On impatience in Markovian M/M/1/N/DWV queue with vacation interruption

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Abstract. In this paper, we establish a cost optimization analysis for an $M/M/1/N$ queuing system with differentiated working vacations, Bernoulli schedule vacation interruption, balking and reneging. Once the system is empty, the server waits a random amount of time before he goes on working vacation during which service is provided with a lower rate. At the instant of the service achievement in the vacation period, if there are customers present in the system, the vacation is interrupted and the server returns to the regular busy period with probability β' or continues the working vacation with probability $1-\beta'$. Whenever the working vacation is ended, the server comes back to the normal busy period. If the system is empty, the server can take another working vacation of shorter duration. In addition, it is supposed that during both busy and working vacation periods, arriving customers may become impatient with individual timers exponentially distributed. Explicit expressions for the steady-state system size probabilities are derived using recursive technique. Further, interesting performance measures are explicitly obtained. Then, we construct a cost model in order to determine the optimal values of service rates, simultaneously, to minimize the total expected cost per unit time by using a quadratic fit search method (QFSM). Finally, numerical illustrations are added to validate the theoretical results.

Keywords: Bernoulli schedule vacation interruption, cost queuing model, differentiated working vacations, impatient customers, QFSM optimization

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1. Introduction

queuing systems subject to working vacation policy have been extensively studied due to their wide applications in computer systems, telecommunications, and production management. Many working vacation policies improve the adaptability for optimal design of queuing systems. These models have been studied by different researchers. As a general rule, in working vacation policy, the server resumes its work at the normal service rate after the end of the vacation, only if customers are waiting in the system. Li and Tian [11] introduced vacation interruptions in an infinite-buffer Markovian single server queue with working vacation policy at which the server

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may come back from the vacation to the regular working level once the number of customers attain a certain value in the vacation period. Then, Li et. al [12] considered a $GI/M/1$ queuing model with working vacation and vacation interruption. Baba [3] dealt with a $M/PH/1$ queuing model under working vacations and vacation interruption. An $M/G/1$ queuing system with single working vacation and vacation interruption under Bernoulli schedule has been examined in Gao and Liu [9]. Lee and Kim [10] presented the sojourn time distribution of an $M/G/1$ queue with a single working vacation and vacation interruption. Later, the transient analysis of an infinite capacity sigle server Markovian queue with differentiated vacations has been done in Vijayashree and Janani [20]. Majid et. al [15] treated a $M/M/1$ queuing model with working vacation and vacation interruption under Bernoulli schedule. Recently, significant results on the subject have been presented, see for instance, Ameur et. al [2], Majid and Manoharan [14], and Rajadurai [16] and the reference therein.

Various queuing situations occur where customers tend to be discouraged by a long queue. As a consequence, customers decide either not to join the queue (balk) or leave after logging into the queue without being served because of their impatience (renege). Numerous researchers have been attracted by the analysis of the impatience behavior in queuing models with vacation and working vacation. Yue et. al [21] dealt with impatience behavior in an $M/M/1/$ queue under multiple working vacation policy. A finite buffer renewal input queuing model with balking and multiple working vacations has been carried out by Vijaya Laxmi and Jyothsna [18]. Then, Vijaya Laxmi and Jyothsna [19] analyzed the impatience behaviour in a queuing model with Bernoulli schedule vacation interruption. Later, Bouchentouf and Yahiaoui [8] investigated a M/M/1 queuing model with feedback, reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption. Majid and Manoharan [13] provided the analysis of an infinite-space Markovian multi-server queuing model with single and multiple synchronous working vacations. Afroun et. al [1] used a Q-matrix method for the study of an unreliable $M/M/1/N$ queuing model with customer's impatience. Recently, Bouchentouf et. al [4], Bouchentouf and Guendouzi [5, 6], Bouchentouf and Medjahri [7], and Sampath and Liu [17] gave some important papers in this area.

In this investigation, we consider a finite-buffer Markovian single server queuing system with waiting server, balking and reneging, under differentiated working vacations and Bernoulli schedule vacation interruption at which the server is subject to two types of working vacation, namely: working vacation after the busy period and working vacation taken immediately after the server has just returned from previous working vacation to find that there are no customers in the queue. During working vacation period, the service is supposed to be interrupted under the Bernoulli schedule.

The rest of the paper is organized as follows. Section 2 presents the description of the queuing model. In Section 3 we derive the stationary distribution of the system. In Section 4 we deduce important characteristics of the system. In Section 5 we construct a cost model in order to determine the optimal values of service rates, simultaneously, to minimize the total expected cost per unit time by using a quadratic fit search method (QFSM). Numerical demonstrations are given in Section 6. Finally, Section 7 concludes the paper.

2. Description of the model

Consider a finite–buffer single server queuing system subject to differentiated working vacations, Bernoulli schedule vacation interruption, waiting server and customers' impatience. Figure 1 shows the state transition diagram of the considered model.

