ASYMPTOTIC BEHAVIOR AND TRAVELING WAVES FOR SOME POPULATION MODELS

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ASYMPTOTIC BEHAVIOR AND TRAVELING WAVES FOR SOME POPULATION MODELS

by

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School of Graduate Studies
in partial fulfilment of the
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Department of Mathematics and Statistics
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To My Parents

Abstract

Since the 1970s, more and more mathematicians have been trying to propose reasonable models for the growth of species in all kinds of environments and for the spread of epidemic diseases, and to understand the long-term behavior of their modelling systems. This thesis, consisting of five chapters, mainly deals with the dynamics of population and epidemic models represented by some time-delayed ordinary and partial differential equations, and reaction-diffusion systems.

In Chapter 1, we present some basic concepts and theorems, which involve the theories of monotone dynamics, uniform persistence, essential spectrum of linear operators, asymptotic speeds of spread and minimal traveling wave speed.

Based on some specific competitive models, we formulate in Chapter 2 a class of asymptotically periodic delay differential equations, which models multi-species competition, and investigate the global dynamics of the model. More precisely, we established the sufficient conditions for competitive coexistence, exclusion and uniform persistence via theories of competitive systems on Banach spaces, uniform persistence, periodic and asymptotically periodic semiflows.

Chapter 3 focuses on a nonlocal reaction-diffusion equation modelling the growth

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I would like to express my great appreciation to my supervisor, Professor Xiaoqiang Zhao, for his careful guidance, unbelievable patience and invaluable suggestions throughout my Ph.D. program. Without his help, this work would not have been possible. My deepest respect to him is not only for his kind encouragement and brilliant insight, but also for his great effort of training me to be a mathematical researcher and serving me as a model of very hard-working and productive scholar.

I wish to express sincere thanks to Professors Peter Booth, Hermann Brunner, Andy Foster and Xingfu Zou for teaching me the theories of topology, numerical solutions of partial differential equations, dynamical systems and delay differential equations. I also thank them for their helpful discussions, frequent advice and encouragement. In addition, I would like to thank Dr. Bruce Watson and Mrs. Gaskill for helping me in comprehensive tests and oral English, respectively.

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of a single species. For this model, we obtain a threshold dynamics and the global attractivity of a positive steady state. We also discuss the effects of spatial dispersal and maturation period on the evolutionary behavior in two specific cases. Our numerical investigation seems to suggest that the model admits a unique positive steady state even without monotonicity conditions.

In Chapter 4, we consider an epidemic model represented by a reaction-diffusion equation coupled with an ordinary differential equation, which is proposed by Capasso et al. Here, the existence, uniqueness (up to translation) and global exponential stability with phase shift of bistable traveling waves are studied by phase plane techniques, monotone semiflow approaches and a detailed spectrum analysis.

In Chapter 5, the asymptotic speeds of spread for solutions and traveling wave solutions to the integral version of the epidemic model in Chapter 4 are investigated. Our results show that the minimal wave speed for monotone traveling waves coincides with the asymptotic speed of spread for solutions with initial functions having compact supports. Some numerical simulations are also provided.

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Chapter 1

Preliminaries

In this chapter, we present some basic theorems which will be used in this thesis. They involve persistence theory, monotone dynamical systems, spectrum analysis, and newly developed theory for asymptotic speeds of spread and traveling waves.

1.1 Uniform Persistence

In population dynamics, uniform persistence is one of important concepts which characterize the long-term existence of species in an ecosystem. Let X be a metric space with metric d, and $f : X \to X$ a continuous map. Suppose X_0 is an open subset of X. Define $\partial X_0 := X \setminus X_0$, and $M_\partial := \{x \in \partial X_0 : f^n(x) \in \partial X_0, \forall n \geq 0\}$.

Definition 1.1.1 *A subset $A \subset X$ is said to be an attractor for f if A is nonempty, compact and invariant ($f(A) = A$), and A attracts some open neighborhood of itself. A global attractor for $f : X \to X$ is an attractor that attracts every point in X.*

Definition 1.1.2 *f is said to be uniformly persistent with respect to $(X_0, \partial X_0)$ if there exists $\eta > 0$ such that $\liminf_{n\to\infty} d(f^n(x), \partial X_0) \geq \eta$ for all $x \in X_0$.*

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Theorem 1.1.1 *([94, Theorem 2.2]) Let $f : X \to X$ be a continuous map with $f(X_0) \subset X_0$. Assume that*

(1) $f : X \to X$ has a global attractor A;

(2) Let $A_\partial = A \bigcap M_\partial$ be the maximal compact invariant set of in ∂X_0. $\tilde{A}_\partial = \bigcup_{x \in A_\partial} \omega(x)$ has an isolated and acyclic covering $\bigcup_{i=1}^k M_i$ in ∂X_0, that is, $A_\partial \subset \bigcup_{i=1}^k M_i$, where $M_1, M_2, \ldots, M_k$ are pairwise disjoint, compact and isolated invariant sets of f in ∂X_0 such that each M_i is also an isolated invariant set in X, and no subset of the M_i's forms a cycle for $f_\partial = f|_{A_\partial}$ in A_∂.

(3) $W^s(M_i) \bigcap X_0 = \emptyset$ for each $1 \leq i \leq k$, where $W^s(M_i) = \{x : \; x \in X, \omega(x) \neq \emptyset$ and $\omega(x) \subset M_i\}$ is the stable set of M_i.

Then f is uniformly persistent with respect to $(X_0, \partial X_0)$.

Theorem 1.1.2 *([94, Theorem 2.3] and [63, Theorem 4.5]) Let $f : X \to X$ be a continuous map with $f(X_0) \subset X_0$, where X is a closed subset of a Banach space, and X_0 is a convex and relatively open subset in X. Assume that*

(1) $f : X \to X$ is point dissipative and uniformly persistent with respect to $(X_0, \partial X_0)$;

(2) f is α-condensing, and f^{n_0} is compact for some integer $n_0 \geq 1$.

Then $f : X_0 \to X_0$ admits a global attractor A_0, and f has a fixed point $x_0 \in A_0$.

For an autonomous semiflow $T(t) : X \to X, t \geq 0$, we can define uniform persistence by replacing f^n with $T(t)$ (see [81]). Furthermore, the continuous-time

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Theorem 1.1.3 *([76, Theorem A.2] and [95, Theorem 1.3.9]) Let (Z, Z^+) be an ordered Banach space with $int(Z^+) \neq \emptyset$ and $T(t) : X \to X, t \geq 0$, be an autonomous semiflow with $T(t)X_0 \subset X_0, t \geq 0$. Assume that*

(1) $T(t) : X \to X$ is point dissipative, compact for $t \geq t_1 > 0$, and is uniformly persistent with respect to $(X_0, \partial X_0)$;

(2) there exists $t_2 > 0$ such that $T(t_2)X_0 \subset int(Z^+)$ and $T(t_2) : X_0 \to int(Z^+)$ is continuous.

Then, for any given $e \in int(Z^+)$, there exists $\beta > 0$ such that for any compact subset B of X_0, there exists $t_0 = t_0(B) \geq t_2$ such that $T(t)B \geq \beta e, \forall t \geq t_0$, in Z.

1.2 Monotone Dynamical Systems

Many types of equations can generate discrete- or continuous-time monotone dynamical systems, i.e., ordered initial values imply ordered subsequences or solutions. These types include difference, ordinary, functional and partial differential equations. Let E be an ordered Banach space with cone P such that $int(P) \neq \emptyset$. For $x, y \in E$, we write $x \geq y$ if $x - y \in P$, $x > y$ if $x - y \in P \setminus \{0\}$, and $x \gg y$ if $x - y \in int(P)$. By an order interval $[a, b]$, we mean that $[a, b] = \{x \in E : , a \leq x \leq b\}$.

Definition 1.2.1 *Let U be a subset of E, and $f : U \to U$ a continuous map. The map f is said to be monotone if $x \geq y$ implies that $f(x) \geq f(y)$; strictly* version of Theorem 1.1.1 and 1.1.2 still hold (see [81, Theorem 4.6], [94, Theorem 2.4] or [95, Theorem 1.3.7], and [63, Theorem 4.7]).

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(2) there exists $t_2 > 0$ such that $T(t_2)X_0 \subset int(Z^+)$ and $T(t_2) : X_0 \to int(Z^+)$ is continuous.

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Definition 1.2.2 *Let U be a nonempty closed and order convex set in P. A continuous map $f : U \to U$ is said to be subhomogeneous (or sublinear) if $f(\alpha x) \geq \alpha f(x)$ for any $x \in U$ and $\alpha \in [0, 1]$; strictly subhomogeneous if $f(\alpha x) > \alpha f(x)$ for any $x \in U$ with $x \gg 0$ and $\alpha \in (0, 1)$; strongly subhomogeneous if $f(\alpha x) \gg \alpha f(x)$ for any $x \in U$ with $x \gg 0$ and $\alpha \in (0, 1)$.*

Definition 1.2.3 *A linear operator L on E is said to be positive if $L(P) \subset P$; strongly positive if $L(P \setminus \{0\}) \subset int(P)$. Denote by $r(L)$ the spectral radius of L.*

Theorem 1.2.1 *([93, Theorem 2.3] or [95, Theorem 2.3.4]) Let either $V = [0, b]$ with $b \gg 0$ or $V = P$. Assume that*

(1) $f : V \to V$ satisfies either

(i) f is monotone and strongly sublinear; or

(ii) f is strongly monotone and strictly sublinear;

(2) $f : V \to V$ is asymptotically smooth, and every positive orbit of f in V is bounded.

(3) $f(0) = 0$, and the Fréchet derivative $Df(0)$ of f at zero is compact and strongly positive.

Then there exist threshold dynamics:

(a) If $r(Df(0)) \leq 1$, then every positive orbit in V converges to zero.

monotone if $x > y$ *implies that* $f(x) > f(y)$; *strongly monotone if* $x > y$ *implies that* $f(x) \gg f(y)$.

Definition 1.2.2 *Let* U *be a nonempty closed and order convex set in* P. *A continuous map* $f : U \to U$ *is said to be subhomogeneous (or sublinear) if* $f(\alpha x) \geq \alpha f(x)$ *for any* $x \in U$ *and* $\alpha \in [0, 1]$; *strictly subhomogeneous if* $f(\alpha x) > \alpha f(x)$ *for any* $x \in U$ *with* $x \gg 0$ *and* $\alpha \in (0, 1)$; *strongly subhomogeneous if* $f(\alpha x) \gg \alpha f(x)$ *for any* $x \in U$ *with* $x \gg 0$ *and* $\alpha \in (0, 1)$.

Definition 1.2.3 *A linear operator* L *on* E *is said to be positive if* $L(P) \subset P$; *strongly positive if* $L(P \setminus \{0\}) \subset int(P)$. *Denote by* $r(L)$ *the spectral radius of* L.

Theorem 1.2.1 *([93, Theorem 2.3] or [95, Theorem 2.3.4]) Let either* $V = [0, b]$ *with* $b \gg 0$ *or* $V = P$. *Assume that*

(1) $f : V \to V$ *satisfies either*

(i) f *is monotone and strongly sublinear; or*

(ii) f *is strongly monotone and strictly sublinear;*

(2) $f : V \to V$ *is asymptotically smooth, and every positive orbit of* f *in* V *is bounded.*

(3) $f(0) = 0$, *and the Fréchet derivative* $Df(0)$ *of* f *at zero is compact and strongly positive.*

Then there exist threshold dynamics:

(a) If $r(Df(0)) \leq 1$, *then every positive orbit in* V *converges to zero.*

(b) *If* $r(Df(0)) > 1$, *then there exists a unique fixed point* $u^* \gg 0$ *in* V *such that every positive orbit in* $V \setminus \{0\}$ *converges to* u^*.

For an autonomous semiflow $T(t)$ on E, we can define monotonicity and sub-linearity in a similar ways. Moreover, there exists a continuous-time version of Theorem 1.2.1. [96, Theorem 3.2] is the version for delay differential equations, and [96, Corollary 3.2] is the version for ordinary differential equations.

Theorem 1.2.2 *([95, Theorem 2.2.4]) Let* U *be a closed convex subset of an ordered Banach space* $\mathcal{X}$, *and* $\Phi(t) : U \to U$ *be a monotone semiflow. Assume that there exists a monotone homeomorphism* h *from* $[0, 1]$ *onto a subset of* U *such that*

(1) *For each* $s \in [0, 1]$, $h(s)$ *is a stable equilibrium for* $\Phi(t) : U \to U$;

(2) *Each orbit of* $\Phi(t)$ *in* $[h(0), h(1)]_{\mathcal{X}}$ *is precompact;*

(3) *One of the following two properties holds:*

(3a) *If* $h(s_0) <_{\mathcal{X}} \omega(\phi)$ *for some* $s_0 \in [0, 1)$ *and* $\phi \in [h(0), h(1)]_{\mathcal{X}}$, *then there exists* $s_1 \in (s_0, 1)$ *such that* $h(s_1) \leq_{\mathcal{X}} \omega(\phi)$;

(3b) *If* $\omega(\phi) <_{\mathcal{X}} h(r_1)$ *for some* $r_1 \in (0, 1]$ *and* $\phi \in [h(0), h(1)]_{\mathcal{X}}$, *then there exists* $r_0 \in (0, r_1)$ *such that* $\omega(\phi) \leq_{\mathcal{X}} h(r_0)$.

Then for any precompact orbit $\gamma^+(\phi_0)$ *of* $\Phi(t)$ *in* U *with* $\omega(\phi_0) \cap [h(0), h(1)]_{\mathcal{X}} \neq \emptyset$, *there exists* $s^* \in [0, 1]$ *such that* $\omega(\phi_0) = h(s^*)$.

The following attractivity theorem is due to M. W. Hirsch ([48]), and is a powerful tool to prove the global attractivity of a unique equilibrium.

(b) If $r(Df(0)) > 1$*, then there exists a unique fixed point* $u^* \gg 0$ *in* V *such that every positive orbit in* $V \setminus \{0\}$ *converges to* u^*.

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Theorem 1.2.3 *([48, Theorem 3.3]) Let $T(t) : E \to E$ be a monotone semiflow. Assume that $T(t)$ admits an attractor K such that K contains only one equilibrium x^*. Then every trajectory attracted to K converges to x^*.*

In the following, we introduce two theorems about competitive systems on ordered Banach spaces, which are one of the main tools in Chapter 2. For $i = 1, 2$, let X_i be ordered Banach spaces with positive cones X_i^+, where $int(X_i^+) \neq \emptyset$. Let $X = X_1 \times X_2$, $X^+ = X_1^+ \times X_2^+$, and $K = X_1^+ \times (-X_2^+)$. Denote by $\leq_K$ the order on X defined by K. The following hypotheses ([50]) are meant to capture the essence of competition between two adequate competitors:

(A1) $f : X^+ \to X^+$ is strictly monotone with respect to $<_K$, and is order compact in the sense that $f([0, x_1] \times [0, x_2])$ is precompact in X for every $(x_1, x_2) \in X^+$.

(A2) 0 is a repelling fixed point of f in the sense that there exists a neighborhood U_0 of 0 in X^+ such that for each $x \in U_0$ with $x \neq 0$, there is an integer n such that $f^n(x) \notin U_0$.

(A3) $f(X_1^+ \times \{0\}) \subset X_1^+ \times \{0\}$, and there exists $\hat{x}_1 \in int(X_1^+)$ such that $f((\hat{x}_1, 0)) = (\hat{x}_1, 0)$ and the omega limit set $\omega((x_1, 0))$ of the orbit $f^n(x_1, 0)$ is $(\hat{x}_1, 0)$ for every $x_1 \in X_1^+ \setminus \{0\}$. The symmetric conditions hold for f on $\{0\} \times X_2$, and the fixed point is denoted by $(0, \tilde{x}_2)$.

(A4) If $x, y \in X^+$ satisfy $x <_K y$ and either x or y belongs to $int(X^+)$, then $f(x) \ll_K f(y)$. If $x = (x_1, x_2) \in X^+$ with $x_i \neq 0, i = 1, 2$, then $f(x) \gg 0$.

Denote the "boundary" fixed points of f by $E_0 = (0, 0), E_1 = (\hat{x}_1, 0), E_2 = (0, \tilde{x}_2)$. Let $I = [E_2, E_1]_K := \{x \in X : E_2 \leq_K x \leq_K E_1\}$. Obviously, $I = [0, \hat{x}_1] \times [0, \tilde{x}_2]$.

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The following result ([50]) says that for a competitive system, either there is a positive fixed point of f, representing coexistence of the two populations, or one population drives the other to extinction.

Theorem 1.2.4 *([50, Theorem A.1]) Let (A1)-(A4) hold. Then the omega limit set of every orbit in X^+ is contained in I, and exactly one of the following holds:*

(a) There exists a positive fixed point E_* of f in I;

(b) $\omega(x) = E_1$ for every $x = (x_1, x_2) \in I$ with $x_i \neq 0, i = 1, 2$;

(c) $\omega(x) = E_2$ for every $x = (x_1, x_2) \in I$ with $x_i \neq 0, i = 1, 2$.

Finally, if (b) or (c) holds and $x = (x_1, x_2) \in X^+ \setminus I$ with $x_i \neq 0, i = 1, 2$, then either $\omega(x) = E_1$ or $\omega(x) = E_2$.

Theorem 1.2.5 *([95, Theorem 2.4.2]) Let (A1)-(A4) hold and assume that E_1 and E_2 are isolated fixed points of f. Let $W^s(E_i)$ be the stable set of E_i for f. If $W^s(E_i) \bigcap int(X^+) = \emptyset$, $i = 1, 2$, then there exist positive fixed points $E_{**} \leq_K E_*$ of f such that $\omega(x) = E_*$ for every $x = (x_1, x_2)$ satisfying $E_* \leq_K x <_K E_1$ and $x_2 \neq 0$, $\omega(x) = E_{**}$ for every $x = (x_1, x_2)$ satisfying $E_2 <_K x \leq_K E_{**}$ and $x_1 \neq 0$, and the order interval $[E_{**}, E_*]_K$ attracts any point in $(X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$.*

1.3 Essential Spectrum

This section presents some results about essential spectrum of certain ordinary differential operators obtained in [46, Page 136-142].

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1.3 Essential Spectrum

This section presents some results about essential spectrum of certain ordinary differential operators obtained in [46, Page 136-142].

Definition 1.3.1 *If L is a linear operator in a Banach space, a normal point for L is any complex number which is in the resolvent set, or is an isolated eigenvalue of L with finite multiplicity. Any other complex number is in the essential spectrum. Denote the resolvent set and spectrum of L by $\rho(L)$ and $\sigma(L)$, respectively.*

Theorem 1.3.1 *([46, Lemma 2]) Suppose the matrices $A_+(\lambda), A_-(\lambda)$ are analytic functions of $\lambda \in \mathbb{C}$, the complex number set. Let*

$$S_\pm = \{\lambda : \ A_\pm(\lambda) \textit{ has an imaginary eigenvalue } \}.$$

Let $A(x,\lambda) = A_+(\lambda)$ for $x > 0$, $A_-(\lambda)$ for $x < 0$, and define the differential operator $L(\lambda)u = \frac{d}{dx}u + A(\cdot,\lambda)u$ in $C_0(\mathbb{R})$, the continuous function set, or $C_{unif}(\mathbb{R})$, the uniformly continuous function set; we may consider $L(\lambda)$ as closed and densely defined. Then if G is any open connected set in $\mathbb{C} \setminus (S_+ \bigcup S_-)$, either

(i) $0 \in \sigma(L(\lambda))$ for all λ in G, or

(ii) $0 \in \rho(L(\lambda))$ for all λ in G except at isolated points, the exceptional points are poles of $L(\lambda)^{-1}$ of finite order.

Also, $0 \in \sigma(L(\lambda))$ whenever $\lambda \in S_+ \bigcup S_-$.

Theorem 1.3.2 *([46, Theorem A.1]) Suppose X is a Banach space, $T : D(T) \subset X \to X$ is a closed linear operator, $S : D(S) \subset X \to X$ is linear with $D(T) \subset D(S)$ and $S(\lambda_0 I - T)^{-1}$ is compact for some λ_0. Let U be an open connected set in $\mathbb{C}$ consisting entirely of normal points of T; then either U consists entirely of normal points of $T + S$, or entirely of eigenvalues of $T + S$.*

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1.4 Spreading Speeds and Traveling Waves

The theory of asymptotic speeds of spread and traveling waves, developed in [8, 7, 9, 26, 28, 27, 79, 80, 90], has been recently generalized to a class of scalar nonlinear integral equations in [83]. In this section, we present some results obtained in [83].

Definition 1.4.1 *A number $c^* > 0$ is called the asymptotic speed of spread for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ if $\lim_{t\to\infty, |x|\geq ct} u(t,x) = 0$ for each $c > c^*$, and if there exists some $\bar{u} > 0$ such that $\lim_{t\to\infty, |x|\leq ct} u(t,x) = \bar{u}$ for each $c \in (0, c^*)$.*

Consider an integral equation

$$u(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^n} F(u(t-s, x-y), s, y)dyds, \tag{1.4.1}$$

where $F : \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u and Borel measurable in (s,y), and $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is Borel measurable and bounded. Assume that

(B) There exists a function $k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that

- (B1) $k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s,x)dxds < \infty$;
- (B2) $0 \leq F(u,s,x) \leq uk(s,x), \forall u, s \geq 0, x \in \mathbb{R}^n$;
- (B3) For every compact interval I in $(0,\infty)$, there exists some $\varepsilon > 0$ such that $F(u,s,x) \geq \varepsilon k(s,x), \forall u \in I, s \geq 0, x \in \mathbb{R}^n$;
- (B4) For every $\varepsilon > 0$, there exists some $\delta > 0$ such that $F(u,s,x) \geq (1-\varepsilon)uk(s,x)$, $\forall u \in [0,\delta]$, $s \geq 0$, $x \in \mathbb{R}^n$;
- (B5) For every $w > 0$, there exists some $\Lambda > 0$ such that

$$|F(u,s,x) - F(v,s,x)| \leq \Lambda|u-v|k(s,x), \forall u, v \in [0,w], s \geq 0, x \in \mathbb{R}^n.$$

The following proposition shows that the existence, uniqueness and some properties of solutions to equation (1.4.1).

Proposition 1.4.1 *([83, Proposition 2.1]) If assumptions (B) hold, then for every Borel measurable, nonnegative and bounded function $u_0(t, x)$, there exists a unique Borel measurable solution $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ of (1.4.1), and u is bounded on $[0, r] \times \mathbb{R}^n$ for every $r > 0$. Furthermore, the following statements hold.*

(a) The solution u is bounded if there exist $c_1, c_2 > 0$ such that $c_1 k^* < 1$ and $F(u, s, x) \le (c_2 + c_1 u) k(s, x), \forall u, s \ge 0, x \in \mathbb{R}^n$.

(b) If $r > 0$ and $\lim\limits_{|x|\to\infty} u_0(t, x) = 0$ uniformly for $t \in [0, r]$, then the solution u has the same property.

To obtain some more properties for equation (1.4.1), we have to make some assumptions on k.

(C) $k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a Borel measurable function such that

(C1) $k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s, y) dy ds \in (1, \infty)$;

(C2) There exists some $\lambda^\circ > 0$ such that $\int_0^\infty \int_{\mathbb{R}^n} e^{\lambda^\circ y_1} k(s, y) dy ds < \infty$;

(C3) There exist numbers $\sigma_2 > \sigma_1 > 0$, $\rho > 0$ such that $k(s, x) > 0$, $\forall s \in (\sigma_1, \sigma_2)$, $|x| \in [0, \rho)$;

(C4) k is isotropic.

Here, a function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is said to be isotropic if for almost all $s > 0$, $f(s, x) = f(s, y)$ whenever $|x| = |y|$. For a fixed $Z \in \mathbb{R}^n$ with $|Z| = 1$, define the

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Here, a function $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is said to be isotropic if for almost all $s > 0$, $f(s,x) = f(s,y)$ whenever $|x| = |y|$. For a fixed $Z \in \mathbb{R}^n$ with $|Z| = 1$, define the transform

$$\mathcal{K}(c,\lambda) := \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs - Z\cdot y)} k(s,y)dyds, \quad \forall c \geq 0,\ \lambda \geq 0,$$

where $\cdot$ means the usual inner product on $\mathbb{R}^n$. Suppose that k is isotropic. Since for any $Z \in \mathbb{R}^n$ with $|Z| = 1$, there exists an orthogonal matrix A with $AZ = -e_1$, where e_1 is the first canonical basis vector of $\mathbb{R}^n$, there holds

$$\mathcal{K}(c,\lambda) = \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs + y_1)} k(s,y)dyds,$$

where y_1 is the first coordinate of y. If (C) holds, then, for every $c > 0$, there exists some $\lambda^\sharp(c) \in (0, \infty]$ such that $\mathcal{K}(c,\lambda) < \infty$ for $\lambda \in [0, \lambda^\sharp(c))$ and $\mathcal{K}(c,\lambda) = \infty$ for $\lambda > \lambda^\sharp(c)$ ([79, Lemma 3.7]). Let $c^* := \inf\{c \geq 0 : \mathcal{K}(c,\lambda) < 1 \text{ for some } \lambda > 0\}$. The following lemma shows the existence of c^*.

Lemma 1.4.1 *([83, Proposition 2.3]) Let (C) hold and assume that* $\liminf\limits_{\lambda \nearrow \lambda^\sharp(c)} \mathcal{K}(c,\lambda) \geq k^*$ *for every* $c > 0$. *Then there exists a unique* $\lambda^* \in (0, \lambda^\sharp(c^*))$ *such that* $\mathcal{K}(c^*, \lambda^*) = 1$ *and* $\mathcal{K}(c^*, \lambda) > 1$ *for* $\lambda \neq \lambda^*$. *Moreover,* c^* *and* λ^* *are uniquely determined as the solutions of the system* $\mathcal{K}(c,\lambda) = 1$, $\frac{d}{d\lambda}\mathcal{K}(c,\lambda) = 0$.

The function u_0 is said to be admissible if for every $c, \lambda > 0$ with $\mathcal{K}(c,\lambda) < 1$, there exits some $\gamma > 0$ such that $u_0(t,x) \leq \gamma e^{\lambda(ct - |x|)}, \forall t \geq 0, x \in \mathbb{R}^n$. The following theorems show that c^* is the asymptotic speed of spread for solutions of (1.4.1).

Theorem 1.4.1 *([83, Theorem 2.1]) Let (B) and (C) hold. then for every admissible* u_0, *the unique solution* $u(t,x)$ *of (1.4.1) satisfies* $\lim\limits_{t\to\infty, |x| \geq ct} u(t,x) = 0$ *for each* $c > c^*$.

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Theorem 1.4.2 *([80, Lemma 3.10] and [83, Theorem 2.3]) Assume that a function f satisfies*

(D) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lipschitz continuous function such that

(D1) $f(0) = 0$ and $f(u) > 0$, $\forall u > 0$;

(D2) f is differential at zero, $f'(0) = 1$ and $f(u) \leq u, \forall u > 0$;

(D3) $\lim\limits_{u\to\infty} \frac{f(u)}{u} = 0$;

(D4) there exists a positive solution u^* of $u = k^* f(u)$ such that $k^* f(u) > u$, $\forall u \in (0, u^*)$, and $k^* f(u) < u, \forall u > u^*$.

Set $F(u, s, x) = f(u)k(s, x)$, and let $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel measurable function with the property that $\lim\limits_{t\to\infty} u_0(t, x) = 0$ uniformly in $x \in \mathbb{R}^n$, and $u_0(t, x) \geq \eta > 0, \forall t \in (t_1, t_2), |x| \leq \eta$, for appropriate $t_2 > t_1 \geq 0, \eta > 0$. Assume that (C) holds and the unique solution u of (1.4.1) is bounded, $u^\infty := \limsup\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} u(t, x)$. If there is no pair v and w such that $0 < v < u^* < w \leq u^\infty$ and $w = k^* f(v), v = k^* f(w)$, then, $\lim\limits_{t\to\infty, |x|\leq ct} u(t, x) = u^*, \forall c \in (0, c^*)$.

For the general case of $F(u, s, x)$, we define $\check{F}(u) = \int_0^\infty \int_{\mathbb{R}^n} F(u, s, y)dyds$. Then

Theorem 1.4.3 *([83, Theorem 2.5]) Let (B) and (C) hold and let $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be a bounded and Borel measurable function with the property that $u_0(t, x) \geq \eta > 0, \forall t \in (t_1, t_2), |x| \leq \eta$, for appropriate $t_2 > t_1 \geq 0, \eta > 0$. Assume that $\frac{F(u,s,x)}{u}$ is monotone decreasing and $uF(u, s, x)$ is monotone increasing in $u > 0$ for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, that the monotonicities are strict for $s \in (\sigma_1, \sigma_2), |x| \in (0, \rho)$ with appropriate constants $\sigma_2 > \sigma_1 > 0, \rho > 0$, and that $\lim\limits_{t\to\infty} u_0(t, x) = 0$ uniformly in*

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In the following, let us consider the limiting equation of (1.4.1) with $n = 1$

$$u(t,x) = \int_0^\infty \int_{\mathbb{R}} F(u(t-s, x-y), s, y)dyds. \tag{1.4.2}$$

A solution $u(t,x)$ of (1.4.2) is said to be a traveling wave solution if it is of the form $u(t,x) = v(x+ct)$. The parameter c is called the wave speed, and the function v is called the wave profile. Here, we require that

$$v(\cdot) \text{ is positive and bounded on } \mathbb{R}, \text{ and } \lim_{\xi\to-\infty} v(\xi) = 0. \tag{1.4.3}$$

In the case where $F(u,s,x) = f(u)k(s,x)$, we make the following modified assumptions on f

(D′) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

- (D1′) $f(0) = 0$, and there exists a positive solution u^* of $u = k^* f(u)$ such that $k^* f(u) > u, \forall u \in (0, u^*)$;
- (D2′) f is differentiable at zero, $f'(0) = 1$ and $f(u) \leq u, \forall u \in [0, u^*]$.

Theorem 1.4.4 *([28, Theorem 6.5] and [83, Theorem 3.2]) Let (C) with* $n = 1$ *and (D′) hold, and set* $F(u,s,x) = f(u)k(s,x)$. *Assume that* $|f(u) - f(v)| \leq |u - v|, \forall u, v \in [0, u^*]$. *Then for* $c > c^*$, *(1.4.2) and (1.4.3) admit at most one monotone increasing traveling wave* $v(x + ct)$ *connecting* 0 *and* u^* *up to translation.*

Theorem 1.4.5 *([83, Theorem 3.3]) Let (B2) and (C) with* $n = 1$ *hold. Assume that* $F(\cdot, s, x)$ *is increasing on* $[0, u^*]$ *for each* $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$, *and* $F(u,s,x) \geq$

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$(u - bu^\sigma)k(s,x), \forall u \in [0,\delta], (s,x) \in \mathbb{R}_+ \times \mathbb{R}$, for appropriate $\delta \in (0,u^*], \sigma > 1$ and $b > 0$, where u^* is the fixed point of $\check{F}$, and $\check{F}(u) > u, \forall u \in (0,u^*)$. Then for each $c > c^*$, there exists a monotone traveling wave solution of (1.4.2) with speed c and connecting 0 and u^*.

Theorem 1.4.6 *([83, Theorem 3.4]) Let $F(u,s,x) = \sum_{i=1}^{m} f_i(u)k_i(s,x)$ and let the assumptions of Theorem 1.4.5 be satisfied. Assume that each $k_i(s,\cdot)$ is continuous on $\mathbb{R}$ for all $s \geq 0$. Then there exists a monotone traveling wave solution of (1.4.2) with speed c^* and connecting 0 and u^*.*

Theorem 1.4.7 *([83, Theorem 3.5]) Let (B) and (C) hold. Then for each $c \in (0,c^*)$, there exists no traveling wave solutions of (1.4.2) and (1.4.3) with speed c.*

Chapter 2

An Asymptotically Periodic Competitive Model

In this chapter, we consider a time-delayed asymptotically periodic system which describes the competition among mature populations. By appealing to theories of monotone dynamical systems, periodic and asymptotically periodic semiflows and uniform persistence, we analyze the evolutionary behavior of the system and establish sufficient conditions for competitive coexistence, exclusion and uniform persistence.

The organization of this chapter is as follows. In Section 2.1, based on some specific population models, we formulate a general periodic competitive system and an asymptotically periodic system. Section 2.2 provides some preliminary results on the spectral radius of the Poincaré map associated with a linear periodic and delayed equation, threshold dynamics in a scalar periodic and delayed system, and the relationship between solutions of an asymptotically periodic system with delays and its limiting system. In Section 2.3, we first analyze the global dynamics in the

$(u - bu^\sigma)k(s,x), \forall u \in [0,\delta], (s,x) \in \mathbb{R}_+ \times \mathbb{R}$, *for appropriate* $\delta \in (0, u^*], \sigma > 1$ *and* $b > 0$, *where* u^* *is the fixed point of* $\check{F}$, *and* $\check{F}(u) > u, \forall u \in (0, u^*)$. *Then for each* $c > c^*$, *there exists a monotone traveling wave solution of (1.4.2) with speed* c *and connecting* 0 *and* u^*.

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2.1 Introduction

Since the 1970s, the population models with stage structure have received extensive investigations (see [57, 84, 11, 44, 66, 73, 36, 2, 56, 24, 77, 91] and references therein). To describe a single species growth, Aiello and Freedman [1] proposed the following system

$$\begin{aligned} \dot{x}(t) &= \alpha e^{-\gamma\tau} x(t-\tau) - \beta x^2(t), \\ \dot{y}(t) &= \alpha x(t) - \gamma y(t) - \alpha e^{-\gamma\tau} x(t-\tau), \end{aligned} \tag{2.1.1}$$

where $x(t)$ and $y(t)$ denote the mature and immature populations, β and γ represent the death rates of the mature and the immature, α denotes the birth rate of the mature, and τ is the maturation age. They showed that there exists an asymptotically stable positive equilibrium, and concluded that the introduction of stage structure does not affect the permanence of the species.

In order to investigate how the stage structure affects the asymptotical behavior of the competitive species, Liu et al. [60] combined the competitive Lotka-Volterra system with system (2.1.1) and obtained a two-species competitive model with stage two-species competitive system by applying theories for competitive systems on Banach spaces [50], and then lift these results to the asymptotically periodic system. In Section 2.4, we first investigate the uniform persistence of multi-species competitive systems by two-side comparison method, and then obtain natural invasibility conditions for the uniform persistence and coexistence states of three-species competitive systems by using the theory of uniform persistence.

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$$\dot{x}_i(t) = b_i e^{-d_i\tau_i} x_i(t-\tau_i) - x_i(t)(a_{i1}x_1(t) + a_{i2}x_2(t)), \tag{2.1.2}$$
$$\dot{y}_i(t) = b_i x_i(t) - d_i y_i(t) - b_i e^{-d_i\tau_i} x_i(t-\tau_i), \quad i = 1, 2,$$

where $x_i(t)$ and $y_i(t)$ denote the mature and immature populations of the ith species, $a_{ij} > 0$, b_i and d_i denote the birth rate of the ith mature population and the death rate of the ith immature population, respectively, τ_i is the maturation age of species i. One of the basic assumptions is that the immature do not compete with the other species. Note that studying only (2.1.2) is enough to know the properties of the whole system. In [60], the authors defined $\xi_i = d_i\tau_i$ as the degree of stage, and concluded that if $\frac{a_{12}}{a_{22}} < \frac{b_1 e^{-\xi_1}}{b_2 e^{-\xi_2}} < \frac{a_{11}}{a_{21}}$, then system (2.1.2) is permanent. Furthermore, Liu et al. generalized the above system to an autonomous competitive system for n species in [59] and a T-periodic competitive system for n species in [61]:

$$\dot{x}_i(t) = B_i(t) x_i(t-\tau_i) - x_i(t)\sum_{j=1}^{n} a_{ij}(t)x_j(t), \tag{2.1.3}$$
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where $b_i(t)$, $a_{ii}(t)$, $d_i(t) > 0$, $a_{ij}(t) \geq 0$, and

$$B_i(t) = b_i(t-\tau_i)e^{-\int_{t-\tau_i}^{t} d_i(s)ds}, \ 1 \leq i \leq n.$$

They concluded that if

$$B_i^l > \sum_{j\neq i} a_{ij}^m B_j^m / a_{jj}^l, \ 1 \leq i \leq n, \tag{2.1.4}$$

then system (2.1.3) is permanent, where

$$a_{ij}^l = \inf_t a_{ij}, \ a_{ij}^m = \sup_t a_{ij}, \ B_j^l = \inf_t B_j, \ B_j^m = \sup_t B_j, \ 1 \leq i, j \leq n.$$

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$$a_{ij}^l = \inf_t a_{ij}, \; a_{ij}^m = \sup_t a_{ij}, \; B_j^l = \inf_t B_j, \; B_j^m = \sup_t B_j, \; 1 \leq i, j \leq n.$$

Note that condition (2.1.4) is very strong. There should exist some more general conditions, such as average integrals of certain functions over the period, which is more natural. Also motivated by system (2.1.1), (2.1.2) and (2.1.3), we consider a more general system of competing mature populations:

$$\begin{aligned} \dot{u}_i(t) &= u_i(t-\tau_i)F_i(t, u_i(t-\tau_i)) - u_i(t)G_i(t, u_1(t), \dots, u_m(t)) \\ &= f_i(t, u_1(t), \dots, u_m(t), u_i(t-\tau_i)), \ 1 \le i \le m, \end{aligned} \tag{2.1.5}$$

where the continuous function $f_i(t, u_1, \dots, u_m, v_i)$ is T-periodic in t, and Lipschitzian in $(u_1, \dots, u_m, v_i)$ in any bounded subset of $\mathbb{R}_+^{m+1}$, $i = 1, 2, \dots, m$. Note that system (2.1.5) is also a general form of Ayala's system (see, e.g., [10] and [55] for the autonomous case, and [34] for the nonautonomous case).

It is known that some parameters in an ecological system are not exactly periodic in time, but asymptotically periodic in time. Based on the periodic system (2.1.5), we then consider the following asymptotically periodic system

$$\begin{aligned} \dot{\tilde{u}}_i(t) &= \tilde{u}_i(t-\tau_i)\tilde{F}_i(t, \tilde{u}_i(t-\tau_i)) - \tilde{u}_i(t)\tilde{G}_i(t, \tilde{u}_1(t), \dots, \tilde{u}_m(t)) \\ &= \tilde{f}_i(t, \tilde{u}_1(t), \dots, \tilde{u}_m(t), \tilde{u}_i(t-\tau_i)), \ 1 \le i \le m, \end{aligned} \tag{2.1.6}$$

with the property that

(A) The continuous function $\tilde{f}_i(t, u_1, \dots, u_m, v_i)$ is Lipschitzian in $(u_1, \dots, u_m, v_i)$ in any bounded subset of $\mathbb{R}_+^{m+1}$; $\tilde{F}_i$ and $\tilde{G}_i$ satisfy $\lim_{t\to\infty} |\tilde{F}_i(t, u_i) - F_i(t, u_i)| = 0$ and $\lim_{t\to\infty} |\tilde{G}_i(t, u_1, \dots, u_m) - G_i(t, u_1, \dots, u_m)| = 0$ uniformly on any bounded subsets of $\mathbb{R}_+$ and $\mathbb{R}_+^m$, respectively, $i = 1, 2, \dots, m$.

The purpose of this chapter is to analyze the global dynamics of system (2.1.5) and (2.1.6). By appealing to the theory of autonomous and nonautonomous semiflows, we establish sufficient conditions for the existence of periodic coexistence, Note that condition (2.1.4) is very strong. There should exist some more general conditions, such as average integrals of certain functions over the period, which is more natural. Also motivated by system (2.1.1), (2.1.2) and (2.1.3), we consider a more general system of competing mature populations:

$$\begin{aligned} \dot{u}_i(t) &= u_i(t-\tau_i)F_i(t, u_i(t-\tau_i)) - u_i(t)G_i(t, u_1(t), \ldots, u_m(t)) \\ &= f_i(t, u_1(t), \ldots, u_m(t), u_i(t-\tau_i)), \ 1 \le i \le m, \end{aligned} \tag{2.1.5}$$

where the continuous function $f_i(t, u_1, \ldots, u_m, v_i)$ is T-periodic in t, and Lipschitzian in $(u_1, \ldots, u_m, v_i)$ in any bounded subset of $\mathbb{R}_+^{m+1}$, $i = 1, 2, \ldots, m$. Note that system (2.1.5) is also a general form of Ayala's system (see, e.g., [10] and [55] for the autonomous case, and [34] for the nonautonomous case).

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The purpose of this chapter is to analyze the global dynamics of system (2.1.5) and (2.1.6). By appealing to the theory of autonomous and nonautonomous semiflows, we establish sufficient conditions for the existence of periodic coexistence, global persistence and extinction in terms of spectral radii of the Poincaré maps associated with linear periodic delay equations. In the case where the delays are integer multiples of the period, these conditions can be determined by the average integrals of certain periodic functions. When applied to system (2.1.3), the obtained conditions are necessary to those in [61], and the results improve those obtained in [61].

2.2 Scalar Delay Differential Equations

In this section, we first present some notation used in this chapter, and then give some results about scalar delay differential equations. Let τ, τ_1 and τ_2 be positive numbers, and

$$Y = C([-\tau, 0], \mathbb{R}),\ Y^+ = C([-\tau, 0], \mathbb{R}_+),\ X_i = C([-\tau_i, 0], \mathbb{R}),$$

$$X_i^+ = C([-\tau_i, 0], \mathbb{R}_+),\ i = 1, 2,\ X = X_1 \times X_2,\ X^+ = X_1^+ \times X_2^+.$$

Then $(Y, Y^+), (X_i, X_i^+)$ and (X, X^+) are ordered Banach spaces. Denote these orders by $\leq, <$ and $\ll$. Let $K = X_1^+ \times (-X_2^+)$. Then (X, K) is also an ordered Banach space. Denote the orders by $\leq_K, <_K$ and $\ll_K$. By an order interval $[\varphi, \psi]_K$ on X, we mean the set

$$[\varphi, \psi]_K = \{\xi \in X :\ \varphi \leq_K \xi \leq_K \psi\}.$$

For a linear operator P, we denote the spectral radius of P by $r(P)$.

Consider a linear scalar equation with delay τ

$$\dot{u}(t) = a(t)u(t) + b(t)u(t - \tau). \tag{2.2.7}$$

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$$\dot{u}(t) = a(t)u(t) + b(t)u(t - \tau). \tag{2.2.7}$$

Assume that

(E) $a(t)$ and $b(t)$ are T-periodic and continuous, and $b(t) > 0$, $\forall t \geq 0$.

Then for any $\varphi \in Y^+$, equation (2.2.7) has a unique solution $u(t, \varphi)$ for $t \geq 0$, with $u(s, \varphi) = \varphi(s), \forall s \in [-\tau, 0]$. Let $u_t(\varphi)$ be the solution semiflow for equation (2.2.7) defined by $u_t(\varphi)(s) = u(t + s, \varphi), \forall s \in [-\tau, 0]$. In this chapter, we always denote by $u(t, \varphi)$ the solution of a certain system, and by $u_t(\varphi)$ the associated solution semiflow. Since $b(t) > 0$, by the positivity theorem ([72, Theorem 5.2.1]), $u_t(\varphi) \geq 0, \forall \varphi \in Y^+, t \geq 0$. Define the Poincaré map $P : Y^+ \to Y^+$ by $P(\varphi) = u_T(\varphi)$. It then follows that $P^n(\varphi) = u_{nT}(\varphi)$ for integer $n \geq 0$.

The following result associates the spectral radius $r(P)$ with an integral of the coefficients of equation (2.2.7).

Proposition 2.2.1 $r = r(P)$ *is positive and is an eigenvalue of* P *with a positive eigenfunction* φ^*. *Moreover, if* $\tau = kT$ *for some integer* $k \geq 0$, *then* $r - 1$ *has the same sign as* $\int_0^T (a(t) + b(t))dt$.

Proof. By assumption (E), [42, Theorem 3.6.1] and [72, Lemma 5.3.2], there exists an integer $mT \geq 2\tau$ such that P^m is compact and strongly positive. By the Krein-Rutman theorem (see, e.g., [47, Theorem 7.2]), $r_m = r(P^m) > 0$ and is an algebraically simple eigenvalue of P^m with an eigenfunction $\varphi_m^* \gg 0$. Since P is a bounded linear operator on Y^+, $r_m = r^m$ (see, e.g., [54, Theorem 7.4-2]). Moreover, the spectrum of P consists of the point spectrum of P and the possible accumulation point being zero (see, e.g., [42, Page 192]). Thus, r is a positive eigenvalue of P. Let $P\varphi^* = r\varphi^*$. Without lose of generality, we assume $\varphi^*(s_0) > 0$ for some $s_0 \in [-\tau, 0]$.

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Since $P^m\varphi^* = r^m\varphi^* = r_m\varphi^*$, we have $\varphi^* = c\,\varphi_m^*$ for some positive constant c. Thus $\varphi^* \gg 0$.

Letting $u(t) = e^{\lambda t}v(t)$, we obtain a linear periodic equation with parameter λ,

$$\dot{v} = (a(t) - \lambda)v(t) + b(t)e^{-\lambda\tau}v(t-\tau). \tag{2.2.8}$$

Define $Q : Y^+ \to Y^+$ by $Q(\varphi) = v_T(\varphi)$, where $v_t(\varphi)$ is the solution semiflow of equation (2.2.8). Let E_λ be a map from Y^+ to Y^+ defined by $[E_\lambda(\varphi)](s) = e^{\lambda s}\varphi(s), \forall s \in [-\tau, 0]$. Then

$$Q(\varphi)(s) = v_T(\varphi)(s) = v(T+s, \varphi) = e^{-\lambda(T+s)}u(T+s, E_\lambda(\varphi)),\ \forall s \in [-\tau, 0],$$

and hence,

$$Q(\varphi) = e^{-\lambda T}E_{-\lambda}(u_T(E_\lambda(\varphi))) = e^{-\lambda T}E_{-\lambda}(P(E_\lambda(\varphi))).$$

Thus, $Q(E_{-\lambda}(\varphi^*)) = e^{-\lambda T}E_{-\lambda}(P(\varphi^*)) = re^{-\lambda T}E_{-\lambda}(\varphi^*)$. Let $\lambda_0 = \frac{1}{T}\ln r$. Then $E_{-\lambda_0}(\varphi^*)$ is a positive fixed point of Q. Thus $v_0(t) = v(t, E_{-\lambda_0}(\varphi^*))$ is a positive T-periodic solution of (2.2.8), and $u(t) = v_0(t)e^{\lambda_0 t} > 0$ for $t \geq -\tau$. In particular, if $\tau = kT$ for some integer $k \geq 0$, then $v_0(t)$ satisfies

$$\frac{\dot{v}_0(t)}{v_0(t)} = a(t) - \lambda_0 + b(t)e^{-\lambda_0\tau},\ \forall t \geq 0.$$

Integrating both sides of the above equation from 0 to T, we get $\lambda_0 = \frac{1}{T}\int_0^T(a(t) + e^{-\lambda_0\tau}b(t))dt$. Note that $G(\lambda) = \frac{1}{T}\int_0^T a(t)dt + \frac{1}{T}e^{-\lambda\tau}\int_0^T b(t)dt$ is strictly decreasing, and λ_0 is the unique solution of $\lambda = G(\lambda)$, we have $\lambda_0 G(0) > 0$, i.e., $(r-1)\int_0^T(a(t)+b(t))dt > 0$. The desired results are established. ∎

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Let us consider a nonlinear T-periodic equation

$$\begin{cases} \dot{u} = f(t, u(t), u(t-\tau)), \\ u(s) = \varphi(s), -\tau \le s \le 0, \end{cases} \tag{2.2.9}$$

where $\varphi \in Y^+$ is an initial function to be specified later.

Assume that the continuous function $f(t, v_1, v_2)$ is T-periodic in t and Lipschitzian in (v_1, v_2) in any bounded subset of $\mathbb{R}^2_+$, and satisfies

(E1) $f(t, 0, 0) = 0, f(t, 0, v_2) \ge 0, \frac{\partial}{\partial v_2} f(t, v_1, v_2) > 0, \forall v_1, v_2 \ge 0$;

(E2) f is strictly sublinear;

(E3) there exists a positive number $L > 0$ such that $f(t, L, L) \le 0$.

Let P_u be the Poincaré map of the linearized equation associated with equation (2.2.9) at $u \equiv 0$, and $r = r(P_u)$. Then we have the following threshold type result on the global dynamics of (2.2.9).

Theorem 2.2.1 *Let (E1)-(E3) hold. Then the following statements hold.*

(i) If $r \le 1$, then zero solution is globally asymptotically stable for equation (2.2.9) with respect to Y^+;

(ii) If $r > 1$, then equation (2.2.9) has a unique positive T-periodic solution $u(t, \varphi_0)$, and $u(t, \varphi_0)$ is globally asymptotically stable with respect to $Y^+ \setminus \{0\}$.

Proof. Let $a(t) = \frac{\partial}{\partial v_1} f(t, 0, 0)$, $b(t) = \frac{\partial}{\partial v_2} f(t, 0, 0)$. Since f is strictly sublinear, $f(t, v_1, v_2) \le a(t)v_1 + b(t)v_2$. Note that $b(t) > 0, f(t, 0, v_2) \ge 0$. By the comparison theorem ([72, Theorem 5.1.1]) and the positivity theorem ([72, Theorem 5.2.1]), each

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By the continuity and differentiability of solutions with respect to initial values, it follows that the Poincaré map S_u is differentiable at zero, and $DS_u(0) = P_u$. Since $b(t) > 0$, as in the proof of Proposition 2.2.1, $(DS_u(0))^n$ is compact and strongly positive for all $nT \geq 2\tau$.

Let us consider $S_u^{n_0}$, where $n_0 T \geq 2\tau$. Then, $S_u^{n_0}$ is strongly monotone, and $(DS_u(0))^{n_0}$ is compact and strongly positive.

For any $\beta \geq 1$, since f is strictly sublinear, we have $f(t, \beta L, \beta L) < \beta f(t, L, L) \leq 0$. Thus, [72, Remark 5.2.1] implies that for any $\beta \geq 1$, the order interval $V_\beta = [0, \beta L] = \{\varphi \in Y^+ : 0 \leq \varphi(s) \leq \beta L, s \in [-\tau, 0]\}$ is a positive invariant set for S_u. By [42, Theorem 3.6.1], $S_u^{n_0} : V_\beta \to V_\beta$ is compact for any fixed $\beta \geq 1$. Then the closure of $S_u^{n_0}([\varphi, \psi])$ is a compact subset of V_β for any $\varphi, \psi \in V_\beta$ with $\varphi \leq \psi$. Furthermore, $DS_u^{n_0}(0) = (DS_u(0))^{n_0}$, which is compact and strongly positive. Note that S_u is strictly sublinear, $S_u^{n_0}$ is strongly monotone, and $r\{(DS_u(0))^{n_0}\} = [r(DS_u(0))]^{n_0} = (r(P_u))^{n_0} = r^{n_0}$. By Theorem 1.2.1, as applied to $S_u^{n_0}$, we have the following conclusions:

(i) If $r \leq 1$, then zero is a globally asymptotically stable fixed point of $S_u^{n_0}$ with respect to V_β.

solution $u(t,\varphi)$ of equation (2.2.9) with initial value $\varphi \in Y^+$ exists globally, and $u(t,\varphi) \geq 0, \forall t \geq -\tau$. Since $\frac{\partial}{\partial v_2} f(t, v_1, v_2) > 0$, the nonautonomous version of [72, Theorem 5.3.4] implies that for any $\varphi, \psi \in Y^+$ with $\varphi \leq \psi$, $u_t(\varphi) \leq u_t(\psi), \forall t \geq 0$; and if $\varphi < \psi$, then $u_t(\varphi) \ll u_t(\psi), \forall t \geq 2\tau$. Define $S_u : Y^+ \to Y^+$ by $S_u(\varphi) = u_T(\varphi)$. Then S_u is monotone, and S_u^n is strongly monotone for $nT \geq 2\tau$. Moreover, the strict sublinearity of f implies that S_u is strictly sublinear (see the proof of [92, Theorem 3.3]).

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(i) If $r \leq 1$, then zero is a globally asymptotically stable fixed point of $S_u^{n_0}$ with respect to V_β.

(ii) If $r > 1$, then $S_u^{n_0}$ has a unique positive fixed point φ_0 in V_β, and φ_0 is globally asymptotically stable with respect to $V_\beta \setminus \{0\}$.

By the arbitrariness of β, the above results hold on the whole space Y^+ for $S_u^{n_0}$. It then follows that zero solution of equation (2.2.9) is globally asymptotically stable in case (i); and equation (2.2.9) admits the unique positive and n_0T-periodic solution $u(t, \varphi_0)$ in case (ii). It remains to prove that $u(t, \varphi_0)$ is T-periodic. By Proposition 2.2.1, we know that there exists a positive eigenfunction φ^* such that $DS_u(0)(\varphi^*) = r\varphi^*$. In the case of $r > 1$, for any small $\varepsilon > 0$, it is easy to find an increasing sequence $0 \ll \varepsilon\varphi^* \ll S_u(\varepsilon\varphi^*) \leq S_u^2(\varepsilon\varphi^*) \leq \cdots \leq S_u^n(\varepsilon\varphi^*) \leq \cdots$ (see the proof of [96, Theorem 2.1]). On the other hand, $S_u^{n_0 n}(\varepsilon\varphi^*) \to \varphi_0$ as $n \to \infty$. Thus, by the monotonicity of the sequence of $S_u^n(\varepsilon\varphi^*)$ and the continuity of S_u, φ_0 is a fixed point of S_u. That is, $u(t, \varphi_0)$ is a T-periodic solution. ∎

In order to study the asymptotically periodic system (2.1.6), we need to understand the relationship between solutions of an asymptotically periodic system and its limiting periodic system. Let $C = C([-\tau, 0], \mathbb{R}^n)$, $f(t, \varphi)$ and $\tilde{f}(t, \varphi)$ be continuous functions on $\mathbb{R} \times C$. Consider the retarded functional differential equations

$$\dot{u} = f(t, u_t), \; u \in \mathbb{R}^n, \tag{2.2.10}$$

$$\dot{\tilde{u}} = \tilde{f}(t, \tilde{u}_t), \; \tilde{u} \in \mathbb{R}^n. \tag{2.2.11}$$

Assume that continuous functions f and $\tilde{f}$ are Lipschitzian in φ in each compact subset of $\mathbb{R} \times C$, and f is T-periodic in t. For any $(\sigma, \varphi) \in \mathbb{R} \times C$, denote by $u(t, \sigma, \varphi)$ and $\tilde{u}(t, \sigma, \varphi)$ the solutions of system (2.2.10) and (2.2.11) satisfying $u_\sigma = \varphi$ and $\tilde{u}_\sigma = \varphi$, respectively. Let $\tilde{\Phi}(t, \sigma, \varphi) = \tilde{u}_t(\sigma, \varphi)$, $\Phi(t, \sigma, \varphi) = u_t(\sigma, \varphi)$, $T(t)\varphi = \Phi(t, 0, \varphi)$.

(ii) If $r > 1$, then $S_u^{n_0}$ has a unique positive fixed point φ_0 in V_β, and φ_0 is globally asymptotically stable with respect to $V_\beta \setminus \{0\}$.

By the arbitrariness of β, the above results hold on the whole space Y^+ for $S_u^{n_0}$. It then follows that zero solution of equation (2.2.9) is globally asymptotically stable in case (i); and equation (2.2.9) admits the unique positive and n_0T-periodic solution $u(t, \varphi_0)$ in case (ii). It remains to prove that $u(t, \varphi_0)$ is T-periodic. By Proposition 2.2.1, we know that there exists a positive eigenfunction φ^* such that $DS_u(0)(\varphi^*) = r\varphi^*$. In the case of $r > 1$, for any small $\varepsilon > 0$, it is easy to find an increasing sequence $0 \ll \varepsilon\varphi^* \ll S_u(\varepsilon\varphi^*) \leq S_u^2(\varepsilon\varphi^*) \leq \cdots \leq S_u^n(\varepsilon\varphi^*) \leq \cdots$ (see the proof of [96, Theorem 2.1]). On the other hand, $S_u^{n_0 n}(\varepsilon\varphi^*) \to \varphi_0$ as $n \to \infty$. Thus, by the monotonicity of the sequence of $S_u^n(\varepsilon\varphi^*)$ and the continuity of S_u, φ_0 is a fixed point of S_u. That is, $u(t, \varphi_0)$ is a T-periodic solution. ∎

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Assume that continuous functions f and $\tilde{f}$ are Lipschitzian in φ in each compact subset of $\mathbb{R}\times C$, and f is T-periodic in t. For any $(\sigma, \varphi) \in \mathbb{R}\times C$, denote by $u(t, \sigma, \varphi)$ and $\tilde{u}(t, \sigma, \varphi)$ the solutions of system (2.2.10) and (2.2.11) satisfying $u_\sigma = \varphi$ and $\tilde{u}_\sigma = \varphi$, respectively. Let $\tilde{\Phi}(t, \sigma, \varphi) = \tilde{u}_t(\sigma, \varphi)$, $\Phi(t, \sigma, \varphi) = u_t(\sigma, \varphi)$, $\mathcal{T}(t)\varphi = \Phi(t, 0, \varphi)$.

Then, we have the following result for the relationship between $\tilde{\Phi}$ and $\mathcal{T}(t)$.

