SPIKES FOR THE GIERER-MEINHARDT SYSTEM WITH MANY SEGMENTS OF DIFFERENT DIFFUSIVITIES

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Abstract. We rigorously prove results on spiky patterns for the Gierer-Meinhardt system [5] with a large number of jump discontinuities in the diffusion coefficient of the inhibitor. Using numerical computations in combination with a Turing-type instability analysis, this system has been investigated by Benson, Maini and Sherratt [1], [3], [9].

We first review results on the case of two segments given in [25], concerning one-spike steady states: the existence of interior spikes located away from the jump discontinuity was established, along with a necessary condition for the position of the spike, namely, the spike can be located in one-and-only-one of the two subintervals separated by the jump discontinuity of the inhibitor diffusivity. This **localization principle** for a spike does not hold for constant inhibitor diffusivities.

Secondly, there also exist spikes whose distance from the jump discontinuity is of the same order as its spatial extent. It turns out that, generically, there either exist two different one-spike steady states near the jump discontinuity or there is none.

In this paper, we prove a conjecture raised in [25]: We show that one of the spikes is stable while the other is unstable, using an eigenfunction constructed by outer and inner expansions. Moreover, since our argument involves only the two immediate segments around the jump discontinuity, the result holds for any number of segments.

Next, we extend the interior spike results on the case of two segments (one jump) to an arbitrary number of segments. By analyzing the derivatives of the regular part of a Green's function, we give a simple classification of interior segments according to the signs at both ends of the segment: There exists a stable spike, an unstable spike or there does not exist any spike in the segment which is away from the jump discontinuities.

We also give explicit formulas of the solutions and conditions for existence for the case of three segments, which has one interior segment.

Finally, we confirm our results by illustrating the long-term dynamical behavior of the system using numerical computations. We observe a moving spike which converges to a stationary interior spike, a spike near a jump discontinuity or a boundary spike.

Dedicated to Professor Masayasu Mimura on the occasion of his 65th birthday

1. Introduction

For systems with piecewise constant diffusion coefficients, Turing instabilities have been computed numerically and investigated analytically by Benson, Maini and Sherratt [1], [3], [9], and results on dispersion relations and typical solution profiles have been obtained. In particular, the authors showed that the spatial variation of diffusion coefficients may produce isolated patterns and asymmetric spatially oscillating patterns which are not seen in standard homogeneous Turing systems.

Biological applications of these effects include the anterior-posterior asymmetry of skeletal elements in the limb and experimental results on double anterior limbs [27], [9]. The fact that for asymmetric solutions different peaks may have different amplitudes is a possible explanation for the common observation that digits vary in length.

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We give a rigorous mathematical proof of the influence of discontinuous diffusion coefficients on the qualitative and quantitative properties of spiky patterns in a reaction-diffusion system.

In particular, we study the Gierer-Meinhardt system [5], which is given by

$$\begin{cases} a_t = \epsilon^2 a_{xx} - a + \frac{a^2}{h}, \\ \tau h_t = (D(x)h_x)_x - h + a^2. \end{cases} \tag{1.1}$$

Note that h acts as an inhibitor to a, whereas a acts as an activator to both a and h. This motivates the name activator-inhibitor system. In this paper, we assume $x \in (-1,1)$, $0 < \epsilon << 1$ is a small diffusion constant, $\tau \geq 0$ is a time relaxation constant, and

$$D(x) = \begin{cases} D_1, & -1 < x < x_1, \\ D_i, & x_{i-1} < x < x_i, \\ D_N, & x_{N-1} < x < 1, \end{cases} \tag{1.2}$$

where $0 < D_i$ and $D_i \neq D_{i+1}$. Note that the inhibitor diffusivity $D(x)$ has a jump discontinuity at x_i. We study the equation (1.1) on the interval $(-1,1)$ with Neumann boundary conditions:

$$a_x(-1) = a_x(1) = 0, \quad h_x(-1) = h_x(1) = 0. \tag{1.3}$$

The matching conditions at x_i are that $D(x)h_x$ is a continuous function at $x = x_i$.

We begin by reviewing results in [25], which corresponds to the case $N = 2$. It was established that there exists two types of spiky solutions to this system: interior spikes, which has $O(1)$ distance from the jump discontinuity of the inhibitor diffusivity (there is only one jump), and spike near the jump, which has $O(\epsilon)$ distance from the jump discontinuity.

For the interior-spike type solution, a precise condition for its existence was found (Theorem 3.1). The condition also implies that the interior spike can be located in one-and-only-one of the two subintervals. This establishes a **localization principle** for the 1-D Gierer-Meinhardt system with a jump in the inhibitor diffusivity. In contrast, with constant diffusivity, the interior spike is always located in the center of the interval. So the presence of the jump, in effect, "moves" the (unique) position away from the center into one of the subintervals. Precise information about the (limit) location (as $\epsilon \to 0$) is also given in Theorem 3.1.

For the second type of one-spike solutions, the spike having same order in spatial extent as its distance from the jump discontinuity, it turns out that either there exist two different one-spike solutions or there is none (Theorem 7). One question left open in [25] is the stability of these spikes. We are now going to prove the stability behavior of these spikes. The result is that one of the spikes is stable, the other unstable. The proof uses a construction of the eigenfunction involving inner and outer expansions. One will see that both the existence and stability results depend on a parameter involving only the two diffusivities and lengths of the immediate segments around the jump in consideration. Therefore, our results for two segments extend easily to the general case with an arbitrary number of segments.

After completing the arguments for the spike at the jump, we will return to interior spikes and show that the **localization principle** for interior spikes also extends elegantly for general N (i.e 1-D bounded domain with N segments, $N-1$ jumps in the inhibitor diffusivity). We will derive a classification of the interior segments which depends on the sign of the derivatives of the diagonal of the regular part of Green's function at both ends of the segments. By the signs, we shall conclude that there exists a stable spike, an unstable spike or there exists no spike in the segment.

We also study the case of three segments, the smallest N such that there is an interior segment, in more detail. We derive explicit formulas for these derivatives of the regular part of Green's function and the closed form conditions necessary for classification.

Finally, we mention some previous works on spiky steady states for the Gierer-Meinhardt system with constant coefficients. Existence and stability of spiky steady-states, for example, have been studied for 1-D in [8] and their instabilities have been investigated in [19]. For 2-D the existence and stability of multiple spikes has been investigated in [21], [22], [23].

The structure of the paper is as follows: In Section 2, we provide some preliminaries. In Section 3, we review previous results in the two-segment case from [25] of the paper. In Section 4, we briefly recall how construct and analyze a spiky steady-state. In Section 5, we recall the main stability results for interior spikes. In Section 6, we recall an outer and inner expansion which is needed to analyze a spike near a jump and will be used in Section 7. In Section 7, we present a new result: We study the small eigenvalues of a spike near a jump. In Section 8, we prove results on the existence of interior spikes for N segments. We give a classification of interior intervals into three cases: Existence of stable interior spike, existence of an unstable interior spike, non-existence of an interior spike. In Section 9, we investigate spikes near a jump in the N segments case. We give a condition on existence which only uses $O(1)$ quantities and does not use quantities of the inner expansion which are of $O(\epsilon)$. In Section 10, we confirm our analytical results by numerical computations. We also consider situations which are not analyzed in this paper, such as multi-spike solutions.

2. Preliminaries

Before we state our main results in Section 3, we introduce some notations that is used throughout the paper and perform some preliminary analysis.

We will always assume that $\Omega = (-1, 1)$. With $L^2(\Omega)$ and $H^2(\Omega)$ we denote the usual Sobolev spaces.

For classical solutions, $D(x)h_x(x)$ is continuous at $x = x_i$ and therefore $h_x(x)$ has a jump discontinuity at $x = x_i$. To account for these jump discontinuities of h, the function spaces have to be chosen very carefully.

We assume that

$$(a, h) \in H^2_N(-1, 1) \times H^{2,*}_N(-1, 1),$$

where

$$H^2_N(-1, 1) := \left\{ a \in H^2(-1, 1) \,:\, a_x(-1) = a_x(1) = 0 \right\},$$

$$H^{2,*}(-1, 1) := \left\{ h \in H^1(-1, 1) \,:\, (D(x)h_x)_x \in L^2(-1, 1) \right\},$$

$$H^{2,*}_N(-1, 1) := \left\{ h \in H^{2,*}(-1, 1) \,:\, h_x(-1) = h_x(1) = 0 \right\},$$

endowed with the norm

$$\|(a, h)\|^2_{2,*} := \|a\|^2_{H^2(-1,1)} + \|h\|^2_{2,*}, \text{ where } \|h\|^2_{2,*} := \|h\|^2_{H^1(-1,1)} + \|(D(x)h_x)_x\|_{L^2(-1,1)}.$$

The variable w will always denote the so-called *canonical spike solution*, i.e. the unique homo-clinic solution of the following problem:

$$\begin{cases} w'' - w + w^2 = 0 & \text{in } R^1, \\ w > 0, \quad w(0) = \max_{y\in R} w(y), & w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.1}$$

Note that w is an even function and $w^{'}(y) < 0$ if $y > 0$. An explicit representation is

$$w(y) = \frac{3}{2}\cosh^{-2}\frac{y}{2}.$$

We set

$$\rho(y) := \int_0^y w^2(z)\,dz. \tag{2.2}$$

Elementary calculations give

$$\alpha := \int_0^\infty w^2(y)\,dy = \int_0^\infty w(y)\,dy = 3, \quad \int_0^\infty w^3(y)\,dy = 3.6,$$

$$\rho(y) = \frac{9}{2}\tanh\frac{y}{2} - \frac{3}{2}\tanh^3\frac{y}{2}, \quad \int_0^\infty w^3(y)\rho(y)\,dy = \frac{297}{64} = 4.640625,$$

$$\int_0^\infty (w')^2\,dy = \int_0^\infty w^3\,dy - \int_0^\infty w^2\,dy = 0.6. \tag{2.3}$$

To conclude this section, we study a nonlocal linear operator. We first recall the following result.

Theorem 2.1. [20]*: Consider the following nonlocal eigenvalue problem*

$$L\phi := \Delta\phi - \phi + 2w\phi - \gamma\frac{\int_R w\phi\,dy}{\int_R w^2\,dy}w^2 = \lambda\phi, \quad \phi \in H^1(R). \tag{2.4}$$

(i) *If $\gamma < 1$, then there is a positive eigenvalue to (2.4).*
(ii) *If $\gamma > 1$, then for any nonzero eigenvalue λ of (2.4), we have*

$$Re(\lambda) \le -c_0 < 0 \quad \text{for some } c_0 > 0.$$

(iii) *If $\gamma \ne 1$ and $\lambda = 0$, then*

$$\phi = c_0\frac{dw}{dy}$$

for some constant c_0.

The conjugate operator of L under the scalar product in $L^2(R)$ ris

$$L^*\psi = \Delta\psi - \psi + 2w\psi - \gamma\frac{\int_R w^2\psi\,dy}{\int_R w^2\,dy}w, \quad H^2(R) \to L^2(R). \tag{2.5}$$

Then we have the following result:

Lemma 2.2. *(Lemma 3.2 of* [24]*.) If $\gamma \ne 1$, then*

$$X_0 := Ker(L^*) = span\left\{\frac{dw}{dy}\right\} \tag{2.6}$$

As a consequence, we have

Lemma 2.3. *The operator*

$$L : H^2(R) \to L^2(R),$$

restricted to the spaces

$$L : X_0^\perp \cap H^2(R) \to X_0^\perp \cap L^2(R),$$

where the $X_0^\perp$ denotes the orthogonal projection with respect to the scalar product of $L^2(R)$, is invertible. Moreover, $L^{-1} : X_0^\perp \cap L^2(R) \to X_0^\perp \cap H^2(R)$ is bounded.

Proof: This follows from the Fredholm Alternative Theorem and Lemma 2.2.

□

3. Review of previous results in the two segment case: interior spike and spike near the jump discontinuity of the diffusion coefficient

In [25], we derived the following two types of one-spike solutions:

1. An **interior spike** located far away from the jump discontinuity of the inhibitor diffusivity (see Theorem 3.1). For this interior spike a **new localization principle** was shown which states that the spike can exist in one-and-only-one of the two sub-intervals divided by the jump discontinuity. Further, we showed that this solution is stable.

2. A **spike near the jump discontinuity** whose center has a distance of order ϵ from the jump discontinuity; this means that its distance from the jump discontinuity is of the same order as the spatial extent of the spike.

We re-scale $\Omega_\epsilon = \Omega/\epsilon$ and define $u(x) \in H^2_\epsilon(\Omega)$ if and only $u\left(\frac{x}{\epsilon}\right) \in H^2(\Omega_\epsilon)$ with the norm of the former space defined by the norm of the latter, i.e.

$$\|u\|_{H^2_\epsilon(\Omega)} := \|u(\cdot/\epsilon)\|_{H^2(\Omega_\epsilon)}.$$

In the same way we introduce this re-scaling to the other function spaces introduced at the beginning of the previous section. We also denote the only jump discontinuity as x_b for the results of two segment case.

