \mu| < \frac{200}{(\log \alpha)\alpha^{t_2}} < \frac{A}{B^{t_2}}, \tag{13}$$

where we take

$$\tau = \frac{\log 2}{\log \alpha}, \quad A = 420, \quad B = \alpha, \quad \mu = \frac{\log(1 + \alpha^{-t_1})}{\log \alpha}, \quad t_1 = 0, 2, \ldots, 290.$$

Note that we did not consider $t_1 = 1$ since t_1 is one of $n - m$ and $r - s$, and we work with Zeckendorf representations of N and M, respectively. In the case $t_1 = 0, 3$, we get

$$\mu = \frac{\log 2}{\log \alpha}, \ \frac{\log 2}{\log \alpha} - 1 \in \{\tau, \tau - 1\}, \qquad \text{respectively},$$

and the argument from the analysis of the bound on t_1 (a continued fraction of τ) shows that $t_2 \leq 290$. For $t_1 \in \{2, 4, \ldots, 290\}$, we use the Baker-Davenport reduction method. We choose the convergent $p/q := p_{119}/q_{119}$ given by

$$\frac{57529387452415566443000382245771696218286604563466592417621 82}{399393120349622064042949127811896413861254596818539608038185 3}.$$

We choose $M := 10^{56}$, so $6M < 3 \times 10^{60} < q$. Then $M\|q\tau\| < 0.00005$, while

$$\|q\mu\| > 0.0023 \qquad \text{for all} \qquad t_1 \in \{2, 4, \ldots, 290\}.$$

Hence, $\|q\mu\| - M\|q\tau\| > \varepsilon := 0.0005$ for our choices of t_1. We thus get that

$$t_2 \leq \frac{\log(Aq\varepsilon^{-1})}{\log B} < 324.$$

2.8. Bounding t_3

Here, we need to increase p/q. We choose $p/q = p_{199}/q_{199}$. It turns out that $q < 1.3 \times 10^{103}$. We compute $M\|\tau q\| < 1.7 \times 10^{-47}$. In the asymmetric case $t_1 = r - s$, $t_2 = r + s$, we have $2r = t_1 + t_2 < 620$, so $r < 310$. We generated all numbers of the form $\mu := (\log(\sqrt{5}(F_r + F_s)))/(\log \alpha)$ with $0 \leq s \leq r - 2 < 310$. They appear in the analog of (13) with t_2 replaced by t_3 which is

$$|a\tau - (n - \delta_U r) \pm \mu| < \frac{200}{(\log \alpha)\alpha^{t_3}} < \frac{A}{B^{t_3}}.$$

In our particular situation, $\delta_U = 0$. Computing $\|q\mu\|$, we get that this number is $> 1.6 \times 10^{-37}$ in all cases. Hence, $1.6 \times 10^{-37} - 1.7 \times 10^{-47} > \varepsilon := 10^{-37}$. Then we get that

$$t_3 < \frac{\log(Aq\varepsilon^{-1})}{\log \alpha} < 683.$$

In the case when $\{t_1, t_2\} = \{n - m, r - s\}$, we computed $(1 + \alpha^{-t_2})/(1 + \alpha^{-t_1})$ for $2 \le t_1 < t_2 < 324$. We ignore the case $t_1 = t_2$ since then $\eta_U = 1$ and $t_3 \le 290$ by using the continued fraction of τ as in the bound for t_1. We also ignore the case $\{t_1, t_2\} = \{2, 6\}$. Indeed, if say $n - m = 6, r - s = 2$, then we have $F_n + F_m = F_n + F_{n-6} = 2(F_{n-2} + F_{n-4})$, so we get $2(F_{n-2} + F_{n-4}) = 2^a(F_r + F_{r-2})$. This gives $F_{n-2} + F_{n-4} = 2^{a-1}(F_r + F_{r-2})$. The case $a = 1$ gives the second known parametric family. The case $a - 1 > 0$ yields a new solution $(n', m', a', r', s') = (n - 2, n - 4, a - 1, r, r - 2)$ with $n' - m' = 2 = r' - s'$, so $\eta_U = 1$, showing that $t_3 \le 290$.

The case $r - s = 6, n - m = 2$ is similar, namely we have $F_n + F_{n-2} = F_n + F_m = 2^a(F_r + F_{r-6}) = 2^{a+1}(F_{r-2} + F_{r-4})$, so we got a new solution $(n', m', a', r', s') = (n, m, a + 1, r - 2, r - 4)$ with $n' - m' = r' - s' = 2$ and we again get $t_3 \le 290$.

So, now we computed all numbers of the form $\|q\mu\|$ for such values of μ obtaining that the minimum exceeds 5.5×10^{-6}. Hence, we can take $\varepsilon := 5 \times 10^{-6}$. We then get

$$t_3 < \frac{\log(Aq\varepsilon^{-1})}{\log \alpha} < 532.$$

To summarise, we have that $t_3 < 683$.

2.8.1. Bounding t_4

There is a lot of work to be done here. First of all, if $n < 683$, we are in good shape. If not $2r = (r + s + r - s) < 683 + 324 < 1100$, so $r < 510$. Having now $s < r < 510$ and $n - m < 683$, we compute an upper bound on the height of the number $h(\eta_U)$ for $U = 4$ appearing in (3). Indeed, by (11) we get that $h(\eta_{U_3}) = h(\eta_4) \le 700$. Using now the upper bound (10) on Λ_4 and Matveev's theorem, we obtain

$$n \log \alpha < \log 100 + C_3(700)(1 + \log n) < 5 \times 10^{14}(1 + \log n)$$

giving

$$n < 1.1 \times 10^{15}(1 + \log n),$$

and so $n < 10^{17}$. With this new upper bound for n we go back to the reductions for t_1, t_2, t_3, and repeating the continued fractions arguments and the Baker-Davenport reductions we get $t_1 < 100$, $t_2 < 115$, $t_3 < 235$.