Proposition 2.2.2 *Assume that* $\lim_{t\to\infty} \|\tilde{f}(t,\varphi) - f(t,\varphi)\| = 0$ *uniformly for* φ *in any bounded set of* C*, and solutions of (2.2.10) and (2.2.11) are uniformly bounded. Then for any integer* $M > 0$ *and real number* $B > 0$,

$$\lim_{n\to\infty} \|\tilde{\Phi}(t + nT, nT, \varphi) - \mathcal{T}(t)\varphi\| = 0$$

uniformly for $t \in [0, MT]$ *and* $\varphi \in C$ *with* $\|\varphi\| \le B$*. In particular,* $\tilde{\Phi}$ *is asymptotic to* $\mathcal{T}(t)$.

Proof. For any $\varphi \in C$ with $\|\varphi\| \le B$, there exists B' such that $\|u_t(\sigma,\varphi)\| \le B'$, $\|\tilde{u}_t(\sigma,\varphi)\| \le B'$ for any $t \ge \sigma \ge 0$. It follows that there exists a compact set $D = D(B) \subset C$ such that for any $\sigma \ge 0$ and any $\varphi \in C$ with $\|\varphi\| \le B$, $u_t(\sigma,\varphi)$, $\tilde{u}_t(\sigma,\varphi) \in D$ for all $t \ge \sigma$. Let c be the Lipschitz constant of f on the set $[0,T] \times D$, and set $u(t) = u(t, nT, \varphi), \tilde{u}(t) = \tilde{u}(t, nT, \varphi), \forall t \ge nT$. Integrating system (2.2.10) and (2.2.11) from nT to t, respectively, we have

$$\begin{aligned} u(t) &= u(nT) + \int_{nT}^{t} f(s, u_s)ds = \varphi(0) + \int_{nT}^{t} f(s, u_s)ds, \\ \tilde{u}(t) &= \tilde{u}(nT) + \int_{nT}^{t} \tilde{f}(s, \tilde{u}_s)ds = \varphi(0) + \int_{nT}^{t} \tilde{f}(s, \tilde{u}_s)ds. \end{aligned}$$

Then,

$$\begin{aligned} \|\tilde{u}(t) - u(t)\| &\le \int_{nT}^{t} \|\tilde{f}(s, \tilde{u}_s) - f(s, u_s)\|ds \\ &\le \int_{nT}^{t} \|\tilde{f}(s, \tilde{u}_s) - f(s, \tilde{u}_s)\| + \int_{nT}^{t} \|f(s, \tilde{u}_s) - f(s, u_s)\|ds \\ &\le MT\|\tilde{f} - f\|_{D_n} + c\int_{nT}^{t} \|\tilde{u}_s - u_s\|ds \end{aligned}$$

Then, we have the following result for the relationship between $\tilde{\Phi}$ and $\mathcal{T}(t)$.

Proposition 2.2.2 *Assume that* $\lim_{t\to\infty}\|\tilde{f}(t,\varphi)-f(t,\varphi)\|=0$ *uniformly for* φ *in any bounded set of* C*, and solutions of (2.2.10) and (2.2.11) are uniformly bounded. Then for any integer* $M>0$ *and real number* $B>0$*,*

$$\lim_{n\to\infty}\|\tilde{\Phi}(t+nT,nT,\varphi)-\mathcal{T}(t)\varphi\|=0$$

uniformly for $t\in[0,MT]$ *and* $\varphi\in C$ *with* $\|\varphi\|\le B$*. In particular,* $\tilde{\Phi}$ *is asymptotic to* $\mathcal{T}(t)$*.*

Proof. For any $\varphi\in C$ with $\|\varphi\|\le B$, there exists B' such that $\|u_t(\sigma,\varphi)\|\le B'$, $\|\tilde{u}_t(\sigma,\varphi)\|\le B'$ for any $t\ge\sigma\ge 0$. It follows that there exists a compact set $D=D(B)\subset C$ such that for any $\sigma\ge 0$ and any $\varphi\in C$ with $\|\varphi\|\le B$, $u_t(\sigma,\varphi)$, $\tilde{u}_t(\sigma,\varphi)\in D$ for all $t\ge\sigma$. Let c be the Lipschitz constant of f on the set $[0,T]\times D$, and set $u(t)=u(t,nT,\varphi),\tilde{u}(t)=\tilde{u}(t,nT,\varphi),\forall t\ge nT$. Integrating system (2.2.10) and (2.2.11) from nT to t, respectively, we have

$$\begin{aligned}
u(t)&=u(nT)+\int_{nT}^{t}f(s,u_s)ds=\varphi(0)+\int_{nT}^{t}f(s,u_s)ds,\\
\tilde{u}(t)&=\tilde{u}(nT)+\int_{nT}^{t}\tilde{f}(s,\tilde{u}_s)ds=\varphi(0)+\int_{nT}^{t}\tilde{f}(s,\tilde{u}_s)ds.
\end{aligned}$$

Then,

$$\begin{aligned}
\|\tilde{u}(t)-u(t)\|&\le\int_{nT}^{t}\|\tilde{f}(s,\tilde{u}_s)-f(s,u_s)\|ds\\
&\le\int_{nT}^{t}\|\tilde{f}(s,\tilde{u}_s)-f(s,\tilde{u}_s)\|+\int_{nT}^{t}\|f(s,\tilde{u}_s)-f(s,u_s)\|ds\\
&\le MT\|\tilde{f}-f\|_{D_n}+c\int_{nT}^{t}\|\tilde{u}_s-u_s\|ds
\end{aligned}$$

for $t \in [nT, (n+M)T]$, where

$$D_n = \{(t, \phi) \in \mathbb{R} \times C : \ \|\phi\| \le B', t \in [nT, (n+M)T]\},$$

and

$$\|\tilde{f} - f\|_{D_n} = \max_{(t,\phi)\in D_n} \|\tilde{f}(t, \phi) - f(t, \phi)\|.$$

Let $v(t) = \|\tilde{u}_t(nT, \varphi) - u_t(nT, \varphi)\|$. Then, it easily follows that

$$v(t) \le MT\|\tilde{f} - f\|_{D_n} + c\int_{nT}^{t} v(s)ds$$

for $t \in [nT, (n+M)T]$. Applying Grownwall inequality to the last inequality, we have $v(t) \le c'\|\tilde{f} - f\|_{D_n}$ for $t \in [nT, (n+M)T]$, where c' is a constant independent of t, n and φ. Replacing t by $t + nT$, the last inequality is changed into $v(t+nT) \le c'\|\tilde{f} - f\|_{D_n}$ for $t \in [0, MT]$. That is,

$$\|\tilde{u}_{t+nT}(nT, \varphi) - u_{t+nT}(nT, \varphi)\| \le c'\|\tilde{f} - f\|_{D_n}$$

for $t \in [0, MT]$. Note that $u_{t+nT}(nT, \varphi) = u_t(0, \varphi) = \mathcal{T}(t)\varphi$, we have

$$\|\tilde{\Phi}(t + nT, nT, \varphi) - \mathcal{T}(t)\varphi\| \le c'\|\tilde{f} - f\|_{D_n}.$$

Thus

$$\lim_{n\to\infty} \|\tilde{\Phi}(t + nT, nT, \varphi) - \mathcal{T}(t)\varphi\| \le \lim_{n\to\infty} c'\|\tilde{f} - f\|_{D_n} = 0$$

uniformly for $t \in [0, MT]$ and $\varphi \in C$ with $\|\varphi\| \le B$.

In particular, for any $(t_0, \varphi_0) \in \mathbb{R}^+ \times C$, choose $M > 0, B > 0$, such that $t_0 \in [0, MT], \|\varphi_0\| \le B$. For any $t \in [0, MT], \|\varphi\| \le B$, by the following triangle inequality,

$$\|\tilde{\Phi}(t + nT, nT, \varphi) - \mathcal{T}(t_0)\varphi_0\| \le \|\tilde{\Phi}(t + nT, nT, \varphi) - \mathcal{T}(t)\varphi\| + \|\mathcal{T}(t)\varphi - \mathcal{T}(t_0)\varphi_0\|,$$

for $t \in [nT, (n+M)T]$, where

$$D_n = \{(t, \phi) \in \mathbb{R} \times C : \ \|\phi\| \leq B', t \in [nT, (n+M)T]\},$$

and

$$\|\tilde{f} - f\|_{D_n} = \max_{(t,\phi)\in D_n} \|\tilde{f}(t,\phi) - f(t,\phi)\|.$$

Let $v(t) = \|\tilde{u}_t(nT,\varphi) - u_t(nT,\varphi)\|$. Then, it easily follows that

$$v(t) \leq MT\|\tilde{f} - f\|_{D_n} + c\int_{nT}^{t} v(s)ds$$

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$$\|\tilde{u}_{t+nT}(nT,\varphi) - u_{t+nT}(nT,\varphi)\| \leq c'\|\tilde{f} - f\|_{D_n}$$

for $t \in [0, MT]$. Note that $u_{t+nT}(nT,\varphi) = u_t(0,\varphi) = \mathcal{T}(t)\varphi$, we have

$$\|\tilde{\Phi}(t+nT, nT, \varphi) - \mathcal{T}(t)\varphi\| \leq c'\|\tilde{f} - f\|_{D_n}.$$

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$$\lim_{n\to\infty} \|\tilde{\Phi}(t+nT, nT, \varphi) - \mathcal{T}(t)\varphi\| \leq \lim_{n\to\infty} c'\|\tilde{f} - f\|_{D_n} = 0$$

uniformly for $t \in [0, MT]$ and $\varphi \in C$ with $\|\varphi\| \leq B$.

In particular, for any $(t_0, \varphi_0) \in \mathbb{R}^+ \times C$, choose $M > 0, B > 0$, such that $t_0 \in [0, MT], \|\varphi_0\| \leq B$. For any $t \in [0, MT], \|\varphi\| \leq B$, by the following triangle inequality,

$$\|\tilde{\Phi}(t+nT, nT, \varphi) - \mathcal{T}(t_0)\varphi_0\| \leq \|\tilde{\Phi}(t+nT, nT, \varphi) - \mathcal{T}(t)\varphi\| + \|\mathcal{T}(t)\varphi - \mathcal{T}(t_0)\varphi_0\|,$$

we have $||\tilde{\Phi}(t+nT, nT, \varphi) - \mathcal{T}(t_0)\varphi_0|| \to 0$ as $(t, \varphi, n) \to (t_0, \varphi_0, +\infty)$. Thus $\tilde{\Phi}$ is asymptotic to the T-periodic semiflow $\mathcal{T}(t)$. ∎

Based on system (2.2.9), we consider a scalar asymptotically periodic system

$$\begin{cases} \dot{\tilde{u}} = \tilde{f}(t, \tilde{u}(t), \tilde{u}(t-\tau)),\ t \geq 0, \\ \tilde{u}(s) = \varphi(s), s \in [-\tau, 0], \varphi \in Y^+. \end{cases} \tag{2.2.12}$$

Assume that the continuous function $\tilde{f}(t, v_1, v_2)$ is Lipschitzian in (v_1, v_2) in any bounded subset of $\mathbb{R}^2_+$, and satisfies

(E1′) $\tilde{f}(t,0,0) = 0, \tilde{f}(t,0,v_2) \geq 0,\ \frac{\partial}{\partial v_2}\tilde{f}(t, v_1, v_2) > 0$ for $t \geq 0, v_i \geq 0, i = 1, 2$;

(E2′) there exists $L > 0$ such that $\tilde{f}(t, l, l) \leq 0$ for all $l \geq L$;

(E3′) $\lim\limits_{t\to\infty} |\tilde{f} - f| = 0$ uniformly on any bounded subset on $\mathbb{R}^2_+$, where f is defined by equation (2.2.9).

Then, by the positivity theorem ([72, Theorem 5.2.1]), the solution $\tilde{u}(t, \varphi)$ of system (2.2.12) is nonnegative on it's existence interval for any $\varphi \in Y^+$. Furthermore, it is easy to see that $\tilde{u}(t, \varphi)$ is bounded by $\max(L, ||\varphi||)$. Thus solutions of system (2.2.12) exist globally and are uniformly bounded.

In view of Theorem 2.2.1 and Proposition 2.2.2, the following result is a straightforward application of the theory of asymptotically periodic semiflows. Here we omit the proof (for a similar argument, see, e.g., the proof of Theorem 2.3.4 and [87, Theorem 2.1]).

we have $\|\tilde{\Phi}(t+nT, nT, \varphi) - \mathcal{T}(t_0)\varphi_0\| \to 0$ as $(t, \varphi, n) \to (t_0, \varphi_0, +\infty)$. Thus $\tilde{\Phi}$ is asymptotic to the T-periodic semiflow $\mathcal{T}(t)$. ∎

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Theorem 2.2.2 *Let (E1)-(E3) and (E1')-(E3') hold. Suppose that r is the spectral radius defined by Theorem 2.2.1 associated with the limiting equation (2.2.9). Then the following statements hold.*

(i) If $r \leq 1$, then zero solution attracts every solution of system (2.2.12).

(ii) If $r > 1$, then $u(t, \varphi_0)$, which is defined by Theorem 2.2.1, attracts each solution of system (2.2.12) except for zero.

2.3 Two-species Competition

For the two-species competition, the solutions of system (2.1.5) preserve a special order so that the theories of competitive systems on Banach spaces (see [50]) can be applied. Therefore, we first apply Theorem 1.2.4 and 1.2.5 to obtain the global dynamics of system (2.1.5) in the first subsection, and then use asymptotically periodic semiflows to analyze the global dynamics of system (2.1.6) in the second subsection.

2.3.1 The Periodic Case

Assume that the periodic system (2.1.5) satisfies

(H1) $F_i(t, u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i F_i(t, u_i)) > 0$, and $\frac{\partial}{\partial u_j} G_i(t, u_1, u_2) > 0$ for $t \geq 0$, $u_i \geq 0$, $1 \leq i \neq j \leq 2$;

(H2) $f_1(t, \cdot, 0, \cdot)$ and $f_2(t, 0, \cdot, \cdot)$ are strictly sublinear on $\mathbb{R}^2_+$, and $f_1(t, L, 0, L) \leq 0$ and $f_2(t, 0, L, L) \leq 0$ for some number $L > 0$.

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Consider the linearization of system (2.1.5) at zero

$$\dot{u}_1(t) = b_1(t)u_1(t-\tau_1) - a_1(t)u_1(t), \tag{2.3.13}$$

$$\dot{u}_2(t) = b_2(t)u_2(t-\tau_2) - a_2(t)u_2(t), \tag{2.3.14}$$

where $b_i(t) = F_i(t,0), a_i(t) = G_i(t,0,0)$. Let $P_1^{(0)}$ and $P_2^{(0)}$ be the Poincaré maps associated with equation (2.3.13) and (2.3.14), $r_{01} = r(P_1^{(0)})$ and $r_{02} = r(P_2^{(0)})$ be the spectral radii of $P_1^{(0)}$ and $P_2^{(0)}$, respectively. Suppose that

(H3) $r_{01} > 1, r_{02} > 1$.

By Theorem 2.2.1, it then follows that there exists a unique positive T-periodic solution $u^{(1)}(t)$ for

$$\dot{u}_1(t) = u_1(t-\tau_1)F_1(t, u_1(t-\tau_1)) - u_1(t)G_1(t, u_1(t), 0) := f_1(t, u_1(t), 0, u_1(t-\tau_1)), \tag{2.3.15}$$

and $u^{(1)}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$. The similar results hold for the following equation

$$\dot{u}_2(t) = u_2(t-\tau_2)F_2(t, u_2(t-\tau_2)) - u_2(t)G_2(t, 0, u_2(t)) := f_2(t, 0, u_2(t), u_2(t-\tau_2)). \tag{2.3.16}$$

Let $u^{(2)}(t)$ be the positive T-periodic solution for equation (2.3.16).

Obviously, $(u^{(1)}(t), 0)$ and $(0, u^{(2)}(t))$ are T-periodic solutions of system (2.1.5). Linearizing system (2.1.5) at $(u^{(1)}(t), 0)$, we have

$$\dot{u}_1(t) = b_1^{(1)}(t)u_1(t-\tau_1) - a_{11}^{(1)}(t)u_1(t) - a_{12}^{(1)}(t)u_2(t), \tag{2.3.17}$$

$$\dot{u}_2(t) = b_2^{(1)}(t)u_2(t-\tau_2) - a_{22}^{(1)}(t)u_2(t), \tag{2.3.18}$$

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By Theorem 2.2.1, it then follows that there exists a unique positive T-periodic solution $u^{(1)}(t)$ for

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and $u^{(1)}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$. The similar results hold for the following equation

$$\dot{u}_2(t) = u_2(t-\tau_2)F_2(t,u_2(t-\tau_2)) - u_2(t)G_2(t,0,u_2(t)) := f_2(t,0,u_2(t),u_2(t-\tau_2)). \tag{2.3.16}$$

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$$\dot{u}_2(t) = b_2^{(1)}(t)u_2(t-\tau_2) - a_{22}^{(1)}(t)u_2(t), \tag{2.3.18}$$

where

$$b_1^{(1)}(t) = u^{(1)}(t-\tau_1)\frac{\partial}{\partial u_1}F_1(t, u^{(1)}(t-\tau_1)) + F_1(t, u^{(1)}(t-\tau_1)),$$
$$b_2^{(1)}(t) = F_2(t,0),\ a_{11}^{(1)}(t) = G_1(t, u^{(1)}(t), 0) + u^{(1)}(t)\frac{\partial}{\partial u_1}G_1(t, u^{(1)}(t), 0),$$
$$a_{12}^{(1)}(t) = u^{(1)}(t)\frac{\partial}{\partial u_2}G_1(t, u^{(1)}(t), 0),\ a_{22}^{(1)}(t) = G_2(t, u^{(1)}(t), 0).$$

Similarly, we have the linearized system of system (2.1.5) at $(0, u^{(2)}(t))$

$$\dot{u}_1(t) = b_1^{(2)}(t)u_1(t-\tau_1) - a_{11}^{(2)}(t)u_1(t), \tag{2.3.19}$$
$$\dot{u}_2(t) = b_2^{(2)}(t)u_2(t-\tau_2) - a_{21}^{(2)}(t)u_1(t) - a_{22}^{(2)}(t)u_2(t), \tag{2.3.20}$$

where

$$b_1^{(2)}(t) = F_1(t,0),\ b_2^{(2)}(t) = u^{(2)}(t-\tau_2)\frac{\partial}{\partial u_2}F_2(t, u^{(2)}(t-\tau_2)) + F_2(t, u^{(2)}(t-\tau_2)),$$
$$a_{11}^{(2)}(t) = G_1(t, 0, u^{(2)}(t)),\ a_{21}^{(2)}(t) = u^{(2)}(t)\frac{\partial}{\partial u_1}G_2(t, 0, u^{(2)}(t)),$$
$$a_{22}^{(2)}(t) = G_2(t, 0, u^{(2)}(t)) + u^{(2)}(t)\frac{\partial}{\partial u_2}G_2(t, 0, u^{(2)}(t)).$$

Let $P_2^{(1)}$ and $P_1^{(2)}$ be the Poincaré maps of equations (2.3.18) and (2.3.19), respectively, and denote their spectral radii by $r_{12} = r(P_2^{(1)}), r_{21} = r(P_1^{(2)})$. Let

$$\varphi^*(s_1) = u^{(1)}(s_1),\ \forall s_1 \in [-\tau_1, 0];\quad \varphi^{**}(s_2) = u^{(2)}(s_2),\ \forall s_2 \in [-\tau_2, 0],$$

and set

$$E_0 = (0,0),\ E_1 = (\varphi^*, 0),\ E_2 = (0, \varphi^{**}).$$

For any $\psi \in X^+$, denote by $u(t, \psi)$ the solution of system (2.1.5). Let $u_t(\psi)$ be the solution semiflow associated with system (2.1.5). For convenience, we set $X^0 = \{(\psi_1, \psi_2) \in X^+ : \psi_i \neq 0, i = 1, 2\}$. Then we have the following result.

where

$$b_1^{(1)}(t) = u^{(1)}(t-\tau_1)\frac{\partial}{\partial u_1}F_1(t, u^{(1)}(t-\tau_1)) + F_1(t, u^{(1)}(t-\tau_1)),$$
$$b_2^{(1)}(t) = F_2(t,0),\ a_{11}^{(1)}(t) = G_1(t, u^{(1)}(t), 0) + u^{(1)}(t)\frac{\partial}{\partial u_1}G_1(t, u^{(1)}(t), 0),$$
$$a_{12}^{(1)}(t) = u^{(1)}(t)\frac{\partial}{\partial u_2}G_1(t, u^{(1)}(t), 0),\ a_{22}^{(1)}(t) = G_2(t, u^{(1)}(t), 0).$$

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Let $P_2^{(1)}$ and $P_1^{(2)}$ be the Poincaré maps of equations (2.3.18) and (2.3.19), respectively, and denote their spectral radii by $r_{12} = r(P_2^{(1)}), r_{21} = r(P_1^{(2)})$. Let

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For any $\psi \in X^+$, denote by $u(t,\psi)$ the solution of system (2.1.5). Let $u_t(\psi)$ be the solution semiflow associated with system (2.1.5). For convenience, we set $X^0 = \{(\psi_1, \psi_2) \in X^+ : \psi_i \neq 0, i = 1, 2\}$. Then we have the following result.

Theorem 2.3.1 *Let (H1)-(H3) hold and suppose that $r_{12} > 1, r_{21} > 1$. Then the following hold for system (2.1.5):*

(i) System (2.1.5) has two positive T-periodic solutions $u(t, \phi^*)$ and $u(t, \phi^{**})$ satisfying $u(t, \phi^{**}) \leq_K u(t, \phi^*), t \geq 0$, where $\phi^*, \phi^{**} \in int(X^+)$ with $\phi^{**} \leq_K \phi^*$.

(ii) $\lim_{t\to\infty} \|u(t, \psi) - u(t, \phi^*)\| = 0$ for every $\psi = (\psi_1, \psi_2) \in X^+$ with $\phi^* \leq_K \psi <_K E_1$ and $\psi_2 \neq 0$. Symmetrically, $\lim_{t\to\infty} \|u(t, \psi) - u(t, \phi^{**})\| = 0$ for every $\psi = (\psi_1, \psi_2) \in X^+$ with $E_2 <_K \psi \leq_K \phi^{**}$ and $\psi_1 \neq 0$.

(iii) $\lim_{t\to\infty} dist(u(t, \psi), [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0$ for any point $\psi \in X^0$.

In particular, in the case where $\tau_i = k_i T$ for some integers $k_i, i = 1, 2$, if assumption (H1) and (H2) hold, and $\int_0^T (b_i(t) - a_i(t))dt > 0, \int_0^T (b_j^{(i)}(t) - a_{jj}^{(i)}(t))dt > 0$ for $i \neq j$ and $i, j = 1, 2$, then the above results hold.

In the rest of this section, we use S to denote the Poincaré map associated with system (2.1.5). In order to prove Theorem 2.3.1, we need the following two lemmas. The first one establishes some properties of S, and the second one implies that E_i $(i = 0, 1, 2)$ are isolated fixed points for S^{n_0}, and that there exist no points in $int(X^+)$ converging to E_i under S^{n_0}, where n_0 is an integer.

Lemma 2.3.1 *The Poincaré map $S : X^+ \to X^+$ is strictly monotone with respect to $\leq_K$, and is a bounded map.*

Proof. For any $\psi \in X^+$, by the positivity theorem ([72, Theorem 5.2.1]) and assumption (H1), the solution $u(t, \psi)$ of system (2.1.5) is nonnegative on its existence interval. Note that assumption (H1) implies the inequalities

$$f_1(t, u_1, u_2, v_1) \leq f_1(t, u_1, 0, v_1) \text{ and } f_2(t, u_1, u_2, v_2) \leq f_2(t, 0, u_2, v_2)$$

Theorem 2.3.1 *Let (H1)-(H3) hold and suppose that* $r_{12} > 1, r_{21} > 1$. *Then the following hold for system (2.1.5):*

(i) System (2.1.5) has two positive T*-periodic solutions* $u(t, \phi^*)$ *and* $u(t, \phi^{**})$ *satisfying* $u(t, \phi^{**}) \leq_K u(t, \phi^*), t \geq 0$, *where* $\phi^*, \phi^{**} \in int(X^+)$ *with* $\phi^{**} \leq_K \phi^*$.

(ii) $\lim_{t\to\infty} \|u(t, \psi) - u(t, \phi^*)\| = 0$ *for every* $\psi = (\psi_1, \psi_2) \in X^+$ *with* $\phi^* \leq_K \psi <_K E_1$ *and* $\psi_2 \neq 0$. *Symmetrically,* $\lim_{t\to\infty} \|u(t, \psi) - u(t, \phi^{**})\| = 0$ *for every* $\psi = (\psi_1, \psi_2) \in X^+$ *with* $E_2 <_K \psi \leq_K \phi^{**}$ *and* $\psi_1 \neq 0$.

(iii) $\lim_{t\to\infty} dist(u(t, \psi), [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0$ *for any point* $\psi \in X^0$.

In particular, in the case where $\tau_i = k_i T$ *for some integers* $k_i, i = 1, 2$, *if assumption (H1) and (H2) hold, and* $\int_0^T (b_i(t) - a_i(t))dt > 0, \int_0^T (b_j^{(i)}(t) - a_{jj}^{(i)}(t))dt > 0$ *for* $i \neq j$ *and* $i, j = 1, 2$, *then the above results hold.*

In the rest of this section, we use S to denote the Poincaré map associated with system (2.1.5). In order to prove Theorem 2.3.1, we need the following two lemmas. The first one establishes some properties of S, and the second one implies that E_i $(i = 0, 1, 2)$ are isolated fixed points for S^{n_0}, and that there exist no points in $int(X^+)$ converging to E_i under S^{n_0}, where n_0 is an integer.

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for $u_i, v_i \geq 0, i = 1, 2$. Since the solutions for equation (2.3.15) and (2.3.16) exist globally, by the comparison theorem ([72, Theorem 5.1.1]), the solution $u(t, \psi)$ for system (2.1.5) globally exists for any $\psi \in X^+$. By assumption (H1), it easily follows that the solution $u_1(t, \varphi_1)$ of equation (2.3.15) is bounded by $B = \max\{L, \|\varphi_1\|\}$, and hence the solution for equation (2.3.15) is uniformly bounded. The same conclusions hold for equation (2.3.16). Therefore, solutions for system (2.1.5) are also uniformly bounded.

Let $u_t(\psi)$ be the solution semiflow of system (2.1.5) with $u_0(\psi) = \psi \in X^+$. Then assumption (H1) implies that, $u_t(\psi) \geq 0$ for all $t \geq 0$ (see [72, Theorem 5.2.1]). Moreover, if $\varphi, \psi \in X^+$ with $\varphi \leq_K \psi$, by the comparison theorem and the transformation $U_1 = u_1, U_2 = -u_2$, it easily follows that $u_t(\varphi) \leq_K u_t(\psi)$ for all $t \geq 0$. Let $S : X^+ \to X^+$ be the Poincaré map associated with system (2.1.5), i.e., $S = u_T(\cdot)$. Then S is monotone with respect to $\leq_K$, and S is a bounded map.

It remains to prove that S is strictly monotone with respect to $\leq_K$, i.e., $S(\varphi) <_K S(\psi)$ if $\varphi <_K \psi$. Suppose, by contradiction, that $S(\varphi) = S(\psi)$. Let $u(t, \varphi) = (u_1(t, \varphi), u_2(t, \varphi)), u(t, \psi) = (u_1(t, \psi), u_2(t, \psi))$. Then $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for all $t_i \in [T - \tau_i, T], i = 1, 2$. Thus,

$$\begin{aligned} 0 &= \dot{u}_i(t_i, \varphi) - \dot{u}_i(t_i, \psi) \\ &= u_i(t_i - \tau_i, \varphi) F_i(t_i, u_i(t_i - \tau_i, \varphi)) - u_i(t_i - \tau_i, \psi) F_i(t_i, u_i(t_i - \tau_i, \psi)) \end{aligned}$$

for $t_i \in (T - \tau_i, T]$. Since $u_i F_i(t, u_i)$ is strictly increasing, $u_i(t_i - \tau_i, \varphi) = u_i(t_i - \tau_i, \psi)$. Therefore, $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for $t_i \in (T - 2\tau_i, T], i = 1, 2$. By induction, we have $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for $t_i \in [-\tau_i, 0]$, i.e., $\varphi = \psi$, which contradicts to $\varphi <_K \psi$. Thus we have $S(\varphi) <_K S(\psi)$. ∎

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It remains to prove that S is strictly monotone with respect to $\leq_K$, i.e., $S(\varphi) <_K S(\psi)$ if $\varphi <_K \psi$. Suppose, by contradiction, that $S(\varphi) = S(\psi)$. Let $u(t, \varphi) = (u_1(t, \varphi), u_2(t, \varphi)), u(t, \psi) = (u_1(t, \psi), u_2(t, \psi))$. Then $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for all $t_i \in [T - \tau_i, T], i = 1, 2$. Thus,

$$
\begin{aligned}
0 &= \dot{u}_i(t_i, \varphi) - \dot{u}_i(t_i, \psi) \\
&= u_i(t_i - \tau_i, \varphi)F_i(t_i, u_i(t_i - \tau_i, \varphi)) - u_i(t_i - \tau_i, \psi)F_i(t_i, u_i(t_i - \tau_i, \psi))
\end{aligned}
$$

for $t_i \in (T - \tau_i, T]$. Since $u_i F_i(t, u_i)$ is strictly increasing, $u_i(t_i - \tau_i, \varphi) = u_i(t_i - \tau_i, \psi)$. Therefore, $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for $t_i \in (T - 2\tau_i, T], i = 1, 2$. By induction, we have $u_i(t_i, \varphi) = u_i(t_i, \psi)$ for $t_i \in [-\tau_i, 0]$, i.e., $\varphi = \psi$, which contradicts to $\varphi <_K \psi$. Thus we have $S(\varphi) <_K S(\psi)$. ∎

Lemma 2.3.2 *Suppose $u^*(t) = (u_1^*(t), u_2^*(t))$ is a T-periodic solution of equation (2.1.5) with $u_i^*(t) \geq 0$ for $1 \leq i \leq 2$, and $u_j^*(t) \equiv 0$ for some j. Let P_j be the Poincaré map of*

$$\dot{u}_j(t) = F_j(t,0)u_j(t-\tau_j) - G_j(t, u_1^*(t), u_2^*(t))u_j(t).$$

If $r_j = r(P_j) > 1$, then for any integer $n_0 \geq 1$, there exists $\delta > 0$ such that $\limsup_{n\to\infty} \|S^{n_0 n}(\psi) - \psi^*\| \geq \delta$ for all $\psi \in int(X^+)$, where $\psi^* \in X^+$ is the initial function of $u^*(t)$.

Proof. Since $u^*(t)$ is also an n_0T-periodic solution of n_0T-periodic system (2.1.5), and $r\{(P_j)^{n_0}\} = [r(P_j)]^{n_0} = r_j^{n_0} > 1$, without loss of generality, we can assume that $n_0 = 1$.

It suffices to prove that there exists $\delta > 0$ such that for any $\psi \in int(X^+)$ with $\|\psi - \psi^*\| < \delta$, there exists $N \geq 1$ such that $\|S^N(\psi) - \psi^*\| \geq \delta$. Let $b_1 = \min\limits_{t\in[0,T]} F_j(t,0)$. For any $\varepsilon \in (0, b_1)$, let r^ε be the spectral radius of the Poincaré map associated with

$$\dot{u}(t) = (F_j(t,0) - \varepsilon)u(t-\tau_j) - (G_j(t, u_1^*(t), u_2^*(t)) + \varepsilon)u(t). \tag{2.3.21}$$

Then $\lim\limits_{\varepsilon\to 0} r^\varepsilon = r_j > 1$. In what follows, we fix a sufficient small $\varepsilon \in (0, b_1)$ such that $r^\varepsilon > 1$. For this fixed ε, assumption (H1) implies that there exists $\delta_1 > 0$ such that

$$F_j(t, u_j) > F_j(t,0) - \varepsilon, \ \forall t \in [0,\infty), \ \forall u_j \in [0, \delta_1).$$

Let $b_2 = \max\limits_{t\in[0,T]} \|u^*(t)\|$. By the uniform continuity of G_j on the set $[0,\infty)\times[0, b_2+1]^2$, there exists $\delta_2 > 0$ such that

$$|G_j(t, u_1, u_2) - G_j(t, u_1', u_2')| < \varepsilon, \ \forall t \in [0,\infty),$$

Lemma 2.3.2 *Suppose $u^*(t) = (u_1^*(t), u_2^*(t))$ is a T-periodic solution of equation (2.1.5) with $u_i^*(t) \geq 0$ for $1 \leq i \leq 2$, and $u_j^*(t) \equiv 0$ for some j. Let P_j be the Poincaré map of*

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If $r_j = r(P_j) > 1$, then for any integer $n_0 \geq 1$, there exists $\delta > 0$ such that $\limsup_{n\to\infty} \|S^{n_0 n}(\psi) - \psi^*\| \geq \delta$ *for all $\psi \in int(X^+)$, where $\psi^* \in X^+$ is the initial function of $u^*(t)$.*

Proof. Since $u^*(t)$ is also an n_0T-periodic solution of n_0T-periodic system (2.1.5), and $r\{(P_j)^{n_0}\} = [r(P_j)]^{n_0} = r_j^{n_0} > 1$, without loss of generality, we can assume that $n_0 = 1$.

It suffices to prove that there exists $\delta > 0$ such that for any $\psi \in int(X^+)$ with $\|\psi - \psi^*\| < \delta$, there exists $N \geq 1$ such that $\|S^N(\psi) - \psi^*\| \geq \delta$. Let $b_1 = \min_{t\in[0,T]} F_j(t,0)$. For any $\varepsilon \in (0, b_1)$, let r^ε be the spectral radius of the Poincaré map associated with

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Then $\lim_{\varepsilon\to 0} r^\varepsilon = r_j > 1$. In what follows, we fix a sufficient small $\varepsilon \in (0, b_1)$ such that $r^\varepsilon > 1$. For this fixed ε, assumption (H1) implies that there exists $\delta_1 > 0$ such that

$$F_j(t, u_j) > F_j(t,0) - \varepsilon, \ \forall t \in [0,\infty), \ \forall u_j \in [0, \delta_1).$$

Let $b_2 = \max_{t\in[0,T]} \|u^*(t)\|$. By the uniform continuity of G_j on the set $[0,\infty)\times[0, b_2+1]^2$, there exists $\delta_2 > 0$ such that

$$|G_j(t, u_1, u_2) - G_j(t, u_1', u_2')| < \varepsilon, \ \forall t \in [0,\infty),$$

for any $u = (u_1, u_2), u' = (u'_1, u'_2) \in [0, b_2 + 1]^2$ with $\|u - u'\| < \delta_2$. By the continuous dependence of solutions on initial values, there exists $\delta > 0$ such that for any $\psi \in int(X^+)$ with $\|\psi - \psi^*\| < \delta$,

$$\|u(t, \psi) - u^*(t)\| < \delta' = \min(1, \delta_1, \delta_2), \ \forall t \in [0, T).$$

Proceeding by contradiction, assume that there exists $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) \in int(X^+)$ with $\|\bar{\psi} - \psi^*\| < \delta$ such that $\|S^n(\bar{\psi}) - \psi^*\| < \delta$ for all $n \geq 1$. For any $t \geq 0$, let $t = nT + t'$, where $t' \in [0, T)$, $n = [t/T]$ is the greatest integer less than or equal to t/T. Then,

$$\|u(t, \bar{\psi}) - u^*(t)\| = \|u(t', S^n(\bar{\psi})) - u^*(t')\| < \delta', \ \forall t \geq 0.$$

Let $u(t, \bar{\psi}) = (\bar{u}_1(t), \bar{u}_2(t))$. Then

$$F_j(t, \bar{u}_j(t)) > F_j(t, 0) - \varepsilon,$$

and

$$|G_j(t, \bar{u}_1(t), \bar{u}_2(t)) - G_j(t, u_1^*(t), u_2^*(t))| < \varepsilon, \ \forall t \geq 0.$$

Thus,

$$\begin{aligned} \dot{\bar{u}}_j(t) &= \bar{u}_j(t - \tau_j) F_j(t, \bar{u}_j(t - \tau_j)) - \bar{u}_j(t) G_j(t, \bar{u}_1(t), \bar{u}_2(t)) \\ &> (F_j(t, 0) - \varepsilon)\bar{u}_j(t - \tau_j) - \bar{u}_j(t)(G_i(t, u_1^*, u_2^*) + \varepsilon), \ \forall t \geq 0. \end{aligned} \tag{2.3.22}$$

As in the proof of Proposition 2.2.1, equation (2.3.21) has a solution $u^0(t) = v_0(t)e^{\lambda_0 t}$, where $v_0(t)$ is a positive, T-periodic and continuous function, $\lambda_0 = \frac{1}{T} \ln r^\varepsilon > 0$. Let $\varphi_0(s) = u^0(s), \ s \in [-\tau_j, 0]$. Then $\varphi_0 \gg 0$. Since $\bar{\psi}_j \gg 0$, there exists $\eta > 0$ such that $\eta\varphi_0 \leq \bar{\psi}_j$. By the comparison theorem and inequality (2.3.22), for any $u = (u_1, u_2), u' = (u_1', u_2') \in [0, b_2+1]^2$ with $\|u - u'\| < \delta_2$. By the continuous dependence of solutions on initial values, there exists $\delta > 0$ such that for any $\psi \in int(X^+)$ with $\|\psi - \psi^*\| < \delta$,

$$\|u(t,\psi) - u^*(t)\| < \delta' = \min(1, \delta_1, \delta_2), \ \forall t \in [0, T).$$

Proceeding by contradiction, assume that there exists $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2) \in int(X^+)$ with $\|\bar{\psi} - \psi^*\| < \delta$ such that $\|S^n(\bar{\psi}) - \psi^*\| < \delta$ for all $n \geq 1$. For any $t \geq 0$, let $t = nT + t'$, where $t' \in [0, T)$, $n = [t/T]$ is the greatest integer less than or equal to t/T. Then,

$$\|u(t, \bar{\psi}) - u^*(t)\| = \|u(t', S^n(\bar{\psi})) - u^*(t')\| < \delta', \ \forall t \geq 0.$$

Let $u(t, \bar{\psi}) = (\bar{u}_1(t), \bar{u}_2(t))$. Then

$$F_j(t, \bar{u}_j(t)) > F_j(t, 0) - \varepsilon,$$

and

$$|G_j(t, \bar{u}_1(t), \bar{u}_2(t)) - G_j(t, u_1^*(t), u_2^*(t))| < \varepsilon, \ \forall t \geq 0.$$

Thus,

$$\begin{aligned} \dot{\bar{u}}_j(t) &= \bar{u}_j(t-\tau_j) F_j(t, \bar{u}_j(t-\tau_j)) - \bar{u}_j(t) G_j(t, \bar{u}_1(t), \bar{u}_2(t)) \\ &> (F_j(t,0) - \varepsilon)\bar{u}_j(t-\tau_j) - \bar{u}_j(t)(G_i(t, u_1^*, u_2^*) + \varepsilon), \ \forall t \geq 0. \end{aligned} \tag{2.3.22}$$

As in the proof of Proposition 2.2.1, equation (2.3.21) has a solution $u^0(t) = v_0(t)e^{\lambda_0 t}$, where $v_0(t)$ is a positive, T-periodic and continuous function, $\lambda_0 = \frac{1}{T}\ln r^{\varepsilon} > 0$. Let $\varphi_0(s) = u^0(s)$, $s \in [-\tau_j, 0]$. Then $\varphi_0 \gg 0$. Since $\bar{\psi}_j \gg 0$, there exists $\eta > 0$ such that $\eta\varphi_0 \leq \bar{\psi}_j$. By the comparison theorem and inequality (2.3.22), we have $\bar{u}_j(t) \geq u_j^\varepsilon(t, \bar{\psi}_j) \geq \eta u^0(t)$, where $u_j^\varepsilon(t, \bar{\psi}_j)$ is the solution of (2.3.21) with $u_j^\varepsilon(s, \bar{\psi}_j) = \bar{\psi}_j(s), \forall s \in [-\tau_j, 0]$. Therefore, $\lim_{t\to\infty} \bar{u}_j(t) \geq \lim_{t\to\infty} \eta u^0(t) = \infty$. Thus $S^n(\bar{\psi})$ is unbounded, a contradiction. ∎

Proof of Theorem 2.3.1. Note that the Poincaré map $S : X^+ \to X^+$ is α-condensing and S^n is compact for sufficiently large n (see, e.g., [42, Theorem 3.6.1]). We then proceed with two steps. The first step is to verify the basic assumptions (A1)-(A4) in Section 1.2 for competitive systems on Banach spaces, and then apply the compression theorem (Theorem 1.2.5) to S^{n_0}, where n_0 is an appropriate positive integer. In the second step, we prove that fixed points ϕ^* and ϕ^{**} of S^{n_0} are actually fixed points of S.

Step 1. So far, we have shown that (1). $u^{(1)}(t)$ and $u^{(2)}(t)$ are stable positive T-periodic solutions for equation (2.3.15) and (2.3.16), respectively, and they attract all of the solutions except for the trivial solution; (2). the Poincaré map S for system (2.1.5) is bounded and strictly monotone with respect to $\leq_K$ (see Lemma 2.3.1).

Let S_{u_1} and S_{u_2} be the Poincaré maps of equation (2.3.15) and (2.3.16), respectively. Since $X_1^+ \times \{0\}$ and $\{0\} \times X_2^+$ are clearly invariant sets for system (2.1.5), we have $S = (S_{u_1}, 0)$ on $X_1^+ \times \{0\}$, $S = (0, S_{u_2})$ on $\{0\} \times X_1^+$. Therefore, $\lim_{n\to\infty} S^n((\varphi_1, 0)) = E_1$ for any $\varphi_1 \in X_1^+ \setminus \{0\}$, and $\lim_{n\to\infty} S^n((0, \varphi_2)) = E_2$ for any $\varphi_2 \in X_2^+ \setminus \{0\}$.

Claim. For any $\varphi = (\varphi_1, \varphi_2) \in X^0$, $u(t, \varphi) \gg 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$. In particular, $S^n(\varphi) \gg 0$ for all $nT \geq 2\tau$.

we have $\bar{u}_j(t) \geq u_j^{\epsilon}(t, \bar{\psi}_j) \geq \eta u^0(t)$, where $u_j^{\epsilon}(t, \bar{\psi}_j)$ is the solution of (2.3.21) with $u_j^{\epsilon}(s, \bar{\psi}_j) = \bar{\psi}_j(s), \forall s \in [-\tau_j, 0]$. Therefore, $\lim_{t\to\infty} \bar{u}_j(t) \geq \lim_{t\to\infty} \eta u^0(t) = \infty$. Thus $S^n(\bar{\psi})$ is unbounded, a contradiction. ∎

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Claim. For any $\varphi = (\varphi_1, \varphi_2) \in X^0$, $u(t, \varphi) \gg 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$. In particular, $S^n(\varphi) \gg 0$ for all $nT \geq 2\tau$.

Indeed, for each $i=1,2$, we assume that $\varphi_i(\theta_i)>0$ for some $\theta_i\in[-\tau_i,0], i=1,2$. Then $u_i(\tau_i+\theta_i,\varphi)>0$. In fact, if $u_i(\tau_i+\theta_i,\varphi)=0$, then

$$\dot{u}_i(\tau_i+\theta_i,\varphi)=u_i(\theta_i,\varphi)F_i(\tau_i+\theta_i,u_i(\theta_i,\varphi))=\varphi_i(\theta_i)F_i(\tau_i+\theta_i,\varphi_i(\theta_i))>0,$$

which implies that $u_i(t_i',\varphi)<0$ for some $t_i'<\tau_i+\theta_i$. However, by the proof of Lemma 2.3.1, $u_i(t,\varphi)\geq 0$ for all $t\geq -\tau_i$, a contradiction. Thus, we have $u_i(\tau_i+\theta_i,\varphi)>0$. On the other hand,

$$\dot{u}_i(t,\varphi)=u_i(t-\tau_i,\varphi)F_i(t,u_i(t-\tau_i,\varphi))-u_i(t,\varphi)G_i(t,u_1,u_2)\geq -u_i(t,\varphi)G_i(t,u_1,u_2).$$

Then

$$u_i(t,\varphi)\geq u_i(\tau_i+\theta_i,\varphi)e^{-\int_{\tau_i+\theta_i}^{t}G_i(s,u_1,u_2)ds}>0,\ \text{ for } t\geq \tau_i+\theta_i.$$

Therefore, $u_i(t,\varphi)>0$ for $t\geq\tau_i+\theta_i$. Thus $u(t,\varphi)\gg 0$ for $t\geq\tau=\max(\tau_1,\tau_2)$.

Given an order interval $I=[0,\alpha_1]\times[0,\alpha_2], \alpha_i\in X_i^+, i=1,2$. Since S^n is compact for $nT\geq\tau$ (see, e.g., [42, Theorem 3.6.1]), $S^n(I)$ is precompact. Thus, for all $nT\geq\tau$, S^n is order compact with respect to $\leq_K$.

At any point $\varphi=(\varphi_1,\varphi_2)\in int(X^+)$, the Jacobi matrix of system (2.1.5) is

$$D(f_1,f_2)=\begin{pmatrix} D_{11} & -\varphi_1(0)\frac{\partial}{\partial u_2}G_1(t,\varphi_1(0),\varphi_2(0)) \\ -\varphi_2(0)\frac{\partial}{\partial u_1}G_2(t,\varphi_1(0),\varphi_2(0)) & D_{22}\end{pmatrix},$$

where

$$D_{ii}=\frac{\partial}{\partial u_i}\left(u_iF_i(t,u_i)\right)\Big|_{u_i=\varphi_i(-\tau_i)}-\frac{\partial}{\partial u_i}\left(u_iG_i(t,u_1,u_2)\right)\Big|_{u_1=\varphi_1(0),u_2=\varphi_2(0)},$$

$i=1,2$. $D(f_1,f_2)$ is irreducible due to assumption (H1). By [72, Theorem 5.3.4], it then easily follows that $S^n(\varphi)\ll_K S^n(\psi), \forall nT\geq 3\tau$ for any $\varphi,\psi\in int(X^+)$ with $\varphi<_K\psi$.

Indeed, for each $i = 1, 2$, we assume that $\varphi_i(\theta_i) > 0$ for some $\theta_i \in [-\tau_i, 0], i = 1, 2$. Then $u_i(\tau_i + \theta_i, \varphi) > 0$. In fact, if $u_i(\tau_i + \theta_i, \varphi) = 0$, then

$$\dot{u}_i(\tau_i + \theta_i, \varphi) = u_i(\theta_i, \varphi)F_i(\tau_i + \theta_i, u_i(\theta_i, \varphi)) = \varphi_i(\theta_i)F_i(\tau_i + \theta_i, \varphi_i(\theta_i)) > 0,$$

which implies that $u_i(t_i', \varphi) < 0$ for some $t_i' < \tau_i + \theta_i$. However, by the proof of Lemma 2.3.1, $u_i(t, \varphi) \geq 0$ for all $t \geq -\tau_i$, a contradiction. Thus, we have $u_i(\tau_i + \theta_i, \varphi) > 0$. On the other hand,

$$\dot{u}_i(t, \varphi) = u_i(t - \tau_i, \varphi)F_i(t, u_i(t - \tau_i, \varphi)) - u_i(t, \varphi)G_i(t, u_1, u_2) \geq -u_i(t, \varphi)G_i(t, u_1, u_2).$$

Then

$$u_i(t, \varphi) \geq u_i(\tau_i + \theta_i, \varphi)e^{-\int_{\tau_i+\theta_i}^{t} G_i(s, u_1, u_2)ds} > 0, \text{ for } t \geq \tau_i + \theta_i.$$

Therefore, $u_i(t, \varphi) > 0$ for $t \geq \tau_i + \theta_i$. Thus $u(t, \varphi) \gg 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$.

Given an order interval $I = [0, \alpha_1] \times [0, \alpha_2], \alpha_i \in X_i^+, i = 1, 2$. Since S^n is compact for $nT \geq \tau$ (see, e.g., [42, Theorem 3.6.1]), $S^n(I)$ is precompact. Thus, for all $nT \geq \tau$, S^n is order compact with respect to $\leq_K$.

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where

$$D_{ii} = \frac{\partial}{\partial u_i}\left(u_i F_i(t, u_i)\right)\Big|_{u_i = \varphi_i(-\tau_i)} - \frac{\partial}{\partial u_i}(u_i G_i(t, u_1, u_2))\Big|_{u_1 = \varphi_1(0), u_2 = \varphi_2(0)},$$

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Let φ, ψ be in X^+ satisfying $\varphi = (\varphi_1, \varphi_2) \gg 0, \psi = (\psi_1, 0) \in X_1^+ \times \{0\}$, and $\varphi_1 \leq \psi_1$. Then $\varphi <_K \psi$. We want to show that $S^n(\varphi) \ll_K S^n(\psi)$ for all large integers n. Let $u(t,\varphi) = (u_1(t,\varphi), u_2(t,\varphi)), u(t,\psi) = (u_1(t,\psi), 0)$. Then $u(t,\varphi) \leq_K u(t,\psi)$, i.e., $0 \leq u_1(t,\varphi) \leq u_1(t,\psi), u_2(t,\varphi) \geq 0$. By the above claim, we have $u_i(t,\varphi) > 0, \forall t \geq \tau$. Thus we only need to prove that $u_1(t,\varphi) < u_1(t,\psi), \forall t > 0$. Assume, by contradiction, that $u_1(t_0,\varphi) = u_1(t_0,\psi)$ for some $t_0 > 0$. Since $\frac{\partial}{\partial u_2} G_1(t, u_1, u_2) > 0$, and $\frac{\partial}{\partial u_1} u_1 F_1(t, u_1) > 0$, we have

$$
\begin{aligned}
&\dot{u}_1(t_0,\varphi) - \dot{u}_1(t_0,\psi) = \\
&\qquad u_1(t_0 - \tau_1, \varphi) F_1(t_0, u_1(t_0 - \tau_1, \varphi)) - u_1(t_0 - \tau_1, \psi) F_1(t_0, u_1(t_0 - \tau_1, \psi)) \\
&\qquad + u_1(t_0, \psi) G_1(t_0, u_1(t_0, \psi), 0) - u_1(t_0, \varphi) G_1(t_0, u_1(t_0, \varphi), u_2(t_0, \varphi)) < 0,
\end{aligned}
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which implies that $u_1(t,\varphi) - u_1(t,\psi) > 0$ for some $t < t_0$. The conclusion contradicts $u_1(t,\varphi) \leq u_1(t,\psi)$ for all $t \geq -\tau_1$. Thus, $u_1(t,\varphi) < u_1(t,\psi), \forall t > 0$, and hence we have $u(t,\varphi) \ll_K u(t,\psi)$ for $t > 0$. In particular, $S^n(\varphi) \ll_K S^n(\psi)$ for all $nT \geq 2\tau$. Similarly, if φ and ψ belong to X^+ and satisfy $\varphi <_K \psi, \psi \in int(X^+)$ and $\varphi \in \{0\} \times X_2^+$, we have $S^n(\varphi) \ll_K S^n(\psi)$ for all $nT \geq 2\tau$.

Let us fix an integer n_0 such that S^{n_0} satisfies

(1) $S^{n_0}(\varphi) \gg 0$ for any $\varphi \in X^0$.

(2) If $\varphi, \psi \in X^+$ satisfy $\varphi <_K \psi$, and either φ or ψ belongs to $int(X^+)$, then $S^{n_0}(\varphi) \ll_K S^{n_0}(\psi)$.

Also, S^{n_0} has the following properties:

(3) S^{n_0} is order compact and strictly monotone with respect to $\leq_K$.

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$$
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&\dot{u}_1(t_0,\varphi) - \dot{u}_1(t_0,\psi) = \\
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(5) Since $r_{12} > 1$, it follows from Lemma 2.3.2 that E_1 is an isolated fixed point of S^{n_0}, and $W^s(E_1) \cap int(X^+) = \emptyset$, where $W^s(E_1)$ is the stable set of E_1 for S^{n_0}. The same results hold for E_0 and E_2. Also, Theorem 2.2.1 implies that E_0 is a repelling fixed point of S^{n_0}.

Therefore, S^{n_0} satisfies the conditions in Theorem 1.2.5. Thus, for the map S^{n_0}, we have the following results.

(i) S^{n_0} has two positive fixed points ϕ^* and ϕ^{**} with $\phi^{**} \leq_K \phi^*$. Then, system (2.1.5) has two positive n_0T-periodic solutions $u(t, \phi^*)$ and $u(t, \phi^{**})$ with $u(t, \phi^{**}) \leq_K u(t, \phi^*)$.

(ii) For every $\psi = (\psi_1, \psi_2) \in X^+$ with $\psi_2 \neq 0$ and $\phi^* \leq_K \psi <_K E_1$, $\lim_{n\to\infty} S^{n_0 n}(\psi) = \phi^*$. It then follows that $\lim_{t\to\infty} \|u(t, \psi) - u(t, \phi^*)\| = 0$. Symmetrically, for every $\psi = (\psi_1, \psi_2) \in X^+$ with $\psi_1 \neq 0$ and $E_2 <_K \psi \leq_K \phi^{**}$, $\lim_{n\to\infty} S^{n_0 n}(\psi) = \phi^{**}$, and hence, $\lim_{t\to\infty} \|u(t, \psi) - u(t, \phi^{**})\| = 0$.

(iii) $\lim_{n\to\infty} dist(S^{n_0 n}(\psi), [\phi^{**}, \phi^*]_K) = 0$ for any point $\psi \in X^0$, and hence

$$\lim_{t\to\infty} dist(u(t, \psi), [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0.$$

Step 2. It remains to prove that $u(t, \phi^*)$ and $u(t, \phi^{**})$ are T-periodic solutions. We only need to show that ϕ^* and ϕ^{**} are fixed points of S. In what follows, we prove that ϕ^{**} is a fixed point for S.

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(iii) $\lim_{n\to\infty} dist(S^{n_0 n}(\psi), [\phi^{**}, \phi^*]_K) = 0$ for any point $\psi \in X^0$, and hence

$$\lim_{t\to\infty} dist(u(t, \psi), [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0.$$

Step 2. It remains to prove that $u(t, \phi^*)$ and $u(t, \phi^{**})$ are T-periodic solutions. We only need to show that ϕ^* and ϕ^{**} are fixed points of S. In what follows, we prove that ϕ^{**} is a fixed point for S.

By Proposition 2.2.1, we have $P_1^{(2)}e_1 = r_{21}e_1$, and $e_1 \gg 0$. Let S^{**} be the Poincaré map of the linearized system (2.3.19)-(2.3.20). We claim that r_{21} is an eigenvalue of S^{**}. Indeed, for any $\varphi \in X_2^+$, suppose that $u(t, \sigma, \varphi)$ solves

$$\dot{u}_2(t) = b_2^{(2)} u_2(t - \tau_2) - a_{22}^{(2)} u_2(t) \tag{2.3.23}$$

with initial values $u_\sigma = \varphi$. Let $W(t, \sigma)\varphi = u_t(\sigma, \varphi)$, then $W(t, \sigma)$ is a continuous linear evolution operator. Let $u_1(t, e_1)$ be the solution of equation (2.3.19) satisfying $u_1(\theta, e_1) = e_1(\theta), \forall \theta \in [-\tau_1, 0]$. By the variation-of-constants formula, the solutions of equation (2.3.20) can be expressed by

$$u_t(\sigma, \varphi) = W(t, \sigma)\varphi + \int_\sigma^t W(t, s)X_0 h(s)ds, \quad t \geq \sigma,$$

where $X_0(\theta) = 0$ for $\theta \in [-\tau_2, 0)$, $X_0(\theta) = 1$ for $\theta = 0$, and $h(s) = -a_{21}^{(2)}(s)u_1(s, e_1) < 0$. Consider the following equation

$$(r_{21} - W(T, 0))e_2 = -\int_0^T W(T, s)X_0 h(s)ds, \ e_2 \in X_2^+. \tag{2.3.24}$$

Since $u^{(2)}(t)$ is a globally asymptotically stable T-periodic solution of equation (2.3.16), and its linearized equation at $u^{(2)}(t)$ coincides with equation (2.3.23), we have $r(W(T, 0)) \leq 1$. Since $W(T, s)X_0 > 0$, $-\int_0^T W(T, s)X_0 h(s)ds > 0$. By the Krein-Rutman theorem (see, e.g., [47, Theorem 7.3]), equation (2.3.24) has a unique solution e_2 and $e_2 \gg 0$. Let $e = (e_1, -e_2)$, then $e \gg_K 0$. Let P_2 be the Poincaré map of equation (2.3.20). Then,

$$P_2(-e_2) = W(T, 0)(-e_2) + \int_0^T W(T, s)X_0 h(s)ds.$$

Thus,

$$S^{**}e = (P_1^{(2)}(e_1), P_2(-e_2)) = r_{21}(e_1, -e_2) = r_{21}e,$$

By Proposition 2.2.1, we have $P_1^{(2)}e_1 = r_{21}e_1$, and $e_1 \gg 0$. Let S^{**} be the Poincaré map of the linearized system (2.3.19)-(2.3.20). We claim that r_{21} is an eigenvalue of S^{**}. Indeed, for any $\varphi \in X_2^+$, suppose that $u(t,\sigma,\varphi)$ solves

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with initial values $u_\sigma = \varphi$. Let $W(t,\sigma)\varphi = u_t(\sigma,\varphi)$, then $W(t,\sigma)$ is a continuous linear evolution operator. Let $u_1(t,e_1)$ be the solution of equation (2.3.19) satisfying $u_1(\theta,e_1) = e_1(\theta), \forall\theta \in [-\tau_1, 0]$. By the variation-of-constants formula, the solutions of equation (2.3.20) can be expressed by

$$u_t(\sigma,\varphi) = W(t,\sigma)\varphi + \int_\sigma^t W(t,s)X_0h(s)ds, \quad t \geq \sigma,$$

where $X_0(\theta) = 0$ for $\theta \in [-\tau_2, 0)$, $X_0(\theta) = 1$ for $\theta = 0$, and $h(s) = -a_{21}^{(2)}(s)u_1(s,e_1) < 0$. Consider the following equation

$$(r_{21} - W(T,0))e_2 = -\int_0^T W(T,s)X_0h(s)ds, \ e_2 \in X_2^+. \tag{2.3.24}$$

Since $u^{(2)}(t)$ is a globally asymptotically stable T-periodic solution of equation (2.3.16), and its linearized equation at $u^{(2)}(t)$ coincides with equation (2.3.23), we have $r(W(T,0)) \leq 1$. Since $W(T,s)X_0 > 0$, $-\int_0^T W(T,s)X_0h(s)ds > 0$. By the Krein-Rutman theorem (see, e.g., [47, Theorem 7.3]), equation (2.3.24) has a unique solution e_2 and $e_2 \gg 0$. Let $e = (e_1, -e_2)$, then $e \gg_K 0$. Let P_2 be the Poincaré map of equation (2.3.20). Then,

$$P_2(-e_2) = W(T,0)(-e_2) + \int_0^T W(T,s)X_0h(s)ds.$$

Thus,

$$S^{**}e = (P_1^{(2)}(e_1), P_2(-e_2)) = r_{21}(e_1, -e_2) = r_{21}e,$$

and hence r_{21} is an eigenvalue of S^{**} with eigenfunction $e \gg_K 0$.

