

BOUNDS ON PACKING DENSITY VIA SLICING

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Submitted to the Graduate Faculty of
the Kenneth P. Dietrich School of Arts and Sciences

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2014

UNIVERSITY OF PITTSBURGH
MATHEMATICS DEPARTMENT

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This document is composed of a series of articles in discrete geometry, each solving a problem in packing density.

- The first proves a local upper bound for the packing density of regular pentagons in \mathbb{R}^2 . By reducing a nonlinear programming problem to a linear one, computational methods show that the conjectured global optimal solution is locally optimal.
- The second proves an upper bound for the packing density of finite cylinders in \mathbb{R}^3 . Using a measure theoretic approach to estimate boundary error, the first bound that is asymptotically sharp with respect to the length of the cylinder is found. This gives the first sharp upper bound for the packing density of half-infinite cylinders as a corollary.
- The third proves an upper bound for the packing density of infinite polycylinders in \mathbb{R}^n . Using transversality and a dimension reduction argument, an existing result for \mathbb{R}^3 is applied to \mathbb{R}^n . This gives the first non-trivial sharp upper bound for the packing density of any object in dimensions four and greater.

Keywords: MSC2010[52C17] (packing in n dimensions), MSC2010[05B40] (packing and covering), MSC2010[11H31] (lattice packing and covering), discrete geometry, nonlinear programming, conical programming, interval arithmetic, measure theory, foliations, slicing.

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PREFACE

To my friends and family

This would not have been possible without the support of many people.

I would like to thank the following: Károly Bezdek, Włodzimierz Kuperberg, Robert Kusner, János Pach, *et alii anonymous*, for their letters and comments. Gregory Constantine, Jason DeBlois, Huiqiang Jiang and Anthony Rollett for their help, time, and service on my committee. Thomas Hales, not only for his mathematical counsel, but for his support.

Finally, I thank the faculty, students and staff of the University of Pittsburgh.

This work was partial supported by NSF grant 1104102.

Wöden Kusner

April 13, 2014, Pittsburgh, PA.

I. BACKGROUND

A. INTRODUCTION

This dissertation is concerned with the density of infinite packings. The overarching (and possibly tautological) philosophy is that the critical points of functions on a geometric configuration space correspond to intrinsically interesting geometric configurations. Based on physical observations and an expectation that nice objects are somehow universal, this trope has inspired the use of the optimization of various geometric quantities¹ to find candidates for critical points of other interesting functions, often with great success. Solutions are expected to be well behaved.² Upon further reflection, there is little reason to expect this. Not only does mathematics stray from physical reality,³ it often fails to satisfy purely mathematical intuition.⁴

The problem of packing objects in containers most efficiently, with maximum density or volume fraction, is easy to conceptualize. The density function can often be considered locally, among some finite collection of objects, and the density of a particular packing is easy to estimate. Still, finding general solutions is extremely hard. A modern motivation is found in Hilbert's 18th, from *Mathematische Probleme* [New02], regarding dense configurations:

I point out the following question, related to the preceding one, and important to number theory and perhaps sometimes useful to physics and chemistry: How can one arrange most densely in space an infinite number of equal solids of given form, e.g., spheres with given radii or regular tetrahedra with given edges (or in prescribed position), that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as great as possible?

¹e.g. energies

²e.g. exhibit exceptional symmetry

³Take for example, the blow up of solutions to various PDEs or the existence of non-measurable sets.

⁴Take for example, Smale's sphere eversion [Lev95] or the failure of Keller's conjecture [LS92].

Aristotle claimed that the regular tetrahedron tiled \mathbb{R}^3 [LZ12], and many mathematicians are still surprised that it does not. After more than two millennia, the first explicit upper bound for tetrahedra appeared in 2011 [GEK11].

Theorem I.A.1 (V. Elser, S. Gravel and Y. Kallus). *The maximum packing density of the regular tetrahedron in \mathbb{R}^3 is less than $1 - 2.6 \times 10^{-25}$.*

In 1611, Kepler conjectured that the cannonball packing was the densest packing of spheres [Kep11]. The problem remained open nearly 400 years, until 1998 [Hal05]. In fact, the sphere is only the third non-trivial class of convex bodies in \mathbb{R}^3 for which a sharp upper bound has been found. The first class was based on infinite circular cylinders [BK90] and the second class was based on truncated rhombic dodecahedra [Bez94].⁵

Theorem I.A.2 (A. Bezdek and W. Kuperberg). *The maximum packing density of the bi-infinite circular cylinder in \mathbb{R}^3 is $\pi/\sqrt{12}$.*

Theorem I.A.3 (T. Hales). *The maximum packing density of the sphere in \mathbb{R}^3 is $\pi/\sqrt{18}$.*

At the time Hilbert posed his 23 problems, even the case of *circle packings in the plane* was still being resolved. The first sharp result for a general packing of \mathbb{R}^2 is attributed to Thue [Thu10],⁶ who claimed in an 1890 lecture:

Theorem I.A.4 (A. Thue). *The maximum packing density of the circle in \mathbb{R}^2 is $\pi/\sqrt{12}$.*

This is now a corollary to a celebrated result of L. Fejes Tóth [FT53], which proved that the densest packing of the plane by congruent centrally-symmetric bodies is achieved by a lattice packing.

Theorem I.A.5 (L. Fejes Tóth). *The maximum packing density of a centrally-symmetric domain in \mathbb{R}^2 is the packing density in a circumscribing, centrally-symmetric hexagon.*⁷

Packings of the plane by non-centrally-symmetric bodies are still not understood. A new bound for regular pentagons, a body that serves as a toy model for harder cases, appeared

⁵This is not really a distinct class, as it is corollary of the sphere packing result. However, it was initially proved by an independent method.

⁶There is some debate as to the content of the initial version of Thue's proof.

⁷The analogous result is not true in higher dimensions. There are explicitly constructed packings of \mathbb{R}^3 by ellipsoids and elliptical cylinders with greater density than their densest lattice packing [BK91], [Wil91].

in 2013 [MdOFV13], but it is still quite far from the conjectured maximum density of $(5 - \sqrt{5})/3 = 0.921311\dots$

Theorem I.A.6 (F. Mário de Oliveira Filho, F. Valentin). *The maximum packing density of the regular pentagon in \mathbb{R}^2 is less than 0.98103\dots*

In dimensions four and greater, almost nothing is known. As sphere packings have been the most studied, there are non-trivial upper and lower bounds in terms of dimension, but they differ significantly, see for example [CE03], [TV91]. There are no general constructions known that achieve even weak versions of the lower density bound. It is conjectured that the densest sphere packings very high dimensions are disordered [SST08].

For more on the history of packing, see for example [BMP05], [Wea99].

B. FRAMEWORK

This document is composed of a series of articles in discrete geometry, each solving a problem in packing density. The articles are related but independent. The first proves a local upper bound for the packing density of regular pentagons in \mathbb{R}^2 . By reducing a nonlinear programming problem to a linear one, computational methods show that the conjectured global optimal solution is locally optimal. The second proves an upper bound for the packing density of finite cylinders in \mathbb{R}^3 . Using a measure theoretic approach to estimate boundary error, the first bound that is asymptotically sharp with respect to the length of the cylinder is found. This gives the first sharp upper bound for the packing density of half-infinite cylinders as a corollary. The third proves an upper bound for the packing density of infinite polycylinders in \mathbb{R}^n . Using transversality and a dimension reduction argument, an existing result for \mathbb{R}^3 is applied to \mathbb{R}^n . This gives the first non-trivial sharp upper bound for the packing density of any object in dimensions four and greater.

The main content is found in [Chapter II](#), [Chapter III](#) and [Chapter IV](#). [Chapter V](#) contains a series of remarks and conjectures. Some material is separated and appended. [Appendix A](#) contains code related to [Chapter II](#). [Appendix B](#) contains computations related

to [Chapter II. Appendix C](#) contains a remark on the properties of the density function. Associated libraries and notebooks are maintained by D-Scholarship@Pitt.

C. GENERAL NOTATION

Throughout this document definitions and notation may conflict across articles, except in their broadest interpretations. However, each article has explicit local definitions and is internally consistent. Generally, script letters refer to families of objects, italic capital letters refer to objects or sets and italic lowercase letters refer to sub-objects or elements.

A packing, in the most general sense, is a domain and a collection of subdomains with disjoint interiors.

Definition I.C.1. A *packing of $K \subseteq \mathbb{R}^n$ by C* is a countable family $\mathcal{C} = \{C_i\}_{i \in I}$ of Euclidean congruent⁸ bodies C_i with mutually disjoint interiors and $C_i \subseteq K$.

One can ask for the “best” or “most efficient packing”; this is taken to mean the densest packing. Of course, the question may not be well formed. It is not obvious that a packing has an associated density.⁹

For a packing \mathcal{C} of a finite volume region K , the density $\rho(\mathcal{C})$ is simply the volume fraction

$$\rho(\mathcal{C}) = \frac{\text{Vol}(\mathcal{C})}{\text{Vol}(K)}$$

which is well defined for measurable sets. In the case of a packing \mathcal{C} in \mathbb{R}^n , the density $\rho_K(\mathcal{C})$ is the limiting density after exhaustion by convex bodies rK , as r tends to infinity,

$$\rho_K(\mathcal{C}) = \lim_{r \rightarrow \infty} \frac{\text{Vol}(\mathcal{C} \cap rK)}{\text{Vol}(rK)}.$$

This limit might not exist or might depend on the choice of K . For a planar packing where concentric annuli are packed with different densities $\rho_1 > \rho_2$ as illustrated in [Figure 1\(a\)](#), the density ρ_{Ball} may oscillate between $\rho_1 - \epsilon$ and $\rho_2 + \epsilon$. For a planar packing where alternating quadrants are packed with different densities $\rho_1 > \rho_2$, with density ρ_1 in quadrants I and III

⁸or larger symmetry groups, e.g. dilations, symplectomorphisms or volume-preserving transformations

⁹e.g. with respect to Lebesgue measure

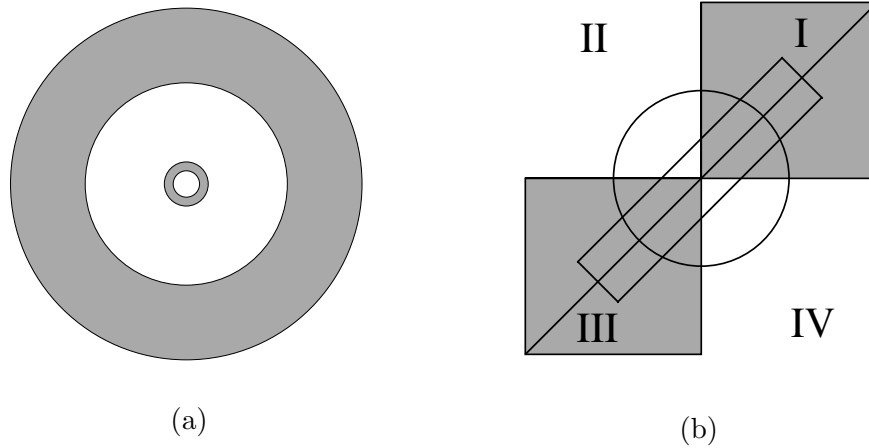


Figure 1: A packing (a) with oscillating packing density. (b) with isotropic packing density.

and ρ_2 in quadrants II and IV, as illustrated in Figure 1(b), the density may depend on the choice of K . For example, $\rho_{Ball} = (\rho_1 + \rho_2)/2$ where as $\rho_{Rectangle} = \rho_1 - \epsilon$.

For most purposes it is enough to consider the **upper density** taken with respect to a ball

$$\delta^+(\mathcal{C}) = \limsup_{r \rightarrow \infty} \frac{\text{Vol}(\mathcal{C} \cap r\mathbb{B})}{\text{Vol}(r\mathbb{B})}$$

or the **maximal packing density** of C over all packings \mathcal{C}

$$\delta^+(C) = \max_{\mathcal{C}} \limsup_{r \rightarrow \infty} \frac{\text{Vol}(\mathcal{C} \cap r\mathbb{B})}{\text{Vol}(r\mathbb{B})}.$$

See Radin [Rad04] or Conway Goodman-Strauss and Sloane [CGSS99] for further discussion of the subtleties of the “best” packings and density.

II. PENTAGON PACKINGS IN THE PLANE

A. INTRODUCTION TO PENTAGON PACKINGS

Finding the densest packing of regular pentagons in the plane is still an open problem. The local analysis of this problem is also important, as experimental evidence [DGT95], [SPMF05], [DRŽ14] indicates pentagon packings exhibit a variety of behaviors near conjectured density-critical configurations.

The best lower bound for the density of pentagon packings and the conjectured maximal density configuration is shown in Figure 2. This packing has a density of $(5 - \sqrt{5})/3 = 0.921311\dots$. Only recently has a reasonable upper bound of $0.98103\dots$ been produced as a corollary to a more general method [MdOFV13], where pentagons serve as an archetype for general non-centrally-symmetric figures.

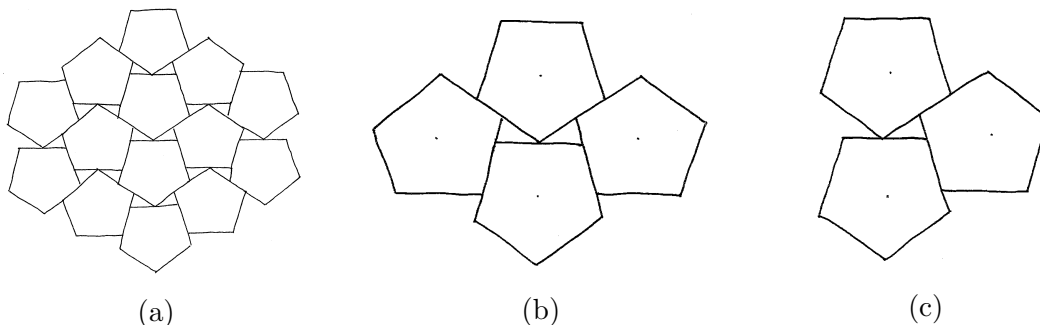


Figure 2: Part of the conjectured densest packing (a) of regular pentagons, \mathcal{P}^* . The configuration (b) of four pentagons \mathcal{P}_4^* . The configuration (c) of three pentagons \mathcal{P}_3^* .

The pentagon packing problem serves as a toy model for other packing problems. Fejes Tóth [FT53] proved that the densest packing of the plane by congruent centrally-symmetric

bodies is achieved by a lattice packing. This is certainly not the case for non-centrally-symmetric bodies.¹

The regular pentagon is the simplest non-centrally-symmetric figure which does not tile. The fact that its interior angle is incommensurate with the circle is analogous to higher-dimensional problems with tetrahedral packings. Packings by other regular $2n + 1$ -gons show similar behavior. The packing of pentagons is well-studied, both experimentally and mathematically. Its packing density with respect to doubly-periodic packings is understood [KK90] and the conjectured optimal configuration is known to be optimal when restricted to that class of packings. Thus, the local behavior of a packing of regular pentagons is reasonable to investigate.

B. SLICING NONLINEAR PROGRAMS

The method presented in this section is motivated by the problem of finding the densest packing of regular pentagons in the plane. Here, a general problem is described locally near the conjectured global optimal by a nonlinear programming problem and can be certified as optimal by a linear programming problem. In Section II.C, the method is used to prove the local optimality of \mathcal{P}^* , the conjectured globally optimal pentagon packing.

This procedure compliments one described in A. Solovyev’s dissertation. Solovyev implements a numerical method for proving linear programming bounds of the form $c^T x \leq K$ for problems

$$\max_{x \in \mathbb{R}^n} c^T x \text{ subject to } Ax \leq b.$$

However, there is a requirement that the inequality $\max_{x \in \mathbb{R}^n} c^T x := M < K$ be strict [Sol12, §3.1], to account for numerical error. This means that, while global linear programming bounds can be proved, a specific value cannot be certified as critical. For an introduction to linear and nonlinear programming bounds, see for example [Dan65].

For the geometric problems considered, there are *a priori* configurations that are con-

¹e.g. equilateral triangles tile the plane, but not as a lattice packing.

tured to be maximal and an assumption that objective and constraint functions are analytic² in a neighborhood of a conjectured optimal point in the configuration space. To address the pentagon problem, there are additional assumptions included to simplify the analysis.

To produce a certificate of local optimality for this type of problem, the procedure is to slice a special type of nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0, r \in I$$

in a neighborhood of 0, with respect to a specially chosen parametrization, giving a family of programs that are subordinate to the linearization of the main program at 0. The following assumptions are required.

