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Research note
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Economic lot sizing with imperfect rework derived without derivatives

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SUMMARY

This paper presents an algebraic method for solving economic production quantity (EPQ) model with imperfect rework. Conventional method for deriving optimal lot size is by using differential calculus on the cost function with the need to prove optimality first. Recent articles proposed algebraic approach to the solution of classic economic order quantity (EOQ) and EPQ model without reference to the use of derivatives. This note extends them to an EPQ model taking into consideration an imperfect rework of defective items. We demonstrate that the optimal lot size and the expected production-inventory cost for such a realistic EPQ model can be derived without derivatives.

Key words: *lot sizing, production, random defects, imperfect rework, algebraic approach.*

1. INTRODUCTION

Since the economic order quantity (EOQ) model was first introduced several decades ago [1], the economic lot size problems have been extensively studied. In the manufacturing sector, when items are produced internally instead of being obtained from an outside supplier, the economic production quantity (EPQ) model is often used to determine the optimal production lot size. The classic EPQ model assumes that the manufacturing process will never produce defective items in a production lot. But in reality, defective items generated during a production run are inevitable. A considerable amount of research has been carried out to address the imperfect quality EPQ model, see Refs. [2-9].

Conventional method for solving the optimal ordering/production quantity model is by using differential calculus on the cost function equation with the need to prove optimality first. Several recent articles, for examples Grubbström and Erdem [10] and Cárdenas-Barrón [11], proposed the algebraic approach to the solution of classic economic order quantity (EOQ) and EPQ model without reference to the use of derivatives (neither applying the first-order nor the second-order differentiations). This note extends the aforementioned works presented in Refs. [10-11] to an EPQ model taking into consideration an imperfect rework of defective items.

2. MATHEMATICAL MODELLING AND ANALYSIS

We reconsider the imperfect quality EPQ model examined by Chiu and Gong in Ref. [7]. The manufacturing process has a production rate P and demand rate λ, where P is much larger than λ. The process randomly generates x proportion of defective items and all of them are assumed to be reworked right after the regular production ends, at a reworking rate of P_1. Rework process itself is assumed to be imperfect either, a portion θ_1 of defective items fail the rework and become scrap items. Since the proposed model does not allow shortages, hence, the replenishment policy is to re-start a new production run whenever on-hand inventories run out. The production rate d of the defective items could be expressed as the product of the production rate times the percentage of defective items produced. Therefore, d can be written as $d=Px$. Let d_1 be the production rate of scrap items during the reworking, then d_1 can be expressed as the product of the reworking rate P_1 times the percentage of scrap items produced during the rework process θ_1. Hence, d_1 can be written as: $d_1=P_1\theta_1$. The following notations are introduced in our analysis:

- Q is production lot size,
- x is the proportion of imperfect quality items produced, a random variable with known probability density function,
- H_1 is the maximum level of perfect quality on-hand inventory in units, when the regular production process stops,
- H is the maximum level of perfect quality on-hand inventory in units, when rework process ends,
- K is setup cost for each production run,
- C is production cost per item ($/item; inspection cost per item is included),
- C_R is repair cost for each imperfect quality item reworked ($/item),
- C_S is disposal cost for each scrap item produced ($/scrap item),
- h is holding cost for each item ($/item/unit time),
- h_1 is holding cost for each reworked item ($/item/unit time),
- $TCU(Q)$ is the total inventory costs per unit time.

The production rate must be always greater than or equal to the sum of the demand rate and the rate at which defective items are produced. Therefore, we must have:

$$P-d-\lambda \geq 0 \quad and \quad \left(1-\frac{\lambda}{P}\right) \geq x \geq 0 \tag{1}$$

The following derivations are similar to the prior works, Refs. [6, 7], referring to Figure 1.

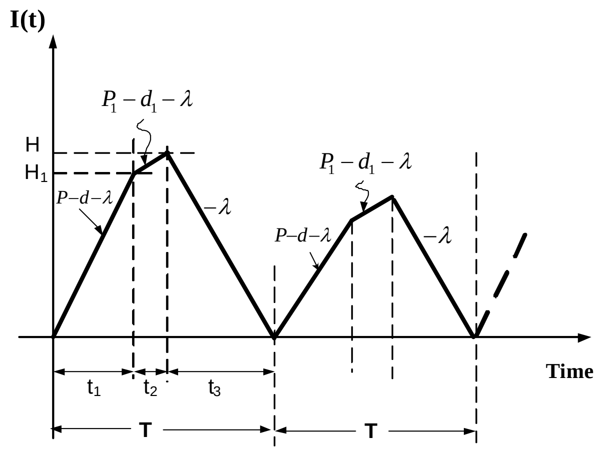

Fig. 1 On-hand inventory of perfect quality items

The expressions of production uptime t_1, the time t_2 needed to rework the defective items, production downtime t_3, on-hand inventory level H_1 and H, and the cycle length are as follows:

$$t_1=\frac{H_1}{P-d-\lambda}=\frac{Q}{P} \tag{2}$$

$$t_2=\frac{x\cdot Q}{P_1}=\frac{d\cdot Q}{P_1 P} \tag{3}$$

$$t_3=\frac{H}{\lambda} \tag{4}$$

$$H_1=(P-d-\lambda)t_1=Q\left(1-x-\frac{\lambda}{P}\right) \tag{5}$$

$$H=H_1+(P_1-d_1-\lambda)t_2=Q\left(1-\frac{\lambda}{P}-\frac{d_1 d}{P_1 P}-\frac{d\lambda}{P_1 P}\right) \tag{6}$$

$$T=\sum_{i=1}^{3} t_i=\frac{Q(1-\theta_1\cdot x)}{\lambda} \tag{7}$$

where $0\leq\theta_1\leq 1$ and Q, θ_1 and x are the scrap items produced during the imperfect rework process. Solving the inventory cost per cycle, we have:

$$\begin{aligned} TC(Q)&=C\cdot Q+C_R\cdot[x\cdot Q]+C_S\cdot[x\cdot Q\cdot\theta_1]+K+\\ &+h\left[\frac{H_1+d\cdot t_1}{2}(t_1)+\frac{H_1+H}{2}(t_2)+\frac{H}{2}(t_3)\right]+\\ &+h_1\cdot\frac{P_1\cdot t_2}{2}(t_2) \end{aligned} \tag{8}$$

Since the percentage of imperfect quality items is a random variable, the expected annual cost is $E[TCU(Q)] = E[TC(Q)] / E[T]$, from Eqs. (1) through (8), one obtains:

$$E\left[TCU(Q)\right]=\lambda\left[C\cdot\frac{1}{1-\theta_1\cdot E[x]}+C_R\cdot\frac{E[x]}{1-\theta_1\cdot E[x]}+C_S\cdot\theta_1\cdot\frac{E[x]}{1-\theta_1\cdot E[x]}\right]+$$
$$+\frac{K\lambda}{Q}\cdot\frac{1}{1-\theta_1\cdot E[x]}+\frac{h\cdot Q}{2}\left(1-\frac{\lambda}{P}\right)\cdot\frac{1}{1-\theta_1\cdot E[x]}+\frac{\lambda Q}{2P_1}\left[h_1-h(1-\theta_1)\right]\cdot\frac{E[x^2]}{1-\theta_1\cdot E[x]}- \tag{9}$$
$$-h\cdot Q\cdot\theta_1\cdot\left(1-\frac{\lambda}{P}\right)\cdot\frac{E[x]}{1-\theta_1\cdot E[x]}+\frac{h\cdot Q\cdot\theta_1^2}{2}\cdot\frac{E[x^2]}{1-\theta_1\cdot E[x]}$$

3. ECONOMIC LOT SIZE DERIVED WITHOUT DERIVATIVES

The expected inventory cost function *E[TCU(Q)]* shown in Eq. (9) yields the same result as given by Chiu and Gong [7]. Now, instead of applying derivatives to the cost function, we propose an algebraic solution procedure to solve the optimal production lot size. From Eq. (9) we have:

$$E\left[TCU(Q)\right]=\lambda\left[\frac{C+C_R\cdot E[x]+C_S\cdot\theta_1\cdot E[x]}{1-\theta_1\cdot E[x]}\right]+\frac{1}{2Q\left(1-\theta_1E[x]\right)}\cdot$$
$$\cdot\left\{2K\lambda+hQ^2\left(1-\frac{\lambda}{P}\right)+\frac{\lambda Q^2}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2hQ^2\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+hQ^2\theta_1^2E[x^2]\right\} \tag{10}$$