Now we review our first main theorem:

Theorem 3.1. *(Existence of an interior-spike solution.) Suppose that the condition*

$$\frac{1}{\theta_1}\tanh\theta_1(1+x_b) > \frac{1}{\theta_2}\tanh\theta_2(1-x_b) \tag{3.1}$$

holds, where $\theta_i = D_i^{-1/2}$. Then there exists a steady state of (1.1) – (1.3) with an interior spike for the activator which is located in the subinterval $(-1, x_b)$. More precisely, we have

$$a_\epsilon(x) \sim \xi_\epsilon w\left(\frac{x-t^\epsilon}{\epsilon}\right) + o(1) \qquad in\ H^2_\epsilon(\Omega), \tag{3.2}$$

where $t^\epsilon \to t_0 \in (-1, x_b)$ and $\xi_\epsilon/h(t^\epsilon) \to 1$ as $\epsilon \to 0$. The limit position t_0 is given by

$$\frac{1}{\theta_1}\tanh\left(\theta_1(2t_0+1-x_b)\right) = \frac{1}{\theta_2}\tanh\left(\theta_2(1-x_b)\right). \tag{3.3}$$

If (3.1) holds then there **do not exist** *any steady states of (1.1) – (1.3) with an interior spike for the activator in the subinterval $(x_b, 1)$.*

Remark 3.2. *(Implications of the condition (3.1))*

(i) *If $x_b = 0$, i.e. if the jump discontinuity is located in the center of the interval, (3.1) implies that there exists a spike on the subinterval with the larger diffusion constant D_1 (and the smaller θ_1) but not on the other subinterval. This follows from the fact that the function $\tanh \alpha/\alpha$ is strictly monotone decreasing for $\alpha > 0$.*

(ii) *Condition (3.1) combines the effects of sub-domain size and diffusion constant. Hence the localization effect is due jointly, and favorably, to relatively large subinterval and large diffusion constant.*

(iii) *The reverse sign of (3.1) does not have to be studied separately. It follows by reflection about the center $x = 0$ of the interval. By this transformation θ_1 and θ_2 are exchanged and the sign of x_b is reversed. An easy calculation shows that the inequality resulting from this transformation is equivalent to (3.1) with reversed sign.*

We now review a stability result for the linear stability of the interior spike.

Theorem 3.3. *(Stability of an interior-spike solution.) The interior spike established in Theorem 3.1 is linearly stable.*

Our second main theorem from [25] establishes the existence of spikes near the jump discontinuity of the inhibitor diffusivity, more precisely at a distance of order ϵ from this discontinuity. (Note that the definition of β is different from that in [25])

Theorem 3.4. *(Existence of spikes near the jump discontinuity x_b of the inhibitor diffusivity.)*

Set

$$\beta = \frac{\theta_2 \tanh \theta_1(1 + x_b) - \theta_1 \tanh \theta_2(1 - x_b)}{\theta_2 \tanh \theta_1(1 + x_b) + \theta_1 \tanh \theta_2(1 - x_b)}. \tag{3.4}$$

(i) *If*

$$\begin{cases} \theta_1 < \theta_2 \quad \textit{and} \\ 0 < \beta < \dfrac{\theta_2^2 - \theta_1^2}{2\theta_1^2} \dfrac{I(L_0)}{3.6} \end{cases} \tag{3.5}$$

then there exist exactly two spikes near the jump discontinuity x_b. They are given by (3.2) with $t^\epsilon = x_b - \epsilon L$ for two possible values of L.

Here we have used

$$I(L) := \int_L^\infty w^3(y)(\rho(y)/\alpha - \beta)\, dy, \tag{3.6}$$

where $\rho(y)$ and $\alpha(= 3)$ are defined in (2.2) and (2.3) respectively, while L_0 is uniquely determined by $\rho(L_0)/\alpha = \beta$.

(ii) *If*

$$\begin{cases} \theta_1 < \theta_2 \quad \textit{and} \\ 0 < \beta = \dfrac{\theta_2^2 - \theta_1^2}{2\theta_1^2} \dfrac{I(L_0)}{3.6} \end{cases} \tag{3.7}$$

then there exists exactly one spike near the jump discontinuity x_b. *It is given by (3.2) with* $t^\epsilon = x_b - \epsilon L_0$.

(iii) *If condition (3.1) holds and* $\theta_1 > \theta_2$ *or if*

$$\begin{cases} \theta_1 < \theta_2 \quad \textit{and} \\ \beta > \dfrac{\theta_2^2 - \theta_1^2}{2\theta_1^2} \dfrac{I(L_0)}{3.6} > 0 \end{cases} \tag{3.8}$$

there is **no** *spike near the jump discontinuity* x_b. *More precisely, there is no spike given by (3.2) with* $|t^\epsilon - x_b| = O(\epsilon)$.

Finally, in [25] we proved the following simple nonexistence result for spikes near the jump discontinuity.

Corollary 3.5. *Suppose that* $\theta_1 < \theta_2$ *and*

$$|\beta| > 0.4296875 \frac{\theta_2^2 - \theta_1^2}{2\theta_1^2}$$

then there is **no** *spike near the jump discontinuity, i.e. a spike which satisfies* $|t^\epsilon - x_b| = O(\epsilon)$.

4. The construction and analysis of spiky steady-state solutions

We briefly review the method of how to construct a spiky steady-state solution to (1.1) – (1.3). We first take a rescaled and translated spike

$$w_0(x) = w\left(\frac{x-t}{\epsilon}\right), \tag{4.1}$$

and let r_0 be such that

$$r_0 = \frac{1}{10}\left(\min\left(t_0 + 1,\, 1 - t_0\right)\right). \tag{4.2}$$

Using this smooth cut-off function $\chi : R \to [0,1]$ such that

$$\chi(x) = 1 \quad \text{for} \quad |x| < 1 \quad \text{and} \quad \chi(x) = 0 \quad \text{for} \quad |x| > 2, \tag{4.3}$$

we get

$$\tilde{w}_0(x) = w_0(x)\chi\left(\frac{x-t}{r_0}\right), \tag{4.4}$$

where $\tilde{w}_0(x)$ satisfies

$$\epsilon^2 \Delta \tilde{w}_0 - \tilde{w}_0 + \tilde{w}_0^2 + \text{e.s.t.} = 0 \quad \text{in } (-1,1), \quad \tilde{w}_0'(-1) = \tilde{w}_0'(1) = 0 \tag{4.5}$$

and"e.s.t." means exponentially small terms.

We proceed to carefully choose the amplitude: for $t \in (-1,1)$ let

$$\hat{\xi}_0(t) = \frac{1}{G(t,t)}, \tag{4.6}$$

where $G(x,y)$ is the Green's function, defined in (8.3), which can be used to represent the solution of the second equation of (1.1). It plays a major role throughout the paper. We will mostly drop the argument of $\hat{\xi}_0(t)$ and write $\hat{\xi}_0$ instead. Set

$$\xi_0 := \hat{\xi}_0 \xi_\epsilon, \tag{4.7}$$

where

$$\xi_\epsilon := \left(\epsilon \int_R w^2(z)\, dz\right)^{-1} = \frac{1}{6\epsilon}. \tag{4.8}$$

Then, finally, we choose the first component of our approximate steady state for (1.1) to be

$$w_{\epsilon,t}(x) = \xi_0 \tilde{w}_0(x). \tag{4.9}$$

For a function $A \in L^2(-1,1)$, we define $T[A]$ to be the solution in $H^{2,*}_N(-1,1)$ of

$$(D(x)(T[A])_x)_x - T[A] + A^2 = 0, \quad -1 < x < 1. \tag{4.10}$$

By standard elliptic theory, the solution $T[A]$ is positive and unique.

For $A = w_{\epsilon,t}$, where $t \in B_{\epsilon^{3/4}}(t_0)$, we choose the function $T[A](x)$ to be the second component of our approximate steady state for (1.1).

Using Liapunov-Schmidt reduction, it has been shown in [25] that there exists an exact steady-state which is close to this approximate solution. The same argument works for the N-segment case with minor modifications. The main changes are replacing the Green's function specific to $N = 2$ by the one for general N for the interior spike, and making a similar replacement suitable for multi-segments for the spikes near the jump discontinuity. We omit the details and refer the reader to [25].

The result can be summarized as follows:

Proposition 4.1. *Suppose that*

$$\nabla_{t_0} H(t_0, t_0) = 0 \quad \text{and} \quad \nabla^2_{t_0} H\,(t_0) \neq 0, \tag{4.11}$$

where H is the regular part of the Green's function. Then, for ϵ sufficiently small, there exists a point $t^\epsilon \in B_{\epsilon^{3/4}}(t_0)$ with $t_\epsilon \to t_0$ such that there are spiky steady-states given, up to leading order, by (4.9), (4.10).

To prove existence of an interior spike in one of the segments we have to check condition (4.11) explicitly by computing the Green's function and its derivatives. For the details in the two-segment case we refer to [25].

We will check in Section 8 (Theorem 8.1) when in the case of N segments with $N = 3, 4, \ldots$ condition (4.11) can be satisfied. We will derive explicit criteria for existence or non-existence of interior spikes.

5. Stability Analysis

The large ($O(1)$) eigenvalues as $\epsilon \to 0$ are studied by reduction to a nonlocal eigenvalue problem (NLEP) and using the results given in Theorem 2.1 (2) with $\gamma = 2$. This approach works for all spikes considered in this paper (both interior spikes and spikes near the jump, with an arbitrary number of jumps in the domain interval) and they are all stable with respect to large eigenvalues.

The small ($o(1)$) eigenvalues are considered by using a projection similar to Liapunov-Schmidt reduction, but now at an exact instead of an approximate solution. For an interior spike the following result, by asymptotics, is derived in [25]:

$$-2\epsilon^2 \hat{\xi}_0^3 a_0^\epsilon \left(\nabla^2_{t^\epsilon} H(t^\epsilon, t^\epsilon)\right) + o(\epsilon^2) = \lambda_\epsilon \hat{\xi}_0^2 a_0^\epsilon\,(1 + o(1)), \tag{5.1}$$

using (2.3). Equation (5.1) shows that the small eigenvalue λ_ϵ of (7.1) satisfies

$$\lambda_\epsilon = -2\epsilon^2\hat{\xi}_0\left(\nabla^2_{t^\epsilon}H(t^\epsilon,t^\epsilon)\right) + o(\epsilon^2)$$

by (2.3). It can then be checked explicitly that for interior spikes in the two-segment case we always have $\nabla^2_{t^\epsilon}H(t^\epsilon,t^\epsilon)$ is positive. This implies the small eigenvalue λ_ϵ satisfies $\mathrm{Re}(\lambda_\epsilon) \le -c\epsilon^2$ for some $c > 0$ which is independent of ϵ and therefore is stable. We will show in Section 8 (Theorem 8.1) that for N segments with $N = 3, 4, ...$ it is possible to have either sign for $\nabla^2_{t^\epsilon}H(t^\epsilon,t^\epsilon)$ and so an interior spike can be stable or unstable. We will also derive easily verifiable criteria for stability or instability.

6. Spikes near the jump discontinuity x_b of the inhibitor diffusivity

The existence of spikes near the jump discontinuity is furnished by Theorem 7, derived in [25]. Stability of these solutions were not treated and we complete the picture for two segments now. As a preparation for the stability proof, we review the proof of existence. It is based on outer and inner expansion of the inhibitor h. We will see later that an outer and inner expansion for the inhibitor part of the eigenfunction is necessary for the stability proofs.

We first compute an approximation to the inhibitor function $h_\epsilon(x)$. Let

$$a_\epsilon(x) = \xi_\epsilon w\left(\frac{x-t^\epsilon}{\epsilon}\right)\chi\left(\frac{x-t^\epsilon}{\epsilon}\right) + O(\epsilon) \quad \text{in } H^2(\Omega_\epsilon),$$

where t_ϵ is the center of the spike, $x_b - t^\epsilon = \epsilon L$ and $\frac{\xi_\epsilon}{\xi_0} \to 1$ as $\epsilon \to 0$.

We decompose the second component, h_ϵ, into two parts:

$$h_\epsilon(x) = \xi_\epsilon\left(\epsilon h_1\left(\frac{x-t^\epsilon}{\epsilon}\right) + h_2(x)\right) + O(\epsilon) \quad \text{in } H^{2,*}(\Omega), \tag{6.1}$$

where the inner expansion $h_1(y)$ for $y = (x-t^\epsilon)/\epsilon$ satisfies

$$\begin{cases} (D(t^\epsilon+\epsilon y)h_{1,y}(y))_y + w^2(y) = 0, \\ h_1(0) = 0,\ h_{1,y}(0) = 0 \end{cases} \tag{6.2}$$

and the outer expansion $h_2(x)$ is given by

$$\begin{cases} (D(x)h_{2,x}(x))_x - h_2(x) - \epsilon h_1(x) = 0, \\ h_{2,x}(\pm 1) = -h_{1,y}(\pm\infty). \end{cases} \tag{6.3}$$

Integrating (6.2), we get

$$h_{1,y}(y) = \begin{cases} -\theta_1^2\rho(y), & -\infty < y < L, \\ -\theta_2^2\rho(y), & L < y < \infty, \end{cases} \tag{6.4}$$

where $\theta_i = D_i^{-1/2}$ and $\rho(y)$ has been defined in (2.2).