Assumptions II.B.1.³

1. Let I be a finite index set.
2. Let e_1 be the standard unit vector $\{1, 0, \dots, 0\}$ in \mathbb{R}^n .
3. For r in I , let f and g_r be analytic functions on a neighborhood of 0.
4. Assume $f(0) = g_r(0) = 0$ for all r in I .
5. Let $F(t) = \nabla f(te_1)$.
6. Let $G_r(t) = \nabla g_r(te_1)$.
7. Assume the linear program

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

has a bounded solution and that the maximum is attained at 0.

8. Assume that the set of solutions in \mathbb{R}^n to

$$F(0) \cdot x = 0 \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is

$$E := \{te_1 : t \in \mathbb{R}\}.$$

9. Let H be the orthogonal complement of E so that $\mathbb{R}^n = E \oplus H$.

²Weaker regularity conditions should suffice, i.e. C^1 or locally Lipschitz or subdifferentiable functions.

³These are the assumptions that are required for the pentagon packing problem. There are a number of ways they might be weakened, e.g. the condition that E be 1-dimensional is not essential.

10. Assume there is an $\epsilon > 0$ so the functions $g_r(te_1) = 0$ for all $t \in (-\epsilon, \epsilon)$, for all r in I .
11. Assume $\frac{\partial}{\partial t} f(0) = 0$, $\frac{\partial^2}{\partial t^2} f(0) < 0$.

Lemma II.B.1. Given *Assumptions II.B.1*, the linear program

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

has a unique maximum at $x = 0$

Proof. By assumptions 7 and 8, the linear program

$$\max_{x \in \mathbb{R}^n} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on E . The feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I \text{ and } x \in H\}$ is a subset of the feasible set $\{x : G_r(0) \cdot x \geq 0, r \in I\}$. Thus, the program

$$\max_{x \in H} F(0) \cdot x \text{ subject to } G_r(0) \cdot x \geq 0, r \in I$$

is maximized exactly on the (non-empty) intersection

$$E \cap \{x : G_r(0) \cdot x \geq 0, r \in I\} \cap H = 0.$$

□

Definition II.B.1. A *finitely generated cone* is a subset of \mathbb{R}^n which is the non-negative span of a finite set of non-zero vectors $\{v_1, \dots, v_m\}$ in \mathbb{R}^n , which are called the **generators** of the cone.

Definition II.B.2. A *conical linear program* is a linear program with a constraint set that is a finitely generated cone.

The linear programs described throughout this section are always constrained to be on the intersection of half-spaces with 0 on the boundary. These are conical programs.

Definition II.B.3. For a cone C , the set $C^p := \{x \in \mathbb{R}^n : v \cdot x \leq 0 \text{ for all } v \in C\}$ is the *polar cone* of C .

Lemma II.B.2. *A conical linear program with $F \neq 0$ given by*

$$\max_{x \in \mathbb{R}^n} F \cdot x \text{ subject to } G_r \cdot x \geq 0, r \in I$$

(a) *has a unique⁴ maximum at $x = 0$ iff F is in the interior of the polar cone C^p of $C = \{x : G_r \cdot x \geq 0, r \in I\}$ (b) has a bounded solution iff F is in the polar cone C^p of $C = \{x : G_r \cdot x \geq 0, r \in I\}$ and attains its maximum exactly on the span of the generators v_i such that $F \cdot v_i = 0$.*

Proof. If F is in the interior of the polar cone C^p , then $F \cdot v_i < 0$ for all generators v_i . Therefore $F \cdot x$ is uniquely maximized in C at the vertex. If F is on the boundary of the polar cone, then $F \cdot x$ is maximized in C exactly on the span of the generators v_i for which $F \cdot v_i = 0$ as $F \cdot v_j < 0$ otherwise. If F is outside the polar cone, then $F \cdot v_i > 0$ for some generator v_i . Then $F \cdot x$ is unbounded in C . \square

Lemma II.B.3. *Given [Assumptions II.B.1](#), there exists $\epsilon > 0$ such that for all t in $(-\epsilon, \epsilon)$, the linear program*

$$\max_{y_t \in H} F(t) \cdot y_t$$

subject to

$$G_r(t) \cdot y_t \geq 0, r \in I$$

has a unique maximum at $y_t = 0$.⁵

Proof. The program for $t \in (-\epsilon, \epsilon)$, for y_t in H , for each fixed t in $(-\epsilon, \epsilon)$, for some $\epsilon > 0$, can be written as a conical program on all of \mathbb{R}^n with a cone C_t in \mathbb{R}^n of co-dimension ≥ 1 by introducing further constraints $e_1 \cdot y_t \geq 0$ and $-e_1 \cdot y_t \geq 0$. By [Lemma II.B.1](#) and [Lemma II.B.2](#), $F(0)$ is in the polar cone of $C_0 = \{y_0 : G_r(0) \cdot y_0 \geq 0, e_1 \cdot y_0 \geq 0, -e_1 \cdot y_0 \geq 0\}$. As $f, g_r \in C^\omega$, the condition of $F(t)$ being in the interior of the polar cone C_t^p is open and the condition of the feasible set $C_t = \{y_t : G_r(t) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0\}$ being conical is open.⁶ Therefore, by [Lemma II.B.2](#) the program has a unique maximum at $y_t = 0$ for each fixed t in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. \square

⁴The maximum satisfies a stronger uniqueness condition. It is stable under perturbations of F and G_k .

⁵Here y_t is a dummy variable and does not depend on t . It is labeled y_t to ease later exposition.

⁶The relationships between the constraint cone, the generators v_i and the constraint gradients G_k is subtle, but the condition being open essentially follows from the continuity of the distance function.

Lemma II.B.4. Given *Assumptions II.B.1* and ϵ as in *Lemma II.B.3*, for all $t \in (-\epsilon, \epsilon)$ there exists $\delta(t) > 0$ and a cube $Q(t) \subset \mathbb{R}^n$ of side length $2\delta(t)$ such that

$$\{(F(t) + Q(t)) \cap (\partial(C_t^p) + Q(t))\} = \emptyset.$$

Proof. This follows from *Lemma II.B.3*, which shows $F(t)$ is in the interior of the polar cone C_t^p . Then $F(t)$ and the boundary of C_t^p can be separated and the existence of Q is trivial. \square

Corollary II.B.1. Given *Assumptions II.B.1* and ϵ as in *Lemma II.B.3*, for all $t \in (-\epsilon, \epsilon)$,

$$(F(t) + \Delta) \cdot y_t \leq 0$$

whenever y_t satisfies

$$(G_r(t) + \Delta_r) \cdot y_t \geq 0, r \in I \text{ and } e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0$$

where Δ and Δ_r are any points in the $2\delta(t)$ -cube $Q(t)$ and y_t is in H .

Proof. By *Lemma II.B.4*, $F(t) + \Delta$ is in the interior of the polar cone $C_{t,\Delta}^p$, where $C_{t,\Delta} = \{y_t : (G_r(t) + \Delta_r) \cdot y_t \geq 0, e_1 \cdot y_t \geq 0, -e_1 \cdot y_t \geq 0, r \in I\}$. \square

Lemma II.B.5. Given *Assumptions II.B.1* and ϵ as in *Lemma II.B.3*, for all $t \in (-\epsilon, \epsilon)$, let $y_t = x - te_1 \in H$. Choose $\Delta = \Delta(y_t)$ and $\Delta_r = \Delta_r(y_t)$ in the $2\delta(t)$ -cube $Q(t)$ to be the corner given by the sign of $x - te_1 = y_t$. Then there is an ϵ_t for which

$$(F(t) + \Delta(y_t)) \cdot y_t \leq 0 \implies f(x) - f(te_1) \leq 0$$

and

$$(G_r(t) + \Delta_r(y_t)) \cdot y_t \leq 0 \implies g_r(x) - g_r(te_1) = g_r(x) \leq 0$$

for all $\|y_t\| \leq \epsilon_t$.

Proof. This follows from the local expansions of the nonlinear program. By this choice of $\Delta(y_t)$ and $\Delta_r(y_t)$,

$$\begin{aligned} f(x) - f(te_1) &= F(t) \cdot (x - te_1) + O(t^2) = F(t) \cdot y_t + O(t^2) \\ &\leq F(t) \cdot y_t + \delta(t)\|y_t\|_1 = (F(t) + \Delta(y_t)) \cdot y_t \end{aligned}$$

and using assumption 10,

$$\begin{aligned} g_r(x) - g_r(te_1) &= G_r(t) \cdot (x - te_1) + O(t^2) = G_r(t) \cdot y_t + O(t^2) \\ &\leq G_r(t) \cdot y_t + \delta(t)\|y_t\|_1 = (G_r(t) + \Delta_r(y_t)) \cdot y_t. \end{aligned}$$

□

By [Lemma II.B.4](#) and [Corollary II.B.1](#), for t in $(-\epsilon, \epsilon)$, the program

$$\max_{y_t \in H} (F(t) + \Delta) \cdot y_t \text{ subject to } (G_r + \Delta_r) \cdot y_t$$

is uniquely maximized at $y_t = 0$ for any choice of Δ, Δ_r in the $2\delta(t)$ cube $Q(t)$. Combined with [Lemma II.B.5](#), there is an ϵ_t neighborhood of 0 where $f(y_t + te_1)$ is less than $f(te_1)$ on $\cup_{\Delta_r \in Q(t)} \{y_t : (G_r + \Delta_r) \cdot y_t \geq 0, r \in I, y_t \in H\}$, which contains the feasible set $\{y_t : g_r(y_t + te_1) \geq 0, r \in I, y_t \in H\}$. Therefore the nonlinear programs $f(y_t + te_1)$ subject to $g_r(y_t + te_1) \geq 0, y_t \in H$, which are parameterized by t in $(-\epsilon, \epsilon)$, have local maxima at $y_t = 0$. This gives the following:

Theorem II.B.1. *Given [Assumptions II.B.1](#), a fixed t in $(-\epsilon, \epsilon)$ and choosing Δ and Δ_r as in [Lemma II.B.5](#), for x satisfying $g_r(x) \geq 0$ for all r in I and $y_t = x - te_1$ in H , there exist linear programs⁷*

$$\max_{y_t \in H} (F(t) + \Delta(y_t)) \cdot y_t \text{ subject to } (G_r(t) + \Delta_r(y_t)) \cdot y_t \geq 0$$

that give solutions to the nonlinear programs

$$\max_{x \in H + te_1} f(x) \text{ subject to } g_r(x) \geq 0$$

in an ϵ_t neighborhood of te_1 in $H + te_1$. □

⁷These programs may depend on a choice of $y_t \in H$, but $f(x)$ is always less than $f(te_1)$ by [Lemma II.B.5](#).

By choice of a sufficiently small ϵ and a minimal⁸ non-zero ϵ_t , [Theorem II.B.1](#) gives an open neighborhood of 0 in which the maximum value of the original nonlinear program occurs on E . The assumptions for the first and second t -derivatives at 0 shows 0 to be a local maximum for the nonlinear program

$$\max_{x \in \mathbb{R}^n} f(x) \text{ subject to } g_r(x) \geq 0.$$

Theorem II.B.2. *A nonlinear program satisfying [Assumptions II.B.1](#) has an isolated local maximum at 0 with $f(0) = 0$.* □

C. PENTAGON PACKING

Using the technique described in [Section II.B](#), the conjectured optimal configuration is verified to be locally optimal. It is a local maximum for the packing density in the configuration space of four pentagons with respect to a Delaunay triangulation.

Definition II.C.1. *Given a set of points S in the plane, a **Delaunay triangulation** $DT(S)$ is a triangulation of S such that no point of S is in the circumcircle of any triangle in $DT(S)$.*

Definition II.C.2. *A **pentagon packing** is a countable family*

$$\mathcal{P} = \{P_i\}_{i \in I}$$

of congruent regular pentagons $P_i \subset \mathbb{R}^2$ with mutually disjoint interiors. Let \mathcal{P}^ be the conjectured optimal packing of the plane.*

Definition II.C.3. *A Delaunay triangulation $DT(\mathcal{P})$ of a pentagon packing \mathcal{P} is a Delaunay triangulation of $DT(S)$ where S is the set of centers of pentagons in \mathcal{P} .*

Definition II.C.4. *The **upper density** $\delta^+(\mathcal{P})$ of a pentagon packing \mathcal{P} is defined to be*

$$\delta^+(\mathcal{P}) = \limsup_{r \rightarrow \infty} \frac{\text{Area}(\mathcal{P} \cap r\mathbb{B}^2)}{\text{Area}(r\mathbb{B}^2)}$$

where \mathbb{B}^2 is the open unit Euclidean ball.

⁸This exists by a compactness argument.

Using a Delaunay triangulation $DT(\mathcal{P})$ obtained from a pentagon packing \mathcal{P} , a bound on density of the packing may be computed as

$$\delta^+(\mathcal{P}) \leq \sup_{T \in DT(\mathcal{P})} \frac{\text{Area}(T \cap \mathcal{P})}{\text{Area}(T)}.$$

In the case of \mathcal{P}^* , symmetry gives the density as

$$\delta^+(\mathcal{P}^*) = \frac{\text{Area}(T \cap \mathcal{P}^*)}{\text{Area}(T)}$$

for any T in $DT(\mathcal{P}^*)$.

Definition II.C.5. Let $FT(\mathcal{P})$ be the finite Delaunay triangles of a packing \mathcal{P} . Let $\mathcal{P}_4 = \{P_1, P_2, P_3, P_4\}$ be a packing of four regular pentagons in the plane. The **Delaunay density** of the packing is

$$\delta(\mathcal{P}_4) = \frac{\text{Area}(FT(\mathcal{P}_4) \cap \mathcal{P}_4)}{\text{Area}(FT(\mathcal{P}_4))}.$$

Let \mathcal{P}_4^* be four pentagons in \mathcal{P}^* configured as in [Figure 2\(b\)](#).

Proposition II.C.1. There is an open neighborhood U of \mathcal{P}_4^* in the configuration space of four regular pentagons in the plane P_1, P_2, P_3, P_4 where the Delaunay density $\delta(\mathcal{P}_4)$ is no greater than $\delta(\mathcal{P}^*) = (5 - \sqrt{5})/3 = 0.921311\dots$ for any packing \mathcal{P}_4 in U .

Remark II.C.1. Given [Proposition II.C.1](#), there exists $\epsilon > 0$, for which it is impossible to rescale the pentagons of \mathcal{P}^* by a factor of $1 - \epsilon_0$, perturb them by ϵ_1 , and rescale them by a factor greater than $1/(1 - \epsilon_0)$ for any ϵ_0 and ϵ_1 where $0 \leq \epsilon_0 < \epsilon$ and $0 \leq \epsilon_1 < \epsilon$.

Proof. Consider \mathcal{P}^* as a tiling by $\mathcal{P}_4^* \cap FT(\mathcal{P}_4^*)$. A perturbation as described would increase the density of the packing and all the tiles would be close to \mathcal{P}^* in the configuration space of four pentagons. If such a perturbation were possible for all ϵ , one could find perturbed tiles where four pentagons were arbitrarily close to \mathcal{P}_4^* in the configuration space of four pentagons and having Delaunay density greater than \mathcal{P}_4^* . This contradicts [Proposition II.C.1](#). \square

To prove [Proposition II.C.1](#) via the methods of [Section II.B](#) the following procedure is used. First, the density problem is described as a nonlinear program $\max_{x \in \mathbb{R}^n} f(x)$ subject to $g_r(x) \geq 0$ where f and g_r are analytic on a neighborhood of 0 and $f(0) = g_r(0) = 0$. Then, using interval arithmetic, a certificate that the program satisfies the linear programming and second derivative conditions is generated.⁹ Finally, the geometric conditions of [Theorem II.B.1](#) are verified.¹⁰

All computations are performed in Mathematica 9 [[Wol12](#)], which supports precision and accuracy control as well as interval arithmetic.¹¹ The relevant code can be found in [Appendix A](#) and [Appendix B](#). Mathematica notebooks are maintained on D-Scholarship@Pitt.

D. NONLINEAR PROGRAM

This section describes the full nonlinear program¹² used to analyze [Proposition II.C.1](#). For the purposes of computation, it is necessary to work with coordinates and to introduce a variety of new functions. Some of these functions are unwieldy. Refer to the [Appendix A](#) for details.

1. Variables

Let $\mathcal{M}(\mathcal{P})$ be the configuration space of four pentagons modulo the Euclidean group. To satisfy the requirements of [Theorem II.B.1](#), the coordinate system introduced on $\mathcal{M}(\mathcal{P})$ is not the naive parameterization where all pentagons are independent, but rather a coupled system.

⁹The linear program at 0 is maximized at 0 and the t -derivative condition is satisfied.

¹⁰The solution set to the linear program is E .