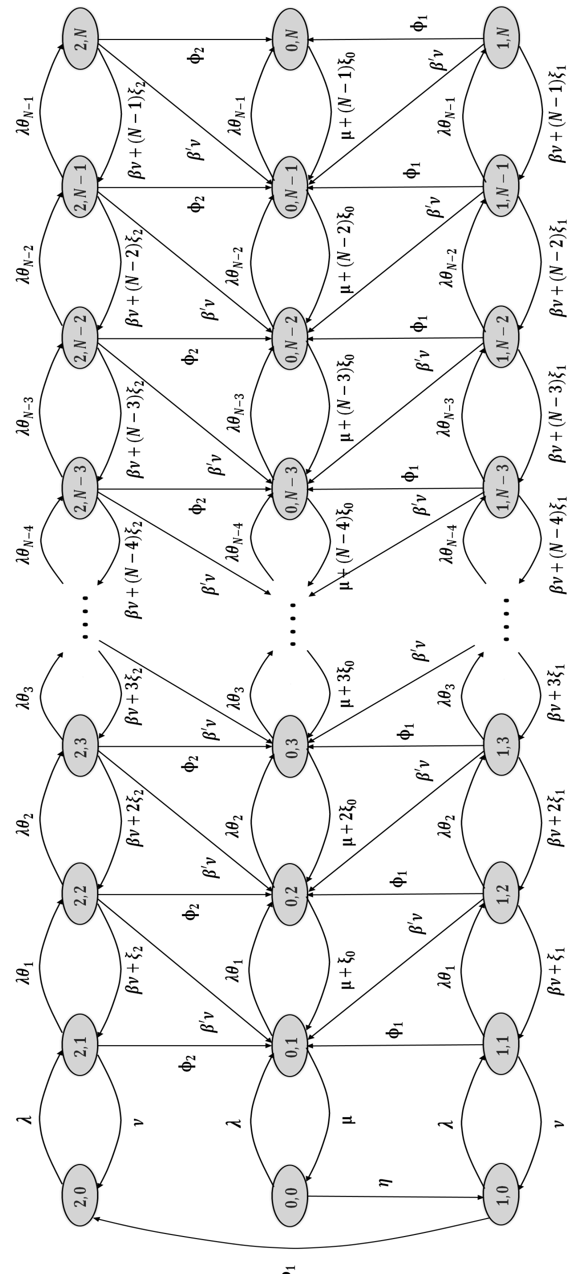

Figure 1: *State–transition diagram*

What follows are the assumptions of the system.

i) Customers join the queue according to a Poisson process with rate λ

ii) Service time during normal busy period follows an exponential distribution with rate μ

iii) On arrival, a customer either decides to join the queue with probability θ_n or balk with probability $\theta'_n = 1 - \theta_n$. Note that $0 \leq \theta_{n+1} \leq \theta_n < 1$ with $1 \leq n \leq N-1$, $\theta_0 = 1$, and $\theta_N = 0$.

iv) Once the normal busy period is over, the server waits for a random duration of time before he goes on working vacation. It is supposed that waiting time duration follows an exponential distribution with rate η.

v) The server begins a type–I working vacation after the waiting time duration is finished. When the working vacation time is ended, and the server finds an empty system, he takes another vacation of shorter duration, named type–II working vacation. Otherwise, he comes back to the normal busy period and starts serving the customers present in the system. Type–I and type–II working vacation times follow exponential distributions with parameter ϕ_1 and ϕ_2, respectively.

vi) During vacation period, customers are served at lower rate. The service time during this period follows an exponential distribution with $\nu < \mu$.

vii) During working vacation period, the server is supposed to be interrupted under the Bernoulli rule, i.e., at a service completion instant during this period if there are customers in the system, the server may interrupt the vacation and switch to normal busy period with probability β' or continue the vacation with probability $\beta = 1 - \beta'$. It is worth noting that the vacation service rate can be only applied to the first customer arrived during a vacation period.

viii) In normal busy period, type–I, and type–II working vacations, each arriving customer activates an individual timer, exponentially distributed with parameters ξ_0, ξ_1 and ξ_2, respectively, such that, if the customer's service has not begun before the customer's timer expires, he abandons the queue and never returns.

ix) The inter–arrival times, service times, waiting times, working vacation times, and impatience times are mutually independent. The service discipline is First–Come, First–Served.

2.1. Practical application of the proposed queuing model

Power consumption and the delay of the data frame are two important points in the design of an effective energy saving mechanism. The considered queuing model is evolved for the data frames of type–I and type–II energy saving classes in IEEE 802.16e. Consider the finite capacity (buffer) fed by the downlink data frames at the base station at which the data frames arrive at the buffer according to Poisson process with arrival rate λ. In active mode (normal busy period), the mobile station and the base station connect with each other for sending and receiving messages in the form of data frame. The service time is supposed to be exponentially distributed with parameter μ. Let θ_n be the probability with which the incoming data frames decide to join the buffer and let $1 - \theta_n$ denote the probability that they decide to balk, when there are n incoming data frames in front of them in the system.

When there is no data frame in the system, the mobile station is allowed to remain in an inactive state for a random period (waiting server). Whenever the idle status is over, it sends a MOB-SLP-REQ message to the base station requesting to enter a type–I sleep mode. Once it received the response message MOB-SLP-RSP from the base station, the mobile station enters the sleep mode which is distributed exponentially with parameter ϕ_1. If there is no data frame on its return, it is authorized to enter type–II sleep mode of shorter duration which is exponentially distributed with parameter ϕ_2. In type–I and type–II sleep modes, the system operates continuously but slowly (working vacation policy) at which the service time is assumed to be exponentially distributed with parameter ν. After completing type–II sleep mode, it comes back to regular busy state even if there is no frame date. At the time of service completion during both type–I and type–II sleep modes, the system can continue operating with probability

β or switch to the normal busy period with $(1-\beta)$. At this time, if some data frames are present in the system, they will be processed immediately (working vacation interruption).

During both busy period, type–I or type–II sleep modes, a data frame (customer) can be recalled by the sender; on its arrival, if the system is on busy, type–I or type–II sleep mode, a customer activates an individual impatience timer T_0, T_1 or T_2, which is exponentially distributed with parameter ξ_0, ξ_1 or ξ_0, respectively. If impatience timers expire before the begging of the processing time, the frame date abandons the system.

3. Steady–state analysis

Let $N(t)$ denote the number of customers in the system and $J(t)$ be the status of the server at time t, such that

$$J(t) = \begin{cases} 0, \text{ when the server is in normal busy period,} \\ 1, \text{ when the server is on type-I working vacation period,} \\ 2, \text{ when the server is on type-II working vacation period.} \end{cases}$$

Clearly, the process $\{(J(t), N(t)), t \geq 0\}$ is a continuous–time Markov process with state space

$$\Omega = \{(j; n) : j = 0, 1, 2; n = 0, 1, ..., N\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, N(t) = n\}, (j, n) \in \Omega$, denote the system state probabilities.

