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SOME WEIGHTED TRAPEZOIDAL TYPE INEQUALITIES VIA h-PREINVEXITY

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Abstract. In this paper, a new identity is given, some weighted trapezoidal type inequalities via h-preinvexity are established, and several known results are derived.

1. Introduction

Let f be a convex function on the finite interval $[a, b]$, then

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int\limits_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}, \tag{1.1}$$

The inequality (1.1) is known in the literature as Hermite-Hadamard inequality.

The above inequality has never ceased to intrigue researchers, several variants, extensions, generalizations and improvements have been established.

In [4], Dragomir and Agarwal established the following Hermite-Hadamard type inequalities

$$\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \tfrac{b-a}{8}\left(|f'(a)| + |f'(b)|\right),$$

and

$$\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \tfrac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\tfrac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In [6], Kirmaci et al. gave the following result connected with Hermite-Hadamard type inequalities

$$\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \tfrac{b-a}{8}\left(\tfrac{1}{2}\right)^{1-\frac{1}{q}}\left(\tfrac{2^s s+1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.$$

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In [12], Pearce and Pečarić showed the following inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In [5] Hua et al. gave the following weighted trapezoidal inequalities for s-convex functions

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b f(x)\,w(x)\,dx\right|\\&\leq\frac{(b-a)^2}{4}\|w\|_\infty\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}}\Big\{\left[\left(2^{s+1}-1\right)|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\\&\quad+\left[|f'(a)|^q+\left(2^{s+1}-1\right)|f'(b)|^q\right]^{\frac{1}{q}}\Big\},\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b f(x)\,w(x)\,dx\right|\\&\leq\frac{b-a}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}\int_0^1\left[\int_{\varphi(t)}^{\Psi(t)}w(x)\,dx\right]dt.\end{aligned}$$

Motivated by the above and others existing in the literature, in this study we start by establishing a new equality as a partial result then we derive some new inequalities of weighted Hermite-Hadamard type for h-preinvex functions. Several known results are also derived.

2. Preliminaries

In this section, we recall some definitions

Definition 2.1 ([8]). *A set $I\subseteq\mathbb{R}^n$ is said to be convex if for any $x,y\in H$, and $\forall t\in[0,1]$, we have*

$$tx+(1-t)\,y\in I.$$

Definition 2.2 ([13]). *A function $f:I\to\mathbb{R}$ is said to be convex on I where I is an interval of $\mathbb{R}$, if*

$$f(tx+(1-t)\,y)\leq tf(x)+(1-t)\,f(y)$$

holds for all $x,y\in I$ and all $t\in[0,1]$.

Definition 2.3 ([3]). *A nonnegative function $f:I\to\mathbb{R}$ is said to be P-convex, if*

$$f(tx+(1-t)\,y)\leq f(x)+f(y)$$

holds for all $x,y\in I$ and all $t\in[0,1]$.

DEFINITION 2.4 ([2]). *A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if*

$$f(tx + (1-t)y) \ \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

DEFINITION 2.5 ([16]). *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function, where $(0,1) \subseteq J$. A nonnegative function $f : I \to \mathbb{R}$ is said to be h-convex function on I, if*

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

DEFINITION 2.6 ([17]). *A set $K \subset \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$, if*

$$x + t\eta\,(y,x) \in K$$

holds for all $x, y \in K$ and $t \in [0,1]$.

DEFINITION 2.7 ([17]). *A function $f : K \subset (0,+\infty) \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f\,(x + t\eta\,(y,x)) \le (1-t)\,f\,(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

DEFINITION 2.8 ([11]). *A nonnegative function $f : K \to \mathbb{R}$ is said to be P-preinvex function with respect to η, if*

$$f\,(x + t\eta\,(y,x)) \le f\,(x) + f(y)$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

DEFINITION 2.9 ([7]). *A nonnegative function $f : K \subset [0,\infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if*

$$f\,(x + t\eta\,(y,x)) \ \le (1-t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

DEFINITION 2.10 ([9]). *Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function $h \neq 0$. A nonnegative function f on the invex set K is said to be h-preinvex function with respect to η, if*

$$f(x + t\eta\,(y,x)) \le h(1-t)f(x) + h(t)f(y)$$

holds for all $x, y \in K$ and $t \in (0,1)$.

