ON CAUSAL GEOMETRIES

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Submitted to the Graduate Faculty of

the Kenneth P. Dietrich Graduate School of Arts and Sciences in

partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2015

UNIVERSITY OF PITTSBURGH

KENNETH P. DIETRICH GRADUATE SCHOOL OF ARTS AND SCIENCES

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University of Pittsburgh, 2015

Causal geometries are geometric structures on manifolds for which a (non-degenerate) null cone exists at every point, such that the null cones satisfy a version of Huygen's principle. Causal geometries are a natural generalization of conformal geometries (in non-Euclidean signature). They appear naturally as incidence geometries for projective geometries in three-dimensions, and third-order ordinary differential equations. These share features with conformal geometries: null geodesics exist, as does the Weyl tensor, and there are Raychaudhuri conditions on the null geodesic deviation.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1.0 INTRODUCTION

Soon after the development of the theory of gravitation by Albert Einstein [22], Hermann Weyl [82] tried to do away with lengths and instead to base the theory entirely on angles (Afriat [1], [2], [3]). In present-day language, Weyl wanted a conformally invariant theory, so a theory invariant under the transformations $g(x) \to \omega(x)^2 g(x)$, where $g(x)$ is the Lorentzian metric of the theory and $\omega(x)$, the conformal factor, is a positive function of the space-time point x. Indeed the purely gravitational degrees of freedom of space-time are represented by the Weyl tensor (Weyl [83]), which is conformally invariant (Schouten [74]). However Weyl's theory did not agree with the experiment, as was quickly pointed out by Einstein [23]. Weyl eventually recast and revived his theory, by turning away from gravity and instead developing gauge theory. In the meantime, the attempt to understand the conformal properties of gravity has led to much progress and has been a constant preoccupation of many researchers.

The present work tries to reconcile the opposing philosophies of Einstein and Weyl. At the same time, we effectively generalize the theory of Einstein. The reconciliation is achieved by moving the entire theory into the tangent bundle of space-time, where the focus is on the bundle of null directions and the associated null geodesic foliation of that bundle, both conformally invariant constructs. In the Einstein theory, the bundle is defined by the vanishing of the homogeneous quadratic form in the velocities, $g(x)(v, v) = 0$, where $g(x)$ is the metric and v is a tangent vector at a point x of the space-time. I generalize by allowing the null cone to be given by a general homogeneous function of the velocities, subject only to the genericity condition that its Hessian with respect to the velocities be non-degenerate (and Lorentzian in the case of space-time). I call this structure a causal geometry.

A key point is that the allowed conformal rescalings are vastly generalized, the function ω being replaced by a homogeneous function on the tangent bundle, so a function of $2n-1$ free variables, as opposed to the standard n variables, for a space-time of $n \geq 3$ dimensions. So the conformal transformations are on a more even footing with the metric, as compared with the standard theory, where there is one function representing the conformal transformation and $2^{-1}n(n+1)$ functions encoding the metric. Our first main result is that there is a natural generalization of the Weyl tensor, which is proven to be invariant under the enlarged class of conformal transformations.

Causal geometries are somewhat similar to the notion of Finsler [28] geometries, however the null geodesics in a causal geometry are inherently non-variational. Our geometries generalize more directly those of Bryant ([9], [10]), who defines a *generalized Finsler geometry* in terms of its unit sphere bundle. This is a smooth hypersurface $\Sigma \subset TM$ for which the projection $\Sigma \to M$ is a submersion, and such that in each fiber Σ is convex, connected, and transverse to the radial direction. Bryant's geodesics are obtained by imposing contact conditions on the geodesic spray, and do not readily admit a variational characterization. The geodesics in our geometry bring a similar machinery to bear on the null geodesics. Furthermore, generalized Finsler geometries can be obtained from suitable causal geometries by symmetry reduction. Conversely, a generalized Finsler geometry defines a causal geometry in one higher dimension in a natural manner.

When extrapolating from an established physical theory, one wants to preserve as much of the structure of the old theory as possible. One motivation for going beyond the Einstein theory is the inevitable presence of singularities in the theory, as first brilliantly proved by Sir Roger Penrose [66], [64], [65]. One might wish to construct a new theory free of singularities. However, at least classically, the intuition behind the Penrose theorem is compelling and depends only on the attractive nature of the gravitational interaction and very little on the details of the theory. This intuition is vital to the generalized theory, so we ultimately wish to generalize the Penrose singularity theorem to our case. Examination of the proof given by Penrose shows that apart from general causal properties described in Kronheimer and Penrose [50] and Geroch, Kronheimer, and Penrose [32], which do not depend on the null cones being quadratic in the velocities, the only other ingredient needed for the proof to go through is apparently the Raychaudhuri–Sachs effect which predicts the existence of conjugate points for congruences of null geodesics, given that a local positive energy condition holds and that there is a point of the congruence where the divergence is negative (Raychaudhuri [69], [68], Sachs [70]). It is proven in Chapter 8 that the Raychauhuri–Sachs theorem naturally generalizes to the new context, as does the Raychauhuri–Sachs effect in the Lorentzian case, subject to a natural generalization of the local positive energy condition, so this main ingredient of the singularity theorem goes through.

While so far this presents the theory as a generalized theory of gravity, we see potential applications in many other areas of mathematics and physics. In our case the motivation for constructing the theory came from two areas studied by us: neither of these areas is concerned with generalizing the Einstein theory. Both involve the construction of a metric that is once degenerate and yet is not generally invariant in the degenerate direction: crudely speaking, a metric of the form $g_{ij}(x,t)dx^i dx^j$, where the (invertible) coefficient matrix $g_{ij}(x,t)$ in general depends non-trivially on the parameter t. The question is what to do with this metric?

This dissertation presents a number of scenarios in which such a metric appears naturally in mathematical problems. The remainder of this introduction serves as a rough guide.

1.1 PRELIMINARIES

Chapter 2 contains preliminaries on differentiable manifolds. We here adopt a somewhat novel perspective in considering C^k manifolds from the point of view of sheaf theory. A C^k manifold is a topological space equipped with a sheaf of rings that is locally isomorphic as a ringed space to a Euclidean space with its sheaf of k-times continuously differentiable functions. Many familiar notions from the theory of smooth manifolds require considerably more care to define in a coordinate-independent manner for C^k manifolds than for smooth manifolds. The root of the reason is the lack of a decent preparation theorem (Hadamard's Lemma 2).

To formulate everything in a coordinate-independent manner, we should like to be able to define differentials and, more generally, jets. In this undertaking, it becomes clear why an adequate preparation lemma is required. We should like to be able to write a function as a Taylor polynomial plus a remainder term that is (schematically) of the form $x^k g(x)$ where the function g is as differentiable as the original function. But in general this is not true. So what can be done? The remarkable fact that makes this work is that it is possible to define the topology of uniform convergence of k-jets on compacta on the sheaf of C^k functions, *without* reference to partial derivatives, coordinate systems, or jets. We do this in a sneaky way that is inspired by the celebrated theorem of Jaak Peetre [62] (in the smooth case). The philosophy underlying this theorem is that it is possible to make sense of the notion of "linear differential operator" without referring to any fine structure of the functions. The sheaf of C^k functions then carries the initial topology with respect to all linear differential operators defined on it. To justify this procedure, it is necessary to prove a slightly modified version of Peetre's theorem that holds for C^k functions. I do not claim any great originality in the proof: the main thrust borrows from the proof of the C^∞ version of the theorem that appears in the book by Kollař, Michor, and Slovák [46].

For the remainder of Chapter 2, many of the standard constructions and theorems on differentiable manifolds are reviewed. This includes constructions of the tangent and cotangent bundle, differential forms, the Poincaré lemma, Frobenius' theorem, and Darboux's theorem.

Some rather less well-trodden topics are also treated briefly, chiefly the Frölicher–Nijenhuis bracket, which is needed in the final chapter.

Chapter 3 is devoted to Weyl geometries, which include conformal and projective geometries. In the sense that we use the term, a Weyl geometry is a family of affine connections on a manifold that is indexed by the cotangent bundle. In good cases, we show that a Weyl geometry defines in a natural manner a kind of Cartan connection on the manifold. This is a non-linear connection on the bundle of affine connections that defines the Weyl structure. In the conformal and projective cases, this Cartan connection coincides with the canonical normal Cartan connection for the structure (after quotienting by appropriate parabolic structure group).

This association of a Weyl geometry to a parabolic structure seems to offer some tantalizing possibilities. Chapter 4 concerns parabolic geometries more extensively. A parabolic structure on a vector bundle is nothing more than a suitably generic connection in the bundle, together with a preferred class of filtrations of the bundle, and a distinguished filtration in this class. Starting from these data, we show how to construct a Weyl geometry, and so the associated connection.

1.2 CAUSAL GEOMETRIES

Chapter 5 contains the definition of causal geometries and their basic properties. A causal geometry consists of an association of a null "cone" to each point of a space that is ruled by unparametrized null "geodesics" in such a way that certain natural conditions are satisfied. The most non-obvious condition is a geometrical version of Huygen's principle, that null cones associated to different points of the same null geodesic must make contact to first order. So to specify a causal geometry, it is sufficient to say what the null geodesics are (and to show that these satisfy the defining conditions).

A natural way to obtain a causal geometry is to give a Lagrangian on the tangent bundle (subject to some non-degeneracy conditions). The null geodesics of the Lagrangian are precisely the null stationary points of the associated energy functional. Conversely, we prove that it is possible to associate a Lagrangian (up to an easily definable equivalence) to *any* causal geometry.

1.3 EXAMPLES OF CAUSAL GEOMETRIES

Chapter 6 presents incidence geometries, based on the parabolic structures previously defined in Chapter 4, as examples of causal geometries. The basic idea is this: if X is a three-dimensional manifold on which there is a notion of "geodesic" (such as a Riemannian or projective manifold), then the space of geodesics $\mathbb{N}$ carries a causal structure that is defined as follows. Two geodesics are said to be incident if they intersect at some point. A null geodesic in $\mathbb{N}$ is a family of geodesics in X that intersect a given geodesic at the same point and so that the two geodesics generate the same tangent plane at the point of intersection. (All considerations here are local.)

We then present two examples in detail: $X = S^2 \times \mathbb{R}$ with its associated Riemannian structure, and X is the Heisenberg group with its associated sub-Riemannian structure. Another example is the "paraconformal" (or, more properly, $GL(2, \mathbb{R})$) geometries of Dunajski [20], [19]; this example is not discussed further.

1.3.1 Third-order differential equations

Chapter 7 presents the next example, which comes from the theory of third-order ordinary differential equations (in general non-linear) under contact equivalence [41]. We may write such a third-order differential equation in terms of the vanishing of an ideal of one-forms in four variables: $\{dy - p dx, dp - q dx, dq - F(x, y, p, q) dx\}$ where $p = y'$, $q = y''$ and the differential equation is $y''' = F(x, y, y', y'')$. Here the prime denotes differentiation with respect to the variable x. By setting $p = y'$ and $q = y''$, the solutions of this equation coincide with curves in the second jet space J^2 with coordinates (x, y, p, q) that are everywhere tangent to the contact distribution annihilated by the one-forms

$$\theta_1 = dy - p dx, \quad \theta_2 = dp - q dx$$

and that are also annihilated by the one-form

$$\omega = dq - F(x, y, p, q) dx.$$

A contact transformation $\Phi : J^2 \to J^2$ is a local diffeomorphism that preserves the contact filtration on J^2, meaning that

$$\Phi^*\theta_1 \equiv 0 \pmod{\theta_1}, \quad \Phi^*\theta_2 \equiv 0 \pmod{\theta_1, \theta_2}.$$

Contact transformations act on the set of differential equations by composition. Chapter 7 is concerned with structures that are invariant with respect to transformations of this form.

An alternative characterization of third-order equations and contact transformations relying only on the first jet space J^1 and the contact structure associated to θ_1 will be given in this chapter.

The equivalence problem for third-order ordinary differential equations has a long history. It was first studied under the class of point transformations by Élie Cartan [12]. In 1940, S. S. Chern [15] focused on the equivalence problem under contact transformations. A certain scalar invariant of the structure, called the Wünschmann invariant, divides the family of third-order equations into two classes. Those with vanishing Wünschmann invariant admit a natural conformal Lorentzian structure on the space of solutions. For Chern, this conformal structure presented itself in the form of a normal $SO(3,2)$ Cartan connection on the solution space or, what is the same, a conformal Lorentzian metric on that space. A geometrical description of this conformal structure was presented much later by Fritelli, Kozameh, and Newman [30], who showed that the general third-order equation with vanishing Wünschmann invariant could be obtained by considering one-parameter families of null hypersurfaces in a 3-dimensional space with a conformal Lorentzian metric. More directly, as discussed in Section 7.0.1, this case can be understood also in terms of the null geodesic spray on the bundle of null rays: null hypersurfaces then being related by means of an envelope construction.

Chern also determined all of the contact invariants of the general third-order equation in which the Wünschmann is nonzero. This was presented in the modern language of bundles and connections by Sato and Yoshikawa [73], who in addition clarified the geometrical interpretation of the structure as a normal $SO(3,2)$ Cartan connection on the space J^2. Nurowski and the second author later showed the existence of a conformal $O(3,3)$ structure on a certain fiber bundle over J^2 (see [60]). This $O(3,3)$ structure is of Fefferman [26] type, in the sense of Graham [34], if and only if the Wünschmann invariant vanishes. Godliński [33] then proved that the associated normal $SO(4,4)$ Cartan connection included the Chern–Sato–Yoshikawa connection as its $\mathfrak{o}(3,2)$ part.

The structure of Chern–Sato–Yoshikawa can doubtless be understood directly in terms of the geometry of the space of solutions of the differential equation. However, whereas when the Wünschmann invariant vanishes, there is a standard geometry underlying the presence of certain connections—namely a conformal Lorentzian metric on the space of solutions—when the Wünschmann invariant is nonzero, such an underlying geometry appears to be missing. We supply the missing geometry and examine its precise relationship with these constructions.

Invariantly associated to the structure on J^2 is a degenerate conformal Lorentzian metric (see Nurowski [60]) whose degeneracy is in the direction of the total derivative vector field coming from the differential equation. We prove in Chapter 7 that this degenerate conformal structure arises naturally from elementary geometrical constructions on a certain curve (the *polar curve*) in the projective cotangent bundle of the space of solutions. The Wünschmann invariant itself is precisely the projective curvature of this curve. Finally, the paper introduces a new conformal invariant of fourth order in the metric that gives a complete geometrical characterization of degenerate metrics that arise in this manner from third-order differential equations.

There is a natural *incidence relation* on the space of solutions, described in Section 7.1.1. An infinitesimal or linearized version of this idea is implicit in Wünschmann's [85] investigations into Monge equations of the second degree (see also [52] and [15]). When the Wünschmann invariant vanishes, two solutions are incident if and only if they lie on the same null geodesic. When the Wünschmann is nonzero, the incidence relation still defines a decent structure in a sense that is axiomatized in Section 7.1: roughly, the sheaf of curves defining the incidence relation is envelope-forming. Such a family of envelope-forming curves is dubbed a *causal geometry*, the terminology suggested by an affinity with structures that typically arise in the study of hyperbolic partial differential equations. The space of incidence curves in the solution space corresponds naturally to the points of the 1-jet space J^1. When the incidence curves through a point are linearized at that point, the resulting cone in the tangent space resembles the null cone associated to a Lorentzian structure. The null cone projects to a curve in the projective tangent space, called the *indicatrix* curve, borrowing terminology from optics [5].

The indicatrix gives rise to a Lagrangian in a natural manner that can be written down in terms of the general solution of the differential equation, as described in Section 7.1.2. The Lagrangian is a function on the tangent bundle which is homogeneous of degree two with respect to the scalar homothety of the bundle. It is not fully contact-invariant, but its locus of zeros in the projective tangent bundle is invariant, and coincides with the indicatrix. Null geodesics—extremals of the Lagrangian along which the Lagrangian vanishes identically—are precisely the incidence curves. The resulting structure is a Finsler [28] analog of conformal Lorentzian geometry in dimension 3.

The Lagrangian is in addition regular at every point of the indicatrix, and therefore gives rise to a Hamiltonian on the cotangent bundle, which is described in Section 7.1.3. The zero locus of the Hamiltonian inside the projective cotangent space is the polar curve of the indicatrix. The total space of the indicatrix or its polar curve defines a 4-dimensional bundle over the space of solutions, and the projective Hamiltonian spray defines a projective vector field on this bundle. The 3-dimensional quotient space under the flow of the vector field inherits a natural contact form from the cotangent bundle. The resulting space is contactomorphic to J^1, the space of 1-jets in the plane, and on it the polar curves descend to a path geometry that defines a third-order differential equation.

The entire procedure sketched here is summarized in the theorem, proven in §7.2:

Theorem 1. *There is a natural local isomorphism between the set of third-order equations under contact equivalence and the set of isomorphism classes of causal geometries.*

It is also of interest to determine when a (degenerate) rank three conformal Lorentzian metric g on a four-dimensional space N. The fundamental invariant associated to this structure is

$$\Gamma = g \wedge \mathscr{L}_V g \wedge \mathscr{L}_V^2 g \wedge \mathscr{L}_V^3 g \wedge \mathscr{L}_V^4 g \in \wedge^5 S^2 \ker(V \lrcorner) \cong S^2(TN/V)$$

where V is the degenerate direction. Here $\ker(V \lrcorner)$ is the space of one-forms annihilated by V, and each of the Lie derivatives $\mathscr{L}_V^k g$ lies in the symmetric square $S^2 \ker(V \lrcorner)$. The space $S^2 \ker(V \lrcorner)$ is six-dimensional, and its fifth exterior power is naturally isomorphic to the symmetric square $S^2(TN/V)$ of the quotient of the tangent bundle of N by the vertical direction V.

The following theorem is the end result of this analysis, which occupies §7.3

Theorem 2. *The degenerate Lorentzian metric g on the four-manifold N arises (locally) from a third-order differential equation if and only if either*

1. *Γ is nonzero and the classical adjoint of Γ vanishes identically (equivalently, Γ has rank* 1*). In this case, there exists a natural conformal isometry of N with J^2 equipped with its invariant degenerate metric coming from a third-order differential equation with non-zero Wünschmann invariant.*
2. *$\mathscr{L}_V g$ is proportional to g. In this case, the Wünschmann invariant vanishes and there exists a conformal isometry of N with J^2 equipped with its invariant degenerate metric coming from a third-order differential equation that is natural up to a gauge transformation of J^2.*

We may pass to the three-dimensional space of solutions of the differential equation, $\mathcal{S}$. Two points of $\mathcal{S}$ are defined to be incident if the corresponding solutions, regarded as curves in the (x, y, p)-space, meet and are mutually tangent. This incidence condition defines the null cones of an ordinary conformal structure on the space $\mathcal{S}$, provided that a certain contact invariant of the differential equation, the Wünschmann [85] invariant, $\mathcal{W}$, vanishes identically. The simplest example with $\mathcal{W} = 0$ is the trivial equation $y''' = 0$, with general solution $y = sx^2 + 2tx + u$, where the null cone is that of a flat Minkowski space with conformal structure given by $dsdu - dt^2$.

Élie Cartan [12] and later Shiing-Shen Chern [15] studied the space, $\mathcal{T}$, with co-ordinates (x, y, p, q). $\mathcal{T}$ carries a canonical direction field, $V = \partial_x + p\partial_y + q\partial_p + F(x, y, p, q)\partial_q$, such that the quotient of $\mathcal{T}$ by V is the space $\mathcal{S}$. Chern showed that $\mathcal{T}$ carries a once-degenerate conformal metric, which is killed by V, such that it is also invariant under V (up to scale), so passes down to $\mathcal{S}$, if and only if $\mathcal{W} = 0$. In the case that $\mathcal{W} \neq 0$, the present author and Sparling showed [41] that one could use the Chern metric to construct on the space $\mathcal{S}$ a null cone structure by the method of envelopes and which reduces to the standard null cone structure in the case $\mathcal{W} = 0$. This null cone structure is exactly that given by the incidence condition. Thus the causal geometry is natural for this case and one wants to develop an analogue of the usual connection theory which applies in this case. Our theory does this, although, ironically, the Weyl curvature vanishes identically, as it does in the standard case of $\mathcal{W} = 0$, for dimensional reasons.

The simplest example with $\mathcal{W} \neq 0$ is the differential equation $y''' = y''$. Its solutions are $y = se^x + tx + u$, where the parameters (s, t, u) are global co-ordinates for $\mathcal{S}$. The incidence conditions are $0 = dy - pdx = e^x ds + xdt + du$ and $0 = dp - qdx = e^x ds + dt$. Eliminating x between these equations gives the causal null cone in the form $e^{1-\frac{du}{dt}} + \frac{ds}{dt} = 0$. This is well defined and has non-singular hessian with respect to the variables (ds, dt, du), provided only that $dt \neq 0$. It is dramatically more complicated than the case of $\mathcal{W} = 0$.

A remaining issue is to relate this work to that of George Sparling and Pawel Nurowski, who built a canonical conformal structure in six dimensions that encodes the geometry of the third-order equation and which, when $\mathcal{W} = 0$ reduces to a conformal structure of the type first given by Charles Feffermann [26]. This structure is described in Nurowski [60] and Godlińksi [33]. When $\mathcal{W} \neq 0$, one needs a generalized Fefferman structure, applied to general parabolic geometries, as shown by Hammerl and Sagerschnig [35].

2.0 DIFFERENTIABLE MANIFOLDS

In this chapter, we present the preliminaries on differentiable manifolds needed for the subsequent investigations. Much of the basic material on sheaf theory can be found in MacLane and Moerdijk [53] and Hartshorne [36]. The material on the Frölicher–Nijenhuis bracket can be found in Kollař, Michor, and Slovák [46].

2.1 SHEAVES

Let X be a topological space. Let $\tau(X)$ be the category whose objects are the open subsets of X and whose arrows are inclusions $U \subset V$ of open sets. Let $\mathcal{C}$ be a complete category. A *presheaf* on X with values in $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \tau(X) \to \mathcal{C}$. If $U, V \in \mathrm{Ob}(\tau(X))$ are two open sets with $U \subset V$, let $\rho_U = \mathcal{F}(U \subset V)$ be the morphism in $\mathcal{C}$ associated to the inclusion of U in V (dependence on V is suppressed). This is called a *restriction map* associated to the presheaf. Functoriality of $\mathcal{F}$ ensures compatibility of the restriction maps: if $U \subset V \subset W$, then $\rho_U = \rho_U \circ \rho_V$.

Let $U \in \mathrm{Ob}(\tau(X))$ be an open subset of X. If $\{U_i\} \subset \mathrm{Ob}(\tau(X))$ is an open cover of U, then the restriction morphisms are

$$\rho_{U_i} = \mathcal{F}(U_i \subset U) : \mathcal{F}(U) \to \mathcal{F}(U_i).$$

Taking a product gives a morphism

$$\rho : \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i).$$

There are a pair of morphisms $p, q : \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$ defined so that the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{F}(U_i) \arrow[r, "\mathcal{F}(U_i\cap U_j\subset U_i)"] & \mathcal{F}(U_i\cap U_j) \\
\mathcal{F}(U) \arrow[ur] \arrow[dr] \arrow[r, "\rho"] & \prod_i \mathcal{F}(U_i) \arrow[u] \arrow[d] \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{F}(U_i\cap U_j) \arrow[u] \arrow[d] \\
 & \mathcal{F}(U_j) \arrow[r, "\mathcal{F}(U_i\cap U_j\subset U_j)"'] & \mathcal{F}(U_i\cap U_j)
\end{tikzcd}$$

A presheaf is called a *sheaf* if, for every $U \in \mathrm{Ob}(\tau)$ and every open cover $\{U_i\}$ of U, the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho"] & \prod_i \mathcal{F}(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{F}(U_i\cap U_j)
\end{tikzcd}$$

is an equalizer. The means that if $f_i \in \mathcal{F}(U_i)$ are such that $\rho_{U_i\cap U_j}(f_i) = \rho_{U_i\cap U_j}(f_j)$ for all pairs of indices i, j, then there is a unique $f \in \mathcal{F}(U)$ such that $f_i = \rho_{U_i}(f)$ for all indices i.

A morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ over the same topological space is a natural transformation of functors. Presheaves over X form a category. A subpresheaf of a presheaf $\mathcal{F}$ on X is a presheaf $\mathcal{F}'$ on X such that $\mathcal{F}'(U)$ is a subobject of $\mathcal{F}(U)$ for all $U \in \mathrm{Ob}(\tau(X))$.

Sheaves over X form a subcategory of the category of presheaves. The left adjoint to the inclusion functor of sheaves in presheaves is called the *sheafification* functor, and is denoted by Γ. An explicit construction is given later.

2.1.1 Examples

- If $\mathcal{C}$ is the category of abelian groups, then a sheaf with values in $\mathcal{C}$ is called a sheaf of abelian groups.
- If $\mathcal{C}$ is the category of rings, then a sheaf with values in $\mathcal{C}$ is called a sheaf of rings.
- Let $\mathcal{C}$ be the category of all real unital commutative algebras and let X be a topological space. Let $\mathscr{C}_X$ be the presheaf that associates to each open subset U of X the algebra of continuous real-valued functions on U, with pointwise addition and multiplication. This is a sheaf, for if $U = \cup_i U_i$ is an open covering of U, and $f_i \in \mathscr{C}_X(U_i)$ are continuous functions such that $f_i|U_j = f_j|U_i$, then there exists a unique function $f : U \to \mathbb{R}$ such that $f|U_i = f_i$ for all i. This function is continuous, since for any open set $O \subset \mathbb{R}$, $f^{-1}O = \cup_i f_i^{-1}O$ is a union of open sets (since the f_i are all continuous), and is therefore open.
- Let $\mathscr{C}_n^k$ be the presheaf that associates to each open subset U of $\mathbb{R}^n$ the algebra of k-times continuously differentiable functions $f : U \to \mathbb{R}$, under pointwise addition and multiplication. This is a sheaf for the same reason as the previous example; it is a

subsheaf of $\mathscr{C}_{\mathbb{R}^n}$. Moreover, $\mathscr{C}^k_n(U)$ can be equipped with the structure of a Fréchet space using the seminorms $\|f\|_{j,K} = \sup_{x\in K}\|D^j f(x)\|$ for $j = 0,\dots,k$, and K ranging over all compact subsets of U. This equips $\mathscr{C}^k_n$ with the structure of a sheaf of topological rings.

- For $U \subset \mathbb{R}^n$ an open set, let $\mathscr{C}^k_{n,c}(U)$ be the algebra of compactly-supported k-times continuously differentiable functions $f : U \to \mathbb{R}$. This is *not* a presheaf in the category $\mathcal{C}$ of algebras since it is not closed under restriction. On the contrary, it is a presheaf in the category $\mathcal{C}^{op}$, the opposite category, because to each inclusion of open sets $U \subset V$ in $\tau(\mathbb{R}^n)$, there is an *extension* operator $\rho_{UV} : \mathscr{C}^k_{n,c}(U) \to \mathscr{C}^k_{n,c}(V)$. The condition for this presheaf to be a sheaf is that any element of $\mathscr{C}^k_{n,c}(U)$ can be expressed as a sum of extensions of elements of $\mathscr{C}^k_{n,c}(U_i)$ for any open cover $U = \cup_i U_i$. This follows by the existence of compactly supported partitions of unity subordinate to any open covering. Spaces of distributions, which are the (topological) duals of the $\mathscr{C}^k_{n,c}$, form sheaves in the more usual sense.

2.1.2 Stalks and germs

Let $\mathcal{F}$ be a presheaf on X and $p \in X$ a point. Define the *stalk* of $\mathcal{F}$ at p to be

$$\mathcal{F}_p = \lim_{\leftarrow} \mathcal{F}(U)$$

where the direct limit is taken over the directed set of open subsets U of X containing p.

To obtain an explicit description, suppose that $\mathcal{C}$ is a complete and cocomplete subcategory (that is, having all limits and colimits) of the category of sets. Then $\mathcal{F}_p$ is the set of equivalence classes of elements of $\coprod_{U\ni p}\mathcal{F}(U)$ such that elements $x \in \mathcal{F}(U)$ and $y \in \mathcal{F}(V)$ are equivalent if there is an open set W with $p \in W \subset U \cap V$ such that $\rho_W(x) = \rho_W(y)$. If $f \in \mathcal{F}(U)$, then the equivalence class in $\mathcal{F}_p$ corresponding to f is denoted by f_p. It is called the *germ* of f at p. If $\mathcal{F}$ is a presheaf on X, the sheafification of $\mathcal{F}$ has the following explicit construction:

- $\Gamma(U,\mathcal{F})$ is the set of all functions $s : U \to \coprod \mathcal{F}_p$ such that for all $p \in U$, $s(p) \in \mathcal{F}_p$ and there exists an open neighborhood V of p contained in U and a $t \in \Gamma(V,\mathscr{F})$ such that $t_q = s(q)$ for all $q \in V$.

Define the *étale space* of a presheaf $\mathcal{F}$ on X to be the set

$$E(\mathcal{F}) = \coprod_{p\in X}\mathcal{F}_p.$$

Associated to any open set $U \subset X$, there is a map $E : U \times \mathcal{F}(U) \to E(\mathcal{F})$ given by $E(p,s) = s_p$. Equip $E(\mathcal{F})$ with the final topology for this family of maps. Sections of $\Gamma(\mathcal{F})$

define in a natural way continuous sections of the bundle $E(\mathcal{F}) \to X$. Moreover, *every* continuous section of this bundle arises in this manner.

2.1.3 Operations with sheaves

Let X, Y be topological spaces and $f : X \to Y$ a continuous function. If $\mathcal{F}$ is a sheaf on X, then the *direct image* sheaf, denoted $f_*\mathcal{F}$, is the sheaf on Y given by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U).$$

If $\mathcal{G}$ is a sheaf on Y, then the *inverse image* sheaf $f^{-1}\mathcal{G}$ is the sheafification of the presheaf

$$U \mapsto \lim_{\substack{V \supset f(U) \\ V \in \tau(X)}} \mathcal{G}(V).$$

The stalk of $f^{-1}\mathcal{G}$ at a point $p \in X$ is just $\mathcal{G}_{f(p)}$. In particular, if $X \subset Y$ is a subspace, and $i : X \to Y$ is the inclusion map, then the sheaf $i^{-1}\mathcal{G}$ is called the *restriction* of $\mathcal{G}$ to X, and is denoted $\mathcal{G}|Z$.

2.1.4 Ringed spaces

A *ringed space* is pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ on X. A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ such that $\mathscr{O}_{X,p}$ is a local ring for all $p \in X$. Denote the (unique) maximal ideal in $\mathscr{O}_{X,p}$ by $\mathfrak{m}_{X,p}$.

In a locally ringed space, the sheaf $\mathscr{O}_X$ is naturally equipped with an *evaluation map*. Let $f \in \mathscr{O}_X(U)$ be a local section and let $p \in U$. Define $\mathrm{ev}_p(f) = f_p \mod \mathfrak{m}_p.$. This coset is an element of the field $\mathscr{O}_{X,p}/\mathfrak{m}_p$. In particular, $\mathfrak{m}_p = \ker \mathrm{ev}_p$. When there is no risk of confusion, denote by $f(p) = \mathrm{ev}_p(f)$.

A morphism from a ringed space $(X, \mathscr{O}_X)$ to a ringed space $(Y, \mathscr{O}_Y)$ is a pair $(f, f^\sharp)$ consisting of a continuous function $f : X \to Y$ and a morphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ of sheaves of rings on Y. A morphism $(f, f^\sharp)$ is a morphism of locally ringed spaces if the induced maps of the stalks $f^\sharp_p : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$ is a local homomorphism, meaning that $f^\sharp_p(\mathfrak{m}_{Y,f(p)}) = \mathfrak{m}_{X,p}$, for all $p \in X$.

Let U be an open subset of X and $f \in \mathscr{O}_X(U)$ a section of the sheaf of rings $\mathscr{O}_X$. The *support* of f is the complement in U of the union of all open subsets $V \subset U$ such that $\rho_V(f) = 0$ in $\mathscr{O}_X(V)$:

$$\operatorname{supp} f = U \setminus \left[\bigcup \{ V \in \tau(U) \mid \rho_V(f) = 0 \} \right].$$

Let $(X, \mathscr{O}_X)$ be a ringed space. A sheaf of $\mathscr{O}_X$-modules on X is a sheaf $\mathcal{F}$ on X such that, for each U, $\mathcal{F}(U)$ has the structure of an $\mathscr{O}_X(U)$-module and such that the restriction maps of the sheaf $\mathcal{F}$ are compatible with the restriction maps of $\mathscr{O}_X$. A morphism of sheaves of $\mathscr{O}_X$-modules is a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ such that for all open sets $U \subset X$, the maps $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathscr{O}_X(U)$-modules. The category of all $\mathscr{O}_X$-modules is denoted by $\mathscr{O}_{\mathrm{X}}$-**Mod**.

If $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces and $\mathcal{F}$ is a sheaf of rings on Y, then the pullback of $\mathcal{F}$ is defined by $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$. The functor $f^* : \mathscr{O}_{\mathrm{Y}}\text{-}\mathbf{Mod} \to \mathscr{O}_{\mathrm{X}}\text{-}\mathbf{Mod}$ is the left-adjoint of the functor $f_* : \mathscr{O}_{\mathrm{X}}\text{-}\mathbf{Mod} \to \mathscr{O}_{\mathrm{Y}}\text{-}\mathbf{Mod}$. In particular, there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

for any $\mathscr{O}_X$-module G and $\mathscr{O}_Y$-module F.

2.2 REALCOMPACTNESS

Henceforth, all algebras have identity and all homomorphisms are identity-preserving. Let A be an algebra over the reals. A *character* of A is an algebra homomorphism $\phi : A \to \mathbb{R}$.

Proposition 1. *Every character of the algebra $C^k(\mathbb{R}^n)$ has the form ev_p for some $p \in \mathbb{R}^n$.*

Proof. Let $\phi : C^k(\mathbb{R}^n) \to \mathbb{R}$ be an algebra homomorphism. First note that there is $f \in \ker \phi$ such that $f^{-1}\{0\}$ is compact. Indeed, let $g \in C^k(\mathbb{R}^n)$ be the function $g(x) = |x|^2$. Then $f = g - \phi(g)1 \in C^k(U)$ has the required property.

We will now show that $Z = \bigcap_{f \in \ker \phi} f^{-1}\{0\}$ is non-empty. Since this intersection is compact, it is sufficient to show that it satisfies the finite intersection property. If $f, g \in \ker \phi$, then $f^{-1}\{0\} \cap g^{-1}\{0\} = h^{-1}\{0\}$ where $h = f^2 + g^2 \in \ker \phi$. So Z is non-empty.

Since Z is non-empty, it contains a point p. If $f \in C^k(\mathbb{R}^n)$ is arbitrary, then $f - \phi(f)1 \in \ker \phi$ and so vanishes at p. That is $\phi(f) = f(p)$ as required. $\square$

This is also true for the algebra of C^k functions on separable Banach spaces (see [4]) and more generally for the algebra of C^k functions on a manifold modeled on a complete separable Mackey space [47], a category which includes all separable Fréchet spaces. If every real-valued homomorphism of an algebra of functions on a space is given by an evaluation map, the space and algebra are called *realcompact*, a notion introduced by Hewitt [38]. Smooth realcompactness was studied (in infinite dimensions) by Kriegl, Michor, and Schachermayer [49], who cite Milnor and Stasheff [55], and later by Kriegl and Michor [47], [48].

2.3 DIFFERENTIABLE MANIFOLDS

Let $\mathscr{C}_n^k$ be the sheaf of k-times continuously differentiable real-valued functions on $\mathbb{R}^n$, $k = 1, 2, \ldots, \infty$. We claim that $(\mathbb{R}^n, \mathscr{C}_n^k)$ is a locally ringed space and $\mathfrak{m}_p$ is the set of germs at p of functions that vanish at p. This is an immediate consequence of Proposition 1.

Definition 1. *A locally ringed space $(X, \mathscr{O}_X)$ is called a C^k-*manifold *if X is a paracompact hausdorff space and there is a covering of X by open sets such that for each open set U of the cover $(U, \mathscr{O}_X|U)$ is isomorphic to the locally ringed space $(\mathbb{R}^n, \mathscr{C}_n^k)$. In that case, n is the* dimension *of the manifold.*

If $k = \infty$, then $(X, \mathscr{O}_X)$ is called a *smooth manifold*. The dependence of the definition on the sheaf $\mathscr{O}_X$ will be suppressed in later chapters. This is equivalent to the usual notion of a C^k-manifold given in terms of charts and transition functions. We prove this after introducing some notation.

A morphism from a C^k manifold $(X, \mathscr{O}_X)$ to a C^ℓ manifold $(Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces, i.e., a pair $(f, f^\sharp)$ consisting of a continuous function $f : X \to Y$ and a morphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ of sheaves of rings whose localization at each point of Y is a morphism of local rings.

If $(X, \mathscr{O}_X)$ is a C^k manifold and $\ell \leq k$, then define a C^ℓ manifold $(X_\ell, \mathscr{O}_{X_\ell})$ by setting $X_\ell = X$ (as topological spaces) and, for each open $U \subset X$,

$$\mathscr{O}_{X_\ell}(U) = \{h(f_1, \ldots, f_N) \mid N \in \mathbb{N}, f_1, \ldots, f_N \in \mathscr{O}_X(U), h \in C^\ell(\mathbb{R}^N)\},$$

equipped with the pointwise operations of addition and multiplication. A more satisfactory definition appears later. This defines a functor from the category of C^k manifolds to the category of C^ℓ manifolds. If $\ell > 0$, this is an equivalence of categories, by a theorem of Hassler Whitney.

2.3.1 Local coordinates

Lemma 1. *Let $(X, \mathscr{O}_X)$ be an n-dimensional C^k manifold. There is a covering of X by open sets U_α such that $\mathscr{O}_X(U_\alpha)$ admits n sections $x_\alpha^1, \ldots, x_\alpha^n$ satisfying the following*

(a) The function $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ defined by $\psi_\alpha(u) = (x_\alpha^1(u), \ldots, x_\alpha^n(u))$ is a homeomorphism of U_α onto an open subset V_α of $\mathbb{R}^n$.

(b) The composition $\psi_\beta \circ \psi_\alpha^{-1} : V_\alpha \cap V_\beta \to V_\alpha \cap V_\beta$ is a C^k diffeomorphism.

Proof. Let $p \in X$ be given. There is an open neighborhood U of p, an open subset V of $\mathbb{R}^n$ and an isomorphism $(\eta, \eta^\sharp) : (U, \mathscr{O}_X|U) \to (V, \mathscr{C}_n^k|V)$. Let e^i be the standard linear coordinates on $\mathbb{R}^n$ and set $x^i = \eta^\sharp(e^i|V)$. Then, in particular, $\eta = (x^1, \ldots, x^n)$. $\square$

Lemma 2 (Hadamard's lemma)**.** *Let $U \subset \mathbb{R}^n$ be an open neighborhood of a point p and let $f \in \mathscr{C}_n^k(U)$. If $f(p) = 0$, then there are functions $g_i \in \mathscr{C}_n^{k-1}(U)$ such that $f(x) = \sum_{i=1}^n (x^i - p^i) g_i(x)$ for all $x \in U$.*

In a star-shaped neighborhood, $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x-p))\, dt$ works. It is also clear how to do this in an open set that does not contain p in its closure. Any neighborhood of p can be expressed as a union of a star-shaped neighborhood and an open set with p in the exterior. So it is enough to take a solution of the problem in these two sets and glue by a partition of unity, a technique that we introduce presently.

2.3.2 Partitions of unity

Let $(X, \mathscr{O}_X)$ be a locally ringed space. A *partition of unity* of the sheaf $\mathscr{O}_X$ is a collection $\{\rho_i \in \mathscr{O}_X(X) | i \in I\}$ of global sections of $\mathscr{O}_X$ indexed by some set I such that:

(a) The supports of the ρ_i are a locally finite collection of sets.

(b) $\displaystyle\sum_i \rho_i(x) = 1$ for all $x \in X$, and $\rho_i(x) \geq 0$ for all $x \in X$ and $i \in I$.

Local finiteness here means that every point $p \in X$ is contained in a neighborhood U_p that intersects only finitely many of the sets $\{\operatorname{supp} \phi_i \mid i \in I\}$. A partition of unity is *subordinate* to an open covering $\{U_\alpha\}$ of X if, for each $i \in I$, there is an α such that $\operatorname{supp} \phi_i \subset U_\alpha$.

Lemma 3. *Let $(X, \mathscr{O}_X)$ be a C^k manifold and $\{U_\alpha\}$ an open cover of X. Then there exists a partition of unity $\{\rho_i\}$ such that the following conditions hold:*

(a) $\rho_i \in \mathscr{O}_X(X)$ for all $i \in I$

(b) $\operatorname{supp} \rho_i$ is compact for all i

(c) $\{\rho_i\}$ is subordinate to the cover $\{U_\alpha\}$

Proof. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is equal to unity on the closed unit ball $\overline{B_0(1)}$, is zero on the complement of the open ball of radius two $B_0(2)^c$, and has support $\overline{B_0(2)}$.[1]

By paracompactness, there is a locally finite refinement $\{V_i\}$ of the original cover $\{U_\alpha\}$. By further refining the cover, we assume without loss of generality that each V_i has compact

[1] One such function is $\rho(x) = \frac{\phi(2-|x|)}{\phi(|x|-1)+\phi(2-|x|)}$ where $\phi(t) = e^{-1/t}$ for $t > 0$ and $\phi(t) = 0$ for $t \leq 0$.

closure contained in some U_α and there is a diffeomorphism $\phi_i : V_i \to B_0(2)$. Let

$$\varrho_i(x) = \begin{cases} \rho \circ \phi_i(x) & x \in V_i \\ 0 & x \notin V_i. \end{cases}$$

Set $\varrho = \sum_i \varrho_i$. This is strictly positive everywhere and so

$$\rho_i = \frac{\varrho_i}{\varrho}$$

is a partition of unity. Since $\operatorname{supp} \rho_i \subset \overline{V_i} \subset$ some V_α this is a partition of unity subordinate to the cover U_α. Since $\overline{V_i}$ is compact, the supports of the ρ_i are compact. □

Because of the existence of partitions of unity, any sheaf of $\mathscr{O}_X$ modules is generated by its *global* sections.

2.3.3 Affine schemes in the smooth category

In algebraic geometry, the *spectrum* of a commutative ring is the set of prime ideals in the ring, equipped with the Zariski topology, and a natural sheaf constructed using localizations of the ring. This definition is not well-suited to smooth manifolds, however. In the smooth category, there are several suitable notions of a spectrum. Moerdijk, Quê, Reyes [56] proceed by defining the spectrum as ∞-radical prime ideals in the ring. This notion is well-behaved from the categorical point of view, but the structure of these prime ideals is very complicated. In fact, even maximal ideals in C^∞ rings can have non-Archimedean quotients. We therefore adopt the definition of Dubuc [18] (see also Joyce [44]): the spectrum of a C^k ring $\mathfrak{C}$ is the set of all real characters of $\mathfrak{C}$ (algebra homomorphisms $\mathfrak{C} \to \mathbb{R}$). The maximal ideal associated to any such homomorphism is called a *real point* of the ring $\mathfrak{C}$.

If $(X, \mathscr{O}_X)$ is a C^k manifold, then evaluation at any point p of X defines a real point of the ring $\mathfrak{C} = \mathscr{O}_X(X)$ of global C^k functions on X. The evaluation map is thus a one-to-one map $X \to \operatorname{spec} \mathscr{C}$. It follows from Proposition 1 and Lemma 3 that this is a surjection as well. This bijection is continuous, since any closed subset of $\operatorname{spec} \mathfrak{C}$ has the form $V(I)$ for some ideal I, and the preimage of this closed set is $\cap_{f\in I} f^{-1}\{0\}$. This bijection is a homeomorphism:

Lemma 4. *If $K \subset X$ is a closed set. Then there exists $f \in \mathscr{O}_X(X)$ such that $K = f^{-1}\{0\}$.*

Proof. Using a partition of unity subordinate to a coordinate atlas, it is sufficient to prove the statement for closed subsets $K \subset \mathbb{R}^n$. Cover $\mathbb{R}^n \setminus K$ with a countable family of balls $B_{r_k}(x_k)$ such that $r_k \to 0$ as $k \to \infty$. Let

$$\phi_k(x) = \frac{r_k^k}{2^k}\,\rho\left(\frac{x - x_k}{2r_k}\right)$$

where ρ is the bump function from Lemma 3. Then $f = \sum_k \phi_k$ converges to a smooth function that is zero precisely on K. □

Define a sheaf $\mathscr{O}_{\mathfrak{C}}$ on $\operatorname{spec}(\mathfrak{C})$ as follows. For each $\mathfrak{m} \in \operatorname{spec}(\mathfrak{C})$, let $\mathfrak{C}_{\mathfrak{m}}$ be the localization of $\mathfrak{C}$ at $\mathfrak{m}$. (That is, the localization of A with respect to the multiplicative subset $\mathfrak{C} \setminus \mathfrak{m}$. This is a local ring.) For U any open subset of $\operatorname{spec}(\mathfrak{C})$, let $\mathscr{O}_A(U)$ be the set of all functions $s : U \to \coprod_{\mathfrak{m}\in U} \mathfrak{C}_{\mathfrak{m}}$ such that the following conditions hold:

- $s(\mathfrak{m}) \in \mathfrak{C}_{\mathfrak{m}}$ for all $\mathfrak{m} \in U$
- For every $\mathfrak{m} \in U$ there is a neighborhood $V \subset U$ of $\mathfrak{m}$ and elements $a, b \in A$ with $b \notin I(V)$ such that, for all $\mathfrak{q} \in V$, $s(\mathfrak{q}) = a/b$ in $\mathfrak{C}_{\mathfrak{m}}$.

Then $\mathscr{O}_{\mathfrak{C}}$ is a sheaf of rings whose stalk at each $\mathfrak{m} \in \operatorname{spec}(\mathfrak{C})$ is the local ring $\mathfrak{C}_{\mathfrak{m}}$. Thus the pair $(\operatorname{spec}\mathfrak{C}, \mathscr{O}_{\mathfrak{C}})$ is a locally ringed space. A locally ringed space that is isomorphic to $(\operatorname{spec}\mathfrak{C}, \mathscr{O}_{\mathfrak{C}})$ for some C^k ring $\mathfrak{C}$ is called a C^k *affine scheme*. In particular, any C^k manifold $(X, \mathscr{O}_X)$ is an affine scheme.

Lemma 5. *Let $U \subset X$ be an open subset of a C^k manifold $(X, \mathscr{O}_X)$ and $f \in \mathscr{O}_X(U)$. Then $p \mapsto \operatorname{ev}_p f$ is a continuous function $U \to \mathbb{R}$.*

2.3.4 Topology on $\mathscr{O}_X$

Let $(X, \mathscr{O}_X)$ be a C^k manifold. Let $\mathscr{C}_X$ denote the sheaf of continuous real-valued functions on X.

Definition 2. *Let $U \subset X$ be open. An operator $P : \mathscr{O}_X(U) \to \mathscr{C}_X(U)$ is called* support non-increasing *if $\operatorname{supp} Pf \subset \operatorname{supp} f$ for all $f \in \mathscr{O}_X(U)$.*

For each open $U \subset X$, let $\mathscr{C}_X(U)$ have the compact-open topology. Equip $\mathscr{O}_X(U)$ with the weakest topology such that every linear support non-increasing operator $P : \mathscr{O}_X(U) \to \mathscr{C}_X(U)$ is continuous. This topology, called here the *natural topology*, is complete and metrizable (Corollary 2 below). The sheaf $\mathscr{O}_X$ is a sheaf of local topological rings equipped with this topology.

A sequence $f_n \in \mathscr{O}_X(U)$ converges to zero if and only if $P(f_n)$ tends uniformly to zero on compact subsets of U for all linear support non-increasing operators $P : \mathscr{O}_X(U) \to \mathscr{C}_X(U)$.

Since this topology is metrizable, sequential convergence completely determines the topology. If $S \subset \mathscr{O}_X(U)$, let $\operatorname{cl} S$ be the closure of S with respect to this topology.

Since $\mathscr{O}_X$ carries the structure of a topological sheaf, each of its stalks $\mathscr{O}_{X,p}$ carries the structure of a topological algebra which is the final topology associated to the restriction map. A function f from $\mathscr{O}_{X,p}$ to a topological space is continuous if and only if it has a continuous extension to a mapping from $\mathscr{O}_X(U)$ to the topological space, for some open neighborhood U of p.

2.3.5 Differentials and tangent vectors

Let A be a commutative topological $\mathbb{R}$-algebra and M a topological A-module. A derivation from A to M is a linear mapping of $\mathbb{R}$-modules $D : A \to M$ such that $D(ab) = aD(b)+bD(a)$.

Let $(X, \mathscr{O}_x)$ be a C^k-manifold, $p \in X$. The space TX_p of all continuous derivations $D : \mathscr{O}_{X,p} \to \mathbb{R}$ is called the *tangent space* of X at p. The vector space $\mathfrak{m}_{X,p}/\operatorname{cl}\mathfrak{m}^2_{X,p}$ is called the space of differentials at p and is denoted by T^*X_p. For any $f_p \in \mathscr{O}_{X,p}$, define the *differential of f* at p by

$$df_p = f_p - f_p(p) \pmod{\operatorname{cl}\mathfrak{m}^2_{X,p}} \in T^*X_p.$$

Then T^*X_p and TX_p are dual to one another. More precisely:

Lemma 6. *For $(\alpha, v) \in T^*X_p \times TX_p$, the bilinear form*

$$\langle \alpha, v\rangle = v(\alpha)$$

is well-defined, and is a non-degenerate pairing of T^*X_p with TX_p.

Proof. That $v(\operatorname{cl}\mathfrak{m}^2_p) = 0$ follows from the continuity of v, so the bilinear form does not depend on the choice of α modulo $\operatorname{cl}\mathfrak{m}^2_p$. For non-degeneracy, in local coordinates v has the form $v^i\partial/\partial x^i$ and $\alpha = \alpha_i dx^i$. The pairing of these is just $v^i\alpha_i$ which is evidently non-degenerate. $\square$

We associate to $v \in TX_p$ a derivation in the algebra $\wedge^\bullet T^*X_p$ called the *interior product* with v, denoted by $v\lrcorner : \wedge^\bullet T^*X_p \to \wedge^{\bullet-1}T^*X_p$, given on simple elements of the algebra by

$$v\lrcorner(\alpha_1 \wedge \cdots \wedge \alpha_r) = \sum_{i=1}^{r} (-1)^{i+1}\langle v, \alpha_i\rangle \alpha_1 \wedge \cdots \widehat{\alpha_i} \wedge \cdots \wedge \alpha_r$$

where the hat denotes omission from the product.

2.3.6 Pullbacks and differentials

Let X and Y be C^k manifolds, and $f : X \to Y$. Let $p \in X$ and $\alpha \in T^*Y_{f(p)}$. Define the pullback of α by f at p, denoted $f^*\alpha_p$, as follows. Let $\tilde{\alpha} \in \mathscr{O}_{Y,f(p)}$ be a representative of α modulo $\mathfrak{m}_{Y,f(p)}$. Then set

$$f^*\alpha = \operatorname{ev}_p f^\sharp \tilde{\alpha} \pmod{\operatorname{cl} \mathfrak{m}_{X,p}^2}.$$

Since $f^\sharp$ is a morphism of local rings, this is independent of the choice of representative $\tilde{\alpha}$ and so is a well-defined linear map $f^* : T^*Y_{f(p)} \to T^*X_p$. Let $df_p : TX_p \to TY_{f(p)}$ be the adjoint of f^* relative to the duality pairing from Lemma 6.

Definition 3. *(a) A morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a* submersion *at p if df_p is a surjection. A morphism is a submersion if it is a submersion at every point $p \in X$.*

(b) A morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is an immersion *at p if df_p is injective. A morphism is an immersion if it is an immersion at every point.*

(c) A morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a local diffeomorphism *if it is both a submersion and immersion. Such a morphism is a* diffeomorphism *if, in addition, f is a homeomorphism of topological spaces.*

(d) A morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is an embedding *if it is an immersion and f is a homeomorphism onto its image.*

Let $\mathbf{Man}_n^k$ be the category whose objects are n-dimensional C^k manifolds and whose morphisms are the C^k local diffeomorphisms. Lemma 1 implies that any manifold $M \in \operatorname{Ob}(\mathbf{Man}_n^k)$ is a coequalizer in $\mathbf{Man}_n^k$ of open subsets of $\mathbb{R}^n$:

$$\coprod U_i \cap U_j \rightrightarrows \coprod U_i \dashrightarrow M$$

2.3.7 Jets

Let $(X, \mathscr{O}_X)$ be a C^k manifold, $p \in X$, and $\ell \leq k$. The ℓ-jet space of X at p is the quotient of $\mathscr{O}_{X,p}$ by the $(\ell+1)$th power of the maximal ideal:

$$J^\ell \mathscr{O}_{X,p} = \mathscr{O}_{X,p} / \operatorname{cl} \mathfrak{m}_{X,p}^{\ell+1}.$$

(If $\ell = \infty$, we set $\mathfrak{m}_{X,p}^\infty = \cap_{i=1}^\infty \mathfrak{m}_{X,p}^i$.) The equivalence class of any germ $f \in \mathscr{O}_{X,p}$ in $J^\ell \mathscr{O}_{X,p}$ is denoted by $j^\ell f_p$, and is called the ℓ-jet of f at p. There is a canonical splitting of the ℓ-th jet space at p:

$$J^\ell \mathscr{O}_{X,p} = \mathbb{R} \oplus \frac{\mathfrak{m}_{X,p}}{\operatorname{cl} \mathfrak{m}_{X,p}^{\ell+1}}$$

into the constants plus the functions that vanish at p. Let $K_{X,p}^\ell = \mathfrak{m}_{X,p} / \operatorname{cl} \mathfrak{m}_{X,p}^{\ell+1}$.

The powers of the maximal ideal $\mathfrak{m}_{X,p}$ in $\mathscr{O}_{X,p}$ define a filtration on $\mathscr{O}_{X,p}$:

$$\operatorname{cl}\mathfrak{m}_p^{k+1} \subset \operatorname{cl}\mathfrak{m}_p^{k} \subset \cdots \subset \mathfrak{m}_p \subset \mathscr{O}_{X,p}. \tag{2.1}$$

This induces a filtration of $\mathscr{O}_{X,p}/\mathfrak{m}_p^{k+1}$ whose associated graded vector space is

$$\operatorname{Gr}(\mathscr{O}_{X,p}/\operatorname{cl}\mathfrak{m}_p^{k+1}) = \begin{cases} \frac{\mathscr{O}_{X,p}}{\mathfrak{m}_p} \oplus \frac{\mathfrak{m}_p}{\operatorname{cl}\mathfrak{m}_p^2} \oplus \cdots \oplus \frac{\operatorname{cl}\mathfrak{m}_p^k}{\operatorname{cl}\mathfrak{m}_p^{k+1}} & k < \infty \\ \frac{\mathscr{O}_{X,p}}{\mathfrak{m}_p} \oplus \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} \oplus \cdots & k = \infty \end{cases}$$

Each direct factor $\operatorname{cl}\mathfrak{m}_p^{\ell}/\operatorname{cl}\mathfrak{m}_p^{\ell+1}$ is canonically isomorphic to the ℓ-th symmetric power of $\mathfrak{m}_p/\operatorname{cl}\mathfrak{m}_p^2$ under the natural surjection

$$\operatorname{Sym}^{\ell}(\mathfrak{m}_p/\operatorname{cl}\mathfrak{m}_p^2) \to \operatorname{cl}\mathfrak{m}_p^{\ell}/\operatorname{cl}\mathfrak{m}_p^{\ell+1}$$

defined on simple elements by $a_1 \odot \cdots \odot a_\ell \mapsto a_1 \cdots a_\ell$.

A splitting of the filtration on $\mathscr{O}_{X,p}/\operatorname{cl}{}_p^{k+1}$ is an isomorphism of filtered vector spaces $\phi_p : \mathscr{O}_{X,p}/\operatorname{cl}\mathfrak{m}_p^{k+1} \to \operatorname{Gr}(\mathscr{O}_{X,p})$. Let $U \subset X$ be open and $x^1, \dots, x^n$ be n sections of $\mathscr{O}_X(U)$ such that $dx_p^1 \wedge \cdots \wedge dx_p^n$ is nonzero throughout U. Then there is a natural splitting associated to this coordinate system: any $f \in \mathscr{O}_{X,p}/\mathfrak{m}_p^{k+1}$ can be written uniquely as

$$f = \sum_{|\alpha| \leq k} f_\alpha dx^\alpha \pmod{\operatorname{cl}\mathfrak{m}_p^{k+1}}$$

the sum extending over multiindices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$, with $dx^\alpha = dx_1^\alpha \cdots dx_n^\alpha$.

Let $\ell \leq k$ be a positive integer, and define the topology of compact convergence of ℓ-jets on $\mathscr{O}_X$ as follows. For each open $U \subset X$, let $f_i, f \in \mathscr{O}_X(U)$. We shall say that $f_i \to f$ if every compact set $K \subset U$ can be covered by a finite number of coordinate neighborhoods U_γ having a refinement $V_\gamma \subset \overline{V}_\gamma \subset U_\gamma$ by open sets with compact closures such that $j^\ell f_i \to j^\ell f$ uniformly on $K \cap V_\gamma$ for each γ. This topology is induced by a countable family of seminorms, and is therefore metrizable. When $\ell = k$, it is also a complete topological vector space.

If $(X, \mathscr{O}_X)$ is a C^k manifold, define the C^ℓ manifold $(X_\ell, \mathscr{O}_{X_\ell})$ by letting $X_\ell = X$ as a topological space and, for each open set $U \subset X_\ell$, $\mathscr{O}_{X_\ell}(U)$ to be the ring of functions of the form $g(p) = h(f_1(p), \dots, f_N(p))$ for some $N \in \mathbb{N}$, $f_i \in \mathscr{O}_X(U)$, and $h \in C^\ell(\mathbb{R}^N)$. Note that each $\mathscr{O}_{X_\ell}(U)$ is the completion of $\mathscr{O}_X(U)$ in the topology of compact convergence of ℓ-jets.

2.3.8 Linear differential operators

Let $(X, \mathscr{O}_X)$ be a C^k manifold. A *linear differential operator* of order r and class $\ell \leq k-r$ is a mapping $\mathrm{P} : \mathscr{O}_X(X) \to \mathscr{O}_{X_\ell}(X)$ that is linear with respect to constants and that factors as

$$\mathscr{O}_X(X) \xrightarrow{j^r} \mathscr{J}^r_X(X) \xrightarrow{i_{\mathrm{P}}} \mathscr{O}_{X_\ell}(X)$$

for some morphism of $\mathscr{O}_{X_{k-r}}(X)$-modules $i_{\mathrm{P}} : \mathscr{J}^r_X(X) \to \mathscr{O}_{X_\ell}(X)$.