Applying the Markov process theory, we obtain the following set of steady–state equations

$$(\lambda + \eta)P_{0,0} = \mu P_{0,1}, \;\; n = 0, \tag{1}$$

$$\begin{aligned}(\lambda\theta_1 + \mu)P_{0,1} = \lambda P_{0,0} + \phi_1 P_{1,1} + \phi_2 P_{2,1} + \beta'\nu P_{1,2} \\ +\beta'\nu P_{2,2} + (\mu + \xi_0)P_{0,2}, \;\; n = 1,\end{aligned} \tag{2}$$

$$\begin{aligned}(\lambda\theta_n + \mu + (n-1)\xi_0)P_{0,n} = \lambda\theta_{n-1}P_{0,n-1} + \phi_1 P_{1,n} + \phi_2 P_{2,n} + \beta'\nu P_{1,n+1} \\ +\beta'\nu P_{2,n+1} + (\mu + n\xi_0)P_{0,n+1}, \; 2 \leq n \leq N-1,\end{aligned} \tag{3}$$

$$(\mu + (N-1)\xi_0)P_{0,N} = \lambda\theta_{N-1}P_{0,N-1} + \phi_1 P_{1,N} + \phi_2 P_{2,N}, \;\; n = N, \tag{4}$$

$$(\lambda + \phi_1)P_{1,0} = \eta P_{0,0} + \nu P_{1,1}, \;\; n = 0, \tag{5}$$

$$(\lambda\theta_1 + \nu + \phi_1)P_{1,1} = \lambda P_{1,0} + (\beta\nu + \xi_1)P_{1,2}, \;\; n = 1, \tag{6}$$

$$\begin{aligned}(\lambda\theta_n + \nu + (n-1)\xi_1 + \phi_1)P_{1,n} = \lambda\theta_{n-1}P_{1,n-1} + (\beta\nu + n\xi_1)P_{1,n+1}, \\ 2 \leq n \leq N-1,\end{aligned} \tag{7}$$

$$(\nu + (N-1)\xi_1 + \phi_1)P_{1,N} = \lambda\theta_{N-1}P_{1,N-1}, \;\; n = N, \tag{8}$$

$$\lambda P_{2,0} = \phi P_{1,0} + \nu P_{2,1}, \;\; n = 0, \tag{9}$$

$$(\lambda\theta_1 + \nu + \phi_2)P_{2,1} = \lambda P_{2,0} + (\beta\nu + \xi_2)P_{2,2}, \;\; n = 1, \tag{10}$$

$$(\lambda\theta_n + \nu + (n-1)\xi_2 + \phi_2)P_{2,n} = \lambda\theta_{n-1}P_{2,n-1} + (\beta\nu + n\xi_2)P_{2,n+1}, \quad 2 \le n \le N-1, \tag{11}$$

$$(\nu + (N-1)\xi_2 + \phi_2)P_{2,N} = \lambda\theta_{N-1}P_{2,N-1}, \quad n = N. \tag{12}$$

The normalization condition is

$$\sum_{n=0}^{N} P_{0,n} + \sum_{n=0}^{N} P_{1,n} + \sum_{n=0}^{N} P_{2,n} = 1. \tag{13}$$

Theorem 1. *1. The steady-state-probabilities of the system size during type-II working vacation period are given by*

$$P_{2,n} = \psi_n P_{2,N}, \tag{14}$$

where

$$\psi_n = \begin{cases} 1, & n = N; \\ \dfrac{\nu + (N-1)\xi_2 + \phi_2}{\lambda\theta_{N-1}}, & n = N-1; \\ \dfrac{\lambda\theta_{n+1} + \nu + n\xi_2 + \phi_2}{\lambda\theta_n}\psi_{n+1} - \dfrac{\beta\nu + (n+1)\xi_2}{\lambda\theta_n}\psi_{n+2}, & 0 \le n \le N-1. \end{cases} \tag{15}$$

2. The steady-state-probabilities of the system size during type-I working vacation period are given by

$$P_{1,n} = \Theta_1\gamma_n P_{2,N}, \tag{16}$$

where

$$\gamma_n = \begin{cases} 1, & n = N; \\ \dfrac{\nu + (N-1)\xi_1 + \phi_1}{\lambda\theta_{N-1}}, & n = N-1; \\ \dfrac{\lambda\theta_{n+1} + \nu + n\xi_1 + \phi_1}{\lambda\theta_n}\gamma_{n+1} - \dfrac{\beta\nu + (n+1)\xi_1}{\lambda\theta_n}\gamma_{n+2}, & 0 \le n \le N-1; \end{cases} \tag{17}$$

and

$$\Theta_1 = \frac{\lambda\psi_0 - \nu\psi_1}{\phi_1\gamma_0}. \tag{18}$$

3. The steady-state-probabilities of the system size during normal busy period $P_{0,n}$ are given by

$$P_{0,n} = (\Theta_2\omega_n - \tau_n)P_{2,N}, \tag{19}$$

where

$$\omega_n = \begin{cases} 1, & n = N; \\ \dfrac{\mu + (N-1)\xi_0}{\lambda\theta_{N-1}}, & n = N-1; \\ \dfrac{\lambda\theta_{n+1} + \mu + n\xi_0}{\lambda\theta_n}\omega_{n+1} - \dfrac{\mu + (n+1)\xi_0}{\lambda\theta_n}\omega_{n+2}, & 0 \le n \le N-1, \end{cases} \tag{20}$$

$$\tau_n = \begin{cases} 0, & n = N; \\ \frac{\Theta_1\phi_1 + \phi_2}{\lambda\theta_{N-1}}, & n = N-1; \\ \frac{\lambda\theta_{n+1} + \mu + n\xi_0}{\lambda\theta_n}\tau_{n+1} - \frac{\mu + (n+1)\xi_0}{\lambda\theta_n}\tau_{n+2} & \\ + \frac{\Theta_1(\phi_1\gamma_{n+1} + \beta'\nu\gamma_{n+2}) + (\phi_2\psi_{n+1} + \beta'\nu\psi_{n+2})}{\lambda\theta_n}, & 0 \leq n \leq N-1, \end{cases} \tag{21}$$

and

$$\Theta_2 = \frac{\Theta_1}{\eta\omega_0}\Big((\lambda + \phi_1)\gamma_0 - \nu\gamma_1\Big), \tag{22}$$

with

$$P_{2,N} = \left\{\sum_{n=1}^{N}\psi_n + \Theta_1\sum_{n=0}^{N}\gamma_n + \Theta_2\sum_{n=0}^{N}\omega_n - \sum_{n=0}^{N}\tau_n\right\}^{-1}. \tag{23}$$

Proof. Solving recursively equations (10)–(12), we get $P_{2,n} = \psi_n P_{2,N}$, where ψ_n is presented by (15). Via equations (6)–(8), we find $P_{1,n} = \gamma_n P_{1,N}$, where γ_n is given by (17). Using equation (9) we get equations (16)–(18). By solving equations (2)–(4), we derive $P_{0,n}$ in terms of $P_{0,N}$ and $P_{2,N}$. Then, using (5) we obtain $P_{0,n}$ in terms of $P_{2,N}$, given by (19). Finally, using the normalization condition (13) we get equation (23). □

4. Performance measures

Once the steady–state probabilities of the system are obtained, we evaluate the system behaviour based on the following performance measures.