3. Main results

Lemma 3.1. *Let $f:[a,a+\eta(b,a)]\rightarrow\mathbb{R}$ be differentiable on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$, and let $w:[a,a+\eta(b,a)]\rightarrow[0,+\infty)$ be continuous function and symmetric to $\frac{2a+\eta(b,a)}{2}$. If $f'\in L([a,a+\eta(b,a)])$, then one has the following equality*

$$\begin{aligned}&\tfrac{f(a)+f(a+\eta(b,a))}{2}\int\limits_a^{a+\eta(b,a)}w(x)\,dx-\int\limits_a^{a+\eta(b,a)}w(x)\,f(x)\,dx\\ &=\tfrac{\eta(b,a)}{2}\int\limits_0^1\left(\int\limits_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)\,dx\right)f'(a+t\eta(b,a))\,dt. \qquad (3.1)\end{aligned}$$

Proof. Integrating by parts the right side of (3.1), using a change of variable and the symmetry of w, we obtain

$$\begin{aligned}&\tfrac{\eta(b,a)}{2}\int\limits_0^1\left(\int\limits_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)\,dx\right)f'(a+t\eta(b,a))\,dt\\
&=\tfrac{1}{2}\left(\int\limits_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)\,dx\right)f(a+t\eta(b,a))\Bigg|_{t=0}^{t=1}\\
&\quad-\tfrac{\eta(b,a)}{2}\int\limits_0^1\left(w(a+t\eta(b,a))+w(a+(1-t)\eta(b,a))\right)f(a+t\eta(b,a))\,dt\\
&=\tfrac{1}{2}\left(\int\limits_a^{a+\eta(b,a)}w(x)\,dx\right)f(a+\eta(b,a))+\tfrac{1}{2}\left(\int\limits_a^{a+\eta(b,a)}w(x)\,dx\right)f(a)\\
&\quad-\tfrac{\eta(b,a)}{2}\int\limits_0^1\left(w(a+t\eta(b,a))+w(a+(1-t)\eta(b,a))\right)f(a+t\eta(b,a))\,dt\\
&=\tfrac{1}{2}\left(\int\limits_a^{a+\eta(b,a)}w(x)\,dx\right)\left(f(a+\eta(b,a))+f(a)\right)\\
&\quad-\tfrac{1}{2}\int\limits_a^{a+\eta(b,a)}\left(w(x)+w(2a+\eta(b,a)-x)\right)f(x)\,dx\end{aligned}$$

$$=\frac{f(a+\eta(b,a))+f(a)}{2}\int_{a}^{a+\eta(b,a)} w(x)\,dx-\int_{a}^{a+\eta(b,a)} w(x)\,f(x)\,dx.$$

The proof is completed. □

In what follows we assume that $h:\ [0,1]\to\mathbb{R}$ be a nonnegative function and $h\neq 0$, $\eta(b,a)>0$, and $K=[a,a+\eta(b,a)]\subset[0,+\infty)$.

THEOREM 3.2. *Let $f:K\to\mathbb{R}$ be differentiable on K° with $f'\in L(K)$ where $a,b\in K^{\circ}$, and let $w:K\to[0,+\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $|f'|$ is h-preinvex, then one has the following inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)} w(x)\,dx-\int_{a}^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^{2}}{2}\|w\|_{\infty}\left(\left(|f'(a)|+|f'(a+\eta(b,a))|\right)\int_{\frac{1}{2}}^{1}(2t-1)\,h(t)\,dt\right.$$
$$\left.+\,2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\int_{0}^{\frac{1}{2}}(1-2t)\,h(t)\,dt\right).$$

PROOF. From Lemma 3.1, properties of modulus, and h-preinvexity of $|f'|$, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)} w(x)\,dx-\int_{a}^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{2}\int_{0}^{1}\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right|\left|f'(a+t\eta(b,a))\right|dt$$
$$=\frac{\eta(b,a)}{2}\left(\int_{0}^{\frac{1}{2}}\left(\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} w(x)\,dx\right)\left|f'(a+t\eta(b,a))\right|dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left(\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right)\left|f'(a+t\eta(b,a))\right|dt\right)$$

$$\leq \tfrac{(\eta(b,a))^2}{2}\left(\int_0^{\frac{1}{2}}\|w(x)\|_{[a+t\eta(b,a),a+(1-t)\eta(b,a)],\infty}(1-2t)\left|f'(a+t\eta(b,a))\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}\|w(x)\|_{[a+(1-t)\eta(b,a),a+t\eta(b,a)],\infty}(2t-1)\left|f'(a+t\eta(b,a))\right|dt\right)$$