Recalling from (2.3) that

$$\alpha = \int_0^\infty w^2(z)\,dz = 3$$

we have

$$h_{1,y}(-\infty) = \alpha\theta_1^2, \quad h_{1,y}(\infty) = -\alpha\theta_2^2.$$

Integrating (6.4) once more, we have (up to order $O(\epsilon)$ which is included into the error term in (6.1))

$$\epsilon h_1\left(\frac{x-t^\epsilon}{\epsilon}\right)=\begin{cases}\theta_1^2\alpha(x-x_b), & -1<x<x_b,\\ -\theta_2^2\alpha(x-x_b), & x_b<x<1.\end{cases} \tag{6.5}$$

Hence by (6.4) h_2 satisfies (up to order $O(\epsilon)$ which is included into the error term in (6.1))

$$\begin{cases}(D(x)h_{2,x}(x))_x-h_2(x)-\epsilon h_1(x)=0,\\ h_{2,x}(-1)=-\theta_1^2\alpha, \quad h_{2,x}(1)=\theta_2^2\alpha.\end{cases} \tag{6.6}$$

Solving (6.6), using (6.5), we get

$$h_2(x)=\begin{cases}-\theta_1^2\alpha(x-x_b)+A\theta_1\dfrac{\cosh\theta_1(x+1)}{\cosh\theta_1(x_b+1)}, & -1<x<x_b,\\ \theta_2^2\alpha(x-x_b)+B\theta_1\dfrac{\cosh\theta_2(x-1)}{\cosh\theta_2(x_b-1)}, & x_b<x<1.\end{cases}$$

Continuity of the function $h_2(x)$ at $x=x_b$ gives $A=B$ and continuity of $D(x)h_{2,x}(x)$ at $x=x_b$ implies

$$0=D_1h_{2,x}(x_b^-)-D_2h_{2,x}(x_b^+)=A\left(\tanh\theta_1(x_b+1)+\frac{\theta_1}{\theta_2}\tanh\theta_2(1-x_b)\right)-2\alpha$$

and so we have

$$A=\frac{2\alpha\theta_2}{\theta_2\tanh\theta_1(x_b+1)+\theta_1\tanh\theta_2(1-x_b)}.$$

Hence

$$D_1h_{2,x}(x_b^-)=D_2h_{2,x}(x_b^+)=A\tanh\theta_1(x_b+1)-\alpha=\alpha\frac{\theta_2\tanh\theta_1(x_b+1)-\theta_1\tanh\theta_2(1-x_b)}{\theta_2\tanh\theta_1(x_b+1)+\theta_1\tanh\theta_2(1-x_b)}$$

which implies

$$h_{2,x}(x_b^-)=\theta_1^2\alpha\beta, \quad h_{2,x}(x_b^+)=\theta_2^2\alpha\beta,$$

where

$$\beta:=\frac{\theta_2\tanh\theta_1(x_b+1)-\theta_1\tanh\theta_2(1-x_b)}{\theta_2\tanh\theta_1(x_b+1)+\theta_1\tanh\theta_2(1-x_b)}.$$

(It is noteworthy that the parameter β satisfy $0<\beta<1$ and is a constant combining the most important information around a jump. One will see later that the equilibrium behaviors of the system is essentially determined by this value.)

Finally, we consider the solvability condition for Liapunov-Schmidt reduction which is given by

$$\begin{aligned}
0 &= \xi_\epsilon^{-1}\int_{-\infty}^{\infty} w^3(y)h_x(t^\epsilon+\epsilon y)\,dy + O(\epsilon)\\
&= \int_{-\infty}^{\infty} w^3(y)(h_{1,y}(y)+h_{2,x}(t^\epsilon+\epsilon y))\,dy + O(\epsilon)\\
&= \int_{-\infty}^{L} w^3(y)(-\theta_1^2\rho(y)+h_{2,x}(x_b^-))\,dy + \int_{L}^{\infty} w^3(y)(-\theta_2^2\rho(y)+h_{2,x}(x_b^+))\,dy + O(\epsilon)\\
&= \theta_1^2\left(\int_{-\infty}^{L} w^3(y)(-\rho(y)+\alpha\beta)\,dy\right) + \theta_2^2\left(\int_{L}^{\infty} w^3(y)(-\rho(y)+\alpha\beta)\,dy\right) + O(\epsilon)\\
&= \theta_1^2\left(\int_{-\infty}^{\infty} w^3(y)(-\rho(y)+\alpha\beta)\,dy - \int_{L}^{\infty} w^3(y)(-\rho(y)+\alpha\beta)\,dy\right)\\
&\quad +\theta_2^2\left(\int_{L}^{\infty} w^3(y)(-\rho(y)+\alpha\beta)\,dy\right) + O(\epsilon)\\
&= \alpha\beta\theta_1^2\int_{-\infty}^{\infty} w^3(y)\,dy + (\theta_2^2-\theta_1^2)\int_{L}^{\infty} w^3(y)(-\rho(y)+\alpha\beta)\,dy + O(\epsilon)
\end{aligned}$$

since $\rho(y)$ is an odd function.

Hence, for given θ_1, θ_2, β, we need to find L such that

$$\beta\theta_1^2\int_{-\infty}^{\infty} w^3(y)\,dy + (\theta_2^2-\theta_1^2)\int_{L}^{\infty} w^3(y)(-\rho(y)/\alpha+\beta)\,dy = 0. \tag{6.7}$$

Remark 6.1. *We remark here that, in general, the form of the condition (6.7), used for the existence argument of a spike near a jump, is independent of the number of segments. That is, although here we show the analysis specifically for $N = 2$, the existence argument for an arbitrary number of segments requires exactly the same condition (6.7). The only difference among different cases is implicitly distinguished through the parameter β. For general domains with multiple segments, one only needs to replace the constants θ_1 and θ_2 by θ_k and θ_{k+1} in the immediate segments around the jump in consideration.*

We now discuss when condition (6.7) can be satisfied. Firstly, note that $\beta = 0$ (6.7) is not possible since we assumed $\theta_1 \neq \theta_2$. Secondly, the case $\beta < 0$ can be reduced to the case $\beta > 0$ by reflection at the center $x = 0$ of the domain: observe that by this reflection θ_1 and θ_2 are exchanged, x_b, t^ϵ, β all change signs, then note that the order of the locations of the jump discontinuity and the spike are reversed so that the equation $x_b = t^\epsilon + \epsilon y$ with $y = L$ changes to $-x_b = -t^\epsilon + \epsilon y$ with $y = -L$. As a result, (6.7) is transformed to

$$-\beta\theta_2^2\int_{-\infty}^{\infty} w^3(y)\,dy + (\theta_1^2-\theta_2^2)\int_{-L}^{\infty} w^3(y)(-\rho(y)/\alpha+\beta)\,dy = 0$$

which is equivalent to (6.7). Therefore, we shall always assume $\beta > 0$ which is equivalent to (3.1) A necessary condition for (6.7) is

$$\theta_1^2 < \theta_2^2,$$

as otherwise (6.7) implies by separating θ_1^2 and θ_2^2 on different sides of the equality

$$\beta\theta_1^2\int_{0}^{\infty} w^3(y)\,dy + \theta_1^2\int_{L}^{\infty} w^3(y)(-\rho(y)/\alpha+\beta)\,dy = \theta_2^2\int_{L}^{\infty} w^3(y)(-\rho(y)/\alpha+\beta)\,dy$$

for which l.h.s. is obviously bigger than r.h.s. if $\theta_1 \geq \theta_2$ which gives a contradiction.

We now study (6.7) in detail. An important observation is that the integrand of

$$\int_L^\infty w^3(y)(-\rho(y)/\alpha+\beta)\,dy$$

changes sign around $\rho(y)=\alpha\beta$. The function ρ has the following properties:

$$\rho(0)=0, \quad \rho^{'}(y)=w^2(y)>0, \quad \rho(-y)=-\rho(y),$$

$$\rho(y)\to\int_0^\infty w^2\,dy=\alpha(=3) \text{ as } y\to\infty \tag{6.8}$$

and β satisfies the inequality

$$0<\beta<1.$$

Thus for all $0<\beta<1$ there is exactly one positive $y=:L_0>0$ such that $\rho(L_0)-\alpha\beta=0$. Further, $\rho(y)-\alpha\beta<0$ if $0<y<L_0$ and $\rho(y)-\alpha\beta>0$ if $y>L_0$.

To give an explicit formula for L_0, using (2.3) we compute

$$\rho(L_0)=\frac{9}{2}\tanh\frac{L_0}{2}-\frac{3}{2}\tanh^3\frac{L_0}{2}=\alpha\beta.$$

From this equation L_0 can be uniquely calculated.

Recall from (3.6) that for any real number L we have defined

$$I(L):=\int_L^\infty w^3(y)(\rho(y)/\alpha-\beta)\,dy.$$

Then

$I(L)\to 0$ as $L\to\infty$, $\quad I(L)\to -7.2\beta<0$ as $L\to-\infty$

$I(L)$ achieves its unique maximum among all real L at $L=L_0>0$, where $I(L_0)>0$.

$I(L)$ is monotone increasing on $(-\infty,L_0)$.

$I(L)$ is monotone decreasing on (L_0,∞).

$I(L)=0$ for a unique $L=L_1<0$.

We give an elementary interpretation of $I(L)$ using the following two graphs.

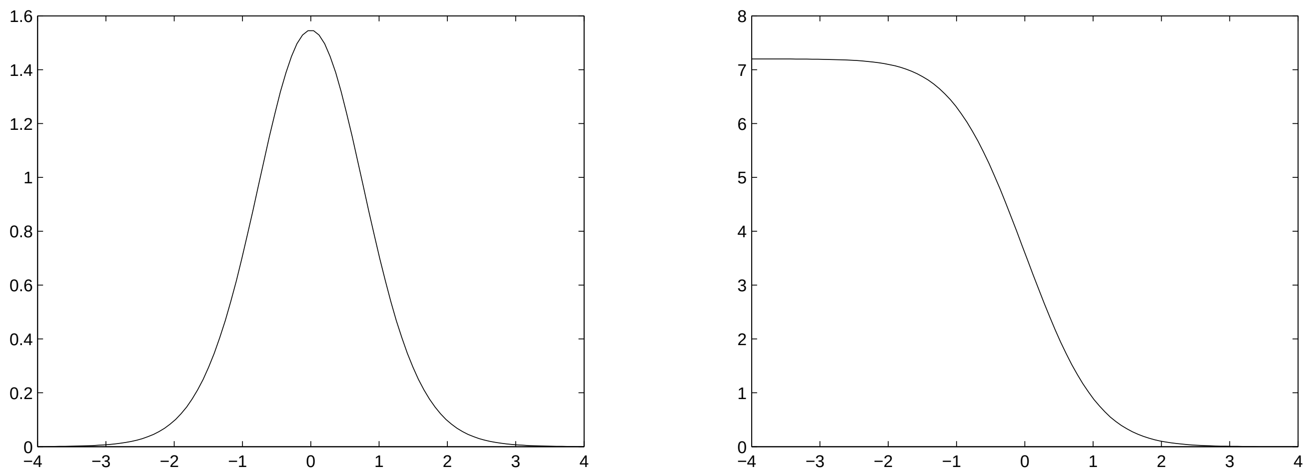

Figure A. *The left and right figures corresponds to the following two integrals depending on L respectively:*

$$\int_L^\infty w^3(y)\rho(y)\,dy \quad \textit{and} \quad \int_L^\infty w^3(y)\,dy.$$

$I(L)$ is simply the linear combination of the two graphs with the weighting $-\beta$ on the latter (note that $0 < \beta, \rho(y)/\alpha < 1$).