For any $\epsilon > 0$, note that $DS(E_2) = S^{**}$, we have

$$S(E_2 + \epsilon e) = S(E_2) + DS(E_2)(\epsilon e) + o(\epsilon) = E_2 + \epsilon(r_{21}e + o(\epsilon)/\epsilon).$$

Since $r_{21} > 1, (r_{21} - 1)e \in int(K)$, there exists $\epsilon_0 > 0$ such that $(r_{21} - 1)e + o(\epsilon)/\epsilon \in int(K)$ for any $\epsilon \in (0, \epsilon_0]$. Hence $S(E_2 + \epsilon e) - (E_2 + \epsilon e) \gg_K 0$; that is, $E_2 + \epsilon e \ll_K S(E_2 + \epsilon e)$. Since S is monotone with respect to $\leq_K$, we have an increasing sequence $E_2 + \epsilon e \ll_K S^n(E_2 + \epsilon e) \leq_K S^{n+1}(E_2 + \epsilon e)$ for all $n \geq 1$. Since $E_2 <_K \phi^{**}$ and $\phi^{**} \gg 0$, we can choose an ϵ such that $E_2 + \epsilon e \leq_K \phi^{**}$. Therefore, $\lim_{n\to\infty} S^{non}(E_2 + \epsilon e) = \phi^{**}$, and hence $\lim_{n\to\infty} S^n(E_2 + \epsilon e) = \phi^{**}$. By the continuity of S, it follows that ϕ^{**} is a fixed point of S. In the same way, it is easy to show that ϕ^* is a fixed point of S.

In the case of $\tau_i = k_iT$, if $\int_0^T (b_i(t) - a_i(t))dt > 0$, $\int_0^T (b_j^{(i)}(t) - b_{jj}^{(i)}(t))dt > 0, 1 \leq i \neq j \leq 2$, Proposition 2.2.1 implies the last statement in the theorem. ∎

Theorem 2.3.1 implies that two species coexist. The following result shows that one species drives the other one to extinction.

Theorem 2.3.2 *Let (H1) and (H2) hold. Assume that system (2.1.5) has no positive T-periodic solution. Then if (H3) holds and $r_{21} > 1$, or in the case where $\tau_i = k_iT$ for some integers k_i, if $\int_0^T (b_i(t) - a_i(t))dt > 0, \forall\, i = 1, 2$, and $\int_0^T (b_1^{(2)}(t) - a_{11}^{(2)}(t))dt > 0$, then for any $\psi \in X^0$, the solution $u(t, \psi)$ of system (2.1.5) satisfies*

$$\lim_{t\to\infty} \|u(t, \psi) - (u^{(1)}(t), 0)\| = 0.$$

A symmetric result holds for $(0, u^{(2)}(t))$.

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Proof. In the case of $r_{21} > 1$, by Lemma 2.3.2, for any $\psi \in X^0$, the omega limit set $\omega(\psi)$ of $S^n(\psi)$ can not be E_2 since $S^n(\psi) \gg 0$ for all $nT \geq 2\tau$ (see the claim in the proof of Theorem 2.3.1). Moreover, just as in the proof of Theorem 2.3.1, we can consider S^{n_0} such that S^{n_0} satisfies assumptions (A1)-(A4) in Section 1.2. Note that system (2.1.5) has no positive T-periodic solutions, and hence S has no positive fixed points. By Theorem 1.2.4, we have $S^{n \cdot n_0}(\psi) \to E_1$ $(n \to \infty)$. Therefore, $\lim_{t\to\infty} \|u(t,\psi) - u(t, E_1)\| = \lim_{t\to\infty} \|u(t,\psi) - (u^{(1)}(t), 0)\| = 0$. A symmetric result holds for $(0, u^2(t))$. ∎

In practice, it is not easy to verify the nonexistence of positive T-periodic solutions. In what follows, we establish some sufficient conditions for the conclusion of Theorem 2.3.2.

Assume that

(H4) $f_1(t, \cdot, u_2, \cdot)$ and $f_2(t, u_1, \cdot, \cdot)$ are strictly sublinear on $\mathbb{R}^2_+$, where $u_1, u_2 \geq 0$, and $f_1(t, L, 0, L) \leq 0$, $f_2(t, 0, L, L) \leq 0$ for some $L > 0$.

Then assumption (H1) implies that $f_1(t, L, u_2, L) \leq 0$, $f_2(t, u_1, L, L) \leq 0$ for all $u_1, u_2 \geq 0$. By Theorem 2.2.1, if $r_{21} > 1$, equation

$$\dot{u}_1(t) = f_1(t, u_1(t), u^{(2)}(t), u_1(t - \tau_1))$$

admits a unique positive T-periodic solution $u_1^{(2)}(t)$, which is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$, where $u^{(2)}(t)$ is the positive T-periodic solution of equation (2.3.16). Let $r_{1,2}^{(2)}$ be spectral radius defined by Theorem 2.2.1 associated with

$$\dot{u}_2(t) = f_2(t, u_1^{(2)}(t), u_2(t), u_2(t - \tau_2)).$$

Proof. In the case of $r_{21} > 1$, by Lemma 2.3.2, for any $\psi \in X^0$, the omega limit set $\omega(\psi)$ of $S^n(\psi)$ can not be E_2 since $S^n(\psi) \gg 0$ for all $nT \geq 2\tau$ (see the claim in the proof of Theorem 2.3.1). Moreover, just as in the proof of Theorem 2.3.1, we can consider S^{n_0} such that S^{n_0} satisfies assumptions (A1)-(A4) in Section 1.2. Note that system (2.1.5) has no positive T-periodic solutions, and hence S has no positive fixed points. By Theorem 1.2.4, we have $S^{n \cdot n_0}(\psi) \to E_1$ $(n \to \infty)$. Therefore, $\lim_{t\to\infty} \|u(t,\psi) - u(t, E_1)\| = \lim_{t\to\infty} \|u(t,\psi) - (u^{(1)}(t), 0)\| = 0$. A symmetric result holds for $(0, u^2(t))$. ∎

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$$\dot{u}_2(t) = f_2(t, u_1^{(2)}(t), u_2(t), u_2(t - \tau_2)).$$

Then we have the following result.

Corollary 2.3.1 *Let (H1), (H3) and (H4) hold. Then if $r_{21} > 1$ and $r_{1,2}^{(2)} < 1$, the conclusion of Theorem 2.3.2 holds.*

Proof. We use the same notation as in Theorem 2.3.2. Assumption (H3) implies $u^{(2)}(t)$ is globally asymptotically stable with respect to $X_2^+ \setminus \{0\}$ for equation (2.3.16). For any $\psi \in X^0$, let $u(t, \psi) = (u_1(t), u_2(t))$. Since assumption (H1) implies

$$\dot{u}_2(t) = f_2(t, u_1(t), u_2(t), u_2(t - \tau_2)) \leq f_2(t, 0, u_2(t), u_2(t - \tau_2)),$$

for any small $\varepsilon > 0$, we have $u_2(t) < u^{(2)}(t) + \varepsilon$ for all $t > t(\varepsilon)$. Therefore,

$$\dot{u}_1(t) = f_1(t, u_1(t), u_2(t), u_1(t - \tau_1)) > f_1(t, u_1(t), u^{(2)}(t) + \varepsilon, u_1(t - \tau_1)) \quad (2.3.25)$$

for $t > t(\varepsilon)$. Let r_{21}^{ε} be the spectral radius defined by Theorem 2.2.1 associated with

$$\dot{u}(t) = f_1(t, u(t), u^{(2)}(t) + \varepsilon, u(t - \tau_1)). \quad (2.3.26)$$

Then $\lim_{\varepsilon \to 0} r_{21}^{\varepsilon} = r_{21} > 1$, and hence $r_{21}^{\varepsilon} > 1$ for all sufficiently small ε. Therefore, by assumption (H4) and Theorem 2.2.1, there exists a unique positive T-periodic solution $u_1^{\varepsilon}(t)$ for equation (2.3.26), and $u_1^{\varepsilon}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$. By inequality (2.3.25), it follows that for any $\varepsilon' > 0$, we have $u_1(t) > u_1^{\varepsilon}(t) - \varepsilon'$ for $t > t(\varepsilon, \varepsilon')$. Therefore, assumption (H1) implies that

$$\dot{u}_2(t) = f_2(t, u_1(t), u_2(t), u_2(t - \tau_2)) < f_2(t, u_1^{\varepsilon}(t) - \varepsilon', u_2(t), u_2(t - \tau_2)) \quad (2.3.27)$$

for $t > t(\varepsilon, \varepsilon')$. Let $r^{\varepsilon'}$ be the spectral radius defined by Theorem 2.2.1 associated with

$$\dot{u}(t) = f_2(t, u_1^{\varepsilon}(t) - \varepsilon', u(t), u(t - \tau_2)). \quad (2.3.28)$$

Then we have the following result.

Corollary 2.3.1 *Let (H1), (H3) and (H4) hold. Then if $r_{21} > 1$ and $r_{1,2}^{(2)} < 1$, the conclusion of Theorem 2.3.2 holds.*

Proof. We use the same notation as in Theorem 2.3.2. Assumption (H3) implies $u^{(2)}(t)$ is globally asymptotically stable with respect to $X_2^+ \setminus \{0\}$ for equation (2.3.16). For any $\psi \in X^0$, let $u(t,\psi) = (u_1(t), u_2(t))$. Since assumption (H1) implies

$$\dot{u}_2(t) = f_2(t, u_1(t), u_2(t), u_2(t-\tau_2)) \leq f_2(t, 0, u_2(t), u_2(t-\tau_2)),$$

for any small $\varepsilon > 0$, we have $u_2(t) < u^{(2)}(t) + \varepsilon$ for all $t > t(\varepsilon)$. Therefore,

$$\dot{u}_1(t) = f_1(t, u_1(t), u_2(t), u_1(t-\tau_1)) > f_1(t, u_1(t), u^{(2)}(t) + \varepsilon, u_1(t-\tau_1)) \tag{2.3.25}$$

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Note that $\lim_{\varepsilon\to 0} u_1^{\varepsilon}(t) = u_1^{(2)}(t)$ uniformly for $t \in [0,T]$ (see, e.g., [95, Theorem 1.4.1] or [74, Theorem 2.1]), we have $\lim_{\varepsilon,\varepsilon'\to 0} r^{\varepsilon'} = r_{1,2}^{(2)} < 1$, and hence $r^{\varepsilon'} < 1$ for all sufficiently small ε and ε'. Therefore, by Theorem 2.2.1, zero solution is globally asymptotically stable for equation (2.3.28). Thus, by inequality (2.3.27), we have $\lim_{t\to\infty} u_2(t) = 0$. That is, system (2.1.5) has no positive T-periodic solutions. Therefore, Theorem 2.3.2 completes the proof. ∎

Remark 2.3.1 Theorem 2.3.1, as applied to system (2.1.3) with $n = 2$, implies that system (2.1.3) is permanent and has at least one positive T-periodic solution. In particular, if there is only one positive T-periodic solution, then it is globally attractive. Therefore, the conclusions of Theorem 2.3.1 are stronger than [61, Theorem 2.2] for system (2.1.3) with $n = 2$. Furthermore, since assumptions (H1)-(H3) are automatically satisfied for system (2.1.3), Theorem 2.3.1 holds if $r_{12} > 1, r_{21} > 1$, or if $\int_0^T (b_2^{(1)}(t) - a_{22}^{(1)}(t))dt > 0$ and $\int_0^T (b_1^{(2)}(t) - a_{11}^{(2)}(t))dt > 0$ in the case of $\tau_i = k_i T, i = 1, 2$.

Remark 2.3.2 For system (2.1.3) with $n = 2$, the conditions of [61, Theorem 2.2] are sufficient for $r_{12} > 1$ and $r_{21} > 1$ (see Lemma 2.3.3 below). Thus, Theorem 2.3.1 is a natural generalization of [61, Theorem 2.2].

Remark 2.3.3 Theorem 2.3.2 and Corollary 2.3.1 imply that one species persists at a positive periodic solution while the other one dies out. The conclusion of Corollary 2.3.1, as applied to system (2.1.3) with $n = 2$, is the same as [61, Corollaries 2.1 and 2.2]. However, by the comparison method in the proofs of Lemma 2.3.3 and Corollary 2.3.1, one can easily conclude that the conditions in [61, Corollaries 2.1 and 2.2] are sufficient for the conditions in Corollary 2.3.1.

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Remark 2.3.4 The spectral radii r_{0i} represent the suitability of the i-th species at the habitat with no competitors. If $r_{0i} > 1$, the i-species is persistent at the habitat with no competitions, and the population is stabilized at the positive periodic quantity $u^{(i)}(t)$, called the carrying capacity of the habitat for the i-th species. Otherwise, the i-species dies out. r_{12} (r_{21} respectively) represents the survival ability of species 2 (1) at the habitat carrying the 1-th (2-th) population capacity, and can also be regarded as a kind of suitability of the species 2 (1) at the habitat with the competitor's population capacity. By equations (2.3.18) and (2.3.19), r_{12} and r_{21} are decreasing as the corresponding population capacities increase. Thus, we can control the population of a species through changing these spectral radii. For example, if we hope to save a species, say species 1, we can enhance the favorite habitat characters of species 1, which leads to increase the population capacity of species 1, and destroy the favorite habitat characters of species 2, which makes the population capacity of species 2 decrease, such that the two species coexist (Theorem 2.3.1), or species 2 is even driven to extinction (Theorem 2.3.2). This consequence can be seen more easily from the case of $T = k_i\tau_i$. When our general model (2.1.5) takes some specific forms, say model (2.1.3), one can easily figure out that the immature population's death rate d_i and the maturation time τ_i have a significant effect on the persistence of species i, even make the species die out. The same biological explanation can be drawn from the next section.

Lemma 2.3.3 *If inequalities (2.1.4) hold, then* $r_{12} > 1$ *and* $r_{21} > 1$.

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Lemma 2.3.3 *If inequalities (2.1.4) hold, then* $r_{12} > 1$ *and* $r_{21} > 1$.

Proof. For system (2.1.3) with $n = 2$, the corresponding equations (2.3.15) and (2.3.18) reduce to

$$\dot{x}_1(t) = B_1(t)x_1(t-\tau_1) - a_{11}x_1^2(t), \tag{2.3.29}$$

$$\dot{x}_2(t) = B_2(t)x_2(t-\tau_2) - a_{21}(t)u^{(1)}(t)x_2(t), \tag{2.3.30}$$

respectively, where $u^{(1)}(t)$ is the positive T-periodic solution for equation (2.3.29). Note that $u^{(1)}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$, and that r_{12} is the spectral radius of the Poincaré map $P_2^{(1)}$ associated with equation (2.3.30). Choosing t^* such that $u^{(1)}(t^*) = \max_{t\in[0,T]} u^{(1)}(t)$, we then have

$$0 = \dot{u}^{(1)}(t^*) = B_1(t^*)u^{(1)}(t^* - \tau_1) - a_{11}(t^*)(u^{(1)}(t^*))^2.$$

Therefore,

$$a_{11}(t^*)(u^{(1)}(t^*))^2 = B_1(t^*)u^{(1)}(t^* - \tau_1) \leq B_1(t^*)u^{(1)}(t^*),$$

and hence $u^{(1)}(t^*) \leq \frac{B_1^m}{a_{11}^l}$, where by the upper indexes we mean the same as these in inequalities (2.1.4).

By inequalities (2.1.4), it is easy to see that for any $\varphi \in X_2^+$ with $\varphi \gg 0$, the solution $x(t,\varphi)$ of the following equation

$$\dot{x}(t) = B_2^l x(t-\tau_2) - a_{21}^m \frac{B_1^m}{a_{11}^l} x(t)$$

satisfies $\lim\limits_{t\to\infty} x(t,\varphi) = \infty$. By the proof of Proposition 2.2.1, it follows that equation (2.3.30) has a positive solution $u^0(t) = v_0(t)e^{\lambda_0 t}$ with $\lambda_0 = \frac{1}{T}\ln r_{12}$ and $v_0(t)$ being continuous and T-periodic. Let $\varphi_0(s) = u^0(s)$, $s \in [-\tau_2, 0]$, then $\varphi_0 \gg 0$. Note that

$$\dot{x}_2(t) = B_2(t)x_2(t-\tau_2) - a_{21}(t)u^{(1)}(t)x_2(t)$$

Proof. For system (2.1.3) with $n = 2$, the corresponding equations (2.3.15) and (2.3.18) reduce to

$$\dot{x}_1(t) = B_1(t)x_1(t - \tau_1) - a_{11}x_1^2(t), \tag{2.3.29}$$

$$\dot{x}_2(t) = B_2(t)x_2(t - \tau_2) - a_{21}(t)u^{(1)}(t)x_2(t), \tag{2.3.30}$$

respectively, where $u^{(1)}(t)$ is the positive T-periodic solution for equation (2.3.29). Note that $u^{(1)}(t)$ is globally asymptotically stable with respect to $X_1^+ \setminus \{0\}$, and that r_{12} is the spectral radius of the Poincaré map $P_2^{(1)}$ associated with equation (2.3.30). Choosing t^* such that $u^{(1)}(t^*) = \max_{t\in[0,T]} u^{(1)}(t)$, we then have

$$0 = \dot{u}^{(1)}(t^*) = B_1(t^*)u^{(1)}(t^* - \tau_1) - a_{11}(t^*)(u^{(1)}(t^*))^2.$$

Therefore,

$$a_{11}(t^*)(u^{(1)}(t^*))^2 = B_1(t^*)u^{(1)}(t^* - \tau_1) \leq B_1(t^*)u^{(1)}(t^*),$$

and hence $u^{(1)}(t^*) \leq \frac{B_1^m}{a_{11}^l}$, where by the upper indexes we mean the same as these in inequalities (2.1.4).

By inequalities (2.1.4), it is easy to see that for any $\varphi \in X_2^+$ with $\varphi \gg 0$, the solution $x(t, \varphi)$ of the following equation

$$\dot{x}(t) = B_2^l x(t - \tau_2) - a_{21}^m \frac{B_1^m}{a_{11}^l} x(t)$$

satisfies $\lim\limits_{t\to\infty} x(t, \varphi) = \infty$. By the proof of Proposition 2.2.1, it follows that equation (2.3.30) has a positive solution $u^0(t) = v_0(t)e^{\lambda_0 t}$ with $\lambda_0 = \frac{1}{T}\ln r_{12}$ and $v_0(t)$ being continuous and T-periodic. Let $\varphi_0(s) = u^0(s), s \in [-\tau_2, 0]$, then $\varphi_0 \gg 0$. Note that

$$\dot{x}_2(t) = B_2(t)x_2(t - \tau_2) - a_{21}(t)u^{(1)}(t)x_2(t)$$

$$\geq B_2^l x_2(t-\tau_2) - a_{21}^m \frac{B_1^m}{a_{11}^l} x_2(t).$$

By the comparison theorem, we have $u^0(t) \geq x(t, \varphi_0)$, and hence $\lim_{t\to\infty} u^0(t) = \infty$. This implies that $\lambda_0 > 0$ and hence $r_{12} > 1$. Similar arguments implies $r_{21} > 1$. ∎

2.3.2 The Asymptotically Periodic Case

In this subsection, we lift the main results in the periodic case to the global dynamics of the asymptotically periodic system (2.1.6).

Assume that system (2.1.6) satisfies

(H1′) $\tilde{F}_i(t, u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i \tilde{F}_i(t, u_i)) > 0$ and $\frac{\partial}{\partial u_j}\tilde{G}_i(t, u_1, u_2) \geq 0$ for $t \geq 0$, $u_i \geq 0$, $1 \leq i \neq j \leq 2$;

(H2′) if $v \geq L$ for some number $L > 0$, then $\tilde{f}_1(t, v, 0, v)$, $\tilde{f}_2(t, 0, v, v) \leq 0$.

It then easily follows that the solution $u_1(t, \varphi_1)$ is bounded by $\max\{L, \|\varphi_1\|\}$ for any $\varphi_1 \in X_1^+$, where $u_1(t, \varphi_1)$ solves equation $\dot{u}_1 = \tilde{f}_1(t, u_1, 0, u_1(t-\tau_1))$. The similar results hold for $\dot{u}_2 = \tilde{f}_2(t, 0, u_2, u_2(t-\tau_2))$. Now, simply following the proof of Lemma 2.3.1, we can conclude that the solutions for system (2.1.6) exist globally and are uniformly bounded. Let $\tilde{u}(t, s, \psi)$ be the solution of system (2.1.6) satisfying $\tilde{u}_s = \psi \in X^+$.

Theorem 2.3.3 *Let (H1′), (H2′), (A) and the conditions in Theorem 2.3.1 hold. Then* $\lim_{t\to\infty} dist(\tilde{u}(t, 0, \psi),\ [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0$ *for any point* $\psi \in X^0$, *where* $u(t, \phi^*)$ *and* $u(t, \phi^{**})$ *are positive* T*-periodic solutions for system (2.1.5) defined by Theorem 2.3.1. In particular, system (2.1.6) is uniformly persistent.*

$$\geq B_2^l x_2(t-\tau_2) - a_{21}^m \frac{B_1^m}{a_{11}^l} x_2(t).$$

By the comparison theorem, we have $u^0(t) \geq x(t, \varphi_0)$, and hence $\lim_{t\to\infty} u^0(t) = \infty$. This implies that $\lambda_0 > 0$ and hence $r_{12} > 1$. Similar arguments implies $r_{21} > 1$. ∎

2.3.2 The Asymptotically Periodic Case

In this subsection, we lift the main results in the periodic case to the global dynamics of the asymptotically periodic system (2.1.6).

Assume that system (2.1.6) satisfies

(H1′) $\tilde{F}_i(t, u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i \tilde{F}_i(t, u_i)) > 0$ and $\frac{\partial}{\partial u_j} \tilde{G}_i(t, u_1, u_2) \geq 0$ for $t \geq 0$, $u_i \geq 0$, $1 \leq i \neq j \leq 2$;

(H2′) if $v \geq L$ for some number $L > 0$, then $\tilde{f}_1(t, v, 0, v)$, $\tilde{f}_2(t, 0, v, v) \leq 0$.

It then easily follows that the solution $u_1(t, \varphi_1)$ is bounded by $\max\{L, \|\varphi_1\|\}$ for any $\varphi_1 \in X_1^+$, where $u_1(t, \varphi_1)$ solves equation $\dot{u}_1 = \tilde{f}_1(t, u_1, 0, u_1(t-\tau_1))$. The similar results hold for $\dot{u}_2 = \tilde{f}_2(t, 0, u_2, u_2(t-\tau_2))$. Now, simply following the proof of Lemma 2.3.1, we can conclude that the solutions for system (2.1.6) exist globally and are uniformly bounded. Let $\tilde{u}(t, s, \psi)$ be the solution of system (2.1.6) satisfying $\tilde{u}_s = \psi \in X^+$.

Theorem 2.3.3 *Let (H1′), (H2′), (A) and the conditions in Theorem 2.3.1 hold. Then* $\lim_{t\to\infty} dist(\tilde{u}(t, 0, \psi),\ [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0$ *for any point* $\psi \in X^0$, *where* $u(t, \phi^*)$ *and* $u(t, \phi^{**})$ *are positive T-periodic solutions for system (2.1.5) defined by Theorem 2.3.1. In particular, system (2.1.6) is uniformly persistent.*

Let $u_t(s,\psi)$ and $\tilde{u}_t(s,\psi)$ be the solution maps for system (2.1.5) and (2.1.6), respectively, and set $\tilde{\Phi}(t,s,\psi) = \tilde{u}_t(s,\psi)$, $\mathcal{T}_n(\psi) = \tilde{u}_{nT}(0,\psi)$, $\mathcal{T}(t)\psi = u_t(0,\psi)$, $S(\psi) = \mathcal{T}(T)\psi$. In order to prove Theorem 2.3.3, we need the following lemma.

Lemma 2.3.4 *Let the assumptions of Lemma 2.3.2 hold. If $r_j = r(P_j) > 1$, then $\widetilde{W}^s(\psi^*)\bigcap int(X^+) = \emptyset$, where $\widetilde{W}^s(\psi^*) = \{\psi \in X^+ : \lim_{n\to\infty} \mathcal{T}_n(\psi) = \psi^*\}$, and ψ^* is the initial value of $u^*(t)$.*

Proof. Since solutions of system (2.1.5) and (2.1.6) are uniformly bounded, by Proposition 2.2.2 and assumption (A), $\tilde{\Phi}(t,s,\psi)$ is asymptotic to the T-periodic semiflow $\mathcal{T}(t)$, and hence $\mathcal{T}_n$ is an asymptotic autonomous discrete dynamical process with the limiting autonomous discrete semiflow S.

Assume, by contradiction, that $\psi \in \widetilde{W}^s(\psi^*)\bigcap int(X^+) \neq \emptyset$. Then, $\lim\limits_{n\to\infty} \mathcal{T}_n(\psi) = \psi^*$. By the reduction theorem ([95, Theorem 3.2.1]), it follows that $\lim\limits_{t\to\infty} \|\tilde{u}(t,0,\psi) - u^*(t)\| = 0$. Let $\tilde{u}(t,0,\psi) = (\tilde{u}_1(t),\tilde{u}_2(t))$. We use the same notation as in Lemma 2.3.2. For any $\varepsilon' \in (0,b_1)$, let $r^{\varepsilon'}$ be the spectral radius of the Poincaré map associated with

$$\dot{u}(t) = (F_j(t,0) - 2\varepsilon')u(t-\tau_j) - (G_j(t,u_1^*(t),u_2^*(t)) + 2\varepsilon')u(t). \tag{2.3.31}$$

Then $\lim\limits_{\varepsilon'\to 0} r^{\varepsilon'} = r_j > 1$. As in the proof of Lemma 2.3.2, in the following, we fix a $\varepsilon' \in (0,b_1)$ such that $r^{\varepsilon'} > 1$. Then, by the analysis in the proof of Lemma 2.3.2, it follows that there exists $\delta_0 < 1$ such that

$$F_j(t,u_j) > F_j(t,0) - \varepsilon',\ \forall t \in [0,\infty), u_j \in [0,\delta_0),$$

and for any $u = (u_1,u_2), u' = (u_1',u_2') \in [0,b_2+1]^2$ with $\|u-u'\| < \delta_0$,

$$|G_j(t,u_1,u_2) - G_j(t,u_1',u_2')| < \varepsilon',\ \forall t \in [0,\infty).$$

Let $u_t(s,\psi)$ and $\tilde{u}_t(s,\psi)$ be the solution maps for system (2.1.5) and (2.1.6), respectively, and set $\tilde{\Phi}(t,s,\psi)=\tilde{u}_t(s,\psi)$, $\mathcal{T}_n(\psi)=\tilde{u}_{nT}(0,\psi), \mathcal{T}(t)\psi=u_t(0,\psi), S(\psi)=\mathcal{T}(T)\psi$. In order to prove Theorem 2.3.3, we need the following lemma.

Lemma 2.3.4 *Let the assumptions of Lemma 2.3.2 hold. If $r_j=r(P_j)>1$, then $\widetilde{W}^s(\psi^*)\bigcap int(X^+)=\emptyset$, where $\widetilde{W}^s(\psi^*)=\{\psi\in X^+ : \lim_{n\to\infty}\mathcal{T}_n(\psi)=\psi^*\}$, and ψ^* is the initial value of $u^*(t)$.*

Proof. Since solutions of system (2.1.5) and (2.1.6) are uniformly bounded, by Proposition 2.2.2 and assumption (A), $\tilde{\Phi}(t,s,\psi)$ is asymptotic to the T-periodic semiflow $\mathcal{T}(t)$, and hence $\mathcal{T}_n$ is an asymptotic autonomous discrete dynamical process with the limiting autonomous discrete semiflow S.

Assume, by contradiction, that $\psi\in\widetilde{W}^s(\psi^*)\bigcap int(X^+)\neq\emptyset$. Then, $\lim\limits_{n\to\infty}\mathcal{T}_n(\psi)=\psi^*$. By the reduction theorem ([95, Theorem 3.2.1]), it follows that $\lim\limits_{t\to\infty}\|\tilde{u}(t,0,\psi)-u^*(t)\|=0$. Let $\tilde{u}(t,0,\psi)=(\tilde{u}_1(t),\tilde{u}_2(t))$. We use the same notation as in Lemma 2.3.2. For any $\varepsilon'\in(0,b_1)$, let $r^{\varepsilon'}$ be the spectral radius of the Poincaré map associated with

$$\dot{u}(t)=(F_j(t,0)-2\varepsilon')u(t-\tau_j)-(G_j(t,u_1^*(t),u_2^*(t))+2\varepsilon')u(t). \tag{2.3.31}$$

Then $\lim\limits_{\varepsilon'\to0}r^{\varepsilon'}=r_j>1$. As in the proof of Lemma 2.3.2, in the following, we fix a $\varepsilon'\in(0,b_1)$ such that $r^{\varepsilon'}>1$. Then, by the analysis in the proof of Lemma 2.3.2, it follows that there exists $\delta_0<1$ such that

$$F_j(t,u_j)>F_j(t,0)-\varepsilon',\ \forall t\in[0,\infty),u_j\in[0,\delta_0),$$

and for any $u=(u_1,u_2),u'=(u_1',u_2')\in[0,b_2+1]^2$ with $\|u-u'\|<\delta_0$,

$$|G_j(t,u_1,u_2)-G_j(t,u_1',u_2')|<\varepsilon',\ \forall t\in[0,\infty).$$

From assumption (A), it follows that there exists an integer N_0 such that

$$\tilde{F}_j(t, u_j) \geq F_j(t, u_j) - \varepsilon' > F_j(t, 0) - 2\varepsilon', \ \forall t \geq N_0 T, \ u_j \in [0, \delta_0),$$

and for any $(u_1, u_2) \in [0, b_2 + 1]^2$,

$$|\tilde{G}_j(t, u_1, u_2) - G_j(t, u_1, u_2)| < \varepsilon', \ \forall t \geq N_0 T.$$

Thus, for any $u = (u_1, u_2), u' = (u_1', u_2') \in [0, b_2 + 1]^2$ with $\|u - u'\| < \delta_0$,

$$\begin{aligned} |\tilde{G}_j(t, u_1', u_2') - G_j(t, u_1, u_2)| &< |\tilde{G}_j(t, u_1', u_2') - G_j(t, u_1', u_2')| \\ &\quad + |G_j(t, u_1', u_2') - G_j(t, u_1, u_2)| < 2\varepsilon' \end{aligned}$$

for all $t \geq N_0 T$. Since $\lim_{t \to \infty} \|\tilde{u}(t, 0, \psi) - u^*(t)\| = 0$, there exists an integer $N \geq N_0$ such that $\|\tilde{u}(t) - u^*(t)\| < \delta_0$ for $t \geq NT$. Therefore,

$$\begin{aligned} \dot{\tilde{u}}_j(t) &= \tilde{u}_j(t - \tau_j)\tilde{F}_j(t, \tilde{u}_j(t - \tau_j)) - \tilde{u}_j(t)\tilde{G}_j(t, \tilde{u}_1(t), \tilde{u}_2(t)) \\ &> (F_j(t, 0) - 2\varepsilon')\tilde{u}_j(t - \tau_j) - (G_i(t, u_1^*(t), u_2^*(t)) + 2\varepsilon')\tilde{u}_j(t) \end{aligned}$$

for all $t \geq NT$. By the comparison theorem, we have

$$\tilde{u}_j(t) \geq u_j^{\varepsilon'}(t, NT, \psi_j') = u_j^{\varepsilon'}(t - NT, 0, \psi_j'), \ t \geq NT,$$

where $\psi_j'(\theta) = \tilde{u}_j(NT + \theta, 0, \psi), \forall \theta \in [-\tau_j, 0]$, and $u_j^{\varepsilon'}(t, NT, \psi_j')$ is the solution of equation (2.3.31) satisfying $u_j^{\varepsilon'}(t, NT, \psi_j') = \psi_j'(t), \forall t \in [NT - \tau_j, NT]$. Simply following the claim in the proof in Theorem 2.3.1, we have $\tilde{u}_j(t) > 0$ for $t \geq \tau_j$. Without loss of generality, we assume that $NT \geq 2\tau_j$. Then $\psi_j' \gg 0$. Now, by the same argument as in the proof of Lemma 2.3.2, it follows that $\lim_{t \to \infty} \tilde{u}_j(t) \geq \lim_{t \to \infty} u_j^{\varepsilon'}(t - NT, 0, \psi_j') = \infty$, which contradicts $\lim_{t \to \infty} \|\tilde{u}(t, 0, \psi) - u^*(t)\| = 0$. ∎

From assumption (A), it follows that there exists an integer N_0 such that

$$\tilde{F}_j(t,u_j) \geq F_j(t,u_j) - \varepsilon' > F_j(t,0) - 2\varepsilon',\ \forall t \geq N_0T,\ u_j \in [0,\delta_0),$$

and for any $(u_1,u_2) \in [0,b_2+1]^2$,

$$|\tilde{G}_j(t,u_1,u_2) - G_j(t,u_1,u_2)| < \varepsilon',\ \forall t \geq N_0T.$$

Thus, for any $u=(u_1,u_2), u'=(u_1',u_2') \in [0,b_2+1]^2$ with $\|u-u'\| < \delta_0$,

$$\begin{aligned}|\tilde{G}_j(t,u_1',u_2') - G_j(t,u_1,u_2)| &< |\tilde{G}_j(t,u_1',u_2') - G_j(t,u_1',u_2')| \\ &\quad +|G_j(t,u_1',u_2') - G_j(t,u_1,u_2)| < 2\varepsilon'\end{aligned}$$

for all $t \geq N_0T$. Since $\lim_{t\to\infty} \|\tilde{u}(t,0,\psi) - u^*(t)\| = 0$, there exists an integer $N \geq N_0$ such that $\|\tilde{u}(t) - u^*(t)\| < \delta_0$ for $t \geq NT$. Therefore,

$$\begin{aligned}\dot{\tilde{u}}_j(t) &= \tilde{u}_j(t-\tau_j)\tilde{F}_j(t,\tilde{u}_j(t-\tau_j)) - \tilde{u}_j(t)\tilde{G}_j(t,\tilde{u}_1(t),\tilde{u}_2(t)) \\ &> (F_j(t,0)-2\varepsilon')\tilde{u}_j(t-\tau_j) - (G_i(t,u_1^*(t),u_2^*(t)) + 2\varepsilon')\tilde{u}_j(t)\end{aligned}$$

for all $t \geq NT$. By the comparison theorem, we have

$$\tilde{u}_j(t) \geq u_j^{\varepsilon'}(t,NT,\psi_j') = u_j^{\varepsilon'}(t-NT,0,\psi_j'),\ t \geq NT,$$

where $\psi_j'(\theta) = \tilde{u}_j(NT+\theta,0,\psi), \forall\theta \in [-\tau_j,0]$, and $u_j^{\varepsilon'}(t,NT,\psi_j')$ is the solution of equation (2.3.31) satisfying $u_j^{\varepsilon'}(t,NT,\psi_j') = \psi_j'(t), \forall t \in [NT-\tau_j,NT]$. Simply following the claim in the proof in Theorem 2.3.1, we have $\tilde{u}_j(t) > 0$ for $t \geq \tau_j$. Without loss of generality, we assume that $NT \geq 2\tau_j$. Then $\psi_j' \gg 0$. Now, by the same argument as in the proof of Lemma 2.3.2, it follows that $\lim_{t\to\infty} \tilde{u}_j(t) \geq \lim_{t\to\infty} u_j^{\varepsilon'}(t-NT,0,\psi_j') = \infty$, which contradicts $\lim_{t\to\infty} \|\tilde{u}(t,0,\psi) - u^*(t)\| = 0$. ∎

Proof of Theorem 2.3.3. From the proof of Lemma 2.3.4, we know that $\mathcal{T}_n, n \geq 0$, is an asymptotically autonomous discrete dynamical process with the limiting discrete semiflow S^n. Note that solutions of system (2.1.6) are uniformly bounded. By [49, Lemma 2.2], it follows that for any $\psi \in X^+$, the omega limit set $\omega(\psi)$ of ψ under $\mathcal{T}_n$ is a compact and internally chain transitive set for S.

Note that S is α-condensing, and S^n is compact for $nT \geq 2\tau$ (see, e.g., the proof of Theorem 2.3.1). Let $\partial X^0 = X^+ \setminus X^0$. Then, by Theorem 2.3.1, S is point dissipative and uniformly persistent with respect to $(X^0, \partial X^0)$. Thus, by Theorem 1.1.2, there exists a global attractor A_0 for S in X^0 which attracts strongly bounded sets in X^0. Then $A_0 \subset I_K$. By Lemma 2.3.2 and Theorem 2.3.1, it follows that E_0, E_1 , E_2 and A_0 are isolated invariant sets for S, and there is no S-cyclic chain among them. By [95, Lemma 1.2.8], $\lim_{n\to\infty} \mathcal{T}_n(\psi) = E_0,\ E_1,\ E_2$ or $\lim_{n\to\infty} dist(\mathcal{T}_n(\psi), A_0) = 0$.

For any $\psi \in X^0$, simply following the claim in the proof of Theorem 2.3.1, we have $\tilde{u}(t, 0, \psi) > 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$. Therefore, $\tilde{u}_{n_0T}(0, \psi) \in int(X^+)$ for $n_0T \geq 2\tau$. By Lemma 2.3.4,

$$\widetilde{W}^s(E_0) \cap int(X^+) = \widetilde{W}^s(E_1) \cap int(X^+) = \widetilde{W}^s(E_2) \cap int(X^+) = \emptyset.$$

Thus

$$\lim_{n\to\infty} \mathcal{T}_n(\psi) = \lim_{n\to\infty} \mathcal{T}_n(\tilde{u}_{n_0T}(0, \psi)) \neq E_0, E_1 \text{ or } E_2.$$

Therefore, $\lim_{n\to\infty} dist(\mathcal{T}_n(\psi), A_0) = 0$. Note that $A_0 \subset I_K$, by the reduction theorem ([95, Theorem 3.2.1]), we have $\lim_{t\to\infty} dist(\tilde{u}(t, 0, \psi), [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0$. ∎

Theorem 2.3.4 *Let (H1′), (H2′), (A) and conditions in Theorem 2.3.2 hold. Then*

$$\lim_{t\to\infty} \|\tilde{u}(t, 0, \psi) - (u^{(1)}(t), 0)\| = 0$$

Proof of Theorem 2.3.3. From the proof of Lemma 2.3.4, we know that $\mathcal{T}_n, n \geq 0$, is an asymptotically autonomous discrete dynamical process with the limiting discrete semiflow S^n. Note that solutions of system (2.1.6) are uniformly bounded. By [49, Lemma 2.2], it follows that for any $\psi \in X^+$, the omega limit set $\omega(\psi)$ of ψ under $\mathcal{T}_n$ is a compact and internally chain transitive set for S.

Note that S is α-condensing, and S^n is compact for $nT \geq 2\tau$ (see, e.g., the proof of Theorem 2.3.1). Let $\partial X^0 = X^+ \setminus X^0$. Then, by Theorem 2.3.1, S is point dissipative and uniformly persistent with respect to $(X^0, \partial X^0)$. Thus, by Theorem 1.1.2, there exists a global attractor A_0 for S in X^0 which attracts strongly bounded sets in X^0. Then $A_0 \subset I_K$. By Lemma 2.3.2 and Theorem 2.3.1, it follows that E_0, E_1 , E_2 and A_0 are isolated invariant sets for S, and there is no S-cyclic chain among them. By [95, Lemma 1.2.8], $\lim_{n\to\infty} \mathcal{T}_n(\psi) = E_0$, E_1, E_2 or $\lim_{n\to\infty} dist(\mathcal{T}_n(\psi), A_0) = 0$.

For any $\psi \in X^0$, simply following the claim in the proof of Theorem 2.3.1, we have $\tilde{u}(t, 0, \psi) > 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$. Therefore, $\tilde{u}_{n_0T}(0, \psi) \in int(X^+)$ for $n_0T \geq 2\tau$. By Lemma 2.3.4,

$$\widetilde{W}^s(E_0) \cap int(X^+) = \widetilde{W}^s(E_1) \cap int(X^+) = \widetilde{W}^s(E_2) \cap int(X^+) = \emptyset.$$

Thus

$$\lim_{n\to\infty} \mathcal{T}_n(\psi) = \lim_{n\to\infty} \mathcal{T}_n(\tilde{u}_{n_0T}(0, \psi)) \neq E_0, E_1 \text{ or } E_2.$$

Therefore, $\lim_{n\to\infty} dist(\mathcal{T}_n(\psi), A_0) = 0$. Note that $A_0 \subset I_K$, by the reduction theorem ([95, Theorem 3.2.1]), we have $\lim_{t\to\infty} dist(\tilde{u}(t, 0, \psi), [u(t, \phi^{**}), u(t, \phi^*)]_K) = 0$. ∎

Theorem 2.3.4 *Let (H1'), (H2'), (A) and conditions in Theorem 2.3.2 hold. Then*

$$\lim_{t\to\infty} \|\tilde{u}(t, 0, \psi) - (u^{(1)}(t), 0)\| = 0$$

for any $\psi \in X^0$. *A symmetric result holds for* $(0, u^{(2)}(t))$.

Proof. By the proof of Theorem 2.3.3, for any $\psi \in X^+$, the omega limit set $\omega(\psi)$ of ψ under $\mathcal{T}_n$ is a compact and internally chain transitive set for S. By Lemma 2.3.2 and Theorem 2.3.2, it follows that E_0, E_1 and E_2 are isolated invariant sets for S, and there is no S-cyclic chain among them. Thus, for any $\psi \in X^0$, by the convergence theorem (see [49, Theorem 3.2] or [95, Theorem 1.2.2]), we have $\omega(\psi) = E_0, E_1$, or E_2. Using the argument similar to the claim in the proof of Theorem 2.3.1, we have $\tilde{u}(t, 0, \psi) > 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$, i.e., $\tilde{u}_{nT}(0, \psi) \in int(X^+)$ for $nT \geq 2\tau$. Note that Lemma 2.3.4 implies that $\omega(E_0) \cap X^0 = \omega(E_2) \cap X^0 = \emptyset$. Thus, we have $\omega(\psi) = E_1$, i.e., $\lim_{n\to\infty} \mathcal{T}_n(\psi) = E_1$. By the reduction theorem [95, Theorem 3.2.1], we have $\lim_{t\to\infty} \|\tilde{\Phi}(t, 0, \psi) - \mathcal{T}(t)(E_1)\| = 0$, i.e., $\lim_{t\to\infty} \|\tilde{u}(t, 0, \psi) - (u^{(1)}(t), 0)\| = 0$. A symmetric result holds for $(0, u^{(2)}(t))$. ∎

2.4 Multi-species Competition

As we have seen in Section 2.3, the monotonicity of the Poincaré map associated with the periodic system (2.1.5) with $m = 2$ plays an important role in obtaining the global dynamics. However, for system (2.1.5) with $m \geq 3$, we are not able to appeal to the powerful theory of monotone dynamical systems. In this section, we use the elementary comparison method to establish a set of conditions for uniform persistence in the asymptotically periodic competitive system (2.1.6) with $m \geq 3$. In virtue of the persistence theory, we further obtain natural invasibility conditions for uniform persistence and the existence of positive periodic solutions in 3-species *for any* $\psi \in X^0$. *A symmetric result holds for* $(0, u^{(2)}(t))$.

Proof. By the proof of Theorem 2.3.3, for any $\psi \in X^+$, the omega limit set $\omega(\psi)$ of ψ under $\mathcal{T}_n$ is a compact and internally chain transitive set for S. By Lemma 2.3.2 and Theorem 2.3.2, it follows that E_0, E_1 and E_2 are isolated invariant sets for S, and there is no S-cyclic chain among them. Thus, for any $\psi \in X^0$, by the convergence theorem (see [49, Theorem 3.2] or [95, Theorem 1.2.2]), we have $\omega(\psi) = E_0, E_1$, or E_2. Using the argument similar to the claim in the proof of Theorem 2.3.1, we have $\tilde{u}(t, 0, \psi) > 0$ for $t \geq \tau = \max(\tau_1, \tau_2)$, i.e., $\tilde{u}_{nT}(0, \psi) \in int(X^+)$ for $nT \geq 2\tau$. Note that Lemma 2.3.4 implies that $\omega(E_0) \cap X^0 = \omega(E_2) \cap X^0 = \emptyset$. Thus, we have $\omega(\psi) = E_1$, i.e., $\lim_{n\to\infty} \mathcal{T}_n(\psi) = E_1$. By the reduction theorem [95, Theorem 3.2.1], we have $\lim_{t\to\infty} \|\tilde{\Phi}(t, 0, \psi) - \mathcal{T}(t)(E_1)\| = 0$, i.e., $\lim_{t\to\infty} \|\tilde{u}(t, 0, \psi) - (u^{(1)}(t), 0)\| = 0$. A symmetric result holds for $(0, u^{(2)}(t))$. ∎

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We first consider m-species competitive system (2.1.5) and (2.1.6). Assume that for $u_i \geq 0, 1 \leq i \neq j \leq m$, we have

(K1) $F_i(t, u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i F_i(t, u_i)) > 0$, $\frac{\partial}{\partial u_j} G_i(t, u_1, \ldots, u_m) > 0$;

(K2) $f_i(t, u_1, \ldots, u_{i-1}, \cdot, u_{i+1}, \ldots, u_m, \cdot)$ is strictly sublinear on $\mathbb{R}^2_+$; and for some $L > 0$, $f_i(t, 0, \ldots, 0, L, 0, \ldots, 0, L) \leq 0$, where the two L are ith and $(m+1)$th components of f_i except for t;

(K3) $\tilde{F}_i(t, u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i \tilde{F}_i(t, u_i)) > 0$, $\frac{\partial}{\partial u_j} \tilde{G}_i(t, u_1, \ldots, u_m) \geq 0$;

(K4) there exists a positive number L such that $\tilde{f}_i(t, 0, \ldots, 0, l, 0, \ldots, 0, l) \leq 0$ for all $l \geq L$;

Then

$$f_i(t, u_1, \ldots, u_{i-1}, L, u_{i+1}, \ldots, u_m, L) \leq 0,$$
$$\tilde{f}_i(t, u_1, \ldots, u_{i-1}, l, u_{i+1}, \ldots, u_m, l) \leq 0,$$

for all $u_i \geq 0, l \geq L, i = 1, 2, \ldots, m$. As analyzed before, assumptions (K1)-(K4) imply that solutions of system (2.1.6) and (2.1.5) are uniformly bounded.

Let $\bar{r}_i$ be the spectral radius defined by Theorem 2.2.2 associated with

$$\dot{\tilde{u}}_i(t) = \tilde{f}_i(t, 0, \ldots, 0, \tilde{u}_i(t), 0, \ldots, 0, \tilde{u}_i(t - \tau_i)). \tag{2.4.32}$$

Assume that

(K5) $\bar{r}_i > 1$, $i = 1, 2, \ldots, m$.

competitive periodic system (2.1.5).

We first consider m-species competitive system (2.1.5) and (2.1.6). Assume that for $u_i \geq 0, 1 \leq i \neq j \leq m$, we have

(K1) $F_i(t,u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i F_i(t,u_i)) > 0$, $\frac{\partial}{\partial u_j} G_i(t,u_1,\ldots,u_m) > 0$;

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(K3) $\tilde{F}_i(t,u_i) > 0$, $\frac{\partial}{\partial u_i}(u_i \tilde{F}_i(t,u_i)) > 0$, $\frac{\partial}{\partial u_j} \tilde{G}_i(t,u_1,\ldots,u_m) \geq 0$;

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Assume that

(K5) $\bar{r}_i > 1$, $i = 1,2,\ldots,m$.

Then for each i, by Theorem 2.2.2, there exists a unique positive T-periodic solution $\bar{u}_i(t, \bar{\phi}_i)$ for

$$\dot{u}_i(t) = f_i(t, 0, \ldots, 0, u_i(t), 0, \ldots, 0, u_i(t - \tau_i)),$$

which attracts every solution of equation (2.4.32) except for zero. Let $\underline{r}_i$ be the spectral radius defined by Theorem 2.2.2 with respect to

$$\dot{\tilde{u}}_i(t) = \tilde{f}_i(t, \bar{u}_1(t, \bar{\phi}_1), \ldots, \bar{u}_{i-1}(t, \bar{\phi}_{i-1}), \tilde{u}_i(t), \bar{u}_{i+1}(t, \bar{\phi}_{i+1}), \ldots, \bar{u}_m(t, \bar{\phi}_m), \tilde{u}_i(t - \tau_i)). \tag{2.4.33}$$

If we assume that $\underline{r}_i > 1$, then there exists a unique positive T-periodic solution $\underline{u}_i(t, \underline{\phi}_i)$ for

$$\dot{u}(t) = f_i(t, \bar{u}_1(t, \bar{\phi}_1), \ldots, \bar{u}_{i-1}(t, \bar{\phi}_{i-1}), u_i(t), \bar{u}_{i+1}(t, \bar{\phi}_{i+1}), \ldots, \bar{u}_m(t, \bar{\phi}_m), u(t - \tau_i)),$$

which attracts all solutions of equation (2.4.33) except for zero.

Let $Z_m^+ = C(\prod_{i=1}^m [-\tau_i, 0], \mathbb{R}_+^m)$, $Z_m^0 = \{\psi = (\psi_i)_{i=1}^m \in Z_m^+ : \psi_i \neq 0, \forall 1 \leq i \leq m\}$. For any $\phi \in Z_m^+$, denote by $\tilde{u}(t, \psi) = (\tilde{u}_i(t))_{i=1}^m$ the solution of system (2.1.6) with $\tilde{u}_0(\psi) = \psi$. The following theorem implies that systems (2.1.5) and (2.1.6) are persistent. We omit the extinction results, which can be obtained by the same arguments.

Theorem 2.4.1 *Let assumption (A) and (K1)-(K5) hold. Suppose that* $\underline{r}_i > 1$, $i = 1, 2, \ldots, m$. *Then for any* $\psi \in Z_m^0$, *the solution* $\tilde{u}(t, \psi)$ *of system (2.1.6) satisfies*

$$\lim_{t \to \infty} dist(\tilde{u}(t, \psi), [\underline{u}(t), \bar{u}(t)]) = 0,$$

where $[\underline{u}(t), \bar{u}(t)] = \{u = (u_i)_{i=1}^m \in \mathbb{R}_+^m : \underline{u}_i(t, \underline{\phi}_i) \leq u_i \leq \bar{u}_i(t, \bar{\phi}_i), \ \forall 1 \leq i \leq m\}$. *In particular, the same result holds for the solution* $u(t, \psi)$ *of system (2.1.5), and system (2.1.5) admits a positive* T*-periodic solution.*

Then for each i, by Theorem 2.2.2, there exists a unique positive T-periodic solution $\bar{u}_i(t,\bar{\phi}_i)$ for

$$\dot{u}_i(t) = f_i(t,0,\ldots,0,u_i(t),0,\ldots,0,u_i(t-\tau_i)),$$

which attracts every solution of equation (2.4.32) except for zero. Let $\underline{r}_i$ be the spectral radius defined by Theorem 2.2.2 with respect to

$$\dot{\tilde{u}}_i(t) = \tilde{f}_i(t,\bar{u}_1(t,\bar{\phi}_1),\ldots,\bar{u}_{i-1}(t,\bar{\phi}_{i-1}),\tilde{u}_i(t),\bar{u}_{i+1}(t,\bar{\phi}_{i+1}),\ldots,\bar{u}_m(t,\bar{\phi}_m),\tilde{u}_i(t-\tau_i)). \tag{2.4.33}$$

If we assume that $\underline{r}_i > 1$, then there exists a unique positive T-periodic solution $\underline{u}_i(t,\underline{\phi}_i)$ for

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Proof. By Theorem 2.2.2 and the standard two-side comparison method similar to that in the proof of Corollary 2.3.1, for any $\psi \in Z_m^0$ and any small $\varepsilon, \varepsilon' > 0$, we have $\underline{u}_i^\varepsilon(t) - \varepsilon' < \tilde{u}_i(t, \psi) < \bar{u}_i(t, \bar{\phi}_i) + \varepsilon$ for $t > t(\varepsilon, \varepsilon')$, where $\underline{u}_i^\varepsilon(t)$ is positive and T-periodic and satisfies that $\lim_{\varepsilon \to 0} \underline{u}_i^\varepsilon(t) = \underline{u}_i(t, \underline{\phi}_i)$ uniformly for $t \in [0, T]$. Let $\varepsilon, \varepsilon' \to 0$, we have

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In particular, the same result holds for the solutions of the limiting system (2.1.5).

Let S be the Poincaré map of system (2.1.5). Then S is bounded, point dissipative, α-condensing and uniformly persistent with respect to $(Z_m^0, \partial Z_m^0)$, where $\partial Z_m^0 = Z_m^+ \setminus Z_m^0$. Furthermore, S^n is compact for $nT \geq 2\tau = 2\max(\tau_1, \tau_2, \ldots, \tau_m)$ (see, e.g., [42, Theorem 3.6.1]). By Theorem 1.1.2, S has a coexistence state $\phi_0 \in Z_m^0$. Thus system (2.1.5) admits a positive T-periodic solution $u(t, \phi_0)$. ∎

As mentioned in [87], for the periodic system (2.1.5) in the case of $m = 2$, $(\bar{u}_1(t, \bar{\phi}_1), 0)$ and $(0, \bar{u}_2(t, \bar{\phi}_2))$ (i.e., $(u^{(1)}(t), 0)$ and $(0, u^{(2)}(t))$ defined in Section 2.3) are the semitrivial periodic solutions. Then $\underline{r}_1 > 1$ and $\underline{r}_2 > 1$ (i.e., $r_{12} > 1$, $r_{21} > 1$ in Theorem 2.3.1) are natural invasibility conditions for uniform persistence. However, for the m-species competition periodic system ($m \geq 3$), the periodic functions

$$(\bar{u}_1(t, \bar{\phi}_1), \ldots, \bar{u}_{i-1}(t, \bar{\phi}_{i-1}), 0, \bar{u}_i(t, \bar{\phi}_i), \ldots, \bar{u}_m(t, \bar{\phi}_m)), \quad 1 \leq i \leq m$$

are not solutions of system (2.1.5), and hence, due to the overestimation of the effect of competition, conditions $\underline{r}_i > 1$ in Theorem 2.4.1 are very strong conditions. In the rest of this section, we use the ideas in [87] to obtain some natural invasibility conditions for uniform persistence and existence of a positive coexistence state in the three-species competition.

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Consider the T-periodic model for the three-species competition

$$\begin{aligned} \dot{u}_i(t) &= u_i(t-\tau_i)F_i(t, u_i(t-\tau_i)) - u_i(t)G_i(t, u_1(t), u_2(t), u_3(t)) \\ &= f_i(t, u_1(t), u_2(t), u_3(t), u_i(t-\tau_i)),\ 1 \le i \le 3, \end{aligned} \tag{2.4.34}$$

which satisfies condition (K1), (K2) and (K5) in the case of $m = 3$. For each i, there is a corresponding 2-species competition system

$$\dot{u}_j(t) = f_j(t, u_1(t), u_2(t), u_3(t), u_j(t-\tau_j)),\ u_i(t) \equiv 0,\ j \neq i,\ 1 \le j \le 3. \tag{R_i}$$

Suppose that each system (R_i) satisfies the conditions either in Theorem 2.3.1 or in Theorem 2.3.2. We consider the following three cases:

(Q1) each (R_i) satisfies Theorem 2.3.1 and admits only one positive T-periodic solution $\hat{u}^{(i)}(t)$;

(Q2) both (R_2) and (R_3) satisfy Theorem 2.3.1, and each of them admits only one positive T-periodic solution. (R_1) satisfies Theorem 2.3.2;

(Q3) (R_3) satisfies Theorem 2.3.1 and admits only one positive T-periodic solution. (R_1) and (R_2) satisfy Theorem 2.3.2.