$$\leq \tfrac{(\eta(b,a))^2}{2}\|w\|_\infty\left(\int_0^{\frac{1}{2}}(1-2t)\left|f'(a+t\eta(b,a))\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(2t-1)\left|f'(a+t\eta(b,a))\right|dt\right)$$

$$\leq \tfrac{(\eta(b,a))^2}{2}\|w\|_\infty\left(\int_0^{\frac{1}{2}}(1-2t)\left(h(1-t)\left|f'(a)\right|+h(t)\left|f'\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|\right)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(2t-1)\left(h(1-t)\left|f'\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|+h(t)\left|f'(a+\eta(b,a))\right|\right)dt\right)$$

$$=\tfrac{(\eta(b,a))^2}{2}\|w\|_\infty\left(\left(\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right)\int_{\frac{1}{2}}^{1}(2t-1)h(t)\,dt\right.$$

$$\left.+2\left|f'\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|\int_0^{\frac{1}{2}}(1-2t)h(t)\,dt\right).$$

□

Corollary 3.3. *In Theorem 3.2, if we choose* $w(x)=\frac{1}{\eta(b,a)}$, *we obtain*

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$

$$\leq \tfrac{\eta(b,a)}{2}\left(\left(\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right)\int_{\frac{1}{2}}^{1}(2t-1)h(t)\,dt\right.$$

$$+ 2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\int_0^{\frac{1}{2}}(1-2t)\,h(t)\,dt\Bigg).$$

COROLLARY 3.4. *In Theorem 3.2, taking $\eta(b,a)=b-a$, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b w(x)\,f(x)\,dx\right|\le\frac{(b-a)^2}{2}\|w\|_\infty$$

$$\times\left((|f'(a)|+|f'(b)|)\int_{\frac{1}{2}}^1(2t-1)\,h(t)\,dt+2\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^{\frac{1}{2}}(1-2t)\,h(t)\,dt\right).$$

Moreover, if we choose $w(x)=\frac{1}{b-a}$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{2}$$

$$\times\left((|f'(a)|+|f'(b)|)\int_{\frac{1}{2}}^1(2t-1)\,h(t)\,dt+2\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^{\frac{1}{2}}(1-2t)\,h(t)\,dt\right).$$

COROLLARY 3.5. *In Theorem 3.2, if we assume that $|f'|$ is P-preinvex function we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}w(x)\,f(x)\,dx\right|$$

$$\le\frac{(\eta(b,a))^2}{8}\|w\|_\infty\left(|f'(a)|+2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+|f'(a+\eta(b,a))|\right).$$

Moreover, if we take $w(x)=\frac{1}{\eta(b,a)}$

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$

$$\le\frac{\eta(b,a)}{8}\left(|f'(a)|+2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+|f'(a+\eta(b,a))|\right).$$

COROLLARY 3.6. *In Corollary 3.5, if we take $\eta(b,a)=b-a$, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b w(x)\,f(x)\,dx\right|$$

$$\le\frac{(b-a)^2}{8}\|w\|_\infty\left(|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right).$$

Moreover, if we take $w(x)=\frac{1}{b-a}$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right|\leq\frac{b-a}{8}\left(|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right).$$

Corollary 3.7. *In Theorem 3.2, if we assume that* $|f'|$ *is preinvex function*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int\limits_a^{a+\eta(b,a)} w(x)\,dx-\int\limits_a^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{48}\|w\|_\infty\left(5|f'(a)|+2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+5|f'(a+\eta(b,a))|\right).$$

Moreover, if we take $w(x)=\frac{1}{\eta(b,a)}$

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{48}\left(5|f'(a)|+2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+5|f'(a+\eta(b,a))|\right). \tag{3.2}$$

Remark 3.8. In inequality (3.2), using the fact that $2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\leq|f'(a)|+|f'(b)|$ and $|f'(a+\eta(b,a))|\leq|f'(b)|$, we obtain Theorem 2.1 from [1].