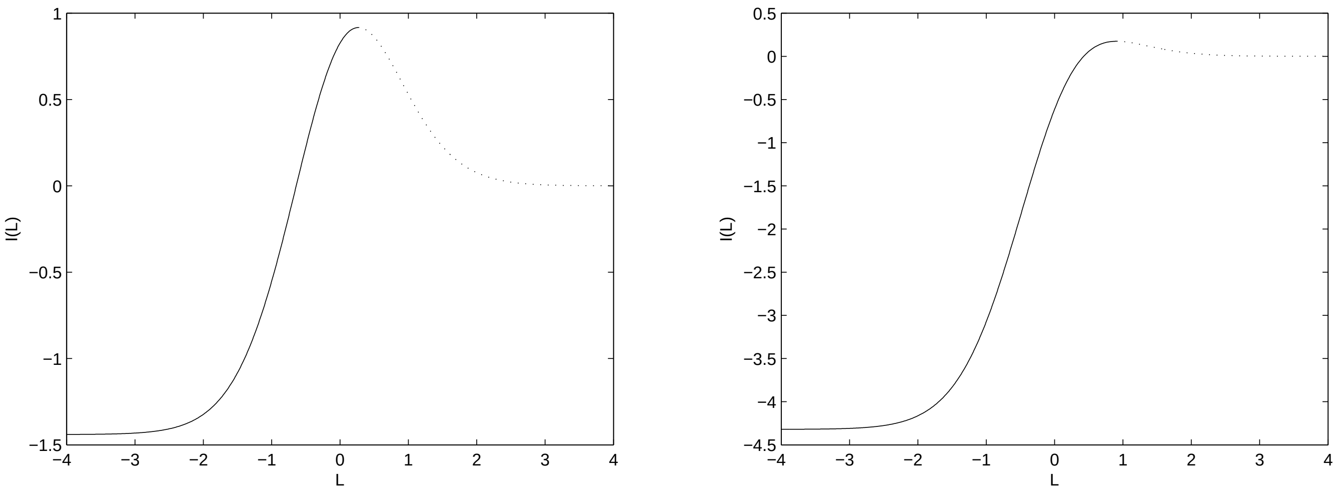

Figure B. *The left and right figures corresponds to $I(L)$ for $\beta = 0.2$ and $\beta = 0.6$, with maximum values about 0.92 and 0.17, located at about 0.26 and 0.90 respectively. As will be shown in the next section, the dotted lines and solid lines corresponds to unstable and stable steady states respectively.*

Therefore, the equation $I(L) = c$ has

$$\begin{cases} \text{two solutions} & \text{if } 0 < c < I(L_0), \\ \text{one solution} & \text{if } c = I(L_0) \text{ or } -7.2\beta < c \le 0, \\ \text{no solution} & \text{if } c > I(L_0) \text{ or } c \le -7.2\beta. \end{cases} \tag{6.9}$$

Because of $\theta_1 < \theta_2$ for (6.7) only the case $c > 0$ is relevant. Combining (6.9) with (6.7), we have

(i) two solutions for (6.7) if

$$0 < \beta < \frac{\theta_2^2 - \theta_1^2}{2\theta_1^2} \frac{I(L_0)}{3.6}.$$

(ii) one solution for (6.7) if

$$\beta = \frac{\theta_2^2 - \theta_1^2}{2\theta_1^2} \frac{I(L_0)}{3.6}.$$

(iii) no solution for (6.7) if

$$\beta > \frac{\theta_2^2 - \theta_1^2}{2\theta_1^2} \frac{I(L_0)}{3.6}.$$

This shows Theorem 7. □

7. Stability Analysis II: Small Eigenvalues of the Spike near the Jump

In this section, we investigate the eigenvalue problem which after re-scaling becomes

$$\epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + 2\frac{\bar{w}_\epsilon}{\bar{h}_\epsilon}\phi_\epsilon - \frac{\bar{w}_\epsilon^2}{\bar{h}_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \tag{7.1}$$

$$(D(x)\psi_{\epsilon,x})_x - \psi_\epsilon + 2\xi_\epsilon\bar{w}_\epsilon\phi_\epsilon = \lambda_\epsilon\tau\psi_\epsilon,$$

where

$$\bar{w}_\epsilon = \xi_\epsilon^{-1}\left[w_{\epsilon,t^\epsilon} + \phi_{\epsilon,t^\epsilon}\right], \quad \bar{h}_\epsilon = T[w_{\epsilon,t^\epsilon} + \phi_{\epsilon,t^\epsilon}], \tag{7.2}$$

$t^\epsilon = x_b - \epsilon L$ is the center of the spike which has been determined in Theorem , and ξ_ϵ is given by (4.8).

In particular, we investigate the small eigenvalue, i.e. we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$.

Let us define

$$\tilde{w}_{\epsilon,0}(x) = \chi\left(\frac{x - t^\epsilon}{r_0}\right)\bar{w}_\epsilon(x), \tag{7.3}$$

where r_0 and $\chi(x)$ are given in (4.2) and (4.3), respectively. We define

$$\mathcal{K}^{new}_{\epsilon,t^\epsilon} := \text{span}\,\{\tilde{w}_{\epsilon,0}^{'}\} \subset H^2(\Omega_\epsilon),$$

$$\mathcal{C}^{new}_{\epsilon,t^\epsilon} := \text{span}\,\{\tilde{w}_{\epsilon,0}^{'}\} \subset L^2(\Omega_\epsilon).$$

Then it is easy to see that

$$\bar{w}_\epsilon(x) = \tilde{w}_{\epsilon,0}(x) + \text{e.s.t.} \tag{7.4}$$

Further

$$\bar{h}_{\epsilon,x}(t^\epsilon + \epsilon y) = h_{1,y}(y) + h_{2,x}(x_b^-) + O(\epsilon)$$

$$= \theta_1^2(-\rho(y) + 3\beta) + O(\epsilon) \quad \text{for } -\infty < y < L \tag{7.5}$$

and

$$\bar{h}_{\epsilon,x}(t^\epsilon + \epsilon y) = h_{1,y}(y) + h_{2,x}(x_b^+) + O(\epsilon)$$

$$= \theta_2^2(-\rho(y) + 3\beta) + O(\epsilon) \quad \text{for } L < y < \infty. \tag{7.6}$$

Note that $\tilde{w}_{\epsilon,0}(x) = \hat{\xi}_0\tilde{w}_0(x) + O(\epsilon)$ in $H^2_\epsilon(-1,1)$ and $\tilde{w}_{\epsilon,0}$ satisfies

$$\epsilon^2\Delta\tilde{w}_{\epsilon,0} - \tilde{w}_{\epsilon,0} + \frac{(\tilde{w}_{\epsilon,0})^2}{\bar{h}_\epsilon} + \text{e.s.t.} = 0.$$

Thus $\tilde{w}_{\epsilon,0}^{'} := \frac{d\tilde{w}_{\epsilon,0}}{dx}$ satisfies

$$\epsilon^2\Delta\tilde{w}_{\epsilon,0}^{'} - \tilde{w}_{\epsilon,0}^{'} + \frac{2\tilde{w}_{\epsilon,0}}{\bar{h}_\epsilon}\tilde{w}_{\epsilon,0}^{'} - \frac{\tilde{w}_{\epsilon,0}^2}{(\bar{h}_\epsilon)^2}\bar{h}_\epsilon^{'} + \text{e.s.t.} = 0. \tag{7.7}$$

Let us now decompose

$$\phi_\epsilon = \epsilon a_0^\epsilon\tilde{w}_{\epsilon,0}^{'} + \phi_\epsilon^\perp \tag{7.8}$$

with complex numbers a_0^ϵ, (the scaling factor ϵ is introduced to ensure $\phi_\epsilon = O(1)$ in $H^2_{loc}(\Omega_\epsilon)$), where $\phi_\epsilon^\perp \perp \mathcal{K}^{new}_{\epsilon,t^\epsilon}$.

Suppose that $\|\phi_\epsilon\|_{H^2_N(\Omega_\epsilon)} = 1$. Then $|a_j^\epsilon| \le C$.

The decomposition of ϕ_ϵ implies the following decomposition of ψ_ϵ:

$$\psi_\epsilon = \epsilon a_0^\epsilon\psi_{\epsilon,0} + \psi_\epsilon^\perp = \epsilon a_0^\epsilon[\psi_1(y) + \psi_2(x)] + \psi_\epsilon^\perp,$$

where the inner expansion $\psi_1(y)$ is given by

$$\begin{cases} (D(t^\epsilon + \epsilon y)\psi_{1,y}(y))_y + 2w(y)w_y(y) = 0, \\ \psi_1(0) = 0,\ \psi_{1,y}(0) = -\theta_1^2 w^2(0) \end{cases} \tag{7.9}$$

and the outer expansion $\psi_2(x)$ satisfies

$$\begin{cases} (D(x)\psi_{2,x}(x))_x - \psi_2(x) - \psi_1(x) = 0, \\ \psi_{2,x}(\pm 1) = -\psi_{1,y}(\pm\infty). \end{cases} \tag{7.10}$$

Integrating (7.9), we get

$$\psi_{1,y}(y) = \begin{cases} -\theta_1^2 w^2(y), & -\infty < y < L, \\ -\theta_2^2 w^2(y), & L < y < \infty, \end{cases} \tag{7.11}$$

$$= \begin{cases} -\frac{9}{4}\theta_1^2(\cosh^{-4}(y/2) - 1), & -\infty < y < L, \\ -\frac{9}{4}\theta_2^2(\cosh^{-4}(y/2) - 1), & L < y < \infty. \end{cases}$$

This implies

$$\psi_{1,y}(-\infty) = O(\epsilon), \quad \psi_{1,y}(\infty) = O(\epsilon).$$

Integrating ψ_1 once more, we have

$$\psi_1(y) = \begin{cases} -\theta_1^2\rho(y), & -\infty < y < L, \\ -\theta_2^2\rho(y) + (\theta_2^2 - \theta_1^2)\rho(L), & L < y < \infty. \end{cases} \tag{7.12}$$

in the case $L \geq 0$ and a similar result holds for $L < 0$. The important observation now is that $\psi_1(y)$ is continuous at $y = L$. (Note that $\psi_{1,y}(y)$ has a jump at $y = L$.)

Hence ψ_2 satisfies (up to order $O(\epsilon)$ which is included into the error term in (7.10))

$$\begin{cases} (D(x)\psi_{2,x}(x))_x - \psi_2(x) - \psi_1(x) = 0, \\ \psi_{2,x}(-1) = 0, \quad \psi_{2,x}(1) = 0, \end{cases} \tag{7.13}$$

which implies $\psi_2 = O(1)$ in $H_N^{2,*}(\Omega)$.

Substituting the decompositions of ϕ_ϵ and ψ_ϵ into (7.1), we have, using (7.7),

$$\begin{aligned} &\epsilon a_0^\epsilon \left(\frac{(\tilde{w}_{\epsilon,0})^2}{\bar{h}_\epsilon^2} \bar{h}_\epsilon' - \frac{(\bar{w}_\epsilon)^2}{\bar{h}_\epsilon^2} \psi_{\epsilon,0} \right) \\ &+\epsilon^2 \Delta\phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\frac{\bar{w}_\epsilon}{\bar{h}_\epsilon}\phi_\epsilon^\perp - \frac{\bar{w}_\epsilon^2}{\bar{h}_\epsilon^2}\psi_\epsilon^\perp - \lambda_\epsilon\phi_\epsilon^\perp + \text{e.s.t.} \quad = \quad \lambda_\epsilon \epsilon a_0^\epsilon \tilde{w}_{\epsilon,0}'. \end{aligned} \tag{7.14}$$

We first compute

$$\begin{aligned} I_4 \quad &:= \quad \epsilon a_0^\epsilon \left(\frac{(\tilde{w}_{\epsilon,0})^2}{\bar{h}_\epsilon^2} \bar{h}_\epsilon' - \frac{(\bar{w}_\epsilon)^2}{\bar{h}_\epsilon^2} \psi_{\epsilon,0} \right) \\ &= \quad \epsilon a_0^\epsilon \frac{(\tilde{w}_{\epsilon,0})^2}{\bar{h}_\epsilon^2} \left[-\psi_{\epsilon,0} + \bar{h}_\epsilon' \right] + \text{e.s.t.}. \end{aligned}$$

Let us also put

$$\tilde{L}_\epsilon \phi_\epsilon^\perp := \epsilon^2 \Delta\phi_\epsilon^\perp - \phi_\epsilon^\perp + \frac{2\bar{w}_\epsilon}{\bar{h}_\epsilon}\phi_\epsilon^\perp - \frac{\bar{w}_\epsilon^2}{\bar{h}_\epsilon^2}\psi_\epsilon^\perp. \tag{7.15}$$

Multiplying both sides of (7.14) by $\tilde{w}'_{\epsilon,0}$ and integrating over $(-1,1)$, we obtain

$$\begin{aligned} \text{r.h.s.} &= \epsilon\lambda_\epsilon a_0^\epsilon \int_{-1}^1 \tilde{w}'_{\epsilon,0}\tilde{w}'_{\epsilon,0}\,dx \\ &= \lambda_\epsilon a_0^\epsilon \hat{\xi}_0^2 \int_R (w_y(y))^2\,dy\ (1+O(\epsilon)) \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} \text{l.h.s.} &= -\epsilon a_0^\epsilon \int_{-1}^1 \frac{\tilde{w}_{\epsilon,0}^2}{\bar{h}_\epsilon^2}\left[\psi_{\epsilon,0} - \bar{h}'_\epsilon\right]\tilde{w}'_{\epsilon,0}\,dx + \int_{-1}^1 \frac{\tilde{w}_{\epsilon,0}^2}{\bar{h}_\epsilon^2}(\bar{h}'_\epsilon\phi_\epsilon^\perp)\,dx - \int_{-1}^1 \frac{\tilde{w}_{\epsilon,0}^2}{\bar{h}_\epsilon^2}(\psi_\epsilon^\perp\tilde{w}'_{\epsilon,0})\,dx \\ &= J_1 + J_2 + J_3 + O(\epsilon^2), \end{aligned}$$

where J_i, $i=1,2,3$, are defined by the last equality. The following is the key lemma.