Let $Z_3^+ = C(\prod_{i=1}^3[-\tau_i, 0], \mathbb{R}_+^3)$, $Z_3^0 = \{(\phi_i)_{i=1}^3 \in Z_3^+ : \phi_i \neq 0, \forall\, 1 \le i \le 3\}$. For any $\phi \in Z_3^+$, denote the solution of system (2.4.34) by $u(t,\phi) = (u_i(t,\phi))_{i=1}^3$, and the solution semiflow by $u_t(\phi)$. We then have

Theorem 2.4.2 *Let (Q1) hold. Denote by* $\underline{r}^{(1)}$ *the spectral radius defined by Theorem 2.2.1 associated with* $\dot{u}(t) = f_1(t, u(t), \hat{u}^{(1)}(t), u(t-\tau_1))$. *In the same way, we can define* $\underline{r}^{(i)}, i = 2, 3$. *Suppose that* $\underline{r}^{(i)} > 1, i = 1, 2, 3$. *Then system (2.4.34) admits*

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which satisfies condition (K1), (K2) and (K5) in the case of $m = 3$. For each i, there is a corresponding 2-species competition system

$$\dot{u}_j(t) = f_j(t,u_1(t),u_2(t),u_3(t),u_j(t-\tau_j)),\ u_i(t) \equiv 0,\ j \ne i,\ 1 \le j \le 3. \tag{R_i}$$

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Let $Z_3^+ = C(\prod_{i=1}^3[-\tau_i,0],\mathbb{R}_+^3)$, $Z_3^0 = \{(\phi_i)_{i=1}^3 \in Z_3^+ : \phi_i \ne 0, \forall 1 \le i \le 3\}$. For any $\phi \in Z_3^+$, denote the solution of system (2.4.34) by $u(t,\phi) = (u_i(t,\phi))_{i=1}^3$, and the solution semiflow by $u_t(\phi)$. We then have

Theorem 2.4.2 *Let (Q1) hold. Denote by $\underline{r}^{(1)}$ the spectral radius defined by Theorem 2.2.1 associated with $\dot{u}(t) = f_1(t,u(t),\hat{u}^{(1)}(t),u(t-\tau_1))$. In the same way, we can define $\underline{r}^{(i)}, i = 2,3$. Suppose that $\underline{r}^{(i)} > 1, i = 1,2,3$. Then system (2.4.34) admits* *a positive T-periodic solution and is permanent in the sense that there exist $\alpha > 0$ and $\beta > 0$ such that for any $\phi \in Z_3^0$, $\beta \leq \liminf_{t\to\infty} u_i(t,\phi) \leq \limsup_{t\to\infty} u_i(t,\phi) \leq \alpha$.*

Proof. For any $\phi \in Z_3^0$, by the argument similar to the claim in the proof of Theorem 2.3.1, $u_i(t,\phi) > 0$ for all $t \geq \tau = \max(\tau_1, \tau_2, \tau_3)$. For any $\phi \in Z_3^+$, let $\mathcal{T}(t)(\phi) = u_t(\phi), S(\phi) = u_T(\phi)$. Then $\mathcal{T}(t)\phi, S^n(\phi) \in int(Z_3^+)$ for $\phi \in Z_3^0$ and $t, nT \geq 2\tau$. By the same argument as in the proof of Corollary 2.3.1 or Theorem 2.4.1, we have $u_i(t,\phi) < \bar{u}_i(t,\bar{\phi}_i) + \varepsilon, \forall t > t(\varepsilon)$. Thus, it is easy to find a number α such that $\limsup_{t\to\infty} u_i(t,\phi) \leq \alpha$ for all i and $\phi \in Z_3^0$. In particular, S is point dissipative and a bounded map (by the same argument of Lemma 2.3.1).

Note that S is α-condensing and orbits of bounded sets are bounded. By [41, Theorem 2.4.7], S admits a connected global attractor $A \subset Z_3^+$. Let $M_1 = (0,0,0), M_2 = (\bar{\phi}_1, 0, 0), M_3 = (0, \bar{\phi}_2, 0), M_4 = (0,0,\bar{\phi}_3), M_5 = (0, \hat{\phi}_2^{(1)}, \hat{\phi}_3^{(1)}), M_6 = (\hat{\phi}_1^{(2)}, 0, \hat{\phi}_3^{(2)}), M_7 = (\hat{\phi}_1^{(3)}, \hat{\phi}_2^{(3)}, 0)$, where $(\hat{\phi}_2^{(1)}, \hat{\phi}_3^{(1)}), (\hat{\phi}_1^{(2)}, \hat{\phi}_3^{(2)})$, and $(\hat{\phi}_1^{(3)}, \hat{\phi}_2^{(3)})$ are initial functions of $\hat{u}^{(1)}(t), \hat{u}^{(2)}(t)$ and $\hat{u}^{(3)}(t)$, respectively. Clearly, all M_i are fixed points of S. For any $\phi \in \partial Z_3^0 = Z_3^+ \setminus Z_3^0$, let $\omega(\phi)$ be the ω-limit set of ϕ with respect to the discrete semiflow $\{S^n\}_{n=0}^{\infty}$. By assumption (Q1) and Theorem 2.3.1, $\bigcup_{\phi\in\partial Z_3^0} \omega(\phi) = \{M_1, M_2, M_3, M_4, M_5, M_6, M_7\}$, and no subset of the M_i's forms a cycle for S in ∂Z_3^0. By assumption (Q1) and (K1), simply following the proof of Lemma 2.3.2, we can obtain that M_i are isolated invariant sets in Z_3^+ for S, and $W^s(M_i) \bigcap int(Z_3^+) = \emptyset$, where $W^s(M_i)$ is the stable set of M_i for S. Then $W^s(M_i) \bigcap Z_3^0 = \emptyset$. By Theorem 1.1.1, it follows that S is uniformly persistent with respect to $(Z_3^0, \partial Z_3^0)$. Note that S^n is compact for $nT \geq 2\tau$ (see, e.g., [42, Theorem 3.6.1]), by Theorem 1.1.2, there exists a global attractor $A_0 \subset Z_3^0$ for S which *a positive T-periodic solution and is permanent in the sense that there exist $\alpha > 0$ and $\beta > 0$ such that for any $\phi \in Z_3^0$, $\beta \leq \liminf_{t\to\infty} u_i(t,\phi) \leq \limsup_{t\to\infty} u_i(t,\phi) \leq \alpha$.*

Proof. For any $\phi \in Z_3^0$, by the argument similar to the claim in the proof of Theorem 2.3.1, $u_i(t,\phi) > 0$ for all $t \geq \tau = \max(\tau_1,\tau_2,\tau_3)$. For any $\phi \in Z_3^+$, let $\mathcal{T}(t)(\phi) = u_t(\phi), S(\phi) = u_T(\phi)$. Then $\mathcal{T}(t)\phi, S^n(\phi) \in int(Z_3^+)$ for $\phi \in Z_3^0$ and $t, nT \geq 2\tau$. By the same argument as in the proof of Corollary 2.3.1 or Theorem 2.4.1, we have $u_i(t,\phi) < \bar{u}_i(t,\bar{\phi}_i) + \varepsilon, \forall t > t(\varepsilon)$. Thus, it is easy to find a number α such that $\limsup_{t\to\infty} u_i(t,\phi) \leq \alpha$ for all i and $\phi \in Z_3^0$. In particular, S is point dissipative and a bounded map (by the same argument of Lemma 2.3.1).

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Let $A_0^* = \bigcup_{0 \leq t \leq n_0 T} \mathcal{T}(t) A_0$, where $n_0 T \geq 2\tau$. Then by the argument given in the claim in the proof of Theorem 2.3.1, $A_0^* \in int(Z_3^+)$, and by [94, Theorem 2.1], it follows that A_0^* is a compact set and attracts strongly bounded sets in Z_3^0. Since $\mathcal{T}(t)\phi \in int(Z_3^+)$ for $t \geq 2\tau$ and $\phi \in Z_3^0$, A_0^* attracts every point in Z_3^0 under $\mathcal{T}(t)$. For every $\phi \in A_0^*$, there exists a number $\beta_\phi > 0$ such that $\phi \gg \beta_\phi I_d$, where $I_d = (1, 1, 1)$. By the compactness of A_0^*, it follows that there exists $\beta(V)$ such that $\phi \gg \beta(V) I_d, \forall \phi \in V$, where V is a neighborhood of A_0^* in $int(Z_3^+)$. Thus for any $\phi \in Z_3^0$, $\mathcal{T}(t)\phi \gg \beta(V) I_d$ for all sufficiently large t. Therefore, $\liminf_{t \to \infty} u_i(t, \phi) \geq \beta(V)$. ∎

Theorem 2.4.3 *Let (Q2) hold, and r_{32} be spectral radius defined by Theorem 2.3.2 for (R_1). Suppose that $r_{32} > 1, \underline{r}^{(i)} > 1, i = 2, 3$. Then the conclusions of Theorem 2.4.2 hold.*

Proof. We use the same notation as in the proof of Theorem 2.4.2. By Theorem 2.3.2, it follows that $\lim_{n \to \infty} S^n(\phi) = (0, \bar{\phi}_2, 0) = M_3$ for any $\phi = (\phi_i)_{i=1}^3 \in \partial Z_3^0$ with $\phi_1 = 0$ and $\phi_2 \neq 0$. By assumption (Q2), Theorem 2.3.1 and 2.3.2, $\bigcup_{\phi \in \partial Z_3^0} = \{M_1, M_2, M_3, M_4, M_6, M_7\}$, and no subset of the M_i's forms a cycle for S^{n_0} in ∂Z_3^0. Thus as in the proof of Theorem 2.4.2, S is uniformly persistent with respect to $(Z_3^0, \partial Z_3^0)$. Now, the same argument given in Theorem 2.4.2 completes the proof. ∎

Theorem 2.4.4 *Let (Q3) hold and r_{31} be spectral radius defined by Theorem 2.3.2 for (R_2). Suppose that $r_{31} > 1$, $r_{32} > 1$ and $\underline{r}^{(3)} > 1$. Then the conclusions of Theorem 2.4.2 hold.*

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Remark 2.4.1 As in Theorem 2.4.1, the permanence for system (2.4.34) in Theorem 2.4.2-2.4.4 can be lifted to asymptotically periodic systems. According to Proposition 2.2.1, conditions for the spectral radii in all of the theorems of this section can be expressed in terms of certain average integrals in the case where $\tau_i = k_i T$.

Chapter 3

A Nonlocal and Delayed Reaction-Diffusion Model

In Chapter 2, we discussed a general model for multi-species competition, which does not include diffusion terms. In reality, most populations always move around. Thus, when we consider species which disperse in a domain, population models should include some kind of diffusion effects. This chapter will investigate a single species model represented by a nonlocal reaction-diffusion equation. For the model, we establish a threshold dynamics and global attractivity of positive steady state in terms of principal eigenvalues, and discuss effects of spatial dispersal and maturation period on the evolutionary behavior in two specific cases. Also, some numerical simulations are provided to illustrate the uniqueness of positive steady states.

The rest of this chapter is arranged as follows. Section 3.1 presents the model, and some related works. In Section 3.2, we establish the global existence and positivity of solutions, and the existence of a global attractor for the associated solution semiflow. In Section 3.3, we first obtain a threshold type result on the global extinction and uniform persistence in terms of the principal eigenvalue of a nonlocal

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3.1 The Model

Recently, an increasing attention has been paid to nonlocal and time-delayed population models in order to study the effects of spatial diffusion and time delay on the evolutionary behavior of biological systems (see, e.g., [82, 77, 40, 5, 89, 83]). In reality, species may drift from one spatial point at a time to another spatial point at another time, and may disperse from a domain to a larger domain. Moreover, the environment is often spatially heterogeneous. To describe the growth of a single species in a multi-patch environment, certain delay differential equation models were proposed and analyzed in [73, 62, 45, 78, 91]. [77, 5, 40] also formulated nonlocal and delayed reaction-diffusion models for a single species with stage structure, and established the existence of a family of traveling wave fronts for these models.

In order to obtain a general nonlocal and time delayed model for a single species in a bounded domain, we let $u(t, a, x)$ be the density of individuals with age a at a spatial point x and a time t, τ be the length of the juvenile period. Denote by elliptic problem, and then obtain sufficient conditions for the global attractivity of the positive steady state. Section 3.4 is devoted to the discussion of the effects of spatial diffusion and time delay on the asymptotic behavior of the model in two specific cases. Some numerical results are also included in the last section. Our simulations seem to suggest that the steady state is globally attractive even without our monotonicity condition.

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In order to obtain a general nonlocal and time delayed model for a single species in a bounded domain, we let $u(t, a, x)$ be the density of individuals with age a at a spatial point x and a time t, τ be the length of the juvenile period. Denote by

$u_m(t,x)$ the density of mature adults. Then we have (see, e.g., [66])

$$\begin{cases} \partial_t u + \partial_a u = d_j(a)\triangle u - \mu_j(a)u, & 0 < a < \tau,\ x \in \Omega \subset \mathbb{R}^N, \\ Bu = 0, & a \in (0,\tau),\ x \in \partial\Omega, \end{cases} \tag{3.1.1}$$

and u_m satisfies

$$\begin{cases} \partial_t u_m = d_m \triangle u_m - g(u_m) + u(t,\tau,x), & t > 0,\ x \in \Omega, \\ Bu = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{3.1.2}$$

with $u(t,0,x) = f(u_m(t,x)), t \geq -\tau, x \in \Omega$, where $f(u_m)$ is the birth rate, $g(u_m)$ is the mortality rate of mature individuals, $\mu_j(a)$ denotes the per capita mortality rate of juveniles at age a, $\triangle$ is the Laplacian operator on $\mathbb{R}^N$, Ω is a bounded and open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, either $Bu = u$ or $Bu = \frac{\partial u}{\partial n} + \alpha u$ for some nonnegative function $\alpha \in C^{1+\theta}(\partial\Omega, \mathbb{R})$, $\theta > 0$, $\frac{\partial}{\partial n}$ denotes the differentiation in the direction of the outward normal n to $\partial\Omega$. In (3.1.2), the term $u(t,\tau,x)$ represents the rate of recruitment to adulthood, being those of maturation age τ. As in [83, Section 7.1] (see also [73, 82, 77, 5, 58]), integrating (3.1.1) along characteristics setting $\varphi(\gamma,a,x) = u(a+\gamma,a,x)$, we have

$$\begin{cases} \partial_a \varphi = d_j(a)\triangle\varphi - \mu_j(a)\varphi, & 0 < a < \tau,\ x \in \Omega, \\ B\varphi = 0, & a \in (0,\tau),\ x \in \partial\Omega, \\ \varphi(\gamma,0,x) = f(u_m(\gamma,x)). \end{cases}$$

Integrating this equation, we get

$$\varphi(\gamma,a,x) = \int_\Omega \Gamma(\eta(a),x,y)\mathcal{F}(a) f(u_m(\gamma,y))dy,$$

$u_m(t,x)$ the density of mature adults. Then we have (see, e.g., [66])

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Integrating this equation, we get

$$\varphi(\gamma,a,x) = \int_\Omega \Gamma(\eta(a),x,y)\mathcal{F}(a) f(u_m(\gamma,y))dy,$$

where Γ is the Green's function associated with the partial differential operator $\triangle$ and boundary condition $Bu = 0$, and

$$\eta(a) = \int_0^a d_j(s)ds, \quad \mathcal{F}(a) = e^{-\int_0^a \mu_j(s)ds}.$$

Therefore,

$$u(t, a, x) = \int_\Omega \Gamma(\eta(a), x, y)\mathcal{F}(a)f(u_m(t-a, y))dy.$$

Thus, $u_m(t, x)$ satisfies

$$\begin{cases} \partial_t u_m = d_m \triangle u_m - g(u_m) + \\ \qquad \int_\Omega \Gamma(\eta(\tau), x, y)\mathcal{F}(\tau)f(u_m(t-\tau, y))dy, \quad t > 0, \ x \in \Omega, \\ Bu_m = 0, \quad t > 0, \ x \in \partial\Omega, \\ u_m(t, x) = \phi(t, x), \quad t \in [-\tau, 0], \ x \in \Omega, \end{cases} \tag{3.1.3}$$

where $\phi(t, x)$ is a positive initial function to be specified later.

In the case where $\Omega = \mathbb{R}^N$, [83] studied traveling wave solutions, minimal wave speed and asymptotic speed of spread for model (3.1.3). In the case of $\Omega = \mathbb{R}$, $g(u) = \beta u$, system (3.1.3) reduces to the model derived in [77], where traveling wave fronts are investigated. In the case where $\Omega = \mathbb{R}$, $f(u) = \alpha u$ and $g(u) = \beta u^2$, system (3.1.3) reduces to the model discussed in [40], where the linear stabilities of two spatially homogeneous equilibrium solutions, and traveling wave fronts are considered. A global convergence theorem in the case of bounded intervals was also obtained in [40]. The threshold dynamics and global convergence were established in [89] for a special case of system (3.1.3). Here, the purpose is to study the global dynamics of model (3.1.3).

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3.2 Existence of Global Attractor

For convenience, we drop the subscript m in (3.1.3), and write it as

$$\begin{cases} \partial_t u(t,x) = d\triangle u(t,x) - g(u(t,x)) + \\ \qquad \int_\Omega \Gamma(\eta(\tau),x,y)\mathcal{F}(\tau)f(u(t-\tau,y))dy, \quad t>0,\ x\in\Omega, \\ Bu(t,x)=0, \quad t>0,\ x\in\partial\Omega, \\ u(t,x)=\phi(t,x)\ge 0, \quad t\in[-\tau,0],\ x\in\Omega\subset\mathbb{R}^N. \end{cases} \tag{3.2.4}$$

We assume that

(S1) $f\in C^1(\mathbb{R}^+,\mathbb{R}^+)$, $f(0)=0, f'(0)>0$, and f is sublinear;

(S2) $g\in C^1(\mathbb{R}^+,\mathbb{R}^+)$, $g(0)=0, g'(0)\ge 0$, and $-g$ is sublinear;

(S3) There exists a number $M\ge 0$ such that for all $L>M$, $\bar{f}(L)-g(L)<0$, where $\bar{f}(u)=\mathcal{F}(\tau)\max\limits_{v\in[0,u]} f(v)$.

Let $p\in(N,\infty)$ be fixed. For each $\beta\in(\frac{1}{2}+\frac{N}{2p},1)$, let $\mathbb{X}_\beta$ be the fractional power space of $L^p(\Omega)$ with respect to (-A,B) (see, e.g., [46]), where $A:=\triangle$. Then $\mathbb{X}_\beta$ is an ordered Banach space with respect to the positive cone $\mathbb{X}_\beta^+$ consisting of all nonnegative functions in $\mathbb{X}_\beta$, and $\mathbb{X}_\beta^+$ has nonempty interior $int(\mathbb{X}_\beta)$. Moreover, $\mathbb{X}_\beta\subset C^{1+\nu}(\bar{\Omega})$ with continuous inclusion for $\nu\in[0,2\beta-1-\frac{N}{p})$. Denote the norm on $\mathbb{X}_\beta$ by $\|\cdot\|_\beta$. Then there exists a constant $k_\beta>0$ such that $\|\phi\|_\infty=\max\limits_{x\in\bar{\Omega}}|\phi(x)|\le k_\beta\|\phi\|_\beta$, $\forall\phi\in\mathbb{X}_\beta$. It is well known that the differential operator A generates an analytic semigroup $T(t)$ on $L^p(\Omega)$. Moreover, the standard parabolic maximum principle (see, e.g., [72, Corollary 7.2.3]) implies that the semigroup $T(t):\mathbb{X}_\beta\to\mathbb{X}_\beta$ is strongly positive in the sense that $T(t)(\mathbb{X}_\beta^+\setminus\{0\})\subset int(\mathbb{X}_\beta^+), \forall t>0$.

3.2 Existence of Global Attractor

For convenience, we drop the subscript m in (3.1.3), and write it as

$$\begin{cases} \partial_t u(t,x) = d\Delta u(t,x) - g(u(t,x)) + \\ \qquad \int_\Omega \Gamma(\eta(\tau),x,y)\mathcal{F}(\tau)f(u(t-\tau,y))dy, \quad t>0,\ x\in\Omega, \\ Bu(t,x) = 0, \quad t>0,\ x\in\partial\Omega, \\ u(t,x) = \phi(t,x) \geq 0, \quad t\in[-\tau,0],\ x\in\Omega\subset\mathbb{R}^N. \end{cases} \tag{3.2.4}$$

We assume that

(S1) $f \in C^1(\mathbb{R}^+,\mathbb{R}^+)$, $f(0)=0, f'(0)>0$, and f is sublinear;

(S2) $g \in C^1(\mathbb{R}^+,\mathbb{R}^+)$, $g(0)=0, g'(0)\geq 0$, and $-g$ is sublinear;

(S3) There exists a number $M \geq 0$ such that for all $L > M$, $\bar{f}(L) - g(L) < 0$, where $\bar{f}(u) = \mathcal{F}(\tau)\max\limits_{v\in[0,u]} f(v)$.

Let $p \in (N,\infty)$ be fixed. For each $\beta \in (\frac{1}{2}+\frac{N}{2p},1)$, let $\mathbb{X}_\beta$ be the fractional power space of $L^p(\Omega)$ with respect to (-A,B) (see, e.g., [46]), where $A := \Delta$. Then $\mathbb{X}_\beta$ is an ordered Banach space with respect to the positive cone $\mathbb{X}_\beta^+$ consisting of all nonnegative functions in $\mathbb{X}_\beta$, and $\mathbb{X}_\beta^+$ has nonempty interior $int(\mathbb{X}_\beta)$. Moreover, $\mathbb{X}_\beta \subset C^{1+\nu}(\bar{\Omega})$ with continuous inclusion for $\nu \in [0, 2\beta - 1 - \frac{N}{p})$. Denote the norm on $\mathbb{X}_\beta$ by $\|\cdot\|_\beta$. Then there exists a constant $k_\beta > 0$ such that $\|\phi\|_\infty = \max\limits_{x\in\bar{\Omega}}|\phi(x)| \leq k_\beta\|\phi\|_\beta$, $\forall\phi\in\mathbb{X}_\beta$. It is well known that the differential operator A generates an analytic semigroup $T(t)$ on $L^p(\Omega)$. Moreover, the standard parabolic maximum principle (see, e.g., [72, Corollary 7.2.3]) implies that the semigroup $T(t): \mathbb{X}_\beta \to \mathbb{X}_\beta$ is strongly positive in the sense that $T(t)(\mathbb{X}_\beta^+\backslash\{0\}) \subset int(\mathbb{X}_\beta^+), \forall t > 0$.

Let $\mathbb{Y} := C([-\tau, 0], \mathbb{X}_\beta)$ and $\mathbb{Y}^+ := C([-\tau, 0], \mathbb{X}_\beta^+)$. For convenience, we will identify an element $\phi \in \mathbb{Y}$ as a function from $[-\tau, 0] \times \bar{\Omega}$ to $\mathbb{R}$ defined by $\phi(s, x) = \phi(s)(x)$, and for each $s \in [-\tau, 0]$, we regard $g(\phi(s))$ as a function on $\bar{\Omega}$ defined by $g(\phi(s))(\cdot) = g(\phi(s, \cdot))$. For any function $y(\cdot) : [-\tau, b) \to \mathbb{X}_\beta$, where $b > 0$, define $y_t \in \mathbb{Y}, t \in [0, b)$ by $y_t(s) = y(t + s), \forall s \in [-\tau, 0]$. Define $F : \mathbb{Y}^+ \to \mathbb{X}_\beta$ by $F(\phi) = -g(\phi(0)) + \mathcal{F}(\tau)T(\eta(\tau))f(\phi(-\tau, \cdot)), \forall \phi \in \mathbb{Y}^+$. Then we can rewrite (3.2.4) as an abstract functional differential equation

$$\begin{aligned} &\frac{du(t)}{dt} = d\,Au(t) + F(u_t), \quad t > 0, \\ &u_0 = \phi \in \mathbb{Y}^+. \end{aligned}$$

Therefore, we can write the above equation as an integral equation

$$u(t) = T(d\,t)\phi(0) + \int_0^t T(d\,(t-s))F(u_s)ds, \quad t \geq 0,$$

whose solutions are called mild solutions for system (3.2.4).

Since $T(t) : \mathbb{X}_\beta \to \mathbb{X}_\beta$ is strongly positive, we have

$$\lim_{h \to 0^+} dist(\phi(0) + hF(\phi), \mathbb{X}_\beta^+) = 0, \quad \forall \phi \in \mathbb{Y}^+.$$

By [64, Proposition 3 and Remark 2.4] (see also [88, Corollary 8.1.3]), for each $\phi \in \mathbb{Y}^+$, system (3.2.4) has a unique non-continuable mild solution $u(t, \phi)$ with $u_0 = \phi$, and $u(t, \phi) \in \mathbb{X}_\beta^+$ for all $t \in (0, \sigma_\phi)$. Moreover $u(t, \phi)$ is a classical solution of (3.2.4) for $t > \tau$ (see [88, Corollary 2.2.5]). We further have the following result.

Theorem 3.2.1 *Let (S1)-(S3) hold. Then for each* $\phi \in \mathbb{Y}^+$, *a unique solution* $u(t, \phi)$ *globally exists on* $[-\tau, \infty)$, *and the solution semiflow* $\Phi(t) = u_t(\cdot) : \mathbb{Y}^+ \to \mathbb{Y}^+, t \geq 0$, *admits a connected global attractor.*

Let $\mathbb{Y} := C([-\tau, 0], \mathbb{X}_\beta)$ and $\mathbb{Y}^+ := C([-\tau, 0], \mathbb{X}_\beta^+)$. For convenience, we will identify an element $\phi \in \mathbb{Y}$ as a function from $[-\tau, 0] \times \bar{\Omega}$ to $\mathbb{R}$ defined by $\phi(s, x) = \phi(s)(x)$, and for each $s \in [-\tau, 0]$, we regard $g(\phi(s))$ as a function on $\bar{\Omega}$ defined by $g(\phi(s))(\cdot) = g(\phi(s, \cdot))$. For any function $y(\cdot) : [-\tau, b) \to \mathbb{X}_\beta$, where $b > 0$, define $y_t \in \mathbb{Y}, t \in [0, b)$ by $y_t(s) = y(t + s), \forall s \in [-\tau, 0]$. Define $F : \mathbb{Y}^+ \to \mathbb{X}_\beta$ by $F(\phi) = -g(\phi(0)) + \mathcal{F}(\tau)T(\eta(\tau))f(\phi(-\tau, \cdot)), \forall \phi \in \mathbb{Y}^+$. Then we can rewrite (3.2.4) as an abstract functional differential equation

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Proof. For any $L \geq M$, let $\Sigma_L = \{\varphi \in \mathbb{X}_\beta^+ : \varphi(x) \leq L, x \in \bar{\Omega}\}$, $\mathbb{Z}_L = C([-\tau, 0], \Sigma_L)$. By [72, Corollary 7.2.4], we have $T(t)\mathbb{Z}_L \subset \mathbb{Z}_L, \forall t \geq 0$, and $\|T(\eta(\tau))\| \leq 1$. Since the function $l - hg(l)$ is increasing on $l \in [0, L]$ for all sufficiently small $h > 0$, we have

$$\begin{aligned} \phi(0) + hF(\phi) &= \phi(0) - hg(\phi(0)) + h\mathcal{F}(\tau)T(\eta(\tau))f(\phi(-\tau)) \\ &\leq L - hg(L) + h\bar{f}(L) \leq L, \quad \forall \phi \in \mathbb{Z}_L. \end{aligned}$$

It then follows that

$$\lim_{h \to 0^+} dist(\phi(0) + hF(\phi), \Sigma_L) = 0, \quad \forall \phi \in \mathbb{Z}_L.$$

By [72, Corollary 7.2.4] and [88, Corollary 8.1.3], $\mathbb{Z}_L$ is a positively invariant set for system (3.2.4). Thus for any $\phi \in \mathbb{Y}^+$, $u(t, \phi)$ globally exists on $[-\tau, \infty)$, and hence (3.2.4) defines a semiflow $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ by $(\Phi(t)\phi)(s, x) = u(t + s, x, \phi), \forall s \in [-\tau, 0], x \in \bar{\Omega}$. Moreover, $\Phi(t)$ is compact for all $t > \tau$ ([88, Theorem 2.2.6]).

Let us consider the delay differential equation

$$\begin{cases} \dot{v}(t) = -g(v(t)) + \bar{f}(v(t - \tau)), \\ v(s) = \varphi(s) \in C([-\tau, 0], \mathbb{R}^+), \quad \forall s \in [-\tau, 0]. \end{cases} \tag{3.2.5}$$

we claim that the function $\bar{f}$ is Lipschitz in any bounded subset of $\mathbb{R}^+$. In fact, by the definition of $\bar{f}$, we know that $\bar{f}$ is monotone. Without loss of generality, we assume that $0 \leq l_1 < l_2$, and $\bar{f}(l_1) < \bar{f}(l_2)$. Then $\bar{f}(l_2) = \mathcal{F}(\tau)f(l_3)$ for some $l_3 \in [l_1, l_2]$, and $0 < \bar{f}(l_2) - \bar{f}(l_1) \leq f(l_3) - f(l_1) \leq f'(\xi)(l_3 - l_1) \leq f'(0)(l_2 - l_1)$. Therefore, for any $\varphi \in C([-\tau, 0], \mathbb{R}^+)$, system (3.2.5) admits a unique solution $v(t, \varphi)$ with $v(s, \varphi) = \varphi(s), \forall s \in [-\tau, 0]$. It is easy to see that $v(t, \varphi)$ is bounded. Hence $v(t, \varphi)$

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$$\begin{aligned} \phi(0) + hF(\phi) &= \phi(0) - hg(\phi(0)) + h\mathcal{F}(\tau)T(\eta(\tau))f(\phi(-\tau)) \\ &\leq L - hg(L) + h\bar{f}(L) \leq L, \quad \forall \phi \in \mathbb{Z}_L. \end{aligned}$$

It then follows that

$$\lim_{h \to 0^+} dist(\phi(0) + hF(\phi), \Sigma_L) = 0, \quad \forall \phi \in \mathbb{Z}_L.$$

By [72, Corollary 7.2.4] and [88, Corollary 8.1.3], $\mathbb{Z}_L$ is a positively invariant set for system (3.2.4). Thus for any $\phi \in \mathbb{Y}^+$, $u(t, \phi)$ globally exists on $[-\tau, \infty)$, and hence (3.2.4) defines a semiflow $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ by $(\Phi(t)\phi)(s, x) = u(t + s, x, \phi), \forall s \in [-\tau, 0], x \in \bar{\Omega}$. Moreover, $\Phi(t)$ is compact for all $t > \tau$ ([88, Theorem 2.2.6]).

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exists globally on $[-\tau, \infty)$. Furthermore, we claim that $\limsup_{t\to\infty} v(t,\varphi) \leq M, \forall \varphi \in C([-\tau,0],\mathbb{R}^+)$. Indeed, for any $\varphi \in C([-\tau,0],\mathbb{R}^+)$, the omega limit set $\omega(\varphi)$ of the orbit $\gamma^+(\varphi)$ is nonempty, compact and invariant, where $\gamma^+(\varphi) = \{v_t(\varphi) : t \geq 0\}$. Let $G = \{\psi(s) : \psi \in \omega(\varphi), s \in [-\tau, 0]\}$. Then G is compact because of the compactness of $\omega(\varphi)$. Therefore, there exist $s_0 \in [-\tau,0]$ and $\psi \in \omega(\varphi)$ such that $\psi(s_0) = \sup G := L_G$. By the invariance of $\omega(\varphi)$, there exists $\psi_1 \in \omega(\varphi)$ such that $v_\tau(\psi_1) = \psi$, i.e., $v(\tau+s,\psi_1) = \psi(s), \forall s \in [-\tau,0]$. Without loss of generality, we can assume that $\psi(0) = L_G$. Thus,

$$\begin{aligned}
\dot{v}(\tau,\psi_1) &= -g(v(\tau,\psi_1)) + \bar{f}(v(0,\psi_1)) \\
&\leq -g(L_G) + \bar{f}(L_G).
\end{aligned}$$

If $L_G > M$, then $\dot{v}(\tau,\psi_1) < 0$, which implies that there exists some $s \in [-\tau, 0)$ such that $\psi(s) > \psi(0) = L_G$, a contradiction. Thus, $\limsup_{t\to\infty} v(t,\varphi) \leq M, \forall \varphi \in C([-\tau,0],\mathbb{R}^+)$.

For any given $\phi \in \mathbb{Y}^+$, let $\hat{L}(s) = \max\{\phi(\theta,x) : \theta \in [-\tau,0], x \in \bar{\Omega}\}, \forall s \in [-\tau,0]$. Then $\limsup_{t\to\infty} v(t,\hat{L}) \leq M$. Note that, for any $\zeta \in \mathbb{Y}^+$ with $\zeta(s,\cdot) \leq v(t+s,\hat{L})$, $\forall s \in [-\tau,0]$, we have

$$\begin{aligned}
v(t,\hat{L}) &- \zeta(0,x) + h(-g(v(t,\hat{L})) + \bar{f}(v(t-\tau,\hat{L})) \\
&- h(-g(\zeta(0,x)) + \textstyle\int_\Omega \Gamma(\eta(\tau),x,y)\mathcal{F}(\tau) f(\zeta(-\tau,y))dy \\
&\geq v(t,\hat{L}) - \zeta(0,x) - h(g(v(t,\hat{L})) - g(\zeta(0,x))) \\
&\geq 0 \quad \text{for} \quad 0 < h \ll 1,\ x \in \Omega.
\end{aligned}$$

By [64, Proposition 3] (see also [88, Theorem 8.1.10]), $u(t,x,\phi) \leq v(t,\hat{L}), \forall x \in \bar{\Omega}, t \geq -\tau$. Thus, $\limsup_{t\to\infty} u(t,x,\phi) \leq M, \forall x \in \bar{\Omega}$. That is, $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ is point exists globally on $[-\tau,\infty)$. Furthermore, we claim that $\limsup\limits_{t\to\infty} v(t,\varphi) \leq M, \forall \varphi \in C([-\tau,0],\mathbb{R}^+)$. Indeed, for any $\varphi \in C([-\tau,0],\mathbb{R}^+)$, the omega limit set $\omega(\varphi)$ of the orbit $\gamma^+(\varphi)$ is nonempty, compact and invariant, where $\gamma^+(\varphi) = \{v_t(\varphi) : t \geq 0\}$. Let $G = \{\psi(s) : \psi \in \omega(\varphi), s \in [-\tau,0]\}$. Then G is compact because of the compactness of $\omega(\varphi)$. Therefore, there exist $s_0 \in [-\tau,0]$ and $\psi \in \omega(\varphi)$ such that $\psi(s_0) = \sup G := L_G$. By the invariance of $\omega(\varphi)$, there exists $\psi_1 \in \omega(\varphi)$ such that $v_\tau(\psi_1) = \psi$, i.e., $v(\tau+s,\psi_1) = \psi(s), \forall s \in [-\tau,0]$. Without loss of generality, we can assume that $\psi(0) = L_G$. Thus,

$$\begin{aligned}
\dot{v}(\tau,\psi_1) &= -g(v(\tau,\psi_1)) + \bar{f}(v(0,\psi_1)) \\
&\leq -g(L_G) + \bar{f}(L_G).
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If $L_G > M$, then $\dot{v}(\tau,\psi_1) < 0$, which implies that there exists some $s \in [-\tau,0)$ such that $\psi(s) > \psi(0) = L_G$, a contradiction. Thus, $\limsup\limits_{t\to\infty} v(t,\varphi) \leq M, \forall \varphi \in C([-\tau,0],\mathbb{R}^+)$.

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$$\begin{aligned}
v(t,\hat{L}) &- \zeta(0,x) + h(-g(v(t,\hat{L})) + \bar{f}(v(t-\tau,\hat{L})) \\
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By [64, Proposition 3] (see also [88, Theorem 8.1.10]), $u(t,x,\phi) \leq v(t,\hat{L}), \forall x \in \bar{\Omega}, t \geq -\tau$. Thus, $\limsup\limits_{t\to\infty} u(t,x,\phi) \leq M, \forall x \in \bar{\Omega}$. That is, $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ is point dissipative. By [41, Theorem 3.4.8], $\Phi(t)$ admits a connected global attractor on $\mathbb{Y}^+$, which attracts each bounded set in $\mathbb{Y}^+$. ∎

3.3 Threshold Dynamics and Positive Steady State

In this section, we present our main results of this chapter in terms of principal eigenvalues. Let us first consider the following nonlocal problem:

$$\begin{cases} \partial_t u(t,x) = d\triangle u(t,x) - g'(0)u(t,x)) + \\ \qquad f'(0)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)u(t,y)dy, \quad x \in \Omega, \\ Bu(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \end{cases}$$

As noted in the previous section (taking delay τ as zero), the system generates a compact, positive solution semigroup on $\mathbb{X}_\beta^+$. By the same arguments as in [72, Theorem 7.6.1], the nonlocal eigenvalue problem

$$\begin{cases} \lambda v(x) = d\triangle v(x) - g'(0)v(x) + \\ \qquad f'(0)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)v(y)dy, \quad x \in \Omega, \\ Bv(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.3.6}$$

admits a principal eigenvalue, denoted by λ_0. Then we have the following threshold dynamics for system (3.2.4), which shows that the linear stability of (3.2.4) at zero implies the extinction of the species while the instability implies the uniform persistence of the species.

Theorem 3.3.1 *Let $e^* \in int(\mathbb{X}_\beta^+)$ be fixed, and (S1)-(S3) hold. For any $\phi \in \mathbb{Y}^+$, denote by $u(t,x,\phi)$ or $u(t,\phi)$ the solution of system (3.2.4).*

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Theorem 3.3.1 *Let $e^* \in int(\mathbb{X}_\beta^+)$ be fixed, and (S1)-(S3) hold. For any $\phi \in \mathbb{Y}^+$, denote by $u(t,x,\phi)$ or $u(t,\phi)$ the solution of system (3.2.4).*

(i) If $\lambda_0 < 0$, $\lim_{t\to\infty} \|u(t,\phi)\|_\beta = 0$ *for every* $\phi \in \mathbb{Y}^+$.

(ii) If $\lambda_0 > 0$, *then system (3.2.4) admits at least one steady state* φ^* *with* $\varphi^*(x) \in (0, M], \forall x \in \bar{\Omega}$, *and there exists* $\delta > 0$ *such that for every* $\phi \in \mathbb{Y}^+$ *with* $\phi(0,\cdot) \not\equiv 0$, *there is* $t_0 = t_0(\phi) > 0$ *such that* $u(t,\cdot,\phi) \geq \delta e^*(\cdot), t \geq t_0$.

Proof. Note that zero is an equilibrium of (3.2.4). The variational equation about zero is given by

$$\begin{cases} \partial_t v(t,x) = d\Delta v(t,x) - g'(0)v(t,x) + \\ \qquad f'(0)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)v(t-\tau,y)dy, \quad t>0,\ x\in\Omega, \\ Bv(t,x) = 0, \quad t>0,\ x\in\partial\Omega, \\ v(s,x) = \phi(s,x) \geq 0, \quad s\in[-\tau,0],\ x\in\Omega. \end{cases} \tag{3.3.7}$$

By [95, Theorem 9.2.1] and a similar argument in the case of Dirichlet boundary condition, it follows that the eigenvalue problem

$$\begin{cases} \lambda v(x) = d\Delta v(x) - g'(0)v(x) + \\ \qquad f'(0)\mathcal{F}(\tau)e^{-\lambda\tau}\int_\Omega \Gamma(\eta(\tau),x,y)v(y)dy, \quad x\in\Omega, \\ Bv(x) = 0, \quad x\in\partial\Omega, \end{cases} \tag{3.3.8}$$

has a principle eigenvalue $\bar{\lambda}_0$, and $\bar{\lambda}_0$ shares the same sign with λ_0.

(i). In the case of $\lambda_0 < 0$, the properties of the principal eigenvalue $\bar{\lambda}_0$ and linear semigroups imply that $\lim_{t\to\infty} \|v(t,\cdot,\phi)\|_\beta = 0, \forall\phi \in \mathbb{Y}$, where $v(t,x,\phi)$ is the unique solution of (3.3.7). Note that a solution $u(t,x)$ of (3.2.4) satisfies

$$\partial_t u(t,x) \leq d\Delta u(t,x) - g'(0)u(t,x) + f'(0)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)u(t-\tau,y)dy, \quad t>0.$$

(i) If $\lambda_0 < 0$, $\lim_{t\to\infty} \|u(t,\phi)\|_\beta = 0$ *for every* $\phi \in \mathbb{Y}^+$.

(ii) If $\lambda_0 > 0$, *then system (3.2.4) admits at least one steady state* φ^* *with* $\varphi^*(x) \in (0, M], \forall x \in \bar{\Omega}$, *and there exists* $\delta > 0$ *such that for every* $\phi \in \mathbb{Y}^+$ *with* $\phi(0,\cdot) \not\equiv 0$, *there is* $t_0 = t_0(\phi) > 0$ *such that* $u(t,\cdot,\phi) \geq \delta e^*(\cdot), t \geq t_0$.

Proof. Note that zero is an equilibrium of (3.2.4). The variational equation about zero is given by

$$\begin{cases} \partial_t v(t,x) \;=\; d\Delta v(t,x) - g'(0)v(t,x) + \\ \qquad f'(0)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)v(t-\tau,y)dy, \quad t>0,\ x\in\Omega, \\ Bv(t,x) = 0, \quad t>0,\ x\in\partial\Omega, \\ v(s,x) = \phi(s,x) \geq 0, \quad s\in[-\tau,0],\ x\in\Omega. \end{cases} \tag{3.3.7}$$

By [95, Theorem 9.2.1] and a similar argument in the case of Dirichlet boundary condition, it follows that the eigenvalue problem

$$\begin{cases} \lambda v(x) \;=\; d\Delta v(x) - g'(0)v(x) + \\ \qquad f'(0)\mathcal{F}(\tau)e^{-\lambda\tau}\int_\Omega \Gamma(\eta(\tau),x,y)v(y)dy, \quad x\in\Omega, \\ Bv(x) = 0, \quad x\in\partial\Omega, \end{cases} \tag{3.3.8}$$

has a principle eigenvalue $\bar{\lambda}_0$, and $\bar{\lambda}_0$ shares the same sign with λ_0.

(i). In the case of $\lambda_0 < 0$, the properties of the principal eigenvalue $\bar{\lambda}_0$ and linear semigroups imply that $\lim_{t\to\infty} \|v(t,\cdot,\phi)\|_\beta = 0, \forall\phi \in \mathbb{Y}$, where $v(t,x,\phi)$ is the unique solution of (3.3.7). Note that a solution $u(t,x)$ of (3.2.4) satisfies

$$\partial_t u(t,x) \leq d\Delta u(t,x) - g'(0)u(t,x) + f'(0)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)u(t-\tau,y)dy, \quad t>0.$$

The comparison theorem for abstract functional differential equations ([64, Proposition 3]) implies that $u(t,\cdot,\phi) \le v(t,\cdot,\phi), \forall t \ge -\tau$. Thus, $\lim_{t\to\infty} \|u(t,\phi)\|_\beta = 0, \forall \phi \in \mathbb{Y}^+$.

(ii). In the case of $\lambda_0 > 0$, let $\mathbb{Y}_0 = \{\phi \in \mathbb{Y}^+ : \phi(0,\cdot) \not\equiv 0\}$, $\partial\mathbb{Y}_0 := \mathbb{Y}^+ \setminus \mathbb{Y}_0$. For any $\phi \in \mathbb{Y}^+$, the solution $u(t,x,\phi)$ of (3.2.4) satisfies

$$\partial_t u(t,x) \ge d\Delta u(t,x) - g(u(t,x)), \quad t > 0, \; x \in \Omega.$$

By the standard parabolic maximum principle, it then follows that $\Phi(t)(\mathbb{Y}_0) \subset int(\mathbb{Y}^+)$, $\forall t > 0$. Let $Z_1 = \{\phi \in \partial\mathbb{Y}_0 : \Phi(t)\phi \in \partial\mathbb{Y}_0, \forall t \ge 0\}$. Then $\bigcup_{\phi\in Z_1} \omega(\phi) = \{0\}$, where $\omega(\phi)$ denotes the omega limit set of the orbit $\gamma^+(\phi) := \{\Phi(t)\phi : \forall t \ge 0\}$. We claim that

Claim. Zero is a uniform weak repeller for $\mathbb{Y}_0$ in the sense that there exists $\delta_0 > 0$ such that $\limsup_{t\to\infty} \|\Phi(t)\phi\|_\beta \ge \delta_0, \forall \phi \in \mathbb{Y}_0$.

Let us consider the following eigenvalue problem

$$\begin{cases} \lambda v(x) \;=\; d\Delta v(x) - (g'(0)+\varepsilon)v(x) + \\ \qquad (f'(0)-\varepsilon)\mathcal{F}(\tau)e^{-\lambda\tau}\int_\Omega \Gamma(\eta(\tau),x,y)v(y)dy, \quad x \in \Omega, \\ Bv(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.3.9}$$

Since (3.3.8) admits a positive principal eigenvalue $\bar{\lambda}_0$, there exists a sufficiently small positive ε such that (3.3.9) admits a positive principal eigenvalue λ_ε. For this ε, there exists $\delta_\varepsilon > 0$ such that for all $u \in (0,\delta_\varepsilon)$, $g(u) < (g'(0)+\varepsilon)u$ and $f(u) > (f'(0)-\varepsilon)u$. Let $\delta_0 = \delta_\varepsilon/k_\beta$. Suppose, by contradiction, that there exists $\phi_0 \in \mathbb{Y}_0$ such that $\limsup_{t\to\infty} \|\Phi(t)\phi_0\|_\beta < \delta_0$. Then there exists $t' > \tau$ such that

The comparison theorem for abstract functional differential equations ([64, Proposition 3]) implies that $u(t,\cdot,\phi) \le v(t,\cdot,\phi), \forall t \ge -\tau$. Thus, $\lim_{t\to\infty} \|u(t,\phi)\|_\beta = 0, \forall \phi \in \mathbb{Y}^+$.

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$\|u(t,\cdot,\phi_0)\|_\infty \le k_\beta \|u(t,\cdot,\phi_0)\|_\beta < \delta_\varepsilon$ for all $t \ge t' - \tau$. Therefore, $u(t,x,\phi_0)$ satisfies

$$\begin{aligned} \partial_t u(t,x) &> d\Delta u(t,x) - (g'(0)+\varepsilon)u(t,x) + \\ & \quad (f'(0)-\varepsilon)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)u(t-\tau,y)dy, \end{aligned} \tag{3.3.10}$$

for all $t \ge t'$, $x \in \Omega$. Let $\varphi \in \mathbb{X}_\beta$ be the positive eigenfunction associated with the principal eigenvalue λ_ε. Then $u_\varepsilon(t,x) = \varphi(x)e^{\lambda_\varepsilon t}$ is a solution to

$$\begin{cases} \partial_t u(t,x) = d\Delta u(t,x) - (g'(0)+\varepsilon)u(t,x) + \\ \qquad (f'(0)-\varepsilon)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)u(t-\tau,y)dy, \quad t>0,\ x\in\Omega, \\ Bu(t,x) = 0, \quad t>0,\ x \in \partial\Omega. \end{cases}$$

Since $u(t,x,\phi_0) > 0, \forall t > 0, x \in \Omega$, there exists $\varsigma > 0$ such that $u(t'+s,x,\phi_0) \ge \varsigma u_\varepsilon(s,x)$ for $s \in [-\tau,0], x \in \bar{\Omega}$. By inequality (3.3.10) and the comparison theorem ([64, Proposition 3]), we have $u(t,x,\phi_0) \ge \varsigma u_\varepsilon(t-t',x) = \varsigma\varphi(x)e^{\lambda_\varepsilon(t-t')}, \forall t \ge t', x \in \bar{\Omega}$. Since $\lambda_\varepsilon > 0$, $u(t,x,\phi_0)$ is unbounded, a contradiction.

By the continuous time version of Theorem 1.1.1 (see [81, Theorem 4.6]), $\Phi(t)$ is uniformly persistent with respect to $\mathbb{Y}_0$ in the sense that there exists $\delta_1 > 0$ such that $\liminf\limits_{t\to\infty} dist(\Phi(t)\phi, \partial\mathbb{Y}_0) \ge \delta_1, \forall \phi \in \mathbb{Y}_0$. Since $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ is compact for each $t > \tau$, Theorem 1.1.3 with $e = e^* \in int(\mathbb{Y}^+)$ implies that there exists $\delta > 0$ such that for any $\phi \in \mathbb{Y}_0$, $u(t,x,\phi) \ge \delta e^*(x)$ for all $t \ge t(\phi), x \in \bar{\Omega}$.

It remains to prove the existence of a positive steady state. We consider

$$\begin{cases} \partial_t u(t,x) = d\Delta u(t,x) - g(u(t,x)) + \\ \qquad \int_\Omega \Gamma(\eta(\tau),x,y)\mathcal{F}(\tau)f(u(t,y))dy, \quad t>0, \quad x\in\Omega, \\ Bu(t,x) = 0, \quad t>0, \ x\in\partial\Omega, \\ u(0,x) = \varphi(x) \ge 0, \quad x \in \Omega. \end{cases}$$

$\|u(t,\cdot,\phi_0)\|_\infty \le k_\beta \|u(t,\cdot,\phi_0)\|_\beta < \delta_\varepsilon$ for all $t \ge t' - \tau$. Therefore, $u(t,x,\phi_0)$ satisfies

$$\begin{aligned} \partial_t u(t,x) &> d\triangle u(t,x) - (g'(0)+\varepsilon)u(t,x) + \\ & \quad (f'(0)-\varepsilon)\mathcal{F}(\tau)\int_\Omega \Gamma(\eta(\tau),x,y)u(t-\tau,y)dy, \end{aligned} \tag{3.3.10}$$

for all $t \ge t'$, $x \in \Omega$. Let $\varphi \in \mathbb{X}_\beta$ be the positive eigenfunction associated with the principal eigenvalue λ_ε. Then $u_\varepsilon(t,x) = \varphi(x)e^{\lambda_\varepsilon t}$ is a solution to

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Since $u(t,x,\phi_0) > 0, \forall t > 0, x \in \Omega$, there exists $\varsigma > 0$ such that $u(t'+s,x,\phi_0) \ge \varsigma u_\varepsilon(s,x)$ for $s \in [-\tau,0], x \in \bar{\Omega}$. By inequality (3.3.10) and the comparison theorem ([64, Proposition 3]), we have $u(t,x,\phi_0) \ge \varsigma u_\varepsilon(t-t',x) = \varsigma\varphi(x)e^{\lambda_\varepsilon(t-t')}, \forall t \ge t', x \in \bar{\Omega}$. Since $\lambda_\varepsilon > 0$, $u(t,x,\phi_0)$ is unbounded, a contradiction.

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Let $\Phi_0(t) : \mathbb{X}_\beta^+ \to \mathbb{X}_\beta^+, t \geq 0$, be the solution semiflow. As proven for $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$, it follows that $\Phi_0(t)$ is point dissipative on $\mathbb{X}_\beta^+$, compact for each $t > 0$, and uniformly persistent with respect to $\mathbb{X}_\beta^+ \setminus \{0\}$. Then, by the continuous version of Theorem 1.1.2, $\Phi_0(t)$ has an equilibrium $\varphi^* \in \mathbb{X}_\beta^+ \setminus \{0\}$, i.e., $\Phi_0(t)\varphi^* = \varphi^*$ for all $t \geq 0$. Clearly, $\varphi^* \in int(\mathbb{X}_\beta^+)$. ∎

Theorem 3.3.2 *Let (S1)-(S3) hold and $\lambda_0 > 0$. Suppose that either f or $-g$ is strictly sublinear on $[0, M]$, and that f is monotone increasing on $[0, M]$. Then (3.2.4) admits a unique positive steady state φ^*, and $\lim_{t\to\infty} \|u(t,\phi) - \varphi^*\|_\beta = 0$ for every $\phi \in \mathbb{Y}^+$ with $\phi(0,\cdot) \not\equiv 0$, where $u(t,\phi)$ is the solution of (3.2.4).*

Proof. We use notations in the proofs of Theorem 3.2.1 and 3.3.1. Note that f is monotone increasing on $[0, M]$. It then follows that

$$\lim_{h\to 0^+} \frac{1}{h} dist(\psi(0) - \varphi(0) + h(F(\psi) - F(\varphi)), \mathbb{X}_\beta^+) = 0,$$

for all $\varphi, \psi \in \mathbb{Z}_M$ with $\varphi(s,x) \leq \psi(s,x), \forall s \in [-\tau, 0], x \in \bar{\Omega}$. By [64, Proposition 3 and Corollary 5] (see also [88, Corollary 8.1.11]), $\Phi(t) : \mathbb{Z}_M \to \mathbb{Z}_M$ is a monotone semiflow with respect to the order on $\mathbb{Y}$ induced by $\mathbb{Y}^+$. By the proof of Theorem 3.2.1, every omega limits set $\omega(\phi)$ of $\Phi(t)$ is contained in $\mathbb{Z}_M$. In particular, every nonnegative steady state φ of (3.2.4) is contained in Σ_M. We further claim that (3.2.4) admits at most one positive steady state. Indeed, it suffices to show that $\Phi_0(t)$ has at most one positive equilibrium in Σ_M. By [64, Corollary 5] with $\tau = 0$, it then follows that $\Phi_0(t) : \Sigma_M \to \Sigma_M$ is a monotone semiflow with respect to the order on $\mathbb{X}_\beta$ induced by $\mathbb{X}_\beta^+$. Moreover, for any $\varphi_1, \varphi_2 \in \Sigma_M$ with $\varphi_1 - \varphi_2 \in \mathbb{X}_\beta^+ \setminus \{0\}$, Let $\Phi_0(t) : \mathbb{X}_\beta^+ \to \mathbb{X}_\beta^+, t \geq 0$, be the solution semiflow. As proven for $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$, it follows that $\Phi_0(t)$ is point dissipative on $\mathbb{X}_\beta^+$, compact for each $t > 0$, and uniformly persistent with respect to $\mathbb{X}_\beta^+ \setminus \{0\}$. Then, by the continuous version of Theorem 1.1.2, $\Phi_0(t)$ has an equilibrium $\varphi^* \in \mathbb{X}_\beta^+ \setminus \{0\}$, i.e., $\Phi_0(t)\varphi^* = \varphi^*$ for all $t \geq 0$. Clearly, $\varphi^* \in int(\mathbb{X}_\beta^+)$. ∎

Theorem 3.3.2 *Let (S1)-(S3) hold and $\lambda_0 > 0$. Suppose that either f or $-g$ is strictly sublinear on $[0, M]$, and that f is monotone increasing on $[0, M]$. Then (3.2.4) admits a unique positive steady state φ^*, and $\lim_{t\to\infty} \|u(t, \phi) - \varphi^*\|_\beta = 0$ for every $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$, where $u(t, \phi)$ is the solution of (3.2.4).*

Proof. We use notations in the proofs of Theorem 3.2.1 and 3.3.1. Note that f is monotone increasing on $[0, M]$. It then follows that

$$\lim_{h\to 0+} \frac{1}{h} dist(\psi(0) - \varphi(0) + h(F(\psi) - F(\varphi)), \mathbb{X}_\beta^+) = 0,$$

for all $\varphi, \psi \in \mathbb{Z}_M$ with $\varphi(s, x) \leq \psi(s, x), \forall s \in [-\tau, 0], x \in \bar{\Omega}$. By [64, Proposition 3 and Corollary 5] (see also [88, Corollary 8.1.11]), $\Phi(t) : \mathbb{Z}_M \to \mathbb{Z}_M$ is a monotone semiflow with respect to the order on $\mathbb{Y}$ induced by $\mathbb{Y}^+$. By the proof of Theorem 3.2.1, every omega limits set $\omega(\phi)$ of $\Phi(t)$ is contained in $\mathbb{Z}_M$. In particular, every nonnegative steady state φ of (3.2.4) is contained in Σ_M. We further claim that (3.2.4) admits at most one positive steady state. Indeed, it suffices to show that $\Phi_0(t)$ has at most one positive equilibrium in Σ_M. By [64, Corollary 5] with $\tau = 0$, it then follows that $\Phi_0(t) : \Sigma_M \to \Sigma_M$ is a monotone semiflow with respect to the order on $\mathbb{X}_\beta$ induced by $\mathbb{X}_\beta^+$. Moreover, for any $\varphi_1, \varphi_2 \in \Sigma_M$ with $\varphi_1 - \varphi_2 \in \mathbb{X}_\beta^+ \setminus \{0\}$,

$u(t,x) := (\Phi_0(t)\varphi_1)(x) - (\Phi_0(t)\varphi_2)(x)$ satisfies

$$\begin{aligned}\partial_t u(t,x) &\geq d\triangle u(t,x) - u(t,x)\int_0^1 g'(s\Phi_0(t)\varphi_1(x) + (1-s)\Phi_0(t)\varphi_2(x))ds \\ &\geq d\triangle u(t,x) - k_s u(t,x), \quad t > 0,\ x \in \Omega,\end{aligned}$$

where $k_s = \sup_{u\in[0,M]} g'(u)$. Then the standard parabolic maximum principle implies that $u(t) \in int(\mathbb{X}_\beta^+), \forall t > 0$. That is, $\Phi_0(t) : \Sigma_M \to \Sigma_M$ is strongly monotone. By the strict sublinearity of f or $-g$, it easily follows that for each $t > 0$, $\Phi_0(t) : \Sigma_M \to \Sigma_M$ is strictly sublinear (see, e.g., [37, Theorem 2.2]). Now fix a real number $t_0 > 0$. Then [93, Lemma 1] implies that the map $\Phi_0(t_0)$ has at most one positive fixed point in Σ_M, and hence the semiflow $\Phi_0(t)$ has at most one positive equilibrium in Σ_M. Note that $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ is compact for $t > \tau$, admits a global attractor in $\mathbb{Y}^+$, and is uniformly persistent with respect to $\mathbb{Y}_0$. By [43, Theorem 3.2], $\Phi(t) : \mathbb{Z}_M \bigcap \mathbb{Y}_0 \to \mathbb{Z}_M \bigcap \mathbb{Y}_0$ has a global attractor A_0. Theorem 3.3.1, together with the uniqueness of the positive steady state, implies that A_0 contains only one equilibrium φ^*. By Hirsch's attractivity theorem (Theorem 1.2.3), it then follows that φ^* attracts every point in $\mathbb{Z}_M \bigcap \mathbb{Y}_0$. Consequently, every orbit in $\mathbb{Z}_M$ converges to either the trivial equilibrium or the positive equilibrium φ^*. Note that the equilibria 0 and φ^* are also isolated invariant sets in $\mathbb{Z}_M$, and there is no cyclic chain of equilibria. By the continuous time version of [95, Theorem 1.2.2], every compact internally chain transitive set for $\Phi(t) : \mathbb{Z}_M \to \mathbb{Z}_M$ is an equilibrium. For any given $\phi \in \mathbb{Y}^+$, by the proof of Theorem 3.2.1, $\omega(\phi) \subset \mathbb{Z}_M$, and hence $\omega(\phi)$ is an equilibrium. If $\phi \in \mathbb{Y}^+$ with $\phi(0,\cdot) \not\equiv 0$, by Theorem 3.3.1 (ii), we then have $\omega(\phi) = \varphi^*$. ∎

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where $k_s = \sup\limits_{u\in[0,M]} g'(u)$. Then the standard parabolic maximum principle implies that $u(t) \in int(\mathbb{X}_\beta^+), \forall t > 0$. That is, $\Phi_0(t) : \Sigma_M \to \Sigma_M$ is strongly monotone. By the strict sublinearity of f or $-g$, it easily follows that for each $t > 0$, $\Phi_0(t) : \Sigma_M \to \Sigma_M$ is strictly sublinear (see, e.g., [37, Theorem 2.2]). Now fix a real number $t_0 > 0$. Then [93, Lemma 1] implies that the map $\Phi_0(t_0)$ has at most one positive fixed point in Σ_M, and hence the semiflow $\Phi_0(t)$ has at most one positive equilibrium in Σ_M. Note that $\Phi(t) : \mathbb{Y}^+ \to \mathbb{Y}^+$ is compact for $t > \tau$, admits a global attractor in $\mathbb{Y}^+$, and is uniformly persistent with respect to $\mathbb{Y}_0$. By [43, Theorem 3.2], $\Phi(t) : \mathbb{Z}_M \bigcap \mathbb{Y}_0 \to \mathbb{Z}_M \bigcap \mathbb{Y}_0$ has a global attractor A_0. Theorem 3.3.1, together with the uniqueness of the positive steady state, implies that A_0 contains only one equilibrium φ^*. By Hirsch's attractivity theorem (Theorem 1.2.3), it then follows that φ^* attracts every point in $\mathbb{Z}_M \bigcap \mathbb{Y}_0$. Consequently, every orbit in $\mathbb{Z}_M$ converges to either the trivial equilibrium or the positive equilibrium φ^*. Note that the equilibria 0 and φ^* are also isolated invariant sets in $\mathbb{Z}_M$, and there is no cyclic chain of equilibria. By the continuous time version of [95, Theorem 1.2.2], every compact internally chain transitive set for $\Phi(t) : \mathbb{Z}_M \to \mathbb{Z}_M$ is an equilibrium. For any given $\phi \in \mathbb{Y}^+$, by the proof of Theorem 3.2.1, $\omega(\phi) \subset \mathbb{Z}_M$, and hence $\omega(\phi)$ is an equilibrium. If $\phi \in \mathbb{Y}^+$ with $\phi(0,\cdot) \not\equiv 0$, by Theorem 3.3.1 (ii), we then have $\omega(\phi) = \varphi^*$. ∎

3.4 Discussion

In this section, we investigate the effects of spatial diffusion and time delay on the global behavior of model (3.1.3) in two specific cases, and provide some numerical simulations, some of which seem to suggest that Theorem 3.3.2 holds even without the monotonicity condition.