The following theorem is due to Peetre [62] when $k = \infty$:

Theorem 3. *Let $\mathrm{P} : \mathscr{O}_X(X) \to \mathscr{O}_{X_\ell}(X)$ be a support non-increasing linear transformation. Then P is a linear differential operator of class ℓ and (finite) order r for some $r \leq k-\ell$.*

We prove the statement for finite k, as this does not seem to be well-known in that case. For $k = \infty$, we refer to the original papers [62], [63], or the book [46]. Our approach to the problem with finite k can be diagonalized to prove the case $k = \infty$ as well.

Lemma 7. *For any $x \in X$ and $C > 0$, there is an open neighborhood U of x such that, for all $y \in U \setminus \{x\}$ and $f \in \mathscr{O}_X(X)$ with $j^k f(y) = 0$, we have $|\mathrm{P}f(y)| < C$.*

Proof. The statement is local, and is invariant under C^k diffeomorphism, so we prove it in $\mathbb{R}^n$. Suppose to the contrary that there exists a sequence $x_j \to x$ and positively separated balls B_j centered at x_j and $f_j \in \mathscr{C}^k(\mathbb{R}^n)$ such that $j^k f_j(x_j) = 0$ but $\mathrm{P}f_j(x_j) \geq C > 0$.

Since $j^k f_j(x_j) = 0$, there is a ball $B_{\delta_j}(x_j) \subset B_j$ in which, for all multiindices $|\alpha| < k$ (and $|\beta| = k$ if $k < \infty$),

$$\sup_{y \in B'_j} |\partial_\alpha f_j(y)| \leq 2^{-j} \delta_j^{|\alpha|}$$

$$\sup_{y \in B'_j} |\partial_\beta f_j(y)| \leq 2^{-j}.$$

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is equal to 1 on $B_{1/2}(0)$ and vanishes identically outside $B_1(0)$, and put

$$\rho_j(y) = \rho\left(\frac{y - x_j}{\delta_j}\right)$$

be a bump function supported in the ball $B_{\delta_j}(x_j)$. Consider the infinite sum

$$g(y) = \sum_{j=1}^{\infty} \rho_{2j}(y) f_{2j}(y).$$

The above estimates guarantee that the k-jets of the partial sums of this series converge uniformly, and so g is C^k.

Now we have on the one hand $\mathrm{P}\, g(x_{2j}) \geq C$ and on the other that $\mathrm{P}\, g(x_{2j+1}) = 0$. But this implies that $\lim_{j\to\infty} \mathrm{P}\, g(x_j)$ does not exist, which contradicts the fact that $\mathrm{P}\, g$ is C^ℓ. $\square$

Lemma 8. *Let $L : \mathscr{C}^k_n(U) \to \mathscr{C}^\ell_n(U)$ be a linear differential operator*

$$L = \sum_{|\alpha|\leq k} p_\alpha(x) \frac{\partial}{\partial x^\alpha}.$$

Then L has order $\leq k - \ell$.

Proof. Let r be the order of L. By evaluating L against polynomials, it follows that each p_α must be C^ℓ. There is a dense open subset of U consisting of all $x \in U$ such that every p_α is either nonzero at x or is identically zero in a neighborhood of x. Pick such an x_0 and let $V \subset \overline{V} \subset U$ be a neighborhood of x_0 such that every p_α is either identically zero in V or does not vanish anywhere in $\overline{V}$. Performing a translation as needed, we can assume that $0 \notin \overline{V}$. Let $f \in \mathscr{C}^k(U) \setminus \mathscr{C}^{k+1}(U)$ be an arbitrary function. For each multi-index α, let

$$F_\alpha = L(x^\alpha f). \tag{2.2}$$

Then $F_\alpha \in \mathscr{C}^\ell(U)$ by hypothesis. Taking the F_α as given functions, (2.2) is an overdetermined system of equations that can be solved for $\partial^{|\beta|} f/\partial x^\beta$ for any β with $|\beta| = r$ such that $p_\beta \neq 0$ by Gauss reduction, but with coefficients in the ring $\mathscr{C}^\ell(U)$. So $\partial^{|\beta|} f/\partial x^\beta \in \mathscr{C}^\ell(U)$. But this is only possible for all f if $|\beta| \leq k - \ell$. $\square$

Proof of Theorem 3. We first show that under the same hypotheses as the lemma, $\mathrm{P}\, f(y) = 0$ for all $y \in U \setminus \{x\}$ such that $j^k f(y) = 0$. Indeed, suppose that $b = \mathrm{P}\, f(y)$ were non-zero. Let $g = 2Cf/b$. Then $j^k g(y) = 0$ that $|\mathrm{P}\, g(y)| < C$. But $\mathrm{P}\, g(y) = 2C$, a contradiction.

So in $U \setminus \{x\}$,

$$\mathrm{P} = \sum_{|\alpha|\leq k} f_\alpha \partial_\alpha \tag{2.3}$$

where the f_α are C^ℓ. Since P sends C^k functions to C^ℓ functions on all of U, the f_α admit continuations to C^ℓ functions on U, so that (2.3) holds throughout U. $\square$

Corollary 1. *Let X be a C^k manifold, and $U \subset X$ an open set. Then any linear support non-increasing operator $L : \mathscr{O}_X(U) \to \mathscr{O}_{X_\ell}(U)$ is a differential operator of order $\leq k - \ell$. There exists a locally finite covering of U by coordinate neighborhoods (U_i, ϕ_i) and an associated partition of unity ρ_i such that for all $f \in \mathscr{O}_{X,c}(U)$*

$$Lf = \sum_i \sum_{0\leq|\alpha|\leq k-\ell} p_i \frac{\partial^\alpha}{\partial x^\alpha}((\rho_i f) \circ \phi_i^{-1})$$

for some coefficients $p_i \in \mathscr{O}_{X_\ell}(U)$.

Proof. For $g \in \mathscr{O}_X(\phi_i(U))$, let $L_i g = L(g \circ \phi_i)$. By Theorem 3, L_i is a differential operator of order at most $k - \ell$. We have $f = \sum_i \rho_i f$, so $Lf = \sum_i L_i((\rho_i f) \circ \phi_i^{-1})$, whence the corollary. $\square$

Corollary 2. *If X is a C^k manifold and $U \subset X$ is open, then the natural topology on $\mathscr{O}_X(U)$ is metrizable and complete.*

Proof. Cover U by a locally finite collection of coordinate neighborhoods (U_i, ϕ_i) admitting a partition of unity ρ_i as in Corollary 1. If $K \subset U$ is a compact set, define a seminorm $|\cdot|_K$ on $\mathscr{O}_X(U)$ by

$$|f|_K = \sum_{U_i \cap K \neq \emptyset} \max_{0 \leq |\alpha| \leq k} \left| \frac{\partial^\alpha (\rho_i f) \circ \phi_i^{-1}}{\partial x^\alpha} \right|.$$

By Corollary 1, the natural topology on $\mathscr{O}_X(U)$ is the initial topology for the family of seminorms $|\cdot|_K$ for $K \subset U$ compact. Now let $U = \bigcup_i K_i$ be a decomposition of U into a countable family of compact sets K_i. The metric

$$d(f,g) = \sum_i \frac{|f-g|_{K_i}}{2^i(1 + |f-g|_{K_i})}$$

defines the natural topology on $\mathscr{O}_X(U)$. This metric is complete by a standard argument. $\square$

2.3.9 Lie groups

Definition 4. *A* Lie group *is a C^∞ manifold G together with the structure of a group such that the map $(x,y) \mapsto xy^{-1}$ is a smooth morphism $G \times G \to G$.*

Real algebraic groups are always Lie groups. Closed subgroups of Lie groups are Lie groups.

The main examples come from the *classical groups*:

- The most fundamental example is the group $\mathrm{GL}(\mathbb{V})$ of linear automorphisms of a real vector space $\mathbb{V}$. If $\mathbb{V}$ has dimension n, we also write $\mathrm{GL}(n)$.
- The subgroup of transformations of determinant 1 is the special linear group $\mathrm{SL}(\mathbb{V}) = \mathrm{SL}(n)$.
- The subgroup of automorphisms of a nondegenerate quadratic form Q on $\mathbb{V}$ is the real orthogonal group $\mathrm{O}(Q)$. The special orthogonal group $\mathrm{SO}(Q) = \mathrm{O}(Q) \cap \mathrm{SL}(\mathbb{V})$ is the group of automorphisms of unit determinant. This is also denoted by $\mathrm{SO}(p,q)$ where the diagonalization of Q has p positive and q negative entries.

- The group of automorphisms of a nondegenerate skew-symmetric form ω (that is, a symplectic form) is the symplectic group $\mathrm{Sp}(\omega)$. In this case, n is always even, and any two symplectic forms are conjugate under the orthogonal group, so the group is written $\mathrm{Sp}(n, \mathbb{R})$.
- A complex structure $J : \mathbb{V} \to \mathbb{V}$ is a linear transformation satisfying $J^2 = -\mathrm{Id}$. Complex structures only exist in even dimensions. The group of transformations preserving a nondegenerate quadratic form Q and commuting with an orthogonal complex structure J is the *pseudo-unitary group* $U(Q, J)$. These too are classified by the signature of the quadratic form, and we denote by $U(p, q)$ the pseudo-unitary group associated to a quadratic form of signature $(2p, 2q)$. A pseudo-unitary transformation whose restriction to an eigenspace of J has unit determinant is called a special pseudo-unitary transformation. The special pseudo-unitary group is denoted $SU(p, q)$.
- A quaternionic structure on $\mathbb{V}$ is a pair of anti-commuting complex structures J, K. There exists a quaternionic structure on $\mathbb{V}$ if and only if $n = 4m$ is a multiple of 4. Denote by $\mathrm{GL}(J, K)$ the group of linear automorphisms of $\mathbb{V}$ that commute with J and K. All quaternionic structures are conjugate under $\mathrm{GL}(\mathbb{V})$, so we can denote the equivalence class of these by $\mathrm{GL}(m, \mathbb{H})$. The subgroup of all transformations whose restrictions to the eigenspaces of J have unit determinant is the *special quaternionic linear group* $\mathrm{SL}(m, \mathbb{H})$.
- The quaternionic symplectic group $\mathrm{Sp}(p, q)$ is the subgroup of $\mathrm{SL}(m, \mathbb{H})$ preserving a pseudo-unitary structure of signature (p, q) with respect to the complex structure J.
- The quaternionic orthogonal group $\mathrm{SO}(m, \mathbb{H})$ is the subgroup of $\mathrm{SL}(m, \mathbb{H})$ preserving a non-degenerate complex-valued quadratic form on an eigenspace of J for which the other complex structure K is orthogonal.

More on the geometry of the classical groups is contained in Chapter 4, as well as a description of the geometry of the real forms of the exceptional groups.

2.3.10 Jet groups

Let $\mathfrak{m}_0$ be the maximal ideal in $\mathscr{C}_n^\infty$ corresponding to the origin, and let $\mathbb{G}_n$ be the set of all germs $(f_{1,0}, \dots, f_{n,0}) \in \mathfrak{m}_0^{\oplus n}$ such that $df_{1,0} \wedge \cdots \wedge df_{n,0} \neq 0$. This set is the group of germs at the origin of local diffeomophisms $\phi = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ from a neighborhood U of the origin to $\mathbb{R}^n$ such that $\phi(0) = 0$. The composition and inverse of two such germs is again such a germ, so $\mathbb{G}_n$ is a group.

For each $\ell \in \mathbb{Z}_{\geq 0}$, let $\mathbb{N}_\ell = I + \mathfrak{m}_0^{\ell+1}$ with I the identity diffeomorphism of $\mathbb{R}^n$, and let $\mathbb{N}_\infty = \bigcap_{\ell=0}^\infty \mathbb{N}_\ell$. Then $\mathbb{N}_\ell$ is a normal subgroup of $\mathbb{G}_n$. The *jet group* of order $\ell = 0, 1, \dots, \infty$

is the quotient group $\mathbb{G}_n^\ell = \mathbb{G}_n/\mathbb{N}_\ell$. This is a Lie group for all $\ell < \infty$. When $\ell = 1$, the inclusion of the linear transformations of $\mathbb{R}^n$ in $\mathbb{G}_n$ gives an isomorphism $\mathrm{GL}(n) \cong G_n^1$. For $\ell > 1$, $\mathbb{G}_n^\ell$ is a semidirect product of $\mathrm{GL}(n)$ with a nilpotent composition series by nilpotent subgroups $\mathbb{N}_r^k = \mathbb{N}_r/\mathbb{N}_k$:

$$\mathbb{N}_\ell \rhd \cdots \rhd \mathbb{N}_1.$$

Moreover, $\mathbb{G}_n^\infty$ is the inverse limit of the G_n^ℓ for $\ell < \infty$, under the directed set of morphisms

$$\mathbb{G}_n^\infty \to \cdots \to G_n^2 \to G_n^1 \to G_n^0 = \{I\}.$$

It is thus a topological group with the inverse limit topology. The following theorem, due to Epstein and Thurston [24] characterizes the actions of $\mathbb{G}_n$ on metrizable spaces:

Theorem 4. *If $\mathbb{G}_n^\infty$ acts continuously on a metrizable topological space of Lebesgue dimension k, then $\mathbb{N}_{2k+1}$ acts trivially.*

As a consequence, the action of $\mathbb{G}_n$ factors through the action of $\mathbb{G}_n^{2k+1}$. In particular, any finite-dimensional continuous linear representation of $\mathbb{G}_n^\infty$ factors through a jet group of finite order.

The group $\mathbb{G}_n$ acts on the stalk $\mathscr{C}_{n,0}^k$ of k-times continuously differentiable functions of $\mathbb{R}^n$ at 0 by composition under which the filtration (2.1) is stable. Hence the action of $\mathbb{G}_n$ descends to the k-th order jet space $\mathscr{C}_{n,0}^k/\mathfrak{m}_0^{k+1}$, and the kernel of this action is $\mathbb{N}_k$. So $\mathbb{G}_n^k$ acts on $\mathscr{C}_{n,0}^k/\mathfrak{m}_0^{k+1}$. Concretely, $\mathscr{C}_{n,0}^k/\mathfrak{m}_0^{k+1}$ is the truncated polynomial algebra of $k[x_1, \dots, x_n]$ modulo the ideal generated by all homogeneous polynomials of degree $k+1$. The action of $\mathbb{G}_n^k$ is given by truncated polynomial composition. This is the *contragredient representation* of the jet group $\mathbb{G}_n^k$. The *standard representation* of $\mathbb{G}_n^k$ is the dual of the contragredient representation.

In addition to linear representations, there is a class of important nonlinear actions of the jet groups on spaces of k-velocities T_n^k, which is the manifold of all k-jets at the origin of smooth curves $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that $\gamma(0) = 0$. Concretely, $T_n^k = (\mathbb{R}[t]/(t^{k+1}))^n$ is the n-th Cartesian power of the polynomial algebra in one variable truncated to degree k. The space T_n^k is an affine space, and so carries a canonical smooth structure. The group $\mathbb{G}_n^k$ acts smoothly by composition on T_n^k, but this action is nonlinear.

Let $\pi_{n+r}^n : \mathbb{R}^{n+r} \to \mathbb{R}^n$ be the projection mapping onto the first n coordinates and let $\mathbb{G}_{n,r}^k$ be the subgroup of $\mathbb{G}_{n+r}^k$ consisting of the k-jets of all local diffeomorphisms $\phi : \mathbb{R}^{n+r} \to \mathbb{R}^{n+r}$ with $\phi(0) = 0$ that cover a local diffeomorphism of $\mathbb{R}^n \to \mathbb{R}^n$. This is most conveniently expressed as a differential equation, that

$$\frac{\partial}{\partial x^{n+1}} \pi_{n+r}^n \circ \phi = \cdots = \frac{\partial}{\partial x^{n+r}} \pi_{n+r}^n \circ \phi = 0.$$

For example, when $k = 1$, $\mathbb{G}^1_{n,r}$ is the subgroup of $\mathbb{G}^1_{n,r} = \mathrm{GL}(n+r)$ consisting of the invertible block upper triangular $(n+r)\times(n+r)$ matrices with an $n \times n$ block and $r \times r$ block down the diagonal.

2.4 VECTOR BUNDLES

Throughout this section $(X, \mathscr{O}_X)$ is a C^k-manifold of dimension n. A C^k vector bundle over X consists of a C^k manifold $(E, \mathscr{O}_E)$ together with surjective a C^k morphism $\pi : E \to X$ and the structure of a real vector space on each of the fibers $\pi^{-1}(p)$ for all $p \in X$ such that the following local triviality condition holds

- For each $p \in X$ there is a neighborhood U of p and a C^k diffeomorphism $\psi : \pi^{-1}U \to U \times \mathbb{R}^r$ for some $r \in \mathbb{Z}_{\geq 0}$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl] \\ & U & \end{tikzcd}$$

  and such that $\psi|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \mathbb{R}^r$ is a linear isomorphism of vector spaces for all $x \in U$.

A vector bundle is trivial if there is a trivialization of the bundle over $U = X$. If $S \subset X$ is any subset, then we define $E|S = \pi^{-1}(S)$. Note that we do not at the moment assume that the dimension of a vector bundle is constant, although in subsequent chapters we do impose this. In that case, the dimension of the fiber is called the *rank* of the vector bundle E, denoted $\operatorname{rank} E$.

Following [46], the category of vector bundles over $\mathbf{Man}^k_n$, denoted by $\mathbf{Vect}^k_n$, is a category whose objects C^k vector bundles $E \xrightarrow{\pi} X$ with X a C^k manifold, and whose morphisms between two bundles $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi} X'$ are pairs of C^k mappings $g : X \to X'$ and $f : E \to E'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

commutes, and such that for each $x \in X$, $f|\pi^{-1}(x) : \pi^{-1}(x) \to \pi^{-1}(g(x))$ is a linear transformation. We say that $f : E \to E'$ covers $g : X \to X'$. There is a functor $B : \mathbf{Vect}^k_n \to \mathbf{Man}^k_n$:

the *base* functor. The category of vector bundles over X, denoted $\mathbf{Vect}_X$, is the subcategory consisting of all vector bundles over X and morphisms that cover the identity map of X.

Let $U \subset X$ be an arbitrary open subset. The space of C^k sections of E over U are denoted by $\Gamma(E, U)$, and defined by

$$\Gamma(E, U) = \{s \in C^k(U, E) \mid \pi \circ s = \mathrm{id}_U\}.$$

Each $\Gamma(E, U)$ is naturally an $\mathscr{O}_X(U)$-module under pointwise multiplication, so $\Gamma(E)$ is a sheaf of $\mathscr{O}_X$-modules.

If E is a vector bundle over X, then the sheaf of *polynomial* functions $\mathscr{P}$ on E makes sense. Over an open set $U \subset X$, $\mathscr{P}(U)$ is the algebra of all function $f : E|U \to \mathbb{R}$ of the form

$$f(x) = \sum_{|\alpha| \leq M} f_\alpha(\pi(x)) \, \mathrm{ev}_{\pi(x)}\langle s_1, x\rangle^{\alpha_1} \cdots \mathrm{ev}_{\pi(x)}\langle s_N, x\rangle^{\alpha_N}$$

for some $M \in \mathbb{Z}_{\geq 0}$, functions $f_\alpha \in \mathscr{O}_X(U)$, and $s_1, \ldots, s_N \in \Gamma(E^*, U)$.

The algebra of C^k functions on E is then recoverable from the algebra of polynomial functions as in §2.3.7: $\mathscr{O}_E(E|U)$ is the algebra of all functions $f : E|U \to \mathbb{R}$ of the form

$$f(x) = g(h_1(x), \ldots, h_N(x))$$

for some C^k function $g : \mathbb{R}^N \to \mathbb{R}$ and $h_1, \ldots, h_N \in \mathscr{P}(U)$. (We can do this by taking some of the h_i to be the local coordinates on X, and the remaining h_i to be a basis of sections of the dual space of E.) So the algebra of polynomial functions determines the C^k structure $(E, \mathscr{O}_E)$.

2.4.1 Locally free sheaves

A sheaf of $\mathscr{O}_X$-modules is *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$. The rank of such a sheaf of modules is the number of copies of which it is the sum. A sheaf $\mathcal{F}$ of $\mathscr{O}_X$-modules is *locally free* if there is an covering of X by open sets U_i such that, for each i, $\mathcal{F}|U_i$ is a free $\mathscr{O}_X|U_i$-module. The rank of a locally free sheaf $\mathcal{F}$ at p is the dimension of the vector space $\mathcal{F}_p/\mathfrak{m}_p\mathcal{F}_p$ over $\mathscr{O}_{X,p}/\mathfrak{m}_{X,p}$.

Locally free sheaves inherit a topology from the topological ring structure on $\mathscr{O}_X$. A sequence s_n of sections of $\mathscr{O}_X(U_i)$ converges if and only if the components of s_n in the trivialization of $\mathscr{O}_X(U_i)$ converge in the topology of $\mathscr{O}_X(U)$.

The sheaf $\Gamma(E, -)$ associated to a rank r vector bundle E on X is a locally free sheaf of rank r. Each point has a neighborhood U such that $E|U$ is trivial, and so $\mathcal{F}(U)$ is generated as an $\mathscr{O}_X(U)$-module by r sections.

Lemma 9. *The sections functor* $\Gamma : \mathbf{Vect}_X \to \mathscr{O}_X - \mathbf{Mod}$ *is an equivalence of categories onto its image, consisting of all locally free finite rank sheaves. That is, there is a functor* $\mathbf{V}$ *that associates to any locally free sheaf* $\mathcal{F}$ *a vector bundle* $\mathbf{V}(\mathcal{F})$ *such that* $\Gamma(\mathbf{V}(\mathcal{F}), X)$ *is naturally isomorphic to* $\mathcal{F}(X)$ *as an* $\mathscr{O}_X(X)$*-module.*

The proof follows the **Spec** construction of Hartshorne [36], page 128. A related theorem of Serre [75] (in the analytic category) and Swan [79] establishes the equivalence of finitely-generated projective modules and vector bundles on compact manifolds.

Proof. We construct first the sheaf of polynomial functions $\mathscr{P}$ on $\mathbf{V}(\mathcal{F})$. For U an open subset of X, let $\mathscr{P}(U)$ be the set of all functions $f : \mathcal{F}(U) \to \mathscr{O}_X(U)$ that can be expressed as

$$f(x) = \sum_{|\alpha| \leq M} h_\alpha \langle s_1, x\rangle^{\alpha_1} \cdots \langle s_N, x\rangle^{\alpha_N}$$

for some $M \in \mathbb{Z}_{\geq 0}$, $h_\alpha \in \mathscr{O}_X(U)$, and sections $s_1, \ldots, s_N$ of the dual sheaf $\mathcal{F}^*(U)$. Then $\mathscr{P}(U)$ carries the structure of an $\mathscr{O}_X(U)$-algebra under the pointwise operations of addition and multiplication. In particular, there is an inclusion $\mathscr{O}_X(U) \subset \mathscr{P}(U)$.

Let $\mathbf{V}(\mathcal{F}) = \operatorname{spec} \mathscr{P}(X)$. Let $\pi : \mathbf{V}(\mathcal{F}) \to \operatorname{spec} \mathscr{O}_X(X) \cong X$ be the mapping that associates to any algebra homomorphism $\phi : \mathscr{P}(X) \to \mathbb{R}$ the algebra homomorphism $\pi(\phi) = \phi|\mathscr{O}_X(X) : \mathscr{O}_X(X) \to \mathbb{R}$. If $p \in X$ is a given point, then points of $\pi^{-1}(p)$ are those algebra homomorphisms $\phi : A \to \mathbb{R}$ such that $\phi|\mathscr{O}_X(X) = \mathrm{ev}_p$ is evaluation at p. Such a homomorphism is determined by a linear function on the real r-dimensional vector space $(\mathcal{F}_p/\mathfrak{m}_p\mathcal{F}_p)^*$. Thus, by the double-duality isomorphism, $\pi^{-1}(p)$ is naturally in one-to-one correspondence with the real vector space $\mathcal{F}_p/\mathfrak{m}_p\mathcal{F}_p$. This defines the linear structure on the fibers of π.

For local triviality, let U be a neighborhood of p such that $\mathcal{F}(U)$ is free. Shrinking U if necessary, we can assume that there are *global* sections $\sigma_1, \ldots, \sigma_r$ of $\mathcal{F}(X)$ such that the $s_i|U$ generate $\mathcal{F}(U)$ (freely) as an $\mathscr{O}_X(U)$-module. A point of $\mathbf{V}(\mathcal{F})|U$ is then an evaluation map of $\mathscr{O}_X(U)$, together with an assignment of a real value for each of the sections $s_1, \ldots, s_r$. Thus there is an isomorphism $\mathbf{V}(\mathcal{F})|U \to \mathbb{R}^r \times U$ that is linear on each fiber. The transition functions are C^k between two overlapping open sets, since the transition between one set of local free generators $s_1, \ldots, s_r$ of $\mathcal{F}(U)$ and another set of local free generators $s'_1, \ldots, s'_r$ of $\mathcal{F}(V)$ is an $r \times r$ matrix with coefficients in $\mathscr{O}_X(U \cap V)$.

A section of the sheaf $\mathcal{F}$ defines a section of the vector bundle $\mathbf{V}(\mathcal{F})$. Conversely, a section of $\mathbf{V}(\mathcal{F})$ defines a local section of $\mathcal{F}$. $\square$

2.4.2 Operations on vector bundles

If $\pi_E : E \to Y$ is a C^k vector bundle on Y and $f : X \to Y$ a C^k map, then there is a unique vector bundle over Y up to isomorphism, the *pullback bundle* $f^{-1}E$, and a bundle map $f^{-1}E \to E$ that covers f such that the diagram commutes

$$\begin{tikzcd} f^{-1}E \arrow[r] \arrow[d, "\pi_{f^{-1}E}"'] & E \arrow[d, "\pi_E"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

At a point $p \in X$, the fiber of $f^{-1}E$ is given by $f^{-1}E_p = E_{f(p)}$. (This is also known as the fibre product of E with X over f.)

In particular, when X is a vector bundle on E and $f = \pi_X$ is the bundle projection, the pullback bundle $\pi_X^{-1}E$ is canonically isomorphic to $\pi_E^{-1}X$. Because of the symmetry, we denote this vector bundle by $X \oplus E$, the *direct sum.* It is a vector bundle over Y whose fiber at each point $p \in Y$ is the direct sum of vector spaces $X_p \oplus E_p$. The associated locally free sheaf is $\Gamma(X) \oplus \Gamma(E)$.

If E, F are C^k vector bundles over a C^k manifold X, the sheaf of all $\mathscr{O}_X$-module homomorphisms from $\Gamma(E)$ to $\Gamma(F)$ is a locally free sheaf of $\mathscr{O}_X$-modules. The associated vector bundle is denoted by $\mathrm{Hom}(E, F) = \mathrm{Hom}_{\mathbf{Vect}_X}(E, F)$, and consists of all vector bundle homomorphisms $T : E \to F$ that cover the identity automorphism of X. A special case is where $F = X \times \mathbb{R}$ is the rank one trivial bundle; then $E^* := \mathrm{Hom}(E, F)$ is called the *dual bundle* associated to E.

The *tensor product bundle* is the bundle associated to the locally free sheaf $\Gamma(E) \otimes \Gamma(F)$. The tensor product is the left adjoint of the Hom functor, and in particular we have $E \otimes F \cong \mathrm{Hom}(E^*, F)$ via a natural isomorphism in both E and F.

2.4.3 Natural vector bundles and sheaves

A *natural vector bundle* is a covariant functor $F : \mathbf{Man}_n^k \to \mathbf{Vect}_n^r$, $r \leq k$, such that the following conditions hold:

- $B \circ F = I$
- If $i : U \to X$ is an inclusion of an open set in $X \in \mathrm{Ob}(\mathbf{Man}_n^k)$, then Fi is an inclusion of $\pi_{FX}^{-1}(U)$ in FX.
- If $\phi : P \times M \to N$ is a C^k map such that for all $p \in P$, $\phi(p, -) : M \to N$ are local diffeomorphisms, then $\widetilde{F}\phi : P \times FM \to FN$ defined by $\widetilde{F}\phi(p, -) = F(\phi(p, -))$ is C^r. That is, F sends smoothly parametrized systems of local diffeomorphisms to smoothly parametrized systems of bundle morphisms.

A natural sheaf is a covariant functor from $\mathbf{Man}_n^k$ to the category of sheaves of modules such that the analogs of these properties are satisfied. Some examples of natural sheaves are the cotangent sheaf, the tangent sheaf, and sheaves of jets.

If F is a natural bundle, then the group $\mathbb{G}_n$ of germs of diffeomorphisms of $\mathbb{R}^n$ preserving the origin carries a linear representation on the fiber $F\mathbb{R}_0^n$. If this representation factors through $\mathbb{G}_n^\infty$, then by virtue of Theorem 4, it must factor through a representation of $\mathbb{G}_n^k$. Conversely, a natural bundle can be associated to any representation of $\mathbb{G}_n^k$ by an associated vector bundle construction (see §2.4.9).

The following theorem of Epstein and Thurston [24] characterizes the natural bundles:

Theorem 5. *A natural bundle $F : \mathbf{Man}_n^k \to \mathbf{Vect}_n^r$ is C^r isomorphic to an associated vector bundle of some jet group $\mathbb{G}_n^s$ for $0 \leq s \leq 2 \operatorname{rank} F + 1$ and $s + r \leq k$.*

2.4.4 Cotangent sheaf

Let $(X, \mathscr{O}_X)$ be a C^k manifold of dimension n and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules on X. A *derivation* $D : \mathscr{O}_X \to \mathcal{F}$ is a morphism of sheaves that defines a derivation of $\mathbb{R}$-modules $D_U : \mathscr{O}_X(U) \to \mathcal{F}(U)$ on each open subset U of X.

Definition 5. *Let $(X, \mathscr{O}_X)$ be a C^k manifold. The sheaf of Kähler differentials on X is a sheaf Ω_X of $\mathscr{O}_X$-modules equipped with a continuous derivation $d : \mathscr{O}_X \to \Omega_X$ of $\mathscr{O}_{X_{k-1}}$-modules such that the following universal property holds. For any sheaf of topological $\mathscr{O}_{X_{k-1}}$-modules $\mathcal{F}$, and any continuous derivation $D : \mathscr{O}_X \to \mathcal{F}$, there exists a unique morphism of $\mathscr{O}_{X_{k-1}}$-modules $i_D : \Omega_X \to \mathcal{F}$ such that $D = i_D \circ d$.*

If U is an open set in X, then $\Omega_X(U)$ is the $\mathscr{O}_{X_{k-1}}(U)$-module obtained by quotienting the free $\mathscr{O}_{X_{k-1}}(U)$-module on the set $\{da \mid a \in \mathscr{O}_X(U)\}$ by the closure of the idea generated by $\{d(1)\} \cup \{d(ab) - ad(b) - bd(a) | a, b, \in \mathscr{O}_X(U)\}$.

Concretely, consider the Cartesian product $(X \times X, \mathscr{O}_{X \times X})$, with projections $\pi_1, \pi_2 : X \times X \to X$ and diagonal embedding $\Delta : X \to X \times X$. The ideal sheaf $\mathscr{I}$ of Δ is the kernel of $\Delta^\sharp : \mathscr{O}_{X\times X} \to \Delta_* \mathscr{O}_X$. Then $\Omega_X = \Delta^* \mathscr{I} / \operatorname{cl} \mathscr{I}^2$. For $f \in \mathscr{O}_X$, define $df \in \Omega_X$ by $df = \Delta^*(\pi_1^* - \pi_2^*) f \pmod{\operatorname{cl} \mathscr{I}^2}$.

The sheaf Ω_X is functorial in X. If $f : X \to Y$ is a morphism of C^k-manifolds, then for any derivation $D : \mathscr{O}_X \to \mathcal{F}$, there is a derivation $f_* D : f^* \mathscr{O}_Y \to \mathcal{F}$ given on sections by $(f_* D) g = D(f^\sharp g)$. In particular, for the derivation $d_X : \mathscr{O}_X \to \Omega_X$, we obtain a derivation $f_* d_X : f^* \mathscr{O}_Y \to \Omega_X$. By the universal property, there is a unique linear map

$$f^* \Omega_Y \xrightarrow{\Omega_f} \Omega_X \tag{2.4}$$

$\Omega_f : f^*\Omega_Y \to \Omega_X$ such that $f_* d_X = \Omega_f \circ d_Y$.

The sheaf Ω_X is a locally free $\mathscr{O}_{X_{k-1}}$-module of rank n. Indeed, let $p \in X$ be given and choose germs $x^1, \ldots, x^n$ in $\mathscr{O}_{X,p}$ that form a basis of $\mathfrak{m}_p / \operatorname{cl} \mathfrak{m}_p^2$. By Hadamard's lemma, any germ in $\Omega_{X,p}$ has the form $f_i dx^i$ for some $f_i \in \mathscr{O}_{X_{k-1},p}$.

The exterior algebra $\Omega_X^\bullet$ is defined as the sheaf of graded unital anticommutative $\mathscr{O}_X$-algebras

$$\Omega_X^\bullet = \bigoplus_j \wedge^j_{\mathscr{O}_{X_{k-1}}} \Omega_X.$$

Denote by $\Omega_X^j = \wedge^j \Omega_X$ the degree j graded part of the algebra. In particular, $\Omega_X^0 = \mathscr{O}_X$ and $\Omega_X^1 = \Omega_X$.

There is a sequence of derivations

$$\Omega^\bullet_{X_k} \xrightarrow{d} \Omega^{\bullet+1}_{X_{k-1}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\bullet+k}_{X_0}$$

of derivations of algebras that satisfies $d^2 = 0$ and that at each level extends the differential $d : \mathscr{O}_{X_\ell} \to \Omega^1_{X_{\ell-1}}$. This defines the exterior derivative.

2.4.5 Cotangent bundle

Let $(X, \mathscr{O}_X)$ be a C^k manifold and $T^*X = \mathbf{V}(\Omega_X)$ be the C^{k-1} vector bundle associated to the sheaf Ω_X. This is the *cotangent bundle* of X. By construction, the fiber at each point $p \in X$ is $T^*X_p = \mathfrak{m}_p / \operatorname{cl} \mathfrak{m}_p^2$.

Theorem 6. *The cotangent bundle defines a natural bundle $\mathbf{Man}_n^k \to \mathbf{Vect}_n^k$. This is the natural bundle associated to the dual of the standard representation of $\mathbb{G}_n^1 = \mathrm{GL}(n)$*

If $f : X \to Y$ is a morphism of C^k manifolds, then (2.4) induces a mapping of bundles

$$f^* : T^*Y \to f^{-1}T^*X.$$

2.4.6 Tangent sheaf and bundle

Let $(X, \mathscr{O}_X)$ be a C^k manifold. The sheaf of derivations $D : \mathscr{O}_X \to \mathscr{O}_{X_{k-1}}$ is is called the *tangent sheaf* of X, denoted $\mathscr{X}_X$. Sections of the sheaf are *vector fields* on X. This is a locally free sheaf, and is dual (as an $\mathscr{O}_X$-module) to the cotangent sheaf Ω_X.

The *tangent bundle* TX of a C^k manifold $(X, \mathscr{O}_X)$ is the bundle associated to the locally free sheaf $\mathscr{X}_X$, the tangent sheaf of X. This is the natural bundle associated to the standard representation of $\mathbb{G}_n^1 = \mathrm{GL}(n)$. The fiber at a point $p \in X$ is TX_p, the space of derivations $\mathscr{O}_{X,p} \to \mathbb{R}$.

There is a C^{k-1} bilinear duality pairing $\langle -,-\rangle : TX \times_X T^*X \to \mathbb{R}$ that in the fiber at each point of X agrees with the duality pairing in Lemma 6.

If $f : X \to Y$ is a morphism of C^k manifolds, the differential of f at p is the map $df_p : TX_p \to TY_{f(p)}$ that maps a derivation $D : \mathscr{O}_{X,p} \to \mathbb{R}$ to the derivation $D \circ f^\sharp_{f(p)} : \mathscr{O}_{Y,f(p)} \to \mathbb{R}$. This defines fiberwise a C^{k-1} bundle map $df : f^{-1}TX \to TY$ that is the pointwise adjoint of the pullback $f^* : T^*Y \to f^{-1}T^*X$ under the duality pairing $\langle -,-\rangle$. There is in general an obstruction to defining the pushforward of a f_*v of a vector field v in $\mathscr{X}_X$. Indeed, if $D : \mathscr{O}_X \to \mathscr{O}_{X_\ell}$ is a derivation, then $f_*D = D \circ f^\sharp : f^*\mathscr{O}_Y \to \mathscr{O}_{X_\ell}$ is not necessarily a derivation $f^*\mathscr{O}_Y \to f^*\mathscr{O}_{Y_\ell}$.

2.4.7 Jet bundles

Let $(X, \mathscr{O}_X)$ be a C^k manifold. As in §2.4.4, let $\mathscr{I}$ be the sheaf of ideals of the diagonal $\Delta(X)$ in the product manifold $(X \times X, \mathscr{O}_{X\times X})$. For $\ell \leq k$, the ℓ-th order jet sheaf of X is the sheaf of $\mathscr{O}_X$ modules on X defined by $\mathscr{J}^\ell_X = \Delta^*\left(\mathscr{O}_{X\times X}/\operatorname{cl}\mathscr{I}^{\ell+1}\right)$. We observe that, for any $p \in X$, $\mathscr{J}^\ell_{X,p}/\mathfrak{m}_p\mathscr{J}^\ell_p \cong \mathscr{O}_{X,p}/\operatorname{cl}\mathfrak{m}_p^{\ell+1}$ is the space of ℓ-jets at p. There is a mapping of sheaves $j^\ell : \mathscr{O}_X \to \mathscr{J}^\ell_X$ that is linear with respect to constants.

The bundle associated to the locally free sheaf $\mathscr{J}^\ell_{X,p}$ of $O_{X_{k-\ell}}$-modules is the ℓ-th order *jet bundle* of X, and is denoted $J^\ell X$. If $U \subset X$ is an open set and $f \in \mathscr{O}_X(U)$ is a C^k function on U, then associated to f is the section $j^\ell f \in \mathscr{J}^\ell_X(U)$.

As in §2.3.7, there is a canonical splitting of $\mathscr{J}^\ell_X = \mathscr{O}_X \oplus \mathscr{K}^\ell_X$ where

$$\mathscr{K}^\ell_X = \Delta^*\left(\mathscr{I}/\operatorname{cl}\mathscr{I}^{\ell+1}\right).$$

And an associated bundle decomposition of $J^\ell X$ into the direct sum of a trivial bundle with $K^\ell X$: $J^\ell X = \mathbb{R} \times K^\ell X$. The distinguished subbundle $K^\ell X$ has fiber at each point $p \in X$ the space $K^\ell_{X,p}$ of §2.3.7 consisting of all jets at the point p of functions that vanish when evaluated at p.

There is a natural bundle map $d : J^\ell X \to T^*X$ defined by setting, for $p \in X$ and $\sigma \in J^\ell X_p$, $d\sigma_p = (df)_p$ where $f \in \mathscr{O}_{X,p}$ is a representative of σ modulo $\operatorname{cl}\mathfrak{m}_p^{\ell+1}$. This induces an action of vectors on jets at a point: if $v \in \mathscr{T}X_p$ is a vector, then $v(\sigma)_p = v \lrcorner d\sigma_p$.

More generally, if $\mathcal{F}$ is a locally free sheaf of $\mathscr{O}_X$-modules, then the ℓ-th order jet sheaf associated to $\mathcal{F}$ is the sheaf

$$\mathscr{J}^\ell\mathcal{F} = \Delta^*(\Delta_*\mathcal{F}/\operatorname{cl}\mathscr{I}^{\ell+1}).$$

For $p \in X$, we have

$$(\mathscr{J}^\ell\mathcal{F})_p/\operatorname{cl}\mathfrak{m}_p(\mathscr{J}^\ell\mathcal{F})_p \cong \mathcal{F}_p/\operatorname{cl}\mathfrak{m}_p^{\ell+1}\mathcal{F}_p$$

If $f : X \to Y$ is a C^k morphism, then there is a $C^{k-\ell}$ bundle map $j^\ell f : J^\ell Y \to f^{-1}J^\ell X$, called the ℓ-jet of f.

2.4.8 Fibered manifolds

A C^k fibered manifold consists of a pair of C^k manifolds X, E, and a surjective submersion $\pi_E : E \to X$. The *vertical bundle* associated to π_E is the subbundle of TE defined by $\ker d\pi_E$. Because π_E is a submersion, the vertical bundle is a C^{k-1} vector bundle in its own right, whose rank is everywhere the (constant) nullity of the linear operator $d\pi_E$. If U is an open subset of X, then $E|U = \pi_E^{-1}U \xrightarrow{\pi_E|U} U$ is a fibered manifold over U.

A morphism of fibered manifolds $\pi_E : E \to X, \pi_{E'} : E' \to X'$ is a pair of C^k maps $f : E \to E'$ and $g : X \to X'$, with g a local diffeomorphism, such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "g"'] & X' \end{tikzcd}$$

As in the case of vector bundles, the morphism f is said to cover the morphism g. A sub-fibered manifold is a submanifold E' of E such that $\pi_E|E' : E' \to X$ remains a surjective submersion. The original space X is trivially a fibered manifold over itself, with $\pi_X = \mathrm{id}$ the identity operator. A section of π_E is a morphism of fibered manifolds $s : (X, \mathrm{id}) \to (E, \pi_E)$; that is, it is a C^k map $s : X \to E$ such that $\pi_E \circ s = \mathrm{id}_X$. The space of sections of π_E over an open subset $U \subset X$ is denoted $\Gamma(\pi_E, U)$. This is a sheaf of sets.

If $\pi_E : E \to X$ and $\pi_F : F \to X$ are two fibered manifolds over the same base X, then the fiber product $\pi_{E\times_X F} : E \times_X F \to x$ is constructed as follows. Let $\Delta : X \to X \times X$ be the diagonal embedding of X in its Cartesian product with itself. Then $\pi_E \times \pi_F : E \times F \to X \times X$ defines a fibered manifold over $X \times X$. Since $\pi_E \times \pi_F$ is a submersion, its restriction to the fiber over any closed submanifold is also a submersion. In particular, set $G = (\pi_E \times \pi_F)^{-1}(\Delta X)$. Then $\pi_E \times \pi_F|G : G \to \Delta(X)$ is a surjective submersion. Precomposing this with the diffeomorphism Δ^{-1} defines the fibered manifold $E \times_X F$.

A fibered manifold is called a fiber bundle if it is locally trivial. That is, there exists a C^k manifold F such that, for all $p \in X$, there exists an isomorphism of fibered manifolds $E|U \cong F \times U$.

2.4.9 Principal bundles

A *right principal bundle* is a C^k fiber bundle $\pi_P : P \to M$ together with a Lie group G that acts on the right of P via a C^k action $R : P \times G \to P$ such that

- The orbits of G are precisely the fibers of π_P: $R(p, G) = \pi^{-1}(\pi(p))$ for all $p \in P$.
- The fibers of π_P are principal homogeneous spaces for the action of G.

For a fixed $g \in G$, denote by $R_g : P \to P$ the right action by g.

A *left principal bundle* is the analogous object with a left action $L : G \times P \to P$ instead of a right action. Any left principal bundle can be given the structure of a right principal bundle by setting $R_g = L_{g^{-1}}$ and vice-versa.

2.4.10 Associated bundles

Let P be a principal bundle on a C^k manifold X with structure group G. Suppose that G acts smoothly and transitively on the left on a smooth manifold F. Then G acts on the space $P \times F$ via $g : (p, f) \mapsto (p \cdot g^{-1}, g \cdot f)$. The quotient space by this action is a C^k fiber bundle over X, denoted $P \times_G F$. This is called the *associated bundle*.

2.4.11 Principal jet bundles

Let $(X, \mathscr{O}_X)$ be a C^k manifold of dimension n and $\ell \leq k$. Let $P^\ell X$ be the space of all invertible ℓ-jets of mappings $X \to \mathbb{R}^n$. Specifically, we construct this as a subset of the n-th fiber product of $K^\ell X$ with itself. If $u : X \to \mathbb{R}^n$ is a C^k mapping such that $u(p) = 0$, the 1-jet of u at a point $p \in X$ is $j^1_p u$ defines an affine mapping of $(\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ to $\mathbb{R}^n$. If this mapping is invertible, then we will say that $j^1_p u$ is an invertible 1-jet. Now, an ℓ jet of $u : X \to \mathbb{R}^n$ is said to be inverible at p if the associated 1-jet is invertible. Then $P^\ell X$ is the space of all invertible ℓ-jets in the n-th fiber product of $K^\ell X$ with itself. Since this is an open condition, $P^\ell X$ inherits the differentiable structure of a $C^{k-\ell}$ manifold from $K^\ell X$. The group $\mathbb{G}^\ell_n$ acts on the left on P^ℓ by composition of jets. The associated bundle $T^\ell X = P^\ell X \times_{\mathbb{G}^\ell_n} T^\ell_n$ is the *bundle of velocities* of order ℓ on X. At any point $p \in X$, the fiber $T^\ell X_p$ consists of all equivalence classes of curves $\gamma : \mathbb{R} \to X$ with $\gamma(0) = p$, where two curves γ_1 and γ_2 are equivalent if they have ℓ-th order contact at 0: $j^\ell\gamma_1(0) = j^\ell\gamma_2(0)$

Let $\pi_E : E \to X$ be a fibered manifold of rank r. Let $\pi^n_{n+r} : \mathbb{R}^{n+r} \to \mathbb{R}^n$ be the projection of $\mathbb{R}^{n+r}$ onto the first n coordinates. Define the bundle $P^\ell\pi_E$ to be the subbundle of $P^\ell E$ consisting of all ℓ-jets of functions $E \to \mathbb{R}^{n+r}$ that locally cover a morphism of fibered manifolds. More precisely, for $p \in E$, $(P^\ell\pi_E)_p$ consists of all jets σ in $P^\ell E_p$ such that

$v(j^\ell \pi^n_{n+r} \circ \sigma) = 0$ for all vertical vectors $v \in (\ker d\pi_E)_p$. This is a closed submanifold of $P^\ell E$ and is a principal bundle for the reduced structure group $\mathbb{G}^\ell_{n,r}$.

2.5 VECTOR FIELDS AND DIFFERENTIAL EQUATIONS

2.5.1 Vector fields

Let $(X, \mathscr{O}_X)$ be a C^k manifold, and for $\ell \leq k-1$, and $\mathscr{X}_{X,\ell}$ be the sheaf of derivations of $\mathscr{O}_X$-modules $\mathscr{O}_X \to \mathscr{O}_{X_\ell}$.The sheaves $\mathscr{X}_{X,\ell}$ for $0 < \ell \leq k-1$ are a decreasing sequence of nested subsheaves of $\mathscr{X}_{X,0}$. The commutator of two derivations of $\mathscr{O}_X$ modules $D, D' \in \mathscr{X}_{X,\ell}$ given by $[D, D'] = D \circ D' - D' \circ D$ lies in $\mathscr{X}_{X,\ell-1}$. With this bracket, each of the sheaves $\mathscr{X}_{X,\ell}$ is a sheaf of Lie algebroids. When $k = \infty$, $\mathscr{X}_X$ is a sheaf of Lie algebras, being closed under the bracket.

The following lemma collects some basic facts about derivations:

Lemma 10. *Let $(X, \mathscr{O}_X)$ be a C^k manifold, $U \subset X$ an open set, and $D \in \mathscr{X}_{X,k-1}(U)$.*

(a) If c is a constant then $D(c) = 0$.
(b) If $f \in \mathscr{O}_X(U)$ and $\operatorname{supp}(f) \subset K$, *then* $\operatorname{supp}(D(f)) \subset K$.

Proof. (a) Note that $D(1) = 0$ for any derivation D, since $D(1) = D(1^2) = 2D(1)$, and so $D(c) = 0$ for any constant c.

(b) We will show that, for any open set $V \subset\subset U \setminus K$, $D(f)|V = 0$. Let $\phi \in \mathscr{O}_X(U)$ be a function that vanishes identically on V and is equal to one on K. Then $\phi f = f$ and so $D(f) = \phi D(f) + f D\phi$. On restriction to V, the right-hand side vanishes.

□

If $U \subset X$ is open, and $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is a C^k coordinate system, then for $i = 1, \ldots, n$, the differential operator $\partial/\partial x^i : \mathscr{O}_X(U) \to \mathscr{O}_{X_{k-1}}(U)$ defined by

$$\frac{\partial}{\partial x^i} f = (\partial_i (f \circ \phi^{-1})) \circ \phi$$

is a vector field. More generally, a linear combination of the form

$$D = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

is in $\mathscr{X}_{X,k}(U)$ provided that all $v^i \in \mathscr{O}_{X_{k-1}}(U)$.

Theorem 7. *Let U be an open subset of the C^k manifold X and $D \in \mathscr{X}_X(U)$. For any $x \in U$, there is a neighborhood V and C^k local coordinates $(x_1, \dots, x_n) : V \to \mathbb{R}^n$ and coefficients $v^i \in \mathscr{O}_{X_{k-1}}(U)$ such that*

$$D = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}. \tag{2.5}$$

It is sufficient to work with vector fields in $\mathbb{R}^n$. It follows by Peetre's theorem that it is sufficient to verify that D has the form (2.5) when evaluated against *smooth* functions. Define $v^i = D(x^i)$. If $f \in C^\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, then by Hadamard's lemma, there is a smooth function $G : \mathbb{R}^n \to (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ such that $f(x) - f(x_0) = \langle df, x - x_0 \rangle + \langle G(x), (x - x_0) \otimes (x - x_0) \rangle$. Applying D to both sides and evaluating at x_0 gives, invoking the derivation property and Lemma 10 (a), that

$$(Df)(x_0) = D(\langle df, x - x_0 \rangle),$$

as required.

2.5.2 Fundamental theorem of ordinary differential equations

Let $\mathbb{X}$ and $\mathbb{P}$ be Banach spaces, $A \subset \mathbb{R} \times \mathbb{X} \times \mathbb{P}$ an open set containing a point (t_0, x_0, p_0), and $f : A \to \mathbb{X}$ a C^k function ($k \geq 1$). We shall here consider the following initial value problem for a function $x : (t_0 - \omega, t_0 + \omega) \to \mathbb{X}$:

$$\begin{aligned} &x'(t) = f(t, x(t), p) \quad \text{for all } t \in (t_0 - \omega, t_0 + \omega) \\ &x(t_0) = y. \end{aligned} \tag{2.6}$$

(Here $p \in \mathbb{P}$ is a parameter, and y is the initial condition.)

Theorem 8. *Then there exists $\omega > 0$ and open neighborhoods U of x_0 and V of p_0 such that (2.6) has a unique solution in the interval $[t_0 - \omega, t_0 + \omega]$ for each $y \in U$ and $p \in V$. Moreover, the mapping $(t, y, p) \mapsto x(t, y, p)$ is C^k on $(t_0 - \omega, t_0 + \omega) \times U \times V$.*

There is a maximal open subset of A on which this unique solution can be continued.

A proof, which can be found in [86], goes as follows. Write $z(s) = x(t_0 + \omega s) - y$ for all $s \in [-1, 1]$. The problem (2.6) reduces to solving

$$
\begin{aligned}
z'(s) - \omega f(t_0 + \omega s, z(s) + y, p) = 0 \quad &\text{for all } s \in [-1.1] \\
z(0) = 0.&
\end{aligned}
\tag{2.7}
$$

Introduce the Banach spaces $\mathbb{Z} = \{x \in C^1([-1, 1], \mathbb{X}) \mid z(0) = 0\}$ and $\mathbb{W} = C([-1, 1], \mathbb{X})$. Then (2.7) has the form

$$F(z, \omega, t_0, y, p) = 0$$

where $F : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times U \times V \to \mathbb{W}$. At the point $Q = (0, 0, t_0, y, p)$ we have

$$D_z F(Q)z = z'.$$

In particular, $D_z F(Q) : \mathbb{Z} \to \mathbb{W}$ is a bijection, since to each $w \in W$, there is a unique z satisfying $z' = w$, namely $z(s) = \int_0^s w(t)\,dt$. Thus the conditions of the Hildebrandt–Graves [39] implicit function theorem are met, and the level set of (2.7) in a neighborhood of Q can be written as a C^k graph $z(\cdot) = z(\cdot; \omega, t_0, y, p)$.

Lemma 11. *Let $V \in \mathscr{X}_X$ be a vector field on X. For each $p \in X$, there is a neighborhood U of p, an $\epsilon > 0$, and a C^{k-1} function $\phi : (-\epsilon, \epsilon) \times U \to X$ such that the following hold:*

(a) For each fixed $t \in (-\epsilon, \epsilon)$, $\phi(t, \cdot) : U \to X$ is a C^{k-1} diffeomorphism onto its image.

(b) $\phi(0, \cdot)$ is the identity diffeomorphism $U \to U$.

(c) If $|s|, |t|, |s + t| < \epsilon$ and $x, \phi(t, x) \in U$, then

$$\phi(s + t, x) = \phi(s, \phi(t, x)).$$

(d) For any $f \in \mathscr{O}_X(U)$, $f \circ \phi(t, \cdot)$ is C^k in t and

$$V(f) = \frac{d}{dt} f \circ \phi(t, \cdot)\bigg|_{t=0}.$$

The function ϕ is called the *flow of the vector field* V, and is denoted by $\phi(t, \cdot) = \mathrm{Fl}_V^t$.

A proof can be found, for instance, in Spivak [78]. Since the problem is local, it is sufficient to work in $\mathbb{R}^n$ with the vector field $V = \sum_i v^i \partial/\partial x^i$. The function $\phi = (\phi^1, \ldots, \phi^n)$ is obtained as a solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\phi^i(t, x) = v^i(\phi(t, x)) \\ \phi(0, x) = x. \end{cases}$$

The initial value problem locally has a C^{k-1} solution. The local semigroup property (b) follows by differentiating both sides and invoking uniqueness of the solution.

There is no way to obtain one degree more of differentiability in the theorem. More precisely, it is only in the direction of v that an additional order of differentiability is obtained, while the order of differentiability in directions transverse to v is in general unaffected. For example, let $f : \mathbb{R} \to \mathbb{R}$ be a C^0 function, and $v = f(x)\partial/\partial y$ be a vector field in $\mathbb{R}^2$. Then the flow of v is $\phi(t, x, y) = (x, y + f(x)e^t)$ which is not C^1 as a function of (t, x, y), but is smooth for fixed x as a function of y, t.

2.5.3 Lie derivatives

Let $V \in \mathscr{X}_X(U)$ and $\alpha \in \Omega^r_X(U)$. Define $\mathscr{L}_V\alpha \in \Omega^r_{X_{k-1}}(U)$ by

$$\mathscr{L}_V\alpha = d(V \lrcorner \alpha) + V \lrcorner d\alpha.$$

This agrees with the limit

$$\mathscr{L}_V\alpha = \lim_{t\to 0} \frac{\phi_t^*\alpha - \alpha}{t}$$

where ϕ_t is the flow of V.

If $V_1, V_2 \in \mathscr{X}_{X,\ell}(U)$, then define $[V_1, V_2] \in \mathscr{X}_{X,\ell-1}(U)$ on functions $f \in \mathscr{O}_X(U)$ by

$$[V_1, V_2](f) = V_1(V_2(f)) - V_2(V_1(f)).$$

We also set $\mathscr{L}_{V_1}V_2 = [V_1, V_2]$. This measures the failure of the derivations V_1 and V_2 to commute. In fact, if $\phi_{1,t}$ is the flow of V_1 and $\phi_{2,s}$ is the flow of V_2, then acting on functions

$$\phi^*_{1,t}\phi^*_{2,s}\phi^*_{1,-t}\phi^*_{2,-s} = \mathrm{Id} + st[V_1, V_2] + o(s, t).$$

2.6 FROBENIUS THEOREM

Definition 6. *Let $(X, \mathscr{O}_X)$ be a C^k manifold, $k \geq 2$. If $\mathcal{I} \subset \Omega_X^\bullet$ is a homogeneous ideal, then denote by $\mathcal{I}^k$ the degree k part of $\mathcal{I}$.*

- *The sheaf of ideals $\mathcal{I}$ is called a* differential ideal *if $d\mathcal{I} \subset \mathcal{I} \otimes_{\mathscr{O}_{X_{k-1}}} \mathscr{O}_{X_{k-2}}$.*
- *The sheaf of ideals $\mathcal{I}$ is called* completely integrable *if each point $x_0 \in X$ has an open neighborhood U such that $\mathcal{I}(U)$ is generated by the differentials of some set of functions in $\mathscr{O}_X(U)$.*

If an ideal is completely integrable, then it is differential. The converse is true for ideals generated by one-forms:

Theorem 9. *Every differential ideal $\mathcal{I}$ that is generated by r linearly independent one-forms at each point is completely integrable.*

For every $p_0 \in X$, there is a neighborhood U together with a local coordinate system $(x^1, \ldots, x^{n-r}, y^1, \ldots, y^r)$ such that $\mathcal{I}^1(U)$ is spanned by one-forms

$$\omega^i = dy^i + \sum_{\alpha=1}^{n-r} f_\alpha^i dx^\alpha, \qquad i = 1, \ldots, r.$$

The condition on the ω^i ensures that

$$\frac{\partial f_\alpha^i}{\partial x^\beta} + \sum_{j=1}^{r} \frac{\partial f_\alpha^i}{\partial y^j} f_\beta^j$$

is symmetric on α and β.

Thus the theorem is a consequence of the following considerations. Let E, F be Banach spaces and $U \subset E \times F$ an open set. Let $f : U \to L(E, F)$ be a C^1 function with values in the Banach space of continuous linear maps $E \to F$. If x is a variable in the space E and y is a variable in F, then a *solution* of the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{2.8}$$

in an open set $V \subset E$ is a function $\phi : V \to F$ such that $D\phi(x) = f(x, \phi(x))$ for all $x \in V$.

Lemma 12. *For a given $(x_0, y_0) \in U$, there exists a neighborhood V of x_0 in E and a unique C^1 solution ϕ of the equation*

$$\frac{dy}{dx} \cdot (x - x_0) = f(x, y) \cdot (x - x_0) \tag{2.9}$$

in V such that $\phi(x_0) = y_0$. The solution to this equation is a solution of (2.8) in V if and only if for every x in V the element

$$D_1 f(x, \phi(x)) + D_2 f(x, \phi(x)) f(x, \phi(x)) \tag{2.10}$$

of $L^2(E, F)$ in symmetric (under the canonical identification of $L(E, L(E, F))$ with the space of bilinear F-valued forms on E).

The proof of the lemma has two steps. The first step is to show that (2.9) has a unique solution in a neighborhood, by using the fundamental existence and uniqueness theorem for ordinary differential equations. The second step is then to show that the integrability condition (2.10) guarantees that the solution for (2.9) is actually a solution of (2.8). For a complete proof, refer to H. Cartan [14].