Corollary 3.9. *In Corollary 3.7, if we take* $\eta(b,a)=b-a$*, we get*

$$\left|\frac{f(a)+f(b)}{2}\int\limits_a^b w(x)\,dx-\int\limits_a^b w(x)\,f(x)\,dx\right|$$
$$\leq\frac{(b-a)^2}{48}\|w\|_\infty\left(5|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+5|f'(b)|\right). \tag{3.3}$$

Moreover, if we take $w(x)=\frac{1}{b-a}$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right|\leq\frac{b-a}{48}\left(5|f'(a)|+2\left|f'\left(\frac{a+b}{2}\right)\right|+5|f'(b)|\right). \tag{3.4}$$

Remark 3.10. In inequality (3.3), using the convexity of $|f'|$, we obtain inequality (2) of Corollary 3.1.1 from [5]. Also Corollary 8 from [15].

Remark 3.11. In inequality (3.4), using the convexity of $|f'|$, we obtain Theoreme 2.2 from [4].

COROLLARY 3.12. *In Theorem 3.2, if we assume that $|f'|$ is s-preinvex function*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{2(s+1)(s+2)}\|w\|_\infty\left(\frac{s2^{1+s}+1}{2^{1+s}}|f'(a)|+\frac{s2^{1+s}+1}{2^{1+s}}|f'(a+\eta(b,a))|\right.$$
$$\left.+\frac{1}{2^s}\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\right).$$

Moreover, if we take $w(x)=\frac{1}{\eta(b,a)}$

(3.5)
$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{2(s+1)(s+2)}\left(\frac{s2^{1+s}+1}{2^{1+s}}|f'(a)|+\frac{1}{2^s}\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+\frac{s2^{1+s}+1}{2^{1+s}}|f'(a+\eta(b,a))|\right).$$

REMARK 3.13. In inequality (3.5), using the preinvexity of $|f'|$, we obtain the correct result of Theorem 2 from [14].

COROLLARY 3.14. *In Corollary 3.12, if we take $\eta(b,a)=b-a$, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b w(x)\,f(x)\,dx\right| \tag{3.6}$$
$$\leq\frac{(b-a)^2}{2(s+1)(s+2)}\|w\|_\infty\left(\frac{s2^{1+s}+1}{2^{1+s}}|f'(a)|+\frac{1}{2^s}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{s2^{1+s}+1}{2^{1+s}}|f'(b)|\right).$$

Moreover, if we take $w(x)=\frac{1}{b-a}$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.7}$$
$$\leq\frac{b-a}{2(s+1)(s+2)}\left(\frac{s2^{1+s}+1}{2^{1+s}}|f'(a)|+\frac{1}{2^s}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{s2^{1+s}+1}{2^{1+s}}|f'(b)|\right).$$

REMARK 3.15. In inequality (3.6), using the convexity of $|f'|$, we obtain inequality (1) of Corollary 3.1.1 from [5].

REMARK 3.16. In inequality (3.7), using the convexity of $|f'|$, we obtain inequality (1) of Corollary 3.1.2 from [5], Also Theorem 2 from [10].

THEOREM 3.17. *Let $f:K\to\mathbb{R}$ be differentiable on K° with $f'\in L(K)$, and let $w:K\to[0,+\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If*

$|f'|^q$ is h-preinvex, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then one has the following inequality

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq\tfrac{\eta(b,a)}{2}\left(\int_0^1\left(\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}$$
$$\times\left(|f'(a)|^q+|f'(a+\eta(b,a))|^q\right)^{\frac{1}{q}}\left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}}.$$

Proof. From Lemma 3.1, properties of modulus, Hölder inequality, and h-preinvexity of $|f'|^q$, we have

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq\tfrac{\eta(b,a)}{2}\int_0^1\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right||f'(a+t\eta(b,a))|\,dt$$
$$\leq\tfrac{\eta(b,a)}{2}\left(\int_0^1\left(\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(a+t\eta(b,a))|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq\tfrac{\eta(b,a)}{2}\left(\int_0^1\left(\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(h(1-t)\,|f'(a)|^q+h(t)\,|f'(a+\eta(b,a))|^q\right)dt\right)^{\frac{1}{q}}$$
$$=\tfrac{\eta(b,a)}{2}\left(\int_0^1\left(\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}$$
$$\times\left(|f'(a)|^q+|f'(a+\eta(b,a))|^q\right)^{\frac{1}{q}}\left(\int_0^1 h(t)\,dt\right)^{\frac{1}{q}}.$$

The proof is achieved. ∎

COROLLARY 3.18. *In Theorem 3.17, if we choose $w(x) = \frac{1}{\eta(b,a)}$, we obtain*