The mean number of customers in the system

$$L_s = \sum_{n=0}^{N} n(P_{0,n} + P_{1,n} + P_{2,n}). \tag{24}$$

The mean number of customers in the queue

$$L_q = \sum_{n=1}^{N}(n-1)(P_{0,n} + P_{1,n} + P_{2,n}). \tag{25}$$

The mean number of customers served per time unit

$$E_{cs} = \mu\sum_{n=1}^{N} nP_{0,n} + \nu\sum_{n=1}^{N} n(P_{1,n} + P_{2,n}). \tag{26}$$

The average balking rate

$$B_r = \lambda\sum_{n=1}^{N}(1-\theta_n)(P_{0,n} + P_{1,n} + P_{2,n}). \tag{27}$$

The mean waiting time of customers in the system

$$W_s = \frac{L_s}{\lambda'}, \text{where } \lambda' = \lambda - B_r. \tag{28}$$

The average reneging rate

$$R_r = \xi_0 \sum_{n=1}^{N}(n-1)P_{0,n} + \xi_1 \sum_{n=1}^{N}(n-1)P_{1,n} + \xi_2 \sum_{n=1}^{N}(n-1)P_{2,n}. \tag{29}$$

The probability that the server is in normal busy period

$$P_b = \sum_{n=0}^{N} P_{0,n}. \tag{30}$$

The probability that the server is on type-I working vacation period (P_{wv1}) (resp. on type-II working vacation period

$$P_{wv1} = \sum_{n=0}^{N} P_{1,n} \text{ and } P_{wv2} = \sum_{n=0}^{N} P_{2,n}. \tag{31}$$

The probability that the server is idle during busy period $P_{id} = P_{0,0}$.

The probability that the server is working during normal busy period

$$P_s = 1 - P_{0,0} - (P_{wv1} + P_{wv2}). \tag{32}$$

5. Cost model and optimization study

To construct the cost model, we consider the following cost (in unit) elements associated with different events:

- C_1 : Cost per unit time when the server is working during normal busy period.
- C_2 : Cost per unit time when the server is idle during normal busy period.
- C_3 : Cost per unit time when the server is on type-I or type-II working vacation period.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per unit time when a customer balks.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Cost per service per unit time during normal busy period.
- C_8 : Cost per service per unit time during type-I or type-II working vacation period.

Using the definition of each cost element and its corresponding system characteristics, the total expected cost per unit time of the system is given by

$$F = C_1P_s + C_2P_{id} + C_3(P_{wv1} + P_{wv2}) + C_4L_q + C_5B_r + C_6R_r + \mu C_7 + \nu C_8.$$

Our objective is to determine the optimal service rates during working vacation periods type-I and type-II ν^* as well as during regular busy period μ^* that minimize the cost function F.

Given a 3-point pattern, we may fit a quadratic function via corresponding functional values that has a unique minimum, x^q, for the given objective function $F(x)$. The unique optimum x^q of the quadratic function agreeing with $F(x)$ at 3-point operation (x^l, x^m, x^u) is given as

$$x^q \cong \frac{1}{2}\left[\frac{F(x^l)(s^m - s^u) + F(x^m)(s^u - s^l) + F(x^u)(s^l - s^m)}{F(x^l)(x^m - x^u) + F(x^m)(x^u - x^l) + F(x^u)(x^l - x^m)}\right],$$

where $s^l = (x^l)^2, s^m = (x^m)^2$, and $s^u = (x^u)^2$.

5.1. Numerical analysis

For the analysis, we fix $C_1 = 45$, $C_2 = 20$, $C_3 = 30$, $C_4 = 40$, $C_5 = 25$, $C_6 = 35$, $C_7 = 40$ and $C_8 = 25$. The tolerance of QFSM is taken as $\epsilon = 10^{-5}$.

To carry out the optimisation study we consider two following cases:

(a) $\lambda = 4.00$, $\eta = 0.20$, $\phi_1 = 0.16$, $\phi_2 = 0.28$, $\beta' = 0.30$, $\xi_0 = 0.20$, $\xi_1 = 1.90$, $\xi_2 = 2.50$, $N = 10$, $\nu = 1.10$ and $\theta'_n = \dfrac{n}{N^2}$. This case is presented by Table 1 and Figure 2.

(b) $\lambda = 4.00$, $\eta = 0.20$, $\phi_1 = 0.16$, $\phi_2 = 0.28$, $\beta' = 0.30$, $\xi_0 = 0.20$, $\xi_1 = 1.90$, $\xi_2 = 2.50$, $N = 10$, $\mu = 4.60$ and $\theta'_n = \dfrac{n}{N^2}$. This case is presented by Table 2 and Figure 3.