Lemma 7.1. *We have*

$$\begin{aligned} J_1 &= \epsilon a_0^\epsilon \hat{\xi}_0^3(\theta_2^2-\theta_1^2)w^3(L)\left(\rho(L)/\alpha - \beta\right) + o(\epsilon) && (7.17)\\ &= -\epsilon a_0^\epsilon \hat{\xi}_0^3(\theta_2^2-\theta_1^2)I'(L) + o(\epsilon), && (7.18) \end{aligned}$$

$$J_2 + J_3 = o(\epsilon), \tag{7.19}$$

Using Lemma 7.1 and comparing l.h.s. with r.h.s., we obtain

$$-\epsilon\hat{\xi}_0^3(\theta_2^2-\theta_1^2)I'(L)a_0^\epsilon + o(\epsilon) = 1.2\lambda_\epsilon a_0^\epsilon \hat{\xi}_0^2\,(1+o(1)), \tag{7.20}$$

using (2.3). Equation (7.20) shows that the small eigenvalue λ_ϵ of (7.1) satisfies

$$\lambda_\epsilon = -\frac{1}{1.2}\epsilon\hat{\xi}_0(\theta_2^2-\theta_1^2)I'(L) + o(\epsilon).$$

This shows that if $I'(L)$ is positive, the small eigenvalue λ_ϵ satisfies $\mathrm{Re}(\lambda_\epsilon) \le -c\epsilon$ for some $c>0$ which is independent of ϵ. On the other hand, if $I'(L)$ is negative, then for ϵ sufficiently the system is unstable due to a Reduction Theorem (Theorem 8.1 of [25]). This, together with the results in Section 5, concludes the proof of the stability theorem for a spike near a jump. The result is given in Theorem 7.2.

□

Theorem 7.2. *The spikes near the jump have a small eigenvalue which satisfies the asymptotic expansion*

$$\lambda_\epsilon = -\frac{1}{1.2}\epsilon\hat{\xi}_0(\theta_2^2-\theta_1^2)I'(L) + o(\epsilon). \tag{7.21}$$

All the other eigenvalues are stable. This implies that the spike with the smaller L (the right one) is stable, the one with the larger L (the left one) is unstable.

Before proving Lemma 7.1, we derive the following estimate for $\phi_\epsilon^\perp$.

Lemma 7.3. *For ϵ sufficiently small, we have*

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = O(\epsilon). \tag{7.22}$$

Proof: As the first step in the proof of Lemma 7.3, we obtain a relation between $\psi_\epsilon^\perp$ and $\phi_\epsilon^\perp$. Note that $\tilde{L}_\epsilon$ is invertible from $(\mathcal{K}_\epsilon^{new})^\perp$ to $(\mathcal{C}_\epsilon^{new})^\perp$ with uniformly bounded inverse for ϵ small enough. By the fact that $\tilde{L}_\epsilon$ is uniformly invertible, we deduce that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = O(\epsilon). \tag{7.23}$$

□

Finally we prove the key lemma – Lemma 7.1.

Proof of Lemma 7.1: The computation of J_1 follows from the outer and inner expansions of h and $\psi_{\epsilon,0}$, respectively. In fact,

$$\begin{aligned}
J_1 &= -\epsilon a_0^\epsilon \int_{-1}^1 \frac{\tilde{w}_{\epsilon,0}^2}{\bar{h}_\epsilon^2}\left(\psi_{\epsilon,0} - \bar{h}_\epsilon'\right)\tilde{w}_{\epsilon,0}'\,dx + o(\epsilon)\\
&= -\epsilon a_0^\epsilon \int_{-1}^1 \frac{\tilde{w}_{\epsilon,0}^2}{\bar{h}_\epsilon^2(x_b)}\left(\psi_{\epsilon,0} - \bar{h}_\epsilon'\right)\tilde{w}_{\epsilon,0}'\,dx + o(\epsilon)\\
&= -\epsilon a_0^\epsilon \hat{\xi}_0^3 \int_{-\infty}^L w^2(y)w_y(y)\left[\psi_1(y) + \psi_2(x_b) - h_{1,y}(y) - h_{2,x}(x_b^-)\right]dy\\
&\quad -\epsilon a_0^\epsilon \hat{\xi}_0^3 \int_L^\infty w^2(y)w_y(y)\left[\psi_1(y) + \psi_2(x_b) - h_{1,y}(y) - h_{2,x}(x_b^+)\right]dy + o(\epsilon)\\
&= \epsilon a_0^\epsilon \hat{\xi}_0^3 \int_{-\infty}^L \frac{1}{3}w^3(y)\left\{\frac{d}{dy}[\psi_1(y) - h_{1,y}(y)] + [\epsilon\psi_{2,x}(x_b^-) - \epsilon h_{2,xx}(x_b^-)]\right\}dy\\
&\quad +\epsilon a_0^\epsilon \hat{\xi}_0^3 \int_L^\infty \frac{1}{3}w^3(y)\left\{\frac{d}{dy}[\psi_1(y) - h_{1,y}(y)] + [\epsilon\psi_{2,x}(x_b^+) - \epsilon h_{2,xx}(x_b^+)]\right\}dy\\
&\quad -\epsilon a_0^\epsilon \hat{\xi}_0^3 \frac{1}{3}w^3(L)\left[(\psi_1(L) + \psi_2(x_b) - h_{1,y}(L^-) - h_{2,x}(x_b^-)\right]\\
&\quad +\epsilon a_0^\epsilon \hat{\xi}_0^3 \frac{1}{3}w^3(L)\left[(\psi_1(L) + \psi_2(x_b) - h_{1,y}(L^+) - h_{2,x}(x_b^+)\right] + o(\epsilon)\\
&= \epsilon a_0^\epsilon \hat{\xi}_0^3 \frac{1}{3}w^3(L)\left[(h_{1,y}(L^-) - h_{1,y}(L^+)) + (h_{2,x}(x_b^-) - h_{2,x}(x_b^+))\right] + o(\epsilon)\\
&= \epsilon a_0^\epsilon \hat{\xi}_0^3 \frac{1}{3}(\theta_2^2 - \theta_1^2)w^3(L)[\rho(L) - 3\beta] + o(\epsilon)\\
&= -\epsilon a_0^\epsilon \hat{\xi}_0^3 (\theta_2^2 - \theta_1^2) I'(L) + o(\epsilon).
\end{aligned}$$

Here we have used

$$\frac{d}{dy}[\psi_1(y) - h_{1,y}(y)] = -\theta_1^2(w^2(y) - w^2(y)) = 0 \quad \text{for } -\infty < y < L,$$

$$\frac{d}{dy}[\psi_1(y) - h_{1,y}(y)] = -\theta_2^2(w^2(y) - w^2(y)) = 0 \quad \text{for } L < y < \infty.$$

Further, we have applied the estimates

$$h_{2,xx}(x_b^-) = O(1), \quad h_{2,xx}(x_b^+) = O(1),$$

$$\psi_{2,x}(x_b^-) = O(1), \quad \psi_{2,x}(x_b^+) = O(1)$$

and the facts that ψ_1 is continuous at $y = L$ and ψ_2 is continuous at $x = x_b$.

Thus we obtain (7.17).

By (7.23) and the equation for $\psi_\epsilon^\perp$ we have:

$$\begin{aligned}\psi_\epsilon^\perp(t_0^\epsilon) &= 2\epsilon\xi_\epsilon \int_{-1}^1 G(t^\epsilon, z)\bar{w}_\epsilon \tilde{\phi}_{\epsilon,0}\, dz \\ &= 2\epsilon G(t^\epsilon, t^\epsilon)\hat{\xi}_0 \frac{\int_R w\phi_\epsilon^\perp\, dy}{\int_R w^2\, dy} + O(\epsilon^2) = O(\epsilon).\end{aligned} \tag{7.24}$$

By (7.23), we have $J_2 = O(\epsilon^2)$ and (7.24) implies that $J_3 = O(\epsilon^2)$. The proof of Lemma 7.1 is finished. □

Remark 1: Large eigenvalues for a spike near the jump are treated in the same way as that of interior spikes. Since we only need to compute $\psi(x_b)$, and no derivatives of ψ are required, the inner expansion is not needed.

Remark 2: In view of Remark 6.1, if we turn to the case with an arbitrary number of segments, similar results as in Section 6 and Section 7 should also hold. Indeed, we will show the existence of a spike near a jump for N segments in Section 9 with minimal effort.

8. Existence of interior spikes for N segments

We now turn to the study of interior spikes for N segments. To show existence, we first compute the Green's function for N segments in $(-1, 1)$, where the Dirac delta distribution is located at one of the jumps:

Let $G(x, x_i)$ be the Green's function which is defined as the unique solution of the problem

$$\begin{cases} (D(x)G(x,x_i)_x)_x - G(x,x_i) + \delta_{x_i} = 0, & G_x(-1,x_i) = G_x(1,x_i) = 0, \\ D_iG_x(x_i^-,x_i) - D_{i+1}G_x(x_i^+,x_i) = 1, & G(x_i^-,x_i) - G(x_i^+,x_i) = 0, \\ D_jG_x(x_j^-,x_i) - D_{j+1}G_x(x_j^+,x_i) = 0, & G(x_j^-,x_i) - G(x_j^+,x_i) = 0,\ j \neq i, \end{cases} \tag{8.1}$$

where δ_{x_i} is the Dirac delta distribution located at x_i and $-1 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots < x_{N-1} < 1$. Let β_i be given by

$$\beta_i = D_iG_x(x_i^-, x_i) + D_{i+1}G_x(x_i^+, x_i). \tag{8.2}$$

Further, for $t_0 \neq x_i$ let $G(x, t_0)$ be the Green's function defined by

$$\begin{cases} (D(x)G(x,t_0)_x)_x - G(x,t_0) + \delta_{t_0} = 0, & G_x(-1,t_0) = G_x(1,t_0) = 0, \\ D(t_0)G_x(t_0^-,t_0) - D(t_0)G_x(t_0^+,t_0) = 1, & G(t_0^-,t_0) - G(t_0^+,t_0) = 0, \\ D(x_j^-)G_x(x_j^-,t_0) - D(x_j^+)G_x(x_j^+,t_0) = 0, & G(x_j^-,t_0) - G(x_j^+,t_0) = 0,\ j = 1,\ldots,N-1, \end{cases} \tag{8.3}$$

where δ_{t_0} is the Dirac delta distribution located at t_0 and $-1 < x_1 < x_2 < \cdots < x_{i-1} < t_0 < x_i < \cdots < x_{N-1} < 1$.

Our first goal is to prove results on the zeros of the derivative of the diagonal of the regular part of Green's function. This will imply existence or nonexistence of a spike.

Our second goal is to determine the sign of the second derivative at such a root. This will answer the question of stability or instability of the spike.

Therefore let us consider the regular part H of G. It is defined by

$$-H(x,t_0) = G(x,t_0) - \frac{\theta_i}{2} e^{-\theta_i |x-t_0|} \qquad \text{for } y \in (x_{i-1}, x_i).$$

If $t_0 = x_i$ the regular part of G is defined by

$$-H(x,y) = G(x,y) - \begin{cases} \dfrac{\theta_i}{2} e^{-\theta_i |x-x_i|} & , \quad x < x_i, \\ \dfrac{\theta_{i+1}}{2} e^{-\theta_{i+1} |x-x_i|} & , \quad x > x_i. \end{cases}$$

Then $H(x,y)$ satisfies the elliptic differential equation

$$(D(x)H_x(x,y))_x - H(x,y) = 0. \tag{8.4}$$

In particular, if $x \in (x_{i-1}, x_i)$ then

$$(D_i H_x(x,y))_x - H(x,y) = 0.$$

Note that by symmetry $H(x,y) = H(y,x)$. Denoting the derivative with respect to the first variable with subscript x, and that of the second variable with subscript y, we have

$$H_x(t_0,t_0) = H_y(t_0,t_0), \quad H'(t_0,t_0) := \frac{d}{dt} H(t_0,t_0) = 2H_x(t_0,t_0),$$
$$H_{xx}(t_0,t_0) = H_{yy}(t_0,t_0), \quad H''(t_0,t_0) = 2H_{xx}(t_0,t_0) + 2H_{xy}(t_0,t_0),$$
$$H_{xxx}(t_0,t_0) = H_{yyy}(t_0,t_0), \quad H_{xxy}(t_0,t_0) = H_{xyy}(t_0,t_0), \text{ etc.}$$

We try to find a solution of

$$H'(t_0,t_0) = 0 \quad \text{with} \quad t_0 \in (x_{i-1}, x_i)$$

and then determine

$$H^{''}(t_0,t_0) := \frac{d^2}{dx_0^2} H(t_0,t_0).$$

By (8.2) we have

$$D_i G_x(x_i^-, x_i) = \frac{\beta_i + 1}{2}, \; D_i G_x(x_{i-1}^+, x_{i-1}) = \frac{\beta_{i-1} - 1}{2}.$$

By the definition of H we have

$$-D_i H_x(x_i^-, x_i) = D_i G_x(x_i^-, x_i) - \frac{1}{2} = \frac{\beta_i}{2},$$
$$-D_i H_x(x_{i-1}^+, x_{i-1}) = D_i G_x(x_{i-1}^+, x_{i-1}) + \frac{1}{2} = \frac{\beta_{i-1}}{2}.$$

This implies

$$D_i H^{'}(x_{i-1}, x_{i-1}) = -\beta_{i-1}, \quad D_i H^{'}(x_i, x_i) = -\beta_i.$$

We consider three cases:

Theorem 8.1. *(i)* $\beta_{i-1} > 0$, $\beta_i < 0$.
Stable interior spike: There exists a unique interior spike which is stable.
(ii) $\beta_{i-1} < 0$, $\beta_i > 0$.
Unstable interior spike: There exists a unique interior spike which is unstable.
(iii) $\beta_{i-1} \cdot \beta_i > 0$.
No interior spike: There is no interior spike.

This gives a complete classification of all interior intervals.

Remark: For boundary intervals, there are two cases according to the signs of β_1 and β_{N-1}, and the existence or nonexistence of a stable interior spike follows from the classification for the two segment case.