¹¹Mathematica is a closed source program. It is not possible to independently verify the correctness of the code for precision and accuracy control or for interval arithmetic. Ideally, the analysis of these types of problems would be performed with bespoke code

¹² Attempts to solve the full nonlinear program crashed the Mathematica kernel on the machines available (PowerMac Dual 2GHz G5 with 6GB RAM running Mathematica 7 [[Wol08](#)], Macbook Pro 2.5 GHz Core i5 with 4GB RAM running Mathematica 9 [[Wol12](#)]). The set of C functions CFSQP [[LZT94](#)] designed for solving constrained nonlinear optimization problems was also used in a separate attempt to run the nonlinear program. This failed, likely due to a combination of user error, unoptimized code and lack of computational power.

Definition II.D.1. *The pentagons are labeled as follows:*

P_1 - upper center pentagon with center c_1 .

P_2 - lower center pentagon with center c_2 .

P_3 - right pentagon with center c_3 .

P_4 - left pentagon with center c_3 .

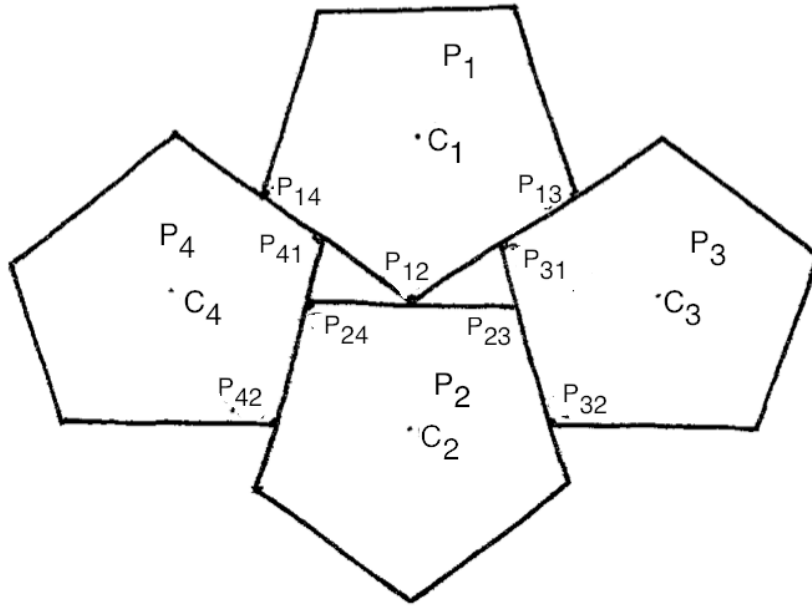


Figure 3: Pentagon labels.

Fix a coordinate system for the plane and fix a point 0 in $\mathcal{M}(\mathcal{P})$ at the configuration illustrated in Figure 3. There is a special parametrization $\{t_2, u_2, \theta_2, t_3, u_3, \theta_3, t_4, u_4, \theta_4\} = \mathbb{R}^9 \rightarrow \mathcal{M}(\mathcal{P})$ that will satisfy the conditions of Theorem II.B.1. First fix P_1 . Define a one-parameter family as illustrated in Figure 4 and parametrize it by the first coordinate t_2 . This gives the horizontal motion t_2 of the lower central pentagon P_2 , which maintains contact with P_1 , and the pentagons P_3 and P_4 are linearly displaced while also maintaining contact with P_1 and P_2 . The centers of the pentagons P_1, P_2 and P_3 in this one-parameter family are non-trivial functions of t_2 .

There is a neighborhood of the one-parameter family consisting of variations in the horizontal components t_3 and t_4 of the centers of the right P_3 and left P_4 pentagons, the

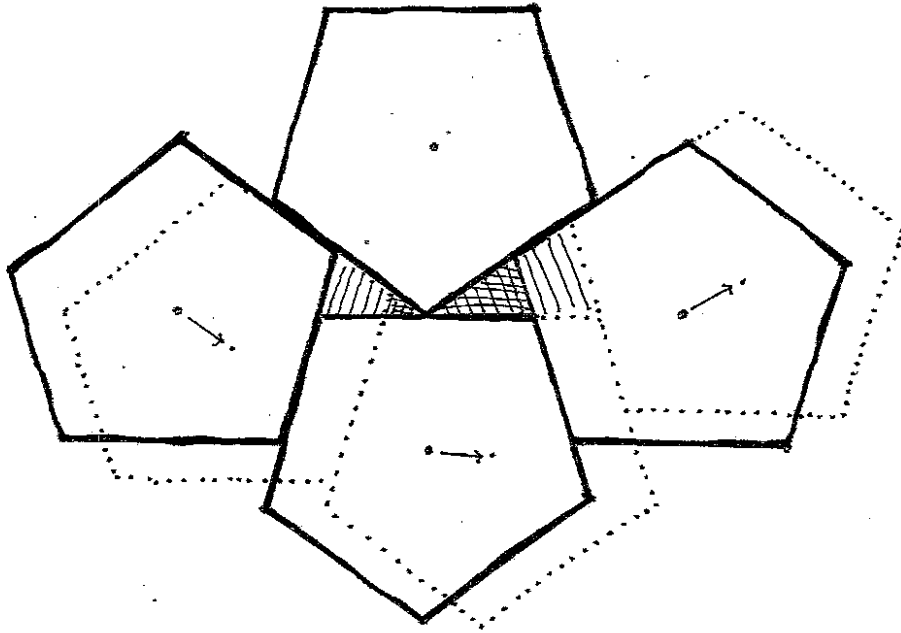


Figure 4: One-parameter family.

vertical components u_2, u_3, u_4 of the centers of the lower central P_2 , right P_3 and left P_4 pentagons, and the counter-clockwise rotations $\theta_1, \theta_2, \theta_3$ of the lower central P_2 , right P_3 and left P_4 pentagons.

In summary, t_2 describes the motion of a one-parameter family in $\mathcal{M}(\mathcal{P})$. The remaining variables $u_2, \theta_2, t_3, u_3, \theta_3, t_4, u_4, \theta_4$ give displacements from that one-parameter family in $\mathcal{M}(\mathcal{P})$. This gives a nine-dimensional local parametrization as in Figure 5. See Table 3 for the explicit construction.

2. Objective function

The objective function for the nonlinear program is defined in terms of the areas and corresponds¹³ to the density function on a neighborhood of \mathcal{P}_4^* . The constituent functions (ob_{ij}) are the areas of various triangular regions of the pentagons as illustrated in Figure 6

¹³It is shifted to be 0 at the conjectured optimum and “double counts” the area of pentagons when they overlap. However, it is constructed to share local maxima with the density function.

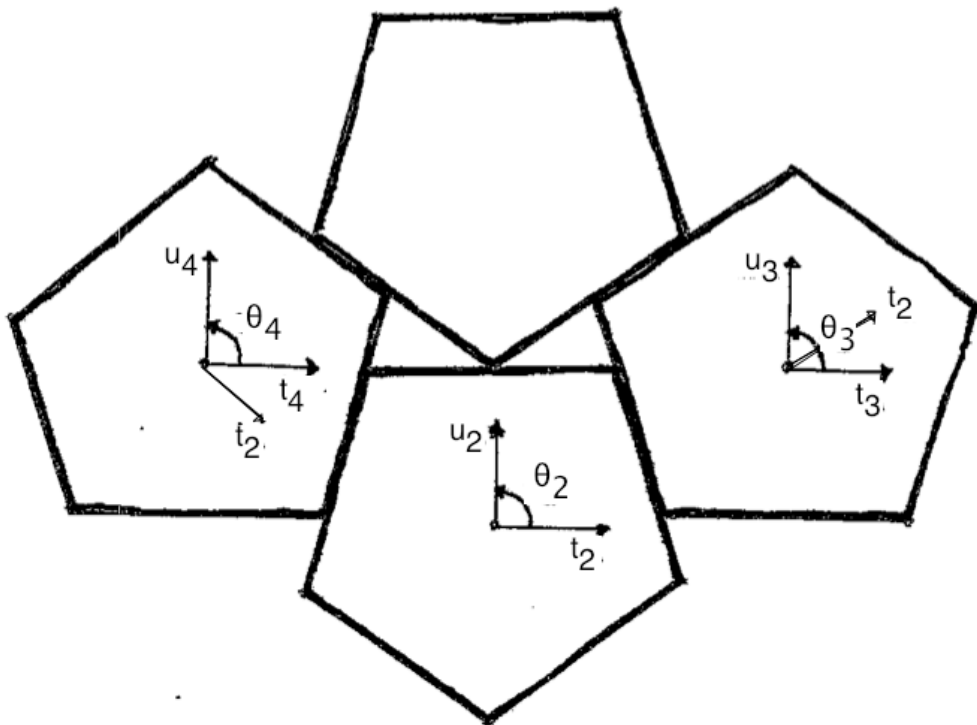


Figure 5: Perturbations about the one-parameter t_2 family.

and the area of the convex hull of $\{c_1, c_2, c_3, c_4\}$. The convex hull of $\{c_1, c_2, c_3, c_4\}$ corresponds to $FT(\mathcal{P})$. The objective function may be written as

$$f = OB = \sum_{i=1}^4 \frac{\text{Area}(FT(\mathcal{P}) \cap P_i)}{\text{Area}(FT(\mathcal{P}))} - \frac{5 - \sqrt{5}}{3}.$$

See [Table 5](#) for details. The objective function is analytic in a neighborhood of the origin in the configuration space, as it may be written in terms of polynomial and trigonometric functions and their inverses away from any singularities.

3. Constraint functions

The constraint functions for the nonlinear program are non-intersection conditions on the pentagons. Locally, it is sufficient to require that no vertex of a pentagon be in the interior of another. Let the vertex of P_i that is in contact with P_j be the p_{ij} . The constraint that

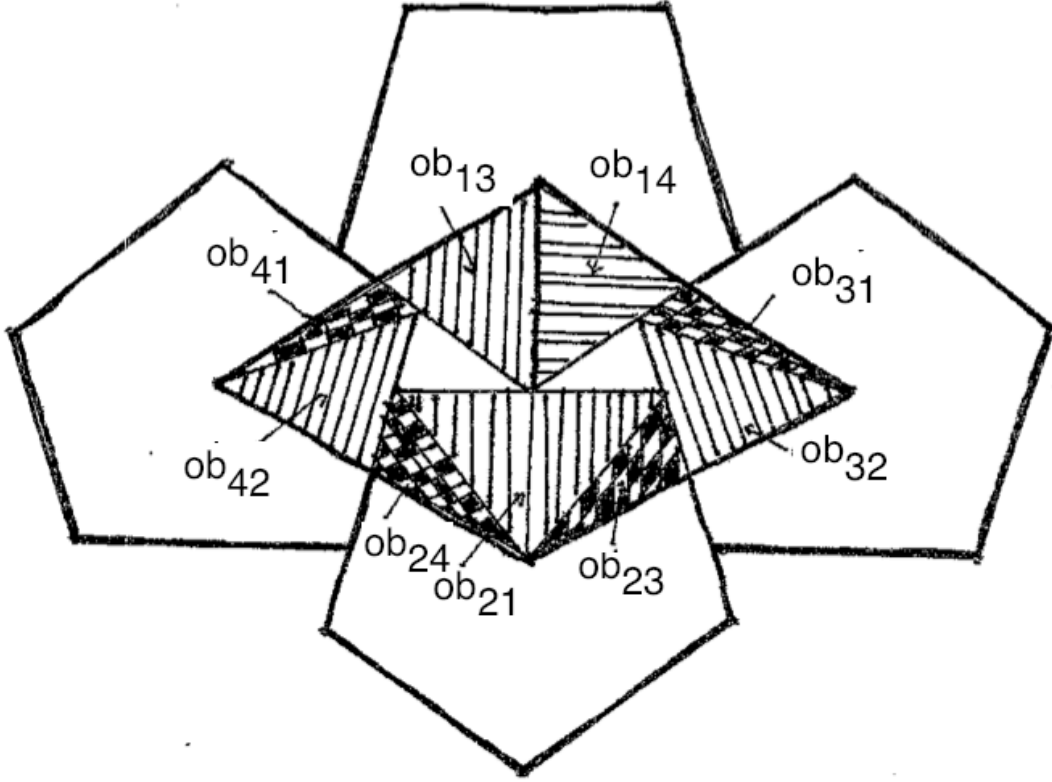


Figure 6: Constituent functions of the objective function.

a vertex of P_k does not lie in P_j may be considered as an angle constraint as illustrated in [Figure 7](#) and written as

$$g_r = \text{Con}_r = \text{Angle}\{c_i - p_{ij}, p_{ki} - p_{ij}\} - \frac{3\pi}{10} \geq 0$$

for appropriate choices of i, j, k in $\{1, 2, 3, 4\}$. See [Table 6](#) for details. The non-intersection constraints are analytic in a neighborhood origin in the configuration space as they may be written in terms of polynomial and trigonometric functions and their inverses, away from any singularities.

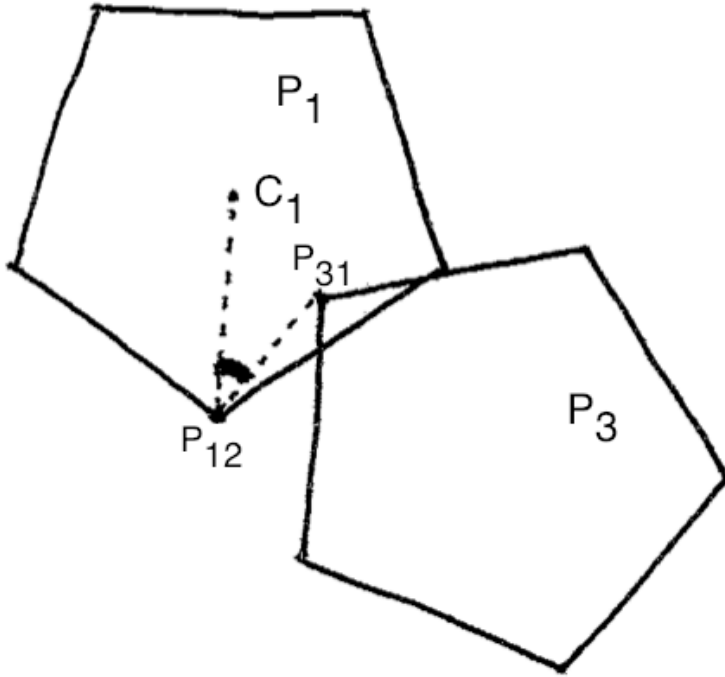


Figure 7: Violation of an angle constraint.

E. CERTIFICATE

The pentagon program meets the assumptions of [II.B](#). Using for the objective and constraint functions f and g_r as defined in [Subsection II.D.2](#) and [Subsection II.D.3](#), [Assumptions II.B.1](#) are satisfied.

1. I is finite by construction.
2. By an appropriate choice of basis where $t_2 = t$, the standard unit vector $e_1 = \{1, 0, \dots, 0\}$ is chosen in a manner consistent with the remaining assumptions.
3. The functions f and g_r are analytic functions on a neighborhood of 0 for all r in I .
4. The functions f and g_r satisfy $f(0) = g_r(0) = 0$ for all r in I by construction.
5. $F(t) = \nabla f(te_1)$ by construction.
6. $G_k(t) = \nabla g_k(te_1)$ by construction.
7. The assumptions that the linear program $\max_{x \in \mathbb{R}^n} F(0) \cdot x$ subject to $G_r(0) \cdot x \geq 0, r \in I$ has a bounded solution and the maximum occurs at 0 are proved by a numerical certificate

and a geometric argument. By geometric arguments,¹⁴ $F(0) \cdot e_1$ and $G_k(0) \cdot e_1$ are equal to 0 and it suffices to consider the program restricted to H . The numerical certificate (Appendix B) proves $F(0)$ is in the interior of the polar cone of the feasible set of the linear constraints restricted to H .

8. By a similar argument, the maximum of the linear program $\max_{x \in \mathbb{R}^n} F(0) \cdot x$ subject to $G_r(0) \cdot x \geq 0, r \in I$ is achieved exactly on E .
9. H is the orthogonal complement of E by construction.
10. From the construction of the constraints, $g_r(te_1) = 0$ for t in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$.
11. The condition that $\frac{\partial^2 f}{\partial t_2^2}(0) < 0$ is proved by the numerical certificate (Appendix B).

Therefore, Proposition II.C.1 holds.