First let us compute the principal eigenvalue λ_0 for problem (3.3.6). In the case of the Neumann boundary condition, it easily follows that the eigenvalue problem (3.3.6) admits the principle eigenvalue $\lambda_0 = -g'(0) + f'(0)\mathcal{F}(\tau)$ (with the eigenfunction $v(\cdot) \equiv 1$). In the case of the Dirichlet boundary condition, we consider (3.3.6) with $\Omega = (0, \pi)$. Let $T_0(t)\varphi = \int_\Omega \Gamma(t, x, y)\varphi(y)dy$, which is the linear semigroup generated by

$$\begin{cases} \partial_t u = \triangle u, \\ u(t, 0) = u(t, \pi) = 0, \\ u(0, x) = \varphi(x) \in \mathbb{X}_\beta^+. \end{cases} \tag{3.4.11}$$

It then follows that $e^{-t}\sin x$ is a solution of (3.4.11) with $\varphi(x) = \sin x$. Thus,

$$T_0(t)\sin(x) = \int_\Omega \Gamma(t, x, y)\sin y dy = e^{-t}\sin x, \forall t \geq 0, x \in (0, \pi).$$

In particular, $T_0(\eta(\tau))\sin x = e^{-\eta(\tau)}\sin x$. It is easy to verify that $\sin x$ is a positive solution of (3.3.6) with $\lambda = -d - g'(0) + f'(0)\mathcal{F}(\tau)e^{-\eta(\tau)}$. Therefore, $\lambda_0 = -d - g'(0) + f'(0)\mathcal{F}(\tau)e^{-\eta(\tau)}$.

Example 1. Consider the model (3.1.3) with $g(u) = \beta u^2, f(u) = \alpha u$ and $\mathcal{F}(\tau) = e^{-\mu_j \tau}$, where α, β, μ_j and the immature diffusion coefficient d_j in (3.1.1) are all positive constants.

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First let us compute the principal eigenvalue λ_0 for problem (3.3.6). In the case of the Neumann boundary condition, it easily follows that the eigenvalue problem (3.3.6) admits the principle eigenvalue $\lambda_0 = -g'(0) + f'(0)\mathcal{F}(\tau)$ (with the eigenfunction $v(\cdot) \equiv 1$). In the case of the Dirichlet boundary condition, we consider (3.3.6) with $\Omega = (0, \pi)$. Let $T_0(t)\varphi = \int_\Omega \Gamma(t, x, y)\varphi(y)dy$, which is the linear semigroup generated by

$$\begin{cases} \partial_t u = \Delta u, \\ u(t,0) = u(t,\pi) = 0, \\ u(0,x) = \varphi(x) \in \mathbb{X}_\beta^+. \end{cases} \tag{3.4.11}$$

It then follows that $e^{-t}\sin x$ is a solution of (3.4.11) with $\varphi(x) = \sin x$. Thus,

$$T_0(t)\sin(x) = \int_\Omega \Gamma(t,x,y)\sin y dy = e^{-t}\sin x, \forall t \geq 0, x \in (0,\pi).$$

In particular, $T_0(\eta(\tau))\sin x = e^{-\eta(\tau)}\sin x$. It is easy to verify that $\sin x$ is a positive solution of (3.3.6) with $\lambda = -d - g'(0) + f'(0)\mathcal{F}(\tau)e^{-\eta(\tau)}$. Therefore, $\lambda_0 = -d - g'(0) + f'(0)\mathcal{F}(\tau)e^{-\eta(\tau)}$.

Example 1. Consider the model (3.1.3) with $g(u) = \beta u^2, f(u) = \alpha u$ and $\mathcal{F}(\tau) = e^{-\mu_j\tau}$, where α, β, μ_j and the immature diffusion coefficient d_j in (3.1.1) are all positive constants.

In the case of the Neumann boundary condition, we have $\lambda_0 = \alpha e^{-\mu_j \tau} > 0$. By Theorem 3.3.2 with $M = \frac{\alpha}{\beta} e^{-\mu_j \tau}$, it follows that for each $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$, $\lim_{t \to \infty} u_m(t, x, \phi) = \varphi^*(x) \equiv \frac{\alpha}{\beta} e^{-\mu_j \tau}$ uniformly for $x \in \Omega$, where $u_m(t, x, \phi)$ is the solution of (3.1.3) with the initial function ϕ. This convergence result is consistent with that in [40]. In this case, we can see that the maturation period τ and the diffusion of the species do not affect the permanence of the species.

In the case of the Dirichlet boundary condition, let $\Omega = (0, \pi)$. Then, $\lambda_0 = -d_m + \alpha e^{-(\mu_j + d_j)\tau}$ (d_m in (3.1.3) is d in (3.3.6)). Note that $\lambda_0 < 0$ if $\alpha < d_m$, and in the case of $\alpha > d_m$, we have $\lambda_0 > 0$ if $\tau \in [0, \tau_0)$, and $\lambda_0 < 0$ if $\tau > \tau_0$, where $\tau_0 = \frac{1}{\mu_j + d_j} \ln \frac{\alpha}{d_m} > 0$. By Theorem 3.3.1 and 3.3.2 with $M = \frac{\alpha}{\beta} e^{-\mu_j \tau}$, we have the following result for this case.

Proposition 3.4.1 *Let $u_m(t, x, \phi)$ be the solution of (3.1.3) subject to the Dirichlet boundary condition and with the initial function $\phi \in \mathbb{Y}^+$.*

(1) If $\alpha < d_m$, then for any $\phi \in \mathbb{Y}^+$, $\lim_{t \to \infty} u_m(t, x, \phi) = 0$ uniformly for $x \in [0, \pi]$.

(2) In the case of $\alpha > d_m$, let $\tau_0 = \frac{1}{\mu_j + d_j} \ln \frac{\alpha}{d_m} > 0$.

(a) If $\tau > \tau_0$, then for any $\phi \in \mathbb{Y}^+$, $\lim_{t \to \infty} u_m(t, x, \phi) = 0$ uniformly for $x \in [0, \pi]$.

(b) If $\tau \in [0, \tau_0)$, then for any $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$, $\lim_{t \to \infty} u_m(t, x, \phi) = \varphi^*(x)$ uniformly for $x \in [0, \pi]$, where φ^* is the unique positive steady state of (3.1.3).

By Proposition 3.4.1, we have the following observations on the model (3.1.3) subject to the Dirichlet boundary condition.

In the case of the Neumann boundary condition, we have $\lambda_0 = \alpha e^{-\mu_j \tau} > 0$. By Theorem 3.3.2 with $M = \frac{\alpha}{\beta} e^{-\mu_j \tau}$, it follows that for each $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$, $\lim_{t \to \infty} u_m(t, x, \phi) = \varphi^*(x) \equiv \frac{\alpha}{\beta} e^{-\mu_j \tau}$ uniformly for $x \in \Omega$, where $u_m(t, x, \phi)$ is the solution of (3.1.3) with the initial function ϕ. This convergence result is consistent with that in [40]. In this case, we can see that the maturation period τ and the diffusion of the species do not affect the permanence of the species.

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Proposition 3.4.1 *Let $u_m(t, x, \phi)$ be the solution of (3.1.3) subject to the Dirichlet boundary condition and with the initial function $\phi \in \mathbb{Y}^+$.*

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By Proposition 3.4.1, we have the following observations on the model (3.1.3) subject to the Dirichlet boundary condition.

Conclusion 1. If all parameters except for d_m are fixed, then the fast mature dispersal in space brings negative effect on persistence of the species.

Conclusion 2. If all parameters except for the delay τ are fixed, then the large maturation time τ brings negative effect on persistence of the species.

Example 2. Consider the model (3.1.3) with $g(u) = \beta u$, $f(u) = pue^{-qu}$ and $\mathcal{F}(\tau) = e^{-\mu_j \tau}$, where β, p, q, μ_j and d_j are all positive constants. A direct computation shows that $f'(u) = pe^{-qu}(1-qu)$, $f''(u) = -pqe^{-qu}(2-qu)$, and $f(u)$ reaches its maximum value $f(\frac{1}{q}) = \frac{p}{q}e^{-1}$.

In the case of the Neumann boundary condition, $\lambda_0 = -\beta + pe^{-\mu_j \tau}$. Therefore, if $\beta > pe^{-\mu_j \tau}$, then Theorem 3.3.1 (i) with $M = 0$ implies that the species goes extinct; if $\beta < pe^{-\mu_j \tau}$, then Theorem 3.3.1 (ii) with $M = \frac{p}{\beta q}e^{-1-\mu_j \tau}$ implies that the species persists. If, in addition, $pe^{-1-\mu_j \tau} \leq \beta < pe^{-\mu_j \tau}$, Theorem 3.3.2 with $M = \frac{1}{q}(\ln \frac{p}{\beta} - \mu_j \tau) > 0$ implies that (3.1.3) admits the unique positive steady state $\varphi^*(x) \equiv \frac{1}{q}(\ln \frac{p}{\beta} - \mu_j \tau)$, which is globally attractive.

The above analysis supports our second conclusion. For various values of the maturation time τ, the species may go to extinction, persist, or stabilize at a positive steady state. However, the diffusion coefficient d_m has no effects on the persistence of the species.

In the case of the Dirichlet boundary condition, $\lambda_0 = -(d_m + \beta) + pe^{-(\mu_j + d_j)\tau}$. By Theorem 3.3.1 and 3.3.2 with $M = 0$, or $M = \frac{p}{\beta q}e^{-1-\mu_j \tau}$ and $\frac{1}{q}(\ln \frac{p}{\beta} - \mu_j \tau)$, we have the following result, which implies the same conclusions about the effects of the maturation period τ and the diffusion coefficient d_m as in Example 1.

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Proposition 3.4.2 *Let $u_m(t,x,\phi)$ be the solution of system (3.1.3) subject to the Dirichlet boundary condition and with the initial function $\phi \in \mathbb{Y}^+$.*

(1) If $p < d_m+\beta$, then for any $\phi \in \mathbb{Y}^+$, $\lim_{t\to\infty} u_m(t,x,\phi) = 0$ uniformly for $x \in [0,\pi]$.

(2) In the case of $p > d_m + \beta$, let $\tau_0 = \frac{1}{\mu_j+d_j}\ln\frac{p}{d_m+\beta}$, $\tau_1 = \frac{1}{\mu_j}(\ln\frac{p}{\beta} - 1)$.

(a) If $\tau > \tau_0$, then for any $\phi \in \mathbb{Y}^+$, $\lim_{t\to\infty} u_m(t,x,\phi) = 0$ uniformly for $x \in [0,\pi]$.

(b) If $\tau_1 \le \tau < \tau_0$, then for any $\phi \in \mathbb{Y}^+$ with $\phi(0,\cdot) \not\equiv 0$, $\lim_{t\to\infty} u_m(t,x,\phi) = \varphi^*(x)$ uniformly for $x \in [0,\pi]$, where φ^* is the unique positive steady state of system (3.1.3).

Numerical simulation. We numerically simulate Example 2 with the domain $\Omega = (0,\pi)$. Model (3.1.3) is discretised by using the finite difference method, where the nonlocal term is approximated by composite integration formulas. Note that in the case of the Neumann boundary condition,

$$\Gamma(\eta(\tau),x,y) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 d_j \tau}\cos nx \cos ny,$$

and in the case of the Dirichlet boundary condition,

$$\Gamma(\eta(\tau),x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 d_j \tau}\sin nx \sin ny$$

(see, e.g., [65, Section 5.1]).

For the case of the Neumann boundary condition, let $p = q = \mu_j = 1, d_m = 0.5, \beta = 0.2, d_j = 0.25$. Then, when $\tau > 1.6094$, every positive solution goes to zero as t goes to infinity (Theorem 3.3.1); when $\tau \in (0.6094, 1.6094)$, model (3.1.3)

Proposition 3.4.2 *Let $u_m(t, x, \phi)$ be the solution of system (3.1.3) subject to the Dirichlet boundary condition and with the initial function $\phi \in \mathbb{Y}^+$.*

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(2) In the case of $p > d_m + \beta$, let $\tau_0 = \frac{1}{\mu_j+d_j} \ln \frac{p}{d_m+\beta}$, $\tau_1 = \frac{1}{\mu_j}(\ln \frac{p}{\beta} - 1)$.

(a) If $\tau > \tau_0$, then for any $\phi \in \mathbb{Y}^+$, $\lim_{t\to\infty} u_m(t, x, \phi) = 0$ uniformly for $x \in [0, \pi]$.

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For the case of the Neumann boundary condition, let $p = q = \mu_j = 1, d_m = 0.5, \beta = 0.2, d_j = 0.25$. Then, when $\tau > 1.6094$, every positive solution goes to zero as t goes to infinity (Theorem 3.3.1); when $\tau \in (0.6094, 1.6094)$, model (3.1.3) admits a globally attractive and positive steady state $\varphi^*(x) \equiv \frac{1}{q}(\ln\frac{p}{\beta} - \mu_j\tau) > 0$ (Theorem 3.3.2). We simulate solutions of system (3.1.3) with the initial function $\phi(t,x) = 1 - \cos 2x$ in the case of $\tau = 1.7$ and $\tau = 1$, which are shown in Figure 3.1 and Figure 3.2, respectively. Clearly, the solution in Figure 3.1 converges to zero, while the solution in Figure 3.2 converges to the unique positive steady state $\varphi^*(x) \equiv 0.6094$. Thus, the numerical results are consistent with our theoretical results. We also simulate solutions of system (3.1.3) when $\tau < 0.6094$, which implies that the monotonicity condition in Theorem 3.3.2 is not satisfied. In Figure 3.3, 3.4, 3.5 and 3.6, the solutions share the parameters with those in Figure 3.1 and 3.2 except for $\tau = 0.3$ and different initial functions. The numerical results shows that all the solutions converge to the same steady state $\varphi^*(x) \equiv \frac{1}{q}(\ln\frac{p}{\beta} - \mu_j\tau) = 1.3094$. Therefore, in this case, the positive steady state $\varphi^*(x)$ may be unique and globally attractive even if the monotonicity condition in Theorem 3.3.2 is not satisfied.

For the case of the Dirichlet boundary condition, let $p = 5, q = \beta = 1, \mu_j = 1.2, d_j = 0.25, d_m = 0.5$. Then, by Proposition 3.4.2, when $\tau > 0.8303$, zero solution attracts every solution of system (3.1.3); when $\tau \in (0.5078, 0.8303)$, model (3.1.3) admits a globally attractive and positive steady state. In Figure 3.7, the solution of system (3.1.3) with initial function $\phi(t,x) = \sin x$ and $\tau = 1$ converges to zero, while in Figure 3.8, the solution with the same initial function and $\tau = 0.65$ converges to the unique steady state $\varphi^*(x)$. Our theoretical results coincide with the numerical simulations. Just as in the case of the Neumann boundary conditions, we also simulate the solutions of system (3.1.3) in the case of $\tau < 0.5078$. The numerical results are shown in Figure 3.9, 3.10, 3.11, 3.12. We can see that the solutions in these figures converge to the same steady state. The numerical simulations, just as admits a globally attractive and positive steady state $\varphi^*(x) \equiv \frac{1}{q}(\ln\frac{p}{\beta} - \mu_j\tau) > 0$ (Theorem 3.3.2). We simulate solutions of system (3.1.3) with the initial function $\phi(t,x) = 1 - \cos 2x$ in the case of $\tau = 1.7$ and $\tau = 1$, which are shown in Figure 3.1 and Figure 3.2, respectively. Clearly, the solution in Figure 3.1 converges to zero, while the solution in Figure 3.2 converges to the unique positive steady state $\varphi^*(x) \equiv 0.6094$. Thus, the numerical results are consistent with our theoretical results. We also simulate solutions of system (3.1.3) when $\tau < 0.6094$, which implies that the monotonicity condition in Theorem 3.3.2 is not satisfied. In Figure 3.3, 3.4, 3.5 and 3.6, the solutions share the parameters with those in Figure 3.1 and 3.2 except for $\tau = 0.3$ and different initial functions. The numerical results shows that all the solutions converge to the same steady state $\varphi^*(x) \equiv \frac{1}{q}(\ln\frac{p}{\beta} - \mu_j\tau) = 1.3094$. Therefore, in this case, the positive steady state $\varphi^*(x)$ may be unique and globally attractive even if the monotonicity condition in Theorem 3.3.2 is not satisfied.

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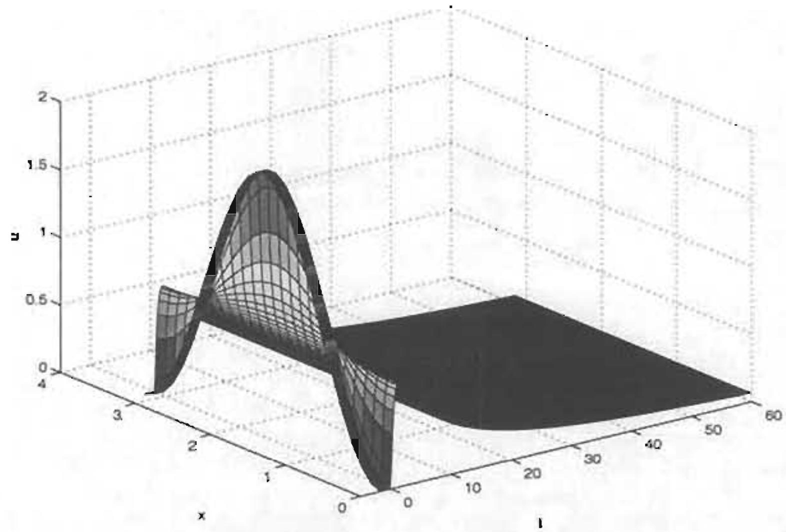


Figure 3.1: The solution of Example 2 in the case of the Neumann boundary condition. The parameters of the system are as follows: $\Omega = (0, \pi)$, $p = q = \mu_j = 1$, $d_m = 0.5$, $\beta = 0.2$, $d_j = 0.25$, $\tau = 1.7$, $\phi(t, x) = 1 - \cos(2x)$.

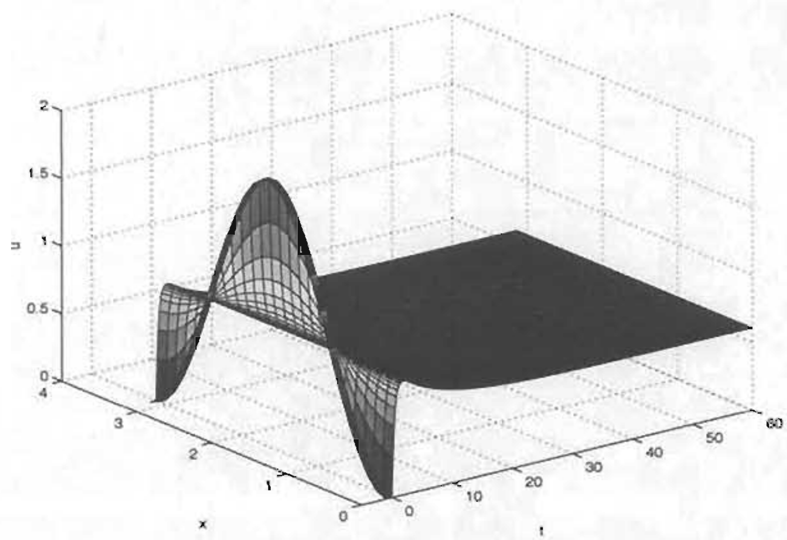
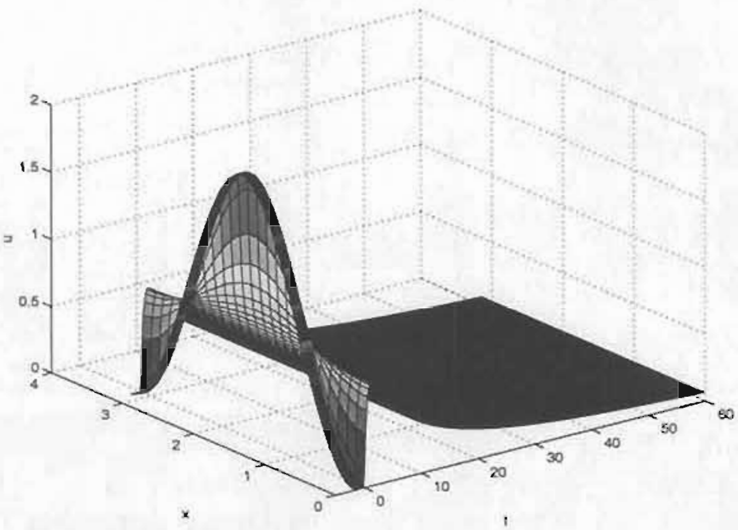


Figure 3.2: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 1$, $\phi(t, x) = 1 - \cos(2x)$.



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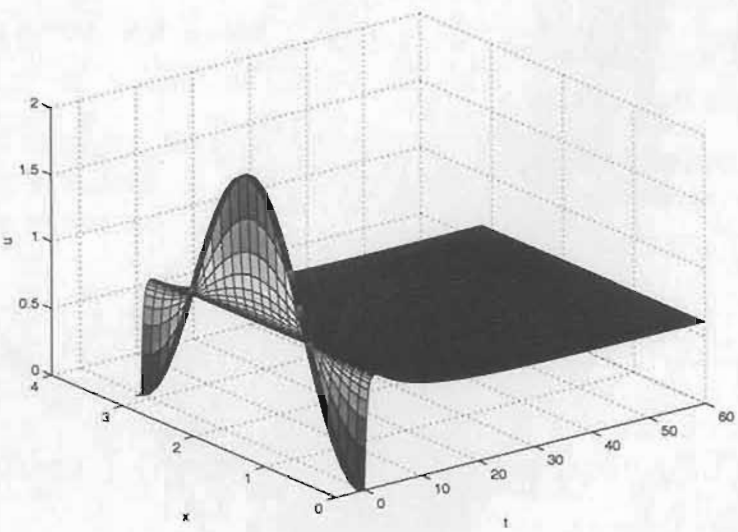

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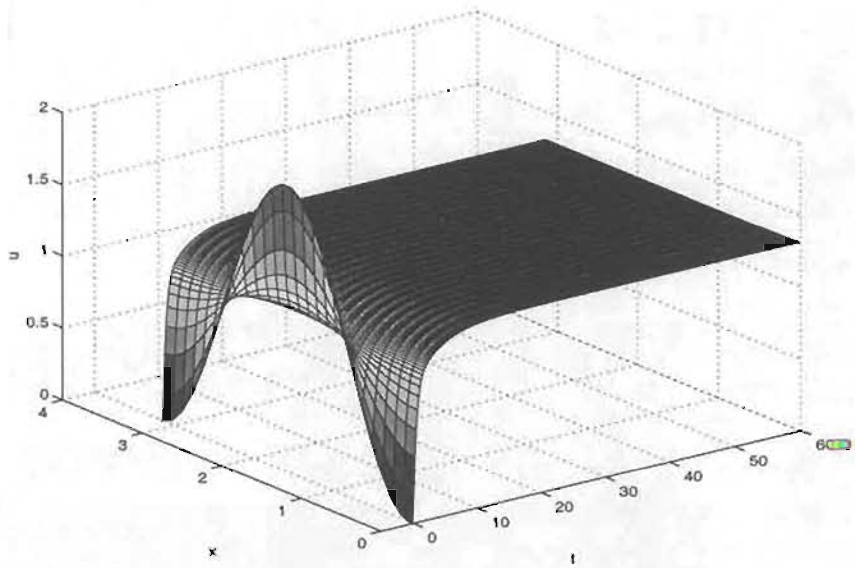


Figure 3.3: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 0.3$, $\phi(t, x) = 1 - \cos(2x)$.

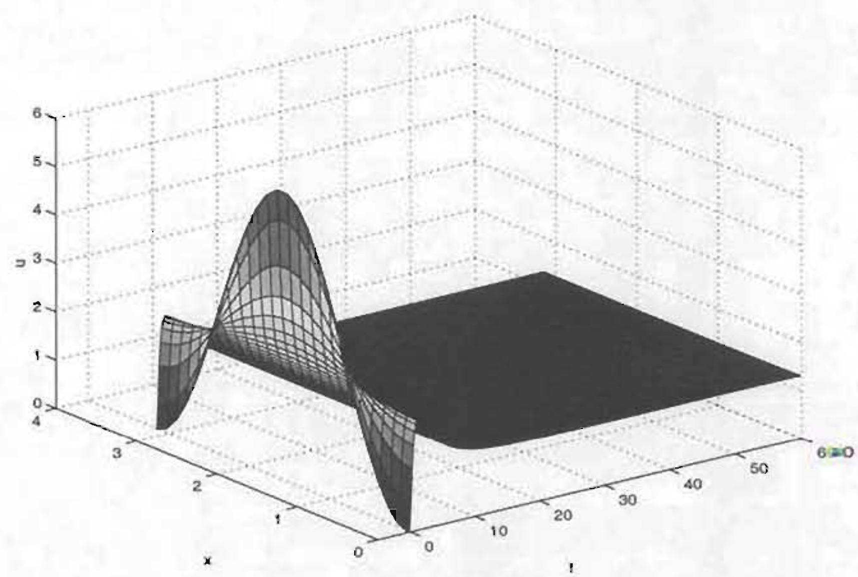


Figure 3.4: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 0.3$, $\phi(t, x) = 3 - 3\cos(2x)$.

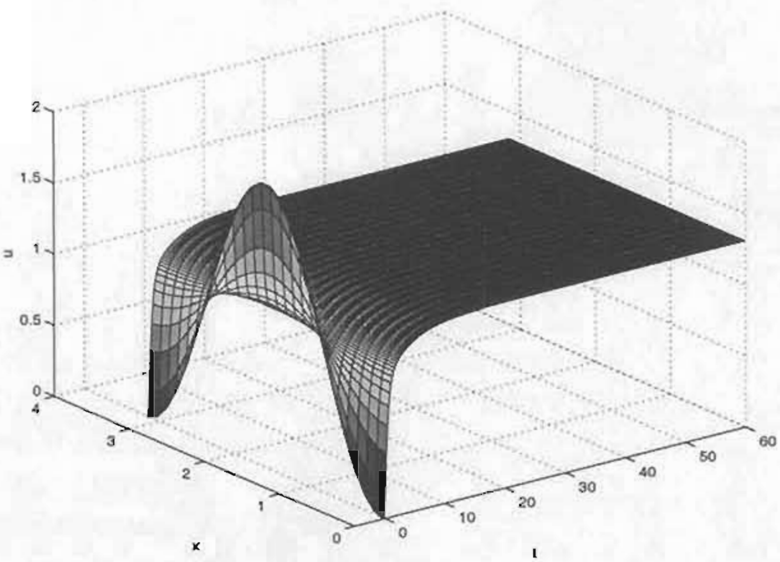


Figure 3.3: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 0.3$, $\phi(t, x) = 1 - \cos(2x)$.

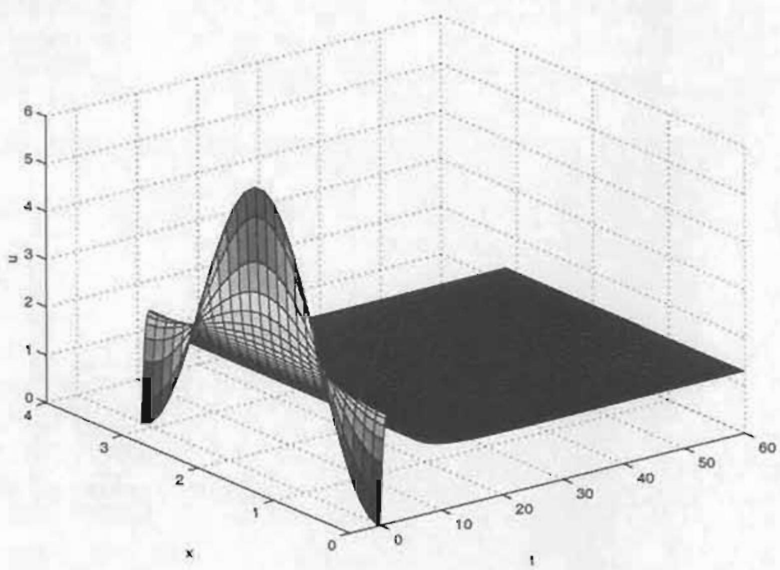


Figure 3.4: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 0.3$, $\phi(t, x) = 3 - 3\cos(2x)$.

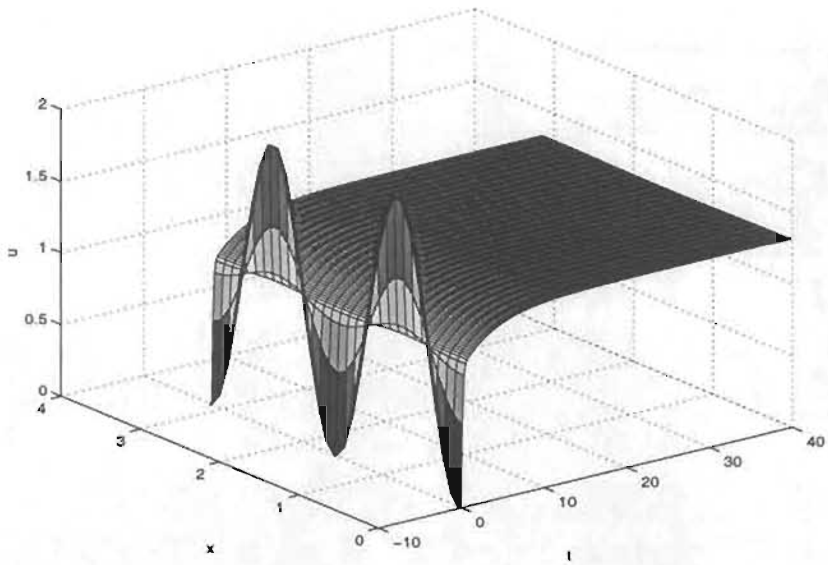


Figure 3.5: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 0.3$, $\phi(t, x) = 1 - \cos(4x)$.

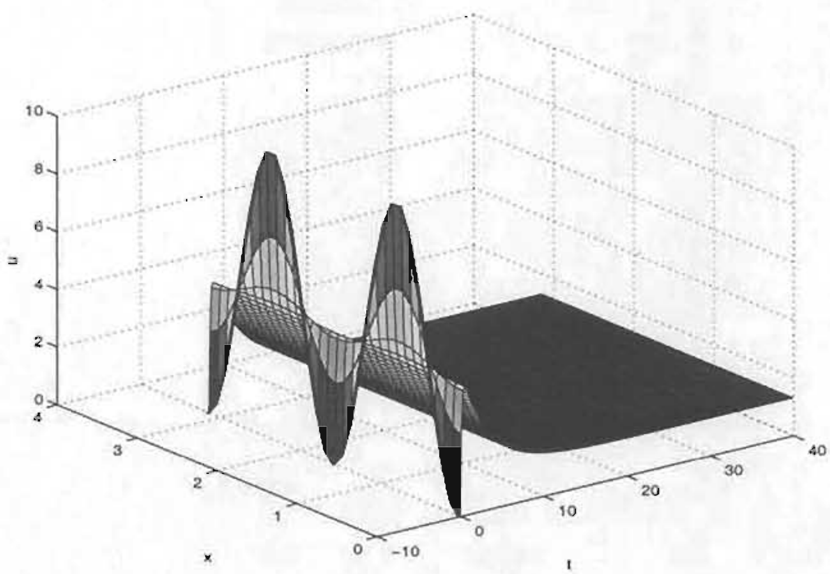
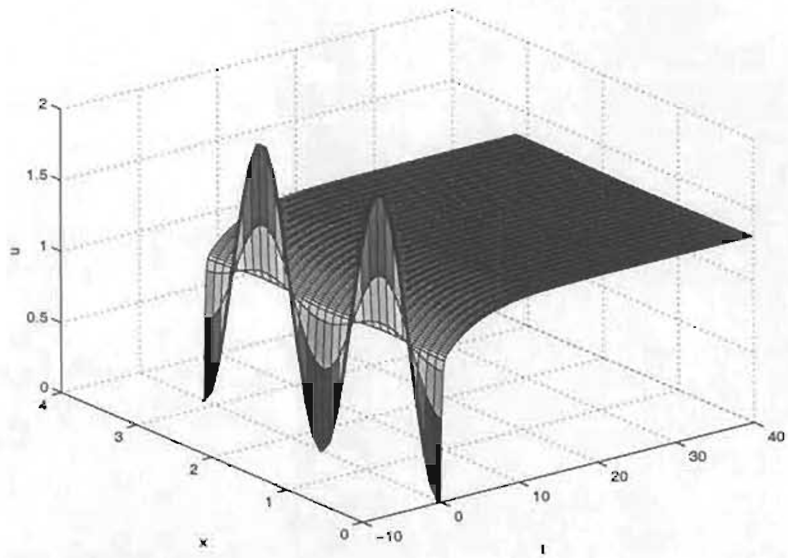


Figure 3.6: The solution of Example 2 with the same condition and parameters as in Figure 3.1, except for $\tau = 0.3$, $\phi(t, x) = 5 - 5\cos(4x)$.



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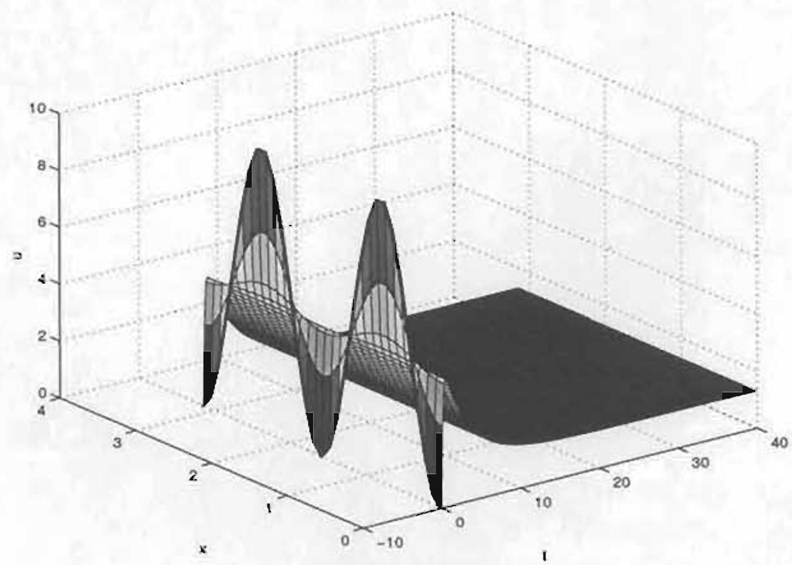

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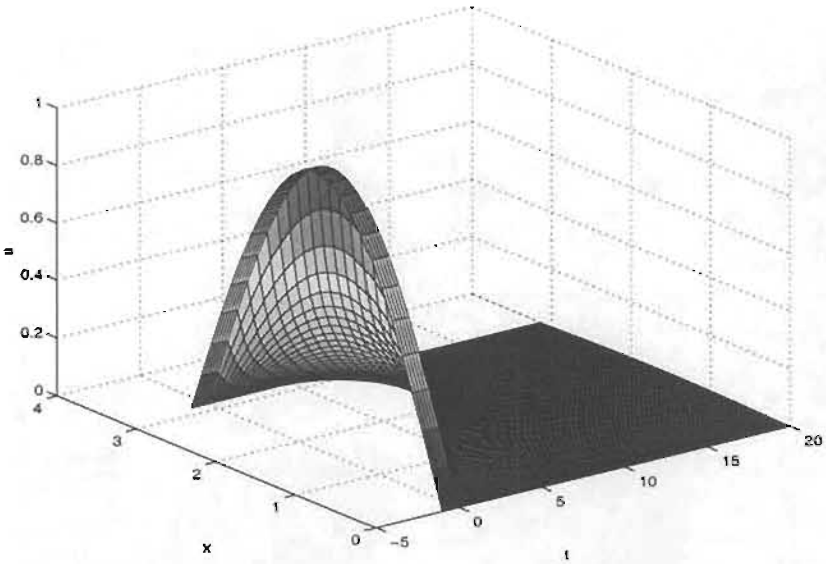


Figure 3.7: The solution of Example 2 in the case of the Dirichlet boundary condition. The parameters of the system are as follows: $\Omega = (0, \pi), p = 5$, $q = \beta = 1$, $\mu_j = 1.2$, $d_j = 0.25$, $d_m = 0.5$, $\tau = 1$, $\phi(t, x) = \sin x$.

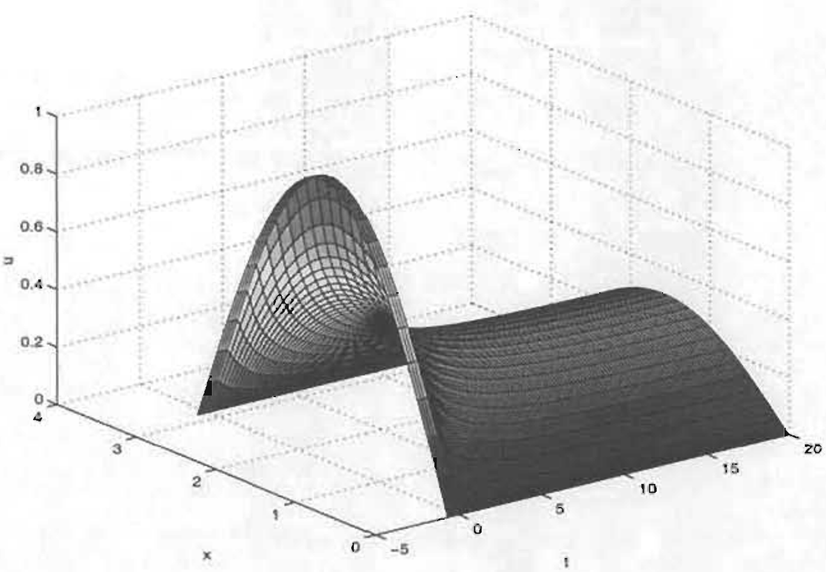
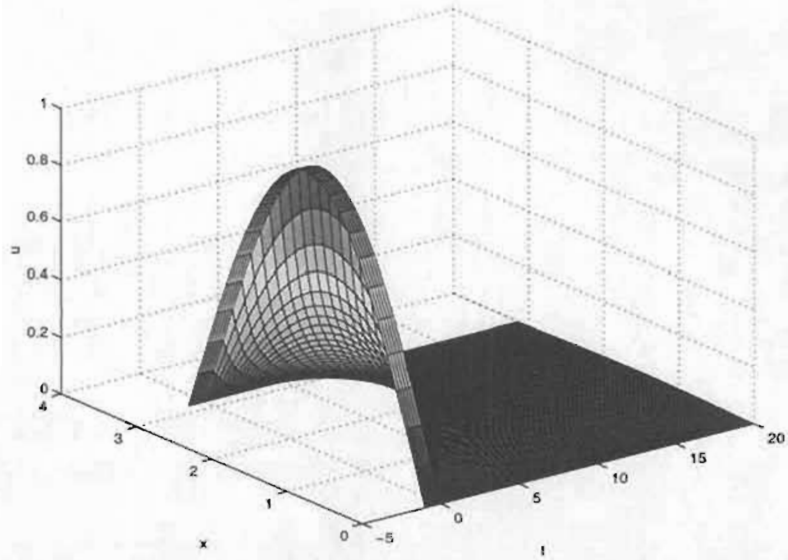


Figure 3.8: The solution of Example 2 with the same condition and parameters as in Figure 3.7, except for $\tau = 0.65$, $\phi(t, x) = \sin x$.



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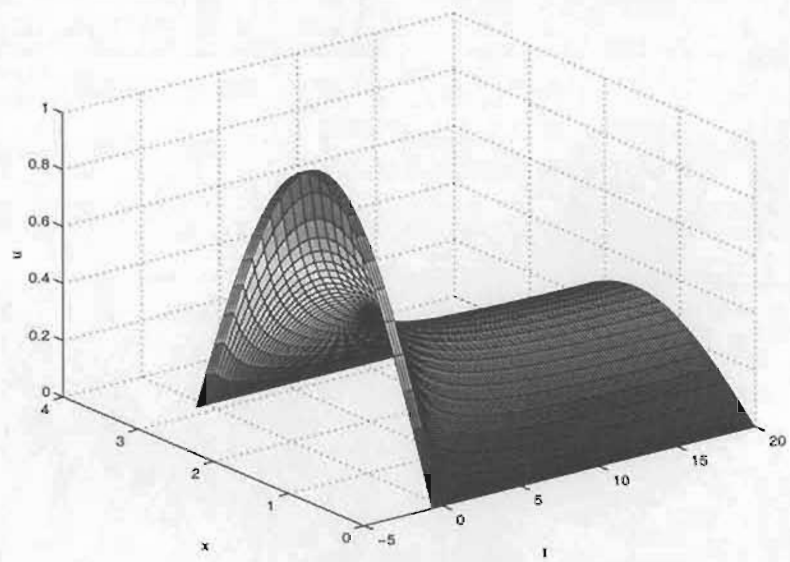

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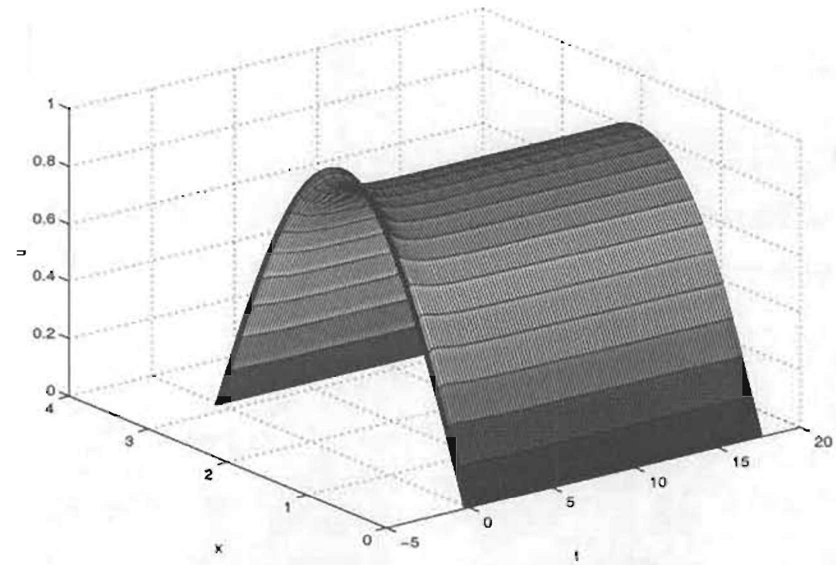


Figure 3.9: The solution of Example 2 with the same condition and parameters as in Figure 3.7, except for $\tau = 0.3$, $\phi(t, x) = \sin x$.

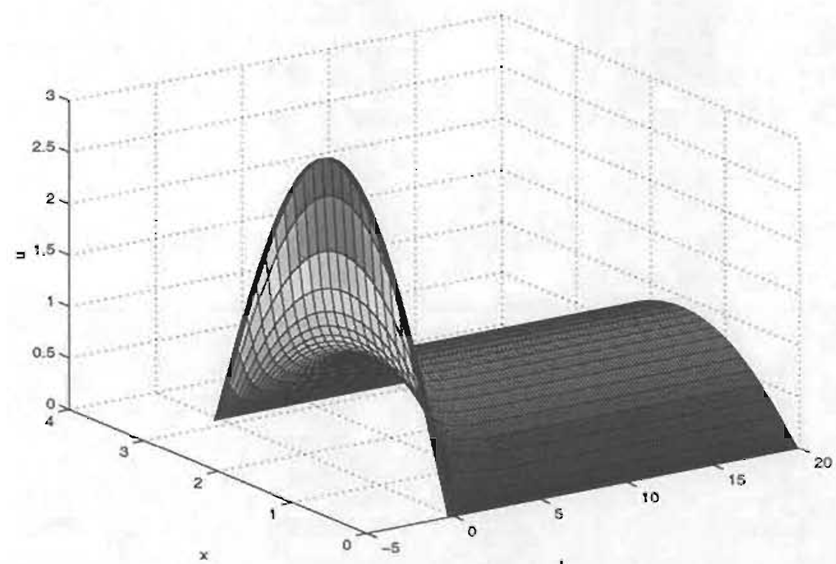
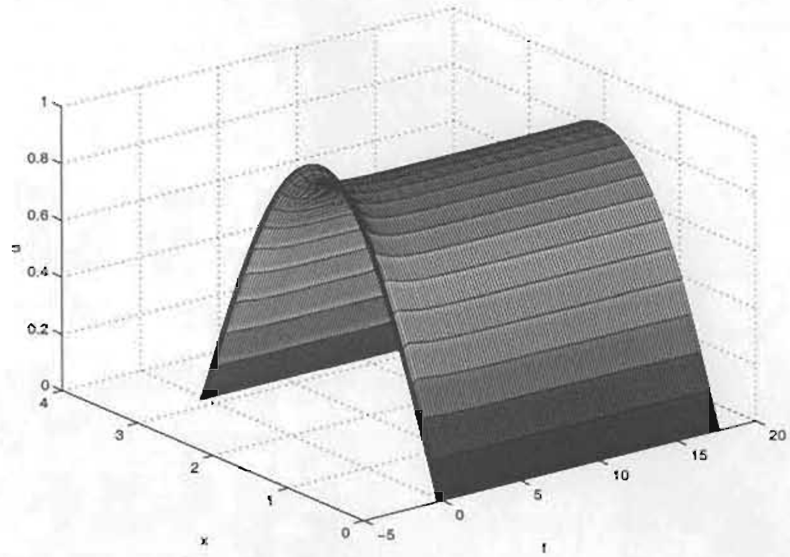


Figure 3.10: The solution of Example 2 with the same condition and parameters as in Figure 3.7, except for $\tau = 0.3$, $\phi(t, x) = 3 \sin x$.



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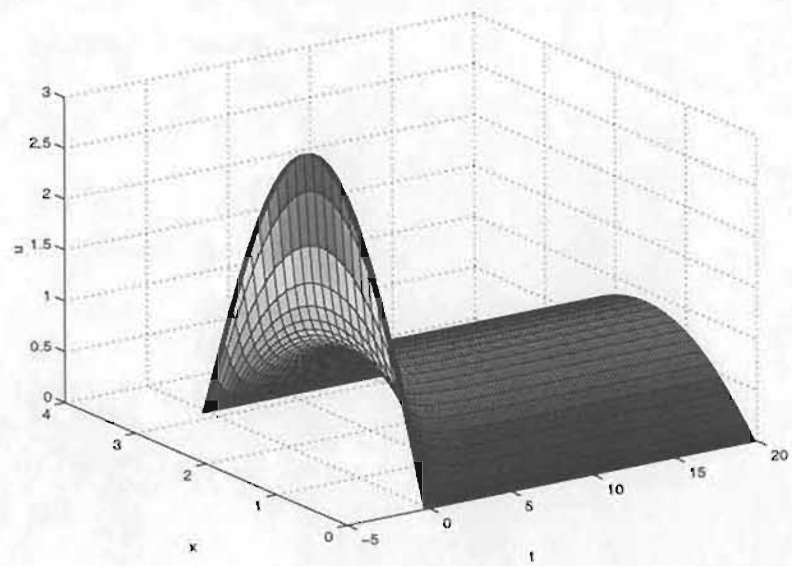

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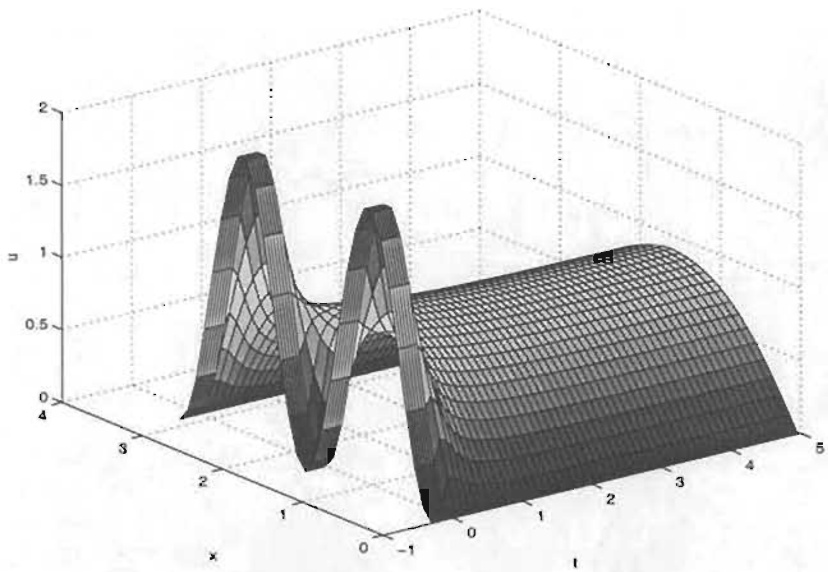


Figure 3.11: The solution of Example 2 with the same condition and parameters as in Figure 3.7, except for $\tau = 0.3$, $\phi(t, x) = 1 - \cos 4x$.

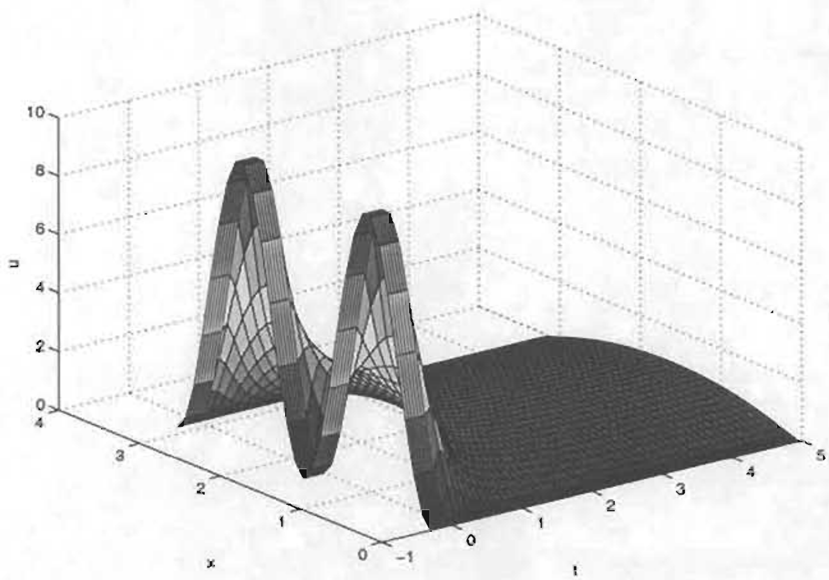
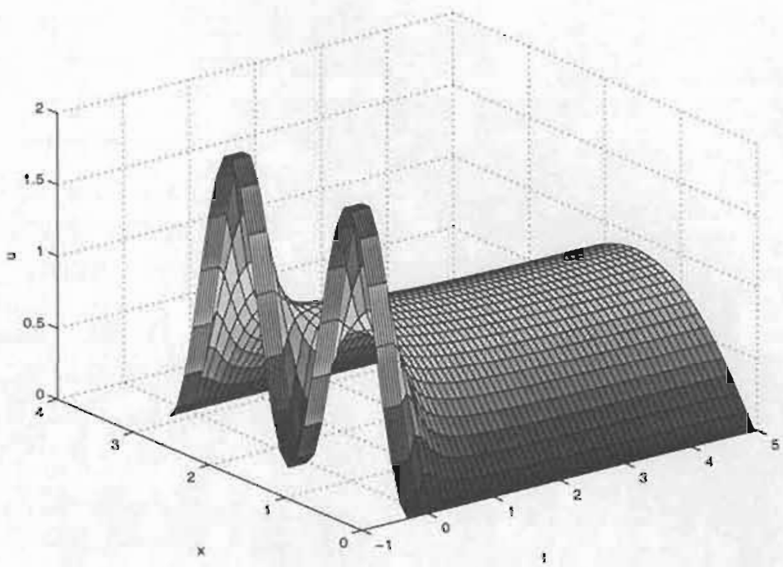


Figure 3.12: The solution of Example 2 with the same condition and parameters as in Figure 3.7, except for $\tau = 0.3, \phi(t, x) = 5 - 5\cos 4x$.



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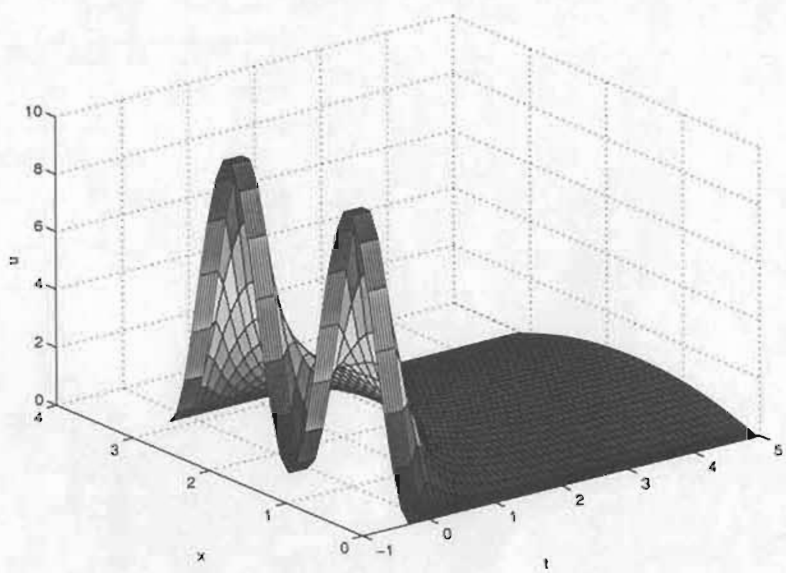

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Chapter 4

Bistable Traveling Waves in an Epidemic Model

Traveling wave solutions are important in epidemic models when investigating the geographic spread of infectious diseases (e.g. [68]). This chapter will focus on bistable traveling waves in an epidemic model proposed by Capasso et al., which models man-environment-man epidemics, while Chapter 5 will be involved in monostable traveling waves in the integral version of the model. In this chapter, the existence, uniqueness up to translation and global exponential stability with phase shift of bistable traveling waves are established. The methods involve phase plane investigation, monotone semiflow approaches and spectrum analysis.