Let us now work on reducing the upper bound for n even more. In fact, if $n < 235$, then we are in good shape. If not, $2r = (r + s + r - s) < 235 + 115 < 350$, so $r < 175$. On the other hand, since $2/\alpha^n < 1/2$, from (10) we have that

$$|a\tau - n + \nu| < \frac{200}{(\log \alpha)\alpha^{t_4}} < \frac{A}{B^{t_4}}, \tag{14}$$

where

$$\nu = \frac{\log(\sqrt{5}(F_r + F_s)/(1 + \alpha^{-(n-m)}))}{\log \alpha}.$$

As mentioned before, Baker-Davenport reduction does not work when μ is a linear combination of 1 and $(\log 2)/(\log \alpha)$ since then $\varepsilon < 0$. In previous cases we identified easily when that was so. That is, when $\mu = (\log(1+\alpha^{-t_1}))/(\log \alpha)$ the only possibility for $t_1 \geq 2$ for which this number was a linear combination of 1 and $(\log 2)/(\log \alpha)$ was for $t_1 = 3$. Similarly, for $\mu = (\log((1+\alpha^{-t_2})/(1+\alpha^{-t_1})))/(\log \alpha)$, the only possibility for $t_3 > t_2 \geq 2$ for which this number was a linear combination of 1 and $(\log 2)/(\log \alpha)$ was for $(t_1, t_2) = (2, 6)$. Here, we have to decide when the number is $\nu = (\log(\sqrt{5}(F_r + F_s)/(1+\alpha^{-t})))/(\log \alpha)$, where $t = n - m = t_i$ for some $i = 1, 2, 3$ a linear combination of 1 and $(\log 2)/(\log \alpha)$. Well, if this is so, then

$$\frac{\sqrt{5}(F_r + F_s)}{1 + \alpha^{-(n-m)}} = \pm 2^b \alpha^c$$

for some integers b, c. Taking norms in $\mathbb{K}$ and absolute values we get

$$\frac{5(F_r + F_s)^2}{L_{n-m} + 1 + (-1)^{n-m}} = 2^{2b}.$$

If $n - m$ is odd, or $4 \mid n - m$, then the denominator on the left–hand side above is L_{n-m} or $L_{(n-m)/2}^2$. Since $5 \nmid L_k$ for any k, the above equation is impossible. So, $2 \| n - m$, and therefore the denominator on the left–hand side above is $5F_{(n-m)/2}^2$. Hence

$$F_r + F_s = 2^b F_{(n-m)/2}.$$

On the other hand, $2^a(F_r + F_s) = F_n + F_m = F_{(n-m)/2} L_{(n+m)/2}$, where the right–hand side factorisation above holds because $2 \| n-m$. Thus, we get $L_{(n+m)/2} = 2^{a+b}$, which implies that $(n+m)/2 = 3$, so $n \leq 6$. So, when doing the last Baker-Davenport reduction we eliminate the above instances.

Finally, applying Lemma 3 to inequality (14), for all choices n, r, s with $0 \leq s \leq r - 2 < 173$ and $2 \leq n - m \leq 235$, we obtain that $n \leq 400$.

For further convenience of the reader we mention that in the computations above we did not consider the cases $(s, r, n-m) = (0, 2, 2)$ and $(s, r, n-m) = (0, r, 2r)$ with r odd since then $\varepsilon < 0$ and so Lemma 3 does not apply. In fact, if $(s, r, n-m) = (0, 2, 2), (0, r, 2r)$ with r odd, we get that $\nu = 1, r$, respectively. In the first case above, we obtain the sporadic solution $F_4 + F_2 = 2^2 F_2$. In the the second case, the original equation is transformed into a simpler equation $L_{m+r} = 2^a$, and so $(m, r, a) = (0, 3, 2)$. Hence, we get the solution $F_6 = 2^2 F_3$.

2.8.2. The final computation

As we saw in the preceding subsection, it is enough to look for solutions to equation (1) for $n \leq 400$. What we did is to generate $F_n + F_m$ for all $m \leq n - 2 \leq 400$. Let $\mathcal{L}_1$ be the set of such numbers. Next, we created a new list $\mathcal{L}_2$ in the following way. For each member N of $\mathcal{L}_1$ for which $4 \mid N$, we put in $\mathcal{L}_2$ the numbers $N/2^k$ for all

$k = 2, 3, \ldots, \nu_2(N)$. Here, $\nu_2(N)$ is the exponent of 2 in the factorisation of N. We computed $\mathcal{L}_1 \cap \mathcal{L}_2$ obtaining

$$\mathcal{L}_1 \cap \mathcal{L}_2 = \{1, 2, 3, 4, 6, 7, 9, 14, 15, 18, 23, 36, 63\}.$$

We also found that $\max\{\nu_2(N) : N \in \mathcal{L}_1\} = 18$. From these facts and the original equation (1), we can conclude that

$$F_n \leq F_n + F_m \leq 63 \cdot 2^{18} < 10^8,$$

and therefore $n \leq 40$. Then a brute force search with *Mathematica* for $n \leq 40$ and $a \geq 2$ gives sporadic solutions from the statement of the theorem. This completes the proof of Theorem 1.

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