Proof.

Case (i) $\beta_{i-1} > 0$, $\beta_i < 0$, Stable interior spike:

By the intermediate value theorem there exists at least one root of $H'(t_0, t_0)$ with $t_0 \in (x_{i-1}, x_i)$ such that $H'(t_0, t_0)$ changes sign at t_0 from negative to positive.

We show that for this root $H''(t_0, t_0) = 0$ is impossible. We argue by contradiction. Using (8.4) we get

$$H_{xxx}(t_0, t_0) = H_x(t_0, t_0) = 0,$$
$$H_{xxy}(t_0, t_0) = H_y(t_0, t_0) = 0$$

and so $H'''(t_0, t_0) = 0$.

Taking the derivatives with xx then with xy in (8.4) and adding up we get

$$H_{xxxx} + H_{xxxy} - (H_{xx} + H_{xy}) = 0.$$

Taking the derivatives with yy then with xy in (8.4) and adding up we get

$$H_{xxyy} + H_{xxxy} - (H_{yy} + H_{xy}) = 0.$$

Taking the sum of the two previous equations gives

$$H_{xxxx} + 2H_{xxxy} + H_{xxyy} = 0$$

which can be rewritten as

$$(H_{xx} + H_{xy})_{xx} + (H_{xx} + H_{xy})_{xy} = 0.$$

This is equivalent to

$$H''''(t_0, t_0) = 0.$$

In the same way it can be shown that

$$\frac{d^k}{dx_0^k} H(t_0, t_0) = 0, \quad k = 1, 2, \ldots$$

This implies $H(t_0, t_0) = c$ which contradicts the boundary conditions

$$H^{'}(x_{i-1}, x_{i-1}) = -\beta_{i-1}, \quad H^{'}(x_i, x_i) = -\beta_i.$$

Therefore $H''(t_0, t_0) = 0$ is impossible. Since H' changes sign from negative to positive at t_0 we necessarily have $H''(t_0, t_0) > 0$ and the spike at t_0 is stable.

We show that there is a unique root of $H'(t_0, t_0)$ in the interval (x_{i-1}, x_i). Assume that this is not the case, then there exist (at least) two such roots $x_a < x_b$. We assume w.l.o.g. that $H'(t_0, t_0) > 0$ for all t_0 with $x_a < t_0 < x_b$. Then obviously we have

$$H''(x_a, x_a) \geq 0, \quad H''(x_b, x_b) \leq 0.$$

Using (8.4) we get

$$H_{xx}(x_a, x_a) = \frac{1}{D_1} H(x_a, x_a) < \frac{1}{D_1} H(x_b, x_b) = H_{xx}(x_b, x_b). \tag{8.5}$$

Here we have used the fact that $H_x(t_0, t_0) > 0$ for $x_a < t_0 < x_b$.

Using (8.4) again, we get

$$H_{xy}(x_b, x_b) - H_{xy}(x_a, x_a)$$
$$= \int_{(x_a,x_a)}^{(x_b,x_b)} [H_{xxy}(x, y)\, dx + H_{xyy}(x, y)\, dy]$$
$$= \int_{(x_a,x_a)}^{(x_b,x_b)} [H_y(x, y)\, dx + H_x(x, y)\, dy] > 0 \tag{8.6}$$

since the last integrand is positive if we choose the path from (x_a, x_a) to (x_b, x_b) sufficiently close to the diagonal (x, x), $x_a < x < x_b$.

Together (8.5) and (8.6) imply that

$$H''(x_a, x_a) < H''(x_b, x_b)$$

which gives a contradiction.

To summarize, we have proved that in Case (i) there is a unique solution of $H'(t_0, t_0) = 0$ with $t_0 \in (x_{i-1}, x_i)$. Further, $H''(t_0, t_0) > 0$. By Proposition 4.1 there exists a unique interior spike. By (5.1) this spike is stable.

This completes the proof in Case (i).

The proofs in Cases (ii) and (iii) are similar and are therefore omitted.

□

We now consider the special case of three segments. An explicit computation of the Green's function for (8.1) which is done in Appendix A (Section 11) yields

$$\beta_1 = \frac{\mathrm{T1T3} + (\mathrm{T1} - \mathrm{T3})\frac{1}{\theta_2}\coth\theta_2(x_2 - x_1) - \frac{1}{\theta_2^2}}{\mathrm{T1T3} + (\mathrm{T1} + \mathrm{T3})\frac{1}{\theta_2}\coth\theta_2(x_2 - x_1) + \frac{1}{\theta_2^2}},$$

$$\beta_2 = -\frac{\mathrm{T1T3} + (\mathrm{T3} - \mathrm{T1})\frac{1}{\theta_2}\coth\theta_2(x_2 - x_1) - \frac{1}{\theta_2^2}}{\mathrm{T1T3} + (\mathrm{T1} + \mathrm{T3})\frac{1}{\theta_2}\coth\theta_2(x_2 - x_1) + \frac{1}{\theta_2^2}},$$

where

$$\mathrm{T1} = \frac{1}{\theta_1}\tanh\theta_1(x_1 + 1), \quad \mathrm{T3} = \frac{1}{\theta_3}\tanh\theta_3(1 - x_2).$$

In particular, we have the following two cases:

Case (i) $\beta_1 < 0$ and $\beta_2 > 0$ for

$$\frac{1}{\theta_1}\tanh\theta_1(x_1 + 1) < \frac{1}{\theta_2}.$$

There exists a stable interior spike in the central interval. There exists no spike in either the left or right interval.

Case (ii) $\beta_1 > 0$ and $\beta_2 < 0$ for

$$\frac{1}{\theta_1}\tanh\theta_1(x_1 + 1) > \frac{1}{\theta_2}.$$

There exists an unstable interior spike in the central interval. There exists a stable spike in both the left or right interval.

We illustrate this behavior in Figures 3 and 4 for $x_1 = -0.4$, $x_2 = 0.4$ and $\epsilon^2 = 0.0001$. We consider the choices $\theta_1 = \theta_3 = 1$, $\theta_2 = \frac{1}{\sqrt{5}}$ which belongs to Case (i) and $\theta_1 = \theta_3 = \frac{1}{\sqrt{5}}$, $\theta_2 = 1$ which belongs to Case (ii).

In Case (i) we show a stable interior spike in the central interval and a stable spike near a jump.

In Case (ii) we show a stable interior spike in one of the boundary intervals and a stable boundary spike.

9. Existence of a spike near a jump for N segments

We now consider a spike near a jump for N segments.

Let β_i be given by (8.2).

Using β_i, θ_i, θ_{i+1} in (6.7) we get

$$\beta_i\theta_i^2 \int_0^\infty w^2\,dy \int_{-\infty}^\infty w^3\,dy = (\theta_{i+1}^2 - \theta_i^2) \int_L^\infty w^3(y) \left(\rho(y) - \beta_i \int_0^\infty w^2\,dz\right) dy.$$

Separating $\frac{\theta_{i+1}^2}{\theta_i^2} > 1$ on the l.h.s. and β_i on the r.h.s. we get

$$\frac{1}{\frac{\theta_{i+1}^2}{\theta_i^2} - 1} \int_0^\infty w^2\,dy \int_{-\infty}^\infty w^3\,dy = \int_L^\infty w^3(y) \left(\frac{1}{\beta_i}\int_0^y w^2(z)\,dz - \int_0^\infty w^2(z)\,dz\right) dy.$$

This implies the necessary and condition condition for existence of a suitable real number L to solve this equality (for given $\frac{\theta_{i+1}}{\theta_i}$ and β_i) which is given in the following theorem:

Theorem 9.1. *Let $\beta_i > 0$ be given by (8.2) and let $\theta_i = D_i^{-1/2}$, $\theta_{i+1} = D_{i+1}^{-1/2}$ be given by the neighboring diffusion constants. Then there exist spikes near the jump if and only if*

$$0 < \frac{1}{\frac{\theta_{i+1}^2}{\theta_i^2} - 1} \int_0^\infty w^2\,dy \int_{-\infty}^\infty w^3\,dy <$$

$$< \max_{L\in(-\infty,\infty)} \int_L^\infty w^3(y) \left(\frac{1}{\beta_i}\int_0^y w^2(z)\,dz - \int_0^\infty w^2(z)\,dz\right) dy$$

$$= \int_{L_0}^\infty w^3(y) \left(\frac{1}{\beta_i}\int_0^y w^2(z)\,dz - \int_0^\infty w^2(z)\,dz\right) dy,$$

where L_0 is given by

$$\frac{1}{\beta_i}\int_0^{L_0} w^2(z)\,dz = \int_0^\infty w^2(z)\,dz.$$

Remark. In the previous characterization the left-hand side is a function of $\frac{\theta_{i+1}}{\theta_i}$ only and the right-hand side is a function of β_i only (note that L has been eliminated). This gives a simple criterion for existence of a spike near a jump involving only these two quantities which both come from the order 1 length-scale. Note that for solvability no quantities of the order ϵ length-scale are required.

An example of a spike near a jump for three segments in presented in Figure 3.

10. Numerical Computations

We now show some numerical computations for system (1.1). We display different types of stable one-spike solutions for the two- and the three-segment case. We choose $\Omega = (-1, 1)$ and varying diffusion value for the coefficients ϵ^2 and $D(x)$.

In each situation we always present the solution for $t = 10^5$. By this time, in all cases, the computation has come to a standstill and this steady state is numerically stable (long-time limit). The first component, a, is shown on the left, the second component, h, on the right.

In the two-segment case we divide Ω at either $x_b = 0$ or $x_b = 0.5$ and choose different constants for $D(x)$ on each of the resulting subintervals.

In the three-segment case we divide Ω at either $x_1 = -0.4$ and $x_2 = 0.4$ and choose different constants for $D(x)$ on the resulting subintervals.

We now show a computation with a spike near the jump discontinuity of the inhibitor diffusion or an interior spike. Note that the spike near the jump discontinuity is located slightly left of it.

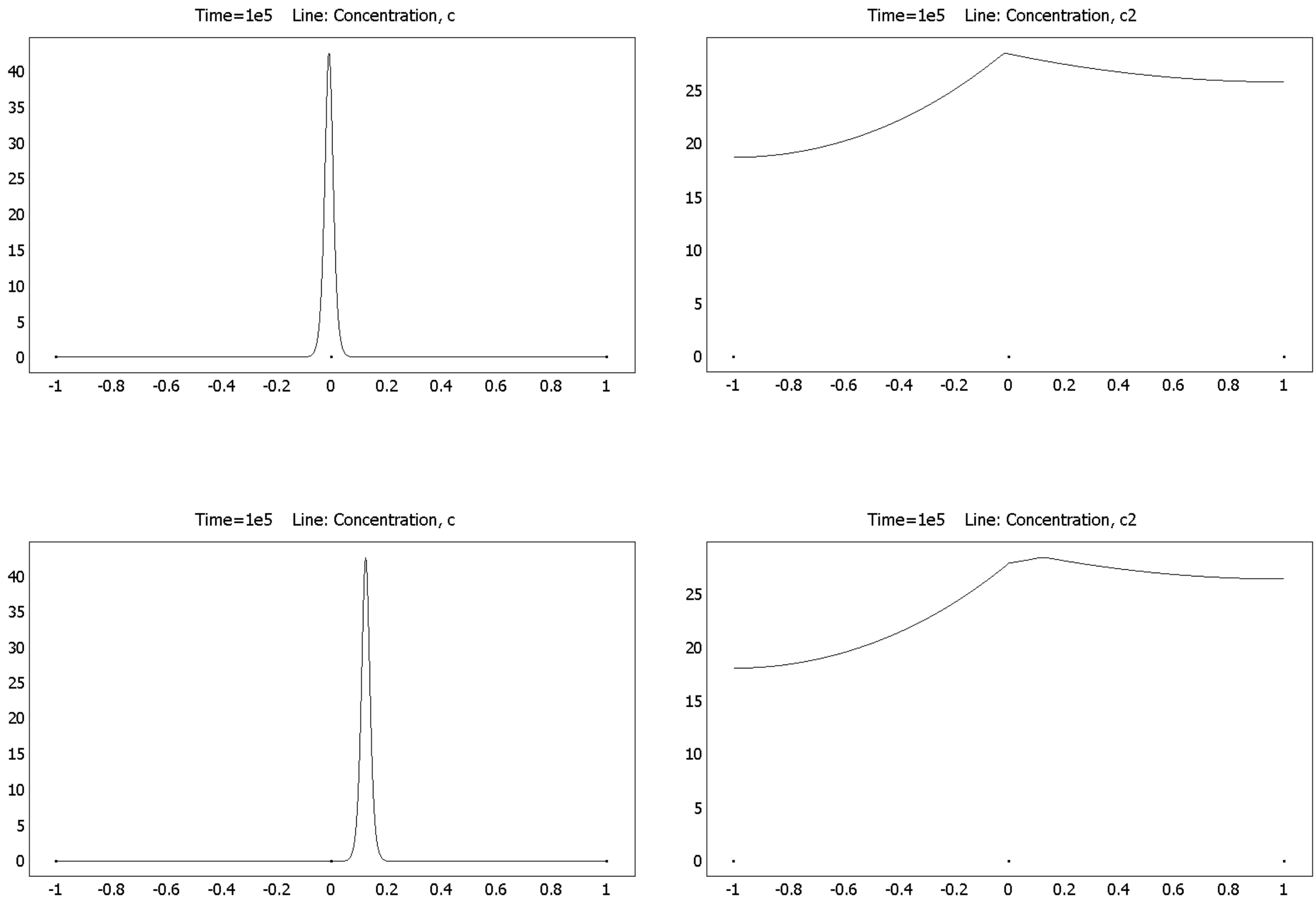

Figure 1. *Long-time limit of the solution to (1.1) – (1.3) with* $\epsilon^2 = 0.0001$ *and* $D(x) = 1$ *for* $-1 < x < 0$, $D(x) = 5$ *for* $0 < x < 1$. *We observe a spike near the jump discontinuity of the inhibitor diffusivity and a spike in the right subinterval, respectively. The conditions (3.1) and (3.5), respectively, are satisfied.*

Moving the jump discontinuity from $x_b = 0.5$ to $x_b = 0$ we obtain an interior spike at a position near the center $x = 0$.