Rearranging Eq. (10) and adding and subtracting an additional term, one has:

$$E\left[TCU(Q)\right]=\lambda\left[\frac{C+C_R\cdot E[x]+C_S\cdot\theta_1\cdot E[x]}{1-\theta_1\cdot E[x]}\right]+\frac{1}{2Q\left(1-\theta_1E[x]\right)}\cdot$$
$$\cdot\left\{\begin{array}{l}\left(\sqrt{2K\lambda}\right)^2+\left(Q\sqrt{h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^2E[x^2]}\right)^2\\-2\sqrt{2K\lambda}\left(Q\sqrt{h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^2E[x^2]}\right)\\+2\sqrt{2K\lambda}\left(Q\sqrt{h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^2E[x^2]}\right)\end{array}\right\} \tag{11}$$

Therefore, the cost function *E[TCU(Q)]* can be expressed as:

$$E\left[TCU(Q)\right]=\lambda\left[\frac{C+C_R\cdot E[x]+C_S\cdot\theta_1\cdot E[x]}{1-\theta_1\cdot E[x]}\right]+\frac{1}{2Q\left(1-\theta_1E[x]\right)}\cdot$$
$$\cdot\left(\sqrt{2K\lambda}-Q\sqrt{h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^2E[x^2]}\right)^2+ \tag{12}$$
$$+\frac{1}{1-\theta_1E[x]}\sqrt{2K\lambda\left(h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^2E[x^2]\right)}$$

From Eq. (12), one notices that in order to minimize E[TCU(Q)] we should have:

$$\left(\sqrt{2K\lambda}-Q\sqrt{h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h(1-\theta_1)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^2E[x^2]}\right)=0 \tag{13}$$

Hence:

$$Q^{*}=\sqrt{\frac{2K\lambda}{h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^{2}E[x^2]}} \tag{14}$$

Equation (14) yields the same result as was derived using differential calculus in Ref. [7]. Suppose that the process produces no defective items, that is $x=0$; one confirms that Eq. (14) becomes the same equation as that given by the classic EPQ model in Refs. [12,13]:

$$Q^{*}=\sqrt{\frac{2K\lambda}{h\left(1-\frac{\lambda}{P}\right)}} \tag{15}$$

Now, substituting Eq. (14) into Eq. (12), the expected inventory cost function $E[TCU(Q)]$ can be obtained immediately as follows:

$$E\left[TCU\left(Q\right)\right]=\lambda\left[\frac{C+C_R\cdot E[x]+C_S\cdot\theta_1\cdot E[x]}{1-\theta_1\cdot E[x]}\right]+ \\ +\frac{1}{1-\theta_1 E[x]}\sqrt{2K\lambda\left(h\left(1-\frac{\lambda}{P}\right)+\frac{\lambda}{P_1}\left[h_1-h\left(1-\theta_1\right)\right]E[x^2]-2h\theta_1\left(1-\frac{\lambda}{P}\right)E[x]+h\theta_1^{2}E[x^2]\right)} \tag{16}$$

4. CONCLUDING REMARKS

This research note extends prior works of Grubbström and Erdem [10] and Cárdenas-Barrón [11] to an EPQ model taking into consideration an imperfect rework of defective items. We demonstrate that the optimal lot size for such a realistic EPQ model with imperfect rework can be solved algebraically and the expected overall inventory cost can be obtained immediately as well. This approach uses algebraic derivation by forming a square term and then setting it to zero to optimize the cost function. It should enable students or practitioners with little or no knowledge of calculus to learn and to manage the realistic production-inventory model.

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EKONOMSKO ODREĐIVANJE VELIČINE PARTIJE S NESAVRŠENOM OBRADOM KOJE SE DERIVIRA BEZ IZVEDENICA

SAŽETAK

Ovaj rad predstavlja algebarsku metodu za rješavanje ekonomskog modela količine proizvodnje (EPQ) s nesavršenom obradom. Uobičajena metoda za deriviranje optimalnog određivanja veličine partije koristi proračun troškovne funkcije sa svrhom dokazivanja optimalnosti. Nedavni članci su predlagali algebarski pristup za rješavanje klasičnog ekonomskog reda veličina (EOQ) ili EPQ modela bez korištenja izvedenica. Ova ih napomena proširuje do EPQ modela uzimajući u obzir nesavršenu obradu manjkavih podataka. Pokazujemo da se optimalno određivanje veličine partije i očekivani trošak proizvodnje zaliha za ovako stvarni EPQ model može izvesti bez izvedenica.

Ključne riječi: *određivanje veličine partije, proizvodnja, nasumični nedostaci, nesavršena obrada, algebarski pristup.*