Definition 7. *An $\mathscr{O}_X$-subalgebroid $\mathcal{A}$ of $\mathscr{X}_X$ is a sheaf of $\mathscr{O}_X$-modules contained in $\mathscr{X}_X$ such that $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A} \otimes_{\mathscr{O}_X} \mathscr{O}_{X_{k-1}}$.*

A distribution D of rank r on X is an r-dimensional subbundle of TX. An integral manifold of D is an r-dimensional immersed submanifold $i : Y \subset X$ such that $di(TY) = D$.

Theorem 10. *Let D be a rank r distribution such that $\Gamma(D)$ be an $\mathscr{O}_X$-subalgebroid of $\mathscr{X}_X$. Then through each $x \in X$, there is a unique maximal integral r-dimensional C^{k-1} submanifold of X.*

2.7 GROWTH VECTOR OF A DISTRIBUTION

Let $(X, \mathscr{O}_X)$ be a C^k manifold, $k \geq 1$, of dimension n. Let $\mathscr{H} \subset \mathscr{X}_X$ be a locally free subsheaf of $\mathscr{O}_{X_{k-1}}$-modules. Let $\mathscr{H}^1 = \mathscr{H}$ and $\mathscr{H}^{r+1} = [\mathscr{H}, \mathscr{H}^r]$. For $p \in X$, let $n_r(p)$ be the dimension of the $\mathscr{O}_{X_{k-r},p}$-module $\mathscr{H}^r_p$. The *growth vector* (see [57]) of the sheaf $\mathscr{H}$ at the point p is the vector-valued function $(n_1(p), \ldots, n_r(p))$. Note that $n_i \leq n_{i+1}$ since $\mathscr{H}^{i+1} \supset \mathscr{H}^i \otimes_{\mathscr{O}_{X_{k-i}}} \mathscr{O}_{X_{k-i-1}}$.

Of special interest is when the sheaf $\mathscr{H}$ generates the entire Lie algebroid of vector fields on X. That is, for each point $p \in X$, there exists an r (depending on p) such that $n_r(p) = n$.

The special case of such structures having growth vector $(2, 3, 5)$ were studied by Cartan [13], and later in the context of general relativity in [17].

A *contact structure* is a distribution $\mathscr{H}$ of hyperplanes such that $n_1 = n - 1$, $n_2 = n$, and such that for $v, w \in \mathscr{X}_X$,

$$(v, w) \mapsto [v, w] \pmod{\mathscr{H} \otimes_{\mathscr{O}_X} \mathscr{O}_{X_{k-1}}}$$

defines a non-singular pairing into $\mathscr{X}_{X_{k-1}}/\mathscr{H} \otimes_{\mathscr{O}_X} \mathscr{O}_{X_{k-1}}$ at each point. A contact structure is equivalently specified by a nonzero differential one-form α on X that annihilates the distribution $\mathscr{H}$. The one-form is determined up to scaling by a nonzero function, and a particular choice of such one-form is called a *contact form*. The non-singularity condition implies that $d\alpha|_{\mathscr{H}}$ has rank $n - 1$ at every point. Being a skew form, this can only occur if n is odd, say $n = 2m + 1$. In that case, the non-singularity condition is equivalent to the requirement

$$\alpha \wedge (d\alpha)^m \neq 0.$$

2.8 POINCARÉ'S LEMMA

Let X be a C^k manifold. Let $i_t : X \to X \times [0, 1]$ be the inclusion of X in the $X \times \{t\}$ slice. Define the fiber integration operator I from differential p-forms on $X \times [0, 1]$ to differential $(p-1)$-forms on X by integration of α up the fiber of $X \times [0, 1] \to X$.

Lemma 13 ([78]). *I is a chain homotopy from $i_0^*\alpha$ to $i_1^*\alpha$, whenever α is a p-form on $X \times [0, 1]$:*

$$i_1^*\alpha - i_0^*\alpha = d(I\alpha) + I(d\alpha)$$

Proof.

$$\begin{aligned} i_1^*\alpha - i_0^*\alpha &= \int_0^1 \frac{d}{dt} i_t^*\alpha \wedge dt \\ &= \int_0^1 (\mathscr{L}_{\partial/\partial t}\alpha) \wedge dt \\ &= \int_0^1 \left(\frac{\partial}{\partial t} \lrcorner d\alpha\right) \wedge dt + \text{‘} \int_0^1 \left(d\frac{\partial}{\partial t} \lrcorner \alpha\right) \wedge dt \\ &= I(d\alpha) + d(I\alpha). \end{aligned}$$

□

In particular, we have the Poincaré lemma:

Theorem 11. *Suppose that X is smoothly contractible to a point. Then any closed p-form on X is exact, for $p > 0$.*

Proof. Let β be the closed form in question, and ϕ_t, $0 \le t \le 1$, be a smoothly parameterized family of maps such that $\phi_0(X) = \{\star\}$ and $\phi_1 = \mathrm{id}$. Define α on $X \times [0,1]$ uniquely by the requirements $i_t^*\alpha = \phi_t^*\beta$, $\frac{\partial}{\partial t} \lrcorner \alpha = 0$. Then α is closed and $i_0^*\alpha = 0$. So by the chain homotopy lemma,

$$\beta = d(I\alpha).$$

□

2.9 DARBOUX RANK

Definition 8. *Let α be a one-form on a manifold X of dimension n.*

- *If $\alpha \wedge (d\alpha)^m \neq 0$ and $(d\alpha)^{m+1} = 0$, then α is said to have* Darboux rank $2m+1$.
- *If $(d\alpha)^m \neq 0$ and $\alpha \wedge (d\alpha)^m = 0$, then α is said to have* Darboux rank $2m$.

Examples.

1. If $n = 2m+1$ and α has Darboux rank m, then α is a contact form on X.
2. If $n = 2m$ and α has Darboux rank m, then $d\alpha$ defines a *symplectic form* on X. In that case α is called a *symplectic potential.*
3. The one form $dx^0 + y_1\, dx^1 + \cdots + y_m\, dx^m$ has Darboux rank $2m+1$ whenever $(x^0, x^1, \dots, x^m, y_1, \dots, y_m)$ are $2m+1$ independent functions.
4. Likewise, the one form $y_1\, dx^1 + \cdots + y_m\, dx^m$ has Darboux rank $2m$ whenever $(x^1, \dots, x^m, y_1, \dots, y_m)$ are $2m$ independent functions.

Definition 9. *Let α be a contact form on X. Then $\alpha \wedge (d\alpha)^m$ trivializes the top exterior power of the bundle T^*M. So there exists a vector field V canonically dual to the $2m$-form $(d\alpha)^{2m}$. This vector field is called the* Reeb vector field *associated to the contact form α, and satisfies $V \lrcorner \alpha = 1$, $V \lrcorner d\alpha = 0$.*

The following theorem is due to Darboux [16]

Theorem 12. *Let X be a C^k manifold of dimension n and α a one-form of constant Darboux rank on X. If the rank of α is $2m$ (even), then in a neighborhood of any point of X, there exist $2m$ independent functions $(x^1, \dots, x^m, y_1, \dots, y_m)$ such that $\alpha = y_1\, dx^1 + \cdots + y_m\, dx^m$. Otherwise, if the rank of α is $2m+1$, then in a neighborhood of any point of X, there exist*

$2m+1$ *independent functions* $(x^0, x^1, \ldots, x^m, y_1, \ldots, y_m)$ *such that* $\alpha = dx^0 + y_1\, dx^1 + \cdots + y_m\, dx^m$.

The following lemma is due to Moser [58]:

Lemma 14. *Let* X *be a smooth manifold,* $x \in X$*, and let* $\omega_t, -T \leq t \leq T$*, be a smoothly parameterized one-parameter family of symplectic forms on a manifold* X *such that the following conditions are met:*

1. $\mathrm{ev}_x\, \omega_t$ *is independent of* t
2. $\frac{d}{dt}\omega_t$ *is exact*

Then there is a neighborhood U *of* x *and a one-parameter family of diffeomorphisms* $\phi_t : U \to \phi_t(U) \subset X$ *such that* $\phi_t(x) = x$ *and* $\phi_t^* \omega_t = \omega_0$*.*

Proof. Write $\frac{d}{dt}\omega_t = d\theta_t$ and let V_t be the unique vector field such that $V_t \lrcorner \omega_t = \theta_t$. By the existence and uniqueness theorem for differential equations, there exists a neighborhood U of x and a unique one-parameter family of local diffeomoprhisms ϕ_t (depending smoothly on t) such that $\phi_0 = \mathrm{id}$ and

$$\frac{d}{dt}\phi_t = d\phi_t(V_t).$$

(A compactness argument shows that such a solution exists for all t in the interval $[-T, T]$, by adjusting the size of the neighborhood U as needed.) Hence,

$$\frac{d}{dt}\phi_t^* \omega_t = \phi_t^* \left(\frac{d}{dt}\omega_t + \mathscr{L}_{V_t}\omega_t \right) = 0.$$

So $\phi_t^* \omega_t = \omega_0$ for all t. $\square$

Corollary 3. *Let* X *be a symplectic manifold of dimension* $n = 2m$ *and* $x \in X$*. Then there there is a neighborhood* U *of* x *and* n *independent functions* $(y_1, \ldots, y_m, x^1, \ldots, x^m)$ *on* U *such that* $\omega | U = \sum_i dy_i \wedge dx^i$

Proof. Since the problem is local, we can work in $\mathbb{R}^n$ and take $x = 0$. Choose a symplectic frame in the tangent space at x. This symplectic frame defines linear coordinates $(x^1, x^2, \ldots, x^m, y_1, \ldots, y_m)$ on $\mathbb{R}^n$, and so a symplectic structure $\omega_0 = \sum_i dy_i \wedge dx^i$. Let $\omega_t = t\omega + (1-t)\omega_0$. Then in a sufficiently small neighborhood of x, ω_t satisfies the hypotheses of Lemma 14. $\square$

Proof of Theorem 12. The strategy of the proof is to reduce to the symplectic case, when $n = 2m$ and α has rank $2m$, and then invoke the corollary.

Even rank: If the rank of α is even, let $\mathscr{H} = \ker d\alpha$ be the set of vector fields satisfying $V \lrcorner d\alpha = 0$. Note that $\mathscr{H}$ is a distribution on X of dimension $n - 2m$. Moreover, because $\alpha \wedge (d\alpha)^m = 0$, the kernel of α is contained in the kernel of $d\alpha$. It follows that $\mathscr{H}$ is involutive: $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. By the Frobenius theorem around any point p of X we can decompose a suitable neighborhood of p into a product $U = \mathbb{R}^{2m} \times \mathbb{R}^{n-2m}$ with $\mathscr{H}$ the vertical distribution for the projection onto the first factor. Also, since $\mathscr{H}$ annihilates α and α Lie derives to zero along any section of $\mathscr{H}$, α itself is a pullback of a one-form on the first factor $\mathbb{R}^{2m}$. This form is then a symplectic form.

Odd rank: Let $\mathscr{H} = \ker \alpha \cap \ker d\alpha$. Then $\mathscr{H}$ is an involutive distribution of dimension $n - 2m - 1$. As is the even case, we split a neighborhood of a point p in X as $U = \mathbb{R}^{2m+1} \times \mathbb{R}^{n-2m-1}$ and α descends to a contact form on the first factor. Hence we may as well assume that α was a contact form to begin with. In that case, let V be the Reeb vector field of α. Then α is also Lie derived along V, so α descends to a symplectic form on the quotient of U by the vector field V.

In either case, it is sufficient to establish that, if α is a symplectic potential on a manifold X of dimension $n = 2m$, then in a neighborhood of each point of X there exist $2m$ independent functions $(x^1, \dots, x^m, y_1, \dots, y_m)$ such that $\alpha = \sum_i y_i dx^i$. Corollary 3 implies that we can choose functions so that

$$d\alpha = \sum_i dy_i \wedge dx^i.$$

So by Poincaré's lemma, $\alpha = df + \sum_i y_i dx^i$ for some function f. By making a linear change in f if necessary, we can assume that $\mathrm{ev}_x\, df = 0$. Then, by the argument of Lemma 14 applied to the one-parameter family $\alpha_t = t\, df + \sum y_i dx^i$, there is a neighborhood of x and a diffeomorphism so that $\phi^*(\sum y_i dx^i) = \alpha$, as required. $\square$

2.10 STRUCTURES ON THE COTANGENT BUNDLE

Let M be a smooth manifold and T^*M its cotangent bundle. Let X denote the total space of the deleted cotangent bundle $T^*M \setminus \{0\}$, with bundle mapping $\pi : X \to M$. The bundle of tensors on X is denoted by $\mathscr{T}X$. The duality pairing between the tangent bundle TX and cotangent bundle T^*X will be denoted by $\lrcorner$, so that if $v \in TX$ and $\alpha \in T^*X$, then $v \lrcorner \alpha$ is a scalar.

Let $E = \pi^{-1}T^*M$ be the pullback of T^*M under π to a bundle over X. Then E is naturally identified with the subbundle $\operatorname{im}\pi^*$ of forms in T^*X that are horizontal with respect to π.[2] Equivalently, E is the subbundle of forms $\alpha \in T^*X$ that factor through $d\pi$: for some $\tilde{\alpha} \in T^*M$

$$\alpha(X) = \tilde{\alpha}(d\pi X), \tag{2.11}$$

for every $X \in TX$. The dual pairing between E and E^*, which is naturally isomorphic to $\pi^{-1}TM$, will be denoted by angle brackets $\langle\cdot,\cdot\rangle_E$. The pairing $\langle\cdot,\cdot\rangle_E : E \times E^* \to \mathbb{R}$ is then extended to tensors in $(\mathscr{T}X \otimes E) \times (\mathscr{T}X \otimes E^*)$ by setting

$$\langle t \otimes e, s \otimes e'\rangle_E = t \otimes s\langle e, e'\rangle_E$$

for all $t, s \in \mathscr{T}X$, $e \in E$, and $e' \in E^*$, and extending by bilinearity. Likewise, we extend the definition of the pairing bilinear pairing $\lrcorner : TX \times T^*X \to \mathbb{R}$ to a bililnear mapping $(TX \otimes E) \times (T^*X \otimes E) \to E \otimes E^*$ by extending bilinearly

$$(v \otimes e)\lrcorner(\alpha \otimes e') = (v \lrcorner \alpha) e \otimes e'.$$

The bundle E has a tautological section (over X) given by $p(x, \alpha) = \pi^*\alpha$ for all $x \in M$ and $\alpha \in T_xM$. This is the diagonal section of E when regarded as the fiber product $E = T^*M \times_M T^*M$.

Let θ be the section of $\Omega^1(X) \otimes E^*$ defined on arbitrary vectors $v \in TX$ by

$$v \lrcorner \theta = d\pi v \in TM.$$

The usual canonical one-form of X is given by

$$\omega = \langle p, \theta\rangle_E.$$

If V is a vector field on M, then define the momentum P_V associated to V to be the function on X given by first lifting V to a vector $\widetilde{V}$ on X and then contracting with the canonical one-form

$$P_V = \widetilde{V} \lrcorner \omega$$

Since ω is horizontal, it is annihilated by vertical directions, and so P_V does not depend on the choice of lift of V.

Let $VX \subset TX$ be the subbundle that is vertical for π; that is $VX = \ker d\pi$. There is a natural isomorphism $VX \cong E$. Abstractly, this identification is represented as a tautological

[2] Throughout the discussion, "naturally" means that there is a natural transformation of bundle functors on the category of smooth manifolds under local diffeomorphism. See [46].

section of $TX \otimes E^*$ denoted by $\partial/\partial p$. This section is equivalently characterized by the pair of equations

$$\frac{\partial}{\partial p} \lrcorner\, \theta = 0$$
$$\frac{\partial}{\partial p} \lrcorner\, dP_V = \pi^{-1}V \in E^*$$

for all vectors V on M. The first of these statements is equivalent to $\frac{\partial}{\partial p} \lrcorner\, \pi^*\alpha = 0$ for all one-forms α on M. Note that dP_X and $\pi^*\alpha$ generate the full cotangent space of X, and so these properties are sufficient to characterize $\partial/\partial p$ by duality.

We define for later use the operator $\mathscr{L}_{\partial/\partial p} : \Gamma\left((E^*)^{\otimes r}\right) \to \Gamma\left((E^*)^{\otimes r+1}\right)$ as follows. On functions (tensors of degree zero), let

$$(\mathscr{L}_{\partial/\partial p} f)(\pi^*\alpha) = \left\langle \frac{\partial}{\partial p} \lrcorner df, \pi^*\alpha \right\rangle_{E^*}.$$

Then, on tensors of degree r, for all $\alpha_1, \alpha_2, \ldots, \alpha_{p+1} \in \Gamma(T^*M)$, put

$$\left(\mathscr{L}_{\partial/\partial p} T\right)(\pi^*\alpha_1, \pi^*\alpha_2, \ldots, \pi^*\alpha_{p+1}) = \mathscr{L}_{\partial/\partial p}[T(\pi^*\alpha_1, \pi^*\alpha_2, \ldots, \pi^*\alpha_p)](\pi^*\alpha_{p+1}).$$

We observe that this defines a tensor of degree $r+1$: this follows since $\partial/\partial p$ acts purely vertically.

2.10.1 Poisson brackets

The manifold X is a symplectic manifold, equipped with the symplectic form $d\omega$ where $\omega = \langle p, \theta \rangle_E$ is the canonical one-form. This is a nonsingular two-form in $\wedge^2 T^*X$, and so it has an inverse, denoted $d\omega^{-1}$, in $\wedge^2 TX$. If f, g are two functions, then define the Poisson bracket of f and g by

$$\{f, g\} = (d\omega)^{-1}(df, dg).$$

In local coordinates,

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x^i}.$$

The Poisson bracket $\{-,-\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ is a skew-symmetric real-bilinear mapping that defines derivation in either argument, with respect to the structure of $C^\infty(X)$ as a multiplicative ring:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

and that satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$$

2.11 STRUCTURES ON THE TANGENT BUNDLE

Let M be a smooth manifold of dimension $n \geq 2$. The tangent bundle TM of M consists of pairs (x, v) with $x \in M$ and $v \in TM_x$, the tangent space to M at x. The bundle projection $\pi_{TM} : TM \to M$ is defined by $\pi_{TM}(x, v) = x$. The double tangent bundle TTM is the tangent bundle of the tangent bundle, and consists of triples (x, v, w) where $(x, v) \in TM$ and $w \in TTM_{(x,v)}$. The bundle projection $\pi_{TTM} : TTM \to TM$ is defined by $\pi_{TTM}(x, v, w) = (x, v) \in TM$.

In local coordinates x^i of M, there are induced linear coordinates v^i in each fiber of TM, defined by

$$v = v^i(v) \left.\frac{\partial}{\partial x^i}\right|_x .$$

Then TTM also carries fiber coordinates in the $2n$-dimensional space $TTM_{(x,v)}$, denoted by ξ^i, ν^i, defined at $w \in TTM_{(x,v)}$ by

$$w = \xi^i(w) \left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + \nu^i(w) \left.\frac{\partial}{\partial v^i}\right|_{(x,v)} .$$

Apart from the bundle projection $\pi_{TTM} : TTM \to TM$ on the second tangent bundle, there is also another natural projection given by the differential $d\pi_{TM} : TTM \to TM$. In the local coordinates described above,

$$\begin{aligned} d\pi_{TM}\frac{\partial}{\partial x^i} &= \frac{\partial}{\partial x^i} \\ d\pi_{TM}\frac{\partial}{\partial v^i} &= 0. \end{aligned}$$

The kernel of $d\pi_{TM}$ is called the *vertical subbundle*, and is denoted by VTM. There is a natural isomorphism between VTM and the pullback bundle $\pi_{TM}^{-1}TM$, given as follows. Let $x \in M$ and $v, w \in TM_x$. The one-parameter group $L_w(s) : (x, v) \mapsto (x, v + sw)$ as $s \in \mathbb{R}$ varies, is a well-defined one-parameter group of diffeomorphisms of TM_x to itself. Denote the generator of this one-parameter group by $\overline{\lambda}_{(x,v)}(w) = L'_w(0)$. Then $\overline{\lambda}_{(x,v)} : TM_x \to VTM_{(x,v)}$. This is a linear isomorphism for each fixed $(x, v) \in TM$, and it depends smoothly on (x, v). So it is an isomorphism $\overline{\lambda} : \pi_{TM}^{-1}TM \to VTM$ of vector bundles over TM. In coordinates,

$$\overline{\lambda}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial v^i}.$$

Since $\pi_{TM}^{-1}TM = TTM/VTM$, it is convenient to compose $\overline{\lambda}$ with the quotient map $q : TTM \to TTM/VTM$ to obtain $\lambda = \overline{\lambda} \circ q : TTM \to VTM$. Then the image and kernel of λ are both the vertical bundle VTM. As a tensor, λ can be identified with a section of $V^0TM \otimes VTM$, In coordinates,

$$\lambda\frac{\partial}{\partial x^i} = \frac{\partial}{\partial v^i}, \qquad \lambda\frac{\partial}{\partial v^i} = 0$$

and, as a tensor, $\lambda = dx^i \otimes \frac{\partial}{\partial v^i}$.

Let X be a vector in TTM. Define a differential operator $D_X : C^\infty(TM) \to C^\infty(TM)$ by

$$D_X(f) = \mathscr{L}_{\lambda X} f.$$

In local coordinates, if $X = \xi^i\frac{\partial}{\partial x^i} + \nu^i\frac{\partial}{\partial v^i}$, then $D_X(f) = \xi^i\frac{\partial f}{\partial v^i}$. Let $D : C^\infty(TM) \to \Gamma_{TM}(T^*TM)$ be the one-form valued operator

$$(Df)(X) = D_X f.$$

If $X, Y \in \Gamma_{TM}(\pi_{TM}^{-1}TM)$ are two vector fields that are lifts of vector fields from M, then

$$D^2_{X,Y} = D_X D_Y = D_Y D_X,$$

and $D^2_{X,Y}(f)$ depends bilinearly on X, Y. Commutativity follows from the commutativity of the one parameter groups L_X and L_Y defined previously.

Let TM' be the tangent bundle with the zero section removed, and $\pi_{TM'} = \pi_{TM}|_{TM'} : TM' \to M$ the induced projection of TM' onto M. Let SM be the space of oriented one-dimensional linear subspaces of TM. Let $\delta_s : TM' \to TM'$ be the scaling $\delta_s(x, v) = (x, sv)$ for $s > 0$, and let $H = \frac{d}{ds}\delta_s|_{s=1}$ be the homogeneity vector field. This defines a group action of $(0, \infty)$ on TM', and SM is the quotient bundle of TM' by the group. Let $\pi_{SM} : SM \to M$ be the projection onto M. There is a factorization $\pi_{TM'} = \pi_{SM} \circ \sigma$ where $\sigma : TM' \to SM$ is the quotient mapping.

2.11.1 Frölicher–Nijenhuis bracket

Let X be a smooth manifold and $\Omega(X) = \oplus_r \Omega^r(X)$ be the graded algebra of smooth differential forms on X. A derivation of degree k of $\Omega(X)$ is a real linear map $D : \Omega(X) \to \Omega(X)$ such that

- $D : \Omega^r(X) \to \Omega^{r+k}(X)$
- For any $\alpha \in \Omega^a(X)$ and $\beta \in \Omega^b(X)$, $D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{ka} \alpha \wedge D\beta$

Let $\mathrm{Der}_k(\Omega(X))$ be the space of derivations of degree k of $\Omega(X)$, and let $\mathrm{Der}(\Omega(X)) = \oplus_{k \in \mathbb{Z}} \mathrm{Der}_k(\Omega(X))$ be the graded vector space of all derivations; this supports the structure of a graded Lie algebra, where the bracket of homogeneous elements $K \in \mathrm{Der}_k(\Omega(X)), L \in \mathrm{Der}_\ell(\Omega(X))$ is defined by

$$[K, L] = K \circ L - (-1)^{k\ell} L \circ K.$$

Extending by bilinearity to all of $\mathrm{Der}(\Omega(X))$, the resulting bracket is easily seen to define a graded Lie algebra:

- The bracket is graded anticommutative:

  $$[K, L] = -(-1)^{k\ell} [L, K]$$

  for $K \in \mathrm{Der}_k(\Omega(X)), L \in \mathrm{Der}_\ell(\Omega(X))$
- The bracket satisfies the graded Jacobi identity:

  $$(-1)^{j\ell} [J, [K, L]] + (-1)^{kj} [K, [L, J]] + (-1)^{\ell k} [L, [J, K]] = 0.$$

Let $\Omega^k(X, TX) = \Omega^k(X) \otimes TX$ denote the sheaf of k-forms on X with values in TX. If $v \in \Gamma_X(TX)$ is a vector field, then the insertion operator $i_v : \Omega(X) \to \Omega(X)$ is a derivation of degree -1. The insertion operator extends to an operator $i_K \in \mathrm{Der}_{k-1}(\Omega(X))$, by defining $i_{\omega \otimes v} = \omega i_v$ for $K \in \Omega^k(X, TX)$ extending by linearity.

Definition 10. *Let $K \in \Omega^k(X, TX)$. Define the Lie derivative along K by*

$$\mathscr{L}_K = [i_K, d] = i_K \circ d + (-1)^k d \circ i_K.$$

The following is proven in [46]:

Theorem 13. *Any derivation $D \in \mathrm{Der}_k(\Omega)$ can be decomposed uniquely as*

$$D = \mathscr{L}_K + i_L$$

for some $K \in \Omega^k(X, TX)$ and $L \in \Omega^{k+1}(X, TX)$.

Proof. Restrict D to the subalgebra of smooth functions $D|C^\infty : C^\infty \to \Omega^k$. There exists a unique $K \in \Omega^k(X, TX)$ such that $D|C^\infty = \mathscr{L}_k$. So, by replacing D by $D - \mathscr{L}_K$, we can assume without loss of generality that $D|C^\infty = 0$. Then $D|\Omega^1 : \Omega^1 \to \Omega^k$ is linear with respect to C^∞ functions, and so there is a unique $L \in \Omega^{k+1}$ such that $D|\Omega^1 = i_L$. Since D is uniquely determined by its action on generators of the algebra Ω, this completes the proof. □

Definition 11. *For $K \in \Omega^k(X, TX)$ and $L \in \Omega^\ell(X, TX)$ define the* Nijenhuis–Richardson bracket $[K, L]^\wedge$ *by*

$$i_{[K,L]^\wedge} = [i_K, i_L].$$

Define the Frölicher–Nijenhuis bracket $[K, L]$ *by*

$$\mathscr{L}_{[K,L]} = [\mathscr{L}_K, \mathscr{L}_L].$$

The algebraic properties of the curvature and related quantities are most easily expressed using the Nijenhuis–Richardson bracket [59] and the Frölicher–Nijenhuis bracket [31].

Lemma 15. $[\mathscr{L}_K, i_L] = i_{[K,L]} - (-1)^{k\ell} \mathscr{L}_{i_L K}$

Proof. On functions $[\mathscr{L}_K, i_L]f = \mathscr{L}_{i_L K} f$. Also by the Jacobi identity, $[[\mathscr{L}_K, i_L], d] = \mathscr{L}_{[K,L]}$ (since d graded-commutes with $\mathscr{L}_K$). Hence by Theorem 13, the lemma holds. □

Lemma 16.

$$\begin{aligned}[K, [L_1, L_2]^\wedge] = [[K, L_1], L_2]^\wedge + (-1)^{k\ell_1}[L_1, [K, L_2]]^\wedge - \\ - \left((-1)^{k\ell_1}[i_{L_1}K, L_2] - (-1)^{(k+\ell_1)\ell_2}[i_{L_2}K, L_1]\right)\end{aligned}$$

Proof. By definition of the two brackets,

$$\begin{aligned}\mathscr{L}_{[K,[L_1,L_2]^\wedge]} &= [\mathscr{L}_K, [[i_{L_1}, i_{L_2}], d]] \\ &= [[[\mathscr{L}_K, i_{L_1}], i_{L_2}], d] + (-1)^{k\ell_1}[[i_{L_1}, [\mathscr{L}_K, i_{L_2}]], d] \qquad \text{by the Jacobi identity} \\ &= \mathscr{L}_{[[K,L_1],L_2]^\wedge} - (-1)^{k\ell_1}\mathscr{L}_{[i_{L_1}K,L_2]} + \\ &\quad + (-1)^{k\ell_1}\mathscr{L}_{[L_1,[K,L_2]]^\wedge} + (-1)^{(k+\ell_1)\ell_2}\mathscr{L}_{[i_{L_2}K,L_1]}\end{aligned}$$

by Lemma 15. □

The following is also useful:

Lemma 17. *If $K, L \in \Omega^1(X, TX)$, then, evaluated on vector fields A, B,*

$$\begin{aligned}[K, L](A, B) = [KA, LB] - [KB, LA] - L([KA, B] - [KB, A]) \\ - K([LA, B] - [LB, A]) + (KL + LK)[A, B].\end{aligned}$$

2.12 CONNECTIONS IN FIBERED MANIFOLDS

Let X be a C^k manifold of dimension n. Let $\pi_E : E \to X$ be a C^k fibered manifold, and denote by $V \subset TE$ the vertical bundle $V = \ker d\pi_E$. Then an Ehresmann connection in E is a surjective bundle homomorphism $P : TE \to V$ such that $P \circ P = P$. The kernel of P is the *horizontal distribution* $H \subset TE$, and since the image of P is a projection $TE = H \oplus V$. In particular, for all $e \in E$, $d\pi_{E,e} : H_e \to T_{\pi_E(e)}M$ is an isomorphism of vector spaces. The inverse is called the *horizontal lift* $h_e : T_{\pi_E(e)}M \to H_e$, which defines a mapping of bundles $h : \pi_E^{-1}TM \to TE$. Conversely, if a distribution H of subspaces of TE is given which splits the exact sequence of bundles

$$0 \to V \to TE \xrightarrow{d\pi_E} \pi_E^{-1}T_M \to 0$$

then the associated projection operator $P : TE \to V$ is an Ehresmann connection in the forgoing sense.[3]

For vector fields X, Y on M, the curvature is defined by

$$R(X,Y) = P[h(X), h(Y)] = [h(X), h(Y)] - h([X,Y]).$$

This is bilinear in X and Y and does not depend on how X and Y are extended away from $\pi_E(e)$, and therefore defines a section of

$$\pi_E^{-1} \wedge^2 T^*M \otimes V.$$

By the tensor-hom adjunction, $\mathrm{Hom}(TE, V) \cong T^*E \otimes V$, the operator P can be regarded as a differential form on E with values in the vertical distribution V. If local coordinates x^i are given on M and fiber coordinates γ_α are given in E, then P can be represented in the form

$$P = \omega_\alpha \otimes \frac{\partial}{\partial \gamma_\alpha}$$

where the one-forms ω_α, called the connection forms, are given by

$$\omega_\alpha = d\gamma_\alpha + N_{\alpha j}\, dx^j$$

where $N_{\alpha j}$ are the connection coefficients. In this language, the horizontal lifts of the basic coordinate vector fields are

$$h_{\partial/\partial x^i} = \frac{\partial}{\partial x^i} - N_{\alpha i}\frac{\partial}{\partial \gamma_\alpha}$$

[3] More details on the general theory of Ehresmann connections can be found in [46].

since these satisfy $\omega_\alpha(h_{\partial/\partial x^i}) = 0$ and $d\pi_{E,e}(h_{\partial/\partial x^i}) = (\partial/\partial x^i)_{\pi_E(e)}$, the two conditions that uniquely characterize the horizontal lift.

Since the coordinate vector fields commute, the curvature is given in coordinates by

$$R_{ij} = [h_{\partial/\partial x^i}, h_{\partial/\partial x^j}] = \left(\frac{\partial N_{\alpha i}}{\partial x^j} - \frac{\partial N_{\alpha j}}{\partial x^i} + N_{\beta i}\frac{\partial N_{\alpha j}}{\partial \gamma_\beta} - N_{\beta j}\frac{\partial N_{\alpha i}}{\partial \gamma_\beta}\right)\frac{\partial}{\partial \gamma_\alpha}.$$

There is an alternative characterization of Ehresmann connections that we shall sometimes employ. Let $J^1\pi_E$ denote the space of 1-jets of local sections of $\pi_E : E \to M$. This space is naturally fibred over E, for if $s : U \to E$ is a the germ of a section of π_E near a point $x \in M$, then $j^1_x s \mapsto s$ gives a fibration $\pi^1_E : J^1\pi_E \to E$. An Ehresmann connection corresponds in a natural way to a section $\sigma : E \to J^1\pi_E$ of this fibration, which we shall denote by $e \mapsto \sigma_e$. Indeed, the horizontal lift $h : \pi_E^{-1}TM \to TE$ is then defined by $h_e(X) = d\sigma_e(X)$.

Conversely, let $\mathrm{Gr}_n(TE)$ denote the Grassman bundle of n-planes in TE. This is naturally a fibre bundle over E with fibre $\mathrm{Gr}_n(\mathbb{R}^{n+\dim V})$. Let $\mathrm{Gr}'_n(TE)$ be the subbundle consisting of horizontal subspaces: those that intersect $V = \ker d\pi_E$ only in the zero subspace. This is an open subset of the full Grassman bundle and so carries a compatible smooth structure. The projection P of an Ehresmann connection is determined by the splitting $TE = H + V$, which in turn is given by a section H of $\mathrm{Gr}'_n(TE)$ over E. There is an isomorphism of bundles over E between $J^1\pi_E$ and $\mathrm{Gr}'_n(TE)$: if $\sigma : U \subset M \to TE$ is a local section in a neighborhood U of a point $x \in M$, then $d\sigma(T_xM) \subset T_{\sigma(x)}E$ is a horizontal subspace which depends only on the jet $j^1_x\sigma$. This defines a natural mapping $J^1\pi_E \to \mathrm{Gr}'_n(TE)$ that associates to each 1-jet of a section at x its tangent space at x. This mapping is a smooth immersion (by looking in local coordinates). Furthermore, it is surjective since any n-plane in $T_{x,e}E$ that does not meet $V_{x,e}$ is tangent to a local section. So $J^1\pi_E \cong \mathrm{Gr}'_n(TE)$. Since a distribution of horizontal subspaces on E is just a section of $\mathrm{Gr}'_n(TE)$, this establishes the converse.

2.12.1 Basic example: Linear connections in vector bundles

Suppose that $\pi_E : E \to M$ carries the structure of a vector bundle. Let $\pi_{TE} : TE \to E$ denote the projection in the tangent bundle of the total space of E. Since E is a vector bundle, affine translation in the fibre gives a canonical identification of the vertical tangent space with the fibre at each point. That is, there is an isomorphism of bundles

$$\nu : \pi_{TE}^{-1}E \to \ker(d\pi_E) \subset TE.$$

A linear connection in E is usually thought of as a mapping $\nabla : \Gamma(TM \times_M E) \to \Gamma(E)$ which, with respect to the $C^\infty(M)$-module structure, is linear in its first argument and is a derivation in its second argument. The action of ∇ on a section (X, σ) is usually denoted by $\nabla_X \sigma$. If $\sigma \in \Gamma(E)$ is a section of E and X is a vector at a point $x_0 \in M$, then $\nabla_X \sigma(x_0) \in E_{x_0}$ does not depend on how X is extended to a vector field away from x_0. Furthermore, since ∇ is a derivation in its second argument, it factors through the 1-jet prolongation to define a mapping of fibre bundles

$$\nabla : TM \times_M J^1 E \to E$$

which is linear in its first argument and a derivation with respect to the $J^1 M$-module structure in its second arguement.

The horizontal distribution can be defined directly in terms of horizontal curves in E. Let x_t be a smooth curve through x_0 with $\dot{x}_0 = X$. Fix $\sigma_0 \in E_{x_0}$. There is a unique curve $\mu(t)$ through σ_0 that lifts x_t such that

$$\nabla_{\dot{x}_t} \mu = 0.$$

The horizontal lift of X is then defined by $h(X) = \mu'(0)$. Alternatively, call a section σ *horizontal at* $\sigma_0 \in E_{x_0}$ if $\nabla_X \sigma(x_0) = 0$ for every x. The horizontal lift to $T_{\sigma_0} E$ is the operator $d\sigma$ for any horizontal section at σ_0. This also furnishes a description of the entire horizontal subspace:

$$H_{\sigma_0} = d\sigma(T_{x_0} M)$$

where σ is any horizontal section at σ_0.

A more convenient approach, that will allow us to introduce some notation, is to define the horizontal lift of a vector field X on M directly as a vector field on E, which we risk confusion and denote by ∇_X. If $\sigma : M \to E$ is a section and $f : E \to \mathbb{R}$ is a smooth function, then define

$$(\nabla_X f) \circ \sigma = X(f \circ \sigma) + \langle df, \nu(\nabla_X \sigma) \rangle$$

where the pairing is the duality between $T^* E$ and TE.

Given a frame e_α of E, the fibre coordinates are the dual coframe $e^\alpha : E \to \mathbb{R}$. Given a coordinate system x^i on M, define the connection coefficients $\Gamma^j_{\alpha i}$, by

$$\nabla_{\partial/\partial x^i} e_\alpha = \Gamma^\beta_{\alpha i} e_\beta.$$

Then

$$\nabla_{\partial/\partial x^i} = \frac{\partial}{\partial x^i} + \Gamma^\beta_{\alpha i} e^\alpha \frac{\partial}{\partial e^\beta}.$$

2.12.2 Ehresmann connections, principal connections, and Cartan connections

Suppose that $E \to M$ is a fiber bundle with structural group G. Then there is an associated principal G-bundle $Q \to M$, unique up to isomorphism, such that $E = Q \times_G F$ where F is the generic fibre of E equipped with its left G-action. A principal connection on Q is a one-form ω on E with values in $\mathfrak{g}$, the Lie algebra of G that is equivariant with respect to the right action ($R_g^*\omega = \operatorname{Ad}_{g^{-1}} \omega$) and reproduces the generators of the right action on vertical vectors ($X_g = dR_g(\omega(X))$).

A principal connection on Q induces an Ehresmann connection on any associated bundle. An Ehresmann connection is called a G-connection if, along any sufficiently small smooth curve $\gamma : [0,1] \to M$, the horizontal displacement $E_{\gamma(0)} \to E_{\gamma(1)}$ (which may only be defined on an open subset of the fiber) is a transformation in G.

Under suitable hypotheses, the G-connections are precisely those that arise by reduction from a principal connection in the associated principal bundle. Let $f \in F$ be an arbitrary point of the fiber. Assume that G acts transitively on F and that the Lie algebra $\mathfrak{g}$ of G acts faithfully on $T_f F$. The principal connection can then be reconstructed by the Lie derivative of the G-connection as follows. Take an admissible local trivialization of E, say $\psi : E|_U \to U \times F$. The trivialization ψ also gives a splitting of the tangent bundle $T(E|_U) \cong TU \oplus TF$. Use this splitting to identify the vector fields $\mathscr{T}_0^1 F$ on F ith vector fields on $T(E|_U)$, and let $(\mathscr{T}_0^1 F)^G$ be the G-invariant vector fields of this kind. Note that the Lie algebra $\mathfrak{g}$ acts on $(\mathscr{T}_0^1 F)^G$ by Lie differentiation. If $X \in TU$ is a tangent vector on $U \subset M$, then the horizontal lift $h(X)$ is an infinitesimal G-action if and only if $L_{h(X)}$ acts on $(\mathscr{T}_0^1 F)^G$ as an element of $\mathfrak{g}$. Since $\mathfrak{g} \to \mathfrak{gl}(T_e F)$ was assumed to be injective, $X \mapsto L_{h(X)}|T_e F$ is a one-form with values in $\mathfrak{g}$. This is the connection form θ_ψ represented in terms of the gauge ψ. Under a change of gauge, $\psi(x) \to g(x) \cdot \psi(x)$, and the G-invariant vertical vector fields change by a factor of g^{-1}. The connection transforms via

$$\theta_{g \cdot \psi} = \operatorname{Ad}_g \theta_\psi + g^* \omega_G$$

where ω_G is the Maurer–Cartan form of G. This is the compatibility condition needed to induce an equivariant connection on the associated principal bundle Q, equipped with the right action of G.

A G-connection in $E \to M$ is said to be of Cartan type if in addition there is a preferred section $\sigma : M \to E$ such that $P \circ d\sigma : TM \to VE$ is an isomorphism.[4] This preferred section is called a *soldering* of M to the fibre, and is a means for regarding the fibres as tangent to M itself. The bundle Q carries a preferred reduction to a principal bundle Q_0 for the

[4] See [21] or [45].

isotropy group G_0 of the fiber F. The principal connection on Q restricts to a principal G_0-equivariant form on Q_0 with values in the full Lie algebra $\mathfrak{g}$ which reproduces the generators of the vertical action. The isomorphism condition on the soldering implies that $\omega : TQ_0 \to \mathfrak{g}$ is an isomorphism of vector spaces. That is, we get a Cartan connection in the usual sense.

3.0 WEYL GEOMETRIES

In [82], Hermann Weyl famously introduced his conformal theory of relativity, in which the connection was replaced by a family of connections containing the information of the null geodesics of the spacetime. While this theory was dismissed as nonphysical by Einstein, decades later it became the basic foundation for gauge theory. Such a family of connections is what we shall call a *Weyl structure*. Weyl structures and their generalizations have been studied as a part of the program of *parabolic geometries* introduced by Charles Fefferman and Robin Graham [27].

Our first significant result is that a Weyl structure on a conformal manifold can be identified in a conformally-invariant way with a bundle of affine spaces each of which carries a canonical conformal structure. Such a bundle of affine spaces can be completed to a bundle of conformal spheres by adding a point at infinity, which is then regarded as being tangent to the manifold at the point of contact. We shall construct a natural Ehresmann connection on the space of Weyl connections whose parallel transport maps are conformal mappings of these spheres. By lifting to a principal bundle of conformal frames on the total space of the bundle of Weyl connections, this recaptures the normal conformal Cartan connection. The approach is significant because it is bottom-up rather than top-down: starting with the space of Weyl connections, one automatically recovers all of the other structural information (including the structure groups themselves). Typically the normal conformal Cartan connection is bottom-up: starting with the structural groups in advance, one is able to pick a canonical connection subject to certain conditions. This logically prior approach in turn generalizes to a large class of geometries, consisting of any suitably generic bundle of affine connections on a manifold.

3.1 WEYL STRUCTURES IN CONFORMAL GEOMETRY

We begin by defining Weyl structures as applied to the study of conformal differential geometry. Let M be a conformal manifold of dimension $n > 2$, and g a representative element of the conformal class. A Weyl structure is a family of torsion-free affine connections ∇^γ, indexed by one-forms γ, such that

$$(\nabla^\gamma_X g)(Y, Z) = -2\gamma(X) g(Y, Z).$$

Observe that, while this definition of the connection ∇^γ depends on the choice of conformal representative, the family of all connections does not so depend, and forms a fiber bundle which we shall denote by W. Upon choosing a connection γ_0, this fiber bundle can be naturally identified with the cotangent bundle. Thus there is a conformally invariant isomorphism of affine bundles between W and the affine cotangent bundle.

The connection with $\gamma = 0$ is the Levi-Civita connection, and the change in connection is given by

$$\nabla^\gamma_X Y = \nabla^0_X Y + \gamma(X)Y + \gamma(Y)X - \langle X, Y\rangle g^{-1}(\gamma).$$

In a coordinate basis, the change in connection has connection coefficients

$$\Gamma^k_{ij} = \gamma_j \delta^k_i + \gamma_i \delta^k_j - \gamma^k g_{ij}$$

where the inverse metric has been used to raise the index in the last term.

Let D^γ denote the exterior covariant derivative. By extending the ring of scalars to the ring of differential forms, D^γ is the unique derivation of $\Omega(M)$-algebras

$$D^\gamma : \Omega(M) \otimes \mathscr{T} \to \Omega(M) \otimes \mathscr{T}$$

that agrees with the exterior derivative on sections of $\Omega(M)$, and such that on sections of $\mathscr{T}$,

$$X \lrcorner (D^\gamma T) = \nabla^\gamma_X T.$$

Note that $D^\gamma g = -2\gamma \otimes g$.

3.1.1 Curvature and invariant decomposition

The curvature R^γ of the connection ∇^γ is defined by

$$R^\gamma(X,Y) = [\nabla^\gamma_X, \nabla^\gamma_Y] - \nabla^\gamma_{[X,Y]}.$$

So R^γ is a two-form with values in $\mathrm{End}(TM)$, which can be identified with the (super)-commutator of derivations

$$R^\gamma = [D^\gamma, D^\gamma] = 2(D^\gamma)^2.$$

The curvature satisfies the Bianchi identity

$$R^\gamma(X,Y)Z + R^\gamma(Y,Z)X + R^\gamma(Z,X)Y = 0.$$

Using the background metric, the curvature R^γ is identified with a covariant rank-four tensor

$$R^\gamma(X,Y,W,Z) = \langle R^\gamma(X,Y)W, Z\rangle_g.$$

This is skew-symmetric on X,Y, but is only skew-symmetric on W,Z when γ is the Levi-Civita connection, since by the Ambrose-Singer theorem skewness on W,Z implies that the connection has $O(n)$ holonomy, but the Levi-Civita connection is the unique torsion-free affine connection with $O(n)$ holonomy.

The Ricci tensor is the trace[1]

$$\mathrm{Ric}^\gamma(X,Y) = \mathrm{tr}(Z \mapsto R^\gamma(Z,X)Y).$$

Since R^γ is generally not skew in X,Y, the Ricci tensor may fail to be symmetric. In fact, the Bianchi identity establishes that

$$2\,\mathrm{Alt}(\mathrm{Ric}^\gamma)(X,Y) = \mathrm{tr}(R^\gamma(X,Y)). \tag{3.1}$$

Now $(D^\gamma)^2 g = -2d\gamma \otimes g$, or equivalently $([\nabla^\gamma_X, \nabla^\gamma_Y] - \nabla^\gamma_{[X,Y]})g = -4\,d\gamma(X,Y)g$. On the other hand, $([\nabla^\gamma_X, \nabla^\gamma_Y] - \nabla^\gamma_{[X,Y]})g = -2\,\mathrm{Sym}(g \circ R^\gamma(X,Y))$. Taking the trace with respect to g gives $-2\,\mathrm{tr}(R^\gamma(X,Y)) = -4n\,d\gamma(X,Y)$. Combined with (3.1), this implies

$$\mathrm{Alt}(\mathrm{Ric}^\gamma)(X,Y) = n\,d\gamma(X,Y).$$

[1] With these curvature conventions, the round sphere has positive Ricci curvature.

In particular, the Ricci tensor is symmetric if and only if $d\gamma = 0$; that is, if γ is the locally the differential of a conformal scale.[2] This calculation shows also that

$$\mathrm{Sym}(g \circ R^\gamma(X,Y)) = \frac{2}{n}\,\mathrm{Alt}(\mathrm{Ric}^\gamma)(X,Y).$$

In what follows, R will denote a generic tensor with the same symmetries as R^γ, and similarly with Ric. Let $\sigma : \wedge^2 T^*M \otimes T^*M \otimes T^*M \to \wedge^2 T^*M \otimes T^*M \otimes T^*M$ be the cyclic permutation of the first three arguments of a covariant tensor:

$$\sigma(R)(X,Y,Z,W) = R(Y,Z,X,W)$$

and let $b = \frac{1}{3}(I + \sigma + \sigma^2)$ be the Bianchi symmetrization map

$$b(R)(X,Y,Z,W) = \frac{1}{3}(R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W)).$$

On covariant rank-two tensors, define the trace tr_g by $\mathrm{tr}_g\, k = \mathrm{tr}(X \to g^{-1}(k(X,-)))$ where $g^{-1} : T^*M \to TM$ is the self-adjoint map defined by duality $\langle g^{-1}(\alpha), X\rangle_g = \alpha(X)$ for all $\alpha \in T^*M$ and $X \in TM$. Let c denote the Ricci trace map: $c(R)(X,Y) = \mathrm{tr}_g\, R(X,-,Y,-)$ so that

$$\mathrm{Ric} = c(R).$$

The scalar curvature is defined by

$$\mathrm{Sc} = \mathrm{tr}_g \mathrm{Ric}\,.$$

Let $s : \wedge^2 T^*M \otimes T^*M \otimes T^*M \to \wedge^2 T^*M \otimes T^*M \otimes T^*M$ be the map

$$\begin{aligned} &s(R)(X,Y,Z,W) = \\ &\qquad = R(X,Y,Z,W) + R(X,Y,W,Z) - \frac{1}{n}(c(R)(X,Y) - c(R)(Y,X))g(Z,W). \end{aligned}$$

The space of curvature tensors associated to Weyl connections is $\ker b \cap \ker s \subset \wedge^2 T^*M \otimes T^*M \otimes T^*M$, and we are interested in the decomposition of this space into irreducible $O(n)$-submodules. Note that $b^2 = b$ and, since σ is a unitary automorphism of order 3, b is self-adjoint, and so $I - b$ is the orthogonal projection of $\wedge^2 T^*M \otimes T^*M \otimes T^*M$ onto $\ker b$.

[2] This is consistent with the fact that the skew part of the Ricci curvature of an affine connection is the curvature two-form of the induced connection on the canonical bundle. Existence of a parallel section of the canonical bundle gives rise to a parallel conformal scale and vice-versa.

If h, k are a pair of covariant rank two tensors, their Kulkarni–Nomizu product is the rank four tensor defined by

$$\begin{aligned} h \owedge k(X,&Y, Z, W) = \\ &= h(X, Z)k(W, Y) + h(Y, W)k(Z, X) - h(X, W)k(Z, Y) - h(Y, Z)k(W, X). \end{aligned}$$

The tensor $h \owedge k$ is skew on X, Y and on Z, W. When h and k are symmetric, it satisfies the Bianchi identity and furthermore, $c(g \owedge h) = (n-2)h + (\operatorname{tr}_g h)g$. It follows that, for $n > 2$, the mapping $h \mapsto g \owedge h : \operatorname{Sym}^2 T^*M \to \ker b$ is an injective homomorphism of $O(n)$-modules.

We claim that $\ker b \cap \ker s$ decomposes (orthogonally) into $O(n)$-modules

$$R = S \oplus \Lambda \oplus Z \oplus W$$

with

$$\begin{aligned} S &= \frac{\mathrm{Sc}}{2n(n-1)} g \owedge g \\ \Lambda &= \frac{1}{n} \left(2 \operatorname{Alt}(\mathrm{Ric}) \otimes g - g \owedge \operatorname{Alt}(\mathrm{Ric})\right) \\ Z &= \frac{1}{n-2} \operatorname{Sym}_0(\mathrm{Ric}) \owedge g \end{aligned}$$

where Alt is the alternating part, Sym_0 is the trace-free symmetric part. It is clear that S, Λ, Z are irreducible. It follows from classical invariant theory that W is irreducible as well (see, e.g., [8]).

It is convenient for later work to define the Schouten tensor by

$$\Sigma \owedge g = S + Z$$

so that

$$\begin{aligned} \Sigma &= \frac{\mathrm{Sc}}{2n(n-1)} g + \frac{1}{n-2} \operatorname{Sym}_0(\mathrm{Ric}) \\ &= \frac{1}{n-2} \left(\operatorname{Sym}\mathrm{Ric} - \frac{\mathrm{Sc}\ g}{2(n-1)} \right). \end{aligned}$$

3.1.2 Curvature of Weyl connections

We return now to the case of the curvature R^γ of the Weyl connection γ. Our next objective shall be to express this in terms of R^0, the curvature of the Levi-Civita connection. For convenience, we shall here use (abstract) indices, and extend the usual ring of scalars of tensorial objects to include differential forms. So there is a natural one-form θ^a expressing the isomorphism of covariant tensors with one-forms (which are scalars) $w_a \mapsto \theta^a w_a$. The change in connection, for $X \in \Omega(M) \otimes \mathscr{T}_0^1 M$, is

$$D^\gamma X^a = D^0 X^a + \Gamma^a_b X^b$$

where

$$\Gamma^a_b = \gamma_b \theta^a + \gamma \delta^a_b - \theta_b \gamma^a.$$

So, the change to the curvature endomorphism is

$$\begin{aligned}
\frac{1}{2}(R^\gamma)^a_b &= \frac{1}{2}(R^0)^a_b + D^0\Gamma^a_b + \Gamma^a_c\Gamma^c_b \\
&= \frac{1}{2}(R^0)^a_b + (D^0\gamma_b)\theta^a + d\gamma\, \delta^a_b + \theta_b D^0\gamma^a - \gamma\theta^a\gamma_b - \theta^a\theta_b\gamma_c\gamma^c - \gamma\theta_b\gamma^a \\
\frac{1}{2}R^\gamma_{ba} &= \frac{1}{2}R^0_{ba} + d\gamma\, g_{ab} - 2\left(\theta_{[a}D\gamma_{b]} - \theta_{[a}\gamma\gamma_{b]} + \frac{1}{2}\theta_{[a}\theta_{b]}\gamma_c\gamma^c\right).
\end{aligned}$$

So, in terms of the $(0,4)$ curvature tensor, the conformal change is

$$\begin{aligned}
R^\gamma &= R^0 + 2(d\gamma)\otimes g - g \owedge \left(D^0\gamma - \gamma\otimes\gamma + \frac{1}{2}\langle\gamma,\gamma\rangle g\right) \\
&= R^0 + (2(d\gamma)\otimes g - g \owedge d\gamma) - g \owedge \left(\operatorname{Sym} D^0\gamma - \gamma\otimes\gamma + \frac{1}{2}\langle\gamma,\gamma\rangle g\right)
\end{aligned}$$

Since the Schouten tensor is the symmetric Ricci part of the curvature decomposition, it transforms according to

$$\Sigma^\gamma = \Sigma^0 - \left(\operatorname{Sym} D^0\gamma - \gamma\otimes\gamma + \frac{1}{2}\langle\gamma,\gamma\rangle g\right).$$

In indices,

$$\Sigma^\gamma_{ij} = \Sigma^0_{ij} - \nabla^0_{(i}\gamma_{j)} + \gamma_i\gamma_j - \frac{1}{2}\gamma_k\gamma^k g_{ij}.$$

It is natural to define the rho tensor to be the $(0,2)$ tensor whose symmetric part is Σ and whose skew part is $-d\gamma$:

$$\mathrm{P}^\gamma = \Sigma^\gamma - d\gamma.$$

Written out in indices

$$\mathrm{P}^\gamma_{ij} = \frac{1}{n}\,\mathrm{Ric}^\gamma_{[ij]} + \frac{1}{n-2}\left(\mathrm{Ric}^\gamma_{(ij)} - \frac{\mathrm{Sc}^\gamma\; g_{ij}}{2(n-1)}\right).$$

By taking traces, the Ricci tensor can be completely recovered from the rho tensor. In particular, one vanishes if and only if the other does. The rho tensor transforms according to

$$\mathrm{P}^\gamma_{ij} = \mathrm{P}^0_{ij} - \nabla^0_i\gamma_j + \gamma_i\gamma_j - \frac{1}{2}\gamma_k\gamma^k g_{ij}.$$

The transformation law can be conveniently re-expressed as

$$\mathrm{P}^\gamma_{ij} = \mathrm{P}^0_{ij} - \nabla^{\gamma/2}_i\gamma_j$$

or, equivalently,

$$\mathrm{P}^\gamma = \mathrm{P}^0 - D^{\gamma/2}\gamma.$$

The $\gamma/2$ comes from our unfortunate factor-of-two conventions.

3.1.3 Connection on the space of Weyl connections

We shall define a natural Ehresmann connection on the total space of W. From the previous section, the Ricci curvature associates to each section γ of W the tensor $\mathrm{Ric}^\gamma \in T^*M \otimes T^*M$. At a point x_0, $\mathrm{Ric}^\gamma(x_0)$ depends only on the 1-jet $j^1_{x_0}\gamma$, so the Ricci tensor factors through the 1-jet prolongation in the sense that there exists a bundle mapping

$$\mathrm{ric} : J^1\pi_W \to \pi_W^{-1}(T^*M \otimes T^*M)$$

such that

$$\mathrm{Ric}^\gamma = \mathrm{ric}(j^1\gamma).$$

The equation $\mathrm{ric}(j^1\gamma) = 0$ defines a submanifold of $J^1\pi_W$, which is the image of a unique section $\sigma : W \to J^1\pi_W$ over W. This, in turn, corresponds to an Ehresmann connection.[3]

To obtain a more explicit description of the Ehresmann connection, we shall implicitly identify W with T^*M. Let x^i be local coordinates on M and γ_i are the induced fiber coordinates of T^*M defined by $\gamma_i(x,\alpha) = \partial/\partial x_i \lrcorner \alpha$. The Levi–Civita connection defines an

[3]In other words, an Ehresmann connection is a distribution of subspaces in TW that are horizontal with respect to the submersion $\pi_W : W \to M$. For $x_0 \in M$, call a section $\sigma : M \to W$ horizontal at x_0 if $\mathrm{Ric}^\gamma(x_0) = 0$. The tangent spaces at x_0 to all horizontal sections at x_0 give the horizontal distribution of the connection.

Ehresmann connection on T^*M as follows. The horizontal lift of $X \in TM$, denoted ∇^0_X, is the vector field on T^*M defined by

$$\nabla^0_X(\pi^*_W f) = X(f)$$
$$\nabla^0_X(\gamma_i) = -\Gamma^k_{ij}\gamma^j X_k$$

where Γ^k_{ij} are the usual Christoffel symbols for the Levi–Civita connection.

Alternatively, the connection is determined by the family of one-forms

$$\omega_j = D\gamma_j - \gamma_{ji}\, dx^i$$

where $D\gamma_j$ is the Levi-Civita connection applied to γ_j (regarded as a one-form), and the coefficients γ_{ji} are obtained by solving $\mathrm{Ric}^\gamma_{ij} = 0$ for $\nabla^0_i\gamma_j$. The horizontal space is the annihilator of the system of one-forms ω_j.

Since Ric vanishes precisely when P does, the transformation law for P gives

$$\gamma_{ji} = \mathrm{P}^0_{ij} + \gamma_i\gamma_j - \frac{1}{2}\gamma^k\gamma_k g_{ij}.$$

Moreover, the pullback of the connection one-form ω_j along a section $\gamma(x)$ is just

$$\gamma^*\omega_j = -P^\gamma_{ji}\, dx^i.$$

The associated horizontal lifts can be represented in the form

$$h(X) = \nabla^0_X + \left(\mathrm{P}^0(\partial/\partial\gamma, X) + \gamma(X)\gamma(\partial/\partial\gamma) - \frac{1}{2}\langle\gamma,\gamma\rangle\,\langle X, \partial/\partial\gamma\rangle\right).$$

Indeed, the Levi–Civita connection, by fixing a section of W, induces an isomorphism $W \cong T^*M$ of vector bundles. Then the Levi–Civita connection also induces a vector field ∇^0_X on T^*M, defined by the two conditions $\nabla^0_X\pi^*_W f(x) = Xf(x)$ and $\nabla^0_X(\gamma_i) = \nabla^0_X\gamma_i$, the Levi–Civita connection applied to the section γ_i of T^*M.