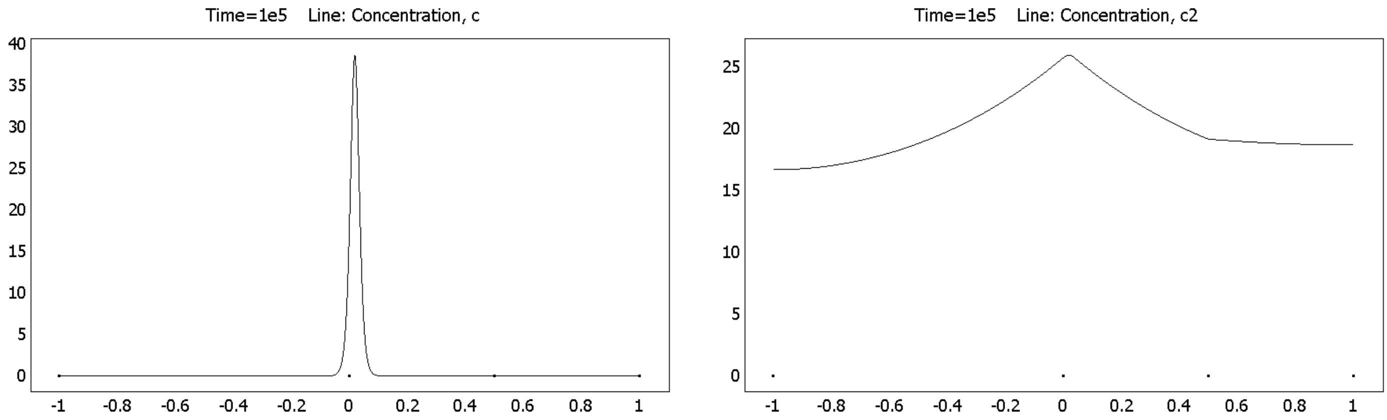

Figure 2. *Long-time limit of the solution to (1.1) – (1.3) with* $\epsilon^2 = 0.0001$ *and* $D(x) = 1$ *for* $-1 < x < 0.5$, $D(x) = 5$ *for* $0.5 < x < 1$. *We observe an interior spike in the left subinterval. The condition (3.5) is satisfied.*

Considering three segments with high values for D in the central segment we get an interior spike in the central segment or at one of the jumps.

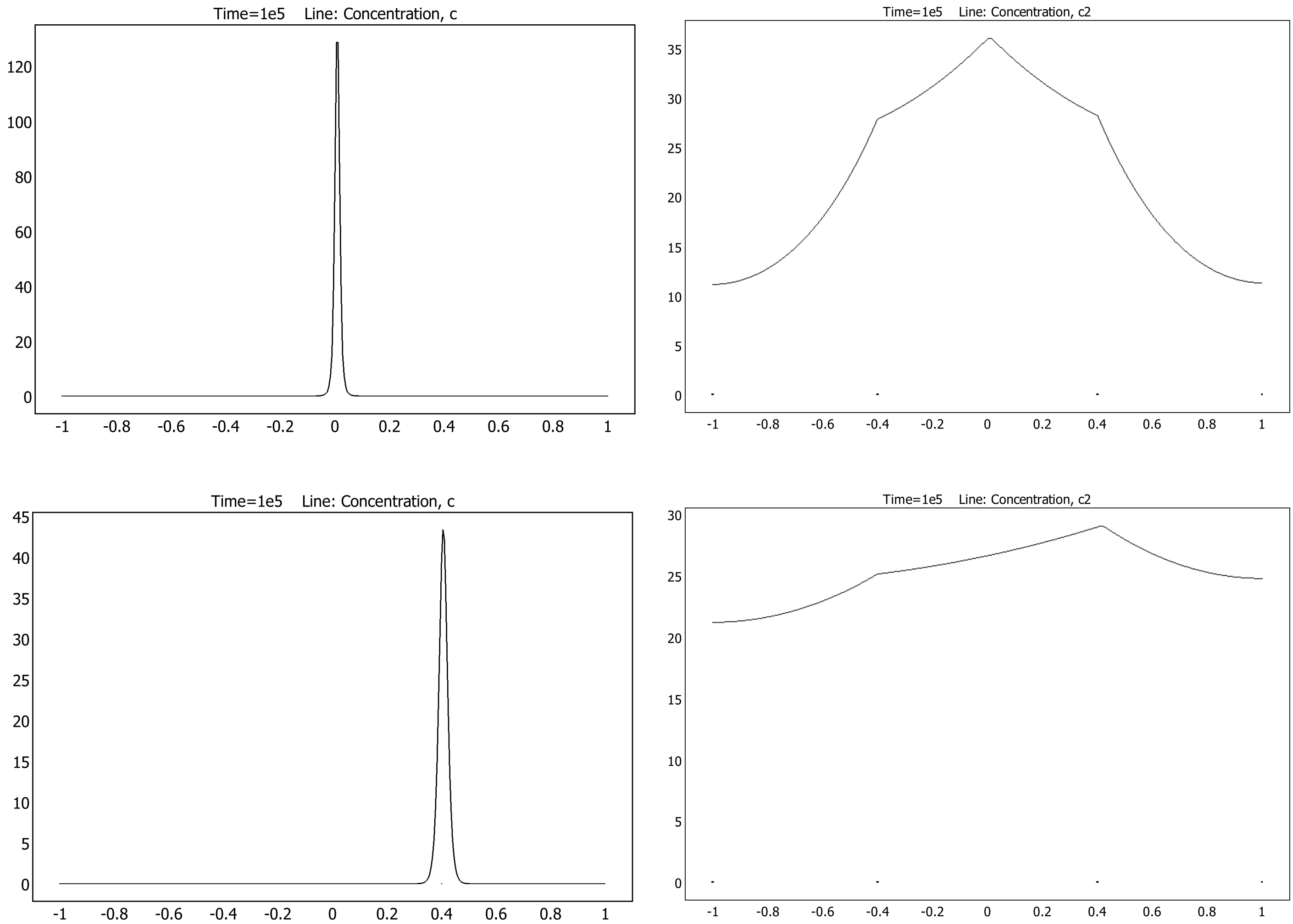

Figure 3. *Long-time limit of the solution to (1.1) – (1.3) with* $\epsilon^2 = 0.0001$ *and* $D(x) = 1$ *for* $-1 < x < -0.4$, $0.4 < x < 1$, $D(x) = 5$ *for* $-0.4 < x < 0.4$. *We observe an interior spike in the central interval and a spike near a jump.*

Considering three segments with high values for D in the left and right segments we get an interior spike in the left or right segment or a boundary spike.

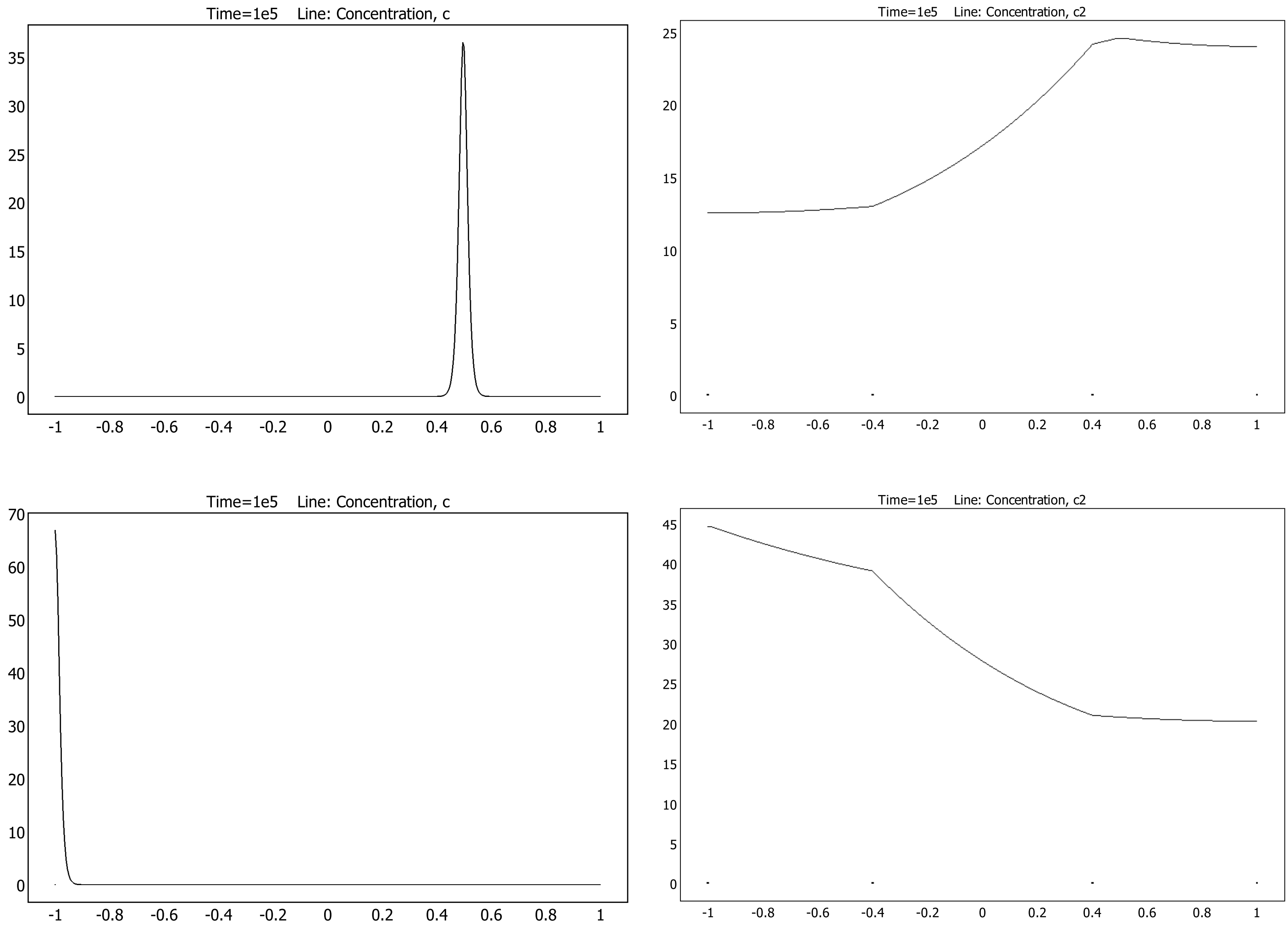

Figure 4. *Long-time limit of the solution to (1.1) – (1.3) with* $\epsilon^2 = 0.0001$ *and* $D(x) = 5$ *for* $-1 < x < -0.4$, $0.4 < x < 1$, $D(x) = 1$ *for* $-0.4 < x < 0.4$. *We observe an interior spike in the right interval and a boundary spike.*

Now we show the computations for some effects not analyzed in this paper. We compute the following situation: $\epsilon^2 = 0.0001$, $D(x) = 0.1$ for $-1 < x < x_b$, $D(x) = 0.5$ for $x_b < x < 1$ for varying x_b. If we make $D(x)$ smaller we expect solutions with multiple spikes. Some examples for this are shown in the following two figures.