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} \left(|f'(a)|^q + |f'(a+\eta(b,a))|^q \right)^{\frac{1}{q}} \left(\int_0^1 h(t)\,dt \right)^{\frac{1}{q}}.$$

COROLLARY 3.19. *In Theorem 3.17, taking $\eta(b,a) = b-a$, we get*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x)\,dx - \int_a^b w(x) f(x)\,dx \right| \leq \frac{b-a}{2} \left(\int_0^1 \left(\left| \int_{at+(1-t)b}^{a(1-t)+tb} w(x)\,dx \right| \right)^p dt \right)^{\frac{1}{p}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} \left(\int_0^1 h(t)\,dt \right)^{\frac{1}{q}}.$$

Moreover, if we choose $w(x) = \frac{1}{b-a}$, we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} \left(\int_0^1 h(t)\,dt \right)^{\frac{1}{q}}.$$

COROLLARY 3.20. *In Theorem 3.17, if we assume that $|f'|^q$ is P-preinvex function*

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x) f(x)\,dx \right| \leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx \right| \right)^p dt \right)^{\frac{1}{p}} \left(|f'(a)|^q + |f'(a+\eta(b,a))|^q \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} \left(|f'(a)|^q + |f'(a+\eta(b,a))|^q \right)^{\frac{1}{q}}.$$

Corollary 3.21. *In Corollary 3.20, if we take $\eta(b,a)=b-a$, we get*

$$\left|\tfrac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b w(x)f(x)\,dx\right|$$
$$\leq\tfrac{b-a}{2}\left(\int_0^1\left(\left|\int_{at+(1-t)b}^{a(1-t)+tb} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x)=\frac{1}{b-a}$

$$\left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\tfrac{b-a}{2(p+1)^{\frac{1}{p}}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.$$

Corollary 3.22. *In Theorem 3.17, if we assume that $|f'|^q$ is preinvex function*

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} w(x)f(x)\,dx\right|$$
$$\leq\tfrac{\eta(b,a)}{2}\left(\int_0^1\left(\left|\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}\left(\tfrac{|f'(a)|^q+|f'(a+\eta(b,a))|^q}{2}\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x)=\frac{1}{\eta(b,a)}$, we get Theorem 2.2 from [1].

Corollary 3.23. *In Corollary 3.22, if we take $\eta(b,a)=b-a$, we get*

$$\left|\tfrac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b w(x)f(x)\,dx\right| \tag{3.8}$$
$$\leq\tfrac{b-a}{2}\left(\int_0^1\left(\left|\int_{at+(1-t)b}^{a(1-t)+tb} w(x)\,dx\right|\right)^p dt\right)^{\frac{1}{p}}\left(\tfrac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x)=\frac{1}{b-a}$, we obtain Theorem 2.3 from [4].

Remark 3.24. In inequality (3.9), using the fact that $w(x)\leq\|w\|_\infty$, we obtain Corollary 13 from [15].

COROLLARY 3.25. *In Theorem 3.17, if we assume that $|f'|^q$ is s-preinvex function*

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x) f(x)\,dx \right|$$

$$\leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left(\left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx \right| \right)^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(a+\eta(b,a))|^q}{s+1} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^q + |f'(a+\eta(b,a))|^q}{s+1} \right)^{\frac{1}{q}}. \tag{3.9}$$

REMARK 3.26. In inequality (3.9), using the fact that $|f'(a+\eta(b,a))| \leq |f'(b)|$, we obtain Theorem 4 from [14].

COROLLARY 3.27. *In Corollary 3.25, if we take $\eta(b,a) = b-a$, we get*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x)\,dx - \int_a^b w(x) f(x)\,dx \right|$$

$$\leq \frac{b-a}{2} \left(\int_0^1 \left(\left| \int_{at+(1-t)b}^{a(1-t)+tb} w(x)\,dx \right| \right)^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 4 from [10].