μ^l	μ^m	μ^u	$F(\mu^l)$	$F(\mu^m)$	$F(\mu^u)$	μ^q	$F(\mu^q)$
3.90000	4.30000	4.70000	347.77245	341.85791	341.52020	4.52422	341.06123
4.30000	4.52422	4.70000	341.85791	341.06123	341.52020	4.52739	341.06148
4.30000	4.52422	4.52739	341.85791	341.06123	341.06148	4.52332	341.06121
4.30000	4.52332	4.52422	341.85791	341.06121	341.06123	4.52322	341.06121
4.30000	4.52322	4.52332	341.85791	341.06121	341.06121	4.52320	341.06121
4.30000	4.52320	4.52322	341.85791	341.06121	341.06121	4.52319	341.06121

Table 1: *Search for optimum service rate during regular busy period* (μ^*)

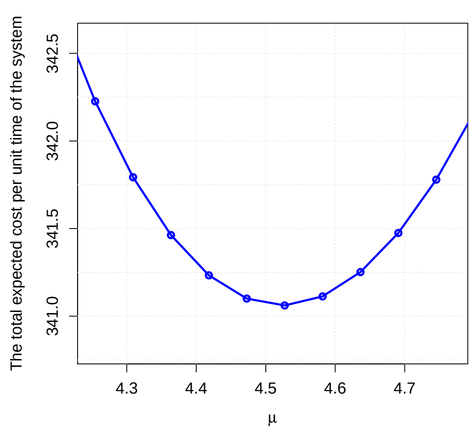

Figure 2: *Effect of* μ *on the total expected cost*

ν^l	ν^m	ν^u	$F(\nu^l)$	$F(\nu^m)$	$F(\nu^u)$	ν^q	$F(\nu^q)$
0.20000	0.30000	0.60000	333.05174	332.70887	334.25941	0.32976	332.71154
0.20000	0.30000	0.32976	333.05174	332.70887	332.71154	0.31323	332.70479
0.30000	0.31323	0.32976	332.70887	332.70479	332.71154	0.31302	332.70479
0.30000	0.31302	0.31323	332.70887	332.70479	332.70479	0.31292	332.70479
0.30000	0.31292	0.31302	332.70887	332.70478	332.70478	0.31292	332.70478
0.30000	0.31292	0.31292	332.70887	332.70478	332.70478	0.31292	332.70478

Table 2: *Search for optimum service rate during working vacation periods* (ν^*)

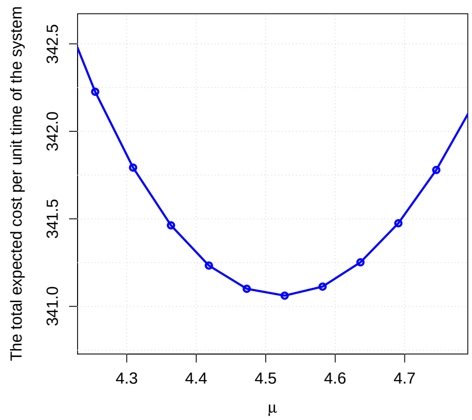

Figure 3: *Effect of ν on the total expected cost*

From Figure 2–3 we clearly observe the convexity of the curves which shows that there exist certain values of the service rates (μ) and (ν) that minimize the total expected cost function for the chosen set of model parameters. Further, via Tables 1–2 respectively, by adopting QFSM and choosing the initial 3-point pattern as $(\mu^l, \mu^m, \mu^u) = (3.90, 4.30, 4.70)$ and $(\nu^l, \nu^m, \nu^u) = (0.20, 0.30, 0.60)$, after finite iterations, we see that the minimum expected operating cost per unit time converges to the solution $F = 341.06121$ at $\mu^* = 4.52319$, and converges to $F = 332.70478$ at $\nu^* = 0.31192$.

5.2. Impact of the arrival rate and the capacity of system

To show the impact of the arrival rate (λ) and the capacity of system (N) on the system performance measures, we consider the following cases:

(a) $\lambda = 3.40 : 0.50 : 5.40$, $\eta = 0.30$, $\phi_1 = 0.10$, $\phi_2 = 0.30$, $\beta' = 0.30$, $\xi_0 = 0.70$, $\xi_1 = 1.10$, $\xi_2 = 1.50$, $N = 12$ and $\theta'_n = \frac{n}{N}$. This case is presented by Table 3.

(b) $\lambda = 4.20$, $\eta = 0.50$, $\phi_1 = 0.05$, $\phi_2 = 0.10$, $\beta' = 0.50$, $\xi_0 = 0.20$, $\xi_1 = 0.30$, $\xi_2 = 0.40$, $N = 4 : 4 : 20$ and $\theta'_n = \frac{n}{N}$. This case is presented by Table 4.

According to Tables 3–4 we observe that along the increasing of λ and N the optimal service rate during regular busy period (μ^*), the optimal service rate during vacation period (ν^*), the minimum expected cost F^*, the probability that the server is working during regular busy period (P_s^*), the mean system size (L_s^*), the average rate of reneging (R_r^*), and the mean number of customers served (E_{cs}^*), monotonically increase. While the probability that the server is idle during normal busy period (P_{id}^*), the probabilities that the server is in type-I (P_{wv1}^*) and type-II (P_{wv2}^*) working vacations, as well as the mean waiting time (W_s^*) all decrease. In addition, the average rate of balking (B_r^*) increases with (λ) and decreases with (N), as intuitively expected. Obviously, the mean system size increases with (λ) and (N) which implies an increase in (P_s^*),, which results in the increasing of the (E_{cs}^*), and a decreasing in (P_{id}^*), (P_{wv1}^*), and (P_{wv2}^*). Further, with the increasing of the system size, the customers become impatient, which leads to the increasing of the average rate of reneging. On the other hand, it is quite obvious that the larger the mean system size, the higher the average balking rate. However, the greater the system capacity, the smaller the mean balking rate.

	$\lambda = 3.40$	$\lambda = 3.90$	$\lambda = 4.40$	$\lambda = 4.90$	$\lambda = 5.40$
μ^*	2.00101	2.26440	2.52243	2.77541	3.02350
ν^*	0.27314	0.28515	0.29377	0.29962	0.30324
P_s^*	0.74176	0.76768	0.78906	0.80702	0.82236
P_{id}^*	0.09022	0.08485	0.07964	0.07472	0.07011
P_{wv1}^*	0.16469	0.14496	0.12934	0.11670	0.10627
P_{wv2}^*	0.00333	0.00251	0.00195	0.00155	0.00126
L_s^*	2.42530	2.57669	2.71753	2.84945	2.97375
W_s^*	1.12658	0.65225	0.61035	0.57518	0.54511
B_r^*	0.68717	0.83742	0.99643	1.16352	1.33819
R_r^*	1.18598	1.28488	1.37685	1.46305	1.54434
E_{cs}^*	4.08661	4.99810	5.95278	6.94498	7.97011
F^*	246.885	271.115	294.568	317.371	339.618