Remark II.E.1. *The three pentagon configuration $\mathcal{P}_3^* = \{P_1, P_2, P_3\}$ as in Figure 2(c) is not a critical point for Delaunay density at the origin in the analogous 6-dimensional parametrization. See Figure 8. There exist packings \mathcal{P}_3 arbitrarily close to \mathcal{P}_3^* in the configuration space of three pentagons that have higher Delaunay density $\delta(\mathcal{P}_3)$, where*

$$\delta(\mathcal{P}_3) = \frac{\text{Area}(\mathcal{P}_3 \cap FT(\mathcal{P}_3))}{\text{Area}(FT(\mathcal{P}_3))}.$$

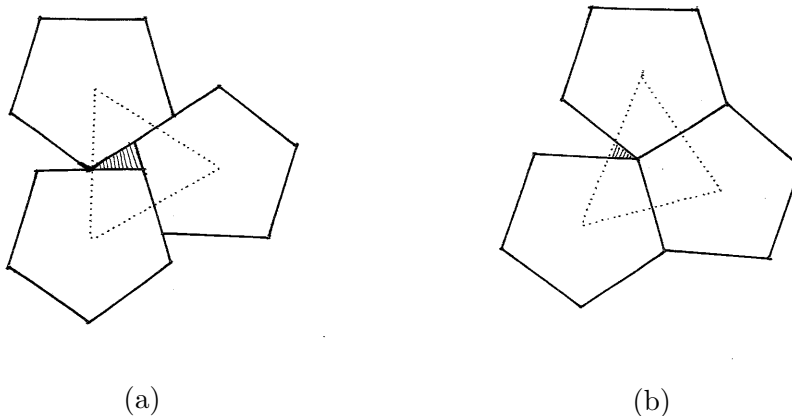


Figure 8: The (a) conjectured optimal (b) denser configuration of three pentagons.

¹⁴The objective function f is symmetric on E , that is $f(te_1) = f(-te_1)$, and thus $F(0) \cdot e_1 = 0$. The angle constraints g_k are constant on E , as the one-parameter t_2 family maintains edge-vertex contact for all the relevant vertices.

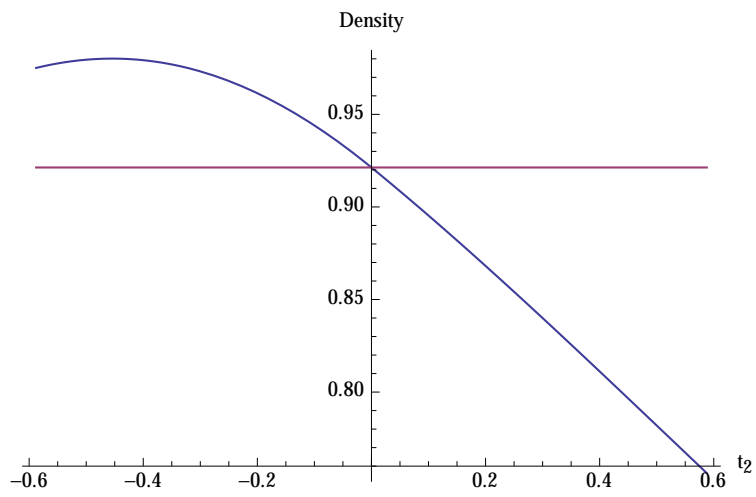


Figure 9: Delaunay density of three pentagons P_1, P_2, P_3 analogous one-parameter family.

III. CYLINDER PACKINGS

A. INTRODUCTION TO CYLINDER PACKINGS

This article¹ proves an upper bound for the packing density of congruent capped circular cylinders in \mathbb{R}^3 . Proved as corollaries are non-trivial upper bounds for packings by congruent circular cylinders, related objects, and the sharp bound for half-infinite circular cylinders.

1. Synopsis

The density bound of A. Bezdek and W. Kuperberg for bi-infinite cylinders is proved in three steps. Given a packing of \mathbb{R}^3 by congruent bi-infinite cylinders, first decompose space into regions closer to the axis of a particular cylinder than to any other. Such a region contains the associated cylinder, so density may be determined with respect to a generic region. Then this region can be sliced perpendicular to the particular axis. Finally, the area of these slices estimated: the area is always large compared to the cross-section of the cylinder.

In the case of a packing of \mathbb{R}^3 by congruent finite-length cylinders, this method fails. The ends of a cylinder may force some slice of a region to have small area. For example, if a cylinder were to abut another, a region in the decomposition might not even wholly contain a cylinder. To overcome this, one shows that these potentially small area slices are always associated to a small neighborhood of the end of a cylinder. For a packing by cylinders of a relatively high aspect ratio, neighborhoods of the end of a cylinder are relatively rare. By

¹The final publication [Kus14b] is available at Springer via <http://dx.doi.org/10.1007/s00454-014-9593-6>. Author retains the right to use his article for his further scientific career by including the final published journal article in other publications such as dissertations and postdoctoral qualifications provided acknowledgement is given to the original source of publication.

quantifying the rarity of cylinder ends in a packing, and bounding the error contributed by any particular cylinder end, the upper density bound is obtained.

2. Objects considered

Definition III.A.1. A ***t-cylinder*** is a closed solid circular cylinder in \mathbb{R}^3 with unit radius and length t .

Definition III.A.2. A ***capped t-cylinder*** (Figure 10) is a closed set in \mathbb{R}^3 composed of a t -cylinder with solid unit hemispherical caps.

Definition III.A.3. A capped t -cylinder C decomposes into the ***t-cylinder body*** C^0 and two ***caps*** C^1 and C^2 . The ***axis*** of the capped t -cylinder C is the line segment of length t forming the axis of C^0 .

The capped t -cylinder C is also the set of points at most 1 unit from its axis.

3. Packings and densities

Definition III.A.4. A ***packing of $X \subseteq \mathbb{R}^3$ by capped t -cylinders*** is a countable family $\mathcal{C} = \{C_i\}_{i \in I}$ of congruent capped t -cylinders C_i with mutually disjoint interiors and $C_i \subseteq X$.

Definition III.A.5. For a packing \mathcal{C} of \mathbb{R}^3 , the ***restriction*** of \mathcal{C} to $X \subseteq \mathbb{R}^3$ is defined to be a packing of \mathbb{R}^3 by capped t -cylinders $\{C_i : C_i \subseteq X\}$.

Let $B(R)$ be the closed ball of radius R centered at 0. In general, let $B_x(R)$ be the closed ball of radius R centered at x .

Definition III.A.6. The ***density*** $\rho(\mathcal{C}, R, R')$ of a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders with $R \leq R'$ is defined as

$$\rho(\mathcal{C}, R, R') = \sum_{C_i \subseteq B(R)} \frac{\text{Vol}(C_i)}{\text{Vol}(B(R'))}.$$

Definition III.A.7. The ***upper density*** ρ^+ of a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders is defined as

$$\rho^+(\mathcal{C}) = \limsup_{R \rightarrow \infty} \rho(\mathcal{C}, R, R).$$

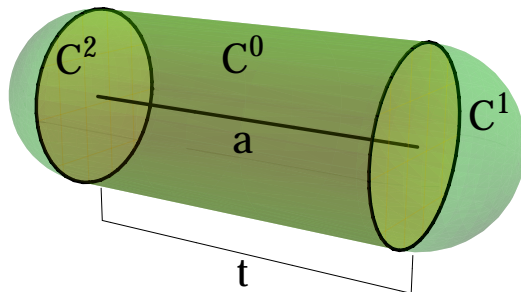


Figure 10: A capped t -cylinder with body C^0 , axis a and caps C^1 and C^2 .

In general,

Definition III.A.8. *a packing of $X \subseteq \mathbb{R}^3$ by a convex body K is a countable family $\mathcal{K} = \{K_i\}_{i \in I}$ of congruent copies of K with mutually disjoint interiors and $K_i \subseteq X$.*

Restrictions and densities of packings by K are defined in an analogous fashion to those of packings by capped t -cylinders.

4. Comparison to existing bounds

The only other bounds for circular cylinders and capped cylinders of finite length are given by G. Fejes Tóth and W. Kuperberg [FTK93], which may be stated as follows.

Theorem III.A.1 (Fejes Tóth and Kuperberg). *Fix a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders.*

Then

$$\rho^+(\mathcal{C}) \leq \frac{3t + 4}{3t \frac{(29-16\sqrt{2})}{6} + 4 \frac{(25-16\sqrt{2})}{2}}.$$

Theorem III.A.2 (Fejes Tóth and Kuperberg). *Fix a packing \mathcal{C} of \mathbb{R}^3 by t -cylinders. Then*

$$\rho^+(\mathcal{C}) \leq \frac{t}{(t-2) \frac{(29-16\sqrt{2})}{6} + \frac{4}{3} \frac{(25-16\sqrt{2})}{2}}.$$

These bounds arise as special cases of a general bound for outer parallel bodies and are explicitly computed in [FTK93] as important cases. For t -cylinders, the bound becomes non-trivial for lengths greater than $8.735\dots$ and gives an asymptotic density bound of $0.941\dots$

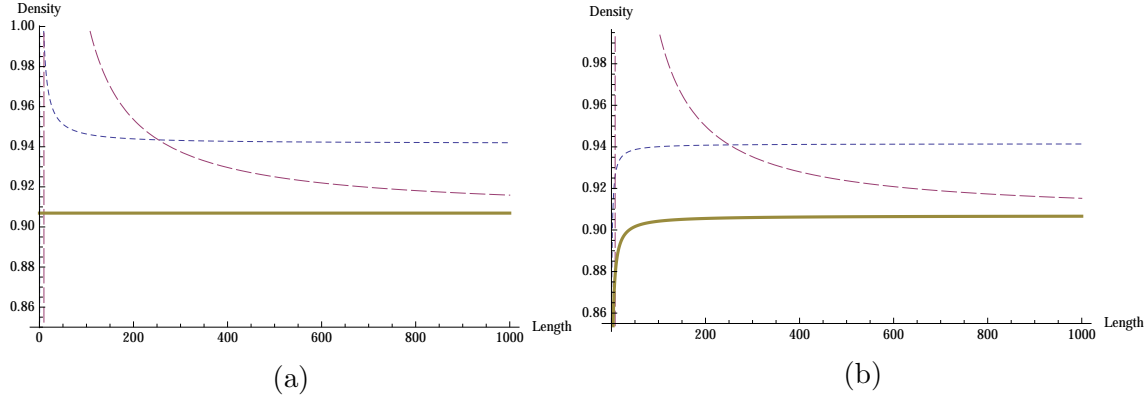


Figure 11: Plot of upper bounds on density of unit radius (a) t -cylinders (b) capped t -cylinders relative to their length.

Blue: W. Kuperberg and G. Fejes Tóth.

Purple: New bound.

Yellow: Conjectured bound.

The bound for capped t -cylinders is similar, giving Blichfeldt's bound of $0.842\dots$ for spheres [Bli29] at length 0 and rapidly approaching $0.941\dots$

The new bounds presented in this paper become non-trivial, i.e. less than 1, for t -cylinders of length greater than $105.147\dots$ and capped t -cylinders of length $96.653\dots$. Both bounds are asymptotic to the known sharp bound of $\pi/\sqrt{12}$ for infinite cylinders. The new bounds for cylinders improve the existing bound for t -cylinders of length greater than $252.751\dots$ and capped t -cylinders of length greater than $250.751\dots$, both very close to where the bounds of Fejes Tóth and Kuperberg flattens out. In this sense, the new and existing bounds are complementary, as illustrated in Figure 11.

B. MAIN RESULTS

Let $t_0 = \frac{4}{3}(\frac{4}{\sqrt{3}} + 1)^3 = 48.3266786\dots$ for the remainder of the paper. This value comes out of the error analysis in Section III.E.

Theorem III.B.1. *Fix $t \geq 2t_0$. Fix $R \geq 2/\sqrt{3}$. Fix a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders.*

Then

$$\rho(\mathcal{C}, R - 2/\sqrt{3}, R) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + 2t_0 + \frac{4}{3}}.$$

This is the content of [Section III.C](#), [Section III.D](#), [Section III.E](#). Note that this upper bound is superseded by the trivial bound of 1 when $t \leq 2t_0$.

Corollary III.B.1. *Fix $t \geq 2t_0$. The upper density of a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders satisfies the inequality*

$$\rho^+(\mathcal{C}) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + 2t_0 + \frac{4}{3}}.$$

Proof. Let V_R and W_R be subsets of the index set I , with $V_R = \{i : C_i \subseteq B(R)\}$ and $W_R = \{i : C_i \subseteq B(R - 2/\sqrt{3})\}$. By definition,

$$\rho^+(\mathcal{C}) = \limsup_{R \rightarrow \infty} \left(\sum_{W_R} \frac{\text{Vol}(C_i)}{\text{Vol}(B(R))} + \sum_{V_R \setminus W_R} \frac{\text{Vol}(C_i)}{\text{Vol}(B(R))} \right).$$

As R grows, the term $\sum_{V_R \setminus W_R} \text{Vol}(C_i)/\text{Vol}(B(R))$ tends to 0. Further analysis of the right-hand side yields

$$\rho^+(\mathcal{C}) = \limsup_{R \rightarrow \infty} \rho(\mathcal{C}, R - 2/\sqrt{3}, R).$$

By [Theorem III.B.1](#), the stated inequality holds. \square

Lemma III.B.1. *Given a packing of t -cylinders with density ρ where t is at least 2, there is a packing of capped $(t - 2)$ -cylinders with packing density $(\frac{t - \frac{2}{3}}{t}) \cdot \rho$.*

Proof. From the given packing of t -cylinders, construct a packing by capped $(t - 2)$ -cylinders by nesting as illustrated in [Figure 12](#). By comparing volumes, this packing of capped $(t - 2)$ -cylinders has the required density. \square

Corollary III.B.2. *Fix $t \geq 2t_0 + 2$. The upper density of a packing \mathcal{Z} of \mathbb{R}^3 by t -cylinders satisfies the inequality*

$$\rho^+(\mathcal{Z}) \leq \frac{t}{\frac{\sqrt{12}}{\pi}(t - 2 - 2t_0) + (2t_0) + \frac{4}{3}}.$$

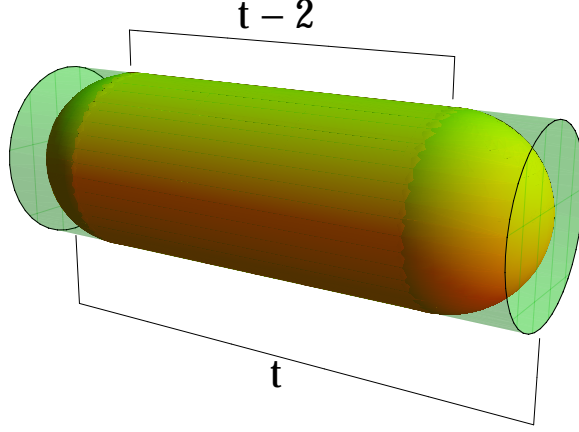


Figure 12: Nesting capped $(t - 2)$ -cylinders in t -cylinders.

Proof. Assume there exists a packing by t -cylinders exceeding the stated bound. Then [Lemma III.B.1](#) gives a packing² of capped $(t - 2)$ -cylinders with density greater than

$$\frac{t - \frac{2}{3}}{t} \cdot \frac{t}{\frac{\sqrt{12}}{\pi}(t - 2 - 2t_0) + (2t_0) + \frac{4}{3}} = \frac{t - 2 + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2 - 2t_0) + (2t_0) + \frac{4}{3}}.$$

This contradicts the density bound of [Theorem III.B.1](#) for capped $(t - 2)$ -cylinders. \square

C. SET UP

For the remainder of the paper, fix the following notation.

Definition III.C.1. Let \mathcal{C}^* be the restriction of \mathcal{C} to $B(R - 2/\sqrt{3})$, indexed by I^* .

To bound the density $\rho(\mathcal{C}^*, R - 2/\sqrt{3}, R)$ for a fixed packing \mathcal{C} and a fixed $R \geq 2/\sqrt{3}$, decompose $B(R)$ into regions D_i with disjoint interiors such that $C_i \subseteq D_i$ for all i in I^* .

Definition III.C.2. For such a packing \mathcal{C}^* with fixed R , define the **Dirichlet cell** D_i of a capped t -cylinder C_i to be the set of points in $B(R)$ no further from the axis a_i of C_i than from any other axis a_j of C_j .

²This method of iterating packings loosens the bound. In this case, it becomes less than 1 only for cylinders of length greater than 105.147... which is itself slightly greater than $2t_0 + 2$.

Definition III.C.3. For any point x on axis a_i , define a plane P_x normal to a_i and containing x .

Definition III.C.4. Define the **Dirichlet slice** d_x to be the set $D_i \cap P_x$.

Definition III.C.5. For a fixed Dirichlet slice d_x , define $S_x(r)$ to be the circle of radius r centered at x in the plane P_x .

Important circles are $S_x(1)$, which coincides with the cross section of the boundary of the cylinder, and $S_x(2/\sqrt{3})$, which circumscribes the regular hexagon in which $S_x(1)$ is inscribed. An *end* of the capped t -cylinder C_i refers to an endpoint of the axis a_i .