The organization of this chapter is as follows. In Section 4.1, we provide an introduction to the epidemic model and a review of the works related to the model and the methods for studying the existence and the stability of traveling waves. In Section 4.2, we establish the existence of bistable waves for the model by a qualitative analysis of a three dimensional ordinary differential system. In Section 4.3, a convergence theorem for monotone semiflows is employed to prove the global

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4.1 Introduction

The geographic spread of infectious diseases is an important subject in mathematical epidemiology. To model the cholera epidemic which spread in the European Mediterranean regions in 1973, Capasso and Paveri-Fontana [19] proposed a system of two ordinary differential equations. As a basic feature, this model involves a positive feedback interaction between the infective human population and the concentration of bacteria. The human population, once infected, has a contribution to the growth rate of bacteria, which is then returned to the environment to increase the infection rate of humans. This kind of mechanism seems to be appropriate to interpret other fecally-orally transmitted epidemics such as typhoid fever, infectious hepatitis, polyometitis etc., with suitable modifications. Under the assumption that the bacteria disperse randomly while the small mobility of the infective human population is neglected, Capasso and Maddalena [17] further obtained a reaction-diffusion system

$$\begin{cases} \frac{\partial}{\partial t}U_1(x,t) = d\frac{\partial^2}{\partial x^2}U_1(x,t) - a_{11}U_1(x,t) + a_{12}U_2(x,t), \\ \frac{\partial}{\partial t}U_2(x,t) = -a_{22}U_2(x,t) + g(U_1(x,t)). \end{cases} \tag{4.1.1}$$

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Here d, a_{11}, a_{12} and a_{22} are positive constants, $U_1(x,t)$ and $U_2(x,t)$, respectively, denote the spatial densities of infectious agent and the infective human population at a point x in the habitat at time $t \geq 0$, $1/a_{11}$ is the mean lifetime of the agent in the environment, $1/a_{22}$ is the mean infectious period of the human infections, a_{12} is the multiplicative factor of the mean infectious agent due to the human populations, and the function g is the infection rate of humans under the assumption that total susceptible human population is constant during the evolution of the epidemic. Note that the second equation of system (4.1.1) has no diffusion terms. That is why we can not directly apply any results in reference when dealing with the existence and the linear stability of traveling waves.

System (4.1.1) and its corresponding reaction system have received extensive investigations. For example, the case in which there is at most one nontrivial endemic equilibrium was studied in [19, 17, 16, 15], and it is known that above some parameter threshold a unique nontrivial state exists and all epidemic outbreaks tend to it (i.e., monostable case), below the parameter threshold, all epidemics tend to extinction. In [18], the bistable case (where the corresponding reaction system of (4.1.1) admits exactly two nontrivial steady states) was obtained by assuming that the infection rate g is sigma-shaped. A saddle point structure was obtained in [18] for (4.1.1) with Neumann boundary conditions and its reaction system, and a complete analysis of the steady states of (4.1.1) subject to Dirichlet boundary conditions and numerical simulations were presented in [20]. It was shown in [52] that system (4.1.1) subject to Dirichlet boundary conditions also admits saddle point behavior.

Recently, the existence of monotone traveling waves and the minimal wave speed were established in [97] for system (4.1.1) in the monostable case. Moreover, it was

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Recently, the existence of monotone traveling waves and the minimal wave speed were established in [97] for system (4.1.1) in the monostable case. Moreover, it was proven in [83] that this minimal wave speed coincides with the asymptotic speed of spread for solutions with initial functions having compact supports. The purpose of this chapter is to study the existence, uniqueness and global exponential stability of traveling waves of system (4.1.1) with bistable nonlinearity.

Various approaches exist for proving the existence of wave solutions of parabolic equations, ranging from topological methods ([38, 39, 23]) to shooting methods based on Wazewski's principle ([29]). For scalar bistable evolution equations, the existence, uniqueness and global exponential stability of traveling waves are well known (see, e.g., [35, 21] for reaction-diffusion equations, [71, 75] for time-delayed reaction-diffusion equations). For quasi-monotone parabolic systems with positive diffusion coefficients, monotone traveling waves were proven to exist via topological methods ([85]). Also by topological methods, the existence and uniqueness of bistable traveling waves were obtained in [67] and [51], respectively, for a reaction-diffusion model of n mutualist species, in which all diffusion coefficients were assumed to be positive.

A standard method to study the local stability of traveling waves is to use the linearization at the waves under study. The stability then splits into two steps. The first step is to prove that the linear stability implies the nonlinear stability. That is, proving that the stability for the linearized system implies the stability for the full nonlinear system. The general results can be found in [46, 12] and references therein. The second step is to analyze the linearized equations. All the information needed is about the spectrum of the corresponding linear operator. This is the key issue for the stability problem. For FitzHugh-Nagumo equations, the spectrum analysis [53] shows that traveling waves are stable. For quasi-monotone parabolic systems proven in [83] that this minimal wave speed coincides with the asymptotic speed of spread for solutions with initial functions having compact supports. The purpose of this chapter is to study the existence, uniqueness and global exponential stability of traveling waves of system (4.1.1) with bistable nonlinearity.

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Evans did a series of works ([30, 31, 32, 33]) for an evolution system of nerve axon equations, where a reaction-diffusion equation is coupled with n ordinary differential equations. In [30], he completed the first step, and the main result in [32], in fact, states that the linearized equations are stable if all spectrum points of the linear operator except for zero lie in an appropriate negative half-plane of the complex plane, and zero is a simple eigenvalue. It then follows that the local stability of bistable waves of system (4.1.1) reduces to the spectral analysis of the linear operator associated with the linearization at the wave profile. That is just what we will do.

4.2 Existence of Traveling Waves

Since we are interested in the bistable case of system (4.1.1), throughout the whole chapter we make the following assumption on the function g.

(R1) $g \in C^2(\mathbb{R}_+), g(0) = 0, g'(0) \geq 0, g'(z) > 0, \forall z > 0, \lim_{z\to\infty} g(z) = 1$, and there is a $\xi > 0$ such that $g''(z) > 0$ for $z \in (0, \xi)$ and $g''(z) < 0$ for $z > \xi$.

Mathematically, we can rescale system (4.1.1) and only study the rescaled system

$$\begin{cases} \frac{\partial}{\partial t}U_1(x,t) = d\frac{\partial^2}{\partial x^2}U_1(x,t) - U_1(x,t) + \alpha U_2(x,t), \\ \frac{\partial}{\partial t}U_2(x,t) = -\beta U_2(x,t) + g(U_1(x,t)), \end{cases} \tag{4.2.2}$$

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where $\alpha = a_{12}/a_{11}^2, \beta = a_{22}/a_{11}$.

Let $\gamma = \beta/\alpha$. Note that the global dynamics of the cooperative system

$$\begin{cases} \dot{U}_1(t) = -U_1(t) + \alpha U_2(t) \\ \dot{U}_2(t) = -\beta U_2(t) + g(U_1(t)) \end{cases} \tag{4.2.3}$$

has been described in detail ([18, 15, 20]). In particular, the following results are known.

Proposition 4.2.1 *There exists $\gamma_{crit} > 0$ such that*

(i) For $\gamma > \gamma_{crit}$, $(0,0) \in \mathbb{R}^2$ is the only equilibrium for system (4.2.3). It is globally asymptotically stable in the positive quadrant of $\mathbb{R}^2$;

(ii) For $\gamma = \gamma_{crit}$ or $0 < \gamma \leq g'(0)$ in the case of $g'(0) > 0$, system (4.2.3) admits a unique nontrivial equilibrium in addition to $(0,0)$;

(iii) For $g'(0) < \gamma < \gamma_{crit}$, system (4.2.3) has three equilibria in the first quadrant of $\mathbb{R}^2$: $E^- = (0,0), E^0 = (a, a/\alpha), E^+ = (b, b/\alpha)$, where $0 < a < b$ are the two positive roots of $g(u) = \frac{\beta}{\alpha}u$. In this case, E^0 is a saddle point, E^- and E^+ are stable nodes.

In order to discuss the existence of bistable waves for (4.2.2), i.e., traveling waves connecting two stable equilibria, we further assume $g'(0) < \gamma < \gamma_{crit}$. See Figure 4.1 for an illustration of three equilibria.

Let $(U_1(x,t), U_2(x,t)) = (u_1(x+ct), u_2(x+ct))$ be a traveling wave solution of (4.2.2). Then the wave front profile $(u_1(\tau), u_2(\tau))$ satisfies the ODE system

$$\begin{cases} cu_1'(\tau) = du_1''(\tau) - u_1(\tau) + \alpha u_2(\tau), \\ cu_2'(\tau) = -\beta u_2(\tau) + g(u_1(\tau)), \end{cases} \tag{4.2.4}$$

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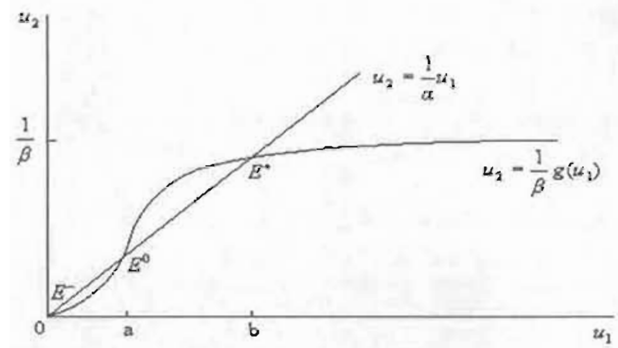


Figure 4.1: Illustration of the three equilibria E^-, E^0 and E^+.

where $'$ denotes the derivatives $\frac{d}{d\tau}$. Since we are interested in traveling wave fronts connecting E^- and E^+, we impose the asymptotic boundary conditions on the system

$$u_1(-\infty) = u_1'(-\infty) = u_2(-\infty) = 0. \tag{4.2.5}$$

Consider the case where $c \neq 0$. By the second equation of system (4.2.4), we have

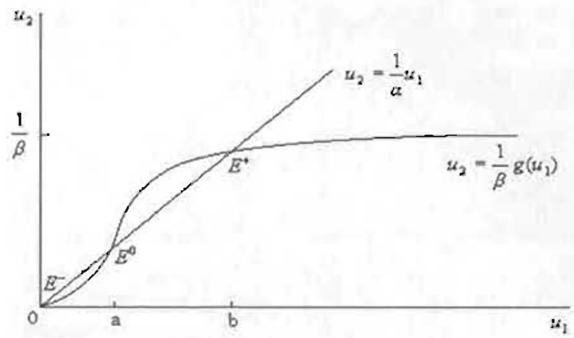
$$u_2(\tau) = e^{-\frac{\beta}{c}(\tau-\tau_0)}u_2(\tau_0) + \frac{1}{c}\int_{\tau_0}^{\tau} e^{-\frac{\beta}{c}(\tau-s)}g(u_1(s))ds.$$

Note that, as $\tau \to -\infty$, $u_2(\tau)$ and $g(u_1(\tau))$ are bounded. By taking $\tau_0 \to -\infty$, we obtain

$$u_2(\tau) = \frac{1}{c}\int_{-\infty}^{\tau} e^{-\frac{\beta}{c}(\tau-s)}g(u_1(s))ds = \frac{1}{c}\int_{-\infty}^{0} e^{\frac{\beta}{c}s}g(u_1(\tau+s))ds, \forall \tau \in \mathbb{R}. \tag{4.2.6}$$

Therefore, if $u_1(\tau)$ is increasing with

$$u_1(-\infty) = 0, \quad u_1(+\infty) = b, \tag{4.2.7}$$



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Therefore, if $u_1(\tau)$ is increasing with

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then $u_2(\tau)$, defined by formula (4.2.6), is also increasing and satisfies

$$u_2(-\infty) = 0, \quad u_2(+\infty) = b/\alpha.$$

Consequently, it suffices to consider positive and increasing solutions $u_1(\tau)$ of system (4.2.4) subject to the boundary conditions (4.2.7).

In the case of $c \neq 0$ in system (4.2.4), let $u_3(\tau) = u_1'(\tau)$. Then system (4.2.4) is equivalent to

$$\begin{cases} u_1'(\tau) = u_3(\tau) \\ u_2'(\tau) = \frac{1}{c}(-\beta u_2(\tau) + g(u_1(\tau))), \\ u_3'(\tau) = \frac{1}{d}(cu_3(\tau) + u_1(\tau) - \alpha u_2(\tau)). \end{cases} \tag{4.2.8}$$

Obviously, system (4.2.8) admits three equilibria: $(E^-, 0), (E^0, 0)$ and $(E^+, 0)$. The Jacobian matrix of (4.2.8) is

$$J = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{c}g'(z) & -\frac{\beta}{c} & 0 \\ \frac{1}{d} & -\frac{\alpha}{d} & \frac{c}{d} \end{pmatrix}.$$

Let $f(\lambda, m) := (\lambda + \frac{\beta}{c})(-\lambda^2 + \frac{c}{d}\lambda + \frac{1}{d}) - m$. Then, at the point $(E^-, 0)$, the eigenvalues of J are given by the roots of $f(\lambda, \frac{\alpha}{cd}g'(0)) = 0$. Note that $f(\lambda, \frac{\beta}{cd})$ admits three real zero points: $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 > 0$, and $f(\lambda, 0)$ also has three zero points: two negative and one positive. By $0 \leq g'(0) < \frac{\beta}{\alpha}$, it follows that at $(E^-, 0)$, J admits a positive eigenvalue $\lambda(c)$ and two negative eigenvalues. Therefore, system (4.2.8) has a one dimensional unstable manifold corresponding to $\lambda(c)$ at $(0, 0, 0)$. Denote by $\mathcal{U}_c$ this manifold. Note that $(1, g'(0)/(\beta + c\lambda(c)), \lambda(c))$ is an eigenvector then $u_2(\tau)$, defined by formula (4.2.6), is also increasing and satisfies

$$u_2(-\infty) = 0, \quad u_2(+\infty) = b/\alpha.$$

Consequently, it suffices to consider positive and increasing solutions $u_1(\tau)$ of system (4.2.4) subject to the boundary conditions (4.2.7).

In the case of $c \neq 0$ in system (4.2.4), let $u_3(\tau) = u_1'(\tau)$. Then system (4.2.4) is equivalent to

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Lemma 4.2.1 *Assume that $c \neq 0$. Then system (4.2.4)-(4.2.5) has exactly one positive solution on $\mathcal{U}_c$ (up to translations). For sufficiently large negative τ, this solution satisfies*

$$u_1'(\tau) = u_3(\tau) = \lambda(c)u_1(\tau) + O(u_1(\tau)),$$
$$u_2(\tau) = \frac{g'(0)}{\beta + c\lambda(c)}u_1(\tau) + O(u_1(\tau)).$$

Remark 4.2.1 If $g'(0) = 0$, we assume that $g''(0) \neq 0$. Then $u_2(\tau)$ in Lemma 4.2.1 can be approximated by

$$u_2(\tau) = \frac{g''(0)}{2\beta + 4c\lambda(c)}u_1(\tau) + O(u_1^2(\tau)), \quad (\tau \to -\infty).$$

In the case where $c = 0$, system (4.2.4) is equivalent to

$$dy_1''(\tau) - y_1(\tau) + \frac{\alpha}{\beta}g(y_1(\tau)) = 0$$

or

$$y_1'(\tau) = y_3(\tau), \quad y_3'(\tau) = \frac{1}{d}(y_1(\tau) - \frac{\alpha}{\beta}g(y_1(\tau))) \tag{4.2.9}$$

with boundary conditions $y_1(-\infty) = y_3(-\infty) = 0$. In what follows, we are only interested in positive and increasing solutions $u_1(\tau)$ and $y_1(\tau)$ of (4.2.4) or (4.2.8) and (4.2.9), respectively. As long as $y_1'(\tau) \geq 0$, for the trajectory $\Psi(\eta) := y_3(y_1^{-1}(\eta))$ we have the following graph equation in the (y_1, y_3) phase space

$$\dot{\Psi}(\eta) = \frac{\eta - \frac{\alpha}{\beta}g(\eta)}{d\,\Psi(\eta)} \qquad \text{for } \eta > 0. \tag{4.2.10}$$

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In the case where $c \neq 0$, as long as $u_1'(\tau) \geq 0$, for $V(\eta) := u_1'(u_1^{-1}(\eta))$ $(\eta = u_1(\tau))$ we have

$$\dot{V}(\eta) = \frac{c}{d} + \frac{\eta - \alpha u_2(\tau)}{d\,V(\eta)} \qquad \text{for } \eta > 0, \tag{4.2.11}$$

where $u_2(\tau) = u_2(u_1^{-1}(\eta))$. For the solutions of (4.2.11) associated with the trajectory for (4.2.4)-(4.2.5), the boundary conditions (4.2.5) and Lemma 4.2.1 provide

$$V(0+) = 0, \qquad \dot{V}(0+) = \lambda(c). \tag{4.2.12}$$

Our proofs involve continuous "switching" between solutions for the graph equation (4.2.11) and the original system (4.2.4) or (4.2.8). So we first give the following lemma on some general properties of trajectories $V(\eta)$ with (4.2.12), which will be frequently used. Let $u(\tau) = (u_1(\tau), u_2(\tau), u_3(\tau))$ be the solution of system (4.2.8) associated with $V(\eta)$.

Lemma 4.2.2 *Let $c > 0$. Then the following statements hold.*

(i) $V(\eta) > 0$, and $\dot{V}(\eta) \geq \frac{c}{d} > 0$ for $\eta \in (0, a]$.

(ii) Let $\bar{\eta} = \inf\{\eta \in (0, b] : V(\eta) = 0\}$. Then $\bar{\eta} > a$, and $\bar{\eta} < b$ implies that $\lim\limits_{\eta \nearrow \bar{\eta}} \frac{V(\eta)}{\eta - \bar{\eta}} = -\infty$.

Proof. (i) As long as $V(\eta)$ is well defined (i.e., $u_1'(\tau) = u_3(\tau) \geq 0$), it follows from (4.2.6) that $u_2(\tau) \leq \frac{1}{\beta} g(u_1(\tau))$ for $u_1(\tau) \in (0, b]$. Therefore, for $u_3(\tau) > 0$ and $u_1(\tau) \in (0, a]$, we have $u_2(\tau) \leq \frac{1}{\beta} g(u_1(\tau)) \leq \frac{1}{\alpha} u_1(\tau)$ (see Figure 4.1). We claim that $u_3'(\tau) > 0$ as long as $u_1(\tau) \in (0, a]$. Indeed, Lemma 4.2.1 implies that $u_3(\tau) > 0$ for small positive $u_1(\tau)$. It follows that for small $u_1(\tau)$, there holds

$$u_3'(\tau) = \frac{1}{d}(c u_3(\tau) + u_1(\tau) - \alpha u_2(\tau)) \geq \frac{1}{d} c u_3(\tau) > 0.$$

In the case where $c \neq 0$, as long as $u_1'(\tau) \geq 0$, for $V(\eta) := u_1'(u_1^{-1}(\eta))$ $(\eta = u_1(\tau))$ we have

$$\dot{V}(\eta) = \frac{c}{d} + \frac{\eta - \alpha u_2(\tau)}{d V(\eta)} \qquad \text{for } \eta > 0, \tag{4.2.11}$$

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Our proofs involve continuous "switching" between solutions for the graph equation (4.2.11) and the original system (4.2.4) or (4.2.8). So we first give the following lemma on some general properties of trajectories $V(\eta)$ with (4.2.12), which will be frequently used. Let $u(\tau) = (u_1(\tau), u_2(\tau), u_3(\tau))$ be the solution of system (4.2.8) associated with $V(\eta)$.

Lemma 4.2.2 *Let $c > 0$. Then the following statements hold.*

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Proof. (i) As long as $V(\eta)$ is well defined (i.e., $u_1'(\tau) = u_3(\tau) \geq 0$), it follows from (4.2.6) that $u_2(\tau) \leq \frac{1}{\beta} g(u_1(\tau))$ for $u_1(\tau) \in (0, b]$. Therefore, for $u_3(\tau) > 0$ and $u_1(\tau) \in (0, a]$, we have $u_2(\tau) \leq \frac{1}{\beta} g(u_1(\tau)) \leq \frac{1}{\alpha} u_1(\tau)$ (see Figure 4.1). We claim that $u_3'(\tau) > 0$ as long as $u_1(\tau) \in (0, a]$. Indeed, Lemma 4.2.1 implies that $u_3(\tau) > 0$ for small positive $u_1(\tau)$. It follows that for small $u_1(\tau)$, there holds

$$u_3'(\tau) = \frac{1}{d}(c u_3(\tau) + u_1(\tau) - \alpha u_2(\tau)) \geq \frac{1}{d} c u_3(\tau) > 0.$$

Suppose, by contradiction, that $\tau_0 \in \mathbb{R}$ is the first point such that $u_3'(\tau_0) = 0$ and $u_1(\tau_0) \in (0, a]$. Then

$$\frac{1}{d}(cu_3(\tau_0) + u_1(\tau_0) - \alpha u_2(\tau_0)) = 0, \text{ and } u_3(\tau_0) > 0.$$

Therefore, $u_1(\tau_0) < \alpha u_2(\tau_0)$, and hence $u_1(\tau_0) > a$, a contradiction. It then follows that $u_3(\tau) > 0$ and $\alpha u_2(\tau) \leq u_1(\tau)$ as long as $u_1(\tau) \in (0, a]$. Thus, for $\eta \in (0, a]$, we have $V(\eta) > 0$ and $\dot{V}(\eta) = \frac{c}{d} + \frac{u_1(\tau) - \alpha u_2(\tau)}{dV(\eta)} \geq \frac{c}{d} > 0$, where $\tau = u_1^{-1}(\eta)$.

(ii) Clearly, $\bar{\eta} > a$. Suppose that $\bar{\eta} < b$. Then $u_3(\bar{\tau}) = 0$ and $u_3'(\bar{\tau}) \leq 0$, where $\bar{\tau} = u_1^{-1}(\bar{\eta})$. We claim that $u_3'(\bar{\tau}) < 0$. Suppose, by contradiction, that $u_3'(\bar{\tau}) = 0$. Then $u_1(\bar{\tau}) - \alpha u_2(\bar{\tau}) = 0$, and $\bar{\tau} < +\infty$. Moreover, we can choose a small $\varepsilon > 0$ such that for $\tau \in (\bar{\tau} - \varepsilon, \bar{\tau})$, $u_3'(\tau) \leq 0$ and $u_3(\tau) > 0$, and hence, $u_1(\tau) - \alpha u_2(\tau) < 0$. Using (4.2.6) and the fact that $u_1'(\tau) > 0$ for $\tau < \bar{\tau}$, we then have

$$\begin{aligned} 0 \leq u_1'(\bar{\tau}) - \alpha u_2'(\bar{\tau}) &= u_3(\bar{\tau}) - \tfrac{\alpha}{c}(-\beta u_2(\bar{\tau}) + g(u_1(\bar{\tau}))) \\ &= -\tfrac{\alpha}{c}(-\beta u_2(\bar{\tau}) + g(u_1(\bar{\tau}))) < 0, \end{aligned}$$

a contradiction. Thus $u_3'(\bar{\tau}) < 0$, and hence $\lim\limits_{\eta \nearrow \bar{\eta}} \frac{V(\eta)}{\eta - \bar{\eta}} = \lim\limits_{\tau \nearrow \bar{\tau}} \frac{u_3'(\tau)}{u_1'(\tau)} = \lim\limits_{\tau \nearrow \bar{\tau}} \frac{u_3'(\tau)}{u_3(\tau)} = -\infty$.

∎

Assume that $\int_0^b (\frac{\alpha}{\beta} g(z) - z) dz > 0$. Let

$$N_k = \{(\eta, \zeta) \in \mathbb{R}^2 : \zeta^2/2 + F(\eta) = k\} \setminus \{(0,0), (a,0), (b,0)\},$$

where $F(\eta) = \frac{1}{d}\int_0^\eta (\frac{\alpha}{\beta} g(z) - z) dz$. Since $z > \frac{\alpha}{\beta} g(z)$ for $z \in (0, a)$ in the case of $g'(0) < \gamma < \gamma_{crit}$ (see Figure 4.1), $k_0 = F(a) < 0$. Note that N_k are exactly the trajectories of solutions to system (4.2.9), and $k \geq 0$ gives exactly the trajectories intersecting ζ-axis. For $k \geq 0$, we define $N_k^+ = N_k \cap \{(\eta, \zeta) \in \mathbb{R}^2 : \zeta > 0\}$ and

Suppose, by contradiction, that $\tau_0 \in \mathbb{R}$ is the first point such that $u_3'(\tau_0) = 0$ and $u_1(\tau_0) \in (0, a]$. Then

$$\frac{1}{d}(cu_3(\tau_0) + u_1(\tau_0) - \alpha u_2(\tau_0)) = 0, \text{ and } u_3(\tau_0) > 0.$$

Therefore, $u_1(\tau_0) < \alpha u_2(\tau_0)$, and hence $u_1(\tau_0) > a$, a contradiction. It then follows that $u_3(\tau) > 0$ and $\alpha u_2(\tau) \le u_1(\tau)$ as long as $u_1(\tau) \in (0, a]$. Thus, for $\eta \in (0, a]$, we have $V(\eta) > 0$ and $\dot{V}(\eta) = \frac{c}{d} + \frac{u_1(\tau) - \alpha u_2(\tau)}{dV(\eta)} \ge \frac{c}{d} > 0$, where $\tau = u_1^{-1}(\eta)$.

(ii) Clearly, $\bar{\eta} > a$. Suppose that $\bar{\eta} < b$. Then $u_3(\bar{\tau}) = 0$ and $u_3'(\bar{\tau}) \le 0$, where $\bar{\tau} = u_1^{-1}(\bar{\eta})$. We claim that $u_3'(\bar{\tau}) < 0$. Suppose, by contradiction, that $u_3'(\bar{\tau}) = 0$. Then $u_1(\bar{\tau}) - \alpha u_2(\bar{\tau}) = 0$, and $\bar{\tau} < +\infty$. Moreover, we can choose a small $\varepsilon > 0$ such that for $\tau \in (\bar{\tau} - \varepsilon, \bar{\tau})$, $u_3'(\tau) \le 0$ and $u_3(\tau) > 0$, and hence, $u_1(\tau) - \alpha u_2(\tau) < 0$. Using (4.2.6) and the fact that $u_1'(\tau) > 0$ for $\tau < \bar{\tau}$, we then have

$$\begin{aligned} 0 \le u_1'(\bar{\tau}) - \alpha u_2'(\bar{\tau}) &= u_3(\bar{\tau}) - \tfrac{\alpha}{c}(-\beta u_2(\bar{\tau}) + g(u_1(\bar{\tau}))) \\ &= -\tfrac{\alpha}{c}(-\beta u_2(\bar{\tau}) + g(u_1(\bar{\tau}))) < 0, \end{aligned}$$

a contradiction. Thus $u_3'(\bar{\tau}) < 0$, and hence $\lim\limits_{\eta \nearrow \bar{\eta}} \frac{V(\eta)}{\eta - \bar{\eta}} = \lim\limits_{\tau \nearrow \bar{\tau}} \frac{u_3'(\tau)}{u_1'(\tau)} = \lim\limits_{\tau \nearrow \bar{\tau}} \frac{u_3'(\tau)}{u_3(\tau)} = -\infty$.
∎

Assume that $\int_0^b (\frac{\alpha}{\beta} g(z) - z)dz > 0$. Let

$$N_k = \{(\eta, \zeta) \in \mathbb{R}^2 : \zeta^2/2 + F(\eta) = k\} \setminus \{(0,0), (a,0), (b,0)\},$$

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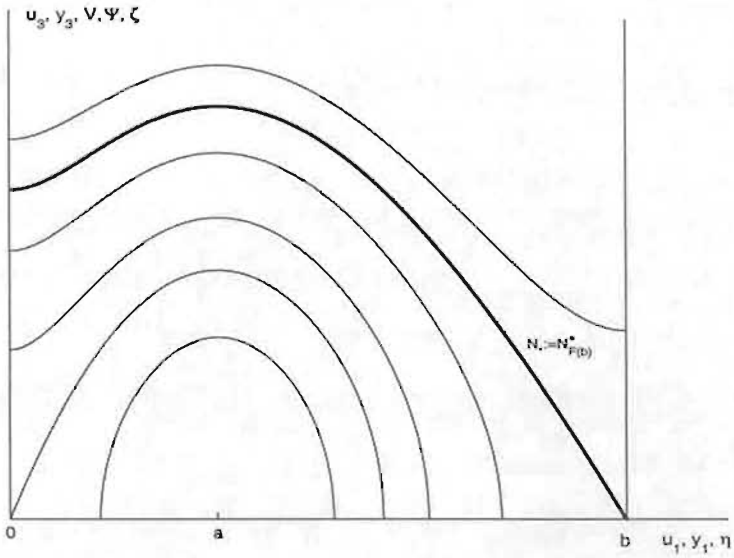


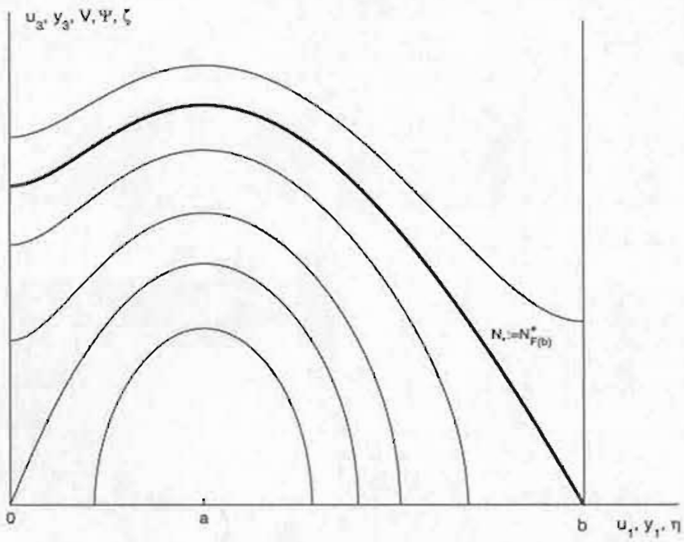
Figure 4.2: Phase portrait of (4.2.9).

Lemma 4.2.3 *For $c > 0$ and $u_3(\tau) > 0$, $(u_1(\tau), u_3(\tau))$ crosses through increasing level sets N_k^+ with increasing τ whenever $u_1(\tau) \in (0, b)$. That is, $V_c(\eta)$ intersects a level set N_k^+ at most once for $\eta \in (0, b)$. Furthermore, at the point of intersection $(\eta_k, V_c(\eta_k)) = (\eta_k, \Psi_k(\eta_k))$, there holds $V_c'(\eta_k) \geq \frac{c}{d} + \Psi_k'(\eta_k)$.*

Proof. Note that $u_2(\tau) \leq \frac{1}{\beta} g(u_1(\tau))$ whenever $u_1(\tau) \in (0, b)$. We then have

$$\begin{aligned} \frac{d}{d\tau} k &= \frac{d}{d\tau}\left(\frac{1}{2} u_3^2(\tau) + F(u_1(\tau))\right) = u_3(\tau) u_3'(\tau) + \frac{1}{d}\left(\frac{\alpha}{\beta} g(u_1(\tau)) - u_1(\tau)\right) u_1'(\tau) \\ &= \frac{c}{d} u_3^2(\tau) + \frac{1}{d} u_3(\tau)\left(\frac{\alpha}{\beta} g(u_1(\tau)) - \alpha u_2(\tau)\right) > 0. \end{aligned}$$

denote by $\Psi_k(\eta)$ the solution of equation (4.2.10) in N_k^+. Let $V_c(\eta)$ be the solution of equations (4.2.11)-(4.2.12) with the velocity of c, and $u_c(\tau) = (u_1(\tau), u_2(\tau), u_3(\tau))$ be the solution of system (4.2.8) with (4.2.5) corresponding to $V_c(\eta)$. Then we have the following result on the relationship between Ψ_k and V_c or u_c.



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Proof. Note that $u_2(\tau) \leq \frac{1}{\beta} g(u_1(\tau))$ whenever $u_1(\tau) \in (0, b)$. We then have

$$\begin{aligned} \frac{d}{d\tau} k &= \frac{d}{d\tau}\left(\frac{1}{2}u_3^2(\tau) + F(u_1(\tau))\right) = u_3(\tau)u_3'(\tau) + \frac{1}{d}\left(\frac{\alpha}{\beta} g(u_1(\tau)) - u_1(\tau)\right)u_1'(\tau) \\ &= \frac{c}{d}u_3^2(\tau) + \frac{1}{d}u_3(\tau)\left(\frac{\alpha}{\beta} g(u_1(\tau)) - \alpha u_2(\tau)\right) > 0. \end{aligned}$$

This proves the first part of the results. The second part follows from a direct computation

$$V_c'(\eta_k) = \frac{c}{d} + \frac{\eta_k - \alpha u_2(\tau_k)}{V_c(\eta_k)} \geq \frac{c}{d} + \frac{\eta_k - \frac{\alpha}{\beta}g(\eta_k)}{\Psi_k(\eta_k)} = \frac{c}{d} + \Psi_k'(\eta_k).$$

∎

Now we are ready to prove the main result of this section.

Theorem 4.2.1 *Assume that* $g'(0) < \gamma < \gamma_{crit}$*. Then there exists a wave speed* c^* *such that system (4.2.2) has a nontrivial strictly monotone wave solution connecting* E^- *and* E^+*, and* c^* *has the same sign as* $\int_0^b \left(\frac{\alpha}{\beta}g(z) - z\right) dz$*. Moreover,* $c^* = 0$ *if and only if the integral vanishes.*

Proof. Without loss of generality, we assume that

$$\int_0^b \left(\frac{\alpha}{\beta}g(z) - z\right) dz \geq 0. \tag{4.2.13}$$

Otherwise, by the change of variables $V_1 = b - U_1, V_2 = \frac{1}{\alpha}b - U_2$, we can transform the original system (4.2.2) into

$$\begin{cases} \frac{\partial}{\partial t}V_1(x,t) = d\frac{\partial^2}{\partial x^2}V_1(x,t) - V_1(x,t) + \alpha V_2(x,t), \\ \frac{\partial}{\partial t}V_2(x,t) = G(V_1(x,t)) - \beta V_2(x,t), \end{cases}$$

where $G(z) = g(b) - g(b-z)$. Since $g(b) = \frac{\beta}{\alpha}b$, $G(z)$ satisfies assumption (R1) on $[0, b)$ and (4.2.13).

If $c^* = 0$, the heteroclinic orbit of system (4.2.9) implies that the integral (4.2.13) vanishes. Conversely, if the integral vanishes, there is a nontrivial wave solution of the ODE system (4.2.4) with velocity $c = 0$. Therefore, we restrict ourselves to the positive integral.

This proves the first part of the results. The second part follows from a direct computation

$$V_c'(\eta_k) = \frac{c}{d} + \frac{\eta_k - \alpha u_2(\tau_k)}{V_c(\eta_k)} \geq \frac{c}{d} + \frac{\eta_k - \frac{\alpha}{\beta}g(\eta_k)}{\Psi_k(\eta_k)} = \frac{c}{d} + \Psi_k'(\eta_k).$$

∎

Now we are ready to prove the main result of this section.

Theorem 4.2.1 *Assume that $g'(0) < \gamma < \gamma_{crit}$. Then there exists a wave speed c^* such that system (4.2.2) has a nontrivial strictly monotone wave solution connecting E^- and E^+, and c^* has the same sign as $\int_0^b \left(\frac{\alpha}{\beta}g(z) - z\right) dz$. Moreover, $c^* = 0$ if and only if the integral vanishes.*

Proof. Without loss of generality, we assume that

$$\int_0^b \left(\frac{\alpha}{\beta}g(z) - z\right) dz \geq 0. \tag{4.2.13}$$

Otherwise, by the change of variables $V_1 = b - U_1, V_2 = \frac{1}{\alpha}b - U_2$, we can transform the original system (4.2.2) into

$$\begin{cases} \frac{\partial}{\partial t}V_1(x,t) = d\frac{\partial^2}{\partial x^2}V_1(x,t) - V_1(x,t) + \alpha V_2(x,t), \\ \frac{\partial}{\partial t}V_2(x,t) = G(V_1(x,t)) - \beta V_2(x,t), \end{cases}$$

where $G(z) = g(b) - g(b-z)$. Since $g(b) = \frac{\beta}{\alpha}b$, $G(z)$ satisfies assumption (R1) on $[0, b)$ and (4.2.13).

If $c^* = 0$, the heteroclinic orbit of system (4.2.9) implies that the integral (4.2.13) vanishes. Conversely, if the integral vanishes, there is a nontrivial wave solution of the ODE system (4.2.4) with velocity $c = 0$. Therefore, we restrict ourselves to the positive integral.

Let $N_* := N^+_{F(b)}$ be the level curve through the critical point $(b, 0)$, and $\Psi_*(\eta) := \Psi_{F(b)}(\eta)$ be the corresponding solution. Define

$$E = \{c > 0 : V_c \text{ and } N_* \text{ intersect at } (\eta_c, V_c(\eta_c)) = (\eta_c, \Psi_*(\eta_c)) \text{ with } \eta_c \in (0, b]\}.$$

In the rest of the proof, we proceed with four steps.

Step 1. $E \neq \emptyset$.

For $c > 0$ and $\eta \in (0, a]$, we have $V_c(\eta) > 0$ and $\dot{V}_c(\eta) \geq c/d > 0$. If $\dot{V}_c(\eta) > 0$ for $\eta \in (a, b)$, then $V_c(\eta)$ must intersect with N_* on $(0, b]$. Suppose that $\dot{V}_c(\eta_0) = 0$ for $\eta_0 \in (a, b)$. Then

$$V_c(\eta_0) = \frac{1}{c}(\alpha u_2(\tau_0) - u_1(\tau_0)) > V_c(a) - V_c(0) = a\dot{V}(\eta_1),$$

where $\eta_1 \in (0, a), \tau_0 = u_1^{-1}(\eta_0)$. We then have $\alpha u_2(\tau_0) - u_1(\tau_0) > \frac{c^2}{d}a$. Note that $0 < u_2(\tau_0) \leq \frac{b}{\alpha}$. Thus, whenever $c^2 \geq \frac{bd}{a}$, we have $\dot{V}_c(\eta) > 0$ for $\eta \in (a, b)$. Then $V_c(\eta)$ intersects N_*. Thus, $E \neq \emptyset$.

Step 2. $\underline{c} := \inf E > 0$.

Let $m > 0$ be a constant. Consider the line $V = -m(\eta - b)$ $(\eta \in [0, b])$. If $V_c(\eta)$ intersects with this line, then, at the intersection we have

$$\dot{V}_c(\eta) = \frac{c}{d} - \frac{\eta - \alpha u_2}{md(\eta - b)} \leq \frac{c}{d} + \frac{b - \eta}{md(\eta - b)} = \frac{1}{d}(c - \frac{1}{m}).$$

For any sufficiently small $c > 0$, we can choose $m \in (0, -c/2 + \sqrt{1 + c^2/4})$ such that $c - 1/m < -m$. Then we must have $V_c(\eta_c) = 0$ for some $\eta_c \in (a, b]$. Thus, by Lemma 4.2.3, V_c does not intersect with N_* on $(0, b]$. Therefore, $\underline{c} > 0$.

Step 3. $\underline{c} \in E$.

Suppose, by contradiction, that $V_{\underline{c}}(\eta)$ does not intersect with N_*. Then Lemma 4.2.2 and 4.2.3 imply that $V_{\underline{c}}(\bar{\eta}) = 0$ for $\bar{\eta} \in (a, b]$. If $\bar{\eta} = b$, we are done.

Let $N_* := N^+_{F(b)}$ be the level curve through the critical point $(b,0)$, and $\Psi_*(\eta) := \Psi_{F(b)}(\eta)$ be the corresponding solution. Define

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Assume that $\bar{\eta} < b$. Let P_1 be a small plane in a small neighborhood of $(0,0,0)$ in (u_1, u_2, u_3) phase space, which is normal to the eigenvector $(1, m(\underline{c}), \lambda(\underline{c}))$ corresponding to the eigenvalue $\lambda(\underline{c})$, where $m(\underline{c}) = \frac{g'(0)}{\beta + \underline{c}\lambda(\underline{c})}$. By Lemma 4.2.1, the trajectory $u_{\underline{c}}(\tau)$ transversely intersects with P_1 at $I_{\underline{c}}$. By the local continuous dependence of $\mathcal{U}_{\underline{c}}$ on $\underline{c}$, for all c in a small neighborhood of $\underline{c}$, $u_c(\tau)$ transversely crosses through P_1 at I_c and $\lim_{c \to \underline{c}} I_c = I_{\underline{c}}$. Without loss of generality, we assume that $u_c(0) = I_c, u_{\underline{c}}(0) = I_{\underline{c}}$. Let $P_2 = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1, u_2 > 0, u_3 = 0\}$. Then, Lemma 4.2.2 implies that $u_{\underline{c}}(\tau)$ transversely intersects P_2 at $(\bar{\eta}, u_2(u_1^{-1}(\bar{\eta})), 0)$. By the continuous dependence of solutions on parameters and initial values, for all I_c in a sufficiently small neighborhood of $I_{\underline{c}}$, $u_c(\tau)$ transversely intersects P_2. Thus, we can choose a $c > \underline{c}$ such that $u_c(\tau)$ intersects P_2. That is, $V_c(\eta) = 0$ for some $\eta \in (a, b)$. By Lemma 4.2.3, $V_c(\eta)$ has no intersection points with N_*. Hence, $\underline{c} < c \notin E$, which contradicts the definition of $\underline{c}$. Therefore, $V_{\underline{c}}$ does intersect with N_*. That is, $\underline{c} \in E$.

Step 4. $\eta_{\underline{c}} = b$.

Suppose that $\eta_{\underline{c}} < b$. Let $P_3 = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : (u_1, u_3) \in N_*, u_2 > 0\}$. Then by Lemma 4.2.3, $u_{\underline{c}}(\tau)$ transversely intersects P_3 at $(\eta_{\underline{c}}, u_2(u_1^{-1}(\eta_{\underline{c}})), V_{\underline{c}}(\eta_{\underline{c}}))$. By the same argument as in Step 2, we obtain that, as $c \to \underline{c}$, $u_c(\tau)$ transversely intersects P_3 at $(\eta_c, u_2(u_1^{-1}(\eta_c)), V_c(\eta_c))$, and $\eta_c \to \eta_{\underline{c}} < b$. It follows that there exists a $\delta > 0$ such that $\underline{c} - \delta \in E$, which contradicts the definition of $\underline{c}$. ∎

Remark 4.2.2 *Note that*

$$\int_0^b \left(\frac{\alpha}{\beta}g(z) - z\right) dz = \alpha \left[\int_a^b \left(\frac{1}{\beta}g(z) - \frac{1}{\alpha}z\right) dz - \int_0^a \left(\frac{1}{\alpha}z - \frac{1}{\beta}g(z)\right) dz\right].$$

By Theorem 4.2.1, it then follows that there exists $\bar{\gamma} \in (g'(0), \gamma_{crit})$ such that $c^* > 0$ if $\gamma \in (g'(0), \bar{\gamma})$, and $c^* < 0$ if $\gamma \in (\bar{\gamma}, \gamma_{crit})$.

Assume that $\bar{\eta} < b$. Let P_1 be a small plane in a small neighborhood of $(0,0,0)$ in (u_1, u_2, u_3) phase space, which is normal to the eigenvector $(1, m(\underline{c}), \lambda(\underline{c}))$ corresponding to the eigenvalue $\lambda(\underline{c})$, where $m(\underline{c}) = \frac{g'(0)}{\beta + \underline{c}\lambda(\underline{c})}$. By Lemma 4.2.1, the trajectory $u_{\underline{c}}(\tau)$ transversely intersects with P_1 at $I_{\underline{c}}$. By the local continuous dependence of $\mathcal{U}_{\underline{c}}$ on $\underline{c}$, for all c in a small neighborhood of $\underline{c}$, $u_c(\tau)$ transversely crosses through P_1 at I_c and $\lim_{c \to \underline{c}} I_c = I_{\underline{c}}$. Without loss of generality, we assume that $u_c(0) = I_c, u_{\underline{c}}(0) = I_{\underline{c}}$. Let $P_2 = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1, u_2 > 0, u_3 = 0\}$. Then, Lemma 4.2.2 implies that $u_{\underline{c}}(\tau)$ transversely intersects P_2 at $(\bar{\eta}, u_2(u_1^{-1}(\bar{\eta})), 0)$. By the continuous dependence of solutions on parameters and initial values, for all I_c in a sufficiently small neighborhood of $I_{\underline{c}}$, $u_c(\tau)$ transversely intersects P_2. Thus, we can choose a $c > \underline{c}$ such that $u_c(\tau)$ intersects P_2. That is, $V_c(\eta) = 0$ for some $\eta \in (a, b)$. By Lemma 4.2.3, $V_c(\eta)$ has no intersection points with N_*. Hence, $\underline{c} < c \notin E$, which contradicts the definition of $\underline{c}$. Therefore, $V_{\underline{c}}$ does intersect with N_*. That is, $\underline{c} \in E$.

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4.3 Attractivity and Uniqueness

In this section, we discuss the global attractivity and uniqueness of traveling waves of system (4.2.2). For convenience, in the rest of this chapter we consider a more general quasi-monotone system

$$\begin{cases} \frac{\partial}{\partial t}U_1(x,t) = d\frac{\partial^2}{\partial x^2}U_1(x,t) + F^1(U_1(x,t),U_2(x,t)), \\ \frac{\partial}{\partial t}U_2(x,t) = -\beta U_2(x,t) + g(U_1(x,t)) := F^2(U_1(x,t),U_2(x,t)). \end{cases} \tag{4.3.14}$$

Assume that

(R2) There exists $l > 0$ such that $F^1 \in C^2((-l,\infty)^2,\mathbb{R})$, and $\frac{\partial}{\partial u_1}F^1(u_1,u_2) < 0$, $\frac{\partial}{\partial u_2}F^1(u_1,u_2) > 0$ for $(u_1,u_2) \in (-l,\infty)^2$.

(R3) $F^1(0,0) = 0$, and for any $l_2 \geq 1/\beta$, there exists $l_1 > 0$ such that $F^1(l_1,l_2) < 0$.

Without loss of generality, we may assume that the function g admits a smooth extension defined on $(-l,\infty)$ with $g'(z) \geq 0$ for $z \in (-l,0)$. In what follows, we use notations

$$F^i_j(u_1,u_2) := \frac{\partial}{\partial u_j}F^i(u_1,u_2), \quad F^i_{jk}(u_1,u_2) := \frac{\partial^2}{\partial u_j \partial u_k}F^i(u_1,u_2), \quad 1 \leq i,j,k \leq 2.$$

Consider the ODE system

$$\begin{cases} w_1'(t) = F^1(w_1(t),w_2(t)), \\ w_2'(t) = F^2(w_1(t),w_2(t)). \end{cases} \tag{4.3.15}$$

Because of our assumptions on F^1 and g, system (4.3.15) is cooperative on $\mathbb{R}^2_+$. Hence the comparison principle implies that every solution to (4.3.15) with nonnegative initial values remains nonnegative. By the standard comparison arguments, it easily follows that solutions of (4.3.15) on $\mathbb{R}^2_+$ are uniformly bounded and ultimately bounded. Thus, each solution of (4.3.15) with nonnegative initial values exists globally on $[0, \infty)$, and the solution semiflow of (4.3.15) is compact, point dissipative, and monotone on $\mathbb{R}^2_+$.

Obviously, E^- is an equilibrium of (4.3.15). We further assume that (4.3.15) admits two nonnegative equilibria in $\mathbb{R}^2_+$. With a little abuse of notations, we denote them by E^0 and E^+. Furthermore, suppose that $E^- \ll E^0 \ll E^+$, and $E^\pm$ are stable nodes and E^0 is a saddle point, where "$\ll$" means that the two vectors satisfy "$<$" elementwise. Let

$$[E^-, E^0] = \{w \in \mathbb{R}^2_+ : E^- \leq w \leq E^0\}$$

and

$$[E^0, \infty) = \{w \in \mathbb{R}^2_+ : E^0 \leq w\}.$$

By the Dancer-Hess connecting orbit lemma (see [25, Proposition 1]) and [72, Theorem 2.3.2], as applied to $[E^-, E^0]$ and $[E^0, \infty)$, respectively, it follows that

$$\lim_{t\to\infty} w(t, w_0) = E^- \text{ for } w_0 \in [E^-, E^0] \setminus \{E^0\},$$

and

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where $w(t, w_0)$ is the solution to (4.3.15) with $w(0, w_0) = w_0 \in \mathbb{R}^2_+$.

Let $\mathbb{X} := BUC(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the usual supreme norm. Let

$$\mathbb{X}_+ = \{(\psi_1, \psi_2) \in \mathbb{X} : \psi_i(x) \geq 0, \forall x \in \mathbb{R}, i = 1, 2\}.$$

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It is easy to see that $\mathbb{X}_+$ is a closed cone of $\mathbb{X}$ and its induced partial ordering makes $\mathbb{X}$ into a Banach lattice. Denote the partial orders by " $\leq_{\mathbb{X}}, <_{\mathbb{X}}, \ll_{\mathbb{X}}$ ". For $\psi^1, \psi^2 \in \mathbb{X}$ with $\psi^1 \leq_{\mathbb{X}} \psi^2$, denote order intervals by $[\psi^1, \psi^2]_{\mathbb{X}} = \{\psi \in \mathbb{X} : \psi^1 \leq_{\mathbb{X}} \psi \leq_{\mathbb{X}} \psi^2\}$.

To prove the global attractivity and uniqueness of traveling waves, we need a series of lemmas. The following lemma shows the existence, uniqueness and the strong monotonicity of solutions to system (4.3.14).

Lemma 4.3.1 *For any $\psi \in \mathbb{X}_+$, system (4.3.14) has a unique, bounded and non-negative solution $U(x,t,\psi)$ with $U(\cdot,0,\psi) = \psi$, and the solution semiflow of (4.3.14) is monotone. Moreover, $U(x,t,\psi^1) \ll U(x,t,\psi^2)$ for $t > 0$ and $x \in \mathbb{R}$ whenever $\psi^1, \psi^2 \in \mathbb{X}_+$ with $\psi^1 <_{\mathbb{X}} \psi^2$.*

Proof. Let $T_1(t)$ be the analytic semigroup on $BUC(\mathbb{R},\mathbb{R})$ generated by $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, and $T_2(t)\psi_2 = e^{-\beta t}\psi_2, \forall \psi_2 \in BUC(\mathbb{R},\mathbb{R})$. Clearly, $T(t) = (T_1(t), T_2(t))$ is a linear semigroup on $\mathbb{X}$. Let $B(\psi)(x) = (F^1(\psi_1(x), \psi_2(x)), g(\psi_1(x))), \forall \psi = (\psi_1, \psi_2) \in \mathbb{X}_+$. Then system (4.3.14) can be rewritten as as the following integral equation

$$U(t) = T(t)U(0) + \int_0^t T(t-s)B(U(s))ds,$$

whose solutions are called mild solutions for system (4.3.14). It is easy to check the quasi-monotonicity of $B(\psi)$. By [64, Corollary 5] (taking delay as zero, also see [88, Corollary 8.1.3]), it then follows that for any $\psi \in \mathbb{X}_+$, system (4.3.14) has a unique nonnegative and noncontinuable mild solution $U(x,t,\psi)$ satisfying $U(\cdot,0,\psi) = \psi$. Moreover, by [88, Corollary 2.2.5], $U(x,t,\psi)$ is a classical solution for $t > 0$. Note that [64, Corollary 5] also implies that the comparison principle holds for system (4.3.14). By the comparison argument, solutions of (4.3.14) on $\mathbb{X}_+$ are uniformly bounded. Therefore, system (4.3.14) defines a monotone solution semiflow on $\mathbb{X}_+$.

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$$U(t) = T(t)U(0) + \int_0^t T(t-s)B(U(s))ds,$$

whose solutions are called mild solutions for system (4.3.14). It is easy to check the quasi-monotonicity of $B(\psi)$. By [64, Corollary 5] (taking delay as zero, also see [88, Corollary 8.1.3]), it then follows that for any $\psi \in \mathbb{X}_+$, system (4.3.14) has a unique nonnegative and noncontinuable mild solution $U(x, t, \psi)$ satisfying $U(\cdot, 0, \psi) = \psi$. Moreover, by [88, Corollary 2.2.5], $U(x, t, \psi)$ is a classical solution for $t > 0$. Note that [64, Corollary 5] also implies that the comparison principle holds for system (4.3.14). By the comparison argument, solutions of (4.3.14) on $\mathbb{X}_+$ are uniformly bounded. Therefore, system (4.3.14) defines a monotone solution semiflow on $\mathbb{X}_+$.

Suppose that $\psi^1, \psi^2 \in \mathbb{X}_+$ with $\psi^1 <_{\mathbb{X}} \psi^2$. Then $U(x,t,\psi^i) \geq 0$, $\forall x \in \mathbb{R}, t \geq 0$. Let $U(x,t) = U(x,t,\psi^2) - U(x,t,\psi^1)$. Then $U(x,t) \geq 0, \forall x \in \mathbb{R}, t \geq 0$, and $U(\cdot, 0) \not\equiv 0$. Note that the first component $U_1(x,t)$ of $U(x,t)$ satisfies

$$\begin{aligned} U_{1,t} &= dU_{1,xx} + \sum_{i=1}^{2} U_i \int_0^1 F_i^1(sU(x,t,\psi^2) + (1-s)U(x,t,\psi^1))ds \quad (4.3.16) \\ &\geq dU_{1,xx} + U_1 \int_0^1 F_1^1(sU(x,t,\psi^2) + (1-s)U(x,t,\psi^1))ds, \quad (4.3.17) \end{aligned}$$

and the second component $U_2(x,t)$ of $U(x,t)$ satisfies

$$U_{2,t}(x,t) = -\beta U_2(x,t) + A(x,t)U_1(x,t),$$

where $U_1(x,t,\psi^1)$, $U_1(x,t,\psi^2)$ are the first components of $U(x,t,\psi^1)$ and $U(x,t,\psi^2)$, respectively, and

$$A(x,t) = \int_0^1 g'(sU_1(x,t,\psi^2) + (1-s)U_1(x,t,\psi^1))ds.$$

It then follows that

$$U_2(x,t) = e^{-\beta t}U_2(x,0) + \int_0^t e^{-\beta(t-s)}A(x,s)U_1(x,s)ds. \quad (4.3.18)$$

In the case where $U_1(\cdot,0) \not\equiv 0$, the strict positivity theorem ([85, Theorem 5.5.4]) and inequality (4.3.17) imply that $U_1(x,t) > 0$, $\forall x \in \mathbb{R}, t > 0$. Since $g'(z) > 0$ for $z > 0$, (4.3.18) implies $U_2(x,t) > 0, \forall x \in \mathbb{R}, t > 0$. Thus, $U(x,t,\psi^1) \ll U(x,t,\psi^2)$ for $x \in \mathbb{R}, t > 0$.