3.1.4 Conformal structure on the space of Weyl connections

The bundle W is naturally isomorphic as an *affine bundle* to T^*M. The differential of this isomorphism is a canonical invertible section ν of the bundle $\mathrm{Hom}(VW, \pi_W^{-1}T^*M)$. Define a metric in the vector bundle $VW \to W$ by

$$g_V(S,T) = g^{-1}(\nu(S), \nu(T)).$$

where g is the given representative of the conformal class on M. In local coordinates,

$$g_V\left(\partial/\partial\gamma_i, \partial/\partial\gamma_j\right) = g^{ij}.$$

From the local coordinate description, this defines a metric that scales in the fiber if g is conformally rescaled. Indeed, the vector field $\partial/\partial\gamma_i$ does not change under a conformal rescaling, and the metric itself just scales by the reciprocal of the conformal factor. Therefore each fibre of $W \to M$ carries naturally the structure of a conformally flat manifold.

The conformal structure is compatible with the connection in the sense that it is preserved under parallel transport. Indeed, we have

$$\begin{aligned}(L_{h(X)}g)(\partial/\partial\gamma_i, \partial/\partial\gamma_j) = g^{-1}(\nabla_X dx^i, dx^j) + g^{-1}(dx^i, \nabla_X dx^j) - \\ - g\left([h(X), \partial/\partial\gamma_i], \partial/\partial\gamma_j\right) - g\left(\partial/\partial\gamma_i, [h(X), \partial/\partial\gamma_j]\right).\end{aligned}$$

Write

$$h(X) = \nabla_X + \left(\mathrm{P}^0(\partial/\partial\gamma, X) + \gamma(X)\gamma(\partial/\partial\gamma) - \frac{1}{2}\langle\gamma,\gamma\rangle \langle X, \partial/\partial\gamma\rangle\right).$$

We have $g([\nabla_X, \partial/\partial\gamma_i], \partial/\partial\gamma_j) = g(\nabla_X dx^i, dx^j)$, so the only terms that remain are

$$\begin{aligned}-g\left([h(X) - \nabla_X, \partial/\partial\gamma_i], \partial/\partial\gamma_j\right) - g\left(\partial/\partial\gamma_i, [h(X) - \nabla_X, \partial/\partial\gamma_j]\right) = \\ = g\left(X_i\gamma_k\partial/\partial\gamma_k + \gamma_k X^k\partial/\partial\gamma_i - \gamma_i g_{k\ell}X^k\partial/\partial\gamma_\ell, \partial/\partial\gamma_j\right) + (i \leftrightarrow j) \\ = X^i\gamma_k g^{kj} + \gamma_k X^k g^{ij} - \gamma^i g_{k\ell}X^k g^{\ell j} + (i \leftrightarrow j) \\ = X^i\gamma^j + \gamma_k X^k g^{ij} - \gamma^i X^j + (i \leftrightarrow j) \\ = 2\gamma_k X^k g^{ij}.\end{aligned}$$

Thus because $L_{h(X)}g$ is proportional to g, $h(X)$ is an infinitesimal conformal map of the fibre.

Note however that the connection is not complete, even locally, so the parallel transport map is only locally defined in each fibre. Indeed, fix a point p of M and a vector X at p. Then lying over p, $L_{h(X)}g$ is unbounded, and so a curve in M with initial velocity X at p has horizontal lifts that escape to infinity in arbitrarily small time. This suggests attempting to conformally compactify the fibres, and to extend the connection if possible to the conformal compactification.

The conformal compactification takes many copies of W, each identified with T^*M, and glues them together smoothly along a diffeomorphism in a way that adds a null cone to the fibre at infinity (or a point in the case of Euclidean signature). Let N be the null cone of W. Let $\phi : W \setminus N \to W \setminus N$ be inversion in the "unit sphere" of the fiber.[4] For each affine translation $\tau : W \to W$, the mapping $\tau^{-1}\phi\tau$ is a *special conformal transformation* from $W \setminus \tau^{-1}N$ to itself. Let W_τ be the family of copies of W, indexed by all affine translations τ, and let W_∞ be an additional copy of W corresponding to the structure at infinity. Define an equivalence relation on the disjoint union $W_\infty \dot\cup \dot{\bigcup}_\tau W_\tau$ by

$$x_\tau \sim x_\infty \iff x_\infty = \tau^{-1}\phi\tau(x_\tau)$$
$$x_\tau \sim x_{\tau'} \iff x_\tau = \tau^{-1}\phi^{-1}\tau(\tau')^{-1}\phi\tau'(x_{\tau'})$$

Then the space

$$\widehat{W} = \left(W_\infty \dot\cup \dot{\bigcup_\tau} W_\tau\right) / \sim$$

is a smooth bundle on M with compact fibres. Since the mappings ϕ and τ are both conformal in the fibres, $\widehat{W}$ carries an induced (flat) conformal structure. The fiber of $\widehat{W}$ is a compact conformal space isomorphic to the conformal sphere $S^{p,q} \cong S^p \times S^q$ in signature p, q.

In coordinates, let γ_i be the local fiber coordinates on W and let $\mu_i = \gamma_i \circ \phi$. Then

$$\mu_i = \frac{\gamma_i}{\gamma_k\gamma^k}$$
$$\gamma_i = \frac{\mu_i}{\mu_k\mu^k}$$

So ϕ is a diffeomorphism. It is conformal, since

$$d\phi\frac{\partial}{\partial\gamma_i} = (\mu_k\mu^k)\frac{\partial}{\partial\mu_i} - 2\mu^i\mu_k\frac{\partial}{\partial\mu_k} \tag{3.2}$$

and so

$$g(d\phi(\partial/\partial\gamma_i), d\phi(\partial/\partial\gamma_j)) = (\mu_k\mu^k)^2 g^{ij}.$$

Each mapping τ is a conformal diffeomorphism as well, being a translation of an affine space with an invariant conformal structure.

[4]In non-Euclidean signature, this is inversion in the hyperboloid $g(\gamma, \gamma) = 1$. In coordinates, $\phi(\gamma)_i = \dfrac{\gamma_i}{\gamma_k\gamma^k}$

The null cone of W_∞ in each fiber defines a natural subset of $\widehat{W}$. The vertex of this null cone defines a canonical section $\sigma : M \to \widehat{W}$. It remains to show that the connection on W extends to a (necessarily unique) connection on $\widehat{W}$, and that the section σ is generic with respect to this connection. It will then follow that the Ehresmann connection in $\widehat{W}$ is of Cartan type.

We shall show that the horizontal lift extends smoothly along the gluing of W_0 and W_∞. The other gluings are handled similarly. In addition to (3.2), we have

$$d\phi(\partial/\partial x^i) = \frac{\partial}{\partial x^i} + \left[\frac{\partial}{\partial x^i}\log(\mu_k\mu^k)\right]\mu_r\frac{\partial}{\partial\mu_r}.$$

Then the horizontal lift of the coordinate vector field $\partial/\partial x^j$ transforms via

$$\begin{aligned}
d\phi(h(\partial/\partial x^j)) &= \frac{\partial}{\partial x^j} + \left[\frac{\partial}{\partial x^i}\log(\mu_k\mu^k)\right]\mu_r\frac{\partial}{\partial\mu_r} + \left[\Gamma^k_{ij}\gamma_k + \mathrm{P}^0_{ij} + \gamma_i\gamma_j - \frac{1}{2}\gamma^k\gamma_k g_{ij}\right]\frac{\partial}{\partial\gamma_i} \\
&= \frac{\partial}{\partial x^j} + \left[\frac{\partial}{\partial x^i}\log(\mu_k\mu^k)\right]\mu_r\frac{\partial}{\partial\mu_r} + \\
&\quad + \left[\Gamma^k_{ij}\frac{\mu_k}{\mu\cdot\mu} + \mathrm{P}^0_{ij} + \frac{\mu_i\mu_j}{(\mu\cdot\mu)^2} - \frac{1}{2}(\mu\cdot\mu)^{-1}g_{ij}\right]\left[(\mu\cdot\mu)\frac{\partial}{\partial\mu_i} - 2\mu^i\mu_r\frac{\partial}{\partial\mu_r}\right] \\
&= \frac{\partial}{\partial x^j} + \left[\Gamma^k_{ij}\mu_k + (\mu\cdot\mu)\,\mathrm{P}^0_{ij} + \frac{\mu_i\mu_j}{\mu\cdot\mu} - \frac{1}{2}g_{ij}\right]\frac{\partial}{\partial\mu_i} - \\
&\qquad - 2\left[\mu^i\Gamma^k_{ij}\frac{\mu_k}{\mu\cdot\mu} + \mu^i\,\mathrm{P}^0_{ij} + \frac{1}{2}\frac{\mu_j}{\mu\cdot\mu}\right]\mu_r\frac{\partial}{\partial\mu_r} \\
&= \frac{\partial}{\partial x^j} + \left[\frac{\partial}{\partial x^i}\log(\mu_k\mu^k)\right]\mu_r\frac{\partial}{\partial\mu_r} + \\
&\quad + \left[\Gamma^k_{ij}\mu_k + (\mu\cdot\mu)\,\mathrm{P}^0_{ij} - \frac{1}{2}g_{ij}\right]\frac{\partial}{\partial\mu_i} - 2\left[\mu^i\Gamma^k_{ij}\frac{\mu_k}{\mu\cdot\mu} + \mu^i\,\mathrm{P}^0_{ij}\right]\mu_r\frac{\partial}{\partial\mu_r} \\
&= \frac{\partial}{\partial x^j} + \left[\Gamma^k_{ij}\mu_k + (\mu\cdot\mu)\,\mathrm{P}^0_{ij} - \frac{1}{2}g_{ij}\right]\frac{\partial}{\partial\mu_i} - 2\mu^i\,\mathrm{P}^0_{ij}\mu_r\frac{\partial}{\partial\mu_r} \\
&\qquad\qquad [\text{since } \mu^i\Gamma^k_{ij}\tfrac{\mu_k}{\mu\cdot\mu} = \tfrac{1}{2}\tfrac{\partial}{\partial x^j}\log(\mu_k\mu^k).]
\end{aligned}$$

This extends smoothly to the null cone $\mu\cdot\mu = 0$ at infinity, and therefore the connection extends to a smooth connection on the bundle $\widehat{W}$.

3.1.5 Curvature of the connection

An Ehresmann connection can be expressed as the simultaneous vanishing of a system of 1-forms on the total space of the fiber bundle. In local coordinates x^i on the base M and fiber coordinates γ_i, such one-forms can be written

$$\omega_i = d\gamma_i + N_{ij}\, dx^j.$$

The horizontal lifts h_i of the coordinate vector fields $\partial/\partial x^i$ can be written

$$h_j = \frac{\partial}{\partial x^j} - N_{ij}\frac{\partial}{\partial \gamma_i}.$$

The curvature of a nonlinear connection is defined in terms of the horizontal lifts h_i by

$$R_{ijk}\frac{\partial}{\partial \gamma_k} = [h_i, h_j].$$

Write

$$\omega_j = d\gamma_j + \left(\Gamma^k_{ij}\gamma_k + \mathrm{P}^0_{ij} + \gamma_i\gamma_j - \frac{1}{2}\gamma^k\gamma_k g_{ij}\right) dx^i$$

where Γ^k_{ij} are the Christoffel symbols for the Levi-Civita connection ∇^0. The horizontal lifts of the coordinate vector fields are

$$h_j = \frac{\partial}{\partial x^j} - \left(\Gamma^k_{ij}\gamma_k + \mathrm{P}^0_{ij} + \gamma_i\gamma_j - \frac{1}{2}\gamma^k\gamma_k g_{ij}\right)\frac{\partial}{\partial \gamma_i}$$

Write

$$\omega_b = D^0\gamma_b - \left(\mathrm{P}^0_{ab} + \gamma_a\gamma_b - \frac{1}{2}\gamma_c\gamma^c g_{ab}\right)\theta^a.$$

Then

$$h_a = \nabla^0_a + \left(\mathrm{P}^0_{ab} + \gamma_a\gamma_b - \frac{1}{2}\gamma_c\gamma^c g_{ab}\right)\frac{\partial}{\partial \gamma_b}$$

3.2 WEYL STRUCTURES IN PROJECTIVE GEOMETRY

Two torsion-free affine connections on a manifold M are *projectively equivalent* if they have the same geodesics, apart from reparameterization. Thus affine connections ∇ and ∇' are projectively equivalent if

$$\nabla'_X Y = \nabla_X Y - \gamma(X)Y - \gamma(Y)X$$

for some one-form γ. A *projective structure* on M is an equivalence class of affine connections. In such cases, the data of M together with this equivalence class of affine connections is called a *projective manifold*. In what follows, fix a background affine connection ∇^0. Then the space of all affine connections in the projective equivalence class is parametrized by a one-form γ.

The associated exterior covariant derivative D^γ is given by

$$D^\gamma X^a = D^0 X^a - \gamma X^a - \theta^a \gamma_c X^c = D^0 X^a + \Gamma^a_b X^b$$

where $\Gamma^a_b = -\gamma\delta^a_b - \theta^a\gamma_b$. The curvature is

$$\begin{aligned}
(D^\gamma)^2 X^a &= D^\gamma(D^0 X^a + \Gamma^a_b X^b) \\
&= (D^0)^2 X^a + \Gamma^a_b D^0 X^b + (D^\gamma \Gamma^a_b) X^b - \Gamma^a_b D^\gamma X^b \\
&= (D^0)^2 X^a + \Gamma^a_b D^0 X^b + (D^0 \Gamma^a_b) X^b - \Gamma^c_b \Gamma^a_c X^b + \Gamma^a_c \Gamma^c_b X^b - \\
&\qquad - \Gamma^a_b D^0 X^b - \Gamma^a_b \Gamma^b_c X^c \\
&= (D^0)^2 X^b + (D^0 \Gamma^a_b) X^a - \Gamma^c_a \Gamma^c_b X^a.
\end{aligned}$$

So the change in curvature is $(D^0\Gamma^a_b) - \Gamma^c_a\Gamma^c_b$. We have

$$D^0 \Gamma^a_b = -d\gamma \delta^b_a + \theta^a D^0 \gamma_b$$

and

$$\Gamma^c_a \Gamma^c_b = (-\gamma\delta^c_a - \theta^c\gamma_a)(-\gamma\delta^b_c - \theta^b\gamma_c) = \gamma\theta^b\gamma_a.$$

So

$$(D^\gamma)^2 X^a = (D^0)^2 X^a + (-d\gamma\delta^a_b + \theta^a D^0\gamma_b - \gamma\theta^b\gamma_a) X^a.$$

The curvature endomorphism transforms by

$$(R^\gamma)^a_b = (R^0)^a_b - 2(d\gamma)\delta^a_b + 2\theta^a D^0 \gamma_b - 2\gamma\theta^b\gamma_a.$$

The Ricci tensor transforms as

$$\mathrm{Ric}^{\gamma}(X,Y) = \mathrm{Ric}^{0}(X,Y) - 2d\gamma(X,Y) - (n-1)(\nabla^{0}_{X}\gamma)(Y) - (n-1)\gamma(X)\gamma(Y).$$

Define the Rho tensor by

$$\mathrm{P}^{\gamma}(X,Y) = \frac{-1}{n^2-1}\left(n\,\mathrm{Ric}^{\gamma}(X,Y) + \mathrm{Ric}^{\gamma}(Y,X)\right)$$

so that

$$\mathrm{P}^{\gamma}(X,Y) = \mathrm{P}^{0}(X,Y) + (\nabla^{0}_{X}\gamma)(Y) + \gamma(X)\gamma(Y).$$

As before, call a section γ of W horizontal at x_0 if $\mathrm{P}^{\gamma}(x_0) = 0$. Horizontal sections are therefore those for which

$$(\nabla^{0}_{X}\gamma)(Y) = -\,\mathrm{P}^{0}(X,Y) - \gamma(X)\gamma(Y).$$

In terms of local coordinates x^i on M, the connection one-forms are

$$\omega_j = d\gamma_j + \left(\Gamma^{k}_{ij}\gamma_k + \mathrm{P}^{0}_{ij} + \gamma_i\gamma_j\right)dx^i$$

where Γ^{k}_{ij} are the connection coefficients for ∇^0 in the coordinate system. The horizontal lifts of the coordinate vector fields are

$$h(\partial/\partial x^i) = \frac{\partial}{\partial x^i} - \left(\Gamma^{k}_{ij}\gamma_k + \mathrm{P}^{0}_{ij} + \gamma_i\gamma_j\right)\frac{\partial}{\partial\gamma_j}. \tag{3.3}$$

The fibre of W carries a natural projective structure, along with the local Lie group action of $\mathrm{PGL}(n+1)$ on the fibre. The infinitesimal generators of the Lie group are vector fields of the form

$$\left(A^{j}_{i}\gamma_j + c_i + b^j\gamma_j\gamma_i\right)\frac{\partial}{\partial\gamma_i}.$$

The vector field in (3.3) is an infinitesimal generator for the projective group. Hence the connection preserves the projective structure on the fibre. This projective structure admits a compactification through a point at infinity, as in the conformal case.

3.3 GENERALIZED WEYL GEOMETRIES

Let $W \to M$ be a fiber bundle of affine connections on M. For a fixed section γ of W, the curvature tensor at a point $x \in M$ is defined by on vectors $X, Y, Z \in T_{x_0}M$ by first extending them to vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ in a neighborhood of x, and then setting

$$R^\gamma(X,Y)Z = [\nabla^\gamma_{\widetilde{X}}, \nabla^\gamma_{\widetilde{Y}}]\widetilde{Z} - \nabla^\gamma_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}.$$

Since the expression on the right-hand side is linear with respect to constant scalings, and is independent of how X, Y, Z are extended away from x, the curvature defines a tensor in $\wedge^2 T^*M \otimes T^*M \otimes TM$.[5] The Ricci tensor is

$$\mathrm{Ric}^\gamma(X,Z) = \mathrm{tr}(Y \to R^\gamma(X,Y)Z).$$

At a point x_0, $\mathrm{Ric}^\gamma(x_0)$ depends only on the 1-jet $j^1_{x_0}\gamma$, so again the Ricci tensor factors through the 1-jet prolongation in the sense that there exists a bundle mapping

$$\mathrm{ric} : J^1\pi_W \to \pi_W^{-1}(T^*M \otimes T^*M)$$

such that

$$\mathrm{Ric}^\gamma = \mathrm{ric}(j^1\gamma).$$

However, the solution of the equation $\mathrm{ric}(j^1\gamma) = 0$ may fail to define a submanifold of $J^1\pi_W$, or even if it does, it may fail to be the image of a section $\sigma : W \to J^1\pi_W$ over W. Call the Weyl geometry *generic* if the solution of $\mathrm{ric}(j^1\gamma) = 0$ is a smooth section of $J^1\pi_W$.

Let x^i be local coordinates on M. A section of W is specified by the connection coefficients $\Gamma^k_{ij} = \Gamma^k_{ij}(\gamma_\alpha, x^i)$ where γ_α are fiber coordinates on W. A nonlinear connection in W is a horizontal distribution for the fibration $W \to M$. Call a section γ horizontal at x_0 if $\mathrm{Ric}^\gamma(x_0) = 0$. This equation involves the first partial derivatives $\partial\gamma_\alpha/\partial x^i$, and if we can always solve uniquely for these second derivatives, then the connection is uniquely determined and so the Weyl geometry is generic. The leading symbol of Ric^γ in γ_α is

$$\partial^\alpha\Gamma^k_{ij}\partial_k\gamma_\alpha - \partial^\alpha\Gamma^k_{ik}\partial_j\gamma_\alpha.$$

The bundle W generic if this linear transformation of the second partials $\partial_k\gamma_\alpha$ is invertible.

We note that this condition is not automatically satisfied. For instance, fix a background connection ∇ and consider the projectively related connections

$$\nabla^\gamma_Y X = \nabla^0_Y X + \gamma(Y)X$$

[5] Here we have implicitly used the tensor-hom adjunction, and we shall do so henceforth without comment.

for γ a one-form. (This family of connections has torsion.) The curvature is given by

$$R^\gamma X = 2(D^\gamma)^2 X = 2(D^0)^2 X + 2(d\gamma)X = R^0 X + 2(d\gamma)X$$

and so the Ricci tensor is

$$\mathrm{Ric}^\gamma = 2d\gamma.$$

But it is clearly not possible to determine all first derivatives of γ from the vanishing of $d\gamma$.

3.4 GENERALIZED CIRCLES

Let $W \to M$ be a generalized Weyl geometry whose curvature is generic in the sense of Section 3.3 and let c be curve in M. Through any initial point $w_o \in \pi^{-1}(c(0))$, there is a unique horizontal lift h_c of c to W under the Ehresmann connection, valid for small time. At each point $c(t)$ of the curve, the horizontal lift $h_c(t)$ is an affine connection in $T_{c(t)}M$. Thus we have a curve and an affine connection in the tangent space to M at each point of the curve. It therefore makes sense to ask whether the curve is a *geodesic* with respect to that affine connection.

Definition 12. *A curve c in M is called a circle if there exists an initial $w_0 \in W$ such that c is an affinely parametrized geodesic with respect to the horizontal connection h_c.*

3.4.1 Conformal case

We consider the calculation in the conformal case. Let g be a background metric representing the conformal class on M. A Weyl connection is any connection ∇^γ that satisfies

$$\nabla^\gamma g = -2\gamma \otimes g$$

where γ is a one-form. The mapping $\gamma \mapsto \nabla^\gamma$ sets up an isomorphism between the bundle W of admissible Weyl connections and the cotangent bundle. This mapping depends on the choice of conformal scale, in terms of which there is a preferred section ∇^0: the Levi-Civita connection. Any other connection ∇^γ can be represented explicitly in terms of ∇^0 by the formula

$$\nabla^\gamma_X Y = \nabla^0_X Y + \gamma(X)Y + \gamma(Y)X - \langle X, Y\rangle g^{-1}(\gamma)$$

which is to say that the change of connection is expressed by the connection coefficients

$$\Gamma^a_{bc} = \gamma_b \delta^a_c + \gamma_c \delta^a_b - \gamma^a g_{bc}.$$

The rho tensor transforms via

$$\mathrm{P}^{\gamma}_{ab} = \mathrm{P}^{0}_{ab} - \nabla^{0}_{a}\gamma_b + \gamma_a\gamma_b - \frac{1}{2}\gamma_c\gamma^c g_{ab}.$$

Suppose that c is a non-null curve, and γ is the horizontal lift of that curve. By definition, this means that $\mathrm{P}^{\gamma} = 0$ along the lifted curve. Equivalently,

$$\nabla^{0}_{a}\gamma_b = \mathrm{P}^{0}_{ab} + \gamma_a\gamma_b - \frac{1}{2}\gamma_c\gamma^c g_{ab}. \tag{3.4}$$

Now, let $v^a = \dot{c}^a$ be the velocity along the curve c and $A^a = v^b\nabla^0_b v^a$ the acceleration (relative to the Levi-Civita connection). The condition for c to be a conformal circle is that $v^b\nabla^{\gamma}_b v^a = 0$, where γ is a lift of c satisfying $P^{\gamma} = 0$. Equivalently,

$$A^a + 2v^b\gamma_b v^a - v_b v^b \gamma^a = 0. \tag{3.5}$$

Taking A^a and v^a as given, we can solve this equation explicitly for γ:

$$\gamma^a = \frac{A^a}{v^c v_c} - 2\frac{v^d A_d}{(v^c v_c)^2} v^a. \tag{3.6}$$

On the other hand, we may differentiate (3.8) once more by applying $v^c\nabla^0_c$, then use (3.7) to reduce the resulting equation to first order in γ, and finally use (3.9) to eliminate γ from the resulting expression. The final equation is thus a third-order condition on the curve c.

Carrying this through, we have

$$\begin{aligned}
v^b\nabla^{\gamma}_b v^i &= v^b\nabla^0_b v^i + 2v^b\gamma_b v^i - v^b v_b \gamma^i \\
&= A^i + 2v^b\gamma_b v^i - v^b v_b\gamma^i \\
v^c\nabla^{\gamma}_c v^b\nabla^{\gamma}_b v^i &= v^c\nabla^0_c v^b\nabla^{\gamma}_b v^i + v^c\gamma_c v^b\nabla^{\gamma}_b v^i - \gamma^i v^c v^b\nabla^{\gamma}_b v_c + v^i\gamma_c v^b\nabla^{\gamma}_b v^c \\
&= v^c\nabla^0_c A^i + 2v^c\nabla^0_c(v^b\gamma_b v^i) - v^c\nabla^0_c(v^b v_b\gamma^i) + \\
&\quad + v^c\gamma_c v^b\nabla^{\gamma}_b v^i - \gamma^i v_c v^b\nabla^{\gamma}_b v^c + v^i\gamma_c v^b\nabla^{\gamma}_b v^c.
\end{aligned}$$

The last three terms involve $v^b\nabla^\gamma_b v^i$, which we are assuming to be zero. Applying (3.7) and (3.8), followed by (3.9) to the remaining two terms gives

$$\begin{aligned}
v^c\nabla^0_c(v^b\gamma_b v^i) &= \left(v_c v^c\gamma^b\gamma_b - 2v^c\gamma_c v^b\gamma_b + v^c v^b\,\mathrm{P}^0_{cb} + v^c\gamma_c v^b\gamma_b - \frac{1}{2}\gamma_d\gamma^d v_c v^c\right)v^i + \\
&\quad + v^b\gamma_b A^i \\
&= \mathrm{P}^0{}_{ab}v^a v^b v^i + \frac{A^2}{2v^2}v^i - \frac{1}{v^4}(v\cdot A)^2 v^i - \frac{1}{v^2}(v\cdot A)A^i \\
v^c\nabla^0_c(v^b v_b\gamma^i) &= -2v_c v^c\gamma_b v^b\gamma^i + v^b v_b\left(v^c\mathrm{P}^0{}_c{}^i + v^c\gamma_c\gamma^i - \frac{1}{2}\gamma_d\gamma^d v^i\right) \\
&= v^2\mathrm{P}^0{}_a{}^i v^a - \frac{A^2}{2v^2}v^i - \frac{2}{v^4}(v\cdot A)^2 v^i + \frac{v\cdot A}{v^2}A^i
\end{aligned}$$

where we have denoted $v^a v_a$ by v^2, $A^a A_a$ by A^2 and $v^a A_a$ by $v\cdot A$. So finally we have

$$\begin{aligned}
v^c\nabla^\gamma_c v^b\nabla^\gamma_b v^i &= v^c\nabla^0_c A^i + 2v^c\nabla^0_c(v^b\gamma_b v^i) - v^c\nabla^0_c(v^b v_b\gamma^i) \\
&= v^b\nabla^0_b A^i - v^2\mathrm{P}^i_a v^a + 2P^0_{ab}v^a v^b v^i + \frac{3A^2}{2v^2}v^i - \frac{3v\cdot A}{v^2}A^i
\end{aligned}$$

Thus a curve c is a conformal circle if and only if

$$v^b\nabla^0_b A^i = v^2\mathrm{P}^i_a v^a - 2P^0_{ab}v^a v^b v^i - \frac{3A^2}{2v^2}v^i + \frac{3v\cdot A}{v^2}A^i$$

which is precisely the way conformal circles are *defined* in [7].

3.4.2 Alternative geometrical characterization

In [7], an alternative geometrical characterization of conformal circles appears. A non-null curve is a conformal circle if and only if it is a geodesic in some conformal scale $e^{2\Upsilon}g$ such that $v^a\,\mathrm{P}^{d\Upsilon}_{ab} = 0$. This is quite different from our definition, which asserts that there is a (not necessarily closed) one-form γ such that $\mathrm{P}^\gamma_{ab} = 0$ along the curve. However the two criteria are related as follows. First suppose that v^a is a geodesic vector field in the conformal scale g_{ab} and $v^a\,\mathrm{P}^0_{ab} = 0$. Along the curve it is possible to write

$$\mathrm{P}^0_{ab} = \nabla^0_a\gamma_b$$

where $\gamma_b = 0$ along the curve. Hence $\mathrm{P}^\gamma_{ab} = 0$ and $v^a\nabla^\gamma_a v^b = v^a\nabla^0_a v^b + 2(v^b\gamma_b)v^a - (v^b v_b)\gamma_a = v^a\nabla^0_a v^b = 0$ along the curve. Conversely, if $\mathrm{P}^\gamma = 0$ along a curve, then there exists locally a closed one-form μ such that $\mu_b = \gamma_b$ and $v^a\nabla^0_a\mu_b = v^a\nabla^0_a\gamma_b$ along the curve.

3.4.3 $SL(2,\mathbb{R})$ structure

We now show how any non-null curve c carries a natural parametrization up to the standard action of $SL(2,\mathbb{R})$ by fractional-linear transformations on the real projective line. Let c be any (unparametrized) curve through p, and suppose that $c(0) = p$. It is always possible to choose a γ such that c is a geodesic for ∇^γ in some parametrization, provided we do not demand that γ be a horizontal section of W. Let L be the subbundle of W consisting of all such connections γ. This is an affine line bundle, since γ is determined by the velocity and acceleration by (3.9).

Fix a parametrization t of c. Then, since (3.9) transforms under reparametrization $\tau(t)$ via

$$\gamma_a \mapsto \gamma_a + \alpha v_a,$$

where $\alpha = -\tau''(t)/\tau'(t)^2$ it follows that L consists of all one-forms of the form $\gamma_a + \alpha v_a$.

The restriction of the Ehresmann connection on W induces an Ehresmann connection on L. We can derive the condition for a section of L to be horizontal (in one dimension, every connection is integrable). In the ambient space, the section is horizontal if $P^{\gamma+\alpha v} = 0$. Thus in the one-dimensional space c, the condition is that $P^{\gamma+\alpha v}_{ab} v^a v^b = 0$:

$$P^0_{ab} v^a v^b - (v\cdot A)\alpha + \frac{v^4}{2}\alpha^2 - v^2 v(\alpha) - v\cdot\nabla^0_v\gamma + v^2\alpha(v\cdot\gamma) - \frac{v^2}{2}\gamma^2 + (v\cdot\gamma)^2 = 0.$$

Contracting (3.9) with v gives $v\cdot\gamma = -\frac{v\cdot A}{v^2}$.. Also $\gamma^2 = A^2/v^4$. Hence

$$P^0_{ab} v^a v^b - 2(v\cdot A)\alpha + \frac{v^4}{2}\alpha^2 - v^2 v(\alpha) - v\cdot\nabla^0_v\gamma - \frac{A^2}{2v^2} + \frac{(v\cdot A)^2}{v^4} = 0.$$

Differentiating (3.9) and contracting with v twice gives

$$v\cdot\nabla^0_v\gamma = -\frac{v\cdot\nabla^0_v A}{v^2} + 4\frac{(v\cdot A)^2}{v^4} - 2\frac{A^2}{v^2}.$$

So

$$P^0_{ab} v^a v^b - 2(v\cdot A)\alpha + \frac{v^4}{2}\alpha^2 - v^2 v(\alpha) + \frac{v\cdot\nabla^0_v A}{v^2} - \frac{5A^2}{2v^2} + 5\frac{(v\cdot A)^2}{v^4} = 0.$$

3.5 THREE-DIMENSIONS

We describe here a metric that is attributed to Sparling in [60] on a natural principal bundle over a conformal three-manifold. We then prove a theorem that the null geodesics of this metric project to conformal circles on M. We begin in arbitrary dimension, then specify to dimension three.

Definition 13. *Let M be a conformal manifold of signature (p, q). Let $CO(M)$ be the bundle of conformal linear mappings $u : \mathbb{R}^{p,q} \to TM$. This is a principal right $CO(p, q)$-bundle over M. Let $PCO(M) = CO(M)/GL(1)$. This is the* bundle of orthonormal frames *of M.*

Let $\theta = (\theta^1, \ldots, \theta^n) : TCO(M) \to \mathbb{R}^n$ be the canonical one-form, defined by $\theta(X_u) = u^{-1}(\pi_* X)$. A Weyl connection $\omega = (\omega_i^j) : TPCO(M) \to \mathfrak{co}(p, q)$ is any torsion-free $\mathfrak{co}(p, q)$ connection on $CO(M)$.

Lemma 18. *Up to scale, the tensor $\theta^{[i} \odot \omega^{jk]}$ does not depend on the choice of Weyl connection. Moreover, it is invariant under the $GL(1)$ action up to an overall scale and is annihilated by the vertical distribution of $CO(M) \to PCO(M)$.*

Proof. Fix a background metric g representing the conformal structure on M. Let ϖ be the Levi-Civita connection of g. Then there exists a one-form γ on M such that

$$\omega_i^j = \varpi_i^j + \gamma\delta_i^j + \gamma_i\theta^j - \gamma^j\theta_i$$

where $(\gamma_i)_u := \gamma(u_i)$ are the components of γ in the frame at each point of the frame bundle, and we have denoted by the same symbol γ the pullback of γ to $O(M)$. Evidently, $\theta^{[i} \odot \omega^{jk]} = \theta^{[i} \odot \varpi^{jk]}$.

Next, the metric g allows a reduction of the structure bundle $\pi : CO(M) \to O(M)$ to an $O(p, q)$. Let λ be the real scaling coordinate of $CO(M)/O(p, q)$. Then

$$\varpi_i^j = \pi^*\varpi_i^j + \frac{d\lambda}{\lambda}\delta_i^j.$$

from which the remaining assertions of the lemma follow. $\square$

3.5.1 The case $n = 3$

As a result of the lemma, if ϵ_{ijk} denotes the canonical skew-symmetric density on $CO(M)$, then the symmetric form $g_S = \epsilon_{ijk}\theta^i\omega^{jk}$ descends to a conformal structure on $PCO(M)$. The signature of this metric is always $(3, 3)$ (regardless of p and q).

The flat case is $SU(2) \times SU(2)$. The metric is $g_S = \omega^i_j \odot \theta^j_i$ where ω^j_i and θ^j_i are the Maurer–Cartan coframe on each factor of $SU(2)$. A null geodesic is a curve $(a(t), b(t))$ such that $a(t)^{-1}b(t)$ and $a(t)b(t)$ are both geodesics. Note that these project to circles on $SU(2)$.

In general, let $\omega^i = \epsilon^{ijk}\omega_{jk}$. The coframe (θ^i, ω^j) defines an absolute parallelism on $O(M)$. The Cartan structure equations are

$$\begin{aligned}
d\theta^i &= -\epsilon^i{}_{jk}\omega^j \wedge \theta^i \\
d\omega^i &= -\epsilon^i{}_{jk}\omega^j \wedge \omega^k + \frac{1}{2}\epsilon^i{}_{jk}G^k_\ell\theta^j \wedge \theta^\ell
\end{aligned}$$

where $G = \mathrm{Ric} - (\mathrm{tr}\, Ric)\,\mathrm{Id}/2$ is the Einstein tensor in dimension 3. These equations can be written in vector notation:

$$\begin{aligned}
d\boldsymbol{\theta} &= -\boldsymbol{\omega} \times \boldsymbol{\theta} \\
d\boldsymbol{\omega} &= -\boldsymbol{\omega} \times \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\theta} \times G(\boldsymbol{\theta}).
\end{aligned}$$

Let (p_i, q_j) be the associated fiber coordinates on $T^*O(M)$ associated to the absolute parallelism, so that the canonical one-form is

$$\alpha = p_i\theta^i + q_i\omega^i = \boldsymbol{p}\cdot\boldsymbol{\theta} + \boldsymbol{q}\cdot\boldsymbol{\omega}.$$

The symplectic form is thus

$$d\alpha = d\boldsymbol{p}\cdot\boldsymbol{\theta} + d\boldsymbol{q}\cdot\boldsymbol{\omega} - \boldsymbol{p}\cdot(\boldsymbol{\omega}\times\boldsymbol{\theta}) - \boldsymbol{q}\cdot(\boldsymbol{\omega}\times\boldsymbol{\omega}) + \frac{1}{2}\boldsymbol{q}\cdot(\boldsymbol{\theta}\times G(\boldsymbol{\theta})).$$

Let $H = \boldsymbol{p}\cdot\boldsymbol{q}$ be the Hamiltonian. We will find the null geodesic spray, so $H = 0$ and

$$dH = \boldsymbol{p}\cdot d\boldsymbol{q} + \boldsymbol{q}\cdot d\boldsymbol{p}.$$

Let

$$V = \frac{1}{2}(\boldsymbol{q}\times G(\boldsymbol{q}))\cdot\frac{\partial}{\partial\boldsymbol{p}} + (\boldsymbol{p}\times\boldsymbol{q})\cdot\frac{\partial}{\partial\boldsymbol{q}} - \boldsymbol{q}\cdot\frac{\partial}{\partial\boldsymbol{\theta}} - \boldsymbol{p}\frac{\partial}{\partial\boldsymbol{\omega}}.$$

Then V is the geodesic spray. Indeed,

$$\begin{aligned}
V \lrcorner d\alpha = \frac{1}{2}&(\boldsymbol{q}\times G(\boldsymbol{q}))\cdot\boldsymbol{\theta} + (\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{\omega} - \boldsymbol{p}\cdot(\boldsymbol{\omega}\times\boldsymbol{q}) - \\
&- \frac{1}{2}\boldsymbol{q}\cdot(\boldsymbol{\theta}\times G(\boldsymbol{q})) + \boldsymbol{p}\cdot d\boldsymbol{q} + 2\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{\omega}) \\
&= \boldsymbol{q}\cdot d\boldsymbol{p} + \boldsymbol{p}\cdot d\boldsymbol{q} = dH.
\end{aligned}$$

3.5.2 Conformal circles

Fix a metric g representing the conformal structure of M. Let ∇^γ be the Weyl connection characterized by

- $\nabla^\gamma g = -2\gamma \otimes g$
- $\nabla^\gamma_X Y - \nabla^\gamma_Y X = [X, Y]$.

A *circle* relative to the connection ∇^γ is a pair $(c(t), A(t))$ consisting of a curve $c(t)$ and an endomorphism $A(t) \in \mathfrak{co}(T_{\gamma(t)}M)$ such that

- The tracefree part of A is covariantly constant along c: $\nabla^\gamma_{\dot c}\left(A - \frac{\operatorname{tr} A}{n}\operatorname{Id}\right) = 0$
- $\dot c$ is an eigenvector of A^2
- $\nabla^\gamma_{\dot c}\dot c = A\dot c$

The trace of A represents a parametrization freedom in the curve.

Let $\omega_i^j \in \mathfrak{co}(p,q)$ be the connection 1-form on $CO(M)$ associated to the connection ∇^γ, and let θ^i be the canonical 1-form. Then (ω_i^j, θ^i) form an absolute parallelism of $CO(M)$, so there are coordinates $\boldsymbol{p} \in \mathbb{R}^n$ and $A \in \mathfrak{co}(p,q)$ in the fibers of $T^*CO(M)$ so that the canonical symplectic potential is given by $\alpha = \boldsymbol{p}\cdot\boldsymbol{\theta} + \operatorname{tr}(A\omega)$.

Define a vector field W on the total space of $T^*O(M)$ by

$$W = (A\boldsymbol{q})\cdot\frac{\partial}{\partial\boldsymbol{q}} - \boldsymbol{q}\cdot\frac{\partial}{\partial\boldsymbol{\theta}} - \operatorname{tr}\left(A\frac{\partial}{\partial\boldsymbol{\omega}}\right).$$

Lemma 19. *The integral curves of W project to circles on M.*

Definition 14. *A curve c in M is called a* conformal circle *if there exists a Weyl connection ∇^γ compatible with the conformal structure such that:*

- *c is a circle for the connection ∇^γ*
- *The Ricci curvature of ∇^γ vanishes identically along c.*

Theorem 14. *The following conditions on a smooth, non-null curve c in M are equivalent:*

1. *There exists a Weyl connection ∇^γ such that c is a circle for ∇^γ and the Ricci tensor of γ vanishes along c*
2. *There exists a Weyl connection ∇^γ such that c is a geodesic for ∇^γ and the Ricci tensor of γ vanishes along c*
3. *There exists a function $\lambda(t)$ such that*

$$\nabla_v A = v^2 P(v) - 2(v\cdot P(v))v - \frac{3A^2}{2v^2}v + \frac{3v\cdot A}{v^2}A + \left(\lambda' - \frac{\lambda^2}{2}\right)v$$

where ∇ is the Levi-Civita connection of any representative metric of the conformal class $v = \dot c$ is the velocity of the curve, $A = \nabla_v v$ is the acceleration, and $P : TM \to TM$ is the Schouten tensor.

3.5.3 Proof of theorem

We first prove the equivalence of (1) and (2). The implication that (1) implies (2) is trivial, since every geodesic is a circle. Conversely, suppose that

$$\nabla^\gamma_v v = Sv + \lambda v$$

where S is a skew-symmetric endomorphism that is parallel transported along c. Choose a conformal scale so that $v(g(v,v)) = \lambda$. Let

$$\gamma' = \frac{\mu v - Sv}{g(v,v)}$$

where μ is a function to be determined. Then

$$\begin{aligned}
\nabla^{\gamma+\gamma'}_v v &= \nabla^\gamma_v v + 2g(\gamma', v)v - g(v,v)\gamma' \\
&= Sv + \lambda v + 2\mu v - \frac{2g(v, Sv)}{v^2} v - \mu v - Sv \\
&= (\lambda + \mu)v.
\end{aligned}$$

So c is a geodesic with respect to the connection $\nabla^{\gamma+\gamma'}$.

Next we shall select μ so that the Ricci curvature vanises along c. We have

$$\begin{aligned}
v^a P^{\gamma+\gamma'}_{ab} &= -v^a \nabla^\gamma_a \gamma'_b + v^a \gamma'_a \gamma'_b - \frac{1}{2}\gamma'_c \gamma'^c v_b \\
&= \lambda\gamma' - \frac{v(\mu)v + \mu Sv + \mu\lambda v - S^2 v - \lambda Sv}{g(v,v)} + \\
&\quad + \frac{\mu^2 v + \mu Sv}{g(v,v)} - \frac{1}{2}\left(\frac{\mu^2}{g(v,v)} + \frac{g(Sv,Sv)}{g(v,v)^2}\right) v \\
&= \frac{-v(\mu)v + S^2 v + (\mu^2/2)v - (2g(v,v))^{-1} g(Sv,Sv)v}{g(v,v)}.
\end{aligned}$$

Since v is an eigenvector of S^2, this is proportional to v. Setting it to zero is a first-order ordinary differential equation in μ that can be solved locally.

Finally, we will prove the equivalence of (2) and (3). A Weyl connection is any connection ∇^γ that satisfies

$$\nabla^\gamma g = -2\gamma \otimes g$$

where γ is a one-form. The mapping $\gamma \mapsto \nabla^\gamma$ sets up an isomorphism between the bundle W of admissible Weyl connections and the cotangent bundle. This mapping depends on the choice of conformal scale, in terms of which there is a preferred section ∇^0: the Levi-Civita connection. Any other connection ∇^γ can be represented explicitly in terms of ∇^0 by the formula

$$\nabla^\gamma_X Y = \nabla^0_X Y + \gamma(X)Y + \gamma(Y)X - \langle X, Y\rangle g^{-1}(\gamma)$$

which is to say that the change of connection is expressed by the connection coefficients

$$\Gamma^a_{bc} = \gamma_b\delta^a_c + \gamma_c\delta^a_b - \gamma^a g_{bc}.$$

The rho tensor transforms via

$$P^\gamma_{ab} = P^0_{ab} - \nabla^0_a\gamma_b + \gamma_a\gamma_b - \frac{1}{2}\gamma_c\gamma^c g_{ab}.$$

Suppose that c is a curve, and γ is the horizontal lift of that curve. By definition, this means that $P^\gamma = 0$ along the lifted curve. Equivalently,

$$\nabla^0_a\gamma_b = P^0_{ab} + \gamma_a\gamma_b - \frac{1}{2}\gamma_c\gamma^c g_{ab}. \tag{3.7}$$

Now, let $v^a = \dot{c}^a$ be the velocity along the curve c and $A^a = v^b\nabla^0_b v^a$ the acceleration (relative to the Levi-Civita connection). The condition for c to be a conformal circle is that $v^b\nabla^\gamma_b v^a = \lambda v^a$, where γ is a lift of c satisfying $P^\gamma = 0$. Equivalently,

$$A^a + 2v^b\gamma_b v^a - v_b v^b\gamma^a = \lambda v^a. \tag{3.8}$$

Taking A^a and v^a as given, we can solve this equation explicitly for γ:

$$\gamma^a = \frac{U^a}{v^c v_c} - 2\frac{v^d U_d}{(v^c v_c)^2}v^a \tag{3.9}$$

where $U^a = A^a - \lambda v^a$.

On the other hand, we may differentiate both sides of (3.8) once more by applying $v^c\nabla^0_c$, then use (3.7) to reduce the resulting equation to first order in γ, and finally use (3.9) to eliminate γ from the resulting expression. The final equation is thus a third-order condition on the curve c.

The right-hand side differentiates to $\lambda' v^a + \lambda A^a$. Carrying through the procedure described for the left-hand side, we have

$$v^c\nabla^0_c v^b\nabla^\gamma_b v^i = v^c\nabla^0_c A^i + 2v^c\nabla^0_c(v^b\gamma_b v^i) - v^c\nabla^0_c(v^b v_b\gamma^i).$$

To each of the terms involving γ, apply (3.7) and (3.8), followed by (3.9). The result is

$$v^c\nabla^0_c v^b\nabla^\gamma_b v^i = v^b\nabla^0_b A^i - v^2 P^{0i}_a v^a + 2P^0_{ab}v^a v^b v^i + \frac{3A^2}{2v^2}v^i - \frac{3v\cdot A}{v^2}A^i + \lambda A + \frac{1}{2}\lambda^2 v^i.$$

Thus a curve c is a conformal circle if and only if

$$v^b\nabla^0_b A^i = v^2 P^i_a v^a - 2P^0_{ab}v^a v^b v^i - \frac{3A^2}{2v^2}v^i + \frac{3v\cdot A}{v^2}A^i + \lambda' v + \frac{1}{2}\lambda^2 v.$$

3.5.4 $n = 3$ revisited

Corollary 4. *In dimension $n = 3$, the null geodesics of the metric g_S project to conformal circles on M.*

Proof. Let C be a null geodesic for g_S and c its projection. Choose a Weyl connection ∇^γ such that the Ricci tensor vanishes identically along c. The curve C is then an integral curve of the vector field

$$V|_{G=0} = (\boldsymbol{p} \times \boldsymbol{q}) \cdot \frac{\partial}{\partial \boldsymbol{q}} - \boldsymbol{q} \cdot \frac{\partial}{\partial \boldsymbol{\theta}} - \boldsymbol{p} \frac{\partial}{\partial \boldsymbol{\omega}}.$$

So C satisfies the differential equation

$$\dot{c} = \boldsymbol{q}$$
$$\nabla^\gamma_{\dot{c}} \dot{c} = -\boldsymbol{p} \times \boldsymbol{q} = A\boldsymbol{q}$$

with $A = \boldsymbol{p} \times -$. In particular, as $\boldsymbol{p}$ is constant along c, so is A. Moreover, at any point of c, we have

$$\begin{aligned} A^2 \dot{c} &= \boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{q}) = (\boldsymbol{p} \cdot \boldsymbol{q})\boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{q} \\ &= H\boldsymbol{q} - (\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{q} = -(\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{q} \end{aligned}$$

since $H = 0$ (the geodesic C is null). So the curve c satisfies the three conditions defining a conformal circle in §3.5.2. □

4.0 PARABOLIC GEOMETRIES

The purpose of this chapter is to describe a large class of geometries that admit Weyl connections in the sense described in the previous chapter. These are the *parabolic geometries* and are precisely those that are associated to Cartan connections modeled on the quotient of a Lie group by a parabolic subgroup. Such geometries include the conformal and projective geometries already discussed, and a host of others. A survey appears in the textbook of [11].

4.1 PARABOLIC GROUPS

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. A Borel subgroup of G is a Lie subgroup B whose Lie algebra $\mathfrak{b}$ is a Borel subalgebra: a maximal solvable subalgebra of $\mathfrak{g}$. A *parabolic subalgebra* $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ that contains a Borel subalgebra, and a subgroup of G is called parabolic if its Lie algebra is.

Suppose that $\mathfrak{t}$ is a maximal toral subalgebra of the Borel subalgebra $\mathfrak{b}$, and let $\Phi \subset \mathfrak{t}^*$ be the root system for $\mathfrak{g}$. Then $\mathfrak{b}$ is specified uniquely by knowing $\mathfrak{t}$ and a subset $\Delta^+ \subset \Phi$ consisting of the associated simple positive roots: it is the direct sum of the non-negative root spaces. A parabolic subalgebra $\mathfrak{p}$ splits into eigenspaces for Φ, and so is completely determined by saying which *negative* root spaces that it contains. These, in turn, are determined by specifying the root spaces corresponding to those negative *simple* roots (elements of $-\Delta^+$) that it contains. The exclusion of a negative simple root from a parabolic subgroup can be annotated in terms of the Dynkin diagram of $\mathfrak{g}$ by crossing out the corresponding node. Thus, in particular, the Borel subalgebra has all the nodes crossed, and the whole of $\mathfrak{g}$ has none of the nodes crossed. The parabolic Lie algebra (and group) obtained in this manner is said to *be associated to* the uncrossed simple roots. The parabolic containing a given Borel is uniquely determined by these root data, and so this characterizes the parabolic up to equivalence.

Associated to a parabolic subgroup P of G is the homogeneous space G/P. This is a compact manifold (in fact, it is a nonsingular projective variety). This homogeneous space is can be identified with an adjoint orbit of the maximal compact subgroup of G, and has the form G_c/P_c where G_c and P_c are the maximal compact subgroups of G and P, respectively.

Suppose now that G is a *real* reductive Lie group containing a maximal torus T. Then T decomposes into a maximal split torus times a compact part. The possibilities here are characterized by the Satake diagram of the group, obtained from the Dynkin diagram of the complexification of G by showing the action of a Cartan involution on the associated root spaces according to the following rules:

- If the involution interchanges two root spaces associated to a pair of simple compact roots, then an arrow is drawn connecting the roots of the Dynkin diagram.
- The (remaining) root spaces that are associated to compact roots are invariant under conjugation, and are shaded.
- The remaining root spaces are all associated to non-compact roots, and these are left unshaded.

Possible Satake diagrams coming from Cartan's classification are given in Tables 1–6 (see also Helgason [37]).

A Borel subgroup of G is obtained first by going up to the complexification $G^{\mathbb{C}}$, and then taking the real points of a Borel group in $G^{\mathbb{C}}$. Likewise, the parabolic groups are obtained first by complexifying and then restricting. So parabolic subgroups of a real reductive group G can be specified by decorating the Satake diagram depending on whether or not a root is included in the complexified parabolic.

In general, the more compact roots are included in the real form, the smaller the Borel group becomes, and the Borel subgroup of the compact form is just the maximal torus itself. To describe the geometry of certain homogeneous spaces, it is convenient to introduce the *fat Borel subgroup*, which is the smallest parabolic subgroup containing all of the compact simple root spaces. (Thus the crossed nodes in the Satake diagram coincide with the unshaded nodes for the fat Borel.)

4.2 LINEAR ALGEBRA

In this section, $\mathbb{F}$ is a field of characteristic different from 2.

4.2.1 Affine spaces

An affine space over $\mathbb{F}$ is a set $\mathbb{A}$ equipped with a free and transitive action of the additive group of some vector space $\mathbb{V}$ over $\mathbb{F}$ on $\mathbb{A}$, denoted by $v.x$ for $v \in \mathbb{V}$ and $x \in \mathbb{A}$. In that case, $\mathbb{A}$ will be called an *affine torsor* for $\mathbb{V}$. By definition, if $p, q \in \mathbb{A}$ then there is a unique element of $\mathbb{V}$ denoted by $\overrightarrow{pq}$ such that

$$\overrightarrow{pq}.p = q.$$

The affine space $\mathbb{A}$ is a principal homogeneous space of the group $\mathbb{V}$ and, once a particular point $p \in \mathbb{A}$ is chosen, the association of each $x \in \mathbb{A}$ the vector $\overrightarrow{px} \in \mathbb{V}$ establishes a $\mathbb{V}$-equivariant isomorphism from $\mathbb{A}$ to $\mathbb{V}$. For such an isomorphism, the point p is said to *witness* an origin for $\mathbb{A}$.

A function $f : \mathbb{A} \to \mathbb{B}$ of affine spaces is called *affine linear* if

$$f((t\overrightarrow{pq}).p) = (t\overrightarrow{f(p)f(q)})).f(p)$$

for all $p, q \in \mathbb{A}$ and $t \in \mathbb{F}$. An affine linear functional is an affine linear function whose target is the underlying affine space of the ground field $\mathbb{F}$.

By the dual space to an affine space $\mathbb{A}$, we shall mean the linear space of all affine-linear functionals on $\mathbb{V}$ *modulo the constant functions.* Thus the dual space of $\mathbb{A}$ is canonically isomorphic to the dual space of $\mathbb{V}$.

4.2.2 Filtered vector spaces

Let $\mathbb{V}$ be a vector space. A *filtration* on $\mathbb{V}$ of length n is an ascending chain of linear subspaces $0 = \mathbb{V}_0 \subset \mathbb{V}_1 \subset \mathbb{V}_2 \subset \cdots \subset \mathbb{V}_n \subset \mathbb{V}_{n+1} = \mathbb{V}$. When we wish to refer to a filtration as a whole, we will denote the n-tuple $(\mathbb{V}_1, \ldots, \mathbb{V}_n)$ by $\mathbb{V}_\bullet$.

The dimensions $d_i = \dim \mathbb{V}_i$ are an important invariant of the filtration, called the *dimension vector* of the filtration and denoted by $d_\bullet$. The set of all dimension vectors forms a lattice under inclusion.

For $i < j$ let $\iota_{i,j} : \mathbb{V}_i \to \mathbb{V}_j$ be the inclusion mappings, and let $\pi_{i,j} : \mathbb{V}/\mathbb{V}_i \to \mathbb{V}/\mathbb{V}_j$ be the associated projections. A *splitting* of the filtration on $\mathbb{V}$ is a family of maps $p_i : \mathbb{V} \to \mathbb{V}$, $i = 1, 2, \ldots, n$, such that the following conditions hold:

(a) $p_1 + p_2 + \cdots + p_n = \mathrm{Id}$

(b) $p_i \circ p_i = p_i$ for all i

(c) $p_i \circ p_j = 0$ for all $i \neq j$

(d) $\pi_{0,i} \circ p_i = 0$ for all i (that is, $\operatorname{im} p_i \subset \mathbb{V}_i$)

It follows from (a) and (d) that $\pi_{0,i} \circ p_{i+1} : \mathbb{V} \to \mathbb{V}_{i+1}/\mathbb{V}_i$ is surjective, and that $\mathbb{V}_i \subset \ker p_{i+1}$.

The kernel of the mapping

$$p_{i+1} + p_{i+2} + \cdots + p_n$$

contains $\mathbb{V}_i$, so it descends to the quotient to give a map $S_i : \mathbb{V}/\mathbb{V}_i \to \mathbb{V}$ that satisfies

$$\pi_{0,i} \circ S_i = \mathrm{id}_{\mathbb{V}/\mathbb{V}_i} . \tag{4.1}$$

Conversely, given an n-tuple of linear maps $S_i : \mathbb{V}/\mathbb{V}_i \to \mathbb{V}$ satisfying (4.1), the maps $p_i = S_i \circ \pi_{0,i+1} : \mathbb{V} \to \mathbb{V}$ satisfy (a)–(d).

The space of all splittings with a given dimension vector is thus identified with the affine space $\mathbb{S}(\mathbb{V}_\bullet)$ of solutions $(S_1, \ldots, S_n)$ to (4.1). Once a particular splitting is chosen to witness an origin for this affine space, it can be identified with the space of all n-tuples of linear maps from from $\mathbb{V}/\mathbb{V}_i \to \mathbb{V}$ whose image is in $\mathbb{V}_i = \ker \pi_{0,i}$.

Hence $\mathbb{S}(\mathbb{V}_\bullet)$ *is an affine torsor for* $\bigoplus_i \mathrm{Hom}(\mathbb{V}/\mathbb{V}_i, \mathbb{V}_i)$.

Dual to the S_i are the projection mappings $P_i : \mathbb{V} \to \mathbb{V}_i$ defined by the partial sums

$$P_i = p_1 + p_2 + \cdots + p_i.$$

The P_i satisfy

$$\iota_{0,i} \circ P_i = \mathrm{id}_{\mathbb{V}_i} \tag{4.2}$$

Conversely, any n-tuple of mappings $P_i : \mathbb{V} \to \mathbb{V}_i$ satisfying (4.2) determines a splitting by setting $p_i = P_i - P_{i-1}$.

4.2.3 Isotropic flags

Let $\mathbb{X}$ be a finite-dimensional vector space over $\mathbb{F}$, fix $\epsilon \in \{+1, -1\}$, and let $h : \mathbb{X} \times \mathbb{X} \to \mathbb{F}$ be a non-degenerate bilinear form satisfying

$$\epsilon h(x, y) = h(y, x) \tag{4.3}$$

for all $x, y \in \mathbb{X}$.

A pair of linear subspaces $\mathbb{U}$ and $\mathbb{V}$ of $\mathbb{X}$ are *orthogonal* under h if $h(u, v) = 0$ for all $u \in \mathbb{U}$ and $v \in \mathbb{V}$. The set of subspaces orthogonal to a given subspace $\mathbb{U}$ is a complete lattice ordered by subspace inclusion, and so has a maximal element: the *orthogonal annihilator* of $\mathbb{U}$, denoted $\mathbb{U}^{\perp_h}$. The linear subspace $\mathbb{U}$ is *isotropic* if $\mathbb{U} \subset \mathbb{U}^{\perp_h}$. That is, $h|_{\mathbb{U}\times\mathbb{U}}$ is identically zero. The meaning of the term *isotropic* in this context is that it is impossible to distinguish between two elements of $\mathbb{U}$ using only h. In view of the non-degeneracy of h, the dimension of an isotropic subspace cannot exceed half the dimension of $\mathbb{X}$. In $\mathbb{X} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$, this necessary condition for the existence of an isotropic subspace is also sufficient: there exist isotropic subspaces of every dimension not exceeding half the dimension of $\mathbb{X}$.

All maximal isotropic subspaces of $\mathbb{X}$ have the same dimension; this dimension is called the (Witt) *index* $s(h)$ of the bilinear form h. For instance, if $\epsilon = -1$, then the dimension of $\mathbb{X}$ is even and $s(h)$ is one half the dimension. If $\mathbb{F}$ is algebraically closed, then the index is $s(h) = \lfloor \frac{\dim \mathbb{X}}{2} \rfloor$. Any maximal isotropic space in $\mathbb{X} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ contains a $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$-invariant subspace of dimension $s(h)$.