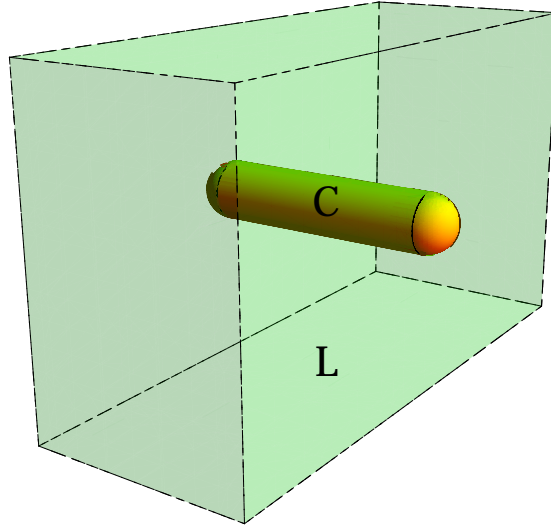


Figure 13: A capped cylinder C and the slab L .

Definition III.C.6. Define the **slab** L_i to be the closed region of \mathbb{R}^3 bounded by the normal planes to a_i through the endpoints of a_i and containing C_i^0 (Figure Figure 13).

Definition III.C.7. The Dirichlet cell D_i decomposes into the region $D_i^0 = D_i \cap L_i$ containing C_i^0 and complementary regions D_i^1 and D_i^2 containing the caps C_i^1 and C_i^2 respectively (Figure Figure 14).

Aside from a few degenerate cases, the set of points equidistant from a point x and line segment a in the affine hull of x and a form a *parabolic spline* (Figure 15).

Definition III.C.8. A **parabolic spline** is a parabolic arc extending in a C^1 fashion to rays at the points equidistant to both the point x and an endpoint of the line segment a .

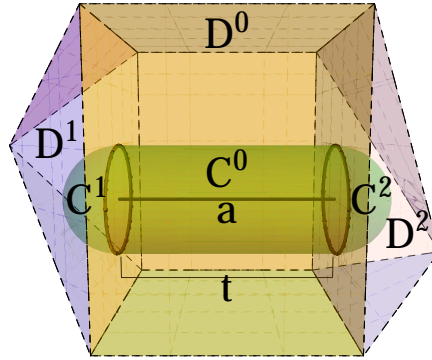


Figure 14: Decomposing a Dirichlet cell.

Call the points where the parabolic arc meets the rays the **Type I** points of the curve. A **parabolic spline cylinder** is a surface that is the cylinder over a parabolic spline.

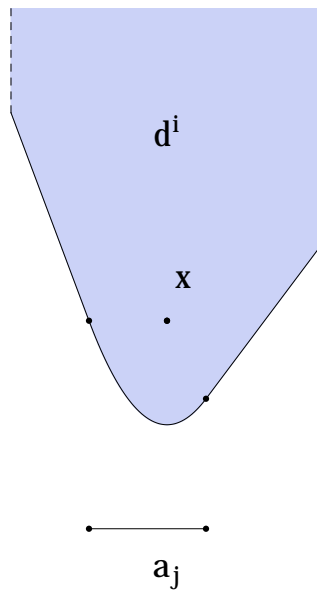


Figure 15: Parabolic spline associated with point x and segment a_j .

D. QUALIFIED POINTS

1. The Dirichlet slice

Definition III.D.1. Fix a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders. Fix $R \geq 2/\sqrt{3}$ and restrict to \mathcal{C}^* . A point x on an axis is **qualified** if the Dirichlet slice d_x has area greater than $\sqrt{12}$, the area of the regular hexagon in which $S_x(1)$ is inscribed.

Proposition III.D.1. Fix a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders. Fix $R \geq 2/\sqrt{3}$ and restrict to \mathcal{C}^* . Let x be a point on an axis a_i , where i is a fixed element of I^* . If $B_x(4/\sqrt{3})$ contains no ends of \mathcal{C}^* , then x is qualified.

The proof of this proposition is a modification of the Main Lemma of [BK90]. A series of lemmas allow for the truncation and rearrangement of the Dirichlet slice. The goal is to construct from d_x a subset d_x^{**} of P_x with the following properties:

- d_x^{**} contains $S_x(1)$.
- The boundary of d_x^{**} is composed of line segments and parabolic arcs with apexes touching $S_x(1)$.
- The non-analytic points of the boundary of d_x^{**} lie on $S_x(2/\sqrt{3})$.
- The area of d_x^{**} is no greater than the area of d_x .

Then the computations of [BK90, §6] apply.

Lemma III.D.1. If a point x satisfies the conditions of [Proposition III.D.1](#), then the Dirichlet slice d_x is a bounded convex planar region, the boundary of which is a simple closed curve consisting of a finite union of parabolic arcs, line segments and circular arcs.

Proof. Without loss of generality, fix a point x on a_i . For each $j \neq i$ in I^* , let d^j be the set of points in P_x no further from a_i than from a_j . The Dirichlet slice d_x is the intersection of $B(R)$ with d^j for all $j \neq i$ in I^* . The boundary of d^j is the set of points in P_x that are equidistant from a_i and a_j . As P_x is perpendicular to a_i at x , the boundary of d^j is also the set of points in P_x equidistant from x and a_j .

This is the intersection of the plane P_x with the set of points in \mathbb{R}^3 equidistant from x and a_j . The set of points in \mathbb{R}^3 equidistant from x and a_j is a parabolic spline cylinder

perpendicular to the affine hull of x and a_j . Therefore the set of points equidistant from x and a_j in P_x is also a parabolic spline, with x on the convex side.

In the degenerate cases where x is in the affine hull of a_j or P_x is parallel to a_j , the set of points equidistant from x and a_j in P_x are lines or is empty.

The region d_x is clearly bounded as it is contained in $B(R)$. The point x lies in the convex side of the parabolic spline so each region d^j is convex. The set $B(R)$ contains x and is convex, so d_x is convex. This is a finite intersection of regions bounded by parabolic arcs, lines and a circle, so the rest of the lemma follows. \square

To apply the results of [BK90], the non-analytic points of the boundary of the Dirichlet slice d_x must be controlled. From the construction of d_x as a finite intersection, the non-analytic points of the boundary of d_x fall into three non-disjoint classes of points: the Type I points of a parabolic spline that forms a boundary arc of d_x , **Type II** points defined to be points on the boundary of d_x that are also on the boundary of $B(R)$, and **Type III** points, defined to be points on the boundary of d_x that are equidistant from three or more axes. The Type III points are the points on the boundary of d_x where the parabolic spline boundaries of various d^j intersect.

Lemma III.D.2. *If a point x satisfies the conditions of Proposition III.D.1, then no non-analytic points of the boundary of d_x are in $\text{int}(\text{Conv}(S_x(2/\sqrt{3})))$, where the interior is with respect to the subspace topology of P_x and $\text{Conv}(\cdot)$ is the convex hull.*

Proof. There are no Type I, Type II, or Type III points in $\text{int}(\text{Conv}(S_x(2/\sqrt{3})))$. By hypothesis, $B_x(4/\sqrt{3})$ contains no ends. The Type I points are equidistant from x and an end. As there are no ends contained in $B_x(4/\sqrt{3})$, there are no Type I points in $\text{int}(\text{Conv}(S_x(2/\sqrt{3})))$.

By hypothesis, x is in $B(R - 2/\sqrt{3})$. Therefore there are no points on the boundary of $B(R)$ in $\text{int}(\text{Conv}(S_x(2/\sqrt{3})))$ and therefore no Type II points in $\text{int}(\text{Conv}(S_x(2/\sqrt{3})))$.

As a Type III point is equidistant from three or more axes, at some distance ℓ , it is the center of a ball tangent to three unit balls. This is because a capped t -cylinder contains a unit ball which meets the ball of radius ℓ centered at the Type III point. These balls do not overlap as the interiors of the capped t -cylinders have empty intersection. Lemma 3 of

[Kup91] states that if a ball of radius ℓ intersects three non-overlapping unit balls in \mathbb{R}^3 , then $\ell \geq 2/\sqrt{3} - 1$. It follows that there are no Type III points in $\text{int}(\text{Conv}(S_x(2/\sqrt{3})))$. \square

Lemma III.D.3. *Fix a packing \mathcal{C} . Then for all $i \neq j$ and $i, j \in I^*$, there is a supporting hyperplane Q of $\text{int}(C_i)$ that is parallel to a_i and separating $\text{int}(C_i \cap L_i)$ from $\text{int}(C_j \cap L_i)$.*

Proof. Extend $C_i \cap L_i$ to an infinite cylinder \bar{C}_i where $C_j \cap L_i$ and \bar{C}_i have disjoint interiors. The sets $C_j \cap L_i$ and \bar{C}_i are convex, so the Minkowski hyperplane separation theorem gives the existence of a hyperplane separating $\text{int}(C_j \cap L_i)$ and $\text{int}(\bar{C}_i)$. This hyperplane is parallel to the axis a_i by construction. Take Q to be the parallel translation to a supporting hyperplane of $\text{int}(C_i)$ that still separates $\text{int}(C_i \cap L_i)$ from $\text{int}(C_j \cap L_i)$. See Figure 16 for an example. \square

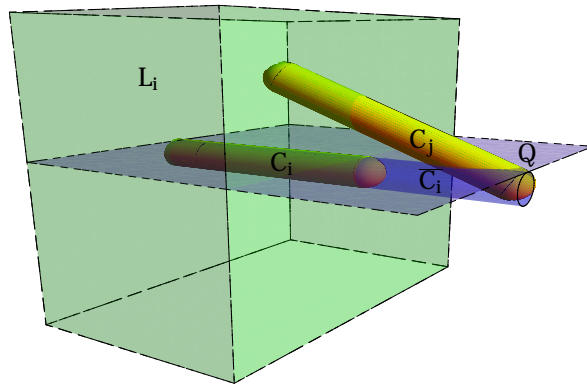


Figure 16: The hyperplane Q separates $\text{int}(C_i \cap L_i)$ from $\text{int}(C_j \cap L_i)$.

Lemma III.D.4. *Fix a packing \mathcal{C} . Fix a point x on the axis a_i of C_i such that $B_x(4/\sqrt{3})$ contains no ends. Let y and z be points on the circle $S_x(2/\sqrt{3})$. If each of y and z is equidistant from C_i and C_j , then the angle yxz is no greater than $2 \arccos(\sqrt{3} - 1) := \alpha_0$, which is approximately 85.88° .*

Proof. By hypothesis, $B_x(4/\sqrt{3})$ contains no ends, including the end of the axis a_i . Therefore any points of C_j that are not in L_i are at a distance greater than $4/\sqrt{3}$ from x . The points of C_i and C_j that y and z are equidistant from must be in the slab L_i , so it is enough to consider y and z equidistant from C_i and $C_j \cap L_i$.

By construction, the hyperplane Q separates all points of $C_j \cap L_i$ from x . Let k be the line of intersection between P_x and Q . As y and z are at a distance of $2/\sqrt{3} - 1$ from both $C_j \cap L_i$ and C_i , they are at most that distance from Q . They are also at most that distance from k . The largest possible angle yxz occurs when y and z are on the x side of k in P_x , each at exactly the distance $2/\sqrt{3} - 1$ from k as illustrated in Figure 17. This angle is exactly $2 \arccos(\sqrt{3} - 1) := \alpha_0$. \square

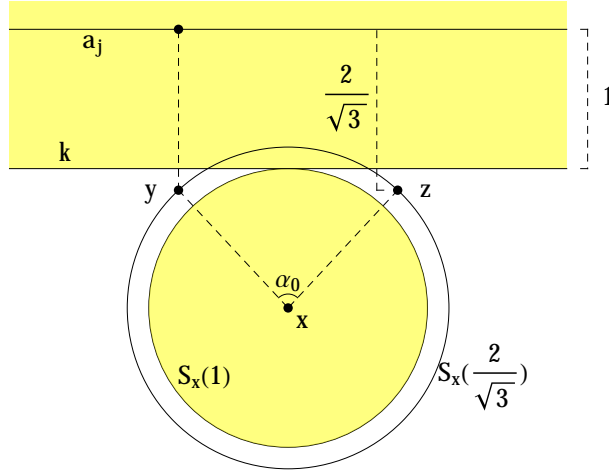


Figure 17: An extremal configuration for the angle α_0 .

The following lemma is proved in [BK90].

Lemma III.D.5. *Let y and z be points on $S_x(2/\sqrt{3})$ such that $60^\circ < yxz < \alpha_0$. For every parabola p passing through y and z and having $S_x(1)$ on its convex side, let $xypzx$ denote the region bounded by segments xy , xz , and the parabola p . Let p_0 denote the parabola passing through y and z and tangent to $S_x(1)$ at its apex.*

$$\text{Area}(xyp_0zx) \leq \text{Area}(xypzx).$$

2. Truncating and rearranging

Consider the Dirichlet slice d_x of a point x satisfying the conditions of Proposition III.D.1. The following steps produce a region with no greater area than that of d_x .

Step 1: The boundary of d_x intersects $S_x(2/\sqrt{3})$ in a finite number of points. Label them $y_1, y_2, \dots, y_n, y_{n+1} = y_1$ in clockwise order. Intersect d_x with $S_x(2/\sqrt{3})$ and call the new region d_x^* .

By [Lemma III.D.2](#), this is a region bounded by arcs of $S_x(2/\sqrt{3})$, parabolic arcs and line segments, with non-analytic points on $S_x(2/\sqrt{3})$.

Step 2: For $i = 1, 2, \dots, n$ if $y_i x y_{i+1} > 60^\circ$ and if the boundary of d_x^* between y_i and y_{i+1} is a circular arc of $S_x(2/\sqrt{3})$, then introduce additional vertices $z_{i_1}, z_{i_2}, \dots, z_{i_k}$ on the circular arc $y_i y_{i+1}$ so that the polygonal arc $y_i z_{i_1} z_{i_2} \dots z_{i_k} y_{i+1}$ does not intersect $S_x(1)$. Relabel the set of vertices in clockwise order to $v_1, v_2, \dots, v_m, v_{m+1} = v_1$.

Step 3: If $v_i x v_{i+1} \leq 60^\circ$ then truncate d_x^* along the line segment $v_i v_{i+1}$ keeping the part of d_x^* which contains $S_x(1)$. This does not increase area by construction. If $v_i x v_{i+1} > 60^\circ$ then $v_i v_{i+1}$ is a parabolic arc. Replace it by the parabolic arc through v_i and v_{i+1} touching $S_x(1)$ at its apex. This does not increase area by [Lemma III.D.4](#). This new region d_x^{**} has no greater area than d_x , contains $S_x(1)$, and bounded by line segments and parabolic arcs touching $S_x(1)$ at their apexes, with all non-analytic points of the boundary on $S_x(2/\sqrt{3})$. If consecutive non-analytic points on the boundary have interior angle no greater than 60° , they are joined by line segments. If they have interior angle between 60° and α_0 , they are joined by a parabolic arc.

The following is a consequence of [[BK90](#), §6], which determines the minimum area of such a region.

Lemma III.D.6. *The region d_x^{**} has area at least $\sqrt{12}$.*

[Proposition III.D.1](#) follows.

E. DECOMPOSITION OF $B(R)$ AND DENSITY CALCULATION

1. Decomposition

Fix a packing \mathcal{C} . Fix $R \geq 2/\sqrt{3}$ and restrict to \mathcal{C}^* .

Definition III.E.1. *Let the set A be the union of the axes a_i over I^* . Let $d\mu$ be the 1-*

dimensional Hausdorff measure on A . Let X be the subset of qualified points of A . Let Y be the subset of A given by $\{x \in A : B_x(\frac{4}{\sqrt{3}}) \text{ contains no ends}\}$. Let Z be the subset of A given by $\{x \in A : B_x(\frac{4}{\sqrt{3}}) \text{ contains an end}\}$.

Notice that $Y \subseteq X \subseteq A$ from [Proposition III.D.1](#) and $Z = A - Y$ by definition.

The sets A , X , Y , and Z are measurable. The set A is just a finite disjoint union of lines in \mathbb{R}^3 . The area of the Dirichlet slice d_x is piecewise continuous on A , so X is a Borel subset of A . Similarly the conditions defining Y and Z make them Borel subsets of A . The ball $B(R)$ is of finite volume, so I^* has some finite cardinality n .

Decompose $B(R)$ into the regions $\{D_i^0\}_{i=1}^n$, $\{D_i^1\}_{i=1}^n$ and $\{D_i^2\}_{i=1}^n$. Further decompose the regions $\{D_i^0\}_{i=1}^n$ into Dirichlet slices d_x , where x is an element of A .