In the case where $U_2(\cdot,0) \not\equiv 0$, it follows from (4.3.18) that $U_2(\cdot,t) \not\equiv 0$ for $t \geq 0$. Since $F_2^1 > 0$ on $\mathbb{R}_+^2$, the equality (4.3.16) implies that $U_1(\cdot,t) \not\equiv 0$ for $t > 0$, and hence by the inequality (4.3.17) and [85, Theorem 1.4.5], we must have $U_1(x,t) > 0$, Suppose that $\psi^1, \psi^2 \in \mathbb{X}_+$ with $\psi^1 <_{\mathbb{X}} \psi^2$. Then $U(x,t,\psi^i) \geq 0$, $\forall x \in \mathbb{R}, t \geq 0$. Let $U(x,t) = U(x,t,\psi^2) - U(x,t,\psi^1)$. Then $U(x,t) \geq 0, \forall x \in \mathbb{R}, t \geq 0$, and $U(\cdot,0) \not\equiv 0$. Note that the first component $U_1(x,t)$ of $U(x,t)$ satisfies

$$\begin{aligned} U_{1,t} &= dU_{1,xx} + \sum_{i=1}^{2} U_i \int_0^1 F_i^1(sU(x,t,\psi^2) + (1-s)U(x,t,\psi^1))ds \quad (4.3.16) \\ &\geq dU_{1,xx} + U_1 \int_0^1 F_1^1(sU(x,t,\psi^2) + (1-s)U(x,t,\psi^1))ds, \quad (4.3.17) \end{aligned}$$

and the second component $U_2(x,t)$ of $U(x,t)$ satisfies

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It then follows that

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In the case where $U_2(\cdot,0) \not\equiv 0$, it follows from (4.3.18) that $U_2(\cdot,t) \not\equiv 0$ for $t \geq 0$. Since $F_2^1 > 0$ on $\mathbb{R}_+^2$, the equality (4.3.16) implies that $U_1(\cdot,t) \not\equiv 0$ for $t > 0$, and hence by the inequality (4.3.17) and [85, Theorem 1.4.5], we must have $U_1(x,t) > 0$,

$\forall x \in \mathbb{R}, t > 0$. Therefore, it follows from (4.3.18) that $U_2(x,t) > 0$, $\forall x \in \mathbb{R}, t > 0$. Thus $U(x,t,\psi^1) \ll U(x,t,\psi^2)$, $\forall x \in \mathbb{R}, t > 0$. ∎

In view of Section 4.2, we suppose that $\varphi(x-ct) = (\varphi_1(x-ct), \varphi_2(x-ct))$ is a strictly increasing traveling wave solution of (4.3.14) connecting E^- and E^+. By the moving coordinate $z = x - ct$, we transform (4.3.14) into the following system

$$\begin{cases} u_{1,t}(z,t) = cu_{1,z}(z,t) + du_{1,zz}(z,t) + F^1(u_1(z,t), u_2(z,t)), \\ u_{2,t}(z,t) = cu_{2,z}(z,t) + F^2(u_1(z,t), u_2(z,t)). \end{cases} \tag{4.3.19}$$

Then $\varphi(z)$ is an equilibrium solution of system (4.3.19). In what follows, we denote by $u(z,t,\psi) = (u_1(z,t), u_2(z,t))$ the solution of system (4.3.19) with $u(\cdot,0,\psi) = \psi \in \mathbb{X}_+$. Clearly, the solution $U(x,t,\psi)$ of (4.3.14) with initial value ψ is given by $U(x,t,\psi) = u(x-ct,t,\psi)$. As noted before, the comparison principle holds for (4.3.14) and hence for (4.3.19). For convenience, we set

$$N_1(u_1,u_2) := u_{1,t}(z,t) - cu_{1,z}(z,t) - du_{1,zz}(z,t) - F^1(u_1(z,t), u_2(z,t)) = 0,$$

$$N_2(u_1,u_2) := u_{2,t}(z,t) - cu_{2,z}(z,t) - F^2(u_1(z,t), u_2(z,t)) = 0.$$

Lemma 4.3.2 *If $\psi = (\psi_1, \psi_2) \in \mathbb{X}_+$ satisfies*

$$\limsup_{\xi \to -\infty} \psi(\xi) \ll E^0 \ll \liminf_{\xi \to \infty} \psi(\xi), \tag{4.3.20}$$

then, for any $\varepsilon > 0$, there exist $\tilde{z} = \tilde{z}(\varepsilon, \psi) > 0$ and a large time $\tilde{t} = \tilde{t}(\varepsilon, \psi)$ such that $\varphi(z - \tilde{z}) - \varepsilon \leq u(z, \tilde{t}, \psi) \leq \varphi(z + \tilde{z}) + \varepsilon$.

Proof. Without loss of generality, we assume that $\psi(\xi) \leq l_1$, $\forall \xi \in \mathbb{R}$ and $\psi(\xi) \leq l_2$, $\forall \xi \leq 0$, where $l_1, l_2 \in \mathbb{R}^2_+$, $l_1 \geq E^+$, $E^- \leq l_2 \ll E^0$. Let $v^+(t) = (v_1^+(t), v_2^+(t)) :=$

$\forall x \in \mathbb{R}, t > 0$. Therefore, it follows from (4.3.18) that $U_2(x,t) > 0$, $\forall x \in \mathbb{R}, t > 0$. Thus $U(x,t,\psi^1) \ll U(x,t,\psi^2)$, $\forall x \in \mathbb{R}, t > 0$. ∎

In view of Section 4.2, we suppose that $\varphi(x-ct) = (\varphi_1(x-ct), \varphi_2(x-ct))$ is a strictly increasing traveling wave solution of (4.3.14) connecting E^- and E^+. By the moving coordinate $z = x - ct$, we transform (4.3.14) into the following system

$$\begin{cases} u_{1,t}(z,t) = cu_{1,z}(z,t) + du_{1,zz}(z,t) + F^1(u_1(z,t), u_2(z,t)), \\ u_{2,t}(z,t) = cu_{2,z}(z,t) + F^2(u_1(z,t), u_2(z,t)). \end{cases} \tag{4.3.19}$$

Then $\varphi(z)$ is an equilibrium solution of system (4.3.19). In what follows, we denote by $u(z,t,\psi) = (u_1(z,t), u_2(z,t))$ the solution of system (4.3.19) with $u(\cdot,0,\psi) = \psi \in \mathbb{X}_+$. Clearly, the solution $U(x,t,\psi)$ of (4.3.14) with initial value ψ is given by $U(x,t,\psi) = u(x-ct,t,\psi)$. As noted before, the comparison principle holds for (4.3.14) and hence for (4.3.19). For convenience, we set

$$N_1(u_1,u_2) := u_{1,t}(z,t) - cu_{1,z}(z,t) - du_{1,zz}(z,t) - F^1(u_1(z,t), u_2(z,t)) = 0,$$

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Lemma 4.3.2 *If $\psi = (\psi_1, \psi_2) \in \mathbb{X}_+$ satisfies*

$$\limsup_{\xi \to -\infty} \psi(\xi) \ll E^0 \ll \liminf_{\xi \to \infty} \psi(\xi), \tag{4.3.20}$$

then, for any $\varepsilon > 0$, there exist $\tilde{z} = \tilde{z}(\varepsilon, \psi) > 0$ and a large time $\tilde{t} = \tilde{t}(\varepsilon, \psi)$ such that $\varphi(z - \tilde{z}) - \varepsilon \le u(z, \tilde{t}, \psi) \le \varphi(z + \tilde{z}) + \varepsilon$.

Proof. Without loss of generality, we assume that $\psi(\xi) \le l_1$, $\forall \xi \in \mathbb{R}$ and $\psi(\xi) \le l_2$, $\forall \xi \le 0$, where $l_1, l_2 \in \mathbb{R}^2_+$, $l_1 \ge E^+$, $E^- \le l_2 \ll E^0$. Let $v^+(t) = (v_1^+(t), v_2^+(t)) :=$

$w(t, 2l_1 - l_2)$ and $v^-(t) := w(t, l_2)$ be the solutions of the reaction system (4.3.15) with $v^+(0) = 2l_1 - l_2, v^-(0) = l_2$. Define $\zeta(s) = \frac{1}{2}(1+\tanh\frac{s}{2})$. Then $\zeta' = \zeta(1-\zeta), \zeta'' = \zeta'(1-2\zeta)$. Let

$$\begin{aligned}\bar{c} &= c + d + \sup\{\frac{(v_1^+(t) - v_1^-(t))^2}{v_i^+(t) - v_i^-(t)}|F_{11}^i(\theta)| + \frac{(v_2^+(t) - v_2^-(t))^2}{v_i^+(t) - v_i^-(t)}|F_{22}^i(\theta)| \\ &\quad +2(v_j^+(t) - v_j^-(t))|F_{12}^i(\theta)| : t \in [0, +\infty), \theta \in (v^-(t), v^+(t)), 1 \le i \ne j \le 2\}.\end{aligned}$$

Set $\tilde{c} \ge \bar{c}$ be a fixed number, and define the function

$$v(z,t) = v^+(t)\zeta(z+\tilde{c}t) + v^-(t)(1 - \zeta(z+\tilde{c}t)), \quad \forall z \in \mathbb{R}, t \ge 0.$$

It easily follows that $v(\cdot, 0) \ge \psi(\cdot)$. We further claim that $v(z,t)$ is a super-solution of system (4.3.19). Indeed, by Taylor's expansion, we have

$$\begin{aligned}\Lambda_i &:= \zeta F^i(v^+) + (1-\zeta)F^i(v^-) - F^i(\zeta v^+ + (1-\zeta)v^-) \\ &= \frac{1}{2}\zeta(1-\zeta)(v_1^+ - v_1^-)^2 F_{11}^i(\theta) + \frac{1}{2}\zeta(1-\zeta)(v_2^+ - v_2^-)^2 F_{22}^i(\theta) \\ &\quad +\zeta(1-\zeta)(v_1^+ - v_1^-)(v_2^+ - v_2^-)F_{12}^i(\theta),\end{aligned}$$

where $\theta \in (v^-(t), v^+(t))$. For each $i = 1, 2$, and $(z,t) \in \mathbb{R} \times \mathbb{R}_+$, we have

$$\begin{aligned}N_i(v(z,t)) &= v_{i,t}(z,t) - cv_{i,z}(z,t) - d_i v_{i,zz}(z,t) - F^i(v(z,t)) \\ &= \zeta F^i(v^+) + (1-\zeta)F^i(v^-) - F^i(v(z,t)) \\ &\quad +\zeta(1-\zeta)[(\tilde{c} - c)(v_i^+ - v_i^-) - d_i(1-2\zeta)(v_i^+ - v_i^-)] \ge 0,\end{aligned}$$

where $d_1 = d, d_2 = 0$, and $v_i(z,t)$ is the i-th component of $v(z,t)$. Therefore $v(z,t)$ is a super-solution of system (4.3.19).

Thus, by the comparison principle we have $u(z,t,\psi) \le v(z,t), \forall t \ge 0$. Note that $\lim_{t\to\infty} v^\pm(t) = E^\pm$. It then follows that for any $\varepsilon > 0$, there exist $\tilde{t} = \tilde{t}(\varepsilon, \psi) > 0$, and

$w(t, 2l_1 - l_2)$ and $v^-(t) := w(t, l_2)$ be the solutions of the reaction system (4.3.15) with $v^+(0) = 2l_1 - l_2, v^-(0) = l_2$. Define $\zeta(s) = \frac{1}{2}(1 + \tanh \frac{s}{2})$. Then $\zeta' = \zeta(1-\zeta), \zeta'' = \zeta'(1-2\zeta)$. Let

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It easily follows that $v(\cdot, 0) \geq \psi(\cdot)$. We further claim that $v(z,t)$ is a super-solution of system (4.3.19). Indeed, by Taylor's expansion, we have

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where $\theta \in (v^-(t), v^+(t))$. For each $i = 1, 2$, and $(z,t) \in \mathbb{R} \times \mathbb{R}_+$, we have

$$\begin{aligned}N_i(v(z,t)) &= v_{i,t}(z,t) - cv_{i,z}(z,t) - d_i v_{i,zz}(z,t) - F^i(v(z,t)) \\ &= \zeta F^i(v^+) + (1-\zeta)F^i(v^-) - F^i(v(z,t)) \\ &\quad +\zeta(1-\zeta)[(\tilde{c} - c)(v_i^+ - v_i^-) - d_i(1-2\zeta)(v_i^+ - v_i^-)] \geq 0,\end{aligned}$$

where $d_1 = d, d_2 = 0$, and $v_i(z,t)$ is the i-th component of $v(z,t)$. Therefore $v(z,t)$ is a super-solution of system (4.3.19).

Thus, by the comparison principle we have $u(z,t,\psi) \leq v(z,t), \forall t \geq 0$. Note that $\lim_{t\to\infty} v^\pm(t) = E^\pm$. It then follows that for any $\varepsilon > 0$, there exist $\tilde{t} = \tilde{t}(\varepsilon, \psi) > 0$, and

$\tilde{z} = \tilde{z}(\varepsilon, \psi) \in \mathbb{R}$ such that $u(z, \tilde{t}, \psi) \leq \varphi(z + \tilde{z}) + \varepsilon, \forall z \in \mathbb{R}$. A similar estimate on the lower bound of the solution completes the proof. ∎

Note that $E^{\pm}$ are stable nodes for the reaction system (4.3.15), i.e., the Jacobian matrices $(F^i_j(E^{\pm}))$ have only negative eigenvalues. Let $A^{\pm} = (\mu^{\pm}_{ij})$ be the constant matrices so that $F^i_j(E^{\pm}) < \mu^{\pm}_{ij}, 1 \leq i, j \leq 2$, and that $A^{\pm}$ are irreducible and have only negative eigenvalues. Denote by $\rho^{\pm} = (\rho^{\pm}_1, \rho^{\pm}_2)$ the positive eigenvectors corresponding to the principle eigenvalues of $A^{\pm}$. Let $\rho_1(\xi), \rho_2(\xi)$ be smooth positive functions such that $\rho(\xi) = (\rho_1(\xi), \rho_2(\xi)) \to \rho^{\pm}$ in $\mathbb{C}^2$-topology as $\xi \to \pm\infty$. Motivated by [69], we have the following result on super- and sub-solution for (4.3.19).

Lemma 4.3.3 *There exist positive numbers σ and ς_0 such that for any $\varsigma \geq \varsigma_0$, any $\hat{z} \in \mathbb{R}$, and $\varepsilon \in (0, \varepsilon_0(\varsigma))$,*

$$w^{\pm}(z,t) = \varphi(z \pm \hat{z} \pm \varsigma\varepsilon(1 - e^{-\sigma t})) \pm \varepsilon\rho(z \pm \hat{z})e^{-\sigma t}, \quad \forall z \in \mathbb{R}, t \geq 0,$$

are super- and sub-solutions of system (4.3.19), respectively.

Proof. Clearly, there exist $\delta, k > 0$ such that

$$\begin{aligned} &F^i_j(u) \leq \mu^{\pm}_{ij} \text{ for } \|u - E^{\pm}\| \leq \delta, u \in \mathbb{R}^2, 1 \leq i, j \leq 2, \\ &\sum_{j=1}^{2} \mu^{\pm}_{ij}\rho_j \leq -k\rho_i \text{ for } \rho = (\rho_1, \rho_2) \in \mathbb{R}^2_+ \text{ with } \|\rho - \rho^{\pm}\| \leq \delta, i = 1, 2. \end{aligned} \tag{4.3.21}$$

Since $\varphi(\xi) \to E^{\pm}, \rho(\xi) \to \rho^{\pm}, \rho'(\xi), \rho''(\xi) \to 0$ as $\xi \to \pm\infty$, there exist $\varepsilon_1, M > 0$ such that

$$k - c\varepsilon_1 - d\varepsilon_1 > 0;$$

$$|\rho'_i(\eta)|, |\rho''_i(\eta)| \leq \varepsilon_1\rho_i(\eta), \ \forall|\eta| \geq M - 1, i = 1, 2;$$

$\tilde{z} = \tilde{z}(\varepsilon, \psi) \in \mathbb{R}$ such that $u(z, \tilde{t}, \psi) \le \varphi(z + \tilde{z}) + \varepsilon, \forall z \in \mathbb{R}$. A similar estimate on the lower bound of the solution completes the proof. ∎

Note that $E^{\pm}$ are stable nodes for the reaction system (4.3.15), i.e., the Jacobian matrices $(F^i_j(E^{\pm}))$ have only negative eigenvalues. Let $A^{\pm} = (\mu^{\pm}_{ij})$ be the constant matrices so that $F^i_j(E^{\pm}) < \mu^{\pm}_{ij}, 1 \le i, j \le 2$, and that $A^{\pm}$ are irreducible and have only negative eigenvalues. Denote by $\rho^{\pm} = (\rho^{\pm}_1, \rho^{\pm}_2)$ the positive eigenvectors corresponding to the principle eigenvalues of $A^{\pm}$. Let $\rho_1(\xi), \rho_2(\xi)$ be smooth positive functions such that $\rho(\xi) = (\rho_1(\xi), \rho_2(\xi)) \to \rho^{\pm}$ in $\mathbb{C}^2$-topology as $\xi \to \pm\infty$. Motivated by [69], we have the following result on super- and sub-solution for (4.3.19).

Lemma 4.3.3 *There exist positive numbers σ and ς_0 such that for any $\varsigma \ge \varsigma_0$, any $\hat{z} \in \mathbb{R}$, and $\varepsilon \in (0, \varepsilon_0(\varsigma))$,*

$$w^{\pm}(z,t) = \varphi(z \pm \hat{z} \pm \varsigma\varepsilon(1 - e^{-\sigma t})) \pm \varepsilon\rho(z \pm \hat{z})e^{-\sigma t}, \quad \forall z \in \mathbb{R}, t \ge 0,$$

are super- and sub-solutions of system (4.3.19), respectively.

Proof. Clearly, there exist $\delta, k > 0$ such that

$$\begin{aligned} &F^i_j(u) \le \mu^{\pm}_{ij} \text{ for } \|u - E^{\pm}\| \le \delta, u \in \mathbb{R}^2, 1 \le i, j \le 2, \\ &\sum_{j=1}^{2} \mu^{\pm}_{ij}\rho_j \le -k\rho_i \text{ for } \rho = (\rho_1, \rho_2) \in \mathbb{R}^2_+ \text{ with } \|\rho - \rho^{\pm}\| \le \delta, i = 1, 2. \end{aligned} \tag{4.3.21}$$

Since $\varphi(\xi) \to E^{\pm}, \rho(\xi) \to \rho^{\pm}, \rho'(\xi), \rho''(\xi) \to 0$ as $\xi \to \pm\infty$, there exist $\varepsilon_1, M > 0$ such that

$$k - c\varepsilon_1 - d\,\varepsilon_1 > 0;$$

$$|\rho_i'(\eta)|, |\rho_i''(\eta)| \le \varepsilon_1\rho_i(\eta), \ \forall |\eta| \ge M - 1, i = 1, 2;$$

$$\begin{aligned}
&\|\rho(\eta)-\rho^+\| \le \delta, \forall \eta \ge M-1;\ \|\rho(\eta)-\rho^-\| \le \delta, \forall \eta \le -M+1; \\
&\|\varphi(\xi)+\varepsilon\rho(\eta)-E^+\| \le \delta,\ \forall \varepsilon \in (0,\varepsilon_1], \xi \ge M-1, \eta \ge M-1; \qquad (4.3.22) \\
&\|\varphi(\xi)+\varepsilon\rho(\eta)-E^-\| \le \delta,\ \forall \varepsilon \in (0,\varepsilon_1], \xi \le -M+1, \eta \le -M+1;
\end{aligned}$$

Let $B_1 > 0$ so that $\|\rho(\eta)\|, \|\rho'(\eta)\|, \|\rho''(\eta)\| \le B_1$ for all $\eta \in \mathbb{R}$. Define

$$B_2 = \sup\{|F^i_j(u)| : u \in [E^- - \delta\vec{e}, E^+ + B_1\vec{e}]\}, \quad B_3 = \inf_{\|\xi\| \le M} \|\varphi'(\xi)\|.$$

Choosing $0 < \sigma \le k - c\varepsilon_1 - d\varepsilon_1$, set

$$\varsigma \ge \varsigma_0 = \frac{B_1}{\sigma B_3}(B_2 + \sigma + c + d), \quad \varepsilon_0 = \min\{\varepsilon_1, \frac{1}{\varsigma}\}.$$

With $q = e^{-\sigma t}$, the argument of φ and φ_i being $\xi = z + \hat{z} + \varsigma\varepsilon(1 - e^{-\sigma t})$ and ρ, ρ_i being $\eta = z + \hat{z}$, for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned}
N_i(w^+(z,t)) &= w^+_{i,t}(z,t) - cw^+_{i,z}(z,t) - d_i w^+_{i,zz}(z,t) - F^i(w^+(z,t)) \\
&= F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) + \varepsilon\varsigma\sigma q\varphi'_i - (\rho_i\sigma + c\rho'_i + d_i\rho''_i)\varepsilon q.
\end{aligned}$$

where $d_1 = d, d_2 = 0$. We distinguish among three cases.

Case (i): $|\xi| \le M$. Note that $F^i_j > 0$ for $i \ne j$, $F^i_j \le 0$ for $i = j$, and $\varphi'_i > 0$. By the choice of ε_0, ς, and σ, we have

$$F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) = -\int_0^1 \varepsilon q \left(\sum_{j=1}^{2} \rho_j F^i_j(\varphi + \varepsilon s\rho q)\right) ds \ge -B_1B_2\varepsilon q,$$

and hence

$$N_i(w^+(z,t)) \ge -B_1B_2\varepsilon q - B_1\varepsilon q(\sigma + c + d_i) + B_3\varepsilon\varsigma\sigma q \ge 0.$$

Case (ii): $\xi \ge M$. Since $\xi - \eta \le \varsigma\varepsilon \le 1$, $\xi > \eta \ge M - 1$. Thus, by the group of inequalities (4.3.22), we have

$$\|\varphi(\xi) + s\varepsilon\rho(\eta) - E^+\| \le \delta,\ \|\rho(\eta) - \rho^+\| \le \delta,\ \forall s \in (0,1).$$

$$\|\rho(\eta)-\rho^+\| \le \delta, \forall \eta \ge M-1; \; \|\rho(\eta)-\rho^-\| \le \delta, \forall \eta \le -M+1;$$
$$\|\varphi(\xi)+\varepsilon\rho(\eta)-E^+\| \le \delta, \; \forall \varepsilon \in (0,\varepsilon_1], \xi \ge M-1, \eta \ge M-1; \tag{4.3.22}$$
$$\|\varphi(\xi)+\varepsilon\rho(\eta)-E^-\| \le \delta, \; \forall \varepsilon \in (0,\varepsilon_1], \xi \le -M+1, \eta \le -M+1;$$

Let $B_1 > 0$ so that $\|\rho(\eta)\|, \|\rho'(\eta)\|, \|\rho''(\eta)\| \le B_1$ for all $\eta \in \mathbb{R}$. Define

$$B_2 = \sup\{|F^i_j(u)| : u \in [E^- - \delta\vec{e}, E^+ + B_1\vec{e}]\}, \quad B_3 = \inf_{\|\xi\| \le M} \|\varphi'(\xi)\|.$$

Choosing $0 < \sigma \le k - c\varepsilon_1 - d\,\varepsilon_1$, set

$$\varsigma \ge \varsigma_0 = \frac{B_1}{\sigma B_3}(B_2 + \sigma + c + d), \quad \varepsilon_0 = \min\{\varepsilon_1, \frac{1}{\varsigma}\}.$$

With $q = e^{-\sigma t}$, the argument of φ and φ_i being $\xi = z + \hat{z} + \varsigma\varepsilon(1 - e^{-\sigma t})$ and ρ, ρ_i being $\eta = z + \hat{z}$, for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned} N_i(w^+(z,t)) &= w^+_{i,t}(z,t) - c w^+_{i,z}(z,t) - d_i w^+_{i,zz}(z,t) - F^i(w^+(z,t)) \\ &= F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) + \varepsilon\varsigma\sigma q\varphi_i' - (\rho_i\sigma + c\rho_i' + d_i\rho_i'')\varepsilon q. \end{aligned}$$

where $d_1 = d, d_2 = 0$. We distinguish among three cases.

Case (i): $|\xi| \le M$. Note that $F^i_j > 0$ for $i \ne j$, $F^i_j \le 0$ for $i = j$, and $\varphi_i' > 0$. By the choice of ε_0, ς, and σ, we have

$$F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) = -\int_0^1 \varepsilon q \left(\sum_{j=1}^2 \rho_j F^i_j(\varphi + \varepsilon s\rho q)\right) ds \ge -B_1 B_2 \varepsilon q,$$

and hence

$$N_i(w^+(z,t)) \ge -B_1B_2\varepsilon q - B_1\varepsilon q(\sigma + c + d_i) + B_3\varepsilon\varsigma\sigma q \ge 0.$$

Case (ii): $\xi \ge M$. Since $\xi - \eta \le \varsigma\varepsilon \le 1$, $\xi > \eta \ge M - 1$. Thus, by the group of inequalities (4.3.22), we have

$$\|\varphi(\xi) + s\varepsilon\rho(\eta) - E^+\| \le \delta, \; \|\rho(\eta) - \rho^+\| \le \delta, \; \forall s \in (0,1).$$

Therefore, by (4.3.21), there holds

$$\begin{aligned} F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) &= -\int_0^1 \varepsilon q \left(\sum_{j=1}^{2} \rho_j F_j^i(\varphi + s\varepsilon\rho q) \right) ds \\ &\geq -\varepsilon q \sum_{j=1}^{2} \mu_{ij}\rho_j \geq k\varepsilon q\rho_i. \end{aligned}$$

Hence,

$$\begin{aligned} N_i(w^+(z,t)) &\geq k\varepsilon\rho_i q - (\rho_i\sigma + c\rho_i' + d_i\rho_i'')\varepsilon q \\ &\geq (k - \sigma - c\varepsilon_1 - d_i\varepsilon_1)\varepsilon q\rho_i \geq 0. \end{aligned}$$

Case (iii): $\xi \leq -M$. By an argument similar to case (ii), we have $N_i(w^+(z,t)) \geq 0$.

Combining cases (i)-(iii), we have $N_i(w^+(z,t)) \geq 0$ for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geq 0$. Thus $w^+(z,t)$ is a super-solution of system (4.3.19). By a similar argument, we can prove that $w^-(z,t)$ is a sub-solution. ∎

Lemma 4.3.4 *The wave profile $\varphi(z)$ is a Liapunov stable equilibrium of (4.3.19).*

Proof. Let ε_0 and $w^\pm(z,t,\varepsilon)$ be defined as in Lemma 4.3.3 with $\hat{z} = 0$ and $\varsigma = \varsigma_0$. It then follows that there exists $K > 0$, independent of ε, such that $\|w^\pm(z,t,\varepsilon) - \varphi(z)\| < K\varepsilon$, $\forall z \in \mathbb{R}, t \geq 0, \varepsilon \in (0, \varepsilon_0)$. For any $\varepsilon \in (0, \varepsilon_0)$, let $\delta = \varepsilon \inf_{z\in\mathbb{R}} \rho(z)$. Thus, for any given $\|\psi - \varphi\| < \delta$, we have

$$w^-(z,0,\varepsilon) = \varphi(z) - \varepsilon\rho(z) \leq \psi(z) \leq \varphi(z) + \varepsilon\rho(z) = w^+(z,0,\varepsilon), \quad \forall z \in \mathbb{R}.$$

Then the comparison principle implies that $w^-(z,t,\varepsilon) \leq u(z,t,\psi) \leq w^+(z,t,\varepsilon)$, $\forall z \in \mathbb{R}, t \geq 0$, and hence $\|u(\cdot,t,\psi) - \varphi(\cdot)\| < K\varepsilon$, $\forall t \geq 0$. ∎

Now we are in a position to prove the main result of this section.

Therefore, by (4.3.21), there holds

$$\begin{aligned} F^i(\varphi) - F^i(\varphi + \varepsilon\rho q) &= -\int_0^1 \varepsilon q \left(\sum_{j=1}^{2} \rho_j F_j^i(\varphi + s\varepsilon\rho q) \right) ds \\ &\geq -\varepsilon q \sum_{j=1}^{2} \mu_{ij}\rho_j \geq k\varepsilon q \rho_i. \end{aligned}$$

Hence,

$$\begin{aligned} N_i(w^+(z,t)) &\geq k\varepsilon\rho_i q - (\rho_i\sigma + c\rho_i' + d_i\rho_i'')\varepsilon q \\ &\geq (k - \sigma - c\varepsilon_1 - d_i\varepsilon_1)\varepsilon q \rho_i \geq 0. \end{aligned}$$

Case (iii): $\xi \leq -M$. By an argument similar to case (ii), we have $N_i(w^+(z,t)) \geq 0$.

Combining cases (i)-(iii), we have $N_i(w^+(z,t)) \geq 0$ for all $\varepsilon \in (0, \varepsilon_0)$ and $t \geq 0$. Thus $w^+(z,t)$ is a super-solution of system (4.3.19). By a similar argument, we can prove that $w^-(z,t)$ is a sub-solution. ∎

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$$w^-(z,0,\varepsilon) = \varphi(z) - \varepsilon\rho(z) \leq \psi(z) \leq \varphi(z) + \varepsilon\rho(z) = w^+(z,0,\varepsilon), \quad \forall z \in \mathbb{R}.$$

Then the comparison principle implies that $w^-(z,t,\varepsilon) \leq u(z,t,\psi) \leq w^+(z,t,\varepsilon)$, $\forall z \in \mathbb{R}, t \geq 0$, and hence $\|u(\cdot,t,\psi) - \varphi(\cdot)\| < K\varepsilon$, $\forall t \geq 0$. ∎

Now we are in a position to prove the main result of this section.

Theorem 4.3.1 *Let $\varphi(x-ct)$ be a monotone traveling wave solution of system (4.3.14) and $U(x,t,\psi)$ be the solution of (4.3.14) with $U(\cdot,0,\psi)=\psi(\cdot)\in\mathbb{X}_+$. Then for any $\psi\in\mathbb{X}_+$ satisfying (4.3.20), there exists $s_\psi\in\mathbb{R}$ such that*

$$\lim_{t\to+\infty}\|U(x,t,\psi)-\varphi(x-ct+s_\psi)\|=0$$

uniformly for $x\in\mathbb{R}$. Moreover, any traveling wave solution of system (4.3.14) connecting E^- and E^+ is a translate of φ.

Proof. We will apply the notations in Lemma 4.3.3. Let $\delta_0=\min_{z\in\mathbb{R}}\rho(z)$, and choose $\varsigma\ge\max\{\varsigma_0,\frac{1}{\delta_0}\}$. For $\varepsilon\in(0,\varepsilon_0(\varsigma))$, by Lemma 4.3.2, there exists $\tilde{t}$ such that

$$\varphi(z-\tilde{z})-\varepsilon\delta_0\le u(z,\tilde{t},\psi)\le\varphi(z+\tilde{z})+\varepsilon\delta_0,\ \forall z\in\mathbb{R}.$$

Let $f(z)=u(z,\tilde{t},\psi)$. Then, from the construction of $w^\pm(z,t)$ in Lemma 4.3.3, we have $w^-(z,0)\le u(z,0,f)\le w^+(z,0),\forall z\in\mathbb{R}$. By the comparison principle, we have $w^-(z,t)\le u(z,t,f)\le w^+(z,t),\ \forall z\in\mathbb{R},t\ge 0$. Note that $u(z,t+\tilde{t},\psi)=u(z,t,u(z,\tilde{t},\psi))$. We then have

$$\begin{aligned}\varphi(z-\tilde{z}-\varepsilon\varsigma) &- \varepsilon\rho(z-\tilde{z})e^{-\sigma t}\le u(z,t+\tilde{t},\psi)\\ &\le \varphi(z+\tilde{z}+\varepsilon\varsigma)+\varepsilon\rho(z+\tilde{z})e^{-\sigma t},\quad\forall t\ge 0.\end{aligned}\tag{4.3.23}$$

Define $\Phi_t(\psi):=u(\cdot,t,\psi),\forall\psi\in\mathbb{X}_+,t\ge 0$. By the estimate (4.3.23), the positive orbit $\gamma^+(\psi):=\{\Phi_t(\psi):\ t\ge 0\}$ is bounded in $C^1(\mathbb{R},\mathbb{R}^2)$. Note that $\lim_{z\to\pm\infty}\varphi(z)=E^\pm$. Consequently, the positive orbit $\gamma^+(\psi)$ is precompact in $\mathbb{X}$, and hence its omega limit set $\omega(\psi)$ is nonempty, compact and invariant.

Letting $z_0=\tilde{z}+\varepsilon\varsigma$ and $t\to\infty$ in (4.3.23), we then have

$$\omega(\psi)\subset I:=[\varphi(\cdot-z_0),\varphi(\cdot+z_0)]_{\mathbb{X}}.$$

Theorem 4.3.1 *Let $\varphi(x-ct)$ be a monotone traveling wave solution of system (4.3.14) and $U(x,t,\psi)$ be the solution of (4.3.14) with $U(\cdot,0,\psi)=\psi(\cdot)\in\mathbb{X}_+$. Then for any $\psi\in\mathbb{X}_+$ satisfying (4.3.20), there exists $s_\psi\in\mathbb{R}$ such that*

$$\lim_{t\to+\infty}\|U(x,t,\psi)-\varphi(x-ct+s_\psi)\|=0$$

uniformly for $x\in\mathbb{R}$. Moreover, any traveling wave solution of system (4.3.14) connecting E^- and E^+ is a translate of φ.

Proof. We will apply the notations in Lemma 4.3.3. Let $\delta_0=\min_{z\in\mathbb{R}}\rho(z)$, and choose $\varsigma\geq\max\{\varsigma_0,\frac{1}{\delta_0}\}$. For $\varepsilon\in(0,\varepsilon_0(\varsigma))$, by Lemma 4.3.2, there exists $\tilde{t}$ such that

$$\varphi(z-\tilde{z})-\varepsilon\delta_0\leq u(z,\tilde{t},\psi)\leq\varphi(z+\tilde{z})+\varepsilon\delta_0,\ \forall z\in\mathbb{R}.$$

Let $f(z)=u(z,\tilde{t},\psi)$. Then, from the construction of $w^\pm(z,t)$ in Lemma 4.3.3, we have $w^-(z,0)\leq u(z,0,f)\leq w^+(z,0),\forall z\in\mathbb{R}$. By the comparison principle, we have $w^-(z,t)\leq u(z,t,f)\leq w^+(z,t)$, $\forall z\in\mathbb{R},t\geq 0$. Note that $u(z,t+\tilde{t},\psi)=u(z,t,u(z,\tilde{t},\psi))$. We then have

$$\begin{aligned}\varphi(z-\tilde{z}-\varepsilon\varsigma)\ &-\ \varepsilon\rho(z-\tilde{z})e^{-\sigma t}\leq u(z,t+\tilde{t},\psi)\\ &\leq\ \varphi(z+\tilde{z}+\varepsilon\varsigma)+\varepsilon\rho(z+\tilde{z})e^{-\sigma t},\quad\forall t\geq 0.\end{aligned}\tag{4.3.23}$$

Define $\Phi_t(\psi):=u(\cdot,t,\psi),\forall\psi\in\mathbb{X}_+,t\geq 0$. By the estimate (4.3.23), the positive orbit $\gamma^+(\psi):=\{\Phi_t(\psi):t\geq 0\}$ is bounded in $C^1(\mathbb{R},\mathbb{R}^2)$. Note that $\lim_{z\to\pm\infty}\varphi(z)=E^\pm$. Consequently, the positive orbit $\gamma^+(\psi)$ is precompact in $\mathbb{X}$, and hence its omega limit set $\omega(\psi)$ is nonempty, compact and invariant.

Letting $z_0=\tilde{z}+\varepsilon\varsigma$ and $t\to\infty$ in (4.3.23), we then have

$$\omega(\psi)\subset I:=[\varphi(\cdot-z_0),\varphi(\cdot+z_0)]_{\mathbb{X}}.$$

Let $h(s) = \varphi(\cdot + s), \forall s \in [-z_0, z_0]$. Then h is a monotone homeomorphism from $[-z_0, z_0]$ onto a subset of I. Let $V = [E^-, E^+]_{\mathbb{X}}$. Then $\Phi_t : V \to V$ is a monotone autonomous semiflow. By Lemma 4.3.4, each $h(s)$ is a stable equilibrium for Φ_t. Clearly, each $\phi \in I$ satisfies condition (4.3.20) and hence, by the above proof, $\gamma^+(\phi)$ is precompact. By Theorem 1.2.2, it suffices to verify the condition 3(a) to obtain the convergence of $\gamma^+(\psi)$.

Assume that for some $s_0 \in [-z_0, z_0)$ and $\phi_0 \in I$, $\varphi(\cdot + s_0) <_{\mathbb{X}} \phi(\cdot)$ for all $\phi \in \omega(\phi_0)$; that is, $\varphi(\cdot + s_0) <_{\mathbb{X}} \omega(\phi_0)$. By Lemma 4.3.1, $\varphi(z + s_0) \ll \Phi_t(\phi)(z), \forall z \in \mathbb{R}, t > 0$, and hence, by the invariance of $\omega(\phi_0)$, $\varphi(z+s_0) \ll \phi(z), \forall \phi \in \omega(\phi_0), z \in \mathbb{R}$.

Since $\lim\limits_{z\to\pm\infty} \varphi'(z) = 0$, we can choose a large positive number $z_1 \in (z_0, +\infty)$ such that $\bar{\delta} = \sup\limits_{|z|\geq z_1 - z_0} \|\varphi'(z)\| \leq \frac{1}{4\varsigma^2}$. By the compactness of $\omega(\phi_0)$, there exists $s_1 \in (s_0, z_0)$ such that $s_1 - s_0 < 2\varepsilon_0\varsigma$, and

$$\varphi(z + s_1) \ll \phi(z),\ \forall z \in [-z_1, z_1], \phi \in \omega(\phi_0).$$

For any fixed $\phi \in \omega(\phi_0)$, there exists a time sequence $\{t_j\}$ such that $\lim\limits_{j\to\infty} t_j = +\infty$, and $\lim\limits_{j\to\infty} \Phi_{t_j}(\phi_0) = \phi$. Fix a t_j such that $\|\Phi_{t_j}(\phi_0) - \phi\| < \bar{\delta}(s_1 - s_0)$. Since $\varphi(z+s_1) \ll \phi(z)$ for $z \in [-z_1, z_1]$, and $\varphi(z + s_0) - \varphi(z + s_1) \ll \phi(z) - \varphi(z + s_1)$ for $\forall z \in \mathbb{R}$, we have

$$\begin{aligned}
\Phi_{t_j}(\phi_0)(z) - \varphi(z + s_1) &= \Phi_{t_j}(\phi_0)(z) - \phi(z) + \phi(z) - \varphi(z + s_1) \\
&> -\bar{\delta}(s_1 - s_0)\vec{e} + \phi(z) - \varphi(z + s_1) \\
&> -\bar{\delta}(s_1 - s_0)\vec{e} - \sup_{|z|\geq z_1} \|\varphi(z + s_0) - \varphi(z + s_1)\|\vec{e} \\
&\geq -\bar{\delta}(s_1 - s_0)\vec{e} - (s_1 - s_0) \sup_{|z|\geq z_1} \|\varphi'(z)\|\vec{e} \\
&\geq -2\bar{\delta}(s_1 - s_0)\vec{e} \geq -\varepsilon_1\rho(z + s_1), \quad \forall z \in \mathbb{R},
\end{aligned}$$

Let $h(s) = \varphi(\cdot + s), \forall s \in [-z_0, z_0]$. Then h is a monotone homeomorphism from $[-z_0, z_0]$ onto a subset of I. Let $V = [E^-, E^+]_{\mathbb{X}}$. Then $\Phi_t : V \to V$ is a monotone autonomous semiflow. By Lemma 4.3.4, each $h(s)$ is a stable equilibrium for Φ_t. Clearly, each $\phi \in I$ satisfies condition (4.3.20) and hence, by the above proof, $\gamma^+(\phi)$ is precompact. By Theorem 1.2.2, it suffices to verify the condition 3(a) to obtain the convergence of $\gamma^+(\psi)$.

Assume that for some $s_0 \in [-z_0, z_0)$ and $\phi_0 \in I$, $\varphi(\cdot + s_0) <_{\mathbb{X}} \phi(\cdot)$ for all $\phi \in \omega(\phi_0)$; that is, $\varphi(\cdot + s_0) <_{\mathbb{X}} \omega(\phi_0)$. By Lemma 4.3.1, $\varphi(z + s_0) \ll \Phi_t(\phi)(z), \forall z \in \mathbb{R}, t > 0$, and hence, by the invariance of $\omega(\phi_0)$, $\varphi(z+s_0) \ll \phi(z), \forall \phi \in \omega(\phi_0), z \in \mathbb{R}$.

Since $\lim_{z \to \pm\infty} \varphi'(z) = 0$, we can choose a large positive number $z_1 \in (z_0, +\infty)$ such that $\bar{\delta} = \sup_{|z| \geq z_1 - z_0} \|\varphi'(z)\| \leq \frac{1}{4\varsigma^2}$. By the compactness of $\omega(\phi_0)$, there exists $s_1 \in (s_0, z_0)$ such that $s_1 - s_0 < 2\varepsilon_0\varsigma$, and

$$\varphi(z + s_1) \ll \phi(z), \ \forall z \in [-z_1, z_1], \phi \in \omega(\phi_0).$$

For any fixed $\phi \in \omega(\phi_0)$, there exists a time sequence $\{t_j\}$ such that $\lim_{j \to \infty} t_j = +\infty$, and $\lim_{j \to \infty} \Phi_{t_j}(\phi_0) = \phi$. Fix a t_j such that $\|\Phi_{t_j}(\phi_0) - \phi\| < \bar{\delta}(s_1 - s_0)$. Since $\varphi(z+s_1) \ll \phi(z)$ for $z \in [-z_1, z_1]$, and $\varphi(z + s_0) - \varphi(z + s_1) \ll \phi(z) - \varphi(z + s_1)$ for $\forall z \in \mathbb{R}$, we have

$$\begin{aligned}
\Phi_{t_j}(\phi_0)(z) - \varphi(z + s_1) &= \Phi_{t_j}(\phi_0)(z) - \phi(z) + \phi(z) - \varphi(z + s_1) \\
&> -\bar{\delta}(s_1 - s_0)\vec{e} + \phi(z) - \varphi(z + s_1) \\
&> -\bar{\delta}(s_1 - s_0)\vec{e} - \sup_{|z| \geq z_1} \|\varphi(z + s_0) - \varphi(z + s_1)\|\vec{e} \\
&\geq -\bar{\delta}(s_1 - s_0)\vec{e} - (s_1 - s_0) \sup_{|z| \geq z_1} \|\varphi'(z)\|\vec{e} \\
&\geq -2\bar{\delta}(s_1 - s_0)\vec{e} \geq -\varepsilon_1 \rho(z + s_1), \quad \forall z \in \mathbb{R},
\end{aligned}$$

where $\vec{e}$ is the unit vector in $\mathbb{R}^2$, $\varepsilon_1 = \frac{s_1 - s_0}{2\varsigma^2\delta_0}$. Note that $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_1\varsigma \leq \frac{1}{2}(s_1 - s_0)$. By the construction of $w^-(z,t)$ in Lemma 4.3.3, we have $w^-(z,0) \leq \Phi_{t_j}(\phi_0)(z)$. It then follows that

$$\begin{aligned}
\Phi_t(\Phi_{t_j}(\phi_0))(z) &\geq w^-(z,t) = \varphi(z + s_1 - \varepsilon_1\varsigma(1 - e^{-\sigma t})) - \varepsilon_1\rho(z + s_1)e^{-\sigma t} \\
&\geq \varphi(z + s_1 - \varepsilon_1\varsigma) - \varepsilon_1\rho(z + s_1)e^{-\sigma t} \\
&\geq \varphi(z + s_1 - \tfrac{1}{2}(s_1 - s_0)) - \varepsilon_1\rho(z + s_1)e^{-\sigma t} \\
&= \varphi(z + \tfrac{1}{2}(s_1 + s_0)) - \varepsilon_1\rho(z + s_1)e^{-\sigma t}, \quad z \in \mathbb{R},\ t > 0.
\end{aligned}$$

Setting $t = t_i - t_j$ and $t_i \to \infty$, we then obtain that $\varphi(\cdot + \frac{1}{2}(s_1 + s_0)) \leq_{\mathbb{X}} \phi(\cdot)$. Denote $s_2 = \frac{1}{2}(s_1 + s_0)$. Then $s_2 \in (s_0, s_1) \subset [s_0, z_0]$ and $\varphi(\cdot + s_2) \leq_{\mathbb{X}} \phi(\cdot)$. Since $\phi \in \omega(\phi_0)$ is arbitrary, we have $\varphi(\cdot + s_2) \leq_{\mathbb{X}} \omega(\phi_0)$.

By Theorem 1.2.2, there exists $s_\psi \in [-z_0, z_0]$ such that $\omega(\psi) = h(s_\psi) = \varphi(\cdot + s_\psi)$. Then $\lim_{t\to\infty} \Phi_t(\psi) = \varphi(\cdot + s_\psi)$. Since $U(x,t,\psi) = u(x - ct, t, \psi) = \Phi_t(\psi)(x - ct)$, we have $\lim_{t\to\infty} \|U(x,t,\psi) - \varphi(x - ct + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$.

Let $\tilde{\varphi}(x - \tilde{c}t)$ be a traveling wave solution of system (4.3.14) connecting E^- and E^+. Then $\tilde{\varphi}$ satisfies condition (4.3.20). By what we have proven above, there exists $\tilde{s}_\psi \in \mathbb{R}$ so that $\lim_{t\to\infty} \|\tilde{\varphi}(\cdot - \tilde{c}t) - \varphi(\cdot - ct + \tilde{s}_\psi)\| = 0$. By a change of variable $z = x - ct$, we have $\lim_{t\to\infty} \|\tilde{\varphi}(\cdot + (c - \tilde{c})t) - \varphi(\cdot + \tilde{s}_\psi)\| = 0$. Since $\tilde{\varphi}(\pm\infty) = E^\pm$, and $\varphi(\cdot)$ is strictly increasing on $\mathbb{R}$, we must have $\tilde{c} = c$, and hence, $\tilde{\varphi}(\cdot) = \varphi(\cdot + \tilde{s}_\psi)$. ∎

4.4 Global Exponential Stability

In the last section we proved that for a large class of initial values, solutions of (4.3.14) converge to translates of the traveling wave front. In this section, we will where $\vec{e}$ is the unit vector in $\mathbb{R}^2$, $\varepsilon_1 = \frac{s_1 - s_0}{2\varsigma^2\delta_0}$. Note that $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_1\varsigma \le \frac{1}{2}(s_1 - s_0)$. By the construction of $w^-(z,t)$ in Lemma 4.3.3, we have $w^-(z,0) \le \Phi_{t_j}(\phi_0)(z)$. It then follows that

$$\begin{aligned}
\Phi_t(\Phi_{t_j}(\phi_0))(z) &\ge w^-(z,t) = \varphi(z + s_1 - \varepsilon_1\varsigma(1 - e^{-\sigma t})) - \varepsilon_1\rho(z + s_1)e^{-\sigma t} \\
&\ge \varphi(z + s_1 - \varepsilon_1\varsigma) - \varepsilon_1\rho(z + s_1)e^{-\sigma t} \\
&\ge \varphi(z + s_1 - \tfrac{1}{2}(s_1 - s_0)) - \varepsilon_1\rho(z + s_1)e^{-\sigma t} \\
&= \varphi(z + \tfrac{1}{2}(s_1 + s_0)) - \varepsilon_1\rho(z + s_1)e^{-\sigma t}, \quad z \in \mathbb{R},\ t > 0.
\end{aligned}$$

Setting $t = t_i - t_j$ and $t_i \to \infty$, we then obtain that $\varphi(\cdot + \frac{1}{2}(s_1 + s_0)) \le_{\mathbb{X}} \phi(\cdot)$. Denote $s_2 = \frac{1}{2}(s_1 + s_0)$. Then $s_2 \in (s_0, s_1) \subset [s_0, z_0]$ and $\varphi(\cdot + s_2) \le_{\mathbb{X}} \phi(\cdot)$. Since $\phi \in \omega(\phi_0)$ is arbitrary, we have $\varphi(\cdot + s_2) \le_{\mathbb{X}} \omega(\phi_0)$.

By Theorem 1.2.2, there exists $s_\psi \in [-z_0, z_0]$ such that $\omega(\psi) = h(s_\psi) = \varphi(\cdot + s_\psi)$. Then $\lim_{t\to\infty} \Phi_t(\psi) = \varphi(\cdot + s_\psi)$. Since $U(x,t,\psi) = u(x - ct, t, \psi) = \Phi_t(\psi)(x - ct)$, we have $\lim_{t\to\infty} \|U(x,t,\psi) - \varphi(x - ct + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$.

Let $\tilde{\varphi}(x - \tilde{c}t)$ be a traveling wave solution of system (4.3.14) connecting E^- and E^+. Then $\tilde{\varphi}$ satisfies condition (4.3.20). By what we have proven above, there exists $\tilde{s}_\psi \in \mathbb{R}$ so that $\lim_{t\to\infty} \|\tilde{\varphi}(\cdot - \tilde{c}t) - \varphi(\cdot - ct + \tilde{s}_\psi)\| = 0$. By a change of variable $z = x - ct$, we have $\lim_{t\to\infty} \|\tilde{\varphi}(\cdot + (c - \tilde{c})t) - \varphi(\cdot + \tilde{s}_\psi)\| = 0$. Since $\tilde{\varphi}(\pm\infty) = E^\pm$, and $\varphi(\cdot)$ is strictly increasing on $\mathbb{R}$, we must have $\tilde{c} = c$, and hence, $\tilde{\varphi}(\cdot) = \varphi(\cdot + \tilde{s}_\psi)$. ∎

4.4 Global Exponential Stability

In the last section we proved that for a large class of initial values, solutions of (4.3.14) converge to translates of the traveling wave front. In this section, we will show that this convergence is also uniformly exponential via the spectrum analysis.

A standard technique for determining stability (exponential) of traveling waves is to use the linearization criterion. As in the last section, we assume that system (4.3.14) admits a strictly increasing traveling wave solution

$$U(x,t) = \varphi(x-ct) = (\varphi_1(x-ct), \varphi_2(x-ct)), \quad c \neq 0.$$

If the right-hand side of (4.3.19) is linearized about its equilibrium solution $\varphi(z)$, the resulting linear operator is

$$Lu = \begin{pmatrix} du_{1,zz} + cu_{1,z} \\ \\ cu_{2,z} \end{pmatrix} + J_{\varphi}(z) \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix},$$

where $J_{\varphi}(z) = (F_j^i(\varphi(z)))$, $u(z) = (u_1(z), u_2(z)) \in \mathbb{X}$.

The linearization criterion for stability of the traveling wave front is that the spectrum $\sigma(L)$ of L (except for zero) lies in a left-half complex plane and is bounded away from the imaginary axis, and zero is a simple eigenvalue. Note that zero is always an eigenvalue of L because of the translation invariance of traveling waves. For the point spectrum $\sigma_p(L)$ of L, we have the following result.

Lemma 4.4.1 *Assume that λ is an eigenvalue of L with eigenfunction $u \in \mathbb{X}_c$, complexified $\mathbb{X}$. If $u \notin span\{\varphi'(\cdot)\}$, then $Re\lambda < 0$.*

Proof. Let $D = diag(d,0), C = diag(c,c), B(z) = (F_j^i(\varphi(z)))$ and $B^{\pm} = (F_j^i(E^{\pm}))$. We claim that there exist positive vectors $q^{\pm}$ such that $B^{\pm}q^{\pm} < 0$. Note that the reaction system (4.3.15) is cooperative and $E^{\pm}$ are stable nodes. Since B^+ is irreducible, we can choose q^+ as a positive eigenvector associated with the negative show that this convergence is also uniformly exponential via the spectrum analysis.

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$$Lu = \begin{pmatrix} du_{1,zz} + cu_{1,z} \\ \\ cu_{2,z} \end{pmatrix} + J_\varphi(z) \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix},$$

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Proof. Let $D = diag(d,0), C = diag(c,c), B(z) = (F_j^i(\varphi(z)))$ and $B^\pm = (F_j^i(E^\pm))$. We claim that there exist positive vectors $q^\pm$ such that $B^\pm q^\pm < 0$. Note that the reaction system (4.3.15) is cooperative and $E^\pm$ are stable nodes. Since B^+ is irreducible, we can choose q^+ as a positive eigenvector associated with the negative principle eigenvalue of B^+. Thus, $B^+q^+ < 0$. If $g'(0) > 0$, then B^- is an irreducible matrix. Therefore, a positive eigenvector q^- can be chosen such that $B^-q^- < 0$. If $g'(0) = 0$, let $q^- = (1, \varepsilon)$. Then $B^-q^- < 0$ for some sufficiently small positive number ε.

Let $z_0 > 0$ be a sufficiently large number so that $B(z)q^+ < 0$ for $z \geq z_0$, and $B(z)q^- < 0$ for $z \leq -z_0$. Set $\epsilon > 0$ be small so that $(\epsilon^2 D + \epsilon C + B(z))q^+ < 0$ for $z \geq z_0$, and $(\epsilon^2 D + \epsilon C + B(z))q^- < 0$ for $z \leq -z_0$. Letting $Q^\pm(z) = e^{\pm \epsilon z} q^\pm$, we have $LQ^+ < 0$ for $z \geq z_0$, and $LQ^- < 0$ for $z \leq -z_0$.

Assume that λ is an eigenvalue of L with eigenfunction $u \in \mathbb{X}_c$ and $u \notin span\{\varphi'(\cdot)\}$. Rewrite $\lambda = \lambda_1 + \lambda_2 i$, $u = u^1 + u^2 i$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, $u^1, u^2 \in \mathbb{X}$, and $u^2 = 0$ if $\lambda_2 = 0$. Consider the Cauchy problem:

$$v_t(z,t) = Lv(z,t) - \lambda_1 v(z,t),\ v(z,0) = u^1(z).$$

The function $v(z,t) = u^1(z)\cos\lambda_2 t - u^2(z)\sin\lambda_2 t$ is a solution of this problem. We require that at least one of the elements of the vector-valued function $v(z,t)$ takes on a positive value (otherwise, we can consider $-v(z,t)$). Let $\psi(z) = \varphi'(z) > 0$. Since $v(z,t)$ is periodic and bounded, we can choose a positive number r such that

$$v(z,t) \leq r\psi(z) \text{ for } |z| \leq z_0 \text{ and } t \geq 0, \tag{4.4.24}$$

where for at least one $k = 1$ or 2, and one $|z_1| \leq z_0$ and $t_1 > 0$, we have the following equality for the k-th components

$$v_k(z_1, t_1) = r\psi_k(z_1). \tag{4.4.25}$$

We proceed the proof by contradiction. Suppose that $\lambda_1 \geq 0$. Then there hold the following two claims.

principle eigenvalue of B^+. Thus, $B^+q^+ < 0$. If $g'(0) > 0$, then B^- is an irreducible matrix. Therefore, a positive eigenvector q^- can be chosen such that $B^-q^- < 0$. If $g'(0) = 0$, let $q^- = (1, \varepsilon)$. Then $B^-q^- < 0$ for some sufficiently small positive number ε.

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Assume that λ is an eigenvalue of L with eigenfunction $u \in \mathbb{X}_c$ and $u \notin span\{\varphi'(\cdot)\}$. Rewrite $\lambda = \lambda_1 + \lambda_2 i$, $u = u^1 + u^2 i$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, $u^1, u^2 \in \mathbb{X}$, and $u^2 = 0$ if $\lambda_2 = 0$. Consider the Cauchy problem:

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We proceed the proof by contradiction. Suppose that $\lambda_1 \geq 0$. Then there hold the following two claims.

Claim 1. $v(z,t) \le r\psi(z)$ for all $z \in \mathbb{R}, t \ge 0$.

Suppose, by contradiction, that there exist some $z > z_0, t \ge 0$ such that $v(z,t) > r\psi(z)$. Since $Q^+(z) = e^{\epsilon z}q^+ \to +\infty$ as $z \to +\infty$, there exists $\tilde{r} > 0$ such that $v(z,t) \le r\psi(z) + \tilde{r}Q^+(z)$ for $z \ge z_0, t \ge 0$, where at least for one j, one $z_2 > z_0$ and $t_2 > 0$, we have the equality for j-th component:

$$v_j(z_2,t_2) = r\psi(z_2) + \tilde{r}Q_j^+(z_2).$$

Let $y(z,t) = r\psi(z) + \tilde{r}Q^+(z) - v(z,t)$. Then the j-th component $y_j(z,t)$ satisfies $y_j(z_2,t_2) = 0, y_j(z_0,t) > 0, y_j(z,t) \ge 0$ for $z \ge z_0, t \ge 0$. Therefore, $y_{j,t}(z_2,t_2) \le 0$, $y_{j,z}(z_2,t_2) = 0$, and if $j = 1$, then $y_{j,zz}(z_2,t_2) \ge 0$. Since $L\psi(z) = 0$, and $LQ^+(z) < 0$ for $z \ge z_0$, $y_j(z,t)$ satisfies

$$\begin{aligned}
y_{j,t} &= -v_{j,t} = -(Lv - \lambda_1 v)_j \\
&> (-Lv + \lambda_1 v + Lr\psi + L\tilde{r}Q^+ - \lambda_1(r\psi + \tilde{r}Q^+))_j \\
&= (Ly - \lambda_1 y)_j \\
&= d_j y_{j,zz} + c y_{j,z} + F_1^j(\varphi(z))y_1 + F_2^j(\varphi(z))y_2 - \lambda_1 y_j,
\end{aligned}$$

where $d_j = d$ if $j = 1$ and $d_j = 0$ if $j = 2$. Evaluating the above inequality at (z_2,t_2) and using the positivity of $F_i^j(\varphi(z))$ for $i \ne j$, we then have a contradiction in signs. Thus $v(z,t) \le r\psi(z), \forall z \ge z_0, t \ge 0$. Using the same argument, we obtain that $v(z,t) \le r\psi(z), \forall z \le -z_0, t \ge 0$. Thus the claim is established.

Claim 2. $v(z,t) \equiv r\psi(z),\ \forall z \in \mathbb{R}, t \ge 0$.

Suppose, by contradiction, that $v(z,t) \not\equiv r\psi(z)$. Then there exist $\bar{t} > 0, \bar{z} \in \mathbb{R}$ and $\bar{k} = 1$ or $= 2$, such that $v_{\bar{k}}(\bar{z},\bar{t}) < r\psi(\bar{z})$. Let $Y(z,t) = r\psi(z) - v(z,t)$. Then $Y(z,t) \ge 0$ for $z \in \mathbb{R}, t \ge 0$, and $Y_{\bar{k}}(\bar{z},\bar{t}) > 0$. Moreover, the components $Y_i(z,t)$ of

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$$\begin{aligned} Y_{i,t} &\geq (LY - \lambda_1 Y)_i \\ &= d_i Y_{i,zz} + cY_{i,z} + F_1^i(\varphi(z))Y_1 + F_2^i(\varphi(z))Y_2 - \lambda_1 Y_i, \end{aligned} \tag{4.4.26}$$

By a similar argument as in Claim 1, it follows from the inequality (4.4.26) that $Y_i(\bar{z},\bar{t}) > 0$ for each $i = 1,2$. Applying the strict positivity theorem ([85, Theorem 5.5.4]), we have $Y_1(z,t) > 0$ for $z \in \mathbb{R}, t > \bar{t}$. By the periodicity of Y in t, we have $Y_1(z,t) > 0$ for $z \in \mathbb{R}, t \geq 0$. Therefore, if $\bar{k} = 1$, defined by (4.4.25), we then have a contradiction. Let us consider the case where $\bar{k} = 2$. Since $Y_2(z_1,t_1) = 0$ and $Y_2(z,t) \geq 0$ for $z \in \mathbb{R}, t \geq 0$, it follows that $Y_{2,t}(z_1,t_1) \leq 0, Y_{2,z}(z_1,t_1) = 0$. Note that $Y_1(z_1,t_1) > 0$. Evaluating (4.4.26) with $i = 2$ at (z_1,t_1), we have a contradiction in signs. This established the claim.