A subspace $\mathbb{U} \subset \mathbb{X}$ is *coisotropic* if $\mathbb{U}^{\perp_h} \subset \mathbb{U}$. That is, if $\mathbb{U}^{\perp_h}$ is isotropic. A space is *totally anisotropic* if $\mathbb{U} \cap \mathbb{U}^{\perp_h}$ is the zero subspace. A subspace is totally anisotropic if and only if $h|_{\mathbb{U}\times\mathbb{U}}$ is itself non-degenerate.

A filtration of $\mathbb{X}$ by linear subspaces

$$0 = \mathbb{X}_0 \subset \mathbb{X}_1 \subset \cdots \mathbb{X}_n \subset \mathbb{X}_{n+1} = \mathbb{X}$$

is called an *isotropic flag* if each of $\mathbb{X}_1, \ldots, \mathbb{X}_n$ is isotropic. There are no isotropic flags at all unless $2d_n \leq \dim \mathbb{X}$, and otherwise all isotropic flags with a given dimension vector can be described by first picking an isotropic subspace $\mathbb{X}_n$ of dimension d_n (if there is one), and then considering all filtrations of $\mathbb{X}_n$ with dimension vector $(d_1, \ldots, d_{n-1})$.

4.2.4 Adjoints, unitary transformations, and projections

The *adjoint* of a linear tranformation $A : \mathbb{X} \to \mathbb{X}$ is the linear transformation $A^{*_h} : \mathbb{X} \to \mathbb{X}$ defined by $h(Ax, y) = h(x, A^{*_h} y)$ for all $x, y \in \mathbb{X}$. In view of the symmetry condition (4.3), there is no distinction between left and right adjoints. The involution $A \mapsto A^{*_h}$ defines an algebra isomorphism $\mathrm{End}(\mathbb{X}) \to \mathrm{End}(\mathbb{X})^{\mathrm{op}}$. The fixed points of this involution are called *self-adjoint* operators.

An operator $U \in \mathrm{End}(\mathbb{X})$ is *unitary* if $U^{*_h} = U^{-1}$. The *unitary group* $\mathbb{U}(h)$ is the group of all unitary mappings.

An operator P is called a *projection* onto a subspace $\mathbb{V} \subset \mathbb{X}$ if $\operatorname{im} P = \mathbb{V}$ and $P^2 = P$. A projection is called *isotropic* if its image is isotropic and coisotropic if its kernel is.[1] Any isotropic projection is unitarily diagonalizable, and in particular all projections onto the same isotropic subspace are conjugate to one another under $\mathbb{U}(h)$. If P is a projection onto a linear subspace $\mathbb{V} \subset \mathbb{X}$, then $\mathrm{Id}_{\mathbb{X}} - P^{*_h}$ is a coisotropic projection onto $\mathbb{V}^{\perp}$.

[1] The image of an operator is isotropic if and only if $A^{*_h} A = A A^{*_h} = 0$.

4.2.5 Duality

Extend the bilinear form h to the exterior powers $\wedge^k \mathbb{X}$ by setting

$$h(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det(h(x_i, y_j))_{1 \le i,j \le k}.$$

Let $n = \dim \mathbb{X}$. Let $\Sigma \in \wedge^n \mathbb{X}$ be an arbitrary volume form such that $h(\Sigma, \Sigma) = d \neq 0$. Modulo squares in $\mathbb{F}$, d is uniquely defined. Its class in the group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$ is an important invariant of h.

Define an operator $\star : \wedge^k \mathbb{X} \to \wedge^{n-k} \mathbb{X}$ by

$$(\star\alpha) \wedge \beta = h(\alpha, \beta)\Sigma$$

for all $\alpha, \beta \in \wedge^k \mathbb{X}$. Acting on p-forms, $\star^2 = (-1)^{p(n-p)} d$.

4.2.6 Even dimensions

In particular, if $n = 2m$ is even, then $\star \in \mathrm{End}(\wedge^m \mathbb{X})$ and, acting on m-forms, $\star^2 = (-1)^m d$. The eigenvalues of $\star$ are $\pm\sqrt{\delta}$ where $\delta = (-1)^m d$, and $\star$ diagonalizes over $\mathbb{F}(\sqrt{\delta})$. The eigenspaces are complementary isotropic subspaces in $\wedge^m \mathbb{X} \otimes \mathbb{F}(\sqrt{\delta})$. If δ is already a square in $\mathbb{F}$, then the eigenspaces are in $\wedge^m \mathbb{X}$. Otherwise, there is a nontrivial field automorphism σ of $\mathbb{F}(\sqrt{\delta})$ fixing $\mathbb{F}$, and a pair of σ-conjugate eigenspaces $\wedge^m_+ \mathbb{X}$ and $\wedge^m_- \mathbb{X}$ in $\wedge^m \mathbb{X} \otimes \mathbb{F}(\sqrt{\delta})$ corresponding, respectively, to the eigenvalues $+\sqrt{\delta}$ and $-\sqrt{\delta}$.

A maximal isotropic linear subspace $\mathbb{U} \subset \mathbb{X} \otimes \overline{\mathbb{F}}$ is called *self-dual* if $\wedge^m \mathbb{U}$ is a (one-dimensional) subspace of $\wedge^m_+ \mathbb{X} \otimes \overline{\mathbb{F}}$, and *antiself-dual* if it is instead a subspace of $\wedge^m_- \mathbb{X} \otimes \overline{\mathbb{F}}$. Any maximal isotropic space is either self-dual or antiself-dual. The mapping $\mathbb{U} \mapsto \wedge^m \mathbb{U}$ from maximal isotropic subspaces into the projective space of $\wedge^m \mathbb{X} \otimes \overline{\mathbb{F}}$ is one-to-one onto a nonsingular projective variety of dimension $m(m-1)/2$ which is the projectivization of the variety of simple elements of the exterior power.

4.2.7 Grassmannians

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. For a fixed dimension vector $d_\bullet$, define the *Grassmannian* of type $d_\bullet$, denoted $\mathrm{Gr}_{d_\bullet}(\mathbb{V})$, to be the set of all filtrations of $\mathbb{V}$ with dimension vector $d_\bullet$. A local coordinate system on $\mathrm{Gr}_{d_\bullet}(\mathbb{V})$ is provided by putting a basis of $\mathbb{V}$ into a normal form under the action of a block upper-triangular subgroup of $\mathrm{GL}(\mathbb{V})$ with blocks down the diagonal of size d_i. Generically, such a normal form is block lower-triangular with the identity matrix down the main diagonal. The Weyl group of $\mathrm{GL}(\mathbb{V})$ moves from one coordinate patch to another, and the transition on the overlap of two coordinate patches is given by polynomial functions in the coordinates. Thus $\mathrm{Gr}_{d_\bullet}(\mathbb{V})$ carries naturally the structure of an algebraic variety over $\mathbb{F}$.

If $d_\bullet \subset d'_\bullet$ are two dimension vectors, one included in the other, then there is a mapping of the Grassmannians $\pi_{d'_\bullet, d_\bullet} : \mathrm{Gr}_{d'_\bullet}(\mathbb{V}) \to \mathrm{Gr}_{d_\bullet}(\mathbb{V})$ given by forgetting the parts of the $d'_\bullet$-filtration that do not have dimensions corresponding to a $d_\bullet$-filtration.

4.2.8 Real, complex, and quaternionic structures

We now specialize to the case of $\mathbb{F} = \mathbb{R}$. An endomorphism $J : \mathbb{V} \to \mathbb{V}$ such that $J^2 = -I$ is called a *complex structure*. This makes $\mathbb{V}$ into a $\mathbb{C}$-module; in particular, $\mathbb{V}$ necessariliy has even dimension. Because the minimal polynomial is an irreducible quadratic $x^2 + 1$, J is diagonalizable over $\mathbb{C}$ with eigenvalues $\pm i$. The eigenspaces of J in the complexification $\mathbb{V} \otimes \mathbb{C}$ are each isomorphic as real vector spaces to $\mathbb{V}$. Denote them by $\mathbb{W}$ and $\overline{\mathbb{W}}$. The complex structure acts by multiplication by i on $\mathbb{W}$ and by $-i$ on $\overline{\mathbb{W}}$.

A *quaternionic structure* on $\mathbb{V}$ is a pair of anticommuting complex structures J and K. Given such a pair, JK is also a complex structure and $\mathbb{V}$ is a module over the quaternion algebra $\mathbb{H}$ via the map $\mathbb{H} \to \mathrm{End}(\mathbb{V})$ that sends $j \mapsto J$, $k \mapsto K$, $i \mapsto JK$. This gives $\mathbb{V}$ the structure of an $\mathbb{H}$-module; in particular, $\mathbb{V}$ has dimension a multiple of 4.

If $\mathbb{V}$ is a complex vector space, then a *real structure* on $\mathbb{V}$ is a complex-linear isomorphism $\sigma : \mathbb{V} \to \overline{\mathbb{V}}$. These spaces are already isomorphic as real vector spaces, and so we identify them as real vector spaces. If $\mathbb{W}$ is the set of fixed points of σ, then $\mathbb{V} = \mathbb{W} \oplus i\mathbb{W}$, so $\mathbb{V}$ is the complexification of $\mathbb{W}$.

Table 1: Lie algebras of type A

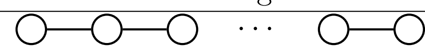
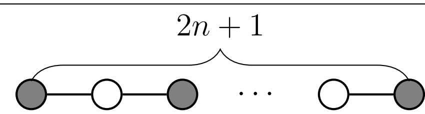
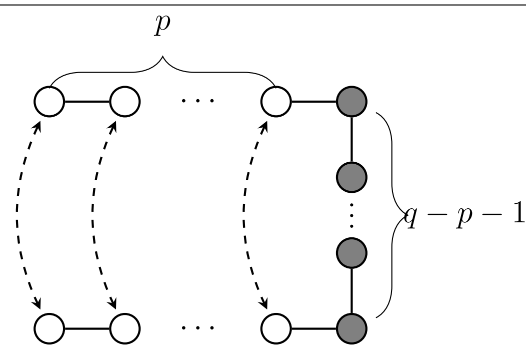
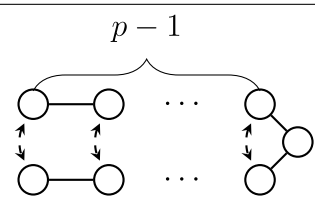
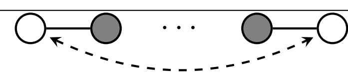
	Cartan type	Satake diagram	Group	Flags
A_n	AI (split)		$\mathrm{SL}_n(\mathbb{R})$	Filtrations of $\mathbb{R}^{n+1}$
	AII		$\mathrm{SL}_n(\mathbb{H})$	Filtrations of $\mathbb{H}^{n+1}$ by quaternionic (resp. complex) subspaces
	AIII		$SU(p,q)$	Flags in $\mathbb{C}^{p+q}$, the first p of which are isotropic and the last p coisotropic
	AIII (semisplit)		$SU(p,p)$	Flags in $\mathbb{C}^{2p}$, the first p of which are isotropic and the last p coisotropic
	AIV		$SU(1,p)$	Flags in $\mathbb{C}^{p+1}$, the first one being isotropic and the last one coisotropic

Table 2: Lie algebras of type B, C

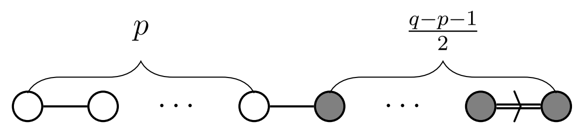
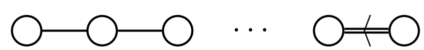
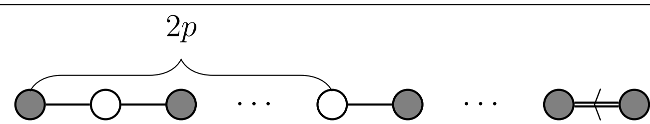
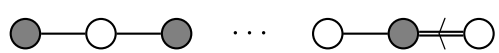
	Cartan type	Satake diagram	Group	Flags
B_n	BI		$\mathrm{SO}(p,q)$	Isotropic flags in $\mathbb{C}^{2n+1}$ the first p of which are complexifications of totally real subspaces, where the underlying real space is equipped with a pseudo-orthogonal structure of Witt index p
B_n	BII		$\mathrm{SO}(1,n-1)$	Isotropic flags in $\mathbb{C}^n$ the first of which is the complexification of a totally real one-dimensional subspace, where the underlying real space is equipped with a Lorentzian pseudo-orthogonal structure
C_n	CI (split)		$\mathrm{Sp}(2n,\mathbb{R})$	Isotropic flags in $\mathbb{R}^{2n}$, equipped with a symplectic structure
	CII		$\mathrm{Sp}(p,q)$	Isotropic $\mathbb{C}\otimes\mathbb{H}$-flags in $\mathbb{C}\otimes\mathbb{H}^{p+q}$, the first p of which are complexifications of $\mathbb{H}$-subspaces.
	CII		$\mathrm{Sp}(p,p)$	Isotropic $\mathbb{C}\otimes\mathbb{H}$-flags in $\mathbb{C}\otimes\mathbb{H}^{2p}$, the first p of which are complexifications of $\mathbb{H}$-subspaces.

Table 3: Lie algebras of type DI

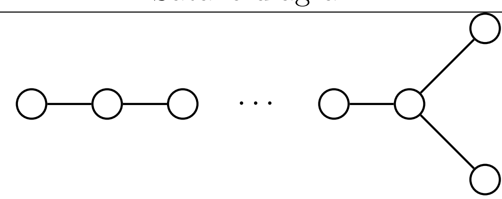
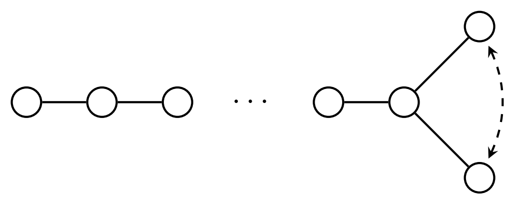
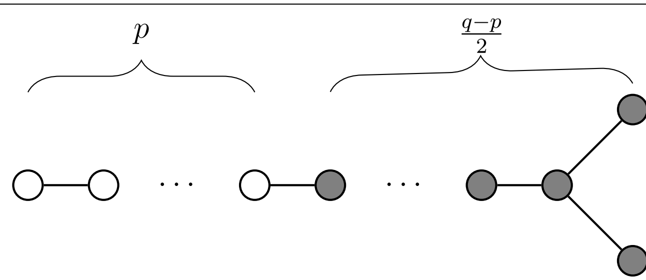
	Cartan type	Satake diagram	Group	Flags
D_n	DI (split)		$\mathrm{SO}(p,p)$	Isotropic flags in $\mathbb{R}^{2p}$, equipped with a pseudoorthogonal structure of Witt index p
	DI (semisplit)		$\mathrm{SO}(p-1,p+1)$	Isotropic flags in $\mathbb{C}\otimes\mathbb{R}^{2p}$ the first $p-1$ of which are the complexifications of totally real subspaces, with $\mathbb{R}^{2p}$ equipped with a pseudoorthogonal structure of Witt index $p-1$
	DI		$\mathrm{SO}(p,q)$	Isotropic flags in $\mathbb{C}\otimes\mathbb{R}^{p+q}$ the first p of which are the complexifications of totally real subspaces, with $\mathbb{R}^{p+q}$ equipped with a pseudoorthogonal structure of Witt index p

Table 4: Lie algebras of type DII and $DIII$

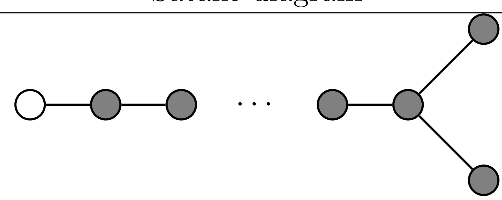
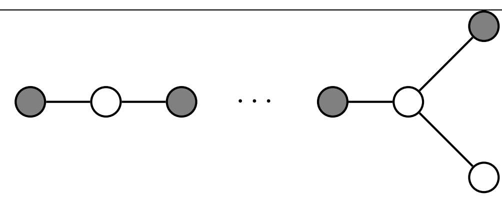
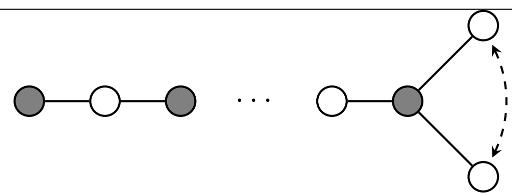
	Cartan type	Satake diagram	Group	Flags
	DII		$\mathrm{SO}(1, n-1)$	Isotropic flags in $\mathbb{C} \otimes \mathbb{R}^n$ the first of which is the complexification of a totally real subspaces, with $\mathbb{R}^n$ equipped with a Lorentzian pseudoorthogonal structure
	DIII		$\mathrm{SO}^*(2n) = \mathrm{SO}(n, \mathbb{H})$	Quaternionic (resp. complex) isotropic flags in $\mathbb{R}^{4n}$, equipped with an orthogonal structure and compatible quaternionic structure
	DIII		$\mathrm{SO}^*(2n) = \mathrm{SO}(n, \mathbb{H})$	

Table 5: Exceptional Lie algebras of type E

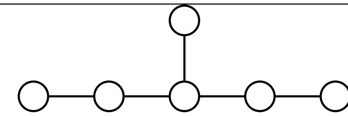
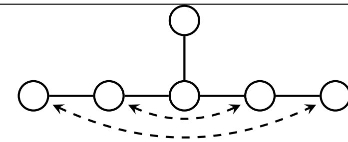
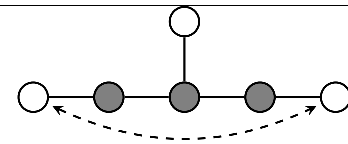
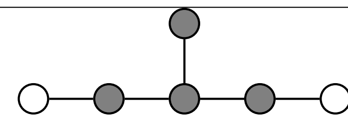
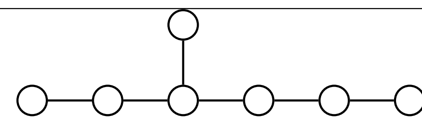
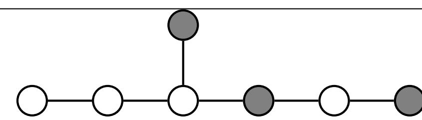
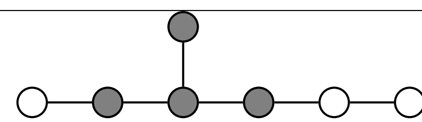
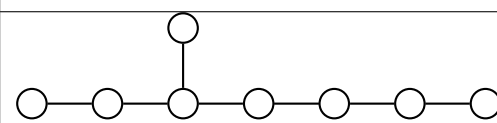
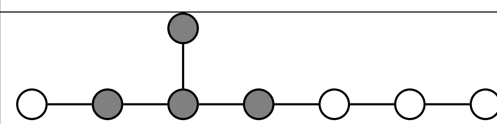
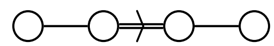
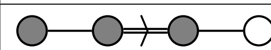
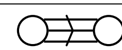
	Cartan type	Satake diagram	Group
E_6	E_I		${}^1E_{6,6}^0$
	E_{II}		${}^2E_{6,4}^2$
	E_{III}		${}^2E_{6,2}^{16}$
	E_{IV}		${}^1E_{6,2}^{28}$
E_7	E_V		$E_{7,7}^0$
E_7	E_{VI}		$E_{7,4}^9$
E_7	E_{VII}		$E_{7,7}^0$
E_8	E_{VIII}		$E_{8,8}^0$
	E_{IX}		$E_{8,4}^{28}$

Table 6: Exceptional Lie algebras of type F, G

	Cartan type	Satake diagram	Group
F_4	FI		$F^0_{4,4}$
	FII		$F^{21}_{4,1}$
G_2	G		$G^0_{2,2}$

4.3 EXCEPTIONAL GROUPS

The real forms of the exceptional Lie groups can also be put into these terms. We refer to the original works of Freudenthal [29], as well as the recent investigations of Landsberg and Manivel [51]. The parabolic homogeneous spaces are organized by Freudenthal's magic square (due to Freudenthal and Tits [80],[81]). The magic square also classifies certain Lie algebras connected with Jordan algebras in dimension 3, and there are many constructions from this perspective; Baez [6] contains a survey.

Table 7: Geometries of Freudenthal's magic square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
2-dimensional elliptic geometry	B_1	A_2	C_3	F_4
2-dimensional projective geometry	A_2	$A_2 + A_2$	A_5	E_6
5-dimensional symplectic geometry	C_3	A_5	D_6	E_7
metasymplectic geometry	F_4	E_6	E_7	E_8

Here the two dimensional elliptic geometries correspond to the projective planes over the four real division algebras:

- $\mathbb{RP}^2$: the symmetric space associated with the Lie group $SO(3)$ of type B_3
- $\mathbb{CP}^2$: the symmetric space associated to the special unitary group $SU(3)$ of type A_2
- $\mathbb{HP}^2$: the symmetric space associated to the quaternionic unitary group $Sp(3)$ of type C_3
- $\mathbb{OP}^2$: the Moufang plane, which is the symmetric space $F_4/\operatorname{Spin}(9)$

The groups are all compact at this level. In elliptic geometry, there is no distinction between points and lines: the polar of every point is a line and vice versa. The distinction between points and lines appears at the level of projective geometry.

At the projective level, there is a larger group of transformations corresponding to collineations in the 2-dimensional elliptic geometries. These groups are non-compact, given (respectively) by

- The group of projective transformations of the real projective plane, $\mathrm{PGL}_3(\mathbb{R})$, a real Lie group of split type A_2.
- The group of projective transformations of the complex projective plane, $\mathrm{PGL}_3(\mathbb{C})$, a *real* Lie group of type $A_2 \times A_2$ (with one factor split and the other compact).

- The group $\mathrm{SL}_3(\mathbb{H}) = \mathrm{SU}^*(6)$, a group of type A_5 (see tables)
- The group E_6

Associated to each of the projective spaces is the cotangent bundle of the vector space underlying the projective space. This symplectic space carries a symplectic action of the group of collineations, but also admits a larger group action by all linear symplectic transformations that respect the underlying vector space structure (real, complex, quaternionic, or octonionic). For the associative division algebras, this is essentially a classical construction. For the non-associative algebra of octonions, the symplectic space is described explicitly in terms of the exceptional Jordan algebra in Freudenthal [29]. For a recent account, with explicit formulas, see Sparling and Tillman [77] and Landsberg and Manivel [51]. The relevant groups that appear on this row of the table are, respectively, $\mathrm{Sp}(6,\mathbb{R})$, $\mathrm{SU}(3,3)$, $\mathrm{SO}^*(12)$, E_7.

The final row consists of the *metasymplectic* geometries. These are geometries characterized by the presence of families of 5-dimensional *symplecta* on which the groups of the previous row act. These geometries, too, admit explicit descriptions in terms of symplectic spaces associated with the Jordan algebra. For details, including a description of the flag structure corresponding to the fourth row of Freudenthal's table, see Landsberg and Manivel [51].

There has been much interest in the last row of Freudenthal's table in string theory and related areas. The group F_4 appears naturally in the work of Sati [71] in the M-theory associated to 11-dimensional supergravity, who also describes a possible topological origin of the theory via bundles with fiber the Moufang plane. Furthermore, the exceptional group E_8 is the standard gauge group for topological M-theory (see Sati [72] for a survey). Recent work of the author and Sparling has shown that the entire structure of this table can be described in terms of a pair of trialities.

4.4 FILTRATIONS ON VECTOR BUNDLES

Let M be a smooth manifold and $\mathbb{T} \to M$ a vector bundle (over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with a linear connection D and a distinguished filtration $\mathbb{T}_\bullet$ of length n by smooth subbundles $\mathbb{T}_i$. For each $i = 1, 2, \ldots, n$, define mappings $A_i : TM \to \mathrm{Hom}(\mathbb{T}_i, \mathbb{T}/\mathbb{T}_i)$ for $v \in TM$ by

$$A_i(v) : T \mapsto D_v T \pmod{\mathbb{T}_i}$$

on sections $T \in \Gamma(\mathbb{T}_i)$. At a point $p \in M$, the right-hand side vanishes on $\mathfrak{m}_p\Gamma(\mathbb{T}_i)$ where $\mathfrak{m}_p$ is the maximal ideal of $C^\infty(M)$ corresponding to the point p, and so descends to a (smooth) linear mapping on the bundle $\mathbb{T}_i$ itself.

The A_i are not independent: they satisfy the compatibility conditions

$$A_j(v) \circ \iota_{i,j} = \pi_{i,j} \circ A_j(v) \tag{4.4}$$

for all $1 \le i < j \le n$. Let $A_\bullet = (A_1, \cdots, A_n)$. Then

$$A_\bullet : TM \to \mathbb{A}$$

where $\mathbb{A}$ is the subbundle of $\bigoplus_i \operatorname{Hom}(\mathbb{T}_i, \mathbb{T}/\mathbb{T}_i)$ consisting of all $a_\bullet$ such that $a_j \circ \iota_{i,j} - \pi_{i,j} \circ a_i = 0$ for each $1 \le i < j \le n$. The connection D is called *generic* if $A_\bullet : TM \to \mathbb{A}$ is an isomorphism.

Given any smooth splitting of the the filtration on $\mathbb{T}$, the operators in $\mathbb{A}$ block decompose, and so $\mathbb{A}$ can be identified with the space $\bigoplus_i \operatorname{Hom}(\mathbb{T}_i/\mathbb{T}_{i-1}, \mathbb{T}/\mathbb{T}_i)$.

Lemma 20. *$\mathbb{A}$ is dual to the space $\mathbb{S}(\mathbb{T}_\bullet)$ of splittings of $\mathbb{T}_\bullet$.*

Proof. Use a particular splitting to witness the origin of $\mathbb{S}(\mathbb{T}_\bullet)$. On the one hand, this splitting permits an identification of $\mathbb{S}(\mathbb{T}_\bullet)$ with $\bigoplus_i \operatorname{Hom}(\mathbb{V}/\mathbb{V}_i, \mathbb{V}_i)$ which can (via the chosen splitting) be further broken into direct summands of the form $\operatorname{Hom}(p_j(\mathbb{T}), p_i(\mathbb{T}))$ with $j > i$. On the other hand, via this selfsame splitting, the direct summands $\operatorname{Hom}(\mathbb{T}_i/\mathbb{T}_{i-1}, \mathbb{T}/\mathbb{T}_i)$ of $\mathbb{A}$ can also be broken down into pieces of the form $\operatorname{Hom}(p_i(\mathbb{T}), p_j(\mathbb{T}))$ for $j > i$. $\square$

Suppose that D is a generic connection on $\mathbb{T}$. Associated to each splitting of $\mathbb{T}_\bullet$ defined by an n-tuple $S_\bullet = (S_1, \ldots, S_n)$, $S_i \in \operatorname{Hom}(\mathbb{T}_i, \mathbb{T}/\mathbb{T}_i)$, there is a natural affine connection ∇^S on M defined by

$$A_i(\nabla^S_v w) = \pi_{0,i} \circ [D_v(S_i \circ A_i(w) \circ P_i)] \circ \iota_{0,i}.$$

Here $P_i \in \operatorname{Hom}(\mathbb{T}, \mathbb{T}_i)$ are the mappings associated to a filtration as described in §4.2.2, and innermost expression is the composite

$$\mathbb{T} \xrightarrow{P_i} \mathbb{T}_i \xrightarrow{A_i(w)} \mathbb{T}/\mathbb{T}_i \xrightarrow{S_i} \mathbb{T}$$

which lies in $\operatorname{End}(\mathbb{T})$. The operator $\nabla_v w$ as defined actually is an affine connection, meaning that it is linear in v, additive in w, and satisfies the Leibniz product rule. Moreover, the association $S_\bullet \mapsto \nabla^S$ is an affine-linear injection from the bundle of splittings (a finite dimensional affine bundle over M) to the bundle of all affine connections on M (also a finite dimensional affine bundle whose fibers are affine torsors of $TM \otimes T^*M \otimes T^*M$).

Thus to any vector bundle $\mathbb{T}$ with a preferred filtration $\mathbb{T}_\bullet$ and generic connection, there is a naturally associated Weyl geometry.

4.5 ISOTROPIC FLAGS ON VECTOR BUNDLES

Suppose now that a vector bundle $\mathbb{T}$ on M is equipped with a smooth unitary structure h (in the style of §4.2.3) and an isotropic flag $\mathbb{T}_\bullet$ of length n by smooth subbundles of $\mathbb{T}$. Suppose also that $\mathbb{T}$ comes equipped with a linear connection D compatible with the unitary structure, meaning that $h(D_v x, y) + h(x, D_v y) = v(h(x, y))$ for all differentiable sections x, y of $\mathbb{T}$ and all $v \in TM$.

In light of the compatibility condition on D, if $\mathbb{U}$ is any smooth subbundle of $\mathbb{T}$, then $D_v : \Gamma(\mathbb{U}) \to \Gamma(\mathbb{U}^{\perp_h})$. Define, for $i = 1, 2, \ldots, n$, a mapping $A_i : TM \to \mathrm{Hom}(\mathbb{T}_i, \mathbb{T}_i^{\perp_h}/\mathbb{T}_i)$ for $v \in TM$ by

$$A_i(v) : x \mapsto D_v x \pmod{\mathbb{T}_i}$$

for all sections x of $\mathbb{T}_i$. As in §4.4, this descends to a mapping on the bundle $\mathbb{T}_i$. The A_i are related by (4.4)

$$A_j(v) \circ \iota_{i,j} = \pi_{i,j} \circ A_j(v)$$

whenever $i < j$. So the image of the n-tuple $A_\cdot(v) = (A_1(v), \ldots, A_n(v))$ is in the subbundle $\mathbb{A}$ of $\bigoplus \mathrm{Hom}(\mathbb{T}_i, \mathbb{T}_i^{\perp_h}/\mathbb{T}_i)$ of solutions of (4.4). The connection D is called *generic* if $A_\bullet : TM \to \mathbb{A}$ is an isomorphism.

Suppose that D is a generic connection on $\mathbb{T}$. Now associated to each splitting of $\mathbb{T}_\bullet$ defined by an n-tuple $S_\bullet = (S_1, \ldots, S_n)$, $S_i \in \mathrm{Hom}(\mathbb{T}_i, \mathbb{T}/\mathbb{T}_i)$, there is a natural affine connection ∇^S on M defined by

$$A_i(\nabla^S_v w) = \pi_{0,i} \circ [(\mathrm{Id}_{\mathbb{T}} - P_i^{*_h}) D_v(S_i \circ A_i(w) \circ P_i)] \circ \iota_{0,i}. \tag{4.5}$$

Here $P_i \in \mathrm{Hom}(\mathbb{T}, \mathbb{T}_i)$ are the mappings associated to a filtration as described in §4.2.2, and innermost expression is the composite

$$\mathbb{T} \xrightarrow{P_i} \mathbb{T}_i \xrightarrow{A_i(w)} \mathbb{T}_i^{\perp_h}/\mathbb{T}_i \xrightarrow{S_i} \mathbb{T}$$

which lies in $\mathrm{End}(\mathbb{T})$, so it makes sense to apply D_v to it giving a section of $\mathbb{T}$. The coisotropic projection $\mathrm{Id}_{\mathbb{T}} - P^{*_h}$ then projects this element of $\mathbb{T}$ onto $\mathbb{T}_i^{\perp_h}$, thus guaranteeing that the right hand side of (4.5) is in the image of A_i.

5.0 CAUSAL GEOMETRIES

5.1 CONES

Let V be an n-dimensional real vector space. The multiplicative group $\mathbb{R}_+ = (0, \infty)$ acts by dilation on V. Denote this action by $\delta_s : v \mapsto sv$. The quotient of $V \setminus \{0\}$ by this action is the $(n-1)$-sphere, denoted by $\mathbb{S}V$. Let $\pi : V \setminus \{0\} \to \mathbb{S}V$ be the associated projection map. Equip $\mathbb{S}V$ with the final topology associated to this projection. Then $\mathbb{S}V$ carries the structure of a smooth manifold. A smooth atlas on $\mathbb{S}V$ is given as follows. For each $\alpha \in V^* \setminus \{0\}$, let $U_\alpha \subset \mathbb{S}V$ be given by

$$U_\alpha = \pi\{x \in V \mid \alpha(x) > 0\}.$$

Let $V_\alpha = \{x \in V \mid \alpha(x) = 1\}$. This is an affine hyperplane in V, and so is diffeomorphic to $\mathbb{R}^{n-1}$. Observe that $U_\alpha = \pi(V_\alpha)$. The pairs (V_α, π) define the charts of the atlas. The transition mappings between a pair of charts are $\psi_{\alpha,\beta}(x) = \frac{\alpha(x)}{\beta(x)}x$. Moreover, if a Euclidean norm is specified on V, then the composition of the inclusion of the Euclidean sphere S^{n-1} with the projection π onto the orbit space $\mathbb{S}V$

$$S^{n-1} \xrightarrow{\subset} V \setminus \{0\} \xrightarrow{\pi} \mathbb{S}V$$

is a diffeomorphism.

A cone K in an n-dimensional real vector space V is a set of vectors in $V \setminus \{0\}$ that is invariant under the group of positive dilations. Associated to a cone K is the subset $\mathbb{S}K \subset \mathbb{S}V$ consisting of all of the rays $\{\alpha v | v \in V, \alpha > 0\}$. Conversely, if $S \subset \mathbb{S}V$ is a subset, then the set

$$\mathrm{Co}(S) = \{x \in V \setminus \{0\} \mid x \equiv s \pmod{\mathbb{R}_+} \text{ for some } s \in S\}$$

is a cone, called the *cone over* S. If $S \subset \mathbb{S}V$ is open, then $\mathrm{Co}(S)$ is also open. If K is a cone, then K is equal to the cone over $\mathbb{S}K$.

Let K be a cone in V. A function $F : K \to \mathbb{R}$ is called *homogeneous of degree* k if $F \circ \delta_s = s^k F$ for all $s \in \mathbb{R}_+$. If $f : S \to \mathbb{R}$, then $f \circ \pi : \mathrm{Co}(S) \to \mathbb{R}$ is a homogeneous function of degree 0. Define a sheaf $\mathscr{O}_V(k)$ on $\mathbb{S}V$ by setting, for each open subset $U \subset \mathbb{S}V$,

$$\Gamma(\mathscr{O}_V(k), U) = \{F : \mathrm{Co}(U) \to \mathbb{R} \mid F \text{ is smooth and homogeneous of degree } k\}.$$

For example, if $f : \mathbb{S}V \to \mathbb{R}$ is a smooth function, the composition $f \circ \pi : \mathrm{Co}(V) \to \mathbb{R}$ is a smooth function that is homogeneous of degree 0. So we can identify the sheaf $\mathscr{O}_V(0)$ with the sheaf $\mathscr{O}$ of smooth functions on $\mathbb{S}V$.

A cone is called *smooth* if $\mathbb{S}K$ is an embedded smooth hypersurface on the $(n-1)$-sphere $\mathbb{S}V$. In general, if $H \subset \mathbb{S}V$ is an embedded smooth hypersurface, then for each $p \in H$, there is a neighborhood U of p in $\mathbb{S}V$ and a smooth function $f : U \to \mathbb{R}$ satisfying the following properties:

1. $H \cap U = f^{-1}\{0\}$
2. df is nonzero throughout U.

Such a function is called a *(smooth) local defining function* of H. The composition $F_0 = f \circ \pi \in \Gamma(\mathscr{O}_V(0), U)$ is a smooth local defining function of $\mathrm{Co}(H)$ in the open set $\mathrm{Co}(U)$ that is homogeneous of degree 0. Assuming without loss of generality that $U \subset U_\alpha$ for some $\alpha \in V^*$, then the function $F_k \in \Gamma(\mathscr{O}_V(k), U)$ defined by $F_k(x) = \alpha(x)^k F_0(x)$ is a smooth local defining function that is homogeneous of degree k, for arbitrary $k \in \mathbb{R}$.

Local defining functions are not unique. If $F_k(x)$ and $F'_\ell(x)$ are smooth defining functions that are homogeneous of degrees k and ℓ, respectively, then $F_k(x)/F'_\ell(x)$ is also smooth and vanishes nowhere. The ambiguity in picking a smooth local defining function in a given open set U is thus

$$F_k(x) = \Omega(x) F'_\ell(x)$$

where $\Omega \in \Gamma(\mathscr{O}_V(k-\ell), U)$ is a non-vanishing function.

There is a topological obstruction to the existence of a global defining function:

Lemma 21. *There exists a global defining function of* $\mathbb{S}K \subset \mathbb{S}V$ *if and only if* $\mathbb{S}K$ *is closed.*

Proof. If f were a global defining function, then $\mathbb{S}K = f^{-1}\{0\}$ is the preimage under a continuous function of a closed subset of $\mathbb{R}$, and is therefore closed.

Conversely, since $\mathbb{S}K$ is compact, it has a finite covering $\mathbb{S}K \subset \bigcup_{i=1}^n U_i$ by open sets U_i with the following properties

- Each U_i can be equipped with a defining function $f_i : U_i \to \mathbb{R}$ (satisfying 1 and 2 above).
- $U_i \setminus \mathbb{S}K$ has two connected components. [These connected components are necessarily the sets $f_i^{-1}(-\infty, 0)$ and $f_i^{-1}(0, \infty)$.]

Since $\mathbb{S}K$ is orientable, the $\mathbb{Z}/2\mathbb{Z}$-valued Čech cocycle $\{\operatorname{sgn}(f_i/f_j)\}_{1\le i<j\le n}$ is exact in a tubular neighborhood of $\mathbb{S}K$ with the zero section deleted. Thus we can write $\operatorname{sgn}(f_i/f_j) = \epsilon_i\epsilon_j$ where each $\epsilon_k = \pm 1$ and is locally constant on $U_k \setminus \mathbb{S}K$.

Now decompose $\mathbb{S}V \setminus \mathbb{S}K = V_1 \cup \cdots \cup V_N$ into a finite number of (open) connected components. Since these are connected, the 0-cocycle defined by the ϵ_i extends over each V_j; denote this extension by ε_j. Let ρ_i, $i = 0, 1, \ldots, n$, ϱ_j, $j = 1, \ldots, N$, be a compactly supported smooth partition of unity subordinate to the cover $\{U_1, \ldots, U_n, V_1, \ldots, V_N\}$ of $\mathbb{S}V$. Then

$$f = \sum_i \epsilon_i \rho_i f_i + \sum_j \varepsilon_j \rho_j$$

is a global defining function for $\mathbb{S}K$. □

5.1.1 Cones in vector bundles

Let M be a smooth manifold and $E \to M$ be a rank r real vector bundle. Denote by E' the submanifold of the total space of E obtained by deleting the zero section. The multiplicative group $\mathbb{R}_+ = (0,\infty)$ acts by dilation on E', $\delta_s : v \mapsto sv$. The quotient of E' by this action is the $(n-1)$-sphere bundle, denoted by $\mathbb{S}E$. Let $\pi : E' \to \mathbb{S}E$ be the associated projection map. As a set, the fiber of $\mathbb{S}E$ over a point $p \in M$ is the sphere $\mathbb{S}E_p$ associated to the vector space E_p. Furthermore, $\mathbb{S}E$ carries the structure of a smooth fibre bundle over M, with fibre S^{n-1}, with the induced local trivializations from those of the bundle E.

A cone in E is a subset of E' that is invariant under the action of the group $\mathbb{R}_+$. A cone in E is smooth if it is an embedded smooth hypersurface of the total space of E'. If $S \subset \mathbb{S}E$ is a subset, then $\operatorname{Co}(S) \subset E'$, the cone over S, is defined as the preimage of S under π. A function $F : \operatorname{Co}(S) \to \mathbb{R}$ is homogeneous of degree k if $F \circ \delta_s = s^k F$ for all $s \in \mathbb{R}_+$. Define a sheaf on $\mathbb{S}E$ by setting for each open $U \subset \mathbb{S}E$,

$$\Gamma(\mathscr{O}_E(k), U) = \{F \in C^\infty(\operatorname{Co}(U)) \mid F \text{ is homogeneous of degree } k\}.$$

5.1.2 Tangent cones

Definition 15 ([25], 3.1.21)**.** *Let $N \subset \mathbb{R}^n$. The* tangent cone *to N at a point $p \in S$, denoted* $\operatorname{Tan}(N,p)$ *is the set of $v \in \mathbb{R}^n$ such that for every $\epsilon > 0$, there exists $r > 0$ and $x \in N$ with $|x-p| < \epsilon$ and $|r(x-p) - v| < \epsilon$.*

This is evidently a (closed) cone. In fact,

$$\operatorname{Tan}(N,p) = (0,\infty) \cdot \bigcap_{\epsilon>0} \operatorname{cl}\left\{ \frac{x-p}{|x-p|} \mid x \in (N \setminus \{p\}) \cap B_\epsilon(p) \right\}.$$

The notion of tangent cone is diffeomorphism-invariant:

Lemma 22 ([25], 3.1.21)**.** *If $N \subset \mathbb{R}^n$, U is an open neighborhood of $p \in N$, and $\phi : U \to U'$ is differentiable at p onto an open subset of $\mathbb{R}^n$ such that $d\phi(p)$ is a linear isomorphism, then $d\phi(p) : T_pM \to T_{\phi(p)}M$ defines a continuous bijection of C_pN to $C_{\phi(p)}\phi(N)$.*

Let M be an n-dimensional differentiable manifold and N a subset of M. The tangent cone $\operatorname{Tan}(N,p)$ at a point $p \in M$ is the subset of T_pM defined so that for any local diffeomorphism $\phi : U \to \mathbb{R}^n$ of an open neighborhood U of p, the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Tan}(N,p) \arrow[r, "\subseteq"] \arrow[d, "d\phi_p"] & T_pM \arrow[d, "d\phi_p"] \\
\operatorname{Tan}(\phi(N),\phi(p)) \arrow[r, "\subseteq"] & T_{\phi(p)}M
\end{tikzcd}$$

We shall be primarily interested in the case where N is a continuous (C^0) hypersurface of M. If $S \subset M$ is an embedded smooth submanifold, then the tangent cone to N along S is the union, in the tangent bundle $TM|S$, of the tangent cones $\operatorname{Tan}(N,p)$ as p varies over S:

$$\operatorname{Tan}(N,S) = \bigcup_{p\in S} \operatorname{Tan}(N,p).$$

We will say that N is smooth with respect to a differentiable submanifold S if $\operatorname{Tan}(N,S)$ is a smooth cone in the vector bundle $TM|S$ over S.

5.1.3 Relative differentials

Our approach to relative differentials is from [25], 3.1.22.

Definition 16. *Let $N \subset M$ be a subset of a differentiable manifold M, and X a normed space. A function f from a subset of M into X is called* differentiable *at a point p* relative to N *if there is an open neighborhood U of p and a differentiable function $g : U \to X$ such that $f|(U \cap N) = g|(U \cap N)$. In that case, $L = dg_p| \operatorname{Tan}(N,p)$ is called the differential of f relative to N at p.*

The differential is well-defined by this construction since if g_1 and g_2 were two differentiable representatives extending f to a common neighborhood U, then $(g_1 - g_2)|(N \cap U) = 0$ and so also $d(g_1 - g_2)_p|\operatorname{Tan}(N, p) = 0$. The notion of relative differentiabiliy is diffeomorphism-invariant. We may thus refer to relative-differentiability of mappings with values in smooth manifolds.

Definition 17. *A mapping $f : K \to M$ from a smooth cone $K \subset V$ to a smooth manifold M is called* smooth *if it is smooth as a mapping of manifolds and there is a neighborhood U of $0 \in V$ and an extension of f to a smooth function $g : U \to M$.*

The hypotheses on f guarantee that f can be extended to a local diffeomorphism g of a neighborhood of $0 \in V$ into M. In particular,

Lemma 23. *Let $f : K \to M$ be a smooth mapping of a smooth cone $K \subset V$ into a smooth manifold M. Let $v \in K \setminus \{0\}$. Then*

$$\lim_{\epsilon \to 0^+} \operatorname{Tan}(f(K), f(\epsilon v)) = f_* \operatorname{Tan}(K, v)$$

where the limit on the left is taken in the topology in the Grassmannian bundle of $(n-1)$-planes in TM.

We would like to be able to talk about conical subsets of manifolds. Roughly, a subset of a manifold is conical (with respect to a point p, its vertex) if there is a local coordinate system in which the subset is a cone. An obvious shortcoming of this definition is that the notion of generators of the cone is no longer meaningful away from the point p. We would like to build in a notion of the generators of a cone. This is supplied by the following definition:

Definition 18. *Let M be a smooth manifold. Let $S \subset M$ be an embedded smooth submanifold of M, and $N \subset M$ a subset that is smooth with respect to S. A* conical fibration *of N over S is a smooth embedding ϕ from a neighborhood of the origin in $\operatorname{Tan}(N, S)$ to N such that the relative differential $d\phi_p : \operatorname{Tan}(N, p) \to \operatorname{Tan}(N, p)$ is the identity for all $p \in S$. Two conical fibrations ϕ and ϕ' are* equivalent *if there is a locally Lipschitz mapping $s : \operatorname{Tan}(N, S) \to (0, \infty)$ such that $\phi(x) = \phi'(s(x)x)$ for all $x \in \operatorname{Tan}(N, S)$.*

5.2 CAUSAL GEOMETRIES

The following definition concerns submanifolds of the product space $M \times M$ of a differentiable manifold M with itself, equipped with the product differential structure, and projections

$\pi_1, \pi_2 : M \times M \to M$. Let $\Delta \subset M \times M$ be the diagonal. If $N \subset M \times M$ and $p \in M$, let $N_p \subset M$ be the *horizontal slice through* p: $N_p = N \cap \pi_2^{-1}(p) = \{x \mid (x,p) \in N\}$.

Definition 19. *A* causal geometry *on a smooth manifold M is an embedded Lipschitz hypersurface $N \subset M \times M$ that is smooth with respect to Δ equipped with a conical fibration over Δ such that the following conditions hold:*

(A) Suppose that $v \in \operatorname{Tan}(N_p, p)$, $q = \phi_p(s_1 v)$ and $r = \phi_p(s_2 v)$ with $0 < s_1 < s_2$. Let $w = \left.\frac{d\phi_p(sv)}{ds}\right|_{s=s_1}$. Then $w \in \operatorname{Tan}(N_q, q)$ and $r = \phi_q((s_2 - s_1)w)$.

(B) In this situation, $\operatorname{Tan}(N_p, r) = \operatorname{Tan}(N_q, r) \subset T_r M$.

This definition requires some discussion. The intuitive picture is a family of submanifolds ("null cones"), one for each point of the manifold M, that are fibered in a way corresponding to the possible trajectories that a luminous signal can travel ("null geodesics"). Condition (A) implies that a null geodesic from a point p to q, if it stops at an intermediate point r, remains a null geodesic if its history prior to the point r is ignored. A signal cannot cheat by first traveling to some intermediate point, and then veering off in some direction that will take it outside the null cone. This is the meaning of (B).

The vector w in (A) is called the *transport* of v along the geodesic. When there is no risk of confusion, we will use the same symbol v to denote the transport of v along the geodesic it generates. We shall need the following technical reformulation of (B):

Lemma 24. *Condition (B) is equivalent to the following property:*

(B') Let $v \in \operatorname{Tan}(N_p, p) \setminus \{0\}$ and $q = \phi_p(v) \in N_p$. Then $\lambda \operatorname{Tan}(N_p, q) = \operatorname{Tan}(\operatorname{Tan}(N_q, q), v)$.

Proof. Assume that condition (B) holds. Let $r_\epsilon = \phi_p((1+\epsilon)v)$, so that by (B) $\operatorname{Tan}(N_p, r_\epsilon) = \operatorname{Tan}(N_q, r_\epsilon)$. By Lemma 23, $\lim_{\epsilon \to 0^+} \operatorname{Tan}(N_q, r_\epsilon) = \lambda \operatorname{Tan}(N_p, q)$. But since $N_p \setminus \{p\}$ is a smooth hypersurface, $\lim_{\epsilon \to 0^+} \operatorname{Tan}(N_p, r_\epsilon) = \operatorname{Tan}(N_p, q)$.

Conversely, suppose that (B') holds. Let $q = \phi_p(v_0)$. Let $v : [-\epsilon, \epsilon] \to \operatorname{Tan}(N_p, p)$ be a smooth function such that $v(0) = v_0$, a one-parameter variation of v_0. Let w_0 be the transport of v_0 to q. By (B'), there is a one-parameter variation of w_0 such that

$$(\phi_p)_* dv = dw.$$

Pushing forward both sides of this equation by ϕ_q establishes (B). $\square$

So, a causal geometry determines the following structure:

Property 1. $N_P \setminus \{P\}$ is a smooth embedded hypersurface in M

Property 2. The singular variety N_P is ruled by a pencil of smooth curves N_{PQ} from P to Q, as Q varies over $N_P \setminus \{P\}$.

Property 3. For every $Q \in N_P \setminus \{P\}$, the curve N_{PQ} on N_P coincides with the curve N_{QP} on N_Q: $N_{PQ} = N_{QP}$.

Property 4. Let C_P be the set of tangents at P to the curves N_{PQ} as Q varies over $N_P \setminus \{P\}$. Then C_P is a regular (smooth) surface in the projective tangent space $\mathbb{P}T_P M$. (The surface C_P is called the *indicatrix*.)

Property 5. Let $Q, R \in N_P \setminus \{P\}$ be given distinct points which are mutually incident. Then the tangent plane to N_P at R is the same as the tangent plane to N_Q at R: $T_R N_P = T_R N_Q$. (*Envelope condition*)

5.2.1 Subgeometries, equivalence, and gluing

Let N be a causal geometry on M. A *subgeometry* of N is an open subset M_1 of M and an open subset N_1 of N that defines, with the induced conical fibration, a causal geometry on M_1. A causal geometry N on M defines a subgeometry on any open subset M_1 by setting $N_1 = N \cap (M_1 \times M_1)$ together with the induced conical fibration. This causal subgeometry is denoted by $N|M_1$.

Two causal geometries N and N' on M are *equivalent* if there is an open neighborhood U of Δ in $M \times M$ such that $N \cap U = N' \cap U$ and the induced conical fibrations are equivalent. (This is in the spirit of the notion of isomorphism of microbundles introduced in Milnor [54].)

Lemma 25. *Let M_i be an open cover of M, and N_i causal geometries in M_i such that the subgeometries $N_i|M_j$ is equivalent to $N_j|M_i$ for all i, j. Then there is a unique causal geometry N on M such that $N|M_i$ is equivalent to N_i for all i.*

The equivalence class of causal geometries obtained in this way in the *gluing* of the subgeometries N_i.

The main invariant of a causal geometry is its null cone bundle:

Definition 20. *Let N be a causal geometry on M. The subbundle $(d\pi_2)\,\mathrm{Tan}(N, \Delta) \subset TM'$ is called the* null cone bundle *and is denoted by $\mathscr{H}$.*

This is the infinitesimal null cone at each point of M. The null cone bundle depends only on the equivalence class of the causal geometry. Moreover, if N_1 is a subgeometry of N and $\mathscr{H}_1$ is the associated null cone bundle, then $\mathscr{H}_1 \subset \mathscr{H}$ as well. Theorem 16, proven later, establishes that the null cone bundle completely characterizes the causal geometry up to equivalence.

5.3 CAUSAL GEOMETRY ASSOCIATED TO A LAGRANGIAN

5.3.1 Lagrangians

Let M be a differentiable manifold, TM its tangent bundle. A C^2 local section L of $\mathscr{O}_{TM}(k)$ is called a *Lagrangian*. A Lagrangian is called *indefinite* if dL is everywhere nonzero, and L assumes both positive and negative values in each tangent space. A Lagrangian is non-degenerate if its Hessian in the vertical directions is a non-singular bilinear form.

Let X be the set of all C^1 embedded curves $\gamma : [a, b] \to M$ with fixed endpoints $\gamma(a)$ and $\gamma(b)$ such that the image of $d\gamma : T[a, b] \to TM$ is in the domain of L. When M is equipped with a smooth metric X becomes a Banach manifold whose connected components are the C^1-isotopy classes of such mappings. The tangent space of X at γ, $T_\gamma X$, is the Banach space of C^1 sections of $\gamma^{-1}TM$ vanishing at a and b with the $W^{1,\infty}$ norm, denoted by $\| \cdot \|_{1,\infty}$. Such a vector field v can be extended to a vector field in a small neighborhood of γ, and the flow of this vector field generates a C^1 map $F(t, s)$, $F : [a, b] \times (-\epsilon, \epsilon) \to M$, such that

- $F(t, 0) = \gamma(t)$
- $(\partial F/\partial s)(t, 0) = v$.

This F is called a variation of γ.

For any Lagrangian L, the Lie derivative $\mathscr{L}_v L$ is well defined along a curve $\gamma \in X$, by

$$\mathscr{L}_v L(\gamma(t)) = \frac{\partial}{\partial s}(L \circ F)(t, 0)$$

where F is any variation of γ.

A Lagrangian L can be pulled back along a C^1 curve $\gamma \in X$ to a continuous function on $[a, b]$, denoted $\gamma^* L$, and defined by

$$(\gamma^* L)(t) = L(\gamma_*(d/dt)).$$

The *energy functional* of a curve $\gamma \in X$ is defined by

$$E[\gamma] = \int_a^b \gamma^* L \, dt.$$

Since γ is an embedded curve, it has a tubular neihborhood (Hirsch [40]); that is, a neighborhood U that is diffeomorphic to the normal bundle of γ such that γ goes to the zero section under this diffeomorphism. There is a global coordinate system on U, say x^i. This is called a *tubular coordinate system*. In that case,

$$\mathscr{L}_v L = \frac{\partial L}{\partial x^i} v^i + \frac{\partial L}{\partial \dot{x}^i} \dot{v}^i.$$

Lemma 26. *$E : X \to \mathbb{R}$ is a C^1 function on X. Moreover, $dE : TX \to \mathbb{R}$ is the* Euler–Lagrange functional

$$dE_\gamma(v) = \int_a^b \mathscr{L}_v L\, dt.$$

Proof. We work in the global coordinate system x^i on the tubular neighborhood U of $\gamma \in X$. If $v \in T_\gamma X$ is sufficiently small in norm, then $\gamma(t) + v(t)$ is also in X. We have

$$\begin{aligned}\left| E[\gamma] - E[\gamma + v] - \int_a^b \mathscr{L}_v L\, dt \right| &\le \int_a^b |\gamma^* L - (\gamma + v)^* L - \mathscr{L}_v L|\, dt \\ &\le C\|v\|_{1,\infty}^2 \sup_\gamma |D^2 L|\end{aligned}$$

where D^2L denotes the Hessian of L with respect to all of the variables x^i and $\dot{x}^i$. So E is differentiable at γ, and its derivative is the Euler–Lagrange functional. The derivative dE_γ depends continuously on γ, again by the smoothness of L. □

A *geodesic* is a critical point of E. This is a curve $\gamma \in X$ such that

$$dE_\gamma(v) = 0$$

for all $v \in T_\gamma X$. In a tubular coordinate system, this reduces to the Euler–Lagrange system

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0.$$

Let L be a non-degenerate Lagrangian. A vector v is called *null* if $L(v) = 0$. A geodesic for L is called null if its tangent vector is null at every point. Two points in M are called *null related* if there is a null geodesic connecting them. The *null cone* of a point $p \in M$ is the subset consisting of all points of M that are null related to p. Let N_p be the null cone of p. For each $p \in M$, let $\phi_p : \mathrm{Tan}(N_p, p) \to N_p$ be the map that associates tangent vector v to N_p at p the null geodesic with initial condition v.

Theorem 15. *Suppose that $L : TM \to \mathbb{R}$ is a non-degenerate Lagrangian. For each $p \in M$ let N_p the null cone through p, and ϕ_p the associated conical fibration. Let $N = \bigcup_{p\in M}(N_p, p)$ and $\phi(n, p) = \phi_p$. Then N, ϕ is a causal geometry on M.*

Proof. For each $P \in M$, the indicatrix $C_P = \{v \in \mathbb{P}T_P M \mid L(P, v) = 0\}$ is smooth. For a curve γ in M, consider the energy functional

$$E[\gamma] = \frac{1}{2} \int_a^b L(\gamma(t), \gamma'(t))\, dt.$$

A null geodesic is a curve γ that is an extremal for E along which $L(\gamma(t), \gamma'(t)) \equiv 0$. The family of null geodesics for a (possibly singular) Lagrangian defines a causal geometry if and only if the Lagrangian is regular at every point of the cone over C_P in $T_P M$ for each P, in the sense that its vertical Hessian is nondegenerate in directions tangent to C_P. In coordinates x^i for M and fiber coordinates $\dot{x}^i$ for TM, this is the condition that the Hessian matrix $\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$ be nonsingular in directions tangent to the null cone. This is also a sufficient condition for L to be conserved along an extremal, and thus the null geodesics are precisely those extremals of the energy for which $L(\gamma(0), \gamma'(0)) = 0$.

The Euler–Lagrange system is a second order ordinary differential equation for the curve $\gamma(t)$. By smooth dependence on initial conditions, for initial conditions lying on the hypersurface $L(P, \gamma'(0)) = 0$ sufficiently near the origin of the tangent space $T_P M$, the null geodesics foliate a smooth hypersurface in M, giving Properties 1–2. Property 3 follows since the null geodesics are critical points of the energy under compactly supported variations, and so in particular are characterized independently of the direction of their parameterization. Property 4 follows from the regularity of the Lagrangian. Finally, for Property 5, it is sufficient to show that, for any point P, and any variation γ_s of null geodesics through P,

$$\left. \frac{\partial L}{\partial \dot{x}^i} \frac{d\gamma^i}{ds} \right|_{s=0} = 0. \tag{5.1}$$

This then establishes that the null cone at P is tangent to the indicatrix at every point, which is equivalent to Property 5 of §5.2. By the Euler–Lagrange equations,

$$\begin{aligned} 0 = \frac{d}{ds} L(\gamma_s, \dot{\gamma}_s) &= \frac{\partial L}{\partial x^i} \frac{d\gamma^i}{ds} + \frac{\partial L}{\partial \dot{x}^i} \frac{d\dot{\gamma}^i}{ds} \\ &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} \right) \frac{d\gamma^i}{ds} + \frac{\partial L}{\partial \dot{x}^i} \frac{d\dot{\gamma}^i}{ds} \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \frac{d\gamma^i}{ds} \right) \end{aligned}$$

So the left-hand side of (5.1) is constant along the curve $\gamma(t)$, as required.

□

If the homogeneity k of the Lagrangian is equal to 1, then the null geodesics have no natural parameterization. Otherwise, if γ is an *unparametrized* null geodesic for a Lagrangian L of homogeneity $k \neq 1$, it satisfies the Euler–Lagrange equation

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \lambda \frac{\partial L}{\partial \dot{x}^i}$$

where λ is a free function of t. This freedom λ can be absorbed by reparametrizing the curve. We henceforth impose that $k \neq 1$ on all Lagrangians. Section 8.1.1 discusses a Legendrian formulation that is independent of the homogeneity of any associated Lagrangian.