Table 3: λ *vs. system characteristics*

	$N = 4$	$N = 8$	$N = 12$	$N = 16$	$N = 20$
μ^*	1.33795	2.13714	2.40345	2.51232	2.57433
ν^*	0.20685	0.52350	0.66346	0.72081	0.74754
P_s^*	0.78373	0.82201	0.84511	0.86353	0.87640
P_{id}^*	0.03834	0.03672	0.03358	0.02969	0.02683
P_{wv1}^*	0.17629	0.13966	0.11984	0.10545	0.09556
P_{wv2}^*	0.00164	0.00162	0.00148	0.00132	0.00121
L_s^*	2.60956	3.47690	4.07565	4.56079	4.94947
W_s^*	0.91540	1.06872	1.12330	1.19678	1.30147
B_r^*	2.74004	1.82537	1.42648	1.19721	1.03939
R_r^*	0.36489	0.54782	0.66700	0.76161	0.83722
E_{cs}^*	2.89298	6.51793	8.84703	10.53426	11.8432
F^*	247.329	306.186	338.751	361.666	379.151

Table 4: N *vs. system characteristics*

5.3. Impact of working vacation rates and the waiting rate of the sever

To study the impact of working vacation rates ϕ_1, ϕ_2, and the waiting rate of the sever η, we consider:

(a) $\lambda = 3.80$, $\eta = 0.80$, $\phi_1 = 0.04 : 0.04 : 0.20$, $\phi_2 = 0.30$, $\beta' = 0.20$, $\xi_0 = 0.30$, $\xi_1 = 0.60$, $\xi_2 = 1.20$, $N = 8$ and $\theta'_n = 1 - \dfrac{1}{n+1}$. This case is presented by Table 5.

(b) $\lambda = 4.10$, $\eta = 0.80$, $\phi_1 = 0.04$, $\phi_2 = 0.12 : 0.04 : 0.28$, $\beta' = 0.20$, $\xi_0 = 0.15$, $\xi_1 = 0.30$, $\xi_2 = 1.20$, $N = 8$ and $\theta'_n = 1 - \dfrac{1}{n+1}$. This case is presented by Table 6.

(c) $\lambda = 4.30$, $\eta = 0.15 : 0.20 : 0.95$, $\phi_1 = 0.04$, $\phi_2 = 0.12$, $\beta' = 0.40$, $\xi_0 = 0.60$, $\xi_1 = 1.20$, $\xi_2 = 0.45$, $N = 8$ and $\theta'_n = 1 - \dfrac{1}{n+1}$. This case is presented by Table 7.

Tables 5–7 depict the impact of type-I and type-II vacation rates as well as the waiting server rate. With the increasing of (ϕ_i), $i = 1, 2$, a decreasing trend is observed in (ν^*), (L_s^*), (W_s^*), (B_r^*), (R_r^*), and (F^*). As intuitively expected, when the mean vacation times decreases, the server switches rapidly to the normal busy period at which customers are served at faster rate. Consequently, the mean system size decreases which results in the decreasing of the

	$\phi_1 = 0.04$	$\phi_1 = 0.08$	$\phi_1 = 0.12$	$\phi_1 = 0.16$	$\phi_1 = 0.20$
μ^*	1.21830	1.26232	1.29821	1.32838	1.35429
ν^*	0.52471	0.44988	0.37691	0.30668	0.23918
P_s^*	0.57433	0.59951	0.61883	0.63437	0.64720
P_{id}^*	0.03432	0.03703	0.03909	0.04070	0.04197
P_{wv1}^*	0.38825	0.35750	0.33332	0.31338	0.29639
P_{wv2}^*	0.00310	0.00597	0.00875	0.01156	0.01445
L_s^*	2.20618	2.19712	2.18934	2.18229	2.17576
W_s^*	0.67237	0.65098	0.63924	0.62850	0.62047
B_r^*	2.38508	2.37837	2.37267	2.36762	2.36309
R_r^*	0.52272	0.51199	0.50375	0.49707	0.49153
E_{cs}^*	2.10512	2.13951	2.16614	2.18806	2.20662
F^*	228.643	228.053	227.254	226.334	225.341

Table 5: ϕ_1 *vs. system characteristics*

	$\phi_2 = 0.12$	$\phi_2 = 0.16$	$\phi_2 = 0.20$	$\phi_2 = 0.24$	$\phi_2 = 0.28$
μ^*	1.75004	1.75039	1.75065	1.75087	1.75104
ν^*	0.50541	0.50536	0.50531	0.50528	0.50526
P_s^*	0.73599	0.73610	0.73617	0.73623	0.73628
P_{id}^*	0.08647	0.08652	0.08656	0.08659	0.08661
P_{wv1}^*	0.17575	0.17587	0.17596	0.17602	0.17607
P_{wv2}^*	0.00178	0.00151	0.00131	0.00116	0.00105
L_s^*	2.19603	2.19572	2.19547	2.19528	2.19512
W_s^*	0.61750	0.61694	0.61607	0.61588	0.61561
B_r^*	2.48420	2.48399	2.48383	2.48370	2.48359
R_r^*	0.24230	0.24215	0.24204	0.24196	0.24190
E_{cs}^*	3.23176	3.23228	3.23266	3.23297	3.23321
F^*	245.028	245.021	245.017	245.013	245.011

Table 6: ϕ_2 *vs. system characteristics*

	$\eta = 0.15$	$\eta = 0.35$	$\eta = 0.55$	$\eta = 0.75$	$\eta = 0.95$
μ^*	1.37287	1.22927	1.14293	1.08299	1.03783
ν^*	0.11612	0.17139	0.19595	0.21078	0.22131
P_s^*	0.75165	0.68826	0.64326	0.60759	0.57802
P_{id}^*	0.07734	0.05523	0.04269	0.03425	0.02803
P_{wv1}^*	0.16983	0.25435	0.31121	0.35480	0.39015
P_{wv2}^*	0.00118	0.00216	0.00283	0.00336	0.00380
L_s^*	1.91357	1.99626	2.04076	2.06873	2.08778
W_s^*	0.48759	0.49874	0.50962	0.51157	0.51922
B_r^*	2.53823	2.61049	2.64926	2.67384	2.69085
R_r^*	0.71047	0.80087	0.85612	0.89540	0.92537
E_{cs}^*	2.18160	1.90269	1.73364	1.61476	1.52495
F^*	226.442	228.900	229.805	230.186	230.323

Table 7: η *vs. system characteristics*

average rates of balking and reneging as well as of the total expected cost of the system. Along the increasing of (ϕ_i), $i = 1, 2$, a increasing trend is seen in (μ^*), (P_{id}^*), (P_s^*), and (E_{cs}^*), as intuitively expected. Also, (P_{wv1}^*) decreases with (ϕ_1) and increases with (ϕ_2). While (P_{wv2}^*) increases with (ϕ_1) and decreases with (ϕ_2), as expected.