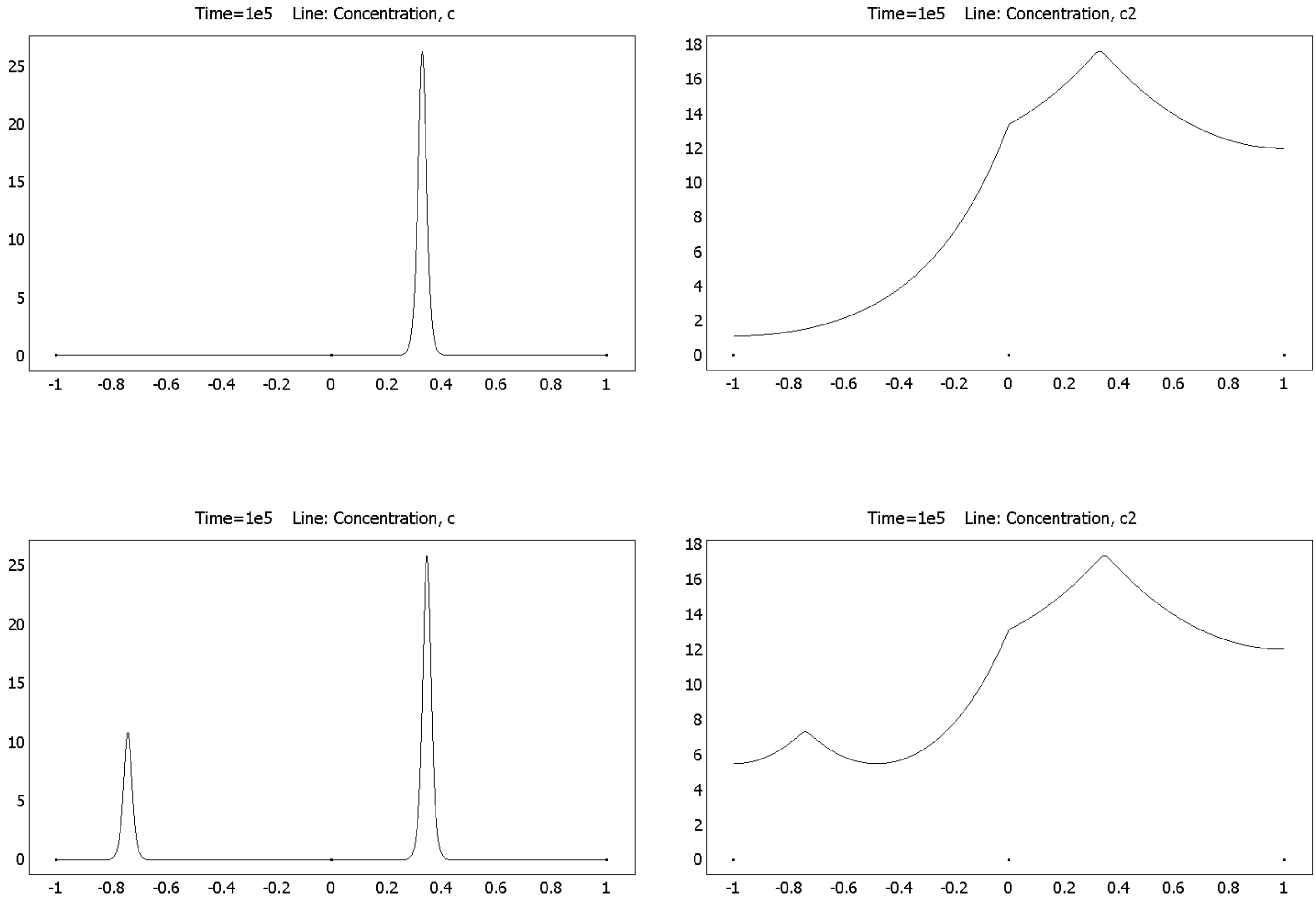

Figure 5. *Long-time limit of the solution to (1.1) – (1.3) with* $\epsilon^2 = 0.0001$ *and* $D(x) = 0.1$ *for* $-1 < x < 0$, $D(x) = 0.5$ *for* $0 < x < 1$*. We observe an interior spike on the right subinterval or two interior spikes on different subintervals, respectively.*

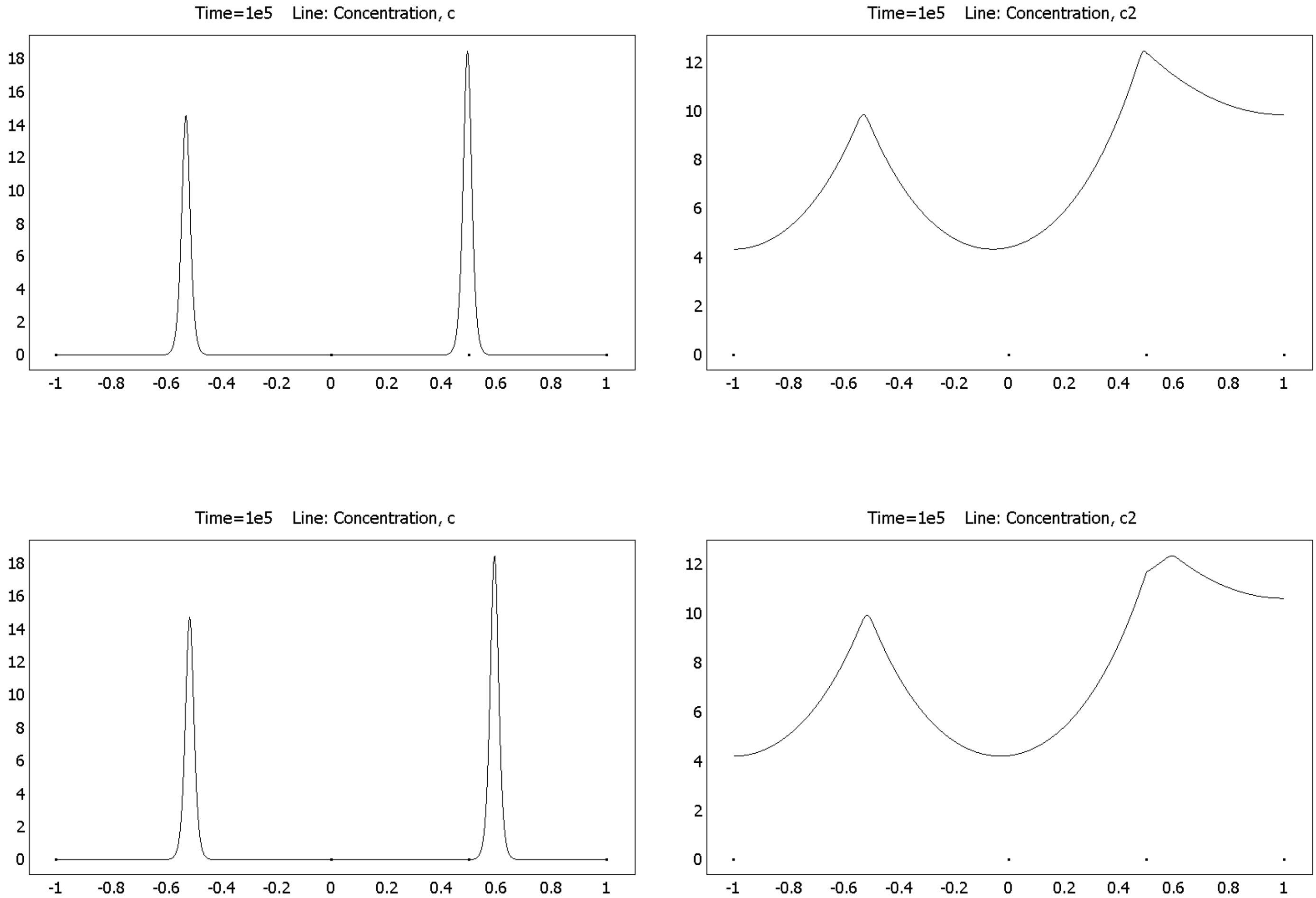

Figure 6. *Long-time limit of the solution to (1.1) – (1.3) with* $\epsilon^2 = 0.0001$ *and* $D(x) = 0.1$ *for* $-1 < x < 0.5$, $D(x) = 0.5$ *for* $0.5 < x < 1$. *We observe an interior spike in the left subinterval and a spike near the jump discontinuity or two interior spikes on different subintervals, respectively.*

11. Appendix: The Green's function for three segments

In this appendix we compute the Green's function $G(x,t_0)$ for $N=3$, i.e. in the case of two jumps or three segments.

First we consider a spike at a jump. We solve the system (8.1) for $t_0 = x_2$ and $N=3$.

$$G(x,t_0) = \begin{cases} A\dfrac{\cosh\theta_1(x+1)}{\cosh\theta_1(x_1+1)}, & -1<x<x_1, \\[2ex] A\dfrac{\sinh\theta_2(x-x_2)}{\sinh\theta_2(x_1-x_2)} + B\dfrac{\sinh\theta_2(x-x_1)}{\sinh\theta_2(x_2-x_1)}, & x_1<x<x_2, \\[2ex] B\dfrac{\cosh\theta_2(x-1)}{\cosh\theta_2(x_2-1)}, & x_2<x<1, \end{cases} \tag{11.1}$$

then $G(x,t_0)$ is automatically continuous at $x=x_1,\ x_2$ and it satisfies the Neumann boundary condition at $x=-1$ and $x=1$.

The jump conditions of G at $x_1,\ x_2$ give the following linear system for (A,B):

$$\left(\frac{1}{\theta_1}\tanh\theta_1(x_1+1) + \frac{1}{\theta_2}\coth\theta_2(t_0-x_1)\right)A - \frac{B}{\theta_2}\frac{1}{\sinh\theta_2(t_0-x_1)} = 1, \tag{11.2}$$

$$\left(\frac{1}{\theta_2}\coth\theta_2(x_2-t_0) + \frac{1}{\theta_3}\tanh\theta_3(1-x_2)+\right)B - \frac{A}{\theta_2}\frac{1}{\sinh\theta_2(x_2-t_0)} = 0. \tag{11.3}$$

We have to compute $G(t_0,t_0)=B$. We get

$$B^{-1} = \frac{\theta_2\tanh\theta_1(x_1+1)\cosh\theta_2(x_2-x_1) + \theta_1\sinh\theta_2(x_2-x_1)}{\theta_2^2\tanh\theta_1(x_1+1)\sinh\theta_2(x_2-x_1) + \theta_1\theta_2\cosh\theta_2(x_2-x_1)} + \frac{1}{\theta_3}\tanh\theta_3(1-x_2). \tag{11.4}$$

Now we assume that $-1<x_1<t_0<x_2<1$, i.e. we investigate a spike in the central segment.

We have to solve the system (8.3) for $N=3$.

Using the ansatz

$$G(x,t_0) = \begin{cases} A\dfrac{\cosh\theta_1(x+1)}{\cosh\theta_1(x_1+1)}, & -1<x<x_1, \\[2ex] A\dfrac{\sinh\theta_2(x-t_0)}{\sinh\theta_2(x_1-t_0)} + B\dfrac{\sinh\theta_2(x-x_1)}{\sinh\theta_2(t_0-x_1)}, & x_1<x<t_0, \\[2ex] B\dfrac{\sinh\theta_2(x-x_2)}{\sinh\theta_2(t_0-x_2)} + C\dfrac{\sinh\theta_2(x-t_0)}{\sinh\theta_2(x_2-t_0)}, & t_0<x<x_2, \\[2ex] C\dfrac{\cosh\theta_2(x-1)}{\cosh\theta_3(x_2-1)}, & x_2<x<1, \end{cases} \tag{11.5}$$

then $G(x,t_0)$ is automatically continuous at $x=x_1,\ t_0,\ x_2$ and it satisfies the Neumann boundary condition at $x=-1$ and $x=1$.

The jump conditions of G at $x_1,\ t_0,\ x_2$ give the following linear system for (A,B,C):

$$\begin{aligned}
\Big(\frac{1}{\theta_1}\tanh\theta_1(x_1+1)+\frac{1}{\theta_2}\coth\theta_2(t_0-x_1)\Big)A-\frac{B}{\theta_2}\frac{1}{\sinh\theta_2(t_0-x_1)} &= 0,\\
-\frac{A}{\theta_2}\sinh\theta_2(x_1-t_0)+\frac{B}{\theta_2}\left(\coth\theta_2(t_0-x_1)+\coth\theta_2(x_2-t_0)\right)-\frac{C}{\theta_2}\frac{1}{\sinh\theta_2(x_2-t_0)} &= 1,\\
\Big(\frac{1}{\theta_2}\coth\theta_2(x_2-t_0)+\frac{1}{\theta_3}\tanh\theta_3(1-x_2)+\Big)C-\frac{B}{\theta_2}\frac{1}{\sinh\theta_2(x_2-t_0)} &= 0.
\end{aligned}$$

We have to compute $G(t_0,t_0)=B$. We get

$$\begin{aligned}
B^{-1}=\frac{1}{\theta_2}\Bigg[&-\frac{1}{\frac{\theta_2}{\theta_1}\sinh^2\theta_2(t_0-x_1)\tanh\theta_1(x_1+1)\sinh\theta_2(t_0-x_1)\cosh\theta_2(t_0-x_1)}\\
&+\coth\theta_2(t_0-x_1)+\coth\theta_2(x_2-t_0)\\
&-\frac{1}{\frac{\theta_2}{\theta_3}\sinh^2\theta_2(x_2-t_0)\tanh\theta_3(1-x_2)+\sinh\theta_2(x_2-t_0)\cosh\theta_2(x_2-t_0)}\Bigg]\\
=\frac{1}{\theta_2}\Bigg[&\frac{\theta_2\tanh\theta_1(x_1+1)\cosh\theta_2(t_0-x_1)+\theta_1\sinh\theta_2(t_0-x_1)}{\theta_2\tanh\theta_1(x_1+1)\sinh\theta_2(t_0-x_1)+\theta_1\cosh\theta_2(t_0-x_1)}\\
&+\frac{\theta_2\tanh\theta_3(1-x_2)\cosh\theta_2(x_2-t_0)+\theta_3\sinh\theta_2(x_2-t_0)}{\theta_2\tanh\theta_3(1-x_2)\sinh\theta_2(x_2-t_0)+\theta_3\cosh\theta_2(x_2-t_0)}\Bigg].
\end{aligned}\tag{11.6}$$

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