5.3.2 Lagrangians associated to a causal geometry

Let N, ϕ be a causal geometry on M. For each $p \in M$, the tangent cone $K = \mathrm{Tan}(N_p, p)$ defines a Lipschitz hypersurface $\mathbb{S}K \subset \mathbb{S}T_pM$. There exists a locally defined Lipschitz function $L : T_pM \to \mathbb{R}$ that is homogeneous of degree two and such that dL is defined at almost every point of $\mathbb{S}K$

Theorem 16. *Let $U \subset \mathbb{S}M$ be an open set and $L \in \Gamma(\mathscr{O}_{TM}(k), U)$ a Lagrangian that gives a defining function for $\mathscr{H}$ in U. The causal geometry defined by L is equivalent to a subgeometry of N. Furthermore, N is equivalent to the gluing of all the causal subgeometries obtained in this manner.*

Proof. Fix $p \in M$ and $v \in \mathrm{Tan}(N_p, p) \cap \mathrm{co}(U)$. Let $\gamma(t) = \phi_p(tv)$, where ϕ is a representative of the conical fibration. We must first show that, for sufficiently small t, $\gamma(t)$ is a null geodesic with respect to the Lagrangian L. First, by Definition 19 (A), γ is null. It remains to show that it is a geodesic.

Let $v_s \in \mathrm{Tan}(N_p, p)$ be a variation of v for $s \in (-1, 1)$, so that $v_0 = v$. Let $\gamma_s(t) = \phi_p(tv_s)$. Then γ_s is a variation of γ along rays of N_p. We will work in a tubular coordinate system around γ. Lemma 24 implies that along $\frac{\partial L}{\partial \dot{x}^i}$ annihilates every vector tangent to N_p along γ. In particular,

$$\left.\frac{\partial L}{\partial \dot{x}^i}\frac{d\gamma_s^i}{ds}\right|_{s=0} = 0.$$

Hence, since $\gamma_s^* L = 0$,

$$0 = \frac{d}{ds}\gamma_s^* L = \frac{\partial L}{\partial x^i}\frac{d\gamma_s^i}{ds} + \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{\gamma}_s^i}{ds}.$$

On the other hand,

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\frac{d\gamma_s^i}{ds}\right) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\frac{d\gamma_s^i}{ds} + \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{\gamma}_s^i}{ds}.$$

Combining these equations gives

$$\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\frac{d\gamma_s^i}{ds} = 0.$$

Hence

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$$

is annihilated by every vector tangent to N_p. So

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \lambda\frac{\partial L}{\partial \dot{x}^i}$$

where λ is a free function of t. Thus γ is a null geodesic for L.

$\square$

6.0 INCIDENCE GEOMETRIES AND TWISTOR THEORY

This chapter presents a simple class of examples of causal geometries based on incidence of geodesics in projective manifolds. The model for these geometries comes from twistor theory, which is the incidence geometry of lines in projective space. The complex version of twistor theory is described in great detail in [67]. The real version, that we study here, is described in [42].

6.1 REAL TWISTOR THEORY

Let $\mathbb{T}$ be a four-dimensional real vector space and let $X = \mathbb{PT}$ be the associated projective space. In any affine patch of X, there is a natural affine connection. These affine connections patch together modulo projective equivalence to define a (flat) projective structure on $\mathbb{PT}$, the geodesics of which are just the images in X of the two-dimensional linear subspaces of $\mathbb{T}$. In particular, the space of lines (geodesics) in X is the Grassmannian $\mathbb{M} = \mathrm{Gr}_2(\mathbb{T})$ of two-dimensional subspaces of $\mathbb{T}$ (or, equivalently, of lines in the projective space $\mathbb{PT}$). Call two distinct points of $\mathbb{M}$ *incident* if the corresponding lines in $\mathbb{PT}$ intersect in a point. For any point $p \in \mathbb{M}$, define the null cone N_p to be the subset of $\mathbb{M}$ consisting of all q that are incident with p. If distinct points p and q are incident, then they determine a point $x \in \mathbb{PT}$ (the point of intersection) as well as a 2-plane in $P \subset \mathbb{PT}$ (the plane tangent to both lines). The null geodesic determined by p and q is the set of $r \in \mathbb{M}$ such that r passes through x and is tangent to P there. If p is fixed, this defines a conical fibration of N_p.

The four-dimensional space $\mathbb{M}$ embeds naturally as the *Klein quadric* in $\mathbb{P}(\wedge^2\mathbb{T})$ given by $\mathbb{M} = \{A \in \mathbb{P}(\wedge^2\mathbb{T}) | A \wedge A = 0\}$. The embedding associates to a two-plane x in $\mathbb{T}$ (that is, a point of $\mathbb{M}$) the exterior square $\wedge^2 x$ in $\wedge^2\mathbb{T}$. Define on $\wedge^2\mathbb{T}$ the line bundle-valued non-degenerate symmetric form $g(A, B) = A \wedge B \in \wedge^4\mathbb{T}$, so that $\mathbb{M}$ is the projective null cone of g. The quadric $\mathbb{M}$ is foliated by two distinct families of planes, called α and β planes. An α plane is the two-dimensional subvariety of $\mathbb{M}$ consisting of all projective lines in $X = \mathbb{PT}$

(that is, points in $\mathbb{M}$) containing a fixed point. For fixed $x \in X$, a point A of the Klein quadric lies on the α plane associated to x if and only if $A = [x \wedge y]$ for some $y \in \mathbb{T}$. In particular, α planes are actually planes. A β plane is the two-dimensional subvariety of $\mathbb{M}$ consisting of all lines in $\mathbb{PT}$ (that is, points of $\mathbb{M}$) contained in a given plane in $\mathbb{PT}$. Duality exchanges α and β planes, so β planes are also planes. Since the quadric is foliated by two families of two-planes, the signature of g is $(3, 3)$.[1]

Let $\alpha \in \mathbb{T}^*$ be a non-zero linear form on $\mathbb{T}$. When restricted to the quadric $\mathbb{M}$, the symmetric form $g_\alpha(dx, dx) = \frac{g(dx,dx)}{\alpha(x)^2}$ is annihilated only by the homogeneity vector field and is Lie derived along this vector field, so it descends to the quotient space $\mathbb{M}$. Thus $\mathbb{M}$ carries a natural conformal structure, and this conformal structure is exactly what defines the causal structure on M described in the previous paragraph. (The metric g_α is actually flat, so this is a flat conformal structure. There is also a global constant curvature metric in the conformal class $\frac{g(dx,dx)}{h(x,x)}$, where h is a positive definite symmetric bilinear form; see Holland and Sparling [43] for an application to cosmology in the case of Lorentzian signature.)

6.1.1 Projective spin bundles

Returning now to the spaces $\mathbb{M}$ and $\mathbb{PT}$ themselves, observe that M is the space of lines in $\mathbb{PT}$ (a three-dimensional projective space), and $\mathbb{PT}$ is the space of α-planes in $\mathbb{M}$, via the double-duality isomorphism. There is therefore a correspondence space in $\mathbb{PT} \times M$, called the *projective primed spin bundle* $\mathbb{PS}'$. This is the subset of $\mathbb{PT} \times \mathbb{M}$ consisting of pairs (x, A), where A is a line in $\mathbb{PT}$, such that x lies on the line A. Or under the double duality isomorphism, this is the subspace of $\mathbb{PT} \times \mathbb{M}$ consisting of pairs (x, A), where A is a point of $\mathbb{M}$, such that x is an α-plane through A.[2]

The dual twistor space is the space of β-planes in $\mathbb{M}$. The *projective spin bundle* is the correspondence space of points in $\mathbb{M}$ and β-planes.

6.1.2 Projective null twistor space

The spaces $\mathbb{PT}$ and $\mathbb{PT}^*$ have a correspondence space between them, denoted by $\mathbb{PN}$, the *projective null twistor space*, contained in $\mathbb{PT} \times \mathbb{PT}^*$ whose points are pairs (z, α) with z a

[1] This can also be seen by introducing a basis e_i of $\mathbb{T}$, the vectors $e_i \wedge e_j$ with $i < j$ define a basis of $\wedge^2\mathbb{T}$. So for $x \in \wedge^2\mathbb{T}$, we have $x = \sum_{1 \le i < j \le 4} x^{ij} e_i \wedge e_j$ for some real coefficients x^{ij}. In these coordinates,

$$g(x, x) = x \wedge x = 2(x^{12}x^{34} - x^{13}x^{24} + x^{14}x^{23})e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

so g has signature $(3, 3)$.

[2] We have suppressed here any mention of the *dual* primed (and unprimed) spin bundles. We are here working projectively, so there is no distinction for us between the spin bundle and its dual.

point on the plane determined by α.

6.1.3 Flag manifolds

We define the following flag manifolds associated to $\mathbb{T}$:

- Projective twistor space: $\mathbb{PT} = \mathrm{Gr}_1(\mathbb{T}) = \{V_1 \subset \mathbb{T} | \dim V_1 = 1\}$
- Spacetime: $\mathbb{M} = \mathrm{Gr}_2(\mathbb{T}) = \{V_2 \subset \mathbb{T} | \dim V_2 = 2\}$
- Dual projective twistor space: $\mathbb{PT}^* = \mathrm{Gr}_3(\mathbb{T}) = \{V_3 \subset \mathbb{T} | \dim V_3 = 3\}$
- Projective null twistor space $\mathbb{PN} = \{(V_1, V_3) | V_1 \subset V_3 \subset \mathbb{T},\ \dim V_1 = 1, \dim V_3 = 3\}$
- Projective primed spin bundle $\mathbb{PS}' = \{(V_1, V_2) | V_1 \subset V_2 \subset \mathbb{T},\ \dim V_1 = 1, \dim V_2 = 2\}$
- Projective spin bundle $\mathbb{PS} = \{(V_2, V_3) | V_2 \subset V_3 \subset \mathbb{T},\ \dim V_2 = 2, \dim V_3 = 3\}$
- The complete flag variety $\mathbb{F} = \{(V_1, V_2, V_3) | V_1 \subset V_2 \subset V_3 \subset \mathbb{T},\ \dim V_1 = 1, \dim V_2 = 2, \dim V_3 = 3\}$.

Let $\mathrm{GL}(\mathbb{T})$ be the group of linear automorphisms of $\mathbb{T}$. Each of these flag manifolds is evidently a homogeneous space for $\mathrm{GL}(\mathbb{T})$ that we now proceed to describe. Let B be a Borel subgroup of $\mathrm{GL}(\mathbb{T})$. Expressed in a suitable basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{T}$, B is the group of all invertible upper-triangular matrices, and so B is just the stabilizer of the complete flag $T_1 \subset T_2 \subset T_3 \subset \mathbb{T}$ of subspaces of $\mathbb{T}$ formed by the span of successive elements of the basis: $T_i = \mathrm{span}\{e_1, \ldots, e_i\}$. Hence $\mathbb{F} = \mathrm{GL}(\mathbb{T})/B$.

A parabolic subgroup of $\mathrm{GL}(\mathbb{T})$ is a subgroup corresponding to a subalgebra of $\mathfrak{gl}(\mathbb{T})$ that contains the Borel algebra (see Chapter 4 for a general discussion). Up to equivalence, any parabolic subalgebra is generated by the Borel subalgebra, along with some of the simple negative root spaces. The parabolic subgroups can be labeled by Dynkin diagrams where a node is crossed if it the corresponding simple negative is not contained in the parabolic algebra, and a Dynkin node is filled in if it is. The various possible groups are listed here. A matrix element is crossed if it is zero in the group.

Each of these parabolic subgroups is a stabilizer of some flag. The Dynkin diagram with a cross in the i position stabilizes the subspace T_i. There are maps going from these various spaces to one another, as summarized in the commutative diagram of Figure 6.1.3.

We now inquire what the situation is when the spacetime $\mathbb{M}$ is replaced by an arbitrary 4-manifold M with an ultrahyperbolic ($(2,2)$-signature) conformal structure. The conformal structure on M is determined by, and uniquely specifies, a normal conformal Cartan connection in the style of [76] and [42]. This consists of the data of a principal $P(\bullet\!\!-\!\!\times\!\!-\!\!\bullet)$-bundle $\mathscr{G}$ over M and a $\mathfrak{gl}(\mathbb{T})$-valued one-form $\omega : \mathscr{G} \to \mathfrak{gl}(\mathbb{T})$ that is an isomorphism in each tangent space that is equivariant under the $P(\bullet\!\!-\!\!\times\!\!-\!\!\bullet)$ action and sends the generators of the vertical action to the corresponding elements of the Lie algebra $\mathfrak{p}(\bullet\!\!-\!\!\times\!\!-\!\!\bullet)$.

Figure 1: Parabolic subgroups of $\mathrm{GL}(\mathbb{T})$

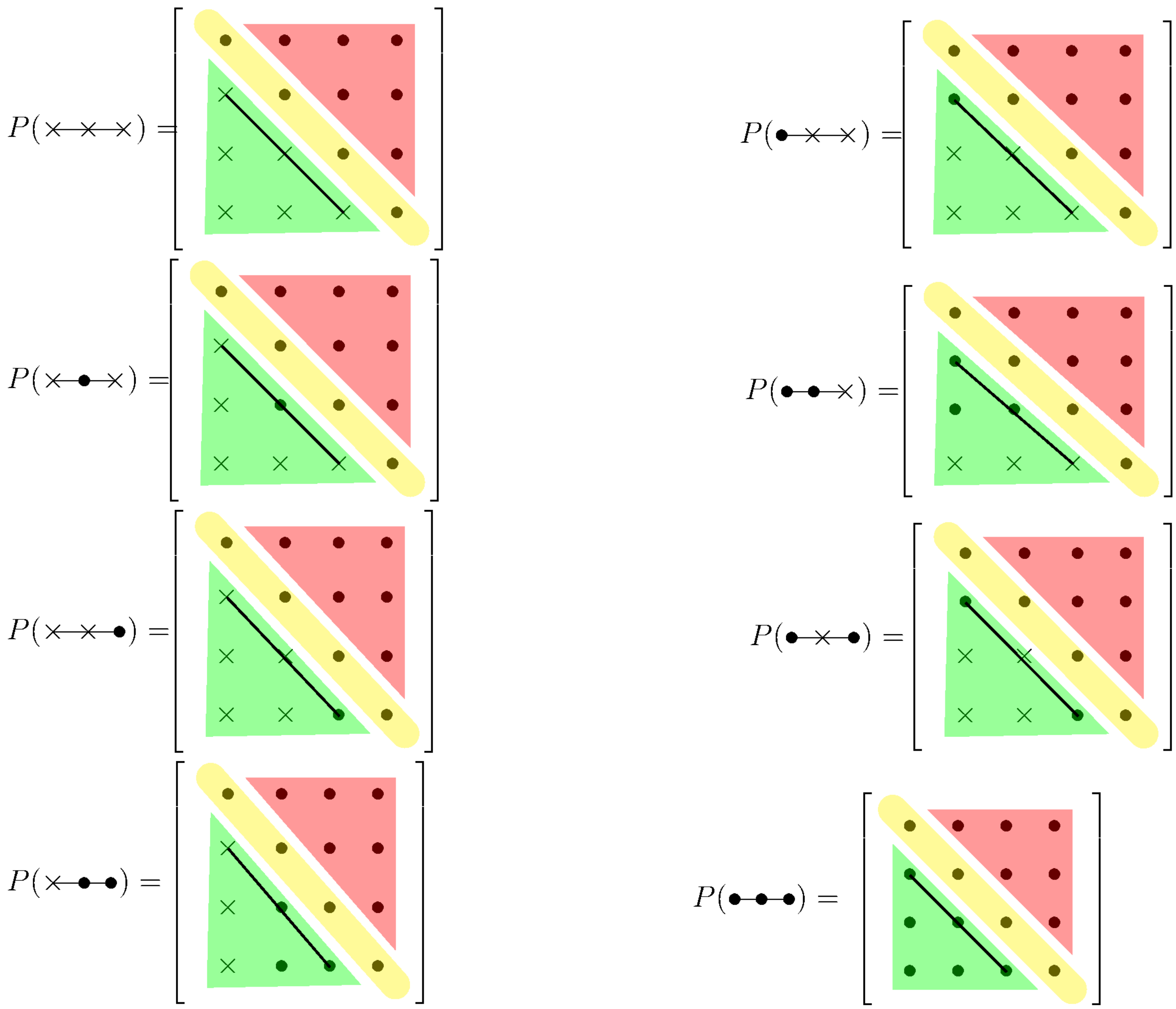
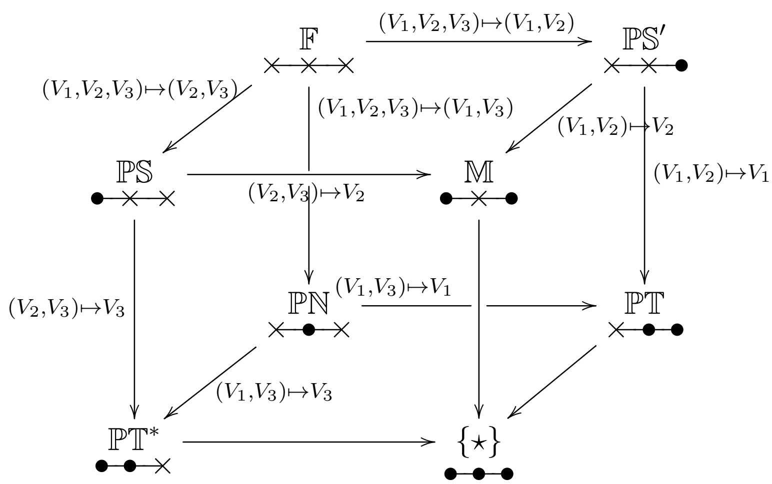


Figure 2: Lattice of parabolic subgroups of $\mathrm{GL}(\mathbb{T})$



The twistor distribution on $\mathscr{G}$ is the codimension 3 distribution on $\mathscr{G}$ defined by $\mathbf{D} = \omega^{-1}(\mathfrak{p}(\times\!\!-\!\!\bullet\!\!-\!\!\bullet)) \subset T\mathscr{G}$. The dual twistor distribution is the distribution $\mathbf{D}' = \omega^{-1}(\mathfrak{p}(\bullet\!\!-\!\!\bullet\!\!-\!\!\times))$. An α-surface in M is the projection to M of an integral surface of $\mathbf{D}$ and a β-surface is the projection of an integral surface of $\mathbf{D}'$. In the flat case, these coincide with the α and β planes, respectively. In general, however, the distributions $\mathbf{D}$ and $\mathbf{D}'$ are not integrable in the sense of Frobenius: a necessary and sufficient condition for integrability of each is the vanishing of the corresponding $\mathrm{SO}(2,2)$ irreducible component of the Weyl tensor.

Assuming integrability of D, the twistor space (denoted $\mathcal{T}(M)$) is defined as the space of leaves of the foliation or, equivalently, the space of α-surfaces in M. We shall assume that this is a decent space: precisely, the quotient by the foliation is a submersion onto $\mathcal{T}(M)$. The twistor space comes equipped with a notion of geodesics: two twistors A and B are incident if the corresponding α surfaces in M intersect. When M is the conformally flat model geometry $\mathrm{Gr}_2(\mathbb{R}^4)$, the twistor space is just $\mathbb{PT}$, and the geodesics in $\mathbb{PT}$ correspond to the geodesics of the natural projective structure on $\mathbb{PT}$. In general, the system of curves in $\mathcal{T}(M)$ defining the geodesics may not correspond to any projective connection on $\mathcal{T}(M)$.

6.2 INCIDENCE IN PROJECTIVE THREE-MANIFOLDS

Let X be a three-dimensional projective manifold. Call two geodesics on X *incident* if they intersect at some point. In a local patch of any point of X, the space of geodesics is a four-dimensional manifold, denoted $\mathcal{M}(X)$. For $p \in \mathcal{M}(X)$, let N_p be the subset of $\mathcal{M}(X)$ consisting of all geodesics incident with p. If $q \in N_p$, then p and q intersect in a unique point x. Define the conical fibration of N_p to be the set of all geodesics that intersect p at the same point x and whose tangents lie in the plane tangent to p and q at x. This defines a causal geometry on $\mathcal{M}(X)$. This is only a conformal structure if X is actually flat (Theorem 17).

If p and q are incident geodesics in X, then the null $\mathcal{M}(X)$-geodesic through pq is the same whether it is regarded as lying on N_p or N_q. Moreover, if p, q, r are mutually incident geodesics, and we move r a small amount to another geodesic incident with q, then that geodesic will be incident to first order with p as well. (This is because a small movement of r can be decomposed into a translation and a twist in the tangent space to X at the common point of intersection, each of which preserves incidence to first order.)

A projective structure on X can be specified by a (unique) normal[3] projective Cartan connection consisting of a principal $P(\times\!\!-\!\!\bullet\!\!-\!\!\bullet)$-bundle $\mathscr{G}$ and equivariant $\mathfrak{sl}_4(\mathbb{R})$-valued one-form $\omega : T\mathscr{G} \to \mathfrak{sl}_4(\mathbb{R})$ that is an isomorphism of each tangent space, and that sends the generators of the vertical action to the corresponding elements of $\mathfrak{p}(\times\!\!-\!\!\bullet\!\!-\!\!\bullet)$. The Cartan connection descends to an isomorphism of $\mathscr{G}/P(\times\!\!-\!\!\times\!\!-\!\!\bullet)$ with the projective tangent bundle of X, denoted $\mathbb{P}TX$ with projection $\pi_{\mathbb{P}TX} \to X$. The distribution $\omega^{-1}\mathfrak{p}(\bullet\!\!-\!\!\times\!\!-\!\!\bullet)$ is invariant under $P(\times\!\!-\!\!\times\!\!-\!\!\bullet)$, and therefore descends to the quotient space. This gives a distribution V on $\mathbb{P}TX$, which is one-dimensional (and so automatically integrable). The space of integral curves of the distribution is the space $\mathcal{M}(X)$ of geodesics in X. Let $\pi_{\mathcal{M}(X)} : \mathbb{P}TX \to \mathcal{M}(X)$ be the associated projection mapping. We shall assume that we are working in a small enough patch of X to ensure that $\mathcal{M}(X)$ is a manifold and π a submersion. The causal cone in $T\mathcal{M}(X)$ is the image of the vertical distribution $V\mathbb{P}TX$ under $d\pi_{\mathcal{M}(X)}$. The image of each fiber of $V\mathbb{P}TX$ is a 2-plane in $T\mathcal{M}(X)$, and this family of 2-planes foliates the null cone.

Theorem 17. *The one-form ω defines a conformal Cartan connection on $\mathcal{M}(X)$ if and only if X is flat.*

Proof. Sufficiency is clear.

For necessity, fix an affine connection $P : TTX \to VTX$ compatible with the projective structure. Let $\lambda : TTX \to VTX$ be the canonical surjective nilpotent operator described in

[3]Normality is a curvature condition that we shall not go into; see Sharpe [76] or Čap and Slovák [11]. The Cartan connection obtained from a generic projective structure by the approach described in Chapter 3 is such a normal connection.

Chapter 1, and H the homogeneity vector field of TX (a section of VTX over TX). The geodesic spray associated to P is the vector field $V = (I - P)(\overline{\lambda}^{-1}(H))$. Let $p \in \mathcal{M}(X)$ be a particular geodesic in X and t be a coordinate along this geodesic. The tangent space to $\mathcal{M}(X)$ at p is the space of sections of the quotient bundle $K = TTX'/\operatorname{span}(H, V)$ that are Lie derived along V. (It makes sense to take the Lie derivative along V of sections of K since the distribution spanned by H and V is integrable and, in particular, is itself Lie derived along V.)

A vector x in K projects to a null vector in $\mathcal{M}(X)$ if and only if $\lambda x \equiv 0 \pmod{H}$: this is the condition that an infinitely near geodesic to p meets p at the point $\pi_X(x) \in X$.

Let $\epsilon : \wedge^3 VTX \to E[1]$ be the canonical map from the top exterior power of VTX to the space of 1-densities on TX. Define a quadratic form on TTX by

$$Q(x) = \epsilon(H, \lambda x, Px).$$

Since $\lambda V = H$, this annihilates the subspace spanned by H and V, and so descends to a quadratic form on K. If Q is Lie derived up to scale along V, then it defines a conformal structure on $\mathcal{M}(X)$. Conversely, if the causal structure on $\mathcal{M}(X)$ comes from a conformal structure, then this conformal structure is of this form.

So for the causal structure of $\mathcal{M}(X)$ to be a conformal structure, a necessary and sufficient condition is for Q to be Lie derived up to scale along the geodesic spray V. Since $\mathscr{L}_V x = 0$ and $\mathscr{L}_V \lambda = P$, we have (up to multiples of Q itself)

$$(\mathscr{L}_V Q)(x) = \epsilon(H, \lambda x, (\mathscr{L}_V P)x)$$

which is precisely the projective curvature function of X. $\square$

6.2.1 Twistor geometry

The null geodesics of the causal geometry on $M = \mathcal{M}(X)$ are labeled by pairs (x, P) with $x \in X$ a point, and P a two-dimensional linear subspace of $T_x X$. That is, the null geodesics in M can be identified with the points of $\mathrm{Gr}_2(TX)$, the Grassmannian of two-planes in TX.

There are a natural family of surfaces in M, called the α surfaces. An α surface consist of all points $p \in M$ such that the corresponding geodesics in X intersect at the same point of X. Any two points of a given α-surface are null-related to one another.

Although globally the structure of the space of geodesics in X can be quite complicated, we can specialize to a more tractable case: let Y be a Riemannian two-manifold, and consider the Riemannian cylinder $X = \mathbb{R} \times Y$ (with the natural product metric). The geodesics in $\mathbb{R} \times Y$ are just Cartesian products of geodesics on $\mathbb{R}$ and geodesics on Y. Let us write a geodesic in the form $(t, \gamma(t))$ with $t \in \mathbb{R}$ and $\gamma(t) \in M$. Assuming that Y is complete, this geodesic is uniquely determined by $\gamma(0)$ and $\gamma'(0)$. So the space of geodesics in X is $M = TY'$, the tangent bundle of Y with the zero section deleted.

6.2.2 Example: The sphere

For example, with $Y = S^2$, the natural product metric on X is $ds^2 = dt^2 + d\sigma^2$ where t is the natural coordinate on $\mathbb{R}$ and $d\sigma^2$ is the round sphere metric with constant curvature 1, which in the standard geodesic polar coordinates has the form $d\sigma^2 = d\phi^2 + \sin^2\phi\, d\theta^2$. A geodesic on X is uniquely determined by its initial ($t = 0$) position $\vec{x}$, unit normal vector $\vec{n}$ tangent to the sphere at x, and the speed $e^r > 0$ at which its projection onto S^2 travels around a great circle there. So the space of geodesics is isomorphic to $\mathbb{S}(TS^2) \times (0, \infty)$.

Suppose that a pair of geodesics $(\vec{x}_1, \vec{n}_1, r_1), (\vec{x}_2, \vec{n}_2, r_2)$ are given. The projection of these geodesics onto S^2 must meet (generically in two points). These geodesics are incident if they intersect at *the same parameter time*. (This is equivalent to the intersection of the lifted geodesics on the cylinder C.)

Suppose that $Z \in S^2$ is the nearest point of intersection of the two geodesics. Then Z lies simultaneously on the planes $\vec{n}_1 \cdot Z = 0$ and $\vec{n}_2 \cdot Z = 0$. So $Z = \vec{n}_1 \times \vec{n}_2 / |\vec{n}_1 \times \vec{n}_2|$. The time taken for the geodesic $(\vec{x}_1, \vec{n}_1, r_1)$ to reach Z is the measure of the angle between $\vec{x}_1$ and Z divided by the speed e^{r_1}: $e^{-r_1}\angle(\vec{x}_1, \vec{n}_1 \times \vec{n}_2)$. Hence the condition for the two geodesics to be incident is

$$e^{r_2}\angle(\vec{x}_1, \vec{n}_1 \times \vec{n}_2) = e^{r_1}\angle(\vec{x}_2, \vec{n}_1 \times \vec{n}_2).$$

Or, equivalently,

$$e^{r_2} \arccos\left(|\vec{n}_1 \times \vec{n}_2|^{-1}[\vec{x}_1, \vec{n}_1, \vec{n}_2]\right) = e^{r_1} \arccos\left(|\vec{n}_1 \times \vec{n}_2|^{-1}[\vec{x}_2, \vec{n}_1, \vec{n}_2]\right)$$

where $[-, -, -]$ denotes the scalar triple-product.

To determine the causal structure, fix a geodesic specified by the triple $(\vec{x}_1, \vec{n}_1, r_1) = (\vec{x}, \vec{n}, r)$, and set $\vec{x}_2 = \vec{x} + d\vec{x}$, $\vec{n}_2 = \vec{n} + d\vec{n}$, $r_2 = r + dr$. We also impose that $\vec{x} \cdot d\vec{x} = \vec{n} \cdot d\vec{n} = \vec{x} \cdot d\vec{n} + \vec{n} \cdot d\vec{x} = 0$. Then $|\vec{n} \times \vec{n}_2| = |d\vec{n}|$ and $[-, \vec{n}, \vec{n}_2] = [-, \vec{n}, d\vec{n}]$. Let $\vec{t} = \vec{n} \times \vec{x}$ so that together $\vec{t}$ and $\vec{x}$ form an orthonormal basis for the plane orthogonal to $\vec{n}$ and $\vec{t}$ and $\vec{n}$ form an orthonormal basis for the plane orthogonal to $\vec{x}$. Then it is possible to write $d\vec{n} = (\vec{t}\cos\theta + \vec{x}\sin\theta)d\rho$ and $d\vec{x} = \vec{t}\, dx + \vec{n}\, dy$. Since $\vec{x} \cdot d\vec{n} = -\vec{n} \cdot d\vec{x}$, we get $\sin\theta\, d\rho = -dy$

so $d\vec{x} = \vec{t}\,dx - \vec{n}\sin\theta\,d\rho$. Observe that $|d\vec{n}| = d\rho$ and $\vec{n} \times d\vec{n} = (\vec{x}\cos\theta - \vec{t}\sin\theta)d\rho$, so $[\vec{x}, \vec{n}, d\vec{n}] = \cos\theta\,d\rho$ and $[d\vec{x}, \vec{n}, d\vec{n}] = -\sin\theta\,dx\,d\rho$. Now consider the incidence condition

$$
\begin{aligned}
e^{r+dr} \arccos\left(|d\vec{n}|^{-1}[\vec{x}, \vec{n}, d\vec{n}]\right) &= e^{r} \arccos\left(|d\vec{n}|^{-1}[\vec{x} + d\vec{x}, \vec{n}, d\vec{n}]\right) \\
\implies e^{dr} \arccos\left(\cos\theta\right) &= \arccos\left(\cos\theta - \sin\theta\,dx\right).
\end{aligned}
$$

Expanding this in a series, keeping the first order terms gives

$$
\cos(dx/dr) = \cos(\theta)
$$

where $\theta = \arctan(\vec{x} \cdot d\vec{n}/\vec{t} \cdot d\vec{n})$.

Specialize now to the case where $\vec{x} = \vec{e}_3$ is the north pole, $\vec{n} = \vec{e}_2$, and $r = 0$, so that the geodesic is the prime meridian in the standard spherical coordinate system travelled with unit speed. Then $\vec{t} = \vec{e}_1$ and we have

$$
\begin{aligned}
d\vec{x} &= \vec{e}_1 dx - \vec{e}_2 \sin\theta d\rho \\
d\vec{n} &= (\vec{e}_1 \cos\theta + \vec{e}_3 \sin\theta) d\rho
\end{aligned}
$$

The space $M \cong \mathbb{S}(TS^2) \times (0, \infty)$ is a principal homogeneous space for $SU(2) \times (0, \infty)$, the Lie group $\mathbb{H}^*$ of non-zero quaternions.[4] The Lie algebra is the quaternion algebra $\mathbb{H}$ under the natural commutator $[x, y] = xy - yx$. To be invariant under the adjoint action, the causal cone in the Lie algebra must be of the form $f(\mathrm{Re}(x), |\,\mathrm{Im}(x)|) = 0$.

[4] Notice that this group is a split form of the compact group $U(2)$, which in turn corresponds to the incidence problem where the geodesics on S^2 carry a $U(1)$ parametrization instead of real parametrization.

6.3 INCIDENCE IN THE HEISENBERG GROUP

Let X be the three dimensional Heisenberg group. This is the group of upper triangular unipotent matrices $\begin{bmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$. Topologically this is $\mathbb{R}^3$. The Lie algebra of X is generated by the vector fields

$$\begin{aligned} v_1 &= \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z} \\ v_2 &= \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z} \\ v_3 &= \frac{\partial}{\partial z} \end{aligned}$$

which satisfy the commutation relations $[v_1, v_2] = v_3$ and $[v_1, v_3] = [v_2, v_3] = 0$. The Heisenberg group is equipped with a canonical invariant contact one-form $\sigma = dz - \frac{1}{2}(x\,dy - y\,dx)$ that is annihilated by v_1 and v_2. The form $v_1^2 + v_2^2$ defines an invariant cometric on X that induces the hamiltonian $H = \frac{1}{2}(\mu(v_1)^2 + \mu(v_2)^2)$ on T^*X, where $\mu = p_x dx + p_y dy + p_z dz$ is the natural moment map associated to the cotangent bundle.

So X carries the following metric structure. Let $P_1 = (x_1, y_1, z_1), P_2 = (x_2, y_2, z_2) \in X$ be fixed. The distance from P_1 to P_2 is obtained by drawing a circular arc in the xy-plane containing the points (x_1, y_1) and (x_2, y_2) such that the chordal area enclosed by the crescent-shaped region bounded by the arc and the chord joining its endpoints is $|z_1 - z_2|$. The distance between the two points is the length of the circular arc. The geodesics in X are helices. They project to circles in the xy plane, and over any arc of the circle the change in height is equal to the signed chordal area enclosed by the arc, measured in a positive sense. (If the geodesic winds round the circle several times, the chordal area is of course taken cumulatively.)

This metric structure comes from the following sub-Riemannian structure on X. Let $A = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}$

The four-dimensional space of geodesics is coordinatized by (x, y, r, ϕ) where (x, y) is the center of the associated circle, r is its radius, and ϕ is a polar angle representing the intersection of the geodesic with the xy-plane.

We now determine the incidence condition. Suppose that two circles with respective centers C_1, C_2 and radii r_1, r_2 are given. Generically these intersect at two points A and B. Let C be the point of intersection of AB and C_1C_2. Let $y = |AC|$; by symmetry $y = |BC|$ as well. Let $\rho = |C_1C_2|$ and $x = |C_1C|$. Since the angle at C is right, the Pythagorean theorem gives the relations

$$\begin{aligned} x^2 + y^2 &= r_1^2 \\ (\rho - x)^2 + y^2 &= r_2^2. \end{aligned}$$

Solving this system gives

$$\begin{aligned} x &= \frac{\rho^2 + r_1^2 - r_2^2}{2\rho} \\ y &= \frac{1}{2\rho}\sqrt{4\rho^2 r_1^2 - (\rho^2 + r_1^2 - r_2^2)^2}. \end{aligned}$$

This gives the equations for the points of intersection of the two circles. The angles $\alpha = \angle C_2C_1A$ and $\beta = \angle AC_2C_1$

The corresponding Heisenberg geodesics intersect if and only if the heights of the geodesics over the intersection points agree. Let ϕ_1 and ϕ_2 be the respective polar angles that each of the geodesic makes at $z = 0$.

For a circle of radius r, the chordal area subtended by an angle γ is $\frac{r^2\gamma}{2} - \frac{r^2 \sin\gamma}{2}$. Let θ be the angle that the ray $\overrightarrow{C_1C_2}$ makes with the positive x axis, and let $\alpha = \angle C_2C_1A$ and $\beta = \angle AC_2C_1$ (both in the range $[0, \pi]$). The height of the first geodesic at the intersection point A is then

$$\frac{r_1^2(\alpha + \theta + \phi_1)}{2} - \frac{r_1^2 \sin(\alpha + \theta + \phi_1)}{2}$$

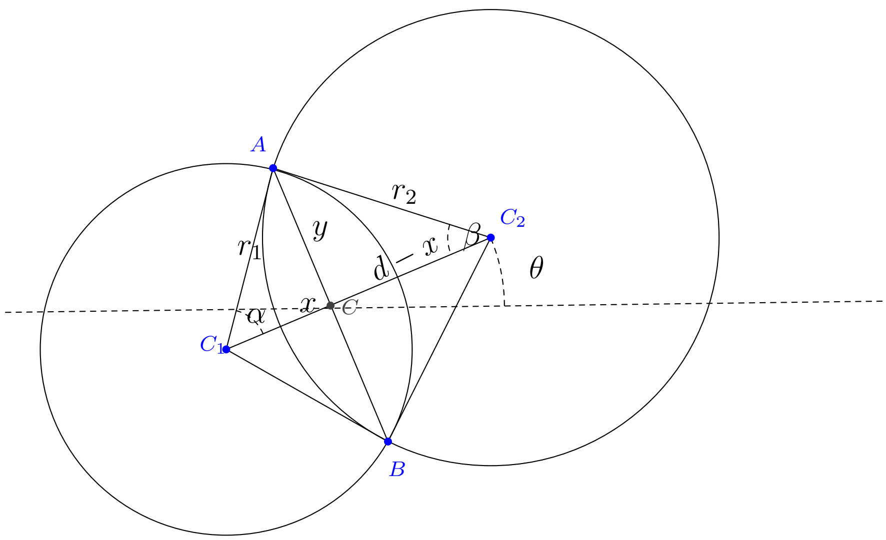
and the height of the second geodesic at the intersection point A is

$$\frac{r_2^2(-\beta + \theta + \phi_2 + \pi)}{2} - \frac{r_2^2(\sin(-\beta + \theta + \phi_2 + \pi))}{2}.$$

The angles α and β are given by

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{\rho^2 + r_1^2 - r_2^2}{2\rho r_1}\right) \\ \beta &= \cos^{-1}\left(\frac{\rho^2 - r_1^2 + r_2^2}{2\rho r_2}\right). \end{aligned}$$

Figure 3: Incidence of geodesics in the Heisenberg group



Now, let $\phi_1 = \phi$, $\phi_2 = \phi + d\phi$, $r_1 = r$, $r_2 = r + dr$, and $\rho = d\rho$. Putting these in to the equation of the causal cone and keeping first order terms gives the causal cone in the tangent space at z given by

$$\begin{aligned} 0 = 2\sin(\theta+\phi)\,dr^2 + \left(2\sin(\theta+\phi) + \sqrt{1-\frac{dr^2}{d\rho^2}}\right) d\rho^2 + \\ + 2\left(\pi + \theta + \phi - \cos^{-1}\left(\frac{dr}{d\rho}\right) - \cos(\theta+\phi)\sqrt{1-\frac{dr^2}{d\rho^2}}\right) dr\,d\rho + \\ + 2r\cos(\theta+\phi)\,dr\,d\phi + r\left(1 + 2\sqrt{1-\frac{dr^2}{d\rho^2}}\,\sin(\theta+\phi)\right) d\phi\,d\rho \end{aligned}$$

7.0 THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

A third-order differential equation under contact equivalence can be conveniently regarded as the following data.

1. A three-dimensional contact manifold J^1.
2. A generic three-parameter family of (unparameterized) contact curves in J^1.

The contact structure on J^1 can represented as a one-form θ which is only invariantly defined up to scale. By Darboux' theorem, there exist coordinates (x, y, p) on J^1 such that $\theta = dy - pdx$. A contact curve is a curve whose tangent annihilates θ at each point. The family of curves is generic if at each given point of J^1 and tangent direction v at x annihilating θ there exists a unique curve through x with tangent along v. These curves are identified with the (prolongation of) solutions of the differential equation. Two such structures are locally equivalent if there is a local diffeomorphism of J^1 to itself that preserves the contact structure and sends one system of curves to the other.

Given a third-order differential equation, it is clear how to generate such a structure by prolongation (see, for instance, [61]), and the resulting structure depends only on the contact-equivalence class of the differential equation, by Bäcklund's theorem. Conversely, suppose we have chosen coordinates (x, y, p) on J^1 such that $\theta = dy - pdx$. The distinguished class of curves is of the form

$$x = \chi(s; a, b, c)$$
$$y = \psi(s; a, b, c)$$
$$p = \pi(s; a, b, c)$$

where a, b, c are the three parameters defining a curve in the class, and s is an evolution parameter of the curve. There is a gauge freedom in selecting the parameterization s of the curve, and so this freedom is eliminated this by imposing the condition $dx/ds = 1$ (that is, by effectively taking x itself to be the parameter). The contact relation takes the form

$p = dy/dx$. Imposing this relation and differentiating ψ three times gives the system of equations

$$y = \psi(x; a, b, c)$$
$$y' = \psi_x(x; a, b, c)$$
$$y'' = \psi_{xx}(x; a, b, c)$$
$$y''' = \psi_{xxx}(x; a, b, c).$$

Solving the first three equations for a, b, c in terms of x, y', y'' and substituting the result into the third equation gives a third-order differential equation $y''' = F(x, y, y', y'')$.

Associated to the jet space J^1 there is naturally a fibration $\pi : J^2 \to J^1$. This is the subbundle of the projective tangent bundle $\mathbb{P}TJ^1$ of J^1 given as the annihilator of θ: $J^2 = \theta^\perp$. Specifically, J^2 is given fiberwise by

$$J^2_x = \left\{ v \in \mathbb{P}T_x J^1 \mid v \lrcorner \theta = 0 \right\}.$$

It is a four-dimensional space fibered over J^1 with S^1 fibers. The space J^2 defined here supports the following contact-invariant structure, independently of the differential equation. This characterization is the four-dimensional analog of structures studied in five dimensions by Doubrov, the author, and Sparling in [17].

Lemma 27. *1. There exists a natural filtration*

$$T^1 \subset T^2 \subset T^3 \subset T^4 = TJ^2$$

of the tangent bundle of J^2. Here T^1 is the vertical distribution for the fibration $J^2 \to J^1$, T^2 is a tautological bundle of 2-planes, and T^3 is the annihilator of the pullback of θ.

2. On sections, $[\Gamma(T^1), \Gamma(T^2)] = [\Gamma(T^1), \Gamma(T^3)] = \Gamma(T^3)$ and $[\Gamma(T^3), \Gamma(T^3)] = \Gamma(T^4)$.

Specifically, T^2 is the tautological 2-plane bundle whose fiber at a point $(x, u) \in J^2_x \subset \mathbb{P}T_x J^1$ consists of all vectors v such that $\pi_* v$ is in the direction of u. In terms of the (x, y, p) coordinates on J^1, any vector field of the form $\partial/\partial x + p\partial/\partial y + q\partial/\partial p$ annihilates the contact form $\theta = dy - pdx$. Therefore this q defines a fiber coordinate for the fibration $J^2 \to J^1$ that allows the vector fields generating $T^2 J^2$ to be expressed as $X = \partial/\partial q$, the vertical vector field for the fibration, and $\partial/\partial x + p\partial/\partial y + q\partial/\partial p$. The lemma follows by taking commutators.

Lemma 28. *Any two four-manifolds equipped with this structure are locally isomorphic: a filtration $T^1 \subset T^2 \subset T^3 \subset T^4$ on the tangent bundle, such that*

$$[\Gamma(T^1), \Gamma(T^2)] = [\Gamma(T^1), \Gamma(T^3)] = \Gamma(T^3) \quad \textit{and} \quad [\Gamma(T^3), \Gamma(T^3)] = \Gamma(T^4).$$

Proof. Let M be a manifold equipped with such a filtration and let θ be a nonvanishing one-form annihilating T^3M. Let N be the 3-manifold obtained by passing to the (locally defined) quotient modulo the flow of T^1M. The distribution T^3M is Lie derived along T^1M, and so descends to a distribution of 2-planes on N. Nowhere is this distribution Frobenius integrable, and so it defines a contact structure on N. By Darboux' theorem, N is locally contactomorphic to J^1 with its standard contact structure and coordinates (x, y, p). Letting q be a fiber coordinate on $M \to N$, $X = \partial/\partial q$ generates T^1M and $\theta = dy - p\,dx$. Plane subbundles of T^3 on which $\mathscr{L}_X$ maps surjectively onto T^3 are all related by a change in the fiber coordinate q. Indeed, as X commutes with $\partial/\partial p$, the latter vector field does not lie in T^2. The bundle T^2 must contain a solution Y of $\mathscr{L}_X Y = \partial/\partial p$. One such solution is $Y = \partial/\partial x + p\partial/\partial y + q\partial/\partial p$, and the ambiguity, modulo T^1, in the solution is a transformation of the form $Y \mapsto Y + \lambda\partial/\partial p$ where $X(\lambda) = 0$. Noting that λ is independent of q, this ambiguity in the choice of Y can be absorbed into a change of coordinates $q \mapsto q + \mu$ where $\mu(x, y, p)$ satisfies the differential equation $\mu - \mu_x - \mu_y - \mu_p = \lambda$. So we are free to choose the fiber coordinate q so that T^2 is generated by X and $Y = \partial/\partial x + p\partial/\partial y + q\partial/\partial p$. The coordinates (x, y, p, q) now defined on M establish a local diffeomorphism with J^2 that sends T^1M, T^2M, and T^3M to T^1J^2, T^2J^2, and T^3J^2, respectively. $\square$

The differential equation is specified in terms of a splitting of the first level of the filtration $T^1J^2 \subset T^2J^2$. This is achieved by means of a vector field V of the form $V = \partial/\partial x + p\partial/\partial y + q\partial/\partial p + F(x, y, p, q)\partial/\partial q$ representing the total derivative. While V itself is not contact-invariant, the splitting direction consisting of all multiples of V is. This splitting can invariantly be described in terms of the system of curves on J^1 that gives the differential equation. Lying over a point $x \in J^1$, the point $u \in J^2_x$ of the fiber is by definition a projective tangent vector at x. Passing through x in the direction defined by u is a distinguished curve of the differential equation. This curve determines a tangent direction at every point which therefore specifies a lift to J^2. The vector $V_{(x,u)}$ is the tangent direction to the lifted curve at the point $(x, u) \in J^2$.

7.0.1 Conformal structure

It is possible to construct from these data a degenerate conformal Lorentzian metric g on J^2. The degenerate direction for the metric is V, and the vector field X is null with respect to the metric. In terms of the coordinates (x, y, p, q) on J^2, this metric is given by

$$g = 2[dy - p\,dx][dq - \tfrac{1}{3}F_q\,dp + K\,dy + (\tfrac{1}{3}qF_q - F - pK)dx] - [dp - q\,dx]^2 \qquad (7.1)$$

where $K = \frac{1}{6}V(F_q) - \frac{1}{9}F_q^2 - \frac{1}{2}F_p$. In Section 7.3, the present paper constructs this metric in a manifestly contact-invariant fashion. The (locally defined) quotient space $\mathbb{M} = J^2/V$ is a three-manifold, but the metric g may not be Lie derived up to scale along V, and so need not pass down to the quotient. The Lie derivative $\mathscr{L}_V g$ is proportional to g if and only if the Wünschmann invariant of the original differential equation vanishes. The Wünschmann invariant is given by

$$W = F_y + (V - \tfrac{2}{3}F_q)K. \tag{7.2}$$

Vanishing of this invariant is a necessary and sufficient condition for $\mathbb{M}$ to possess an invariant Lorentzian conformal structure.

Going the other way, let $\mathbb{M}$ be a conformal Lorentzian 3-manifold. Define $\mathbb{S}$ to be the bundle over $\mathbb{M}$ with fiber S^1 that, at each point P, consists of all null directions in $\mathbb{P}T_P^*\mathbb{M}$. The pullback metric is degenerate in the vertical direction, and so $\mathbb{S}$ supports the structure of a degenerate conformal Lorentzian 4-manifold for which the degenerate direction is a conformal Killing symmetry. There is a canonical symplectic potential ψ defined on the total space of the cotangent bundle of $\mathbb{M}$. The form ψ is annihilated by the scaling in the fiber, and is Lie derived up to scale, and so descends to give a form θ up to scale on $\mathbb{S}$. The condition $\theta \wedge d\theta \neq 0$ follows since $\psi \wedge d\psi$ on $T^*\mathbb{M}$ does not vanish when pulled back to nonzero sections $\mathbb{M} \to T^*\mathbb{M}$. Indeed, in local coordinates $\mathbf{x} = (x^1, x^2, x^3)$ on $\mathbb{M}$ with fiber coordinates $\mathbf{p} = (p_1, p_2, p_3)$ on $T^*\mathbb{M}$, $\psi \wedge d\psi = (\mathbf{p} \cdot d\mathbf{x})(d\mathbf{p} \cdot d\mathbf{x}) = -\frac{1}{2}(\mathbf{p} \times d\mathbf{p}) \cdot (d\mathbf{x} \times d\mathbf{x})$. Since $d\mathbf{x} \times d\mathbf{x}$ has three linearly independent components, and $\mathbf{p} \times d\mathbf{p}$ vanishes only on vectors parallel to the generator of scalings in $T^*\mathbb{M}$, $\psi \wedge d\psi$ does not vanish when pulled back along any section of the projective cotangent bundle, and so *a fortiori* it does not vanish when pulled back to $\mathbb{S}$.

The vector field X is the given by the null geodesic spray in $\mathbb{S}$. To describe this, fix a metric g in the conformal class on $\mathbb{M}$. On $T^*\mathbb{M}$ the geodesic Hamiltonian is $H = \pi^* g^{-1}(\psi, \psi)$, which gives rise to the Hamiltonian vector field $\widehat{H}$, defined by $\widehat{H} \lrcorner d\psi = dH$. An integral curve μ of X projects to a geodesic of M, and the fiber component of μ is the covelocity of the geodesic. The image of $\mathbb{S}$ under the map $\mathbb{S} \to T^*\mathbb{M}$ is the null cone at every point of $\mathbb{M}$. This is everywhere tangent to the spray $\widehat{H}$, because a geodesic being initially null will always remain null. Furthermore, $\widehat{H}$ scales quadratically in the cotangent bundle, and descends to a direction field on the projective cotangent bundle $\mathbb{P}T^*\mathbb{M}$. This direction field is everywhere tangent to $\mathbb{S}$, and so restricts to a direction field X on $\mathbb{S}$.

Passing to the (locally defined) quotient by X gives the space of null geodesics—the twistor space $\mathbb{T}$. It follows from the definition of X and θ that $X \lrcorner \theta = 0$ and $X \lrcorner d\theta = 0$. The second assertion follows by pulling back $X \lrcorner d\psi = dH$ to the null cone, and using the fact that H vanishes identically there. Thus θ descends to a contact structure on the twistor space. The fibers of $\mathbb{S} \to \mathbb{M}$ project to the distinguished curves of the twistor space.

To prove that these distinguished curves are suitably generic, we verify that $\mathbb{S}$ admits a structure satisfying the conditions of Lemma 28. Let T^1 be the bundle spanned by the null geodesic spray X and T^3 be annihilator of θ. From $\theta \wedge d\theta \neq 0$, it follows that T^3 is not Frobenius integrable, and so $[\Gamma(T^3), \Gamma(T^3)] = \Gamma(T\mathbb{S})$. Moreover, since θ is annihilated by X and Lie derived along it, $[\Gamma(T^1), \Gamma(T^3)] = \Gamma(T^3)$.

It remains to identify T^2 and to show that $[\Gamma(T^1), \Gamma(T^2)] = \Gamma(T^3)$. Let V be a nonvanishing vertical vector field for $\mathbb{S} \to \mathbb{M}$ and let $T^2 = \operatorname{span}\{X, V\}$. One such vector field V can be given in terms of the angular momentum operator $\mathbf{L} = \mathbf{p} \times \partial/\partial\mathbf{p}$. Then $\mathbf{L}$ is tangent to the null cone, because it annihilates H. On homogeneous functions of degree 0, the angular momentum factors through a scalar operator $\mathbf{L}(f) = V(f)\mathbf{p}$, which defines the vector field V. Because θ is horizontal, any such vector field, being vertical, annihilates θ. Furthermore, $[X, V] \lrcorner \theta = \mathscr{L}_X(V \lrcorner \theta) = 0$ as well, so $[X, V] \in T^3$. It remains only to show that $X, V, [X, V]$ are linearly independent. Each of these vector fields is at most first order in the metric, and therefore independence follows by a calculation in normal coordinates. In these coordinates, $X = -\mathbf{p} \cdot \partial/\partial\mathbf{x}$ and so on the one hand

$$[X, \mathbf{L}] = \mathbf{p} \times \frac{\partial}{\partial \mathbf{x}}$$

and on the other hand, on the null cone this acts as $\mathbf{p}[X, V]$ on functions homogeneous of degree zero. Finally,

$$d\theta\left([X, \mathbf{L}], \mathbf{L}\right) = (\mathbf{p} \cdot \mathbf{p})g - \mathbf{p} \otimes \mathbf{p}$$

which vanishes nowhere on the null cone. Thus $[\Gamma(T^1), \Gamma(T^2)] = \Gamma(T^3)$.

So modulo the explicit construction of the degenerate metric and the assertion that specifically it is the Wünschmann invariant that governs whether the degenerate metric descends to the 3-manifold, we have proven the following theorem (proven in Fritelli, Kozameh, Newman [30] by entirely different methods):

Theorem 18. *There is a natural local equivalence between third-order differential equations under contact transformations with vanishing Wünschmann invariant and conformal Lorentzian 3-manifolds.*

7.1 CAUSAL GEOMETRIES ON THE SPACE OF SOLUTIONS

This section develops the natural geometric structure associated to the space of solutions $\mathbb{M}$ to a third-order equation. This structure reduces to a conventional Lorentzian conformal structure if and only if the Wünschmann invariant vanishes. To motivate this discussion, the previous section establishes the basic properties of the (possibly locally defined) double fibration

$$\begin{tikzcd} & \mathbb{S} \arrow[dl] \arrow[dr] & \\ \mathbb{T} & & \mathbb{M} \end{tikzcd}$$

An individual fiber of the submersion $\mathbb{S} \to \mathbb{M}$ projects down to give a trajectory solving the differential equation in $\mathbb{T}$: this is a re-expression of the notion that $\mathbb{M}$ is a *space of solutions* of the differential equation. The fibers for the other submersion $\mathbb{S} \to \mathbb{T}$ also project down to the space of solutions $\mathbb{M}$, although their precise meaning has heretofore not been identified in general.

When the Wünschmann invariant vanishes, $\mathbb{M}$ carries a natural conformal Lorentzian metric by Theorem 18, and $\mathbb{S}$ is canonically identified with the null cone bundle associated to this metric. The space $\mathbb{T}$ is then the twistor space: the quotient of $\mathbb{S}$ by the null geodesic spray. The fibration $\mathbb{S} \to \mathbb{M}$ can be understood as the subbundle of the projective tangent bundle $\mathbb{P}T\mathbb{M}$ of null directions. Under this correspondence, a null cone with vertex at $P \in \mathbb{M}$ corresponds to a one-parameter family of null geodesics, which in turn is identified with the trajectory defining the solution P.

This structure can be axiomatized in a manner that allows construction of the spaces $\mathbb{S}$ and $\mathbb{T}$, along with their natural contact structure. The one parameter families of null geodesics starting at each point P give rise to a cone in the tangent space $T_P\mathbb{M}$ with vertex at the origin. Equivalently, such a cone is the affine cone over some curve in the projective space $\mathbb{P}T_P\mathbb{M}$. Here locality considerations may mean that the cone may fail to close up completely, or the associated curve may have one or more singular points. Henceforth, we shall work only near regular points of the curve.[1]

7.1.1 Incidence relation and the indicatrix

Two solutions y_1 and y_2 to the differential equation $y''' = F(x, y, y', y'')$ are said to be *incident* if, at some point x, one has

$$y_1(x) = y_2(x), \quad y_1'(x) = y_2'(x).$$

That is to say, two curves are incident if and only if the associated solution curves in the J^1 intersect. In the latter interpretation, the incidence relation is manifestly contact-invariant.

[1] We can localize near a point of $\mathbb{T}$ and a point of $\mathbb{M}$ simultaneously. Thus all solutions can be assumed to be fully regular, but cannot necessarily be continued for all time. The geometrical implication is that we may isolate a smooth part of the null cone at each point of $\mathbb{M}$, but the "cone" need not then close up.

The set of all solutions incident with a given solution $P \in \mathbb{M}$ cuts out a surface N_P with a conical type singularity at the vertex P. The generators of this cone can be described as follows. If Q is a solution incident with P, then the curve N_{PQ} consisting of solutions R incident with P at the same point of J^1 as Q. (Our localization assumption implies that two distinct solutions are incident at most at a single point of J^1.) The surface N_P is thus ruled by the pencil of curves N_{PQ} as Q varies over solutions incident with P.

Theorem 19 establishes that the conical fibration associated to this incidence relation is a causal geometry on the space of solutions of a third-order differential equation. In this setting, the first four Properties described in §5.2 are fairly natural assumptions that make precise the notion that the N_P should be a cone based at P. The envelope condition, in terms of the bundle $J^1 \to \mathbb{M}$, guarantees that the one-form θ, whose annihilator is the 3-plane bundle that lifts the tangent planes through the vertices of the cones N_P, is Lie derived along the fibers of the fibration $J^2 \to J^1$.

7.1.2 The causal geometry associated to a differential equation

At each point P of $\mathbb{M}$, denote by

$$y = f(x; P)$$

the solution to the differential equation represented by P. We break contact invariance in specifying the differential equation, but shall ultimately be concerned only with the contact-invariant information that can be extracted from f. Locally, this solution depends smoothly on x and P. The underlying assumption under which $\mathbb{M}$ is a differentiable manifold smoothly parameterizing a space of solutions is that in local coordinates $P = (p_1, p_2, p_3)$ on $\mathbb{M}$, the Wronskian determinant

$$\begin{vmatrix} \frac{\partial f}{\partial p_1} & \frac{\partial f}{\partial p_2} & \frac{\partial f}{\partial p_3} \\ \frac{\partial f_x}{\partial p_1} & \frac{\partial f_x}{\partial p_2} & \frac{\partial f_x}{\partial p_3} \\ \frac{\partial f_{xx}}{\partial p_1} & \frac{\partial f_{xx}}{\partial p_2} & \frac{\partial f_{xx}}{\partial p_3} \end{vmatrix} \neq 0. \tag{7.3}$$

To obtain a more concrete description of the cone N_P through a particular solution $P \in \mathbb{M}$ and the associated curves N_{PQ} that rule the cone defined in the previous section, suppose that $\gamma(t)$ is a parameterization of the curve N_{PQ} such that $\gamma(0) = P$. Since all points along $\gamma(t)$ are incident with P at the same point (x, y, y'), the following two equations must hold along γ:

$$\begin{aligned} f(x;\gamma(t)) &= f(x;\gamma(0)) \\ f_x(x;\gamma(t)) &= f_x(x;\gamma(0)). \end{aligned} \tag{7.4}$$

Under generic conditions, the second equation can be used to solve for x in terms of $\gamma(0)$ and $\gamma(t)$, and then substituted into the first equation which may then be solved for the admissible values $\gamma(t)$. Although ostensibly this system of equations is not invariant under contact transformations, by Bäcklund's theorem a contact transformation will preserve the space of solutions $\gamma(t)$. This is geometrically evident because the causal geometry itself is contact-invariant.