THEOREM 3.28. *Under the assumptions of Theorem 3.17, one has*

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x) f(x)\,dx \right|$$

$$\leq \frac{(\eta(b,a))^2}{2^{2-\frac{1}{q}}} \|w\|_\infty \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} \left(\int_0^1 |2t-1|\, h(t)\,dt \right)^{\frac{1}{q}}.$$

PROOF. Using Lemma 3.1, properties of modulus, Power mean inequality, and h-preinvexity of $|f'|^q$, we have

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} \int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x) f(x)\,dx \right|$$

$$\leq \frac{\eta(b,a)}{2} \int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx \right| \left| f'(a+t\eta(b,a)) \right| dt$$

$$\leq \frac{\eta(b,a)}{2} \left(\int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx \right| dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left| \int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)} w(x)\,dx \right| \left| f'(a+t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^2}{2} \|w\|_\infty \left(\int_0^1 |2t-1|\,dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |2t-1| \left| f'(a+t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^2}{2} \|w\|_\infty \left(\int_0^1 |2t-1|\,dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 |2t-1| \left[h(1-t) \left| f'(a) \right|^q + h(t) \left| f'(b) \right|^q \right] dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^2}{2^{2-\frac{1}{q}}} \|w\|_\infty \left(\left| f'(a) \right|^q + \left| f'(b) \right|^q \right)^{\frac{1}{q}} \left(\int_0^1 |2t-1|\, h(t)\,dt \right)^{\frac{1}{q}}.$$

□

Corollary 3.29. *In Theorem 3.28, if we choose* $w(x) = \frac{1}{\eta(b,a)}$*, we obtain*

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right|$$

$$\leq \frac{\eta(b,a)}{2^{2-\frac{1}{q}}} \left(\left| f'(a) \right|^q + \left| f'(b) \right|^q \right)^{\frac{1}{q}} \left(\int_0^1 |2t-1|\, h(t)\,dt \right)^{\frac{1}{q}}.$$

Corollary 3.30. *In Theorem 3.28, taking* $\eta(b,a) = b-a$*, we get*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(x)\,dx - \int_a^b w(x) f(x)\,dx \right|$$

$$\leq \frac{(b-a)^2}{2^{2-\frac{1}{q}}} \|w\|_\infty \left(\left| f'(a) \right|^q + \left| f'(b) \right|^q \right)^{\frac{1}{q}} \left(\int_0^1 |2t-1|\, h(t)\,dt \right)^{\frac{1}{q}}.$$

Moreover, if we choose $w(x) = \frac{1}{b-a}$*, we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}\left(\int_0^1 |2t-1|\,h(t)\,dt\right)^{\frac{1}{q}}.$$

COROLLARY 3.31. *In Theorem 3.28, if we assume that* $|f'|^q$ *is* P*-preinvex function*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{4}\left\|w\right\|_\infty\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{4}\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.$$

COROLLARY 3.32. *In Corollary 3.31, if we take* $\eta(b,a) = b-a$*, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx - \int_a^b w(x)\,f(x)\,dx\right| \leq \frac{(b-a)^2}{4}\left\|w\right\|_\infty\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.$$

COROLLARY 3.33. *In Theorem 3.28, if we assume that* $|f'|^q$ *is preinvex function*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x)\,f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{4}\left\|w\right\|_\infty\left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{4}\left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.34. *In Corollary 3.33, if we take $\eta(b,a) = b - a$, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx - \int_a^b w(x) f(x)\,dx\right| \le \frac{(b-a)^2}{4}\|w\|_\infty \left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 1 from [12].

Corollary 3.35. *In Theorem 3.28, if we assume that $|f'|^q$ is s-preinvex function*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} w(x) f(x)\,dx\right|$$
$$\le \frac{(\eta(b,a))^2}{2^{2-\frac{1}{q}}}\|w\|_\infty \left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} \left(\frac{1+s2^s}{(1+s)(2+s)2^s}\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{\eta(b,a)}$, we obtain the correct result of Theorem 7 from [14].

Corollary 3.36. *In Corollary 3.35, if we take $\eta(b,a) = b - a$, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx - \int_a^b w(x) f(x)\,dx\right|$$
$$\le \frac{(b-a)^2}{2^{2-\frac{1}{q}}}\|w\|_\infty \left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} \left(\frac{1+s2^s}{(1+s)(2+s)2^s}\right)^{\frac{1}{q}}.$$

Moreover, if we take $w(x) = \frac{1}{b-a}$, we obtain Theorem 1 from [6].

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Neke nejednakosti težinskog trapezoidnog tipa preko h-preinveksnosti

B. Meftah i K. Mekalfa

Sažetak. U ovom članku, dan je jedan novi identitet, dobivene su neke nejednakosti težinskog trapezoidnog tipa te su izvedeni neki poznati rezultati.

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