2. Density computation

From the definition of density,

$$\rho(\mathcal{C}, R - 2/\sqrt{3}, R) = \frac{\sum_{I^*} \text{Vol}(C_i^0) + \sum_{I^*} \text{Vol}(C_i^1) + \sum_{I^*} \text{Vol}(C_i^2)}{\sum_{I^*} \text{Vol}(D_i^0) + \sum_{I^*} \text{Vol}(D_i^1) + \sum_{I^*} \text{Vol}(D_i^2)}$$

as $C_i^j \subseteq D_i^j$, and $\text{Vol}(C_i^0) = t\pi$, and $\text{Vol}(C_i^1) = \text{Vol}(C_i^2) = \frac{2}{3}\pi$, it follows that

$$\rho(\mathcal{C}, R - 2/\sqrt{3}, R) \leq \frac{nt\pi + n\frac{4}{3}\pi}{\sum_{I^*} \text{Vol}(D_i^0) + n\frac{4}{3}\pi}. \quad (\text{III.1})$$

Finally, $\rho(\mathcal{C}, R - 2/\sqrt{3}, R)$ is explicitly bounded by the following lemma.

Lemma III.E.1. *For $t \geq 2t_0$,*

$$\sum_{I^*} \text{Vol}(D_i^0) \geq n(\sqrt{12}(t - 2t_0) + \pi(2t_0)).$$

Proof. The sum $\sum_{I^*} \text{Vol}(D_i^0)$ may be written as an integral of the area of the Dirichlet slices d_x over A

$$\sum_{I^*} \text{Vol}(D_i^0) = \int_A \text{Area}(d_x) d\mu.$$

Using the area estimates from [Proposition III.D.1](#), there is an inequality

$$\int_A \text{Area}(d_x) d\mu \geq \int_X \sqrt{12} d\mu + \int_{A-X} \pi d\mu.$$

As $\sqrt{12} > \pi$ and the integration is over a region A with $\mu(A) < \infty$, passing to the subset $Y \subseteq X$ gives

$$\int_X \sqrt{12} d\mu + \int_{A-X} \pi d\mu \geq \int_Y \sqrt{12} d\mu + \int_{A-Y} \pi d\mu = \int_{A-Z} \sqrt{12} d\mu + \int_Z \pi d\mu.$$

The measure of Z is the measure of the subset of A that is contained in all the balls of radius $4/\sqrt{3}$ about all the ends of all the cylinders in the packing. This is bounded from above by considering the volume of cylinders contained in balls of radius $4/\sqrt{3} + 1$. If the cylinders completely filled the ball, they would contain at most axis length $\frac{4}{3}(\frac{4}{\sqrt{3}} + 1)^3 = t_0$. As each cylinder has two ends, there are at worst $2n$ disjoint balls to consider. Therefore $2nt_0 \geq \mu(Z)$.

Provided $t \geq 2t_0$, it follows that

$$\int_{A-Z} \sqrt{12} d\mu + \int_Z \pi d\mu \geq (nt - 2nt_0)\sqrt{12} + 2n(t_0)\pi.$$

□

By combining [Equation III.1](#) with [Lemma III.E.1](#) and simplifying, it follows that

$$\rho(\mathcal{C}, R - 2/\sqrt{3}, R) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}$$

for an arbitrary packing \mathcal{C} of \mathbb{R}^3 by capped congruent t -cylinders.

Item	Length	Diameter	t	Density \leq
Broomstick	1371.6mm	25.4mm	108	0.9956...
20' PVC Pipe	6096mm	38.1mm	320	0.9353...
Capellini	300mm	1mm	600	0.9219...
Carbon Nanotube	-	-	2.64×10^8 [WLY ⁺ 09]	0.9069...

Table 1: Bound For Various Cylinders

F. FURTHER RESULTS

1. A rule of thumb

For $t \geq 0$, the upper bounds for the density of packings by capped and uncapped t -cylinders are dominated by curves of the form $\frac{\pi}{\sqrt{12}} + N/t$. Numerically, one finds a useful rule of thumb:

Theorem III.F.1. *The upper density ρ^+ of a packing \mathcal{C} of \mathbb{R}^3 by capped t -cylinders satisfies*

$$\rho^+(\mathcal{C}) \leq \frac{\pi}{\sqrt{12}} + \frac{10}{t}.$$

Theorem III.F.2. *The upper density ρ^+ of a packing \mathcal{C} of \mathbb{R}^3 by t -cylinders satisfies*

$$\rho^+(\mathcal{C}) \leq \frac{\pi}{\sqrt{12}} + \frac{10}{t}.$$

2. Examples

While the requirement that t be greater than $2t_0$ for a non-trivial upper bound is inconvenient, the upper bound converges rapidly to $\pi/\sqrt{12} = 0.9069\dots$ and is nontrivial for tangible objects, as illustrated in the table below.

3. Additional results

There are other conclusions to be drawn. For example: Consider the density of a packing $\mathcal{C} = \{C_i\}_{i \in I}$ of \mathbb{R}^3 by congruent unit radius circular cylinders C_i , possibly of infinite length. The upper density γ^+ of such a packing may be written

$$\gamma^+(\mathcal{C}) = \limsup_{r \rightarrow \infty} \sum_I \frac{\text{Vol}(C_i \cap B_0(r))}{\text{Vol}(B_0(r))}$$

and coincides with $\rho^+(\mathcal{C})$ when the lengths of C_i are uniformly bounded.

Theorem III.F.3. *The upper density γ of half-infinite cylinders is exactly $\pi/\sqrt{12}$.*

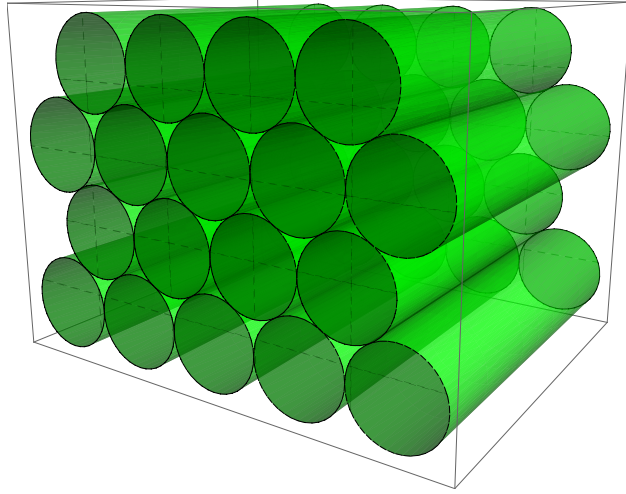


Figure 18: Packing cylinders with high density.

Proof. The lower bound is given by the obvious packing \mathcal{C}' with parallel axes (Figure 18) and $\gamma^+(\mathcal{C}') = \pi/\sqrt{12}$. Since a packing $\mathcal{C}(\infty)$ of \mathbb{R}^3 by half-infinite cylinders also gives a packing $\mathcal{C}(t)$ of \mathbb{R}^3 by t -cylinders, the inequality

$$\frac{t}{\frac{\sqrt{12}}{\pi}(t - 2 - 2t_0) + (2t_0) + \frac{4}{3}} \geq \rho^+(\mathcal{C}(t)) = \gamma^+(\mathcal{C}(t)) \geq \gamma^+(\mathcal{C}(\infty))$$

holds for all $t \geq 2t_0$. □

Theorem III.F.4. *Given a packing $\mathcal{C} = \{C_i\}_{i \in I}$ by non-congruent capped unit cylinders with lengths constrained to be between $2t_0$ and some uniform upper bound M , the density satisfies the inequality*

$$\rho^+(\mathcal{C}) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}$$

where t is the average cylinder length given by $\liminf_{r \rightarrow \infty} \mu(a_i)/|J|$, where J is the set $\{i \in I : C_i \subseteq B(r)\}$.

Proof. None of the qualification conditions require a uniform length t . [Equation III.1](#) may be considered with respect to the total length of A rather than nt . □

It may be easier to compute a bound using the following

Corollary III.F.1. *Given a packing $\mathcal{C} = \{C_i\}_{i \in I}$ by non-congruent capped unit cylinders with lengths constrained to be between $2t_0$ and some uniform upper bound M , the density satisfies the inequality*

$$\rho^+(\mathcal{C}) \leq \frac{t + \frac{4}{3}}{\frac{\sqrt{12}}{\pi}(t - 2t_0) + (2t_0) + \frac{4}{3}}$$

where t is the infimum of cylinder length.

Proof. The stated bound is a decreasing function in t , so this follows from the previous theorem. □

IV. POLYCYLINDER PACKINGS

A. INTRODUCTION TO POLYCYLINDER PACKINGS

G. Fejes Tóth and W. Kuperberg [FTK93] describe a method for computing an upper bound for packings by infinite polycylinders – objects isometric to $\mathbb{D}^2 \times \mathbb{R}^n$ in \mathbb{R}^{n+2} . This article [Kus14a] explicitly computes their bound and then proves the sharp upper bound of $\pi/\sqrt{12}$ for the packing density of infinite polycylinders in any dimension.

Open and closed Euclidean unit n -balls will be denoted \mathbb{B}^n and \mathbb{D}^n respectively. The closed unit interval is denoted \mathbb{I} . A general polycylinder C is a set isometric to $\prod_{i=1}^m \lambda_i \mathbb{D}^{k_i}$ in $\mathbb{R}^{k_1+\dots+k_m}$, where λ_i is in $[0, \infty]$. For this article, the term polycylinder refers to the special case of an infinite polycylinder over a two-dimensional disk of unit radius.

Definition IV.A.1. A *polycylinder* is a set isometric to $\mathbb{D}^2 \times \mathbb{R}^n$ in \mathbb{R}^{n+2} .

The following are standard definitions.

Definition IV.A.2. A *polycylinder packing* is a countable family $\mathcal{C} = \{C_i\}_{i \in I}$ of polycylinders $C_i \subset \mathbb{R}^{n+2}$ with mutually disjoint interiors.

Definition IV.A.3. The *upper density*¹ $\delta^+(\mathcal{C})$ of a packing \mathcal{C} of \mathbb{R}^n is defined to be

$$\delta^+(\mathcal{C}) = \limsup_{r \rightarrow \infty} \frac{\text{Vol}(\mathcal{C} \cap r\mathbb{B}^n)}{\text{Vol}(r\mathbb{B}^n)}.$$

Definition IV.A.4. The *upper packing density* $\delta^+(C)$ of an object C is the supremum of $\delta^+(\mathcal{C})$ over all packings \mathcal{C} of \mathbb{R}^n by C .

The definition of density is equivalent to a number of other definitions under some mild assumptions. For more on density, see for example [BMP05, Fed69, PA11].

¹This notion can be generalized further by replacing \mathbb{B}^n with an arbitrary convex body K .

B. COMPUTING A BOUND OF FEJES TÓTH AND KUPERBERG

In [FTK93], G. Fejes Tóth and W. Kuperberg describe a method for computing upper bounds on a broad class of objects in any dimension: Blichfeldt-type results for balls [Bli29, Ran47] extend to results for outer parallel bodies. Fejes Tóth and Kuperberg compute upper bounds for the density of packings by cylinders $\mathbb{D}^{n-1} \times t\mathbb{I}$ and also for packings by outer parallel bodies of line segments, i.e. capped cylinders, in \mathbb{R}^n but do not address the case of polycylinders. This section recalls that work and explicitly computes the [FTK93] bound to be $\delta^+(\mathcal{C}) \leq 0.941533\dots$ for polycylinder packings in any dimension.

1. Background

Given a packing of \mathbb{R}^n by congruent objects $\mathcal{C} = \{C_i\}_{i \in I}$, there are a fixed body $C \subset \mathbb{R}^n$ and isometries $\{\phi_i\}_{i \in I}$ of \mathbb{R}^n such that $C_i = \phi_i C$ for all i in I .

Definition IV.B.1. *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a **Blichfeldt gauge** for a convex body $C \subset \mathbb{R}^n$ if for any collection of isometries $\Phi = \{\phi_i\}_{i \in I}$ of \mathbb{R}^n where $\mathcal{C} = \{\phi_i C\}_{i \in I}$ is a packing and for all x in \mathbb{R}^n ,*

$$\sigma_\Phi(f)(x) := \sum_{i \in I} f(\phi_i^{-1}x) \leq 1.$$

Notice that the characteristic function $\mathbf{1}_C$ of C is a Blichfeldt gauge for C . Replacing $\mathbf{1}_C$ with a more general Blichfeldt gauge f lets one replace the characteristic function of the packing $\mathbf{1}_\mathcal{C}$ with a diffuse version $\sigma_\Phi(f)$. This new function $\sigma_\Phi(f)$ has the same general characteristics as $\mathbf{1}_\mathcal{C}$, is still bounded pointwise by 1 in the ambient space and is uniformly bounded independent of Φ in the moduli space of packings. As f may have greater mass than $\mathbf{1}_C$, this allows one to estimate the volume of the interstices of a packing and thereby bound the packing density.

Example 1. Blichfeldt initially uses the radial function $2f_0$ where

$$f_0(r) = \begin{cases} \frac{1}{2}(2 - r^2) & : 0 \leq r \leq \sqrt{2} \\ 0 & : r > \sqrt{2} \end{cases}$$

and showed that f_0 is a Blichfeldt gauge. Then, for a packing $\mathcal{C} = \{\phi_i C\}_{i \in I}$ of a cube $t\mathbb{I}^n$ by spheres, the support of $\sigma_{\Phi}(f)$ is contained in a slightly larger cube $(t + 2\sqrt{2} - 2)\mathbb{I}^n$. A bound on sphere packing density can then be extracted as follows. From the definition of the Blichfeldt gauge and integrating in spherical coordinates, one finds

$$(t + 2\sqrt{2} - 2)^n \geq |I| \int_{\mathbb{R}^n} f_0 \, dV = \frac{|I| \text{Vol}(\mathbb{B}^n) 2^{\frac{n+2}{2}}}{n+2}.$$

When density is measured relative to a cube,

$$\delta^+(\mathcal{C}) = \frac{|I| \text{Vol}(\mathbb{B}^n)}{t^n} \leq \frac{n+2}{2^{\frac{n+2}{2}}} \left(1 + \frac{2\sqrt{2} - 2}{t}\right)^n.$$

It is easy to see that the same method works when $t\mathbb{I}^n$ is replaced with $\mathbb{B}_{t/2}^n$. By passing to the limit, the bound

$$\delta^+(\mathcal{C}) \leq \frac{n+2}{2^{\frac{n+2}{2}}}$$

holds for any sphere packing of \mathbb{R}^n .

2. Blichfeldt-type bound for polycylinders

[Example 1](#) motivates the following general observations.

Theorem IV.B.1 (Blichfeldt). *If g is a Blichfeldt gauge for a body C , then $\delta^+(\mathcal{C}) \leq \text{Vol}(C)/J(g)$ where*

$$J(g) = \int_{\mathbb{R}^n} g \, dV.$$

Theorem IV.B.2 (Fejes Tóth–Kuperberg). *If $f(\alpha)$, $\alpha \geq 0$, is a real valued function such that $f(|x|)$ is a Blichfeldt gauge for the unit ball, and C is a convex body with inradius $r(C)$, then for any $\varrho \leq r(C)$*

$$g(x) = f\left(\frac{d(x, C_{-\varrho})}{\varrho}\right)$$

is a Blichfeldt gauge for C , where $C_{-\varrho}$ is the inner parallel body of C at distance ϱ .

For their more general results, Fejes Tóth and Kuperberg do not use f_0 , but rather Blichfeldt's modified version.

Definition IV.B.2. The *modified Blichfeldt gauge* [Bli29] for \mathbb{D}^n is the radial function

$$f_1(r) = \begin{cases} 1 & : 0 \leq r \leq 2 - \sqrt{2} \\ \frac{1}{2}(2 - r)^2 & : 2 - \sqrt{2} \leq r \leq 1 \\ \frac{1}{2}(2 - r^2) & : 1 \leq r \leq \sqrt{2} \\ 0 & : r > \sqrt{2} \end{cases}$$

Definition IV.B.3 (Fejes Tóth–Kuperberg). For the two-dimensional gauge f_1 defined above,

$$A_2 := J(f_1) / \text{Vol}(\mathbb{D}^2) = (29 - 16\sqrt{2})/6.$$

From the previous theorems and definitions, the results of [FTK93] give a estimate for the maximal density of infinite polycylinders as follows. Consider $C(t) = \mathbb{D}^{n+2} + t\mathbb{I}^n$ in \mathbb{R}^{n+2} and the gauge $g_t(x) = f_1(d(x, C(t)_{-1}))$, where f_1 is the modified Blichfeldt gauge and $C(t)_{-1}$ is the inner parallel body at distance 1, i.e. an n -cube of height t . From [Theorem IV.B.1](#) an estimate of the integral $\int_{\mathbb{R}^{n+2}} g_t \, dV$ gives a density bound. By integrating g_t over $C(t)_{-1} \times \mathbb{R}^2$ and noticing that contribution from the complement $\mathbb{R}^{n+2} \setminus (C(t)_{-1} \times \mathbb{R}^2)$ are of strictly lower order – it is bounded above by a constant times the $(n - 1)$ -Hausdorff measure of the boundary $\partial C(t)_{-1} \subset C(t)_{-1}$, it follows that

$$\delta^+(C(t)) \leq \frac{\pi t^n}{\pi A_2 t^n + O(t^{n-1})}.$$

In the limit, as t tends to infinity, this gives a bound of $1/A_2 = .941533\dots$ for infinite polycylinders in any dimension.