Solutions of these equations arise as first integrals of the differentiated forms

$$\begin{aligned} \frac{d}{dt}f(x;\gamma(t)) &= 0 \\ \frac{d}{dt}f_x(x;\gamma(t)) &= 0 \end{aligned} \tag{7.5}$$

with initial conditions $\gamma(0), \gamma'(0)$. The initial conditions are not completely arbitrary. Rather if $\gamma(0)$ is fixed, then $\gamma'(0)$ is constrained by the requirement that

$$\begin{aligned} \left.\frac{d}{dt}f(x;\gamma(t))\right|_{t=0} &= 0 \\ \left.\frac{d}{dt}f_x(x;\gamma(t))\right|_{t=0} &= 0 \end{aligned} \tag{7.6}$$

The second equation of (7.6) can be solved to obtain the parameter x in terms of the initial conditions $\gamma(0)$, $\gamma'(0)$, by (7.3). The following Lagrangian is homogeneous of degree two in $\gamma'(0)$:

$$L(\gamma(0),\gamma'(0)) = \left.\frac{d}{dt}f(x(\gamma(0),\gamma'(0));\gamma(t))\frac{d}{dt}f_{xx}(x(\gamma(0),\gamma'(0));\gamma(t))\right|_{t=0} \tag{7.7}$$

The second factor ensures that the Lagrangian is regular (Lemma 29). For fixed $\gamma(0)$, the values of the tangent $\gamma'(0)$ satisfying equation (7.6) cut out an affine cone over a curve in the projective tangent space at $\gamma(0)$. This cone is the "null cone" for the Lagrangian L, and it coincides with the tangent cone to $N_{\gamma(0)}$ at the vertex $\gamma(0)$. As in Property 4 of §7.1.1, for $P \in \mathbb{M}$, denote by $C_P \subset \mathbb{P}T_P\mathbb{M}$ the curve cut out in the projective tangent space by the equation $L(P, v) = 0$.

Although the Lagrangian L is not itself contact-invariant, by the argument already given its locus of zeros is contact-invariant. Under contact transformations, the Lagrangian is determined up to rescaling by a nonvanishing function of $\gamma(0)$ and $\gamma'(0)$,

$$L(\gamma(0), \gamma'(0)) \to \Omega(\gamma(0), \gamma'(0)) L(\gamma(0), \gamma'(0)).$$

The curves N_{PQ} that generate the causal cone N_P are extremals for the energy functional

$$E[\gamma] = \frac{1}{2} \int_a^b L(\gamma(t), \gamma'(t))\, dt.$$

Theorem 19. *A third-order differential equation determines a causal geometry.*

Proof. Property 1 of §7.1.1 reflects our running localization assumption that the third-order differential equation under consideration is regular. In local coordinates on $\mathbb{M}$, the system of equations

$$f(x; q_1, q_2, q_3) = f(x; p_1, p_2, p_3), \quad f_x(x; q_1, q_2, q_3) = f_x(x; p_1, p_2, p_3) \tag{7.8}$$

for unknowns $x, Q = (q_1, q_2, q_3)$ and fixed $P = (p_1, p_2, p_3)$ has Jacobian matrix

$$\begin{bmatrix} f_x(x; q_1, q_2, q_3) - f_x(x; p_1, p_2, p_3) & \frac{\partial f}{\partial q_1} & \frac{\partial f}{\partial q_2} & \frac{\partial f}{\partial q_3} \\ \\ f_{xx}(x; q_1, q_2, q_3) - f_{xx}(x; p_1, p_2, p_3) & \frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} & \frac{\partial f_x}{\partial q_3} \end{bmatrix}.$$

The matrix always has rank two by (7.3). Moreover, at a solution $Q \neq P$ of the original system, the lower left-hand corner cannot be zero, by uniqueness of solutions. Thus the first column, along with one of the remaining three columns must yield an invertible 2×2 submatrix. The implicit function theorem then implies that the solution is a smooth submanifold away from the vertex $Q = P$.

For Property 2, N_{PR} is the set of solutions $Q = (q_1, q_2, q_3)$ of (7.8) for a given value of $x, P = (p_1, p_2, p_3)$. Because (7.8) has rank two in the q_i variables, the space of solutions is a smoothly embedded curve. These curves clearly cover N_P. Furthermore, two distinct curves meet only at the vertex P, by uniqueness of solutions of differential equations.

Property 3 is obvious. Property 4 is equivalent to the assertion that L is a regular Lagrangian, which is proven in Lemma 29 of the next section.

Finally, suppose without loss of generality that in the statement of Property 5, R is a point between P and Q on N_{PQ}. Along the curve N_{PQ} the value of the parameter x is fixed, and N_{PQ} itself consists of all points R such that

$$f(x; R) = f(x; P)$$
$$f_x(x; R) = f_x(x; P).$$

The tangent plane to N_P at the point R is the annihilator of $d_R f(x, R)$.[2] This is the same tangent plane as that obtained by interchanging the roles of P and Q. $\square$

7.1.3 Hamiltonian formulation

As in the previous section, for $P \in \mathbb{M}$, let $y = f(x; P)$ denote the solution of the differential equation corresponding to P. Once again, contact-invariance is broken, but ultimately we will only be concerned with contact-invariant information contained in the solution. Let $d_P f$ denote the exterior derivative of f regarding x as constant. In local coordinates (p_1, p_2, p_3) on $\mathbb{M}$,

$$d_P f(x; p_1, p_2, p_3) = \frac{\partial f}{\partial p_1} dp_1 + \frac{\partial f}{\partial p_2} dp_2 + \frac{\partial f}{\partial p_3} dp_3.$$

Then $x \mapsto d_P f(x; P)$ defines a curve in the cotangent space $T^*_P\mathbb{M}$. Denote the associated projective curve by $\tilde{C}_P \subset \mathbb{P}T^*\mathbb{M}$.

This curve is linked to the curve C_P cut out by the Lagrangian via the following construction. Let V be a three-dimensional vector space and C a smooth curve in the projective plane $\mathbb{P}V$. The dual projective plane $\mathbb{P}V^*$ is naturally identified with the space of lines in $\mathbb{P}V$. The dual curve C^* is the curve in $\mathbb{P}V^*$ defined by locus of lines tangent to C. Suppose that C is a nondegenerate curve in $\mathbb{P}V$, with parameterization $t \mapsto \gamma(t)$. At a point $\gamma(t)$ of C, the corresponding point of the dual curve is obtained by solving for $\gamma^*(t) \in \mathbb{P}V^*$ the equations

$$\langle \gamma^*(t), \gamma(t) \rangle = 0$$
$$\langle \gamma^*(t), \gamma'(t) \rangle = 0.$$

Properly speaking, to make sense of the second equation, it is necessary to choose a lift of γ to a curve in V. Modulo the first equation, the second equation does not depend on the choice of lift, and so there is no ambiguity in speaking of the solution of the system of equations.

[2] Here and elsewhere, the notation $d_R f(x; R)$ is the exterior derivative of f with respect to the variable R only. Equivalently, it is the exterior derivative of f modulo the relation $dx = 0$.

Proposition 2. *$\tilde{C}_P \subset \mathbb{P}T^*_P\mathbb{M}$ and $C_P \subset \mathbb{P}T_P\mathbb{M}$ are mutually dual.*

Proof. The curve $\tilde{C}_P$ is characterized as the image of the map

$$x \mapsto d_P f(x; P).$$

The dual curve to $\tilde{C}_P$ is defined by the equations

$$\begin{aligned} \langle \gamma^*(x), d_P f(x; P) \rangle &= 0 \\ \langle \gamma^*(x), d_P f_x(x; P) \rangle &= 0. \end{aligned} \tag{7.9}$$

But these two equations are identical with the equations that characterize C_P. □

Alternatively, choose local coordinates at P and linearize the differential equation at the solution defined by P. Then

$$f(x; p_1, p_2, p_3) = \phi_1(x)p_1 + \phi_2(x)p_2 + \phi_3(x)p_3$$

and so

$$d_P f = \phi_1(x)dp_1 + \phi_2(x)dp_2 + \phi_3(x)dp_3.$$

The incidence relation between $f(x; p_1, p_2, p_3)$ and a nearby solution $f(x; p_1+dp_1, p_2+dp_2, p_3+dp_3)$ is then precisely

$$\begin{aligned} \phi_1(x)dp_1 + \phi_2(x)dp_2 + \phi_3(x)dp_3 &= 0 \\ \phi_1'(x)dp_1 + \phi_2'(x)dp_2 + \phi_3'(x)dp_3 &= 0 \end{aligned}$$

but these are the same equations that characterize the dual curve of $\tilde{C}_P$.

Lemma 29. *Let $L : T\mathbb{M} \to \mathbb{R}$ be a Halitonian (7.10). Then L is regular in a neighborhood of each point of C_P.*

Indeed, it is sufficient to show that the Hessian matrix $\partial^2 L/\partial\dot{q}_i\partial\dot{q}_j$ is nonsingular at each point of C_P. The Lagrangian is defined by

$$L(q,\dot{q}) = \left(\dot{q}_i \frac{\partial f}{\partial q_i}(x(q,\dot{q}),q)\right)\left(\dot{q}_j \frac{\partial f_{xx}}{\partial q_j}(x(q,\dot{q}),q)\right). \tag{7.10}$$

The function $x(q,\dot{q})$ is defined by

$$\dot{q}_i \frac{\partial f_x}{\partial q_i}(x(q,\dot{q}),q) = 0. \tag{7.11}$$

In particular, by implicit differentiation,

$$\frac{\partial x}{\partial \dot{q}_i} = -\frac{\partial f_x/\partial q_i}{\dot{q}_k \partial f_{xx}/\partial q_k}. \tag{7.12}$$

At a point of C_P, in addition the following holds:

$$\dot{q}_i \frac{\partial f}{\partial q_i}(x(q,\dot{q}),q) = 0. \tag{7.13}$$

The Hessian at a point of C_P is computed by differentiating (7.10), imposing (7.11) and (7.13) along the way, and then finally substituting (7.12). In detail, denote $f_i = \partial f/\partial q_i$, $f_{xi} = \partial f_x/\partial q_i$, etc., and $x_i = \partial x/\partial \dot{q}_i$. Then

$$\begin{aligned}
L &= \dot{q}^k f_k \dot{q}^\ell f_{xx\ell} \\
L_i &= f_i \dot{q}^\ell f_{xx\ell} + \dot{q}^k f_k f_{xxi} + \dot{q}^k f_{xk} x_i \dot{q}^\ell f_{xx\ell} + \dot{q}^k f_k \dot{q}^\ell f_{xxx\ell} x_i \\
L_{ij} &= f_{xi} x_j \dot{q}^\ell f_{xx\ell} + f_i f_{xxj} + f_i \dot{q}^\ell f_{xxx\ell} x_j + f_j f_{xxi} + \\
&\qquad f_{xj} x_i \dot{q}^\ell f_{xx\ell} + \dot{q}^k f_{xxk} x_i x_j \dot{q}^\ell f_{xx\ell} + f_j \dot{q}^\ell f_{xxx\ell} x_i \qquad \pmod{(7.11), (7.13)} \\
&= -f_{xi} f_{xj} + f_i f_{xxj} + f_{xxi} f_j - f_i f_{xj}\left(\frac{\dot{q}^\ell f_{xxx\ell}}{\dot{q}^k f_{xxk}}\right) - f_{xi} f_j \left(\frac{\dot{q}^\ell f_{xxx\ell}}{\dot{q}^k f_{xxk}}\right)
\end{aligned}$$

by (7.12). Thus the Hessian matrix of L has the form

$$\operatorname{Hess} L = -AA^T + BC^T + CB^T$$

where the column vectors A, B, C are defined by

$$\begin{aligned}
A_i &= f_{xi} \\
B_i &= f_i \\
C_i &= f_{xxi} - \frac{\dot{q}^\ell f_{xxx\ell}}{\dot{q}^k f_{xxk}} f_{xi}.
\end{aligned}$$

By the hypothesis (7.3), A, B, C are linearly independent, and so

$$\det \operatorname{Hess} L = -(\det[A\ B\ C])^3 \neq 0,$$

which establishes the lemma.

The Legendre transformation associated to the Lagrangian L is a function $\mathscr{L} : T\mathbb{M} \to T^*\mathbb{M}$ covering the projection onto M. Over a point $P \in \mathbb{M}$, $L : T_P\mathbb{M} \to \mathbb{R}$, and $\mathscr{L} = DL$, the vertical exterior derivative of L. The Hamiltonian on the cotangent bundle associated to the degree 2 homogeneous function L is defined by $H = L \circ \mathscr{L}^{-1}$. Here $\mathscr{L}^{-1}$ is the inverse, possibly only locally defined near points of C_P, of the function $\mathscr{L} : T_PM \to T_P^*M$.

By Lemma 29, H is well-defined in a neighborhood of the preimage of C_P under $\mathscr{L}$. The Hamiltonian, where it is defined, vanishes precisely on the dual curve $\tilde{C}_P$. Indeed, in local coordinates,

$$\begin{aligned}
\mathscr{L}_i = \frac{\partial L}{\partial \dot{q}^i} &= f_i \dot{q}^\ell f_{xx\ell} + \dot{q}^k f_k f_{xxi} + \dot{q}^k f_{xk} x_i \dot{q}^\ell f_{xx\ell} + \dot{q}^k f_k \dot{q}^\ell f_{xxx\ell} \\
&= f_i \dot{q}^\ell f_{xx\ell}
\end{aligned}$$

when evaluated at any point of C_P. Thus $\mathscr{L}$ is proportional to $d_p f$ at each point of C_P. Inverting, we conclude that $L \circ \mathscr{L}^{-1}(d_P f) = 0$, so H vanishes along $\tilde{C}_P$.

The following alternative construction of the Hamiltonian also applies, by linearizing the problem at P. In local coordinates at P, $\phi_i(x) = \frac{\partial f}{\partial p_i}$, $i = 1, 2, 3$ define independent solutions of the linearized ordinary differential equation. In terms of these three solutions, the linearized equation itself can be recovered by solving the 3×3 system for the unknown coefficients h_i:

$$\phi_i(x) h_0(x) + \phi_i'(x) h_1(x) + \phi_i''(x) h_2(x) = \phi_i'''(x), \quad i = 1, 2, 3. \tag{7.14}$$

Cramer's rule gives

$$h_0(x) = \frac{\begin{vmatrix} \phi_1''' & \phi_1' & \phi_1'' \\ \phi_2''' & \phi_2' & \phi_2'' \\ \phi_3''' & \phi_3' & \phi_3'' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_1' & \phi_1'' \\ \phi_2 & \phi_2' & \phi_2'' \\ \phi_3 & \phi_3' & \phi_3'' \end{vmatrix}}, \qquad h_1(x) = \frac{\begin{vmatrix} \phi_1 & \phi_1''' & \phi_1'' \\ \phi_2 & \phi_2''' & \phi_2'' \\ \phi_3 & \phi_3''' & \phi_3'' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_1' & \phi_1'' \\ \phi_2 & \phi_2' & \phi_2'' \\ \phi_3 & \phi_3' & \phi_3'' \end{vmatrix}}$$

$$h_2(x) = \frac{\begin{vmatrix} \phi_1 & \phi_1' & \phi_1''' \\ \phi_2 & \phi_2' & \phi_2''' \\ \phi_3 & \phi_3' & \phi_3''' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_1' & \phi_1'' \\ \phi_2 & \phi_2' & \phi_2'' \\ \phi_3 & \phi_3' & \phi_3'' \end{vmatrix}}.$$

The Lagrangian of the original equation localized at the point P is equal to the Lagrangian of the linearized equation. It is given first by solving

$$\phi_1'(x)q_1 + \phi_2'(x)q_2 + \phi_3'(x)q_3 = 0$$

for x as a function of q_1, q_2, q_3. In that case,

$$L(q) = (\phi_1(x(q))q_1 + \phi_2(x(q))q_2 + \phi_3(x(q))q_3)\,(\phi_1''(x(q))q_1 + \phi_2''(x(q))q_2 + \phi_3''(x(q))q_3)\,.$$

The associated Hamiltonian is obtained by the same construction, but applied to solutions $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3$ of the adjoint equation to (7.14):

$$y(x)h_0(x) - (y(x)h_1(x))' + (y(x)h_2(x))'' = -y'''(x).$$

The following theorem is due to Wilczynski [84]; cf. also Olver [61]:

Theorem 20. *The projective curves* $x \mapsto \phi_1(x)\,\partial/\partial q_1 + \phi_2(x)\,\partial/\partial q_2 + \phi_3(x)\,\partial/\partial q_3$ *in* $\mathbb{P}T_P\mathbb{M}$ *and* $x \mapsto \tilde{\phi}_1(x)\,dq_1 + \tilde{\phi}_2(x)\,dq_2 + \tilde{\phi}_3(x)\,dq_3$ *are mutually dual.*

7.2 HAMILTONIAN SPRAY

As the point P varies, the dual curve $\tilde{C}_P$ cut out by the Hamiltonian defines a subfibration of the projective cotangent bundle $\mathbb{P}T^*\mathbb{M}$. The four-manifold defined by the total space of this fibration is denoted here by $\mathbb{S}$, and the cone over $\mathbb{S}$ in the cotangent bundle $T^*\mathbb{M}$ is denoted by $\tilde{\mathbb{S}}$. The canonical one-form θ on $T^*\mathbb{M}$ is homogeneous of degree one, and pulls back to a natural one-form defined up to scale on $\tilde{\mathbb{S}}$.

For a fixed choice of Hamiltonian homogeneous of degree two, define the Hamiltonian spray on $T^*\mathbb{M}$ as the unique vector field X such that

$$X \lrcorner d\theta = dH.$$

The Hamiltonian spray is invariant up to scale under rescalings of H. Moreover, it is tangent to the variety $\tilde{\mathbb{S}}$ cut out by $H = 0$ since $X \lrcorner dH = 0$. The vector field X is homogeneous of degree one: if $\mu_t : T^*\mathbb{M} \to T^*\mathbb{M}$ denotes the dilation mapping in the fibers, then $X_{t\alpha} = t(\mu_t)_* X_\alpha$.

The Hamiltonian spray will now be used to define a filtration on $T\mathbb{S}$ in a manner analogous to the proof of Theorem 18. Let V be a nonvanishing vector field that is vertical for the fibration $\mathbb{S} \to \mathbb{M}$. Let $T^1 \subset T\mathbb{S}$ be the subbundle spanned by X, T^2 the subbundle spanned by X, V. Let $T^3 \subset T\mathbb{S}$ be the annihilator of θ. Since X and V both annihilate θ, $T^2 \subset T^3$. Moreover, since θ is annihilated by X and is Lie derived along it, $[\Gamma(T^1), \Gamma(T^2)] = \Gamma(T^3)$. Because $\theta \wedge d\theta$ does not vanish when pulled back along on any section of the cotangent bundle, it also does not vanish on $\mathbb{S}$ and so T^3 is not Frobenius integrable at any point, and thus $[\Gamma(T^3), \Gamma(T^3)] = \Gamma(T\mathbb{S})$

It remains only to show that $[\Gamma(T^1), \Gamma(T^2)] = \Gamma(T^3)$. It is sufficient to prove that $X, V, [X, V]$ are linearly independent. As in the proof of Theorem 18, it is convenient to work with a particular choice of vector field V. Define the tensor $h \in \operatorname{Sym}^2 T(T^*\mathbb{M})$ to be the vertical Hessian of H. In coordinates,

$$h^{ij} = \frac{\partial^2 H}{\partial p_i \partial p_j},$$

and let h_{ij} be the inverse of h^{ij}. These two tensors can be used to raise and lower indices. Let ϵ be the associated volume tensor in the fiber. In coordinates

$$\epsilon = \epsilon^{ijk} dp_i \otimes dp_j \otimes dp_k.$$

The angular momentum

$$L^i = \epsilon^{ijk} p_j h_{k\ell} \frac{\partial}{\partial p_\ell}$$

kills H, since $\partial H/\partial p_\ell = p^\ell$, and so is tangent to the null cone bundle $\widetilde{\mathbb{S}}$. Furthermore, on homogeneous functions of degree zero along the null cone, L^i factors through a scalar operator $L^i = p^i V$, because p^i and $\partial/\partial p_i$ are an orthogonal basis for the orthogonal complement of p_i. This defines a vertical vector V. To show that $X, V, [X, V]$ are linearly independent, it is enough to show that $d\theta([X, L^i], L^j) \neq 0$. The Lie bracket is given by

$$[X, L^i] = -\epsilon^{ijk} p_j \frac{\partial}{\partial x^k} + T^i_j \frac{\partial}{\partial p_j}$$

for some tensor T. So

$$d\theta([X, L^i], L^j) = \epsilon^{imn} p_m h_{kn} \epsilon^{jkn} p_k = p^i p^j - p^k p_k h^{ij}$$

which does not vanish when restricted to the null cone.

Thus by Lemma 28, $\mathbb{S}$ equipped with its geodesic spray and vertical vector field is locally isomorphic to the space J^2. Because of the preferred direction V, the inclusion of bundles $T^1 \subset T^2$ splits, and thus gives rise to a third-order differential equation the distinguished curves of which are the fibers of $\mathbb{S} \to \mathbb{M}$. The entire procedure is reversible, by the construction of the preceding section, which establishes Theorem 1.

7.3 RECOVERING THE DEGENERATE METRIC

The degenerate metric on $\mathbb{S}$ is defined as follows. A point of $\mathbb{S}$ consists of a point $P \in \mathbb{M}$ and $v \in C_P$. Now, through $v \in C_P$, there is a uniquely defined *osculating conic* to C_P at v. This osculating conic in turn defines a unique conformal metric $h_{P,v} : T^*\mathbb{M} \times T^*\mathbb{M} \to \mathbb{R}$. A degenerate conformal Lorentzian metric is defined by pullback on the subspace of the cotangent bundle of $\mathbb{S}$ that annihilates the vertical direction:

$$g_{P,v}(\alpha, \beta) = h_{P,v}(\pi_*\alpha, \pi_*\beta).$$

A proper degenerate conformal Lorentzian metric is obtained by dualizing.[3] In order to derive the formula for the metric (7.1), it is necessary to obtain explicit formulas for the osculating conic of a projective curve. The overall program is inspired by the work of Wilczynski [84].

[3] If V is a vector space and $W \subset V$, and B is a nondegenerate bilinear form on W, then B gives rise to a linear isomorphism $T_B : W \to W'$. The dual (degenerate) form on V' is given by the mapping $T_{\tilde{B}} : V' \to V'/W^\perp \xrightarrow{\cong} W' \xrightarrow{T_B^{-1}} W \xrightarrow{\subset} V$ where the first is the quotient map, the second is the natural isomorphism, the third is the inverse of T_B, and the last is the inclusion map.

Suppose that $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ parametrically specifies the homogeneous coordinates of a projective curve, with $\det(\phi, \phi', \phi'') \neq 0$. The projective curve is a conic provided that there exists a 3×3 symmetric non-singular matrix A such that

$$\phi^T A \phi = 0.$$

Supposing that ϕ is given, the task is to determine a matrix A such that at a given point $t = t_0$ this holds to as many orders in the expansion in powers of $t - t_0$ as possible. Since A is regarded projectively, it has 5 independent numerical components. These are obtained by solving the system of 5 equations linear in the entries of A:

$$\begin{aligned}
(\phi^T A \phi)(t_0) &= 0 \\
(\phi^T A \phi)'(t_0) &= 0 \\
(\phi^T A \phi)''(t_0) &= 0 \\
(\phi^T A \phi)'''(t_0) &= 0 \\
(\phi^T A \phi)^{(4)}(t_0) &= 0.
\end{aligned}$$

Once such a matrix is found, the obstruction to continuing to the fifth order is the derivative $(\phi^T A \phi)^{(5)}(t_0)$, and is the so-called "projective length element" associated to the curve.

By the first two equations, $A\phi(t_0)$ is proportional to the cross product $\phi(t_0) \times \phi'(t_0)$, and since A is taken projectively, we can fix a scale by taking

$$A\phi(t_0) = \frac{\phi(t_0) \times \phi'(t_0)}{\det(\phi(t_0), \phi'(t_0), \phi''(t_0))}$$

or, equivalently, $\phi(t_0)^T A \phi''(t_0) = 1$. The third and fourth equations then give, respectively

$$\begin{aligned}
\phi'^T(t_0) A \phi'(t_0) &= -1 \\
\phi'^T(t_0) A \phi''(t_0) &= -\frac{1}{3} \frac{\det(\phi, \phi', \phi''')(t_0)}{\det(\phi, \phi', \phi'')(t_0)}.
\end{aligned}$$

The final equation now gives

$$3\phi''^T(t_0) A \phi''(t_0) + 4\phi'^T(t_0) A \phi'''(t_0) = -\frac{\det(\phi, \phi', \phi^{(4)})(t_0)}{\det(\phi, \phi', \phi'')(t_0)}.$$

Now the coordinates of the curve $\phi(t)$ satisfy a third-order differential equation

$$\phi'''(t) = h_0(t)\phi(t) + h_1(t)\phi'(t) + h_2(t)\phi''(t)$$

where h_i are given explicitly in terms of determinants of ϕ and its first three derivatives as in (7.14). The above equations reduce to

$$\begin{aligned}
\phi^T(t_0)A\phi(t_0) &= 0 \\
\phi^T(t_0)A\phi'(t_0) &= 0 \\
\phi^T(t_0)A\phi''(t_0) &= 1 \\
\phi'^T(t_0)A\phi'(t_0) &= -1 \\
\phi'^T(t_0)A\phi''(t_0) &= -\frac{1}{3}h_2(t_0) \\
3\phi''^T(t_0)A\phi''(t_0) + 4\phi'^T(t_0)A\phi'''(t_0) &= -h_2^2(t_0) - h_2'(t_0) - h_1(t_0).
\end{aligned}$$

The last equation simplifies by substituting (7.14) for ϕ''' and then using the remaining equations to give

$$3\phi''^T(t_0)A\phi''(t_0) = \frac{1}{3}h_2^2(t_0) - h_2'(t_0) + 3h_1(t_0).$$

The obstruction $(\phi^T A\phi)^{(5)}(t_0)$ can now be calculated by expanding any terms involving $\phi''', \phi^{(4)}, \phi^{(5)}$ in terms of lower order and then using the above equations. We find that

$$(\phi^T A\phi)^{(5)}(t_0) = 12h_0(t_0) + 4h_1(t_0) + \frac{8}{9}h_2(t_0)^3 - 6h_1'(t_0) - 4h_2(t_0)h_2'(t_0) + 2h_2''(t_0), \quad (7.15)$$

which is *precisely* the Wünschmann invariant for the equation (7.14).

The matrix A obtained from this procedure is also of interest, because it gives the conformal Lorentzian structure. In the basis (ϕ, ϕ', ϕ''),[4] the symmetric 2-tensor A is given by

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -\frac{1}{3}h_2(t_0) \\ 1 & -\frac{1}{3}h_2(t_0) & \frac{1}{9}\left(h_2^2(t_0) - 3h_2'(t_0) + 9h_1(t_0)\right) \end{bmatrix}.$$

When, as in section 7.1.3, the curve $\phi(t) \in T_P\mathbb{M}$ is the linearization of the solution $f(x;P)$ to the differential equation at a point $P \in \mathbb{M}$, then (7.14) is the linearization of the differential equation at P:

$$h_0(x) = F_y(x, f(x;P), f_x(x;P), f_{xx}(x;P)), \quad h_1(x) = F_p(x, f(x;P), f_x(x;P), f_{xx}(x;P)),$$

[4] That this basis is "nonholonomic" (i.e., nonconstant) is significant for the inverse construction, discussed presently.

$$h_2(x) = F_q(x, f(x;P), f_x(x;P), f_{xx}(x;P))$$

The projective length element obtained from (7.15) agrees with the Wünschmann invariant (7.2) of the original equation. Substituting in for the components of the 2-tensor A gives the metric

$$g = -\left(\frac{\partial}{\partial p}\right)^2 + 2\frac{\partial}{\partial q}\frac{\partial}{\partial y} - \frac{2}{3}F_q\frac{\partial}{\partial p}\frac{\partial}{\partial q} + \frac{F_q^2 - 3V(F_q) + 9F_p}{9}\left(\frac{\partial}{\partial q}\right)^2.$$

The dual degenerate conformal metric, defined on the full tangent bundle of $\mathbb{S}$, agrees with (7.1).

7.4 INVERSE PROBLEMS

By Theorem 1, every causal geometry gives rise to a third-order differential equation. A more subtle inverse problem is, given a rank three degenerate conformal Lorentzian metric on the tangent bundle of a four manifold, when is there a causal geometry from which it arises? More precisely, given a one-parameter family of plane conics, defined by 3×3 nondegenerate symmetric 2-tensors $A(t)$, when is there a plane curve $\phi(t)$ such that, as $t\to t_0$,

$$\phi(t)^T A(t_0)\phi(t) = O(t-t_0)^5?$$

Interchanging t and t_0, a necessary and sufficient condition is that

$$\phi(t_0)^T A(t)\phi(t_0) = O(t-t_0)^5.$$

So at each t_0, the point $\phi(t_0)$ must satisfy five equations

$$\begin{aligned}\phi(t_0)^T A(t_0)\phi(t_0) &= 0\\ \phi(t_0)^T A'(t_0)\phi(t_0) &= 0\\ \phi(t_0)^T A''(t_0)\phi(t_0) &= 0\\ \phi(t_0)^T A'''(t_0)\phi(t_0) &= 0\\ \phi(t_0)^T A^{(4)}(t_0)\phi(t_0) &= 0\end{aligned}$$

In this system, the matrices $A(t_0), A'(t_0), A''(t_0), A'''(t_0), A^{(4)}(t_0)$ should be regarded as given, and the 3-vector $\phi(t_0)$ as unknown homogeneous coordinates. The system is clearly overdetermined: the two (projective) degrees of freedom in $\phi(t_0)$ must satisfy five equations. Geometrically the point $\phi(t_0)$ must simultaneously lie on five plane conics, but the intersection of more than two plane conics is generically empty.

The overdetermined system gives rise to a consistency condition on A and its first four derivatives, which we now describe. In the generic case, we can solve the linear equations for a 3×3 symmetric matrix X

$$\begin{aligned} \operatorname{tr} A(t_0)X &= 0 \\ \operatorname{tr} A'(t_0)X &= 0 \\ \operatorname{tr} A''(t_0)X &= 0 \\ \operatorname{tr} A'''(t_0)X &= 0 \\ \operatorname{tr} A^{(4)}(t_0)X &= 0. \end{aligned} \tag{7.16}$$

This can be solved uniquely for X, up to scaling, provided the system has rank five. The consistency condition is then that the solution X has rank one and so splits as an outer product

$$X = \phi(t_0)\phi(t_0)^T.$$

This happens if and only if every 2×2 minor of X vanishes.[5]

7.4.1 Intermediate cases

When (7.16) has rank one, the Wünschmann invariant vanishes. When it has full rank, then it gives rise to a causal curve provided the 2×2 minors of the solution X all vanish. A calculation done in *Mathematica* shows that, for structures coming from third-order equations, these are the only two possibilities: either the system has full rank (and thus nonzero Wünschmann) or it has rank one (and zero Wünschmann).

However, *a priori* such a system, coming from an arbitrary degenerate conformal Lorentzian structure in four dimensions, can have any rank between 1 and 5. It is interesting to understand why these intermediate cases do not lead to causal curves.

Rank 2. Suppose that (7.16) has rank two (in an interval around t_0). Then $A''(t) = f(t)A(t) + g(t)A'(t)$ for some functions f and g. The initial matrices $A(t_0), A'(t_0)$ can be brought simultaneously into diagonal form by a transformation of the form

$$A(t_0) \mapsto M^T A(t_0) M, \quad A'(t_0) \mapsto M^T A'(t_0) M.$$

[5] The minors are not independent, however. The variety in $\mathbb{P}(\mathrm{Sym}^2(\mathbb{R}^3))$ on which the 2×2 minors of a symmetric 3×3 matrix vanish is the well-known Veronese surface, which is not a complete intersection. Locally it is the zero locus of any three minors coming from distinct rows and columns.

Relative to this fixed initial basis, $A(t)$ and $A'(t)$ remain diagonal throughout the interval of existence. It is convenient to put

$$\vec{A}(t) = \begin{bmatrix} A_{11}(t) \\ A_{22}(t) \\ A_{33}(t) \end{bmatrix}, \quad \vec{X} = \begin{bmatrix} X_{11} \\ X_{22} \\ X_{33} \end{bmatrix}.$$

The two equations of (7.16) reduce to

$$\operatorname{tr} A(t)X = \vec{A}(t) \cdot \vec{X} = 0$$

$$\operatorname{tr} A'(t)X = A'_{11}(t)X_{11} + A'_{22}(t)X_{22} + A'_{33}(t)X_{33} = 0$$

Solving:

$$\vec{X} = \vec{A}(t) \times \vec{A}'(t)$$

up to an overall scale. Now if $\phi(t)$ is a curve solving

$$\phi(t)^T A(t)\phi(t) = 0 \quad \phi(t)^T A'(t)\phi(t) = 0$$

then the entries of ϕ must square to the entries of $\vec{X}$, so that

$$\phi(t) = \begin{bmatrix} \pm\sqrt{\vec{A} \times \vec{A}' \cdot e_1} \\ \pm\sqrt{\vec{A} \times \vec{A}' \cdot e_2} \\ \pm\sqrt{\vec{A} \times \vec{A}' \cdot e_3} \end{bmatrix}.$$

Differentiating gives

$$\phi_i'(t) = \pm\frac{g(t)}{2}\sqrt{\vec{A} \times \vec{A}' \cdot e_i}$$

and so ϕ' is proportional to ϕ. Thus the range of $\phi(t)$ is a projective point (in the complex sense).[6] Therefore there are one or no solutions, depending on whether the square roots are all real.

Rank 3 and 4. We argue indirectly that, if (7.16) has rank 3 or 4 throughout an interval and $\phi(t)$ is a C^3 solution in that interval, then $\phi(t)$ parameterizes either a projective point (as in the rank 2 case) or a line (which is degenerate from our point of view). We were unable to devise a direct argument analogous to the rank 2 case. In general, because $\phi(t)$ is a three-vector, there must be a non-trivial linear relation between $\phi(t)$ and its first three

[6] Indeed, if $\phi(0)$ and $\phi'(0)$ are linearly independent, then $\phi(0) \times \phi'(0) \cdot \phi(t)$ satisfies a second order ode with both initial conditions zero, so $\phi(0) \times \phi'(0) \cdot \phi(t) \equiv 0$ which implies that $\phi(t)$ is constrained to a projective line. If $\phi(0)$ and $\phi'(0)$ are linearly dependent, then smoothness of dependence on initial conditions gives the result.

derivatives. This must either give a proper third-order equation in an interval, or else there is a linear relation between ϕ, ϕ', ϕ''. In the latter case, $\phi(t)$ does indeed parameterize a line (if it satisfies a second order equation) or a point (if the equation is first order). In the former case, the coefficients of $A(t)$ can be given in terms of $\phi(t), \phi'(t), \phi''(t)$ and the coefficients of the third-order equation $h_1(t), h_2(t), h_3(t)$, as in the previous section. But, as already indicated, this implies that the system (7.16) has rank either 1 or 5, a contradiction.

8.0 STRUCTURE OF CAUSAL GEOMETRIES

This chapter discusses the structure of causal geometries. The main theorems are the theorem that there is an analog of the Weyl tensor depending only on the causal structure, and that the Raychaudhuri–Sachs theorem holds for causal geometries. The main ingredients are therefore in place to derive the singularity theorem along the lines of Penrose [65].

This chapter is organized as follows. It begins with a discussion of the null geodesic dynamics, in which the causal geometry is specified by a subbundle of the sphere bundle over a manifold M. For any Lagrangian representing the causal geometry, it is possible to associate an Ehresmann connection on the null cone bundle. This connection depends on the choice of Lagrangian, but its dependence can be described fairly explicitly. The associated curvature also depends on the choice of Lagrangian, but a suitable tracefree part of it does not. This trace-free part agrees exactly with the Weyl tensor when the causal geometry comes from a quadratic Lagrangian.

8.1 NULL GEODESIC DYNAMICS

8.1.1 Legendrian dynamics

Let M be a smooth manifold of dimension $n \geq 3$. We here recall the notation of §2.11: TM' denotes the tangent bundle with the zero section deleted; $SM = \mathbb{S}TM$ is the sphere bundle; $\sigma : TM' \to SM$ is the projection map. The dynamics is specified by a smooth hypersurface $\mathscr{G}$ (of dimension $2n-2$) in the unit sphere bundle SM, with $\pi(\mathscr{G}) = M$, which has the property that for any $x \in M$, the intersection $\mathscr{G} \cap SM_x$ is a smooth submanifold of the sphere SM_x of dimension $n-2$. Note that the space $\mathscr{H} = \sigma^{-1}\mathscr{G}$, a smooth hypersurface in TM' invariant under the scaling $(x, v) \to (x, tv)$ for $t > 0$, also specifies the dynamics.

If $(x, v) \in \mathscr{H}$, the vertical tangent space $V\mathscr{H}_{(x,v)}$ to $\mathscr{H}$ at (x, v) is the intersection of the tangent space to $\mathscr{H}$ at (x, v) with the vertical space $VTM_{(x,v)}$. So $V\mathscr{H}_{(x,v)}$ has dimension

$n-1$. The image of $V\mathscr{H}_{(x,v)}$ under the map $\overline{\lambda}^{-1}_{(x,v)} : VTM_{(x,v)} \to TM_x$ is then a subspace of TM_x, also of dimension $n-1$. There is then a unique maximal subspace of $T\mathscr{H}_{(x,v)}$ that projects down under the map $d\pi_{TM'}$ to $\overline{\lambda}^{-1}_{(x,v)}(V\mathscr{H}_{(x,v)})$. This subspace is a codimension one distribution within $T\mathscr{H}$, and therefore defines a distribution of hyperplanes on $\mathscr{H}$:

$$\Lambda_{\mathscr{H}} = T\mathscr{H} \cap d\pi^{-1}_{TM'}\left(\overline{\lambda}^{-1}(V\mathscr{H})\right).$$

This entire construction is invariant under the scalar homothety δ, and so $\Lambda_{\mathscr{H}}$ descends to a distribution of $\Lambda_{\mathscr{G}}$ on $\mathscr{G}$ as well.

Definition 21. *A* contact symmetry *of a distribution Λ on a manifold X is a one-parameter local group of diffeomorphisms of X that preserves Λ and whose generators are everywhere tangent to Λ.*

Definition 22. *The* dynamics *of $\mathscr{G}$ is the space of contact symmetries of $\Lambda_{\mathscr{G}}$.*

8.1.2 Lagrangian approach to the dynamics

Let $\mathscr{H}$ have local defining equation $G(x,v) = 0$ where G is a smooth function defined over an open set U of TM' that is invariant under δ satisfying:

- $DG \neq 0$ throughout U
- G homogeneous of some real degree k: $G(x,tv) = t^k G(x,v)$ for all $t > 0$ and all $(x,v) \in U$. For convenience, we shall henceforth assume that $k \neq 1$.

There is a bilinear form g_h on $\pi^{-1}_{TM'}TM$ defined for vector fields X and Y that lift vector fields on M by

$$g_h(X,Y) = D_X D_Y G = D_y D_X G = D^2_{X,Y} G.$$

The definition is independent of the choice of lift of X and Y, and it is bihomogeneous under rescalings $X \to (\pi^*_{TM'}a)X$ and $Y \to (\pi^*_{TM'}b)Y$ where a, b are functions on M, and so it gives rise to a bilinear form. The h here stands for "horizontal", a reflection of the fact that g_h is a section of $V^0TM \otimes V^0TM$. Applying $\overline{\lambda}^{-1}$ yields a bilinear form g_v in $V^*TM \otimes V^*TM$:

$$g_v(X,Y) = g_h(\overline{\lambda}^{-1}(X), \overline{\lambda}^{-1}(Y)).$$

Here the subscript v means "vertical", since g_v is a bilinear form on VTM. In coordinates,

$$g_v = \frac{\partial^2 G}{\partial v^i \partial v^j} dv^i \otimes dv^j = g_{ij} dv^i \otimes dv^j$$
$$g_h = \frac{\partial^2 G}{\partial v^i \partial v^j} dx^i \otimes dx^j = g_{ij} dx^i \otimes dx^j.$$

Lemma 30. *Let $\alpha \in \Gamma_{TM'}(T^*TM')$ be the differential form $\alpha = DG$. Then, on restricting to $\mathscr{H}$, the distribution $\Lambda_{\mathscr{H}}$ is the annihilator of α in $T\mathscr{H}$.*

Proof. By the assumption that $DG \neq 0$, the image of $T\mathscr{H}$ under α is always one-dimensional, and so the annihilator of α is a distribution of hyperplanes in $T\mathscr{H}$. Suppose $\alpha(X) = 0$ for $X \in T\mathscr{H}$. Then, by definition of the D operator, $\lambda X \lrcorner dG = 0$. So $\lambda X \in V\mathscr{H}$. That is, $X \in \lambda^{-1}(V\mathscr{H})$ as required. □

Lemma 31. $g_h(X, Y) = 2d\alpha(\lambda X, Y)$

Proof. Both sides vanish if either X or Y is vertical, so it is sufficient to establish the lemma under the additional assumption that X and Y are lifts of vector fields from M. Since $\alpha(\lambda X) = 0$,

$$\begin{aligned}2d\alpha(\lambda X, Y) &= (\mathscr{L}_{\lambda X}\alpha)(Y) = \lambda X(\alpha(Y)) - \alpha([\lambda X, Y]) \\ &= D^2_{X,Y}G - \alpha([\lambda X, Y]) = g_h(X, Y) - \alpha([\lambda X, Y])\end{aligned}$$

But if X and Y are lifts of vector fields, then $[\lambda X, Y]$ is vertical, and so $\alpha([\lambda X, Y]) = 0$. □

Assume henceforth that the bilinear form g_v is nondegenerate. This assumption is justified in part by

Lemma 32. *The bilinear form g_v is nondegenerate if and only if $d\alpha$ is a symplectic form on a neighborhood of $\mathscr{H}$ in TM'.*

Proof. The subspace VTM' is an isotropic space for $d\alpha$. Choose a complementary space HTM' in TTM'. Then $d\alpha$ induces a bilinear form on $VTM' \times HTM'$ and $d\alpha(X, Y) = 2g_v(X, \lambda Y)$. □

In coordinates,

$$\alpha = \frac{\partial G}{\partial v^i} dx^i.$$

Set $p_i = \partial G/\partial v^i$. By the nondegeneracy of g, the Jacobian matrix $\partial p_i/\partial v^j$ is nonsingular, and so this defines a new set of (local) coordinates on TM'. In the new coordinates,

$$\alpha = p_i dx^i.$$

These are the "canonical coordinates" for the dynamical system.

The symplectic form $d\alpha$ allows us to define the Poisson bracket of two functions f_1, f_2 (in a neighborhood of $\mathscr{H}$) by

$$\{f_1, f_2\} = (d\alpha)^{-1}(df_1, df_2).$$

This satisfies the usual rules:

- $\{f_1, f_2\} = -\{f_2, f_1\}$
- $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$
- $\{f_1, c\} = 0$ if c is constant
- $\{f_1, f_2 + f_3\} = \{f_1, f_2\} + \{f_1, f_3\}$
- $\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + \{f_1, f_3\} f_2$

The last three properties imply that the operator $\{f_1, -\} : f_2 \mapsto \{f_1, f_2\}$ is a derivation on smooth functions, and therefore corresponds to a vector field on M.

In the canonical coordinates,

$$\{f, -\} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial p_i}.$$

As we are interested in the intrinsic geometry of $\mathscr{H}$, we shall consider the pullback of α to $\mathscr{H}$.

Lemma 33. *When pulled back to $\mathscr{H}$, α has Darboux rank $2n-3$:*

$$\alpha \wedge (d\alpha)^{n-2} \neq 0, \quad (d\alpha)^{n-1} = 0.$$

Proof. The fibers of $\mathscr{H} \to M$ are $n-1$ dimensional, and the bilinear form g_v on $V\mathscr{H}$ is annihilated by the generators of scaling up the fiber. So on $V\mathscr{H}$, g_v has rank $n-2$. By the argument in the previous lemma, $(d\alpha)^{n-1} = 0$. However, applying the previous argument to $\Lambda_{\mathscr{H}} = \alpha^0$, and choosing a complement for this in $T\mathscr{G}$ gives $\alpha \wedge (d\alpha)^{n-2} \neq 0$. □

The first main result uses the Darboux theorem:

Lemma 34. *Let M be a manifold of dimension $2n-1$ and α a one-form of Darboux rank $2r-1$. Then the space of vector fields X such that*

$$X \lrcorner\, \alpha = 0, \quad \alpha \wedge \mathscr{L}_X \alpha = 0 \tag{8.1}$$

forms an integrable distribution of rank $2(n-r)$.

Proof. Using the usual version of Darboux' theorem, there exists a coordinate system $x, y^1, \ldots, y^{n-1}, p_1, \ldots, p_{n-1}$ on M such that

$$\alpha = dx + \sum_{i=1}^{r-1} p_i \, dy^i,$$

with the last $(n-r)$ p's and y's not participating. Hence the vector fields $\partial/\partial p_i$ and $\partial/\partial y^i$ for $i = r, \ldots, n-1$ form an integrable distribution of rank $2(n-r)$ satisfying (8.1).

Now, note that any X satisfying (8.1) must also satisfy

$$X \lrcorner (\alpha \wedge (d\alpha)^{r-1}) = 0.$$

But $\alpha \wedge (d\alpha)^{r-1} = \pm dx \wedge dp_1 \wedge \cdots \wedge dp_{r-1} \wedge dy^1 \wedge \cdots \wedge dy^{r-1}$ is annihilated by X if and only if X is a linear combination of $\partial/\partial p_i$ and $\partial/\partial y^i$ for $i = r, \ldots, n-1$. $\square$

Theorem 21. *The dynamical vector fields on $\mathscr{H}$ are spanned as a C^∞ module by H (the generator of the scaling symmetry of $\mathscr{H}$) and the vector field $V = (k-1)\{G, -\}$ restricted to $\mathscr{H}$.*

The particular normalization of V ensures that it defines a spray; see Lemma 35 below.

Proof. By Lemma 33, there are exactly two linearly independent dynamical vector fields at every point. Note that V is tangent to $\mathscr{H}$ since $V(G) = (k-1)\{G, G\} = 0$, and H is tangent to $\mathscr{H}$ since $\mathscr{H}$ is invariant under the scaling action. These are linearly independent, since $V(\pi^*_{\mathscr{H}} f) = (k-1)\{G, \pi^*_{\mathscr{H}} f\}$ is nonzero for some smooth function f on M, but $H(\pi^*_{\mathscr{H}} f) = 0$ for all such f.

Now, note that H satisfies $H \lrcorner \alpha = 0$ (since $\lambda H = 0$). If X is a lift of a vector field on M, then $\delta_s \lambda X = s^{-1} \lambda X$. Differentiating gives $\mathscr{L}_H(\lambda X) = -\lambda X$. For such a vector field X, $[H, X]$ is vertical and so $\lambda[H, X] = 0$. It follows that $\alpha([H, X]) = 0$, and therefore

$$\begin{aligned}
(\mathscr{L}_H \alpha)(X) &= \mathscr{L}_H(\lambda X \lrcorner dG) - \alpha([H, X]) \\
&= -\lambda X \lrcorner dG + \lambda X \lrcorner \mathscr{L}_H dG \\
&= (k-1)\lambda X \lrcorner dG = (k-1)\alpha(X).
\end{aligned}$$

Finally, in a neighborhood of $\mathscr{H}$, V is characterized by

$$V \lrcorner d\alpha = -(k-1) dG.$$

Pulling back to $G = 0$ gives $V \lrcorner d\alpha = 0$. By the previous calculation, $H \lrcorner d\alpha = (k-1)\alpha$. Hence

$$
\begin{aligned}
V \lrcorner \alpha = (k-1)^{-1} V \lrcorner H \lrcorner d\alpha &= -(k-1)^{-1} H \lrcorner V \lrcorner d\alpha \\
= H \lrcorner dG = kG &
\end{aligned}
$$

which also vanishes on $\mathscr{H}$. $\square$

In coordinates, the dynamical vector fields are

$$
\begin{aligned}
H &= v^i \frac{\partial}{\partial v^i} \\
V &= v^i \frac{\partial}{\partial x^i} + u^i \frac{\partial}{\partial v^i}, \quad u^i g_{ij} = \frac{\partial G}{\partial x^j} - v^i \frac{\partial^2 G}{\partial x^i \partial v^j}.
\end{aligned}
$$

The integral curve of the vector field V through a point (x, v) projects to a curve in M whose initial velocity is v. That is, V is a semispray.

Lemma 35. *V is a spray:*

- $[H, V] = V$
- $H = \lambda V$

Proof. The calculations in the proof of the preceding lemma give

$$
\begin{aligned}
[H, V] \lrcorner d\alpha &= \mathscr{L}_H (V \lrcorner d\alpha) - V \lrcorner (\mathscr{L}_H d\alpha) \\
&= k(V \lrcorner d\alpha) - (k-1) V \lrcorner d\alpha = V \lrcorner d\alpha,
\end{aligned}
$$

so $[H, V] = V$.

For the second property, the definition of g_h implies

$$
d\alpha(X, \lambda Y) = \frac{1}{2} g_h(X, Y) = d\alpha(Y, \lambda X) = -d\alpha(\lambda X, Y).
$$

In particular, with $X = V$,

$$
d\alpha(\lambda V, Y) = -d\alpha(V, \lambda Y) = (k-1) dG(\lambda Y) = (k-1)\alpha(Y) = d\alpha(H, Y).
$$

This is true for all Y and so $H = \lambda V$ by nondegeneracy of $d\alpha$. $\square$

Passing down to the sphere bundle SM, only the dynamical vector field V survives, up to an overall positive scale, since H is in the kernel of $d\sigma : TTM \to TSM$. This gives a foliation of $\mathscr{G}$ by the dynamical curves, the (maximally extended) trajectories of V. These dynamical curves are called *null geodesics*. The space of null geodesics, denoted by $\mathscr{N}$, has dimension $2n-3$. The distribution $\Lambda_{\mathscr{G}}$ is Lie derived along the dynamical vector fields, and so descends to a codimension one distribution on $\mathscr{N}$. This distribution is a contact structure since the relation $\alpha_{\mathscr{G}} \wedge (d\alpha_{\mathscr{G}})^{n-2} \neq 0$, valid for any nonzero $\alpha_{\mathscr{G}}$ in the annihilator of $\Lambda_{\mathscr{G}}$, also descends to the quotient.

The null geodesics are naturally oriented, since at each point p of $\mathscr{G}$, the vector field V descends to a ray through the origin in $T_p\mathscr{G}$, which is oriented. The bundle $\mathscr{G}$ is *time oriented* if and only if the space of oriented null geodesics is the disjoint union of two components, $\mathscr{N} = \mathscr{N}^+ \cup \mathscr{N}^-$, such that the oriented null geodesic through (x, v) lies in $\mathscr{N}^\pm$ if and only if the oriented null geodesic through $(x, -v)$ lies in $\mathscr{N}^\mp$. Then the elements of $\mathscr{N}^+$ are called future oriented and the elements of $\mathscr{N}^-$ are called past oriented.

On $\mathscr{H}$, the integral curves of V are called *affinely parametrized* null geodesics. These carry a natural parametrization up to a translation, since they are the integral curves of a single vector field. This natural parametrization requires having a particular defining function G for $\mathscr{H}$, although the definition of (unparametrized, oriented) null geodesics on $\mathscr{G}$ does not.

8.2 EHRESMANN CONNECTION

8.2.1 The Ehresmann connection on $\mathscr{H}$

The purpose of this section is to establish the following:

Theorem 22. *There exists a unique operator $P : TTM' \to VTM'$ satisfying for all $X, Y \in TTM$:*

1. $(\mathscr{L}_V g_h)(X, Y) = g_v(PX, \lambda Y) + g_v(PY, \lambda X)$
2. $2d\alpha(X, Y) = g_v(PX, \lambda Y) - g_v(PY, \lambda X)$

This operator defines an Ehresmann connection on TM', meaning that it has maximal rank and satisfies $P = P \circ P$. Furthermore,

$$P = \frac{1}{2}\left(\mathrm{Id}_{TTM'} + \mathscr{L}_V \lambda\right). \tag{8.2}$$

The restriction of P to $\mathscr{H}$ is also an Ehresmann connection on $\mathscr{H}$: $P(T\mathscr{H}) = V\mathscr{H}$.

(Recall that λ is a section of $V^0TM' \otimes VTM'$, g_h is a section of $V^0TM' \otimes V^0TM'$, and g_v is a section of $V^*TM' \otimes V^*TM'$.)

The proof is broken down into several lemmas.

Lemma 36. *The Frölicher–Nijenhuis bracket of λ with itself is zero: $[\lambda, \lambda] = 0$. Thus if $X, Y \in \Gamma_{TM'}(TTM')$, then*

$$\lambda([\lambda X, Y] + [X, \lambda Y]) = [\lambda X, \lambda Y].$$

Moreover $[V, \lambda Y] = -Y \pmod{VTM'}$ for all vector fields Y.

Proof. If $X, Y \in \Gamma_{TM'}(TTM')$, then

$$\frac{1}{2}[\lambda, \lambda](X, Y) = [\lambda X, \lambda Y] - \lambda([\lambda X, Y] + [X, \lambda Y])$$

The right-hand side vanishes if X or Y is a section of VTM', since VTM' is an integrable distribution on which λ vanishes. Thus it suffices to prove that it vanishes if X and Y are both lifts of vector fields from M. In that case, if f is a function on M, then

$$[\lambda X, Y]\pi^*_{TM'} f = (\lambda X) Y \pi^*_{TM'} f = (\lambda X)\pi^*_{TM'}((d\pi_{TM'} Y) f) = 0.$$

Hence $[\lambda X, Y] \in \Gamma_{TM'}(VTM')$; likewise $[X, \lambda Y] \in \Gamma_{TM'}(VTM')$. So $\lambda([\lambda X, Y] + [X, \lambda Y]) = 0$. Finally, $[\lambda X, \lambda Y] = 0$ as well for X, Y lifts of vector fields on M, since the one-parameter groups L_X and L_Y (defined in §2.11) commute in that case.

It remains only to show that $[V, \lambda Y] = -Y \pmod{VTM'}$ for all vector fields Y. Taking $X = V$ in the first part gives

$$\lambda([\lambda V, Y] + [V, \lambda Y]) = [\lambda V, \lambda Y].$$

But $\lambda V = H$, so rearranging gives

$$\lambda[V, \lambda Y] = (\mathscr{L}_H \lambda)(Y) = -Y$$

as claimed. ∎

Lemma 37. *Any operator $P : TTM' \to VTM'$ satisfying property (1) has maximal rank and satisfies $P \circ P = P$.*

Proof. In view of the fact that $\operatorname{im} P \subset VTM'$ by assumption, it is enough to show that $P(\lambda(X)) = \lambda(X)$ for all X. This then proves that the range of P is equal to the vertical tangent space, and that P acts as the identity on its range. Therefore $P \circ P = P$, and P has maximal rank.