An increasing trend is seen in (ν^*), (L_s^*), (W_s^*), (P_{wv1}^*), (P_{wv2}^*), (B_r^*), (R_r^*), and (F^*) with η. Obviously, the probability that the server switches to the working vacation period increases with (η). This implies an increase in (L_s^*). Thus, the mean number of customers in the system increases with (η), which results in the increasing in the average rates of balking and reneging.

Further, a decreasing trend is observed in (μ^*), (P_s^*), (P_{id}^*), and (E_{cs}^*). This makes a perfect sense; the smaller the mean waiting server time, the greater the probability of busy period, and the larger the mean number of customers served.

5.4. Impact of impatience rates

To show the impact of impatience rates ξ_0, ξ_1 and ξ_2 on the performance measures of the system, we consider the following cases:

(a) $\lambda = 4.50$, $\eta = 0.40$, $\phi_1 = 0.10$, $\phi_2 = 0.80$, $\beta' = 0.50$, $\xi_0 = 0.20 : 0.30 : 1.40$, $\xi_1 = 1.50$, $\xi_2 = 2.50$, $N = 15$, and $\theta'_n = \dfrac{n}{N^2}$. This case is presented by Table 8.

(b) $\lambda = 4.50$, $\eta = 0.30$, $\phi_1 = 0.07$, $\phi_2 = 0.14$, $\beta' = 0.50$, $\xi_0 = 0.40$, $\xi_1 = 0.60 : 0.30 : 1.80$, $\xi_2 = 2.50$, $N = 15$, and $\theta'_n = \dfrac{n}{N^2}$. This case is presented by Table 9.

(c) $\lambda = 4.50$, $\eta = 0.30$, $\phi_1 = 0.07$, $\phi_2 = 0.14$, $\beta' = 0.50$, $\xi_0 = 0.40$, $\xi_1 = 0.90$, $\xi_2 = 1.30 : 0.30 : 2.50$, $N = 15$, and $\theta'_n = \dfrac{n}{N^2}$. This case is presented by Table 10.

	$\xi_0 = 0.20$	$\xi_0 = 0.50$	$\xi_0 = 0.80$	$\xi_0 = 1.10$	$\xi_0 = 1.40$
μ^*	4.99107	4.04238	3.32898	2.76214	2.29614
ν^*	0.78815	0.61576	0.50711	0.43165	0.37837
P_s^*	0.67768	0.71916	0.74584	0.76522	0.78073
P_{id}^*	0.12901	0.10686	0.09206	0.08102	0.07196
P_{wv1}^*	0.19066	0.17196	0.16043	0.15233	0.14604
P_{wv2}^*	0.00265	0.00203	0.00167	0.00143	0.00127
L_s^*	3.13486	3.05444	2.95059	2.84738	2.75133
W_s^*	0.72861	0.71563	0.70042	0.68951	0.64122
B_r^*	0.06554	0.06140	0.05905	0.05695	0.05503
R_r^*	0.91707	1.43442	1.88256	2.26785	2.60027
E_{cs}^*	13.44693	10.62928	8.44975	6.75811	5.42674
F^*	383.392	355.653	335.703	320.326	308.004

Table 8: ξ_0 *vs. system characteristics*

From Tables 8–10, we see that along the increasing of the impatience rates (ξ_0), (ξ_1), and (ξ_2), the optimal service rate during vacation period (ν^*), the optimum expected cost (F^*), and the performance measures (L_s^*), (W_s^*), (B_r^*), and (E_{cs}^*) all decreases. While (R_r^*) increases with (ξ_0), (ξ_1), and (ξ_2). It is quite clear that the mean system size decreases with the impatience rates which implies a decrease in the mean number of customers served. In addition, as expected, (μ^*), (P_{id}^*), (P_{wv1}^*), and (P_{wv2}^*) decrease with (ξ_0) and increase with (ξ_1) and (ξ_2). Whereas (P_s^*) increases with (ξ_0) and decreases with (ξ_1) and (ξ_2), as it should be.

	$\xi_1 = 0.60$	$\xi_1 = 0.90$	$\xi_1 = 1.20$	$\xi_1 = 1.50$	$\xi_1 = 1.80$
μ^*	3.95322	4.24402	4.36186	4.41578	4.44369
ν^*	0.95056	0.81692	0.70978	0.63677	0.58531
P_s^*	0.79932	0.75398	0.72899	0.72013	0.69968
P_{id}^*	0.09568	0.11613	0.12497	0.12581	0.13073
P_{wv1}^*	0.10285	0.12736	0.14329	0.15122	0.16648
P_{wv2}^*	0.00214	0.00253	0.00275	0.00284	0.16648
L_s^*	3.66408	3.28129	3.09512	3.02621	2.90107
W_s^*	0.85584	0.80123	0.77389	0.73955	0.69877
B_r^*	0.07503	0.06648	0.06254	0.06114	0.05854
R_r^*	1.17725	1.13883	1.16401	1.20845	1.24147
E_{cs}^*	13.2511	12.3664	11.8044	11.5310	11.1348
F^*	376.899	368.304	363.573	360.393	358.332