C. TRANSVERSALITY

This section introduces the required transversality arguments in affine geometry.

Definition IV.C.1. A ***d-flat*** is a *d*-dimensional affine subspace of \mathbb{R}^n .

Definition IV.C.2. Given a collection of flats $\{F, \dots, G\}$, the ***parallel dimension***, written $\dim_{\parallel}\{F, \dots, G\}$, is the dimension of their maximal parallel sub-flats.

The notion of parallel dimension can be interpreted in several ways, allowing a modest abuse of notation.

For a collection of flats $\{F, \dots, G\}$, consider their tangent cones at infinity $\{F_{\infty}, \dots, G_{\infty}\}$. The parallel dimension of $\{F, \dots, G\}$ is the dimension of the intersection of these tangent cones. This may be viewed as the limit of a rescaling process $\mathbb{R}^n \rightarrow r\mathbb{R}^n$ as r tends to 0, leaving only the scale-invariant information.

For a collection of flats $\{F, \dots, G\}$, consider each flat as a system of linear equations. The corresponding homogeneous equations define a collection of linear subspaces $\{F_{\infty}, \dots, G_{\infty}\}$. The parallel dimension is the dimension of their intersection $F_{\infty} \cap \dots \cap G_{\infty}$.

Definition IV.C.3. Two disjoint *d*-flats are ***parallel*** if their parallel dimension is *d*, that is, if every line in one is parallel to a line in the other.

Definition IV.C.4. Two disjoint *d*-flats are ***skew*** if their parallel dimension is less than *d*.

Lemma IV.C.1. A pair of disjoint *n*-flats in \mathbb{R}^{n+k} with $n \geq k$ has parallel dimension strictly greater than $n - k$.

Proof. By homogeneity of \mathbb{R}^{n+k} , let $F = F_{\infty}$. As F_{∞} and G are disjoint, G contains a non-trivial vector \mathbf{v} such that $G = G_{\infty} + \mathbf{v}$ and \mathbf{v} is not in $F_{\infty} + G_{\infty}$. It follows that

$$\begin{aligned} \dim(\mathbb{R}^{n+k}) &\geq \dim(F_{\infty} + G_{\infty} + \text{span}(\mathbf{v})) > \dim(F_{\infty} + G_{\infty}) \\ &= \dim(F_{\infty}) + \dim(G_{\infty}) - \dim(F_{\infty} \cap G_{\infty}). \end{aligned}$$

Count dimensions to find $n + k > n + n - \dim_{\parallel}(F_{\infty}, G_{\infty})$. □

Corollary IV.C.1. A pair of disjoint *n*-flats in \mathbb{R}^{n+2} has parallel dimension at least $n - 1$.



Figure 19: Disjoint 2-flats in \mathbb{R}^4 with (a) parallel dimension 2 (b) parallel dimension 1.

D. RESULTS

1. Pairwise foliations and dimension reduction

Definition IV.D.1. The **core** a_i of a polycylinder C_i isometric to $\mathbb{D}^2 \times \mathbb{R}^n$ in \mathbb{R}^{n+2} is the distinguished n -flat defining C_i as the set of points at most distance 1 from a_i .

There might not be a common foliation product foliations coming *all* the cores in a packing, so there might not be a way to realize an arbitrary packing of polycylinders as a product bundle over a packing by circular cylinders in \mathbb{R}^3 . An example of three nonintersecting skew 2-flats in \mathbb{R}^4 are the flats $\text{span}(e_1, e_2)$, $\text{span}(e_1, e_3) + e_4$, $\text{span}(e_2, e_3) - e_4$, where e_i is the i th standard basis vector in \mathbb{R}^4 . This is visualized in [Figure 21](#) and illustrates how a packing that is not a product bundle could be constructed.

In a packing \mathcal{C} of \mathbb{R}^{n+2} by polycylinders, [Corollary IV.C.1](#) shows that, for every pair of polycylinders C_i and C_j , one can choose parallel $(n-1)$ -dimensional subflats $b_i \subset a_i$ and $b_j \subset a_j$ and define a product foliation

$$\mathcal{F}^{b_i, b_j} : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^3$$

with \mathbb{R}^3 leaves that are orthogonal to b_i and to b_j . Given a point x in a_i , there is a distinguished \mathbb{R}^3 leaf $F_x^{b_i, b_j}$ that contains the point x . The foliation \mathcal{F}^{b_i, b_j} restricts to foliations of

C_i and C_j with right-circular-cylinder leaves.

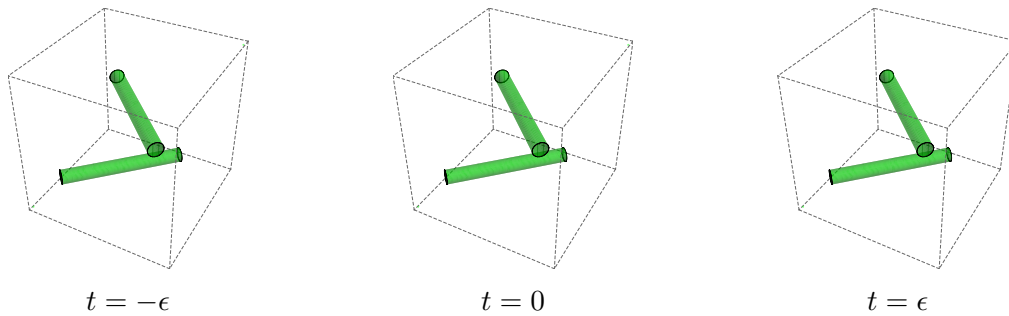


Figure 20: The leaves of a pair of polycylinders in \mathbb{R}^4 indexed by the common parallel direction t are identical.

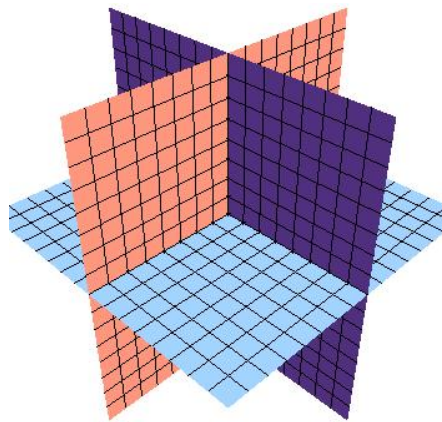


Figure 21: Projection of three skew 2-flats in \mathbb{R}^4 with no overall common parallel 1-flats.

2. The Dirichlet slice

Definition IV.D.2. In a packing \mathcal{C} of \mathbb{R}^{n+2} by polycylinders, the **Dirichlet cell** D_i associated to a polycylinder C_i is the set of points in \mathbb{R}^{n+2} no further from C_i than from any other polycylinder in \mathcal{C} .

The Dirichlet cells of a packing partition \mathbb{R}^{n+2} , because $C_i \subset D_i$ for all polycylinders C_i . To bound the density $\delta^+(\mathcal{C})$, it is enough to fix an i in I and consider the density of C_i in D_i . For the Dirichlet cell D_i , there is a slicing as follows.

Definition IV.D.3. Given a fixed a polycylinder C_i in a packing \mathcal{C} of \mathbb{R}^{n+2} by polycylinders

and a point x on the core a_i , the plane p_x is the 2-flat orthogonal to a_i and containing the point x . The **Dirichlet slice** d_x is the intersection of D_i and p_x .

Note that p_x is a sub-flat of $F_x^{b_i, b_j}$ for all j in I .

3. Bezdek–Kuperberg bound

For any point x on the core a_i of a polycylinder C_i , the results of A. Bezdek and W. Kuperberg [BK90] apply to the Dirichlet slice d_x .

Lemma IV.D.1. *A Dirichlet slice is convex and, if bounded, a parabola-sided polygon.*

Proof. Construct the Dirichlet slice d_x as an intersection. Define d^j to be the set of points in p_x no further from C_i than from C_j . Then the Dirichlet slice d_x is realized as

$$d_x = \{\cap_{j \in I} d^j\}.$$

Each arc of the boundary of d_x in p_x is given by an arc of the boundary of some d^j in p_x . The boundary of d^j in p_x is the set of points in p_x equidistant from C_i and C_j . Since the foliation \mathcal{F}^{b_i, b_j} is a product foliation, the arc of the boundary of d^j in p_x is also the set of points in p_x equidistant from the leaf $C_i \cap F_x^{b_i, b_j}$ of $\mathcal{F}^{b_i, b_j}|_{C_i}$ and the leaf $C_j \cap F_x^{b_i, b_j}$ of $\mathcal{F}^{b_i, b_j}|_{C_j}$. This reduces the analysis to the case of a pair cylinders in \mathbb{R}^3 . From [BK90], it follows that d^j is convex and the boundary of d_j in p_x is a parabola; the intersection of such sets d^j in p_x is convex, and a parabola-sided polygon if bounded. \square

Let $S_x(r)$ be the circle of radius r in p_x centered at x .

Lemma IV.D.2. *The vertices of d_x are not closer to $S_x(1)$ than the vertices of a regular hexagon circumscribed about $S_x(1)$.*

Proof. A vertex of d_x occurs where three or more polycylinders are equidistant, so the vertex is the center of a $(n+1)$ -ball B tangent to three polycylinders. Thus B is tangent to three disjoint unit $(n+2)$ -balls B_1, B_2, B_3 . By projecting into the affine hull of the centers of B_1, B_2, B_3 , it is immediate that the radius of B is no less than $2/\sqrt{3} - 1$. \square

Lemma IV.D.3. *Let y and z be points on the circle $S_x(2/\sqrt{3})$. If each of y and z is equidistant from C_i and C_j , then the angle yxz is smaller than or equal to $2 \arccos(\sqrt{3}-1) = 85.8828\dots^\circ$.*

Proof. Following [BK90, Kus14b], the existence of a supporting hyperplane of C_i that separates $\text{int}(C_i)$ from $\text{int}(C_j)$ suffices. \square

In [BK90], it is shown that planar objects satisfying Lemma IV.D.1, Lemma IV.D.2 and Lemma IV.D.3 have area no less than $\sqrt{12}$. As the bound holds for all Dirichlet slices, it follows that $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ in \mathbb{R}^{n+2} . The product of the dense disk packing in the plane with \mathbb{R}^n gives a polycylinder packing in \mathbb{R}^{n+2} that achieves this density. Combining this with the results of Thue [Thu10] for $n = 0$ and A. Bezdek and W. Kuperberg [BK90] for $n = 1$, it follows that

Theorem IV.D.1. $\delta^+(\mathbb{D}^2 \times \mathbb{R}^n) \leq \pi/\sqrt{12}$ for all natural numbers n .

V. SUMMARY

A. FURTHER WORK

1. Programming bounds and packing pentagons

The method of approximation and slicing linear programming problems introduced in [Chapter II](#) applies not only to the pentagon packing problem but to more general nonlinear programming problems. The assumptions used are stronger than required. For example, the assumption that the objective and constraint functions are analytic. It would be reasonable to assume C_1 , sub-differentiable or local Lipschitz regularity and derive similar results. Also, the assumption that the solution set is one-dimensional can be dropped.

In the case of packing problems, this programming method could be used to prove that other configurations of bodies in the plane are critical with respect to various functions e.g. showing that the conjectured densest packing of the plane by regular $2n + 1$ -gons is a local maximum. The techniques of [Chapter II](#) could also apply to critical configurations of other geometric problems.

With regard to local-global criticality in the pentagon packing problem, the size of the neighborhood on which 0 is the unique maximum is lost when the objective and constraint functions are replaced with their first order approximations. An analysis of the higher order terms should provide an lower bound on the size of the neighborhood. Doing a similar analysis at other points in the configuration space may give a covering of a sufficiently large portion of the configuration space to conclude that the conjectured optimum is the true global optimum.

Conjecture V.A.1. *The packing density of regular pentagons in the plane is no greater*

than $(5 - \sqrt{5})/3 = 0.921311$.

2. Packing cylinders

The theorems of [Chapter III](#) can give density bounds for other objects. For example, the packing density of curved tubes can be bounded by them as containers for cylinders. Better bounds on tubes would come from better bounds on t -cylinders for small values of t . This leads to the following conjectures.

Conjecture V.A.2. *The packing density of unit tori with major radius r is no greater than $\pi/\sqrt{12} + f(r)$, where $\lim_{r \rightarrow \infty} f(r) = 0$.*

Conjecture V.A.3. *The packing density of unit tori of any and possibly distinct radii is no greater than $\pi/\sqrt{12}$.*

Conjecture V.A.4 (Wilker, [[Wil85](#)]). *The expected packing density of congruent unit radius circular cylinders of any length is exactly the planar packing density of the circle.*

Another idea is to parametrize the densities for capped t -cylinders from the sphere to the infinite cylinder by controlling the density of the ends. In [Chapter III](#), the analysis assumes anything in a neighborhood of an end packs with density 1, whereas it is expected that the ends and nearby sections of tubes would pack with density closer to $\pi/\sqrt{18}$.

Conjecture V.A.5 (Torquato, [[TJ12](#)]). *The densest packing by capped cylinders is obtained from extending a dense sphere packing perpendicular to the triangular layers, giving a density bound of*

$$\rho^+(\mathcal{C}(t)) = \frac{\pi}{\sqrt{12}} \frac{t + \frac{4}{3}}{t + \frac{2\sqrt{6}}{3}}.$$

The structure of high density cylinder packings is unclear. For infinite circular cylinders, there are nonparallel packings with positive density [[Kup90](#)]. In the case of finite length t -cylinders, there exist nonparallel packings with density close to $\pi/\sqrt{12}$, obtained by laminating large uniform cubes packed with parallel cylinders, shrinking the cylinders and perturbing their axes. It is not obvious how or if the alignment of cylinders correlates with density. Finally, as the upper bound presented is not sharp, it is not possible to control the defects of packings achieving the maximal density.

Conjecture V.A.6. *For a packing of \mathbb{R}^3 by t -cylinders to achieve a density of $\pi/\sqrt{12}$, the packing must contain arbitrarily large regions of t -cylinders with axes arbitrarily close to parallel.*

3. Packing polycylinders

The theory of Blichfeldt gauges is not fully developed, and a function-theoretic exploration of the area seems warranted. Finding the “best” Blichfeldt gauge is a problem in the calculus of variations of the form

$$\text{maximize } J(g) \text{ subject to } \sigma_{\Phi}(g) \leq 1 \text{ for all } \Phi \text{ where } \Phi(C) \text{ is a packing.}$$

However, it is not clear if a solution even exists. The structure of a “best” Blichfeldt gauge is not obvious, but, by an averaging argument, it seems reasonable to restrict to the case of radially symmetric functions. Then the discrepancy between the symmetry of the gauge and the symmetries of the interstices of the packing would be of some importance. It would also be interesting to prove Blichfeldt-type results where the symmetry group is larger, e.g. symplectomorphisms.

From dimension reduction arguments of the type developed in [Chapter IV](#), it seems possible to find an asymptotic bound for objects of the form

$$\mathbb{D}^2 \times_{i \in \{1, \dots, n\}} \lambda_i \mathbb{I}$$

with $\lambda_i \gg 1$ or of the form $\mathbb{D}^2 \times \lambda K$ for a larger class of objects K with $\lambda \gg 1$. Using methods from [\[Kus14b\]](#), the error contributed by ∂K could be bounded subject to a technical analysis of Dirichlet slices.

The dimension reduction arguments work for cylinders $K \times \mathbb{R}^n$, as long as the core is middle dimensional or higher. In practice, it seems quite hard to apply this technique. Even for the case of $\mathbb{D}^3 \times \mathbb{R}^n$, it reduces to understanding a quadratic version of the dodecahedral conjecture for three-dimensional Dirichlet slices of $\mathbb{D}^3 \times \mathbb{R}^2$. It may be possible to prove the following conjectures using techniques from [Chapter III](#) and [Chapter IV](#).

Conjecture V.A.7. For all $n \geq 0$, the packing density of \mathbb{R}^{n+2} by objects isometric to $\mathbb{D}^2 \times \lambda \mathbb{I}^n$ is no greater than $\pi/\sqrt{12} + f(\lambda)$, where $\lim_{\lambda \rightarrow \infty} f(\lambda) = 0$.