To prove the claimed identity, (1) gives

$$\begin{aligned}(\mathscr{L}_V g_h)(\lambda X, Y) &= g_v(P\lambda X, \lambda Y) + g_v(PY, \lambda\lambda X)\\ &= g_v(P\lambda X, \lambda Y)\end{aligned}$$

since $\lambda\lambda X = 0$. Expanding the left-hand side,

$$\begin{aligned}(\mathscr{L}_V g_h)(\lambda X, Y) &= V(g_h(\lambda X, Y)) - g_h([V, \lambda X], Y) - g_h(\lambda X, [V, Y])\\ &= -g_h([V, \lambda X], Y)\end{aligned}$$

since every vertical direction lies in the kernel of g_h. But, by Lemma 35, $[V, \lambda X] = -X$ (mod VTM), and therefore

$$-g_h([V, \lambda X], Y) = g_h(X, Y).$$

Putting these together,

$$g_h(X, Y) = g_v(P\lambda X, \lambda Y). \tag{8.3}$$

This is true for all X, Y, and so $P\lambda X = \lambda X$, as claimed. □

Lemma 38. *There exists a unique $P \in \Gamma_{TM'}(T^*TM' \otimes VTM')$ satisfying conditions (1) and (2).*

Proof. If (1) and (2) hold, then

$$g_v(PX, \lambda Y) = \frac{1}{2}\left((\mathscr{L}_V g_h)(X, Y) + d\alpha(X, Y)\right).$$

By the non-degeneracy of g_v and the fact that $\lambda : TTM' \to VTM'$ has maximal rank, this admits at most a unique solution $P(X)$ valid for all Y. To prove existence, it is enough to show that the kernel of

$$Y \mapsto (\mathscr{L}_V g_h)(X, Y) + d\alpha(X, Y)$$

contains the kernel of λ, which is also the image of λ, VTM'. So consider

$$\begin{aligned}(\mathscr{L}_V g_h)(X, \lambda Y) + d\alpha(X, \lambda Y) &= -g_h(X, [V, \lambda Y]) + d\alpha(X, \lambda Y)\\ &= g_h(X, Y) - g_h(X, Y) = 0\end{aligned}$$

where we have used the fact that V is a spray in simplifying the first term, and the definition of g_h in simplifying the second term. □

Lemma 39. *The unique connection P satisfying (1) and (2) is given explicitly by*

$$P = \frac{1}{2}\left(\mathrm{Id}_{TTM'} + \mathscr{L}_V\lambda\right).$$

Proof. Let P be the connection characterized by (1) and (2) and let $P_1 = \frac{1}{2}\left(\mathrm{Id}_{TTM'} + \mathscr{L}_V\lambda\right)$. We will show that P_1 acts as the identity on VTM', and that $\ker P_1 = \ker P$. The first claim follows at once from

$$(\mathscr{L}_V\lambda)(\lambda X) = [V, \lambda^2 X] - \lambda[V, \lambda X] = \lambda X.$$

For the second claim, $X \in \ker P_1$ if and only if $(\mathscr{L}_V\lambda)(X) = -X$, or, equivalently,

$$-X = [V, \lambda X] - \lambda[V, X].$$

Now $X \in \ker P$ if and only if

$$(\mathscr{L}_V g_h)(X, Y) = -2d\alpha(X, Y)$$

for all $Y \in TTM'$. Note

$$\begin{aligned}(\mathscr{L}_V g_h)(X, Y) &= V(g_h(X, Y)) - g_h([V, X], Y) - g_h(X, [V, Y]) \\ &= V(d\alpha(\lambda X, Y)) - 2d\alpha(\lambda[V, X], Y) - 2d\alpha(\lambda X, [V, Y]) \\ &= 2d\alpha([V, \lambda X], Y) - 2d\alpha(\lambda[V, X], Y) \\ &= 2d\alpha((\mathscr{L}_V\lambda)(X), Y).\end{aligned}$$

If $X \in \ker P_1$, then this last display reduces to $-2d\alpha(X, Y)$, and so $X \in \ker P$. Conversely, if $X \in \ker P$, then the same calculation shows that $2d\alpha((\mathscr{L}_V\lambda)(X), Y) = -2d\alpha(X, Y)$ for all Y, and hence $(\mathscr{L}_V\lambda)(X) = -X$ by the nondegeneracy of $d\alpha$, and so $X \in \ker P_1$. $\square$

In coordinates,

$$P = \left(dv^i + U^i_j dx^j\right) \otimes \frac{\partial}{\partial v^i} \qquad U^i_j = -\frac{1}{2}\frac{\partial u^i}{\partial v^j}, \qquad u^i g_{ij} = \frac{\partial G}{\partial x^j} - v^i \frac{\partial^2 G}{\partial x^i \partial v^j}.$$

The horizontal lift of the coordinate vector fields $\partial/\partial x^i$ are

$$h(\partial/\partial x^i) = (I - P)(\partial/\partial x^i) = \frac{\partial}{\partial x^i} - U^j_i \frac{\partial}{\partial v^j}.$$

Lemma 40. *The operator $P : TTM' \to VTM'$ satisfying (1) and (2) is such that on $\mathscr{H}$, $P(T\mathscr{H}) = V\mathscr{H}$.*

Proof. Let X be a vector field in $T\mathscr{H}$ that Lie commutes with V. Since $\lambda V = H$,

$$g_v(P(X), H) = g_v(P(X), \lambda V) = \frac{1}{2}\left(d\alpha(X, V) + (\mathscr{L}_V g_h)(X, V)\right).$$

It is sufficient prove that both terms of the right-hand side are zero. By the calculations in the proof of Theorem 21

$$d\alpha(X, V) = (k-1)dG(X) = 0$$

since X is tangent to $\mathscr{H}$. Also, since X and V commute by hypothesis,

$$(\mathscr{L}_V g_h)(X, V) = V\left(g_h(X, V)\right).$$

Now $g_h(X, V) = -2d\alpha(X, \lambda V) = -2d\alpha(X, H) = (k-1)\alpha(X)$, and thus

$$V\left(g_h(X, V)\right) = (k-1)V(\alpha(X)) = (k-1)(\mathscr{L}_V \alpha)(X) = (k-1)dG(X) = 0.$$

□

8.3 CURVATURE

8.3.1 Tidal force

Theorem 23. *Let P be the connection of Lemmas 37 and 38. Then there exists $S \in \Gamma_{TM'}(V^0TM' \otimes V^0TM')$ such that $\frac{1}{2}(\mathscr{L}_V^2 g_h)(X, Y) = g_v(P(X), P(Y)) + S(X, Y)$ for all $X, Y \in TTM'$. Conversely, P is the unique operator such that*

1. *$(\mathscr{L}_V g_h)(X, Y) = g_v(P(X), Y) + g_v(X, P(Y))$ for all $X, Y \in TTM'$.*
2. *There exists $S \in V^0TM' \otimes V^0TM'$ such that $\frac{1}{2}(\mathscr{L}_V^2 g_h)(X, Y) = g_v(P(X), P(Y)) + S(X, Y)$ for all $X, Y \in TTM'$.*

   The symmetric tensor S is called the *tidal force tensor*.

Proof. For the first claim, it is enough to show:

1. $(\mathscr{L}_V^2 g_h)(\lambda X, \lambda Y) = 2g_v(\lambda X, \lambda Y)$
2. $(\mathscr{L}_V^2 g_h)(\lambda X, Y) = 0$ for all X and Y in the kernel of P.

Indeed, assuming these are both true, decomposing two vectors $X = X_h + X_v$ and $Y = Y_h + Y_v$ into $\ker P$ and $\operatorname{im} P$ components,

$$(\mathscr{L}_V^2 g_h)(X_h + X_v, Y_h + Y_v) - g_v(X_v, Y_v) = (\mathscr{L}_V^2 g_h)(X_h, Y_h)$$

which defines $S(X, Y)$.

For (1), since $(\mathscr{L}_V g_h)(\lambda X, \lambda Y) = 0$ and $g_h(\lambda X, Z) = g_h(Z, \lambda Y) = 0$ for all Z,

$$\begin{aligned}(\mathscr{L}_V^2 g_h)(\lambda X, \lambda Y) &= -(\mathscr{L}_V g_h)([V, \lambda X], \lambda Y) - (\mathscr{L}_V g_h)(\lambda X, [V, \lambda Y]) \\ &= 2g_h([V, \lambda X], [V, \lambda Y]) = 2g_h(X, Y) = 2g_v(\lambda X, \lambda Y).\end{aligned}$$

For (2), Y is in the kernel of P if and only if

$$(\mathscr{L}_V g_h)(Y, Z) + 2d\alpha(Y, Z) = 0$$

for all Z. Hence

$$\begin{aligned}(\mathscr{L}_V^2 g_h)(\lambda X, Y) &= V((\mathscr{L}_V g_h)(\lambda X, Y)) - (\mathscr{L}_V g_h)([V, \lambda X], Y) - (\mathscr{L}_V g_h)(\lambda X, [V, Y]) \\ &= V(g_v(P(\lambda X), \lambda Y)) + d\alpha(Y, [V, \lambda X]) - g_v(P(\lambda X), \lambda[V, Y]) \\ &= V(g_h(X, Y)) + d\alpha(Y, [V, \lambda X]) - g_h(X, [V, Y]) \\ &= (\mathscr{L}_V g_h)(X, Y) + g_h([V, X], Y) + d\alpha(Y, [V, \lambda X]) \\ &= g_h([V, X], Y) + 2d\alpha(Y, [V, \lambda X])\end{aligned}$$

since $d\alpha(X, Y) = 0$ for $X, Y \in \ker P$. Now,

$$\begin{aligned}[V, \lambda X] &= (\mathscr{L}_V \lambda)(X) + \lambda[V, X] = (2P - \mathrm{Id})X + \lambda[V, X] \\ &= -X + \lambda[V, X].\end{aligned}$$

So, continuing the above calculation gives

$$(\mathscr{L}_V^2 g_h)(\lambda X, Y) = g_h([V, X], Y) + d\alpha(Y, \lambda[V, X]) = 0$$

where we have used again the fact that $d\alpha(X, Y) = 0$ along with Lemma 31.

For the converse statement, let $\mathscr{P}$ be the affine space consisting of all operators $P : TTM' \to VTM'$ satisfying

$$(\mathscr{L}_V g_h)(X, Y) = g_v(P(X), \lambda(Y)) + g_V(\lambda(X), P(Y)).$$

By Lemma 37, any such P satisfies $P \circ P = P$, and so defines a projection onto VTM'. Any such operator is completely determined by its kernel. But for the P satisfying Lemma 38, it follows from the first part of the lemma that

$$\ker P = \bigcap_{X \in VTM'} \ker[(\mathscr{L}_V^2 g_h)(X, -)].$$

□

8.3.2 Curvature of the connection

The curvature of the Ehresmann connection P is the section of $V^0TM' \otimes V^0TM' \otimes VTM'$ defined by

$$R(X, Y) = P[(\operatorname{Id} - P)(X), (\operatorname{Id} - P)(Y)].$$

Since the vertical bundle is integrable, this evaluates to

$$R(X, Y) = [PX, PY] - P([PX, Y] + [X, PY]) + P[X, Y].$$

Equivalently, this can be re-expressed in terms of the Fröhlicher–Nijenhuis bracket [46], by

$$R = \frac{1}{2}[P, P].$$

Lemma 41. *The Bianchi identity holds:* $[P, R] = 0$. *That is,*

$$[PX, R(Y, Z)] + P[R(X, Y), Z] + R([X, Y], Z) - R([PX, Y], Z) + R([PY, X], Z) + \text{cyclic} = 0.$$

Define $S^\lambda \in T^*TM' \otimes VTM'$ to be the unique tensor such that

$$g_v(S^\lambda X, \lambda Y) = S(X, Y)$$

for all X, Y, where S is the tidal force tensor of Theorem 23. The curvature determines the tidal force, and vice versa:

Theorem 24. *The curvature and tidal force are related by*

$$S^\lambda X = -R(V, X)$$

for all $X, Y \in TTM'$. *Moreover,*

$$[\lambda, S^\lambda] = -\frac{3}{2} R.$$

The following lemma is of interest in its own right:

Lemma 42. *The tidal force tensor satisifies*

$$S(X,Y) = \frac{1}{2} g_v(P(\mathscr{L}_V^2 \lambda)X, \lambda Y).$$

So $S^\lambda = \frac{1}{2}P(\mathscr{L}_V^2\lambda)$. *Moreover, if* $X, Y \in \ker P$, *then*

$$S(X,Y) = -g_v(P[V,X], \lambda Y). \tag{8.4}$$

Proof. Note first the operator identities

$$P(\mathscr{L}_V \lambda) = P, \quad (\mathscr{L}_V\lambda)\lambda = \lambda. \tag{8.5}$$

Thus

$$P(\mathscr{L}_V^2\lambda)\lambda X = P[V, (\mathscr{L}_V\lambda)\lambda X] - P(\mathscr{L}_V\lambda)[V, \lambda X] = 0.$$

So it is sufficient to establish the first statement of the lemma under the additional hypothesis that $X, Y \in \ker P$. In that case

$$X = -(\mathscr{L}_V\lambda)X$$

so

$$\begin{aligned} P(\mathscr{L}_V^2\lambda)X &= P[V, (\mathscr{L}_V\lambda)X] - P(\mathscr{L}_V\lambda)[V, X] \\ &= -P[V,X] - P[V,X] = -2P[V,X]. \end{aligned}$$

So to prove the first part of the lemma, it is enough to show (8.4).

Since $X, Y \in \ker P$, it follows by Theorem 23 that $2S(X,Y) = (\mathscr{L}_V^2 g_h)(X,Y)$. Now,

$$\begin{aligned} 2S(X,Y) = (\mathscr{L}_V^2 g_h)(X,Y) &= V\left(\mathscr{L}_V g_h(X,Y)\right) - \mathscr{L}_V g_h([V,X],Y) - \mathscr{L}_V g_h(X,[V,Y]) \\ &= -2d\alpha([V,X],Y) + 2d\alpha(X,[V,Y]) \\ &= 2V(d\alpha(X,Y)) - 4d\alpha([V,X],Y) \\ &= -4d\alpha([V,X],Y) \end{aligned}$$

since $d\alpha(X,Y) = 0$ for $X, Y \in \ker P$ by Theorem 22. Thus, applying Theorem 22 twice more,

$$2S(X,Y) = -4d\alpha([V,X],Y) = -2(\mathscr{L}_V g_h)([V,X],Y) = -2g_v(P[V,X], \lambda Y).$$

$\square$

Proof of Theorem 24. For the first identity, it is enough to show that $(\mathscr{L}_V^2 g_h)(X,Y) = -2g_v(R(V,X),\lambda Y)$ for all $X, Y \in \ker P$. Since $V \in \ker P$ as well, $R(V,X) = P[V,X]$, and so the last part of Lemma 42 gives

$$S(X,Y) = -g_v(R(V,X),\lambda Y),$$

as claimed.

Now, by the first part of the theorem, $S^\lambda = -\frac{1}{2} i_V R$. So, by Lemma 16,

$$\begin{aligned} 2[\lambda, S^\lambda] &= -[\lambda, [V,R]^\wedge] \\ &= -\left([[\lambda,V],R]^\wedge + [V,[\lambda,R]]^\wedge - [i_V\lambda, R] + [i_R\lambda, V]\right). \end{aligned}$$

Now $i_R\lambda = 0$, $i_V\lambda = \lambda V = H$, and

$$\begin{aligned} [H,R] &= [[H,P],P] + [P,[H,P]] = 0 \\ [[\lambda,V],R]^\wedge &= -[\mathscr{L}_V\lambda, R]^\wedge = -[2P - \mathrm{Id}, R]^\wedge \\ &= -2[P,R]^\wedge + [\mathrm{Id},R]^\wedge = 2R + R = 3R. \end{aligned}$$

So

$$[\lambda, S^\lambda] = -(3R + [V,[\lambda,R]]^\wedge). \tag{8.6}$$

Now, we claim that $[\lambda, R] = 0$. By the graded Jacobi identity,

$$[\lambda, R] = \frac{1}{2}[\lambda,[P,P]] = -[P,[\lambda,P]].$$

From $P = \frac{1}{2}(\mathrm{Id} + \mathscr{L}_V\lambda)$,

$$[\lambda, P] = \frac{1}{2}\left([\lambda,\mathrm{Id}] + [\lambda,[V,\lambda]]\right) = \frac{1}{2}[\lambda,[V,\lambda]]$$

since $[\lambda, \mathrm{Id}] = 0$ by Lemma 17. The graded Jacobi identity applied once more gives

$$-[\lambda,[V,\lambda]] + [V,[\lambda,\lambda]] + [\lambda,[\lambda,V]] = 0.$$

But $[\lambda,\lambda] = 0$ by Lemma 36, and $[V,\lambda] = -[\lambda, V]$. So $[\lambda,[V,\lambda]] = 0$, and therefore $[\lambda, R] = 0$, as claimed.

So (8.6) becomes $[\lambda, S^\lambda] = -\frac{3}{2}R$ as required.

□

Since R is skew-symmetric in its arguments, the first part of the theorem implies immediately

Corollary 5. *For any $X \in TTM'$, $S(V,X) = 0$.*

8.4 CONFORMAL TRANSFORMATIONS

A generalized conformal transformation is the transformation from the Lagrangian $G(x, v)$ to the Lagrangian $\widehat{G}(x, v)$ where

$$G(x,v) = \widehat{G}(x,v)J(x,v)^{-1}.$$

Here $J(x, v)$ is a non-zero function, smooth and defined in a neighborhood of the cone $G(x, v) = 0$. We have the homogeneities, valid for any real $t > 0$:

$$G(x,tv) = t^k G(x,v)$$

$$\widehat{G}(x,tv) = t^p \widehat{G}(x,v)$$

$$J(x,tv) = t^q J(x,v)$$

where

$$p - q = k \neq 1, \quad p \neq 1.$$

Denote by $\widehat{V}$ the dynamical vector field with respect to the transformed Lagrangian $\widehat{G}$.

Lemma 43. *The dynamical vector field V transforms via $\widehat{V} = V + bH + GT$ where $b = -(p-1)^{-1}J^{-1}V(J) = (k-1)^{-1}J^{-1}\hat{V}(J)$ and T is some vertical vector field.*

Proof. Since both V and $\widehat{V}$ satisfy Lemma 35, $\lambda(V - \widehat{V}) = 0$, and therefore the two vector fields can only differ by a vertical vector field. But working modulo $G = 0$, since $\widehat{V}$ are dynamical vector fields on $\mathscr{H}$, $\widehat{V}$ must be in the span of V and H, and so

$$\widehat{V} = V + bH \pmod{G = 0}$$

for some function b.

It remains to show that $b = -(p-1)^{-1}J^{-1}V(J) = -(k-1)^{-1}J^{-1}\widehat{V}(J)$. By definition of α, $\widehat{\alpha} = J\alpha + G\,DJ$, and so pulling back to $\mathscr{H}$ gives

$$d\widehat{\alpha} \equiv dJ \wedge \alpha + J d\alpha \pmod{G, dG}.$$

Contracting with V gives on the one hand $V \lrcorner d\widehat{\alpha} \equiv V(J)\alpha$ since $V \lrcorner \alpha = 0$ and $V \lrcorner d\alpha = -(k-1)dG \equiv 0$. On the other hand, using $V = \widehat{V} - bH$ gives

$$V \lrcorner d\widehat{\alpha} \equiv -b(p-1)\widehat{\alpha} \equiv -b(p-1)J\alpha$$

because $\widehat{V} \lrcorner d\widehat{\alpha} = -(p-1)d\widehat{G} \equiv 0$ and $H \lrcorner d\widehat{\alpha} = (p-1)\widehat{\alpha}$. Combining these two calculations gives $b = -(p-1)^{-1}J^{-1}V(J)$, as claimed. By symmetry, $-b = -(k-1)^{-1}J\widehat{V}(J^{-1}) = (k-1)^{-1}J^{-1}\widehat{V}(J)$, and so $b = -(k-1)^{-1}J^{-1}\widehat{V}(J^{-1})$ as well. $\square$

Lemma 44. $\mathscr{L}_{\widehat{V}}\lambda = \mathscr{L}_V\lambda - b\lambda - Db \otimes H - \alpha \otimes T + G\mathscr{L}_T\lambda$.

Proof.

$$\begin{aligned}\mathscr{L}_{\widehat{V}}\lambda(X) &= [\widehat{V}, \lambda X] - \lambda[\widehat{V}, \lambda X] \\ &= [V + bH + GT, \lambda X] - \lambda[V + bH + GT, X] \\ &= \mathscr{L}_V\lambda(X) + b\mathscr{L}_H\lambda(X) - (D_X b)H + G\mathscr{L}_T\lambda(X) - (D_X G)T + X(G)\lambda T \\ &= (\mathscr{L}_V\lambda - b\lambda - Db \otimes H + G\mathscr{L}_T\lambda - \alpha \otimes T)(X)\end{aligned}$$

where the last equality follows since $\mathscr{L}_H\lambda = -\lambda$, $D_X G = \alpha(X)$ by definition, and $\lambda T = 0$ since T is vertical. □

8.4.1 Weyl tensor

Recall that $\Lambda_{\mathscr{H}}$ is the subbundle of $T\mathscr{H}$ consisting of vectors X that annihilate α: $\alpha(X) = 0$. Then $V\mathscr{H} \subset \Lambda_{\mathscr{H}}$ and also the dynamical vector field V is a section of $\Lambda_{\mathscr{H}}$.

Definition 23. *The* umbral bundle *is the vector bundle E over $\mathscr{H}$ defined as the quotient of $\Lambda_{\mathscr{H}}$ by the kernel of $g_h|_{\Lambda_{\mathscr{H}} \times \Lambda_{\mathscr{H}}}$.*

The umbral bundle is a rank $n - 2$ vector bundle over $\mathscr{H}$. It is so named because in §8.5, the pullback of E along sections of $\mathscr{H}$ can be regarded as a space of infinitesimal screens onto which an object, placed into the null geodesic spray, will cast a shadow. This interpretation is due to Sachs [70].

The kernel of $g_h|_{\Lambda_{\mathscr{H}} \times \Lambda_{\mathscr{H}}}$ is the subspace of $\Lambda_{\mathscr{H}}$ spanned by $V\mathscr{H}$ and the dynamical vector field V. Indeed, $g_h(V, X) = (k-1)\alpha(X)$, which vanishes if $X \in \Lambda_{\mathscr{H}}$. Thus $\ker(g_h|_{\Lambda_{\mathscr{H}} \times \Lambda_{\mathscr{H}}})$ contains V. It also contains $V\mathscr{H}$, since the kernel of the bilinear form g_h on the full tangent space TTM is VTM. From rank considerations, $g_h|_{\Lambda_{\mathscr{H}} \times \Lambda_{\mathscr{H}}}$ has degree of degeneracy at most n, and so the kernel must in fact be equal to $V\mathscr{H} \oplus \operatorname{span} V$. Thus

$$E = \frac{\Lambda_{\mathscr{H}}}{V\mathscr{H} \oplus \operatorname{span} V}.$$

If $X \in \Lambda_{\mathscr{H}}$, denote by $[X]$ the equivalence class of X in E. The tensor g_h descends to a non-degenerate metric on E, via

$$g_E([X], [Y]) = g_h(X, Y).$$

When it is restricted to $\Lambda_{\mathscr{H}} \times \Lambda_{\mathscr{H}}$, S vanishes if either argument is in the kernel of g_h (by Corollary 5). Thus by restriction S defines a section of $E^* \otimes E^*$. Define an endomorphism $S_E^\sharp : E \to E$ by setting $g_E(S_E^\sharp X, Y) = S(X, Y)$ for all $X, Y \in E$.

Definition 24. *The* Weyl tensor $W \in \Gamma_{\mathscr{H}}(E^* \otimes E^*)$ *is the trace-free part of the restriction of the tidal force tensor* $S(X,Y)$ *to* $(X,Y) \in E \times E$*. That is,*

$$W(X,Y) = S(X,Y) - \left(\frac{1}{n-2}\operatorname{tr} S_E^\sharp\right) g_E(X,Y)$$

for $(X,Y) \in E \times E$*.*

Notice that we are now working on $\mathscr{H}$ exclusively, and so $G = 0$.

Theorem 25. *The Weyl tensor depends only on* $\mathscr{H} \subset TTM'$*, not on the choice of defining function* G*.*

In other words, the Weyl tensor is conformally invariant with respect to the class of conformal transformations described at the beginning of §8.4.

Proof. By Lemma 42, $S_E^\sharp[X] = \frac{1}{2}\left[\overline{\lambda}^{-1} P(\mathscr{L}_V^2\lambda)X\right]$ for $X \in \Lambda_{\mathscr{H}}$. So to prove that W is conformally invariant, it is sufficient to compute $\widehat{P}(\mathscr{L}_{\widehat{V}}^2\lambda)$ on E, and then to neglect terms that are proportional to λ, since these will only modify the trace. We shall therefore compute $\mathscr{L}_{\widehat{V}}^2\lambda$ modulo terms involving G, dG, α, since these are zero on E, modulo H since $\widehat{P}H = H$ which is in VTM' and so also zero in E in which the vertical space is quotiented, and modulo V since $\widehat{P}V \equiv \widehat{P}\widehat{V} \pmod{H} \equiv 0$. We treat each term of Lemma 44 in turn:

$$\begin{aligned}
\left(\mathscr{L}_{\widehat{V}}\mathscr{L}_V\lambda\right)X &= [\widehat{V}, (\mathscr{L}_V\lambda)X] - (\mathscr{L}_V\lambda)[\widehat{V}, X] \\
&= [V + bH + GT, (\mathscr{L}_V\lambda)X] - (\mathscr{L}_V\lambda)[V + bH + GT, X] \\
&\equiv \mathscr{L}_V^2\lambda(X) + b(\mathscr{L}_H\mathscr{L}_V\lambda)(X) + \mathscr{L}_{GT}(\mathscr{L}_V\lambda)(X) \pmod{G, dG, \alpha, H, V} \\
&\equiv \mathscr{L}_V^2\lambda(X) + \mathscr{L}_{GT}(\mathscr{L}_V\lambda)(X).
\end{aligned}$$

by homogeneity of V and λ. Now $\mathscr{L}_{GT}(\mathscr{L}_V\lambda)(X)$ vanishes modulo G for $X \in \ker dG$. Hence

$$\left(\mathscr{L}_{\widehat{V}}\mathscr{L}_V\lambda\right)X \equiv \mathscr{L}_V^2\lambda(X).$$

The second term is

$$\begin{aligned}
\left(\mathscr{L}_{\widehat{V}}(b\lambda)\right)X &\equiv \widehat{V}(b)\lambda X + b(\mathscr{L}_{\widehat{V}}\lambda)X \\
&\equiv \widehat{V}(b)\lambda X + b(\mathscr{L}_V\lambda)X - b^2\lambda X.
\end{aligned}$$

The third term is

$$\left(\mathscr{L}_{\widehat{V}}(Db \otimes H)\right)X \equiv -D_X b \otimes V \equiv 0.$$

The remaining terms are zero, since they involve α and G:

$$\mathscr{L}_{\widehat{V}}(G\mathscr{L}_T\lambda) \equiv 0$$
$$\left(\mathscr{L}_{\widehat{V}}(\alpha \otimes T)\right) \equiv 0.$$

Thus we have

$$\mathscr{L}^2_{\widehat{V}}\lambda \equiv \mathscr{L}^2_V\lambda - b(\mathscr{L}_V\lambda) + (b^2 - \widehat{V}(b))\lambda.$$

We now compute $\widehat{P}\mathscr{L}^2_{\widehat{V}}\lambda$. By the transformation law for $\mathscr{L}_{\widehat{V}}\lambda$,

$$\widehat{P} = P - \frac{1}{2}(b\lambda + Db \otimes H + \alpha \otimes T - G\mathscr{L}_T\alpha).$$

Note that since $\mathscr{L}_v\lambda = 2P - \mathrm{Id}$, $P\mathscr{L}_v\lambda = P$ and $\lambda\mathscr{L}_V\lambda = -\lambda$. Among the remaining terms are those involving the $\alpha \otimes T$ term of $\widehat{P}$ contracted with a term of $\mathscr{L}^2_{\widehat{V}}\lambda$. Of these, it follows from $\alpha \circ P = 0$ that $\mathscr{L}_V\lambda$ preserves the annihilator of α, and so the term $(\alpha \otimes T)(b\mathscr{L}_V\lambda)$ term vanishes when restricted to $\Lambda_{\mathscr{H}}$. Since $\alpha \circ \lambda = 0$, the $(\alpha \otimes T)\lambda$ terms vanish. Finally, the term involving $(\alpha \otimes T)(\mathscr{L}^2_V\lambda)$ vanishes on $\Lambda_{\mathscr{H}} = \ker \alpha$ since the kernel of α is Lie derived along V.

So, applying $\widehat{P}$ to $\mathscr{L}^2_{\widehat{V}}\lambda$ gives

$$\begin{aligned}\widehat{P}\mathscr{L}^2_{\widehat{V}}\lambda &\equiv P\mathscr{L}^2_V\lambda - bP + (b^2 - \widehat{V}(b))\lambda - \frac{1}{2}b\lambda\mathscr{L}^2_V\lambda - \frac{1}{2}b^2\lambda \\ &\equiv P\mathscr{L}^2_V\lambda + \left(\tfrac{1}{2}b^2 - \widehat{V}(b)\right)\lambda - \frac{1}{2}b\lambda\mathscr{L}^2_V\lambda - bP\end{aligned}$$

It is now sufficient to show that the last two terms cancel; that is:

$$\lambda\mathscr{L}^2_V\lambda = -2P.$$

We have

$$\begin{aligned}\lambda\mathscr{L}^2_V\lambda(X) &= \lambda\left([V,[V,\lambda X]] - 2[V,\lambda[V,X]] + \lambda[V,[V,X]]\right) \\ &= \lambda[V,[V,\lambda X]] - 2\lambda[V,\lambda[V,X]] \\ &= \lambda[V,[V,\lambda X]] + 2\lambda[V,X].\end{aligned}$$

If X is in the image of P, the first term vanishes because $\operatorname{im} P = \ker \lambda$, leaving only the second term which is $-2X$. If instead $X \in \ker P$, then $\mathscr{L}_V X = -X$. So

$$\begin{aligned}\lambda[V,[V,\lambda X]] + 2\lambda[V,X] &= \lambda[V, -X + \lambda[V,X]] - 2\lambda X \\ &= -\lambda[V,X] + \lambda[V, -\lambda X] - 2\lambda X \\ &= 2\lambda X - 2\lambda X = 0\end{aligned}$$

as required. Thus, in summary

$$\widehat{P}\mathscr{L}^2_{\widehat{V}}\lambda \equiv P\mathscr{L}^2_V\lambda + \left(\tfrac{1}{2}b^2 - \widehat{V}(b)\right)\lambda.$$

Since the term multiplying λ only modifies the trace of S, this completes the proof. □

8.5 RAYCHAUDHURI–SACHS EQUATIONS

We review the Raychaudhuri–Sachs equation of standard general relativity. Let M be a spacetime manifold of dimension $n \geq 3$, equipped with an indefinite metric, g of signature (p, q). Denote the Levi-Civita connection of g by ∇.

Let k be a null vector field that is nowhere zero and satisfies the equation of an affinely parametrized geodesic $\nabla_k k = 0$. The integral curves of k are null geodesics that foliate M: that is, they constitute a null geodesic congruence. Associated to the vector field k is a natural vector bundle K of dimension $n-2$ with a metric of signature $(p-1, q-1)$ (so Euclidean in the case where g is Lorentzian). This bundle consists of the $(n-2)$-plane elements (or "screens") onto which the infinitely near curves of the congruence would cast the shadow of an object. This bundle was introduced in this way by Sachs [70]. A precise definition of this bundle is in section 8.5.2.

The Raychaudhuri–Sachs equation then governs the rate at which this shadow expands (or contracts) as the screens advance along a particular geodesic of the congruence. A principal ingredient in the derivation of the equation is the notion of the divergence of k, of which there are potentially several candidates (that turn out to agree):

- The Lie derivative of the volume element of M along k.
- The Lie derivative of a natural volume element for the bundle K.
- The trace of the endomorphism ∇k.

8.5.1 Notation and conventions

The curvature tensor $R \in \Gamma_M(\wedge^2 T^*M \otimes \mathrm{End}(TM))$ is defined by the relation

$$R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z.$$

The Ricci tensor is given by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y).$$

The metric defines an isomorphism between the tangent and cotangent bundles of M: define $g : TM \to T^*M$ by

$$g(X) : Y \mapsto g(X,Y).$$

This is a self-adjoint transformation (by the symmetry of g) that is invertible (by the non-degeneracy of g). The inverse $g^{-1} : T^*M \to TM$ defines a metric g^{-1} on T^*M by

$$g^{-1}(\alpha,\beta) = \beta(g^{-1}(\alpha)).$$

The metric will be used to convert vectors into covectors systematically using the "musical isomorphism":

- If X is a vector, define $X^\flat = g(X)$.
- If α is a covector, define $\alpha^\sharp = g^{-1}(\alpha)$.

The volume element of M is a density on M that is defined on a collection of vectors $v_1,\dots,v_n$ by

$$|\Omega(v_1,\dots,v_n)|^2 = |\det[g(v_i,v_j)]_{i,j=1,\dots,n}|.$$

This is a section of the density bundle $|\wedge^n T^*M|$. If an orientation is given on M, then it is possible to choose a representative volume form, denoted Ω, for the density $|\Omega|$. In a distinguished oriented local orthonormal basis[1] of vector fields $X_1,\dots,X_n$,

$$\Omega(X_1,\dots,X_n) = |\Omega(X_1,\dots,X_n)| = 1,$$

extended by multilinearity.

Since a local one-parameter group of diffeomorphisms must preserve orientation, the Lie derivative of $|\Omega|$ along any differentiable vector field is well-defined. The divergence of a differentiable vector field X is defined by

$$(\mathrm{div}\, X)|\Omega| = \mathscr{L}_X|\Omega|.$$

Now, for the Raychaudhuri–Sachs equations, assume in addition that k is hypersurface orthogonal. This is equivalent to the condition that the distribution $k^\perp = (k^\flat)^0 \subset TM$ of $(n-1)$-planes annihilating $k^\flat$ be integrable in the sense of Frobenius: $k^\flat \wedge dk^\flat = 0$.

[1]For a metric of indefinite signature, an orthonormal basis is any basis such that $g(X_i,X_j) = \pm\delta_{ij}$.

8.5.2 The umbral bundle of the congruence

Let $k^\perp$ denote the distribution of $(n-1)$-planes orthogonal to k. Thus, at a point $x \in M$,

$$k_x^\perp = \{v \in T_xM \mid g(k,v) = 0\} = (k_x^\flat)^0.$$

Lemma 45. *The distribution $k^\perp$ is Lie-derived along k. That is, if $v \in \Gamma_M(k^\perp)$, then $\mathscr{L}_k v \in \Gamma_M(k^\perp)$.*

Proof. If v is a section of $k^\perp$, then $g(k,v) = 0$. So

$$\begin{aligned} 0 = k(g(k,v)) &= g(\nabla_k k, v) + g(k, \nabla_k v) = g(k, \nabla_k v) \\ &= g(k, \mathscr{L}_k v) + g(k, \nabla_v k) \\ &= g(k, \mathscr{L}_k v) + \frac{1}{2} v(g(k,k)) = g(k, \mathscr{L}_k v) \end{aligned}$$

so $\mathscr{L}_k v$ is also in $k^\perp$, as required. □

Note that k is a section of $k^\perp$, since it is null. Therefore the following definition makes sense:

Definition 25. *Let K be the quotient bundle $K = k^\perp / \operatorname{span} k$.*

If a small object is placed in the path of the congruence k, then the bundle K naturally describes a family of screens onto which the shadow of an object is cast. Hence, this is the *umbral bundle* for the null geodesic congruence k. It is the pullback of the umbral bundle defined in §8.4.1 by the section k of the null cone bundle $\mathscr{H}$; see §8.6 for more details.

Let $[v]$ denote the equivalence class of $v \in k^\perp$ modulo k. Since $k^\perp$ and $\operatorname{span} k$ are both Lie derived along k, the Lie derivative $\mathscr{L}_k$ descends to a differential operator on the quotient K, by setting

$$\mathscr{L}_k [v] = [\mathscr{L}_k v].$$

The Lie derivative extends to a unique derivation on the tensor algebra of K that commutes with tensor contraction.

The metric g in TM induces a bilinear form $g_{k^\perp}$ on $k^\perp$, and the vector k is in the kernel of $g_{k^\perp}$. Hence $g_{k^\perp}$ descends to a bilinear form on K via the rule

$$g_K([X],[Y]) = g_{k^\perp}(X,Y).$$

The bilinear form g_K is a metric of signature $(p-1, q-1)$ on K.

The tidal force along k is the endomorphism $S^\sharp : TM \to TM$ given on vectors X by

$$S^\sharp X = R(k, X)k.$$

Since $S^\sharp k = 0$ and the image of $S^\sharp$ is orthogonal to k, $S^\sharp$ induces an endomorphism of K via

$$S^\sharp_K[X] = [S^\sharp X].$$

The bilinear form S on TM and S_K on K given by

$$S(X, Y) = g(S^\sharp X, Y), \quad S_K([X], [Y]) = g_K(S^\sharp_K[X], [Y])$$

are both symmetric, by the symmetries of the Riemann tensor.

8.5.3 Divergence

Definition 26. *Let X be a vector field. The divergence of X, denoted* $\operatorname{div} X$*, is defined by the equation*

$$(\operatorname{div} X)|\Omega| = \mathscr{L}_X|\Omega|$$

For the vector field X, define the endomorphism ∇X of TM by $\nabla X : Y \mapsto \nabla_Y X$.

Lemma 46. *The divergence of X is the trace of $\nabla X \in \Gamma_M(\operatorname{End}(TM))$*

$$\operatorname{div} X = \operatorname{tr} \nabla X.$$

Proof. Let $v_1, \ldots, v_n$ be a local basis of smooth sections of TM, and let $\alpha^1, \ldots, \alpha^n$ be the dual basis of T^*M, defined by $\alpha^i(v_j) = \delta^i_j$. Let Ω be the local section of $\wedge^n T^*M$ representing $|\Omega|$ obtained by declaring the basis v_i to be positively oriented. First note that if α is a one-form and Y a tangent vector, then

$$0 = Y \lrcorner (\alpha \wedge \Omega) = \alpha(Y)\Omega - \alpha \wedge Y \lrcorner \Omega$$

so

$$\alpha \wedge Y \lrcorner \Omega = \alpha(Y)\Omega. \tag{8.7}$$

By Cartan's identities,

$$\begin{aligned}
\mathscr{L}_X\Omega = d(X\lrcorner\,\Omega) &= \sum_i \alpha^i \wedge \nabla_{v_i}(X\lrcorner\,\Omega) \\
&= \sum_i \alpha^i \wedge (\nabla_{v_i}X)\lrcorner\,\Omega + \sum_i \alpha^i \wedge X\lrcorner\,\nabla_{v_i}\Omega \\
&= \sum_i \alpha^i \wedge (\nabla_{v_i}X)\lrcorner\,\Omega \\
&= \sum_i \alpha^i(\nabla_{v_i}X)\Omega = \operatorname{tr}(\nabla X)\Omega
\end{aligned}$$

by (8.7). □

Fixing an orientation on TM equips the bundle K with an induced orientation, and the associated volume forms are related by

$$k\lrcorner\,\Omega = k^\flat \wedge \Omega_K$$

where $\lrcorner$ is the interior product. The validity of this equation does not depend on the choice of coset representative of Ω_K modulo the ideal generated by $k^\flat$, and so defines Ω_K uniquely as a section of $\wedge^{n-2}K^*$. If no orientation on M is specified, then this only defines a density $|\Omega_K|$ in the determinant bundle $|\wedge^{n-2} K^*|$.

Definition 27. *Define* $\operatorname{div}_K k$ *by*

$$(\operatorname{div}_K k)|\Omega_K| = \mathscr{L}_k|\Omega_K|.$$

Lemma 47. $\operatorname{div}_K k = \operatorname{div} k$

Proof. Working locally with an orientation on M, we have $k\lrcorner\,\Omega = k \wedge \Omega_K$. So

$$\mathscr{L}_k(k\lrcorner\,\Omega) = \mathscr{L}_k(k \wedge \Omega_K)$$

$$k\lrcorner\,\mathscr{L}_k\Omega = k \wedge \mathscr{L}_k\Omega_K$$

$$(\operatorname{div} k)k\lrcorner\,\Omega = (\operatorname{div}_K k)k \wedge \Omega_K.$$

□

The image of the endomorphism ∇k lies in $k^\perp$, since $g(\nabla_X k, k) = \frac{1}{2}X(g(k,k)) = 0$. Furthermore, k lies in the kernel of ∇k, since k is an affinely parametrized geodesic vector field. Therefore, ∇k descends to an endomorphism $\nabla k|_K : K \to K$.

Lemma 48. *For any* $p = 1, 2, \ldots,$

$$\operatorname{tr}((\nabla k)^p) = \operatorname{tr}((\nabla k|_K)^p)$$

Proof. In general, if A is an endomorphism of a vector space V whose image lies in a subspace W, then $\operatorname{tr} A = \operatorname{tr}(A|_W)$. Since ∇k is a linear operator whose image lies in $k^\perp$, $\operatorname{tr}(\nabla k)^p = \operatorname{tr}((\nabla k)^p|_{k^\perp})$. But $(\nabla k)^p|_{k^\perp} = (\nabla k|_{k^\perp})^p$, and so $\operatorname{tr}(\nabla k)^p = \operatorname{tr}(\nabla k|_{k^\perp})^p$. Now, since $\operatorname{span} k$ lies in the kernel of $\nabla k|_{k^\perp}$, $\operatorname{tr}(\nabla k|_{k^\perp})^p = \operatorname{tr}(\nabla k|_K)^p$, as required. ☐

8.5.4 Rate of change of the divergence

The purpose of this section is to compute the rate of change of the divergence of k. Let $R(k,-)k$ denote the endomorphism $R(k,-)k : X \mapsto R(k,X)k$. Then:

Lemma 49. $\nabla_k \nabla k = -(\nabla k)^2 + R(k,-)k$

Proof. For a vector field X,

$$\begin{aligned}
(\nabla_k \nabla k)(X) &= \nabla_k \nabla_X k - \nabla_{\nabla_k X} k \\
&= R(k,X)k + \nabla_X \nabla_k k + \nabla_{[k,X]} k - \nabla_{\nabla_k X} k \\
&= R(k,X)k - \nabla_{\nabla_X k} k \\
&= [-(\nabla k)^2 + R(k,-)k](X)
\end{aligned}$$

☐

Lemma 50.

$$\begin{aligned}
k(\operatorname{div} k) &= -\operatorname{tr}[(\nabla k)^2] + \operatorname{Ric}(k,k) \\
&= -\operatorname{tr}[(\nabla k|_K)^2] + \operatorname{Ric}(k,k)
\end{aligned}$$

Proof. The first equation follows by taking a trace from the previous lemma. The second equation follows from $\operatorname{tr}(\nabla k)^2 = \operatorname{tr}(\nabla k|_K)^2$. ☐

Lemma 51. $\operatorname{tr} S^\sharp = \operatorname{tr} S^\sharp_K$

Proof. The image of $S^\sharp$ lies in $k^\perp$ and the kernel of $S^\sharp$ contains k. Thus the lemma follows by the argument of Lemma 48. ☐

8.5.5 Invariant decomposition

Let

$$\nabla k|_K = \operatorname{Alt}(\nabla k|_K) + \operatorname{Sym}_0(\nabla k|_K) + \frac{1}{n-2}(\operatorname{div} k)\operatorname{Id}_K$$

be the decomposition of $\nabla k|_K$ into its irreducible components for the action of $O(p-1, q-1)$: the alternating, symmetric trace-free, and trace parts. Here the metric g_K is used to identify $\operatorname{End}(K)$ with $K^* \otimes K^*$ in order to define the symmetric and alternating parts.

For the next theorem, introduce the following notation, standard in the relativity literature when $n = 4$:

- $\theta = \operatorname{div} k$ is called the *expansion* of the congruence k in the relativity literature
- $\sigma = \operatorname{Sym}_0(\nabla k|_K)$ is the *shear tensor*
- $\rho = \operatorname{Alt}(\nabla k|_K)$ is the *rotation tensor*

Theorem 26.

$$\begin{aligned} k(\theta) &= -\operatorname{tr}(\rho^2) - \operatorname{tr}(\sigma^2) - \frac{\theta^2}{n-2} + \operatorname{Ric}(k,k) \\ &= -\operatorname{tr}(\rho^2) - \operatorname{tr}(\sigma^2) - \frac{\theta^2}{n-2} + \operatorname{tr} S^\sharp. \end{aligned}$$

Proof. This is a restatement of Lemma 50 under the decomposition

$$\nabla k|_K = \rho + \sigma + \frac{1}{n-2}\theta \operatorname{Id}_K.$$

The absence of cross-terms owes to the orthogonality of the different irreducible representations of $O(p-1, q-1)$. The second equality follows from the definition of $S^\sharp$. □

8.5.6 Hypersurface orthogonality

If k is hypersurface orthogonal, then the distribution $k^\perp = (k^\flat)^0$ is integrable in the sense of Frobenius, and therefore $dk^\flat \equiv 0 \pmod{k^\flat}$.

Lemma 52. *If k is hypersurface orthogonal, then* $\operatorname{tr}(\rho^2) = 0$.

Proof. If k is hypersurface orthogonal, then there exists locally a one-form μ such that $dk^\flat = \mu \wedge k^\flat$. Since k is a geodesic vector field, $k \lrcorner dk^\flat = 0$, and since k is also null $k^\flat(k) = 0$, so $\mu(k) = 0$ as well. Now

$$\operatorname{tr}(\rho^2) = dk^\flat(k, \mu^\sharp) = 0$$

as claimed. □

Theorem 26 becomes the *Raychaudhuri–Sachs equations*:

Corollary 6. *If k is hypersurface orthogonal, then*

$$k(\theta) = -\operatorname{tr}(\sigma^2) - \frac{\theta^2}{n-2} + \operatorname{tr} S^\sharp.$$

8.5.7 Raychaudhuri effect

Lemma 53. *Suppose that J is a vector field that Lie commutes with k. Then J is a Jacobi field along any integral curve of k.*

Proof. Covariantly differentiating $0 = [k, J] = \nabla_k J - \nabla_J k$ along k gives

$$\begin{aligned} 0 &= \nabla_k^2 J - \nabla_k \nabla_J k \\ &= \nabla_k^2 J - R(k, J)k \end{aligned}$$

which is the Jacobi equation □

In particular, since k is hypersurface orthogonal, there are $n-2$ (Jacobi) vector fields $J_1, \dots, J_{n-2}$ that are orthgononal to k, Lie commute with k, and are linearly independent of k. On passing to the quotient, these Jacobi fields define a basis of K. Pick such a basis, and let $\lambda_K = |\Omega_K(J_1, \dots, J_{n-2})|$.

In the Lorentzian case of a space-time of n-dimensions, the signature of the metric g_E of the bundle E is either positive or negative definite, according as g has signature $(n-1, 1)$ or $(1, n-1)$. Thus in the Raychaudhuri–Sachs equations, the trace $\operatorname{tr}(\sigma^2)$ is non-negative, and it is zero if and only if $\sigma = 0$. Thus Corollary 6 gives

$$k(\theta) \le \operatorname{tr} S^\sharp,$$

or equivalently,

$$\mathscr{L}_k^2 \lambda_K \le \operatorname{tr} S^\sharp \lambda_K.$$

The null positive energy condition is the condition

$$\operatorname{Ric}(n, n) \le 0 \quad \text{for all null vectors } n.$$

So when the null positive energy condition holds,

$$\mathscr{L}_k^2 \lambda_K \le 0$$

Note that this equality only requires that $\operatorname{Ric}(k, k) \le 0$ be valid for the particular tangent vectors along the given null geodesic.

Now suppose that $\theta < 0$ at some point x_0 of the congruence. By definition of θ, at that point $\mathscr{L}_k \lambda_K = \theta \lambda_K < 0$. Then λ_K will become zero along the geodesic tangent to k through x_0 at some time prior to the finite affine parameter $t = -(n-2)/\theta(x_0)$. Since k is hypersurface orthogonal, the vectors $J_1, \ldots, J_{n-2}$ span the tangent space of this hypersurface up to the point where the volume λ_K degenerates to zero. At or before that point, the geodesic in question must have a conjugate point. The existence of this conjugate point is the key to the proof by Sir Roger Penrose [65] of his singularity theorem.

8.6 THE GEOMETRIC RAYCHAUDHURI–SACHS THEOREM

In this section, we lift the geometry underlying the Raychaudhuri–Sachs theorem to the bundle $\mathscr{H}$ and at the same time generalize it to regular causal geometries. We first recall some basic sheaf theory.

Let $p : \mathbb{Y} \to \mathbb{X}$ be a (continuous) map of topological spaces.

- A point $y \in \mathbb{Y}$ is said to be a sheaf point if and only if there exists an open set U_y in $\mathbb{Y}$, such that $y \in U_y$ and such that the restriction of p to U_y is a homeomorphism onto its open image $p(U_y)$ in $\mathbb{X}$.
- The sheaf space $\mathcal{S}_p \subset \mathbb{Y}$ of p is the collection of all its sheaf points, with the induced topology. Note that $\mathcal{S}_p$ is an open subset of $\mathbb{Y}$.
- The triple $(\mathbb{Y}, \mathbb{X}, p)$ is said to be a sheaf if and only if p is surjective and $\mathcal{S}_p = \mathbb{Y}$.
- The triple $(\mathbb{Y}, \mathbb{X}, p)$ is said to be a stack if and only if p is surjective and $\mathcal{S}_p$ is dense in $\mathbb{Y}$, i.e. the closure $\overline{\mathcal{S}_p} = Y$.
- The triple $(\mathbb{Y}, \mathbb{X}, p)$ is said to be a branched cover if and only if it is a stack and both $\mathbb{Y}$ and $\mathbb{X}$ are Hausdorff topological spaces.

For example:

- Put $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, the unit circle in the plane. Then the map $e : \mathbb{R} \to \mathbb{S}^1$ given by the formula $e(t) = (\cos(t), \sin(t))$, for any $t \in \mathbb{R}$ makes $(\mathbb{R}, \mathbb{S}^1, e)$ a sheaf.
- Consider the complex parabola $\mathbb{Y} = \{(x, y) \in \mathbb{C}^2 : y^2 = x\}$ and let $p(x, y) = x \in \mathbb{C}$ for any $(x, y) \in \mathbb{Y}$. Then the triple $(\mathbb{Y}, \mathbb{C}, p)$ is a stack, with $\mathcal{S}_p = \mathbb{Y} - \{(0, 0)\}$ and is a branched cover.

If $(\mathbb{Y}, \mathbb{X}, p)$ is a stack, and if U is an open subset of $\mathbb{X}$, then a section s of the stack over U is a map $s : U \to \mathbb{X}$, such that $p \circ s = id_U$.

The key concept we need is that of a Sachs manifold. Let M be a manifold of dimension n and let $\mathscr{G} \subset \mathbb{SM}$ be a regular causal geometry. Denote the natural surjection from $\mathscr{G}$ to M by p. Denote a representative one-form of the contact structure of $\mathscr{G}$ by $\alpha_{\mathscr{G}}$. Put $\mathscr{H} = \sigma^{-1}(\mathscr{G})$ and denote by $\alpha_{\mathscr{H}}$ a representative one-form of the contact structure of $\mathscr{H}$. Also denote by q the natural surjection from $\mathscr{H}$ to M.

- A Sachs manifold for the causal geometry $\mathscr{G}$ is a smooth submanifold $\mathcal{S}$ of $\mathscr{G}$ of dimension n, such that:
  - The triple $(\mathcal{S}, M, p|_{\mathcal{S}})$ is a branched cover.
  - $\mathcal{S}$ is ruled by (unparametrized) null geodesics: i.e. the null geodesic spray V of $\mathscr{G}$ is everywhere tangent to $\mathcal{S}$.
  - $\mathcal{S}$ is hypersurface orthogonal: the restriction of the three-form $\alpha_{\mathscr{G}} d\alpha_{\mathscr{G}}$ to $\mathcal{S}$ vanishes identically.
- An affine Sachs manifold for the causal geometry $\mathscr{G}$ is a submanifold $\mathcal{T}$ of $\mathscr{H}$ of dimension n, such that:
  - The triple $(\mathcal{T}, M, q|_{\mathcal{T}})$ is a branched cover.
  - $\mathcal{T}$ is ruled by affinely parametrized null geodesics: i.e. the null geodesic spray V of $\mathscr{H}$ is everywhere tangent to $\mathcal{T}$.
  - $\mathcal{T}$ is hypersurface orthogonal: the restriction of the three-form $\alpha_{\mathscr{H}} d\alpha_{\mathscr{H}}$ to $\mathcal{T}$ vanishes identically.
- A Sachs section for a causal geometry $\mathscr{H}$ over an open set $U \subset M$ is a section of a given Sachs manifold, whose domain is U.
- A Sachs congruence on an open subset $U \subset M$ is the foliation of U by the null geodesics giving the foliation of a Sachs section. Note that the Sachs congruence is automatically hypersurface orthogonal, with normals the (null) tangent vectors to the foliation and the congruence and the section determine each other uniquely.

In the special case of a standard space-time, the Sachs congruence exactly agrees with the congruence needed for the Raychaudhuri–Sachs equation and we see that in that case the affine Sachs manifold is simply the natural lift to the tangent bundle of the Sachs congruence, so we have a natural generalization.

Now let a Sachs section $s : M \to \mathscr{H}$ be given. The connection $P : T\mathscr{H} \to V\mathscr{H}$ defines an endomorphism $P_s \in \mathrm{End}(TM)$ given by

$$P_s(X) = \overline{\lambda}^{-1} P(s_* X).$$

Let k be the tangent vector field of the congruence, so $s_* k = V$. The tensor g_h pulls back under s to a metric $g_s = s^* g_h$ on M. Moreover, $k^\flat = s^* \alpha$. Since $g_h(V, V) = k(k-1)G$, it follows that $g_s(k, k) = 0$ since s is a section of $\mathscr{H}$ where $G = 0$.

The bundle K is defined as before as $k^\perp / \operatorname{span} k$, where $k^\perp$ is the orthogonal complement of k with respect to the metric g_s. This is naturally isomorphic to the pullback under s of the umbral bundle E defined in section 8.4.1. The metric g_s induces a metric g_K on K, which is of definite signature if g_v has Lorentzian signature. Let ∇ denote the Levi-Civita connection of g_s. The Lie derivative $\mathscr{L}_k$ preserves $\ker k^\flat$, by Lemma 45. Likewise the Lie derivative extends to all associated tensor bundles.

Lemma 54. *$P_s = \nabla k$ where ∇ is the Levi-Civita connection associated with the metric g_s. In particular k is an affinely parametrized geodesic with respect to the connection ∇. Moreover, the pullback of the tidal force tensor along s is the sectional curvature of ∇ in the direction of k:*

$$S(s_*X, s_*Y) = g_s(R(k,X)k, Y)$$

where R is the Riemann tensor associated to ∇.

Proof. The proof of the first claim proceeds by verifying that the two tensors have the same skew and symmetric parts. On the one hand,

$$\begin{aligned}
(\mathscr{L}_k g_s)(X,Y) &= k(g_s(X,Y)) - g_s(\nabla_k X - \nabla_X k, Y) - g_s(X, \nabla_k Y - \nabla_Y k) \\
&= g_s(\nabla_k X, Y) + g_s(X, \nabla_k Y) - g_s(\nabla_k X - \nabla_X k, Y) - g_s(X, \nabla_k Y - \nabla_Y k) \\
&= g_s(\nabla_X k, Y) + g_s(X, \nabla_Y k).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathscr{L}_k g_s)(X,Y) = (\mathscr{L}_V g_h)(s_*X, s_*Y) &= g_v(Ps_*X, \lambda s_*Y) + g_v(\lambda s_*X, Ps_*Y) \\
&= g_h(\overline{\lambda}^{-1} Ps_*X, s_*Y) + g_v(s_*X, \overline{\lambda}^{-1} Ps_*Y) \\
&= g_s(P_sX, Y) + g_s(X, P_sY).
\end{aligned}$$

This shows that ∇k and P_s have the same symmetric part.

For the skew part, on the one hand

$$\begin{aligned}
2(s^*d\alpha)(X,Y) &= 2dk^\flat(X,Y) \\
&= g_s(\nabla_X k, Y) - g_s(X, \nabla_Y k)
\end{aligned}$$

and on the other hand

$$\begin{aligned}2(s^*d\alpha)(X,Y) &= 2d\alpha(s_*X, s_*Y) = g_v(Ps_*X, \lambda s_*Y) - g_v(\lambda s_*X, Ps_*Y)\\ &= g_h(\overline{\lambda}^{-1}Ps_*X, s_*Y) - g_h(s_*X, \overline{\lambda}^{-1}Ps_*Y)\\ &= g_s(P_sX, Y) - g_s(X, P_sY).\end{aligned}$$

Since $P_s = \nabla k$, $\nabla_k k = P_s k = \overline{\lambda}^{-1}PV = 0$ since V is horizontal for the Ehresmann connection P. Hence k is an affinely parametrized geodesic.

For the final claim, Theorem 23 implies that it is sufficient to prove

$$\frac{1}{2}(\mathscr{L}_k^2 g_s)(X,Y) = g_s(\nabla_X k, \nabla_Y k) - g_s(R(k,X)k, Y)$$

since $P_s = \nabla k$ by the first part of the lemma. The identity

$$\mathscr{L}_k \nabla k = \nabla_k \nabla k$$

holds, so

$$\begin{aligned}\frac{1}{2}(\mathscr{L}_k^2 g_s)(X,Y) &= g_s(\nabla_X k, \nabla_Y k) + \frac{1}{2}g_s((\nabla_k\nabla k)(X), Y) + \frac{1}{2}g_s(X, (\nabla_k\nabla k)(Y)) +\\ &\qquad + \frac{1}{2}g_s(\nabla_{\nabla_X k}k, Y) + \frac{1}{2}g_s(X, \nabla_{\nabla_Y k}k)\\ &= g_s(\nabla_X k, \nabla_Y k) + \frac{1}{2}\left(g_s(R(k,X)k, Y) + g_s(X, R(k,Y)k)\right)\\ &= g_s(\nabla_X k, \nabla_Y k) + g_s(R(k,X)k, Y)\end{aligned}$$

by the symmetries of the Riemann tensor. □

The operator $S_E^\sharp : \pi_{TM'}^{-1}TM \to \pi_{TM'}^{-1}TM$ defined in §8.4.1, when restricted to the section s defines an operator $S_s^\sharp : TM \to TM$. By the previous lemma, $S_s^\sharp(X) = R_s(k,X)k$. Moreover, as in §8.5, the image of $S_s^\sharp$ lies in $k^\perp$ and its kernel contains k, so $S_s^\sharp$ descends to and operator $S_K^\sharp : K \to K$. Moreover, $\operatorname{tr} S_s^\sharp = \operatorname{tr} S_K^\sharp = \operatorname{Ric}(k,k)$

As in §8.5.3, the divergence of k can be defined in several equivalent ways. If $|\Omega|$ is the canonical density associated to the metric g_s, then

$$\mathscr{L}_k|\Omega| = (\operatorname{div} k)|\Omega|.$$

If $|\Omega_E|$ is the canonical section of the determinant line bundle $|\wedge^{n-2}E|$, then

$$\mathscr{L}_k|\Omega_E| = (\operatorname{div}_E k)|\Omega_E|.$$

Alternatively, the divergence can be defined as the trace of $\nabla k = P_s$, or the trace of $\nabla k|_E = P_s|_E$. The results of §8.5.3 imply that these are equal:

Lemma 55. $\theta = \operatorname{div} k = \operatorname{div}_E k = \operatorname{tr}(\nabla k) = \operatorname{tr}(P_s) = \operatorname{tr}(\nabla_k|_E) = \operatorname{tr}(P_s|_E)$

The proof of Theorem 26 goes through as in §8.5:

Theorem 27. *Let*

$$\begin{aligned} P_s|_E &= \operatorname{Alt} P_s|_E + \operatorname{Sym}_0 P_s|_E + \frac{\operatorname{tr} P_s|_E}{n-2} \operatorname{Id}_E \\ &= \rho + \sigma + \frac{\theta}{n-2} \operatorname{Id}_E \end{aligned}$$

be the decomposition of P_s into its irreducible $O(p-1, q-1)$ components. Then

$$k(\theta) = -\operatorname{tr}(\rho^2) - \operatorname{tr}(\sigma^2) - \frac{\theta^2}{n-2} + \operatorname{tr} S^\sharp.$$

8.6.1 The Lorentzian case: the geometric Raychaudhuri–Sachs effect

Now consider the case that the fibre metric g_v in VTM' is Lorentzian, which implies in turn that g_s is also Lorentzian, and so the metric g_K of the bundle K has positive or negative definite signature. Then the quantity $\operatorname{tr}(\sigma^2)$ of the Raychaudhuri–Sachs equation is non-negative. Also impose the positive energy condition: $\operatorname{tr} S^\sharp \leq 0$. As in §8.5, let $J_1, \ldots, J_{n-2}$ be a collection of vector fields orthgonal to k that commute with k, and set $\lambda_K = |\Omega_K(J_1, \ldots, J_{n-2})|$. Then

$$\mathscr{L}_k \lambda_K = \theta \lambda_K, \quad \mathscr{L}_k^2 \lambda_K \leq 0.$$

Now if at a point of the congruence we have $\theta < 0$, then it follows that the graph of λ_K along the (affinely parametrized) null geodesic through the point is decreasing and concave down, so λ_K reaches zero in finite affine parameter time in the future. So we have the theorem:

Theorem 28. *Let X be a given null geodesic in M that is future complete, so its affine parameter ranges to positive infinity. Suppose that everywhere along X the positive energy condition $\operatorname{tr} S^\sharp \leq 0$ holds. Suppose there is a section of a Sachs manifold, defined in a neighborhood of X, such that X is a member of the congruence foliating the Sachs manifold. Then the divergence of the congruence is everywhere non-negative along X.*

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APPENDIX A

INDEX

APPENDIX B

INDEX OF NOTATION