Table 9: ξ_1 *vs. system characteristics*

	$\xi_2 = 1.30$	$\xi_2 = 1.60$	$\xi_2 = 1.90$	$\xi_2 = 2.20$	$\xi_2 = 2.50$
μ^*	4.24202	4.24288	4.24341	4.24377	4.24402
ν^*	0.81812	0.81764	0.81733	0.81710	0.81692
P_s^*	0.75456	0.75436	0.75421	0.75409	0.75398
P_{id}^*	0.11596	0.11603	0.11608	0.11611	0.11613
P_{wv1}^*	0.12721	0.12727	0.12731	0.12734	0.12736
P_{wv2}^*	0.00227	0.00233	0.00240	0.00246	0.00253
L_s^*	3.28593	3.28420	3.28298	3.28204	3.28129
W_s^*	0.79854	0.78264	0.78009	0.77897	0.77875
B_r^*	0.06657	0.06654	0.06651	0.06649	0.06648
R_r^*	1.13795	1.13814	1.13836	1.13859	1.13883
E_{cs}^*	12.3795	12.3749	12.37145	12.3687	12.3664
F^*	368.413	368.371	368.342	368.321	368.304

Table 10: ξ_2 *vs. system characteristics*

5.5. Impact of balking function and interruption probability

To examine the effect of balking function θ'_n and interruption probability β' on the characteristics of the considered queuing system, we consider the following cases:

(a) $\lambda = 4.50$, $\eta = 0.80$, $\phi_1 = 0.10$, $\phi_2 = 0.50$, $\beta' = 0.10 : 0.20 : 0.90$, $\xi_0 = 0.20$, $\xi_1 = 0.90$, $\xi_2 = 1.50$, $N = 14$, and $\theta'_n = \dfrac{n}{N}$. This case is presented by Table 11.

(a) $\lambda = 4.80$, $\eta = 0.25$, $\phi_1 = 0.20$, $\phi_2 = 0.80$, $\beta' = 0.45$, $\xi_0 = 0.15$, $\xi_1 = 0.55$, $\xi_2 = 0.95$, $N = 12$, and $\theta'_n = \dfrac{n}{N^2} : \dfrac{n}{N} : 1 - \dfrac{1}{n+1}$. This case is presented by Table 12.

From Table 11, we observe that with the increases of (β'), the probabilities of type-I and type-II working vacations (P_{wv1}^*) and (P_{wv2}^*), the mean system size (L_s^*), the mean waiting time (W_s^*) as well as average rates of balking and reneging (B_r^*) and (R_r^*) monotonically decrease. Obviously, the increasing of the interruption probability implies a decrease in the vacation periods which in turn implies a decrease in reneging and balking average rates. This results in the increasing of the mean number of customers served (E_{cs}^*). Consequently, the optimal expected cost (F^*) decreases. While the probability that the server is working during normal busy period (P_s^*) and the probability that the server is idle during this period (P_{id}^*) increase, which is quite reasonable, the greater the interruption probability, the smaller the probability of working vacation and the higher the probability of busy period.

	$\beta' = 0.10$	$\beta' = 0.30$	$\beta' = 0.50$	$\beta' = 0.70$	$\beta' = 0.90$
μ^*	3.10718	3.36279	3.51371	3.61821	3.69641
ν^*	0.86824	0.86353	0.82789	0.79306	0.76167
P_s^*	0.65587	0.70040	0.71881	0.72910	0.73579
P_{id}^*	0.03560	0.04965	0.05855	0.06504	0.07014
P_{wv1}^*	0.30458	0.24604	0.21875	0.20204	0.19033
P_{wv2}^*	0.00395	0.00391	0.00389	0.00382	0.00374
L_s^*	3.63470	3.38497	3.23954	3.13910	3.06396
W_s^*	0.81992	0.79245	0.73207	0.69408	0.67852
B_r^*	1.16830	1.08802	1.04128	1.00900	0.98485
R_r^*	1.04115	0.86062	0.77012	0.71188	0.67005
E_{cs}^*	9.04293	9.50768	9.69296	9.79422	9.85805
F^*	358.666	351.732	347.384	344.265	341.852

Table 11: β' *vs. system characteristics*

	$\theta'_n = \frac{n}{N^2}$	$\theta'_n = \frac{n}{N}$	$\theta'_n = 1 - \frac{1}{n+1}$
μ^*	5.37564	4.04876	2.19541
ν^*	1.04026	0.49766	0.13724
P_s^*	0.75040	0.77730	0.76538
P_{id}^*	0.14131	0.11448	0.09992
P_{wv1}^*	0.10505	0.10589	0.13267
P_{wv2}^*	0.00324	0.00232	0.00203
L_s^*	3.46201	2.93719	2.08750
W_s^*	0.71920	0.63551	0.54395
B_r^*	0.14949	1.17488	2.83595
R_r^*	0.51961	0.42893	0.26626
E_{cs}^*	16.8522	10.4906	3.8767
F^*	407.532	341.741	259.743

Table 12: θ'_n *vs. system characteristics*

From Table 12 it is well shown that $\theta_n = 1 - \frac{n}{N^2}$ yields the lowest the average rate of balking (B_r^*). In addition, with the increasing of (θ'_n), the probability that the server is idle during normal busy period (P_{id}^*), the probability of type-II working vacation (P_{wv2}^*), the mean system size (L_s^*), and the mean waiting time (W_s^*) monotonically decrease. Nevertheless, the probability of type-I working vacation (P_{wv1}^*) increases with θ_n. This implies a decrease in the average rate of reneging (R_r^*) as well as the mean number of customers served (E_{cs}^*). Therefore, the minimum expected cost (F^*) decreases, as intuitively expected. On the other hand, the probability that the server is working during normal busy period (P_s^*) is not monotone with (θ_n), this can be due to the choice of the system parameters.

6. Conclusion

In this paper, we considered a $M/M/1/N$ queuing model subject to differentiated working vacations, Bernoulli schedule vacation interruption, waiting server and customer' impatience during both busy and working vacation periods. We obtained the steady-state solution of the considered system. Then, we derived useful characteristics of the queuing model. After that, we constructed a cost model in order to determine the optimal values of service rates, simultaneously, to optimize the total expected cost per unit time via a quadratic fit search method (QFSM). Finally, we presented numerical results showing the impact of system parameters on key performance measures. For further works, it will be interesting to generalize the current model to more complex queuing systems such as $M/M/c/N$ and $MAP/M/c/N$ queues with differentiated working vacations, Bernoulli schedule vacation interruption, waiting servers, and impatient customers.

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