Conjecture V.A.8. The packing density of \mathbb{R}^4 by objects isometric to $\mathbb{D}^2 \times \lambda \mathbb{D}^2$ is no greater than $\pi/\sqrt{12} + f(\lambda)$, where $\lim_{\lambda \rightarrow \infty} f(\lambda) = 0$.

Conjecture V.A.9. For all $n \geq 0$, the packing density of \mathbb{R}^{n+3} by objects isometric to $\mathbb{D}^3 \times \mathbb{R}^n$ is no greater than the density of the sphere in the regular circumscribing dodecahedron.

Conjecture V.A.10. For all $n \geq 0$, the packing density of \mathbb{R}^{n+3} by objects isometric to $\mathbb{D}^3 \times \mathbb{R}^n$ is no greater than $\pi/\sqrt{18}$.

APPENDIX A

LIBRARY OF FUNCTIONS

Notebooks and libraries are maintained on D-Scholarship@Pitt.

```

1 NM[u_] := Sqrt[u.u]
2 VAA[u_, v_] := ArcCos[u.v/(NM[u] NM[v])]
3 Angle[u_, v_, w_] := VAA[u - v, w - v]
4 AreaASA[f_] := 1/(2 Cot[f] + 2 Cot[3 Pi/10])
5 AP = Sin[\[Pi]/5] Tan[(3 \[Pi])/10]

```

Table 2: Basic Functions

```

1 (*b describes the one-parameter family, c describes the centers in the configuration space*)
2 b1 = c1 := {0, 0}
3 b2[t2_] := {t2, 0}
4 c2[t2_, u2_] := b2[t2] + {0, u2 - AP - 1}
5
6 b3[t2_] := {Cos[-\[Pi]/10] + ((t2 + Sin\[Pi]/5) ((AP* Sin\[Pi]/5) + Sin[(2 \[Pi])/5])
7 - Cos[-\[Pi]/10))/Sin\[Pi]/5},
8 (-1 + Sin[-\[Pi]/10]) + ((t2 + Sin\[Pi]/5) ((-AP* Cos\[Pi]/5) - Cos[(2 \[Pi])/5])
9 - (-1 + Sin[-\[Pi]/10]))/Sin\[Pi]/5 }
10
11 c3[t2_, t3_, u3_] := b3[t2] + {t3, u3}
12 b4[t2_] := {-(Cos[-\[Pi]/10] + ((-t2 + Sin\[Pi]/5) ((AP* Sin\[Pi]/5) + Sin[(2 \[Pi])/5])
13 - Cos[-\[Pi]/10))/Sin\[Pi]/5},
14 (-1 + Sin[-\[Pi]/10]) + ((-t2 + Sin\[Pi]/5) ((-AP* Cos\[Pi]/5) - Cos[(2 \[Pi])/5])
15 - (-1 + Sin[-\[Pi]/10]))/Sin\[Pi]/5}
16
17 c4[t2_, t4_, u4_] := b4[t2] + {t4, u4}

```

Table 3: Coordinates

```

1 p12 := c1 + {0, -1};
2
3 p13 := c1 + RotationMatrix[2 Pi/5].{0, -1};
4
5 p14 := c1 + RotationMatrix[-2 Pi/5].{0, -1};
6
7 p23[t2_, u2_, th2_] :=
8   c2[t2, u2] + RotationMatrix[4 Pi/5 + th2].{0, -1};
9
10 p24[t2_, u2_, th2_] :=
11   c2[t2, u2] + RotationMatrix[-4 Pi/5 + th2].{0, -1};
12
13 p31[t2_, t3_, u3_, th3_] :=
14   c3[t2, t3, u3] + RotationMatrix[2 Pi/5 + th3].{0, 1};
15
16 p32[t2_, t3_, u3_, th3_] :=
17   c3[t2, t3, u3] + RotationMatrix[4 Pi/5 + th3].{0, 1};
18
19 p41[t2_, t4_, u4_, th4_] :=
20   c4[t2, t4, u4] + RotationMatrix[-2 Pi/5 + th4].{0, 1};
21
22 p42[t2_, t4_, u4_, th4_] :=
23   c4[t2, t4, u4] + RotationMatrix[-4 Pi/5 + th4].{0, 1};

```

Table 4: Relevant Pentagon Vertices

```

1 ob13[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
2   AreaASA[Angle[ c3[t2, t3, u3], c1, {0, -1}]];
3
4 ob14[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
5   AreaASA[Angle[c4[t2, t4, u4], c1, {0, -1}]];
6
7 ob23[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
8   AreaASA[Angle[p23[t2, u2, th2], c2[t2, u2], c3[t2, t3, u3 ]]];
9
10 ob24[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
11   AreaASA[Angle[p24[t2, u2, th2], c2[t2, u2], c4[t2, t4, u4 ]]];
12
13 ob32[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
14   AreaASA[Angle[p31[t2, t3, u3, th3], c3[t2, t3, u3], c2[t2, u2 ]]];
15
16 ob42[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
17   AreaASA[Angle[p41[t2, t4, u4, th4], c4[t2, t4, u4], c2[t2, u2 ]]];
18
19 ob31[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
20   AreaASA[Angle[p31[t2, t3, u3, th3], c3[t2, t3, u3], c1 ]]];
21
22 ob41[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
23   AreaASA[Angle[p41[t2, t4, u4, th4], c4[t2, t4, u4], c1 ]]];
24
25 ob21 [t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
26   1/(2 Cot[2 Pi/5] + 2 Cot[3 Pi/10]);
27
28 Denom[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
29   ((1/2) (Det[{c2[t2, u2] - c1, c3[t2, t3, u3] - c1}])) + ((1/
30     2) (Det[{c4[t2, t4, u4] - c1, c2[t2, u2] - c1}]));
31
32 OB[t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
33   (ob13[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
34     ob14[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
35     ob23[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
36     ob24[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
37     ob32[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
38     ob42[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
39     ob31[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
40     ob41[t2, u2, th2, t3, u3, th3, t4, u4, th4] +
41     +ob21[t2, u2, th2, t3, u3, th3, t4, u4, th4])/
42   Denom[t2, u2, th2, t3, u3, th3, t4, u4, th4];

```

Table 5: Objective Functions

```

1 Con[1, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
2 Angle[c2[t2, u2], p23[t2, u2, th2], p12];
3
4 Con[2, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
5 Angle[c3[t2, t3, u3], p31[t2, t3, u3, th3], p23[t2, u2, th2]];
6
7 Con[3, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
8 Angle[c1, p12, p31[t2, t3, u3, th3]];
9
10 Con[4, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
11 Angle[c2[t2, u2], p23[t2, u2, th2], p32[t2, t3, u3, th3]];
12
13 Con[5, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
14 Angle[c3[t2, t3, u3], p31[t2, t3, u3, th3], p13];
15
16 Con[6, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
17 Angle[c4[t2, t4, u4], p41[t2, t4, u4, th4], p24[t2, u2, th2]];
18
19 Con[7, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
20 Angle[c1, p12, p41[t2, t4, u4, th4]];
21
22 Con[8, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
23 Angle[c2[t2, u2], p24[t2, u2, th2], p42[t2, t4, u4, th4]];
24
25 Con[9, t2_, u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_] :=
26 Angle[c4[t2, t4, u4], p41[t2, t4, u4, th4], p14];

```

Table 6: Constraint Functions

APPENDIX B

COMPUTATIONS

```
1 Dim := 8;
2
3 l1 := Interval[{-0.0000001, .0000001}];
4
5 l2 := {t2 -> l1, u2 -> l1, th2 -> l1, t3 -> l1, u3 -> l1, th3 -> l1,
6   t4 -> l1, u4 -> l1, th4 -> l1};
7
8 Block[{$MinPrecision = 10000, $MaxPrecision = 10000},
9 GradObl =
10  D[OB[t2, u2, th2, t3, u3, th3, t4, u4,
11    th4], {{u2, th2, t3, u3, th3, t4, u4, th4}, 1}] /. l2;
12
13 GradConl =
14  Table[D[Con[i, t2, u2, th2, t3, u3, th3, t4, u4,
15    th4], {{u2, th2, t3, u3, th3, t4, u4, th4}, 1}], {i, 1, 9}] /.
16  l2;
17
18 HessObl =
19  D[OB[t2, u2, th2, t3, u3, th3, t4, u4,
20    th4], {{t2, u2, th2, t3, u3, th3, t4, u4, th4}, 2}] /. l2;]
```

Table 7: Interval Derivatives

```

1 GenCand1 := Flatten /@ Map[NullSpace, Subsets[GradConl, {Dim - 1}]];
2
3 GenCandl2 := Join[GenCandl, -GenCandl];
4
5 Er0 := Max[
6   Flatten[GenCandl // Map[Abs[Max[#] - Min[#]] &, #, {2}] &]]
7 Er1 := Max[GenCandl // Map[Norm[#, 1] &, #] &];
8
9 Test[{u2_, th2_, t3_, u3_, th3_, t4_, u4_, th4_} :=
10  Union[Table[
11    GradConl[[i ]].{u2, th2, t3, u3, th3, t4, u4, th4} >= -Er0*
12    Er1*2, {i, 1, 9}]] == {True}
13 GenCandl3 = Select[GenCandl2, Test];

```

Table 8: Candidates For Generators

```

1 Cert = GenCandl3.GradObl;
2
3 Er2 = .01;
4
5 If[Select[Cert, Not[ TrueQ[# < - Er2]] &] == {} &&
6   TrueQ[HessObl[[1, 1]] < -Er2],
7   Print["Valid:-)_Gradient_in_ interior _of_H_polar_cone_by_Er2.\
8   Positive _Er2_better_._Second_order_test_passed."], Print[" Error" ] ]

```

Table 9: Linear Program And Certificate

APPENDIX C

A. REMARKS ON THE DENSITY FUNCTION

This appendix describes some observations on the behavior of density in the moduli space of convex centrally symmetric bodies. Two pointwise continuity results are established. The method is to show continuity at a fixed centrally symmetric body via the Minkowski gauge of that body, and that those gauges induce the same topology in the moduli space of convex centrally-symmetric bodies.

Definition C.A.1. *Given a metric space (M, d) , define the Hausdorff distance d_H , an extended pseudometric, on the collection of non-empty subsets of M .*

$$d_H(X, Y) = \inf\{\epsilon > 0 : X \subseteq Y_\epsilon \text{ and } Y \subseteq X_\epsilon\}$$

where

$$X_\epsilon = \cup_{x \in X} \{z \in \mathbb{R}^n : d(z, x) \leq \epsilon\}.$$

If $F(M)$ is the set of compact, convex, nonempty subsets of M , then $(F(M), d_H)$ is a metric space.

Definition C.A.2. *Define $F_{CS}(\mathbb{R}^n)$ and $F_{CSB}(\mathbb{R}^n)$ to be the collections of centrally symmetric sets and centrally symmetric bodies in $F(\mathbb{R}^n)$ centered at 0.*

Definition C.A.3 (Bourbaki, [Bou66]). *If (X, d) is a metric space, there is a uniformity generated by d . A system of entourages (the elements of the uniformity) for this uniformity is given by all sets containing sets of the form*

$$\{(x, y) \in X \times X : d(x, y) < \epsilon\}, \epsilon > 0.$$

Lemma C.A.1 (Theorem 3.87: Aliprantis and Border, [AB99]). *Suppose X is metrizable with compatible metrics d and d' that generate the same uniformity \mathfrak{U} . Then the corresponding Hausdorff metrics h_d and $h_{d'}$ are equivalent on $F(X)$.*

Proof. Let E belong to $F(X)$. It suffices to show that for every $\epsilon > 0$ there is $\delta > 0$ so that the h_d ball of radius 2ϵ includes the $h_{d'}$ ball of radius δ at E . Let

$$U_d(\epsilon) = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

be an entourage in \mathfrak{U} . Since d' also generates \mathfrak{U} , there is some $\delta > 0$ such that

$$U_{d'}(2\delta) \subset U_d(\epsilon).$$

Suppose $d'_H(E, C) < \delta$. Then $E \subset C_{d'}(2\delta)$ and $C \subset E_{d'}(2\delta)$. Also, $E_\epsilon = \{y \in X : (x, y) \in U_d(\epsilon) \text{ for some } x \in E\}$. Therefore $E \subset C_d(\epsilon)$ and $C \subset E_d(\epsilon)$. Therefore our assumption holds. [AB99] □

Lemma C.A.2. *The metrics generated by the norms induced by centrally symmetric bodies generate the same uniformity.*

Proof. Metric balls in X nest, therefore the generators of the entourage of \mathfrak{U} in $X \times X$ nest. For every $\epsilon > 0$, there exists $\delta, \lambda > 0$, such that $B_d(\lambda) \subset B'_d(\delta) \subset B_d(\epsilon)$. This inclusion is translation invariant, therefore $U_d(\lambda) \subset U'_d(\delta) \subset U_d(\epsilon)$. □

This gives the following theorem.

Theorem C.A.1. *The topologies induced by the Hausdorff metrics generated by norms induced by centrally symmetric bodies are equivalent. That is, the topology of $F(M)$ is independent of the topology induced by this specific collection of metrics.*

Remark C.A.1. *It is not always true that equivalent metrics generate the same Hausdorff metric [AB99, Example 3.86]. However, they do when the uniform structures of the metrics are the same. For $F_{CSB}(\mathbb{R}^n)$, this is true, as there is an equivalence of bases.*

Theorem C.A.2. *The functions $\text{Vol}(\cdot)$ and $\rho_{\max}(\cdot) : F_{CSB}(\mathbb{R}^n) \rightarrow \mathbb{R}$ are continuous with respect to d_H . The function $\text{Vol}(\cdot)$ fixes a scaling of the the Lebesgue measure μ , and sends an element E of $F_{CSB}(\mathbb{R}^n)$ to $\mu(E)$. The function $\rho_{\max}(\cdot)$ assigns to an element E of $F_{CSB}(\mathbb{R}^n)$ the maximal packing density of E in \mathbb{R}^n .*

Step 1: Use the correspondence between norms and centrally symmetric convex bodies and the equivalence of bases to get an equivalence of uniform structures to induce the same topology on $F_{CSB}(\mathbb{R}^n)$ for the different induced metrics. Then continuity at E in $F_{CSB}(\mathbb{R}^n)$ can be checked with respect to the metric d^E induced by E . This is the content of [Theorem C.A.1](#).

Step 2: $\text{Vol}(\cdot)$ is continuous. Fix an E in $F_{CSB}(\mathbb{R}^n)$, and $\epsilon > 0$. Show for all $\epsilon > 0$ there exists a $1 \gg \delta > 0$ with respect to the gauge such that for all C in $F_{CSB}(\mathbb{R}^n)$,

$$d_H^E(E, C) < \delta \implies |\text{Vol}(E) - \text{Vol}(C)| < \epsilon.$$

By definition of $d_H^E(E, C)$,

$$Y \subseteq E_{2\delta} \text{ and } E \subseteq C_{2\delta}.$$

Furthermore, $E_{2\delta} = (1 + 2\delta)E$ with respect to d^E .

As δ is small with respect to the gauge, $(1 - 2\delta)E \subset C$. Otherwise, the largest ball $kE \subset C$ is strictly smaller than $(1 - 2\delta)E$. Then, considering a shared supporting half-plane \mathbb{H} for C and kE , it follows that

$$C_{2\delta} \subset \mathbb{H}_{2\delta}$$

but $\mathbb{H}_{2\delta}$ does not contain E , by the maximality of kE . This contradicts the assumption that $E \subseteq C_{2\delta}$. Therefore the following containments hold:

$$(1 - 2\delta)E \subseteq C \subseteq (1 + 2\delta)E.$$

As $\text{Vol}(\cdot)$ is homogeneous, it follows that

$$0 \leq (1 - 2\delta)^n \text{Vol}(E) \leq \text{Vol}(C) \leq (1 + 2\delta)^n \text{Vol}(E).$$

The continuity of $\text{Vol}(\cdot)$ at E follows.

Step 3: Using the containments above, compare packing densities by fixing a packing, rescaling and replacing one body with another this give the following inequalities.

$$\rho_{max}(C) \geq \rho_{max}(E) \frac{\text{Vol}(C)}{(1 + 2\delta)^n \text{Vol}(E)}$$

$$\rho_{max}(E) \geq \rho_{max}(C) \frac{\text{Vol}(E)}{\left(\frac{1}{1-2\delta}\right)^n \text{Vol}(C)}$$

which are rearranged into the bounds

$$\rho_{max}(E) \frac{\left(\frac{1}{1-2\delta}\right)^n \text{Vol}(C)}{\text{Vol}(E)} \geq \rho_{max}(C) \geq \rho_{max}(E) \frac{\text{Vol}(C)}{(1 + 2\delta)^n \text{Vol}(E)}$$

The continuity of $\rho_{max}(\cdot)$ at E follows.

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