For $\lambda_2 \neq 0$, Claim 2 implies that $Lv(z,t) = Lr\psi(z) = 0$, i.e., $Lu^1(z)\cos\lambda_2 t - Lu^2(z)\sin\lambda_2 t = 0, \forall t \geq 0$. Hence $Lu^1(z) = 0$ and $Lu^2(z) = 0$. Therefore, $Lu = 0$, which contradicts the fact that $Lu = \lambda u \neq 0$. For $\lambda_2 = 0$, we have $u_2 \equiv 0$ and hence $u(z) = u^1(z) = v(z,t) = r\psi(z)$, which contradicts our assumption that $u \notin span\{\psi\}$. Therefore, $\lambda_1 = Re\lambda < 0$. ∎

To show that the essential spectrum $\sigma_e(L)$ of L satisfies the linearization criterion, we will use the results in Section 1.3 developed in [46].

Let T be the following linear operator:

$$Tu = \begin{pmatrix} du_{1,zz} + cu_{1,z} \\ cu_{2,z} \end{pmatrix} + J(z)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $J(z) = (F_j^i(E^+))$ for $z \geq 0$, $J(z) = (F_j^i(E^-))$ for $z < 0$, and $u(z) =$

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$(u_1(z), u_2(z)) \in \mathbb{X}$. Consider the eigenvalue problem of T

$$(T - \lambda I)\begin{pmatrix} p \\ r \end{pmatrix} = 0, \tag{4.4.27}$$

where $\begin{pmatrix} p \\ r \end{pmatrix}(z) \in \mathbb{X}_c$, complexified $\mathbb{X}$. Rewrite (4.4.27) as a system

$$\begin{aligned}
p'(z) &= q, \\
q'(z) &= -\tfrac{1}{d}(cq + J_{11}(z)p + J_{12}(z)r - \lambda p), \\
r'(z) &= -\tfrac{1}{c}(J_{21}(z)p + J_{22}(z)r - \lambda r),
\end{aligned}$$

where $J(z) = (J_{ij}(z))$. Let $y = (p, q, r) \in \mathbb{C}^3$, and write the above system as

$$y' = A(z, \lambda)y, \tag{4.4.28}$$

where

$$A(z,\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{d}(J_{11}(z) - \lambda) & -\frac{c}{d} & -\frac{1}{d}J_{12}(z) \\ -\frac{1}{c}J_{21}(z) & 0 & -\frac{1}{c}(J_{22}(z) - \lambda) \end{pmatrix}.$$

Define $A^+(\lambda) := A(1, \lambda)$, $A^-(\lambda) := A(-1, \lambda)$, and $S^\pm = \{\lambda \in \mathbb{C} : A^\pm = A^\pm(\lambda)$ have imaginary eigenvalues$\}$, which will provide the necessary information about $\sigma_e(L)$.

Lemma 4.4.2 *$\mathbb{C} \setminus S^\pm$ has an open connected set G for which there exists a $\lambda_0 < 0$ such that $\{\lambda : Re\lambda > \lambda_0\} \subset G$.*

$(u_1(z), u_2(z)) \in \mathbb{X}$. Consider the eigenvalue problem of T

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Proof. Let $P(\lambda) = det(A^{\pm} - \mu i I)$. Then

$$
\begin{aligned}
P(\lambda) &= \begin{pmatrix} -\mu i & 1 & 0 \\ -\frac{1}{d}(F_1^1(E^{\pm}) - \lambda) & -\frac{c}{d} - \mu i & -\frac{1}{d}F_2^1(E^{\pm}) \\ -\frac{1}{c}F_1^2(E^{\pm}) & 0 & -\frac{1}{c}(F_2^2(E^{\pm}) - \lambda) - \mu i \end{pmatrix} \\
&= (\tfrac{2}{d}\mu i - \tfrac{1}{c}\mu^2 + \tfrac{1}{cd}(F_1^1(E^{\pm}) + F_2^2(E^{\pm})))\lambda - \tfrac{1}{cd}\lambda^2 + \mu^2(\mu i + \tfrac{c}{d}) + \tfrac{1}{c}\mu^2 F_2^2(E^{\pm}) \\
&\quad -\tfrac{1}{d}\mu i(F_1^1(E^{\pm}) + F_2^2(E^{\pm})) + \tfrac{1}{cd}(F_1^2(E^{\pm})F_2^1(E^{\pm}) - F_1^1(E^{\pm})F_2^2(E^{\pm})).
\end{aligned}
$$

Setting $P(\lambda) = 0$, we have

$$\lambda = \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm}) + 2c\mu i - d\mu^2 \pm \sqrt{\Delta}),$$

where $\Delta = (F_2^2(E^{\pm}) - F_1^1(E^{\pm}) + d\mu^2)^2 + 4F_1^2(E^{\pm})F_2^1(E^{\pm})$, which is positive since $F_j^i(u_1, u_2) \geq 0$ for $i \neq j, 1 \leq i, j \leq 2$. Let $\lambda = \lambda_1 + \lambda_2 i$, where $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\lambda_1 = \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm}) - d\mu^2 \pm \sqrt{\Delta}), \ \lambda_2 = c\mu.$$

Eliminating the parameter μ, we have

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm}) - \frac{d}{c^2}\lambda_2^2) \\
&\quad \pm\frac{1}{2}\sqrt{(F_2^2(E^{\pm}) - F_1^1(E^{\pm}) + \frac{d}{c^2}\lambda_2^2)^2 + 4F_1^2(E^{\pm})F_2^1(E^{\pm})}.
\end{aligned}
$$

Thus the set $S^{\pm}$ is symmetric about the real axis in the complex plane. It is easy to obtain that the derivative $d\lambda_1/d\lambda_2 \leq 0$ for $\lambda_2 \geq 0$. Therefore, the maximal real part of the point in $S^{\pm}$ is one of the following values

$$\lambda^{\pm} = \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm})) + \frac{1}{2}\sqrt{(F_2^2(E^{\pm}) - F_1^1(E^{\pm}))^2 + 4F_1^2(E^{\pm})F_2^1(E^{\pm})}.$$

Proof. Let $P(\lambda) = det(A^{\pm} - \mu i I)$. Then

$$
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P(\lambda) &= \begin{pmatrix} -\mu i & 1 & 0 \\ -\frac{1}{d}(F_1^1(E^{\pm}) - \lambda) & -\frac{c}{d} - \mu i & -\frac{1}{d}F_2^1(E^{\pm}) \\ -\frac{1}{c}F_1^2(E^{\pm}) & 0 & -\frac{1}{c}(F_2^2(E^{\pm}) - \lambda) - \mu i \end{pmatrix} \\
&= (\tfrac{2}{d}\mu i - \tfrac{1}{c}\mu^2 + \tfrac{1}{cd}(F_1^1(E^{\pm}) + F_2^2(E^{\pm})))\lambda - \tfrac{1}{cd}\lambda^2 + \mu^2(\mu i + \tfrac{c}{d}) + \tfrac{1}{c}\mu^2 F_2^2(E^{\pm}) \\
&\quad -\tfrac{1}{d}\mu i(F_1^1(E^{\pm}) + F_2^2(E^{\pm})) + \tfrac{1}{cd}(F_1^2(E^{\pm})F_2^1(E^{\pm}) - F_1^1(E^{\pm})F_2^2(E^{\pm})).
\end{aligned}
$$

Setting $P(\lambda) = 0$, we have

$$\lambda = \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm}) + 2c\mu i - d\mu^2 \pm \sqrt{\Delta}),$$

where $\Delta = (F_2^2(E^{\pm}) - F_1^1(E^{\pm}) + d\mu^2)^2 + 4F_1^2(E^{\pm})F_2^1(E^{\pm})$, which is positive since $F_j^i(u_1, u_2) \geq 0$ for $i \neq j, 1 \leq i, j \leq 2$. Let $\lambda = \lambda_1 + \lambda_2 i$, where $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\lambda_1 = \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm}) - d\mu^2 \pm \sqrt{\Delta}), \ \lambda_2 = c\mu.$$

Eliminating the parameter μ, we have

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm}) - \frac{d}{c^2}\lambda_2^2) \\
&\quad \pm\frac{1}{2}\sqrt{(F_2^2(E^{\pm}) - F_1^1(E^{\pm}) + \frac{d}{c^2}\lambda_2^2)^2 + 4F_1^2(E^{\pm})F_2^1(E^{\pm})}.
\end{aligned}
$$

Thus the set $S^{\pm}$ is symmetric about the real axis in the complex plane. It is easy to obtain that the derivative $d\lambda_1/d\lambda_2 \leq 0$ for $\lambda_2 \geq 0$. Therefore, the maximal real part of the point in $S^{\pm}$ is one of the following values

$$\lambda^{\pm} = \frac{1}{2}(F_1^1(E^{\pm}) + F_2^2(E^{\pm})) + \frac{1}{2}\sqrt{(F_2^2(E^{\pm}) - F_1^1(E^{\pm}))^2 + 4F_1^2(E^{\pm})F_2^1(E^{\pm})}.$$

Note that $\lambda^{\pm}$ are exact eigenvalues of the Jacobian matrix of the reaction system (4.3.15) at $E^{\pm}$. Thus $\lambda^{\pm} < 0$. Therefore, the curves $S^{\pm}$ are bounded uniformly away from the imaginary axis. This proves the lemma. ∎

The implication of Lemma 4.4.2 is that there is no essential spectrum point of L in G.

Lemma 4.4.3 $\sigma(L) \cap G \subset \sigma_p(L)$.

Proof. Define the differential operator $L(\lambda)y = y' - A(z,\lambda)y$. Then, by Theorem 1.3.1, one of the following cases holds: (i). $0 \in \sigma(L(\lambda))$ for all $\lambda \in G$ (defined by Lemma 4.4.2); (ii). $0 \in \rho(L(\lambda))$ for all $\lambda \in G$ except for isolated points, and the exception points are poles of $L(\lambda)^{-1}$ of finite order. Therefore, the set G consists either entirely of spectral points $\sigma(T)$ of T (case (i)), or entirely of normal points of T (case (ii)). Here a normal point is a resolvent point or an isolated eigenvalue of T with finite multiplicity. It is not difficult to see that large positive numbers are not eigenvalues of T (see, e.g., the proof of Lemma 4.4.1). Thus, G consists entirely of normal points of T. Let $S = J_{\varphi}(z) - J(z)$. Then $L = T + S$. It is easy to show that $S(\lambda_0 I - T)^{-1}$ is compact for large positive λ_0. By Theorem 1.3.2, G consists either entirely of normal points of L, or entirely of eigenvalues of L. Hence, Lemma 4.4.1 implies that $\sigma(L) \bigcap G \subset \sigma_p(L)$. ∎

Now we know that $\sigma_e(L)$ causes no problem for linear stability. Hence, we can draw the following conclusion about the global exponential stability.

Theorem 4.4.1 *Let $\varphi(x - ct)$ be a monotone traveling wave solution of (4.3.14) with $c \neq 0$. Then there exists a positive constant $\mu > 0$ such that for every $\psi \in \mathbb{X}_+$*

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satisfying (4.3.20), the solution $U(x,t,\psi)$ *of (4.3.14) satisfies*

$$\|U(x,t,\psi) - \varphi(x - ct + s_\psi)\| \leq C_\psi e^{-\mu t}, \quad \forall x \in \mathbb{R}, t \geq 0,$$

for some constant $s_\psi \in \mathbb{R}$ *and* $C_\psi > 0$.

Proof. By Lemma 4.4.1 and 4.4.3, it follows that zero is a simple eigenvalue of L and the rest of the spectrum $\sigma(L)$ lies in the left-hand complex plane and is bounded away from the imaginary axis. Thus, by the main theorem in [32], zero solution is stable for the linearized PDE system of (4.3.14) at the traveling wave solution. Then by the result in [30], the traveling wave solutions are locally exponentially stable for the original system (4.3.14), and hence, Theorem 4.3.1 completes the proof. ∎

4.5 Numerical Simulations

By Theorems 4.2.1, 4.3.1 and 4.4.1, we know that the epidemic model (4.2.2) admits a unique monotone bistable traveling wave solution (up to translation), which is globally exponentially stable with phase shift. In order to check this result, we numerically simulate solutions of system (4.2.2). Assume that $d = 0.2, \alpha = 2.3, \beta = 1$ and $g(z) = \frac{z^2}{1+z^2}$. Then, $a = 0.5821, b = 1.7179$, and the integral (4.2.13) is $0.07521 > 0$. Hence, Theorem 4.2.1 implies that the wave speed c^* is positive. System (4.2.2) is discretised by using the finite difference method on a finite spatial interval $[-L, L]$ with the Neumann boundary condition, where $L > 0$ is sufficiently large in comparison with the domain in which the solutions rapidly change shapes. The numerical wave profile is shown as solid lines in Figure 4.3 and 4.4. Figure 4.5 and 4.6 provide the evolution of the solution with initial function being the dashed *satisfying (4.3.20), the solution $U(x,t,\psi)$ of (4.3.14) satisfies*

$$\|U(x,t,\psi) - \varphi(x - ct + s_\psi)\| \leq C_\psi e^{-\mu t}, \quad \forall x \in \mathbb{R}, t \geq 0,$$

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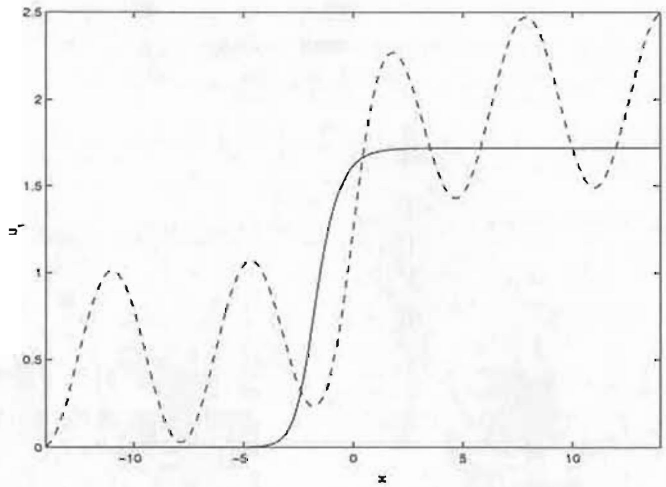

Figure 4.3: The initial function for u_1 component.

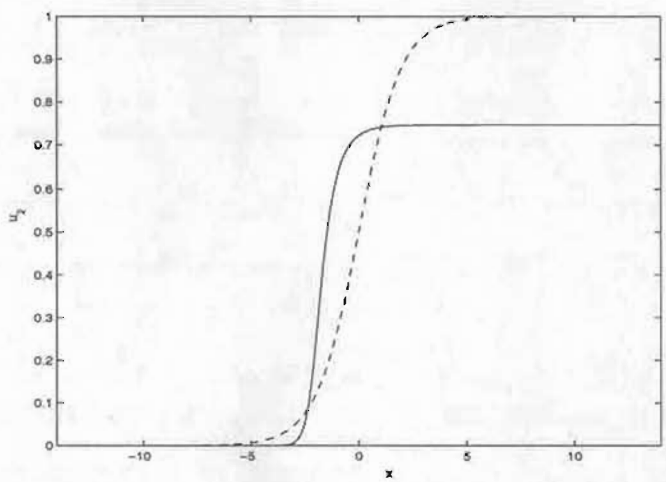

Figure 4.4: The initial function for u_2 component.

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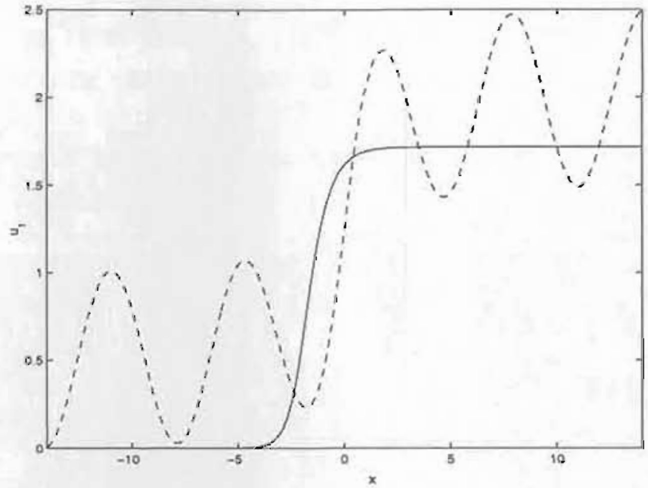


Figure 4.3: The initial function for u_1 component.

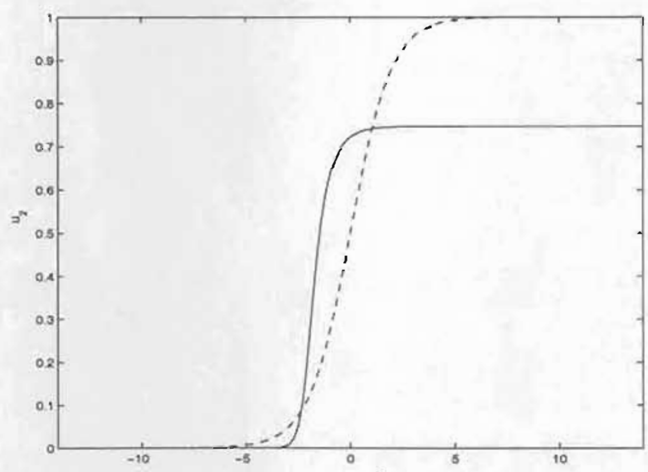


Figure 4.4: The initial function for u_2 component.

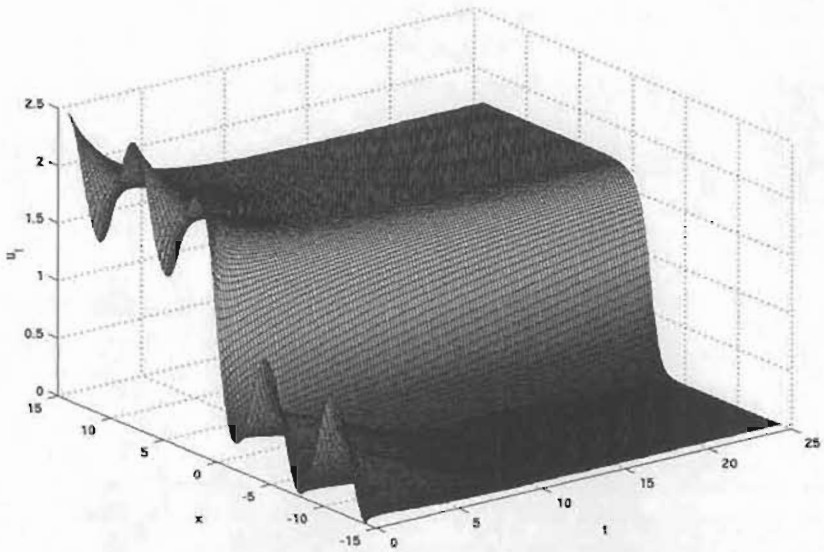


Figure 4.5: The evolutionary graph of $u_1(x, t)$.

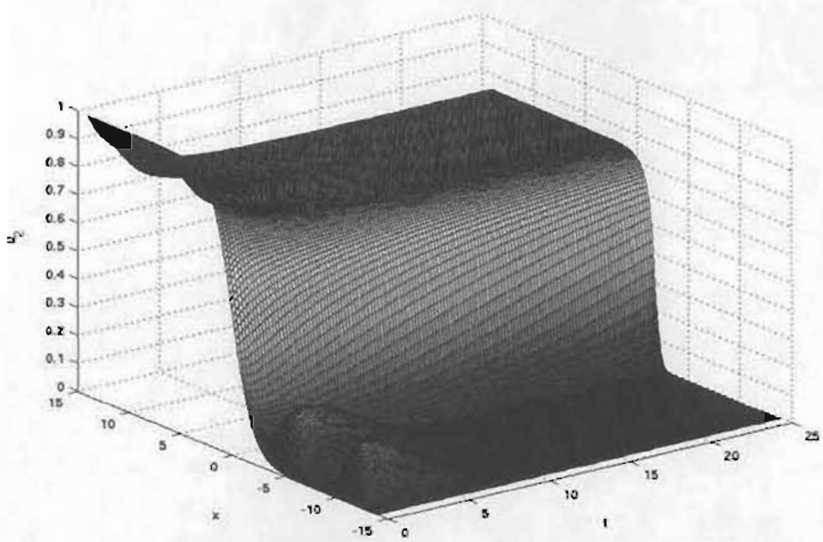


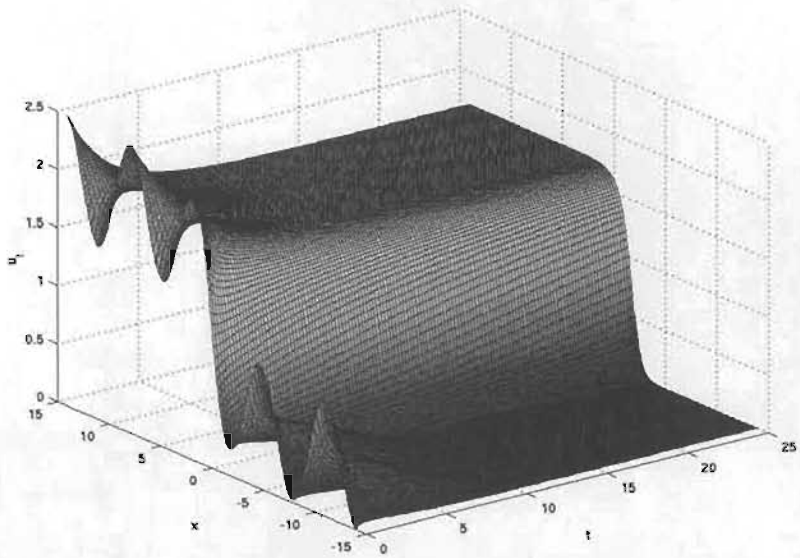
Figure 4.6: The evolutionary graph of $u_2(x, t)$.

Chapter 5

Spreading Speed and Traveling Waves for a Nonlocal Epidemic Model

This chapter will investigate the asymptotic speeds of spread for solutions and traveling wave solutions to the integral version of the epidemic model studied in Chapter 4. We will establish the existence of minimal wave speed for monotone traveling waves, and show that it coincides with the asymptotic speed of spread for solutions with initial functions having compact supports.

This chapter is organized as follows. Section 5.1 presents the nonlocal epidemic model. In Section 5.2, we first reduce the system into an integral equation, and then obtain the asymptotic speed of spread under appropriate assumptions. Section 5.3 is devoted to the existence and nonexistence of monotone traveling wave solutions. Our results show that the asymptotic speed of spread is exactly the minimal wave speed for monotone traveling waves. Finally, some numerical simulations are provided to illustrate the asymptotic speed of spread and monotone traveling waves.



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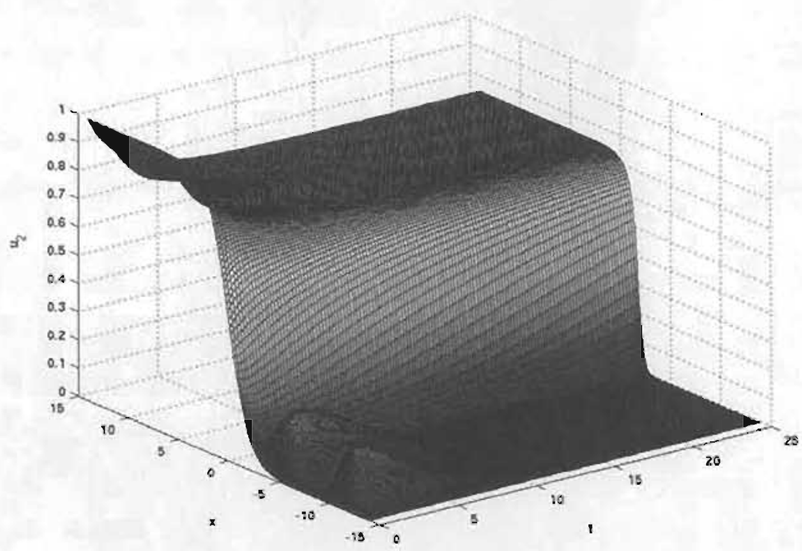

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5.1 Introduction

Recall that, in Chapter 4, we investigated the existence, uniqueness and exponential stability of bistable traveling waves for the epidemic model:

$$\begin{cases} \partial_t u_1(t,x) &= d\triangle u_1(t,x) - a_{11}u_1(t,x) + a_{12}u_2(t,x), \\ \partial_t u_2(t,x) &= -a_{22}u_2(t,x) + g(u_1(t,x)), \end{cases} \tag{5.1.1}$$

where $u_1(t,x)$ and $u_2(t,x)$ denote the spatial densities of infectious agents and the infective human population at time $t \geq 0$, respectively, d, a_{11}, a_{12} and a_{22} are positive constants. This model has some basic assumptions: (i) the total susceptible human population is large enough, with respect to the infective population, to be considered as constant; (ii) the infectious agents diffuse randomly in the habitat Ω due to a particular transmission mechanism; (iii) the infective population at $x \in \Omega$ only contributes to the infectious agents at the same spatial point.

As mentioned in [6], to deal with indirect transmission diseases, typical of infectious diseases transmitted via the pollution of the environment due to the infective population (typhoid fever, schistosomiasis, malaria, etc.), a different approach should be used to model the mechanism of production of the pollutants. A possible model is the one proposed in [14]. Assume that the growth rate of bacteria or pollutants due to the infective population can be modeled by

$$\int_\Omega K(x,y)u_2(t,y)dy, \quad t \geq 0, \ x \in \Omega,$$

where $K(x,y)$ describes the transfer kernel of infectious agents produced by the infective humans located at y and made available at x. From the viewpoint of statistics, normal distribution is one of the most common probability distributions.

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Many phenomena generate random variables with probability distributions which are very well approximated by a normal distribution. Therefore, it is natural to assume that the transfer kernel $K(x, y)$, just like the standard normal density function, is only contingent on the distance between the two spatial points x and y, i.e., $K(x, y) = K(x - y)$, and $K(u) = K(v)$ if $|u| = |v|, \forall u, v \in \Omega$, where $|\cdot|$ denotes the usual norm on $\mathbb{R}^n, n = 1, 2, 3$. This kind of function is said to be isotropic. A typical isotropic function is the standard normal density function. The whole model system is then governed by

$$\begin{cases} \partial_t u_1(t,x) &= d\triangle u_1(t,x) - a_{11}u_1(t,x) + \int_\Omega K(x-y)u_2(t,y)dy, \\ \partial_t u_2(t,x) &= -a_{22}u_2(t,x) + g(u_1(t,x)). \end{cases} \tag{5.1.2}$$

For the monotone increasing infection rate g and a general kernel $K(x, y)$, the stabilities of trivial solution and the unique nontrivial equilibrium solution of (5.1.2) were studied in [14], and [6] provides conditions for exponential decay of the epidemics for (5.1.2). Here, we want to study the asymptotic speed of spread, traveling waves and the minimal wave speed for system (5.1.2) with $\Omega = \mathbb{R}^n$.

The existence of Fisher type monotone traveling waves and minimal wave speed of (5.1.1) were obtained in [97] via the method of upper and lower solutions. In Chapter 4, bistable monotone traveling waves of (5.1.1) were established. Recently, the theory of asymptotic speeds of spread and monotone traveling waves, developed in [8, 3, 13, 7, 9, 26, 28, 27, 79, 80, 71, 90], has been generalized to a large class of scalar nonlinear integral equations in [83]. As an application example, a time-delayed version of (5.1.1) was also analyzed in [83]. We will use this theory to obtain the asymptotic speed of spread for solutions and the minimal wave speed of

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5.2 The Asymptotic Speed of Spread

Recall that a number $c^* > 0$ is called the asymptotic speed of spread for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ if $\lim_{t\to\infty, |x|\geq ct} u(t,x) = 0$ for every $c > c^*$, and there exists some $\bar{u} > 0$ such that $\lim_{t\to\infty,|x|\leq ct} u(t,x) = \bar{u}$ for every $c \in (0, c^*)$, where $|\cdot|$ denotes the usual norm in $\mathbb{R}^n$. In this section, we will find such c^* for solutions of system (5.1.2).

For system (5.1.2), scaling the space variable, we can assume that $d = 1$. Scaling time and absorbing the appropriate constants into u_2 and g, we can rewrite system (5.1.2) as

$$\begin{cases} \partial_t u_1(t,x) &= \triangle u_1(t,x) - u_1(t,x) + \int_{\mathbb{R}^n} K(x-y)u_2(t,y)dy, \\ \partial_t u_2(t,x) &= -\beta u_2(t,x) + g(u_1(t,x)), \quad x \in \mathbb{R}^n, \end{cases} \tag{5.2.3}$$

where $\beta = \frac{a_{22}}{a_{11}}$, and g is the $\frac{1}{a_{11}}$ times of g in system (5.1.2). System (5.2.3) is supplemented by initial conditions

$$u_1(0,x) = \phi_1(x) \geq 0, \;\; u_2(0,x) = \phi_2(x) \geq 0, \;\; x \in \mathbb{R}^n. \tag{5.2.4}$$

In what follows we reduce system (5.2.3)-(5.2.4) into an integral equation for u_1. Let $\Gamma(t,x)$ and $\Gamma_1(t,x)$ be the Green's functions associated with the parabolic equations $\partial_t u = \triangle u$ and $\partial_t u = \triangle u - u$, respectively. Then $\Gamma_1(t,x) = \Gamma(t,x)e^{-t}$.

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$$\begin{cases} \partial_t u_1(t,x) &= \Delta u_1(t,x) - u_1(t,x) + \int_{\mathbb{R}^n} K(x-y) u_2(t,y)dy, \\ \partial_t u_2(t,x) &= -\beta u_2(t,x) + g(u_1(t,x)), \quad x \in \mathbb{R}^n, \end{cases} \tag{5.2.3}$$

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Integrating system (5.2.3) together with (5.2.4), we have

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \\ & \int_0^t ds \int_{\mathbb{R}^n} \Gamma_1(t-s, x-y) \int_{\mathbb{R}^n} K(y-z)u_2(s,z)dzdy, \qquad (5.2.5) \\ u_2(t,x) &= e^{-\beta t}\phi_2(x) + \int_0^t e^{-\beta(t-r)} g(u_1(r,x))dr. \qquad (5.2.6) \end{aligned}$$

Changing the order of spatial integration in (5.2.5),

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \\ & \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z) \int_{\mathbb{R}^n} \Gamma_1(t-s, x-y)K(y-z)dydz. \end{aligned}$$

After a substitution,

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \\ & \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z) \int_{\mathbb{R}^n} \Gamma_1(t-s, x-z-y)K(y)dydz. \end{aligned}$$

Let

$$k_1(t,\xi) = \int_{\mathbb{R}^n} \Gamma_1(t, \xi - y)K(y)dy. \qquad (5.2.7)$$

Then

$$u_1(t,x) = \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z)k_1(t-s, x-z)dz.$$

Inserting (5.2.6) into the above equation,

$$u_1(t,x) = u_0(t,x) + \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s, x-z) \int_0^s e^{-\beta(s-r)} g(u_1(r,z))drdz,$$

where

$$u_0(t,x) = \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s, x-z)e^{-\beta s}\phi_2(z)dz. \quad (5.2.8)$$

Integrating system (5.2.3) together with (5.2.4), we have

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \\ &\int_0^t ds \int_{\mathbb{R}^n} \Gamma_1(t-s, x-y) \int_{\mathbb{R}^n} K(y-z)u_2(s,z)dzdy, \qquad (5.2.5) \\ u_2(t,x) &= e^{-\beta t}\phi_2(x) + \int_0^t e^{-\beta(t-r)} g(u_1(r,x))dr. \qquad (5.2.6) \end{aligned}$$

Changing the order of spatial integration in (5.2.5),

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \\ &\int_0^t ds \int_{\mathbb{R}^n} u_2(s,z) \int_{\mathbb{R}^n} \Gamma_1(t-s, x-y)K(y-z)dydz. \end{aligned}$$

After a substitution,

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \\ &\int_0^t ds \int_{\mathbb{R}^n} u_2(s,z) \int_{\mathbb{R}^n} \Gamma_1(t-s, x-z-y)K(y)dydz. \end{aligned}$$

Let

$$k_1(t,\xi) = \int_{\mathbb{R}^n} \Gamma_1(t, \xi-y)K(y)dy. \qquad (5.2.7)$$

Then

$$u_1(t,x) = \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy + \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z)k_1(t-s, x-z)dz.$$

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Changing the order of the time integration,

$$\begin{aligned} u_1(t,x) &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_r^t k_1(t-s, x-z) e^{-\beta(s-r)} g(u_1(r,z)) ds dz \\ &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_0^{t-r} k_1(t-r-s, x-z) e^{-\beta s} g(u_1(r,z)) ds dz. \end{aligned}$$

After a substitution, we have

$$\begin{aligned} u_1(t,x) &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_0^{t-r} k_1(t-r-s, y) e^{-\beta s} g(u_1(r, x-y)) ds dy \\ &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_0^{r} k_1(r-s, y) e^{-\beta s} g(u_1(t-r, x-y)) ds dy. \end{aligned}$$

Letting $k_2(t,x) = \int_0^t k_1(t-s,x)e^{-\beta s} ds$, we have

$$u_1(t,x) = u_0(t,x) + \int_0^t ds \int_{\mathbb{R}^n} g(u_1(t-s, x-y)) k_2(s,y) dy, \tag{5.2.9}$$

where

$$k_2(t,x) = \int_0^t \int_{\mathbb{R}^n} \Gamma_1(t-s, x-y) K(y) e^{-\beta s} dy ds. \tag{5.2.10}$$

Before making some assumptions on system (5.2.3), we need to compute some Laplace-like transforms of integral kernels. Define $k(t,x) = g'(0)k_2(t,x), f(u) = \frac{g(u)}{g'(0)}$. For any function $\psi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{K}_\phi(c,\lambda) := \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)} \phi(s,y) dy ds, \quad c, \lambda \geq 0,$$

where y_1 is the first coordinate of y. By [83, Proposition 4.2], we have

$$\begin{aligned} \mathcal{K}_{\Gamma_1}(c,\lambda) &= \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)} \Gamma_1(s,y) dy ds \\ &= \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)-s} \Gamma(s,y) dy ds \\ &= \int_0^\infty e^{\lambda^2 s - \lambda c s - s} ds. \end{aligned}$$

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Letting $k_2(t,x) = \int_0^t k_1(t-s,x) e^{-\beta s} ds$, we have

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By [83, Proposition 4.1 (1)], we further obtain

$$\begin{aligned}
\mathcal{K}_k(c,\lambda) &= g'(0)\mathcal{K}_{\Gamma_1}(c,\lambda)\int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)}K(y)e^{-\beta s}dyds \\
&= \frac{g'(0)}{\lambda c+\beta}\mathcal{K}(\lambda)\mathcal{K}_{\Gamma_1}(c,\lambda), \qquad (5.2.11)
\end{aligned}$$

where $\mathcal{K}(\lambda) = \int_{\mathbb{R}^n} e^{-\lambda y_1}K(y)dy$. In particular, $\mathcal{K}_k(0,\lambda) = \frac{g'(0)}{(1-\lambda^2)\beta}\mathcal{K}(\lambda)$. Let $k^* = \mathcal{K}_k(c,0) = \frac{g'(0)}{\beta}\mathcal{K}(0)$. We now can make the following assumptions on system (5.2.3).

(M1) $K : \mathbb{R}^n \to \mathbb{R}_+$ is continuous, and K is isotropic, i.e., $K(x) = K(y)$ if $|x| = |y|$, where $|\cdot|$ is the usual norm on $\mathbb{R}^n$.

(M2) $\mathcal{K}(0) > 0$, and there exists some $\lambda_0 > 0$ such that $\mathcal{K}(\lambda_0) = \infty$, and $\mathcal{K}(\lambda) < \infty$ for all $\lambda \in [0,\lambda_0)$, where λ_0 may be infinity.

(M3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous with $g(0) = 0$, differentiable at 0, and satisfies $g'(0)\mathcal{K}(0) > \beta$, $0 < g(u) \le g'(0)u, \forall u > 0$.

Since $\mathcal{K}(0) > 0$ and $\Gamma_1(t,\cdot) > 0, \forall t > 0$, $k(t,\cdot) > 0$ for all $t > 0$. One can easily check that (M1)-(M3) imply the assumptions (B) and (C) in Section 1.1 with $F(u,s,y) = f(u)k(s,y)$. Our assumptions also imply that system (5.2.3) is quasi-monotone. By [64, Corollary 5] (see also [88, Corollary 8.1.3]) and [88, Corollary 2.2.5], for any bounded, uniformly continuous and nonnegative function $\phi(x) = (\phi_1(x),\phi_2(x))$, system (5.2.3) with (5.2.4) admits a unique and nonnegative mild solution $u(t,x) = (u_1(t,x),u_2(t,x))$, and it is a classic solution for $t > 0$. Note that $u_1(t,x)$ is also a solution of (5.2.9).

With assumption (M2), the expression (5.2.11) shows that if $\lambda^\sharp(c) = \min(\frac{c}{2} + \sqrt{\frac{c^2}{4}+1},\lambda_0)$, then $\mathcal{K}_k(c,\lambda) < \infty$ for all $\lambda \in [0,\lambda^\sharp(c))$, and $\lim\limits_{\lambda\nearrow\lambda^\sharp(c)} \mathcal{K}_k(c,\lambda) = \infty$ for

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$$c^* := \inf\{c \geq 0 : \mathcal{K}_k(c, \lambda) < 1 \text{ for some } \lambda > 0\}.$$

According to Lemma 1.4.1, c^* can be uniquely determined as the solution of the system

$$\mathcal{K}_k(c, \lambda) = 1, \quad \frac{d}{d\lambda}\mathcal{K}_k(c, \lambda) = 0.$$

That is, (c^*, λ^*) is the unique positive solution of the system

$$\begin{aligned}
&(\beta + \lambda c)(1 + \lambda c - c^2) = g'(0)\mathcal{K}(\lambda), \\
&\frac{2\lambda - c}{1 + \lambda c - c^2} - \frac{c}{\beta + \lambda c} = \frac{1}{\mathcal{K}(\lambda)} \int_{\mathbb{R}^n} y_1 e^{-\lambda y_1} K(y) dy.
\end{aligned}$$

The following theorem shows that c^* is the asymptotic speed of spread for solutions of system (5.2.3) with initial functions having compact supports. In order to obtain the convergence for $0 < c < c^*$, we need the following additional conditions.

(M4) $\lim\limits_{u\to\infty} \frac{g(u)}{u} = 0$, and there exists $u^* > 0$ such that g is increasing on $[0, u^*]$, $g(u)\mathcal{K}(0) > \beta u$ for $u \in (0, u^*)$, and $g(u)\mathcal{K}(0) < \beta u$ for $u > u^*$.

(M5) $\limsup\limits_{u\to\infty} \frac{g(u)}{u} < \frac{\beta}{\mathcal{K}(0)}$, $\frac{g(u)}{u}$ is strictly decreasing, and $ug(u)$ is strictly increasing for $u > 0$.

Theorem 5.2.1 *Let (M1)-(M3) hold and c^* be defined as above. Denote by $u(t, x, \phi)$ the unique solution of system (5.2.3)-(5.2.4). Then the following statements are valid:*

(i) For any continuous function $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ with the property that $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, and that for every $\kappa_1 > 0$, there exists $\kappa_2 > 0$ such that

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(i) For any continuous function $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ with the property that $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, and that for every $\kappa_1 > 0$, there exists $\kappa_2 > 0$ such that

$\phi_1(y) + \phi_2(y) \le \kappa_2 e^{-\kappa_1 |y|}, \forall y \in \mathbb{R}^n$, we have

$$\lim_{t\to\infty,\,|x|\ge ct} u(t,x,\phi) = (0,0), \quad \forall c > c^*.$$

(ii) Assume in addition that either (M4) or (M5) holds. Then for any bounded and uniformly continuous function $\phi = (\phi_1,\phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, we have

$$\lim_{t\to\infty,|x|\le ct} u(t,x,\phi) = (u^*,v^*), \quad \forall c \in (0,c^*),$$

where u^* is the unique solution of $g(u)\mathcal{K}(0) = \beta u$, and $v^* = \frac{g(u^*)}{\beta}$.

Proof. Let $\phi = (\phi_1,\phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ be a bounded continuous function with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$. For convenience, we let $u(t,x,\phi) = (u_1(t,x), u_2(t,x))$. Note that $\Gamma_1(t,\cdot) > 0, \forall t > 0$, and $\mathcal{K}(0) > 0$. We have $u_0(t,\cdot) > 0$ for $t > 0$. Let $u_0(t,x) = u_{01}(t,x) + u_{02}(t,x)$, where

$$\begin{aligned} u_{01}(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy, \\ u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s,x-y)e^{-\beta s}\phi_2(y)dy. \end{aligned}$$

In what follows, we show that $\lim_{t\to\infty} u_0(t,x) = 0$ uniformly in $x \in \mathbb{R}^n$. In view of (5.2.7) and the fact that $\int_{\mathbb{R}^n} \Gamma(t,x-y)dy = 1, \forall t \ge 0, x \in \mathbb{R}^n$, we have

$$\begin{aligned} u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma_1(t-s,x-y-z)K(z)e^{\beta s}\phi_2(y)dzdy \\ &= \int_0^t ds \int_{\mathbb{R}^n} e^{-\beta s}K(z) \int_{\mathbb{R}^n} e^{-(t-s)}\Gamma(t-s,x-y-z)\phi_2(y)dydz \\ &\le M_1 \int_0^t e^{-t-\beta s+s}ds \int_{\mathbb{R}^n} K(z)dz \\ &= M_1\mathcal{K}(0) \int_0^t e^{-t-\beta s+s}ds, \end{aligned}$$

$\phi_1(y) + \phi_2(y) \le \kappa_2 e^{-\kappa_1|y|}, \forall y \in \mathbb{R}^n$, we have

$$\lim_{t\to\infty,\,|x|\ge ct} u(t,x,\phi) = (0,0), \quad \forall c > c^*.$$

(ii) Assume in addition that either (M4) or (M5) holds. Then for any bounded and uniformly continuous function $\phi = (\phi_1,\phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, we have

$$\lim_{t\to\infty,|x|\le ct} u(t,x,\phi) = (u^*,v^*), \quad \forall c \in (0,c^*),$$

where u^* is the unique solution of $g(u)\mathcal{K}(0) = \beta u$, and $v^* = \frac{g(u^*)}{\beta}$.

Proof. Let $\phi = (\phi_1,\phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ be a bounded continuous function with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$. For convenience, we let $u(t,x,\phi) = (u_1(t,x), u_2(t,x))$. Note that $\Gamma_1(t,\cdot) > 0, \forall t > 0$, and $\mathcal{K}(0) > 0$. We have $u_0(t,\cdot) > 0$ for $t > 0$. Let $u_0(t,x) = u_{01}(t,x) + u_{02}(t,x)$, where

$$\begin{aligned}
u_{01}(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy, \\
u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s,x-y)e^{-\beta s}\phi_2(y)dy.
\end{aligned}$$

In what follows, we show that $\lim_{t\to\infty} u_0(t,x) = 0$ uniformly in $x \in \mathbb{R}^n$. In view of (5.2.7) and the fact that $\int_{\mathbb{R}^n} \Gamma(t,x-y)dy = 1, \forall t \ge 0, x \in \mathbb{R}^n$, we have

$$\begin{aligned}
u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma_1(t-s,x-y-z)K(z)e^{\beta s}\phi_2(y)dzdy \\
&= \int_0^t ds \int_{\mathbb{R}^n} e^{-\beta s}K(z) \int_{\mathbb{R}^n} e^{-(t-s)}\Gamma(t-s,x-y-z)\phi_2(y)dydz \\
&\le M_1 \int_0^t e^{-t-\beta s+s}ds \int_{\mathbb{R}^n} K(z)dz \\
&= M_1\mathcal{K}(0)\int_0^t e^{-t-\beta s+s}ds,
\end{aligned}$$

where $M_1 = \sup_{y \in \mathbb{R}^n} \phi_2(y)$. Therefore, $\lim_{t\to\infty} u_{02}(t,x) = 0$ uniformly in $x \in \mathbb{R}^n$. Since $\int_{\mathbb{R}^n} \Gamma_1(t,y)dy = e^{-t}$, it follows that $\lim_{t\to\infty} u_{01}(t,x) = 0$, and hence $\lim_{t\to\infty} u_0(t,x) = 0$ uniformly in x. By Proposition 1.4.1, $u_1(t,x)$ is the unique solution of (5.2.9).

(i). For given $c, \lambda > 0$ with $\mathcal{K}_k(c,\lambda) < 1$, $\mathcal{K}_{\Gamma_1}(c,\lambda)$ and $\mathcal{K}(\lambda)$ are finite numbers. Therefore, $\lambda^2 - \lambda c - 1 < 0$. Note that for every $w \in \mathbb{R}^n$ with $|w| = 1$, $-|y| \leq w \cdot y \leq |y|, \forall y \in \mathbb{R}^n$, where $\cdot$ is the inner product on $\mathbb{R}^n$. By the assumption on ϕ_1 and ϕ_2, there exists $\gamma > 0$ such that $\phi_i(y) \leq \gamma e^{-\lambda|y|} \leq \gamma e^{\lambda w\cdot y}, \forall y \in \mathbb{R}^n, i = 1,2$. In the following, we show that $u_0(t,x)$ is admissible in the sense that there exists a constant $\gamma' > 0$ such that $u_0(t,x) \leq \gamma' e^{\lambda(ct-|x|)}, \forall t \geq 0, x \in \mathbb{R}^n$. Note that

$$\int_{\mathbb{R}^n} \Gamma(t,y)e^{-\lambda w\cdot y}dy = \int_{\mathbb{R}^n} \Gamma(t,y)e^{-\lambda y_1}dy = e^{\lambda^2 t}.$$

We then have

$$\begin{aligned}
u_{01}(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy \leq \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\gamma e^{\lambda w\cdot y}dy \\
&= \gamma \int_{\mathbb{R}^n} \Gamma_1(t,y)e^{\lambda w\cdot(x-y)}dy = \gamma e^{\lambda w\cdot x - t}\int_{\mathbb{R}^n} \Gamma(t,y)e^{-\lambda w\cdot y}dy \\
&= \gamma e^{\lambda w\cdot x}e^{(\lambda^2-1)t}.
\end{aligned}$$

Letting $w = -\frac{x}{|x|}$, and using the inequality $\lambda^2 - 1 < \lambda c$, we obtain

$$u_{01}(t,x) \leq \gamma e^{\lambda(ct-|x|)}, \quad \forall t \geq 0,\ x \in \mathbb{R}^n.$$

Applying the similar arguments to $u_{02}(t,x)$, we have

$$\begin{aligned}
u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\beta s}\phi_2(y)\Gamma_1(t-s,x-y-z)K(z)dzdy \\
&= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\beta s}\phi_2(y)\Gamma_1(t-s,x-y-z)K(z)dydz \\
&\leq \gamma \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\beta s}e^{\lambda w\cdot y}\Gamma_1(t-s,x-y-z)K(z)dydz
\end{aligned}$$

where $M_1 = \sup_{y\in\mathbb{R}^n} \phi_2(y)$. Therefore, $\lim_{t\to\infty} u_{02}(t,x) = 0$ uniformly in $x \in \mathbb{R}^n$. Since $\int_{\mathbb{R}^n} \Gamma_1(t,y)dy = e^{-t}$, it follows that $\lim_{t\to\infty} u_{01}(t,x) = 0$, and hence $\lim_{t\to\infty} u_0(t,x) = 0$ uniformly in x. By Proposition 1.4.1, $u_1(t,x)$ is the unique solution of (5.2.9).

(i). For given $c, \lambda > 0$ with $\mathcal{K}_k(c,\lambda) < 1$, $\mathcal{K}_{\Gamma_1}(c,\lambda)$ and $\mathcal{K}(\lambda)$ are finite numbers. Therefore, $\lambda^2 - \lambda c - 1 < 0$. Note that for every $w \in \mathbb{R}^n$ with $|w| = 1$, $-|y| \leq w \cdot y \leq |y|, \forall y \in \mathbb{R}^n$, where $\cdot$ is the inner product on $\mathbb{R}^n$. By the assumption on ϕ_1 and ϕ_2, there exists $\gamma > 0$ such that $\phi_i(y) \leq \gamma e^{-\lambda|y|} \leq \gamma e^{\lambda w\cdot y}, \forall y \in \mathbb{R}^n, i = 1, 2$. In the following, we show that $u_0(t,x)$ is admissible in the sense that there exists a constant $\gamma' > 0$ such that $u_0(t,x) \leq \gamma' e^{\lambda(ct-|x|)}, \forall t \geq 0, x \in \mathbb{R}^n$. Note that

$$\int_{\mathbb{R}^n} \Gamma(t,y)e^{-\lambda w\cdot y}dy = \int_{\mathbb{R}^n} \Gamma(t,y)e^{-\lambda y_1}dy = e^{\lambda^2 t}.$$

We then have

$$\begin{aligned}
u_{01}(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy \leq \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\gamma e^{\lambda w\cdot y}dy \\
&= \gamma \int_{\mathbb{R}^n} \Gamma_1(t,y)e^{\lambda w\cdot(x-y)}dy = \gamma e^{\lambda w\cdot x - t}\int_{\mathbb{R}^n} \Gamma(t,y)e^{-\lambda w\cdot y}dy \\
&= \gamma e^{\lambda w\cdot x}e^{(\lambda^2-1)t}.
\end{aligned}$$

Letting $w = -\frac{x}{|x|}$, and using the inequality $\lambda^2 - 1 < \lambda c$, we obtain

$$u_{01}(t,x) \leq \gamma e^{\lambda(ct-|x|)}, \quad \forall t \geq 0,\ x \in \mathbb{R}^n.$$

Applying the similar arguments to $u_{02}(t,x)$, we have

$$\begin{aligned}
u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\beta s}\phi_2(y)\Gamma_1(t-s,x-y-z)K(z)dzdy \\
&= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\beta s}\phi_2(y)\Gamma_1(t-s,x-y-z)K(z)dydz \\
&\leq \gamma \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\beta s}e^{\lambda w\cdot y}\Gamma_1(t-s,x-y-z)K(z)dydz
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma \int_0^t ds \int_{\mathbb{R}^n} e^{-\beta s} e^{\lambda w\cdot(x-z)} e^{(\lambda^2-1)(t-s)} K(z) dz \\
&\leq \gamma e^{\lambda(ct+w\cdot x)} \int_0^t e^{-(\lambda c+\beta)s} ds \int_{\mathbb{R}^n} e^{-\lambda z_1} K(z) dz \\
&\leq \frac{\mathcal{K}(\lambda)}{\lambda c+\beta} e^{\lambda(ct+w\cdot x)}.
\end{aligned}
$$

Letting $w = -\frac{x}{|x|}$,

$$u_{02}(t,x) \leq \frac{\mathcal{K}(\lambda)}{\lambda c+\beta} e^{\lambda(ct-|x|)}, \quad \forall t \geq 0,\ x \in \mathbb{R}^n.$$

Therefore, $u_0(t,x)$ is admissible. By Theorem 1.4.1, it follows that

$$\lim_{t\to\infty, |x|\geq ct} u_1(t,x) = 0, \quad \text{for each } c > c^*,$$

and hence (5.2.6) implies the result.

(ii). Assume in addition that either (M4) or (M5) holds. Then we can find some constants $c_1, c_2 \geq 0$ such that $c_1 k^* < 1$ and $g(u) \leq g'(0)(c_2 + c_1 u), \forall u \geq 0$. Therefore, Proposition 1.4.1 implies that every solution of (5.2.9) is bounded. Note that the monotonicity of g on $[0, u^*]$ implies that there is no pair $w > u^* > v > 0$ such that $\beta w = \mathcal{K}(0) g(v)$ and $\beta v = \mathcal{K}(0) g(w)$. Thus, by Theorem 1.4.2 and 1.4.3, we have

$$\lim_{t\to\infty, |x|\leq ct} u_1(t,x) = u^*, \quad \forall c \in (0, c^*).$$

Therefore,

$$\lim_{t\to\infty, |x|\leq ct} u_2(t,x) = g(u^*) \int_0^\infty e^{-\beta s} ds = \frac{g(u^*)}{\beta} = v^*, \quad \forall c \in (0, c^*).$$

This completes the proof. ∎

Remark 5.2.1 Theorem 5.2.1 implies that c^* is the asymptotic speed for solutions of system (5.2.3) with initial functions having compact supports. Let $u(t,x) =$

$$\begin{aligned}
&\leq \gamma \int_0^t ds \int_{\mathbb{R}^n} e^{-\beta s} e^{\lambda w\cdot(x-z)} e^{(\lambda^2-1)(t-s)} K(z)dz \\
&\leq \gamma e^{\lambda(ct+w\cdot x)} \int_0^t e^{-(\lambda c+\beta)s} ds \int_{\mathbb{R}^n} e^{-\lambda z_1} K(z)dz \\
&\leq \frac{\mathcal{K}(\lambda)}{\lambda c+\beta} e^{\lambda(ct+w\cdot x)}.
\end{aligned}$$

Letting $w = -\frac{x}{|x|}$,

$$u_{02}(t,x) \leq \frac{\mathcal{K}(\lambda)}{\lambda c+\beta} e^{\lambda(ct-|x|)}, \quad \forall t \geq 0,\ x \in \mathbb{R}^n.$$

Therefore, $u_0(t,x)$ is admissible. By Theorem 1.4.1, it follows that

$$\lim_{t\to\infty, |x|\geq ct} u_1(t,x) = 0, \quad \text{for each } c > c^*,$$

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(ii). Assume in addition that either (M4) or (M5) holds. Then we can find some constants $c_1, c_2 \geq 0$ such that $c_1 k^* < 1$ and $g(u) \leq g'(0)(c_2 + c_1 u), \forall u \geq 0$. Therefore, Proposition 1.4.1 implies that every solution of (5.2.9) is bounded. Note that the monotonicity of g on $[0, u^*]$ implies that there is no pair $w > u^* > v > 0$ such that $\beta w = \mathcal{K}(0)g(v)$ and $\beta v = \mathcal{K}(0)g(w)$. Thus, by Theorem 1.4.2 and 1.4.3, we have

$$\lim_{t\to\infty, |x|\leq ct} u_1(t,x) = u^*, \quad \forall c \in (0, c^*).$$

Therefore,

$$\lim_{t\to\infty, |x|\leq ct} u_2(t,x) = g(u^*) \int_0^\infty e^{-\beta s} ds = \frac{g(u^*)}{\beta} = v^*, \quad \forall c \in (0, c^*).$$

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$(u_1(t,x), u_2(t,x))$ be such a solution. For any given $\rho \in (0, u^*)$, denote by $x_+^\rho(t)$ and $x_-^\rho(t)$ the most right and left points with $u_1(t, x_\pm^\rho(t)) = \rho$, respectively. Clearly, $x_+^\rho(t)$ and $x_-^\rho(t)$ are well defined for all large t because of the two limit formulas in Theorem 5.2.1. We claim that $\lim\limits_{t\to\infty} \frac{x_+^\rho(t)}{t} = c^*$. Indeed, by Theorem 5.2.1, it follows that for any $0 < \varepsilon < \min(\rho, u^* - \rho)$, there exists some $t_0 = t_0(\varepsilon) > 0$ such that $u_1(t,x) < \varepsilon$ for all $t \geq t_0, |x| \geq (c^* + \varepsilon)t$, and $|u_1(t,x) - u^*| < \varepsilon$ for all $t \geq t_0, |x| \leq (c^* - \varepsilon)t$. Therefore, $x_+^\rho(t) < (c^* + \varepsilon)t$ and $x_+^\rho(t) > (c^* - \varepsilon)t$, and hence, $|\frac{x_+^\rho(t)}{t} - c^*| < \varepsilon$, for all $t > t_0$. By a similar argument, we can prove that $\lim\limits_{t\to\infty} \frac{|x_-^\rho(t)|}{t} = c^*$. We will use this observation to compute c^* numerically.

5.3 Traveling Wave Solutions

In this section, we consider the traveling wave solutions of system (5.2.3) with $n = 1$. Recall that a solution $u(t,x)$ of system (5.2.3) is said to be a traveling wave solution if it is of the form $u(t,x) = U(x+ct)$. The parameter c is called the wave speed, and the function $U(\cdot)$ is called the wave profile. We will impose the following conditions on the wave profile:

$$U(\cdot) \text{ is positive and bounded on } \mathbb{R}, \text{ and } \lim_{\xi\to-\infty} U(\xi) = 0. \tag{5.3.12}$$

Consider the system

$$u_1(t,x) = \int_0^\infty ds \int_{\mathbb{R}} f(u_1(t-s, x-y))k(s,y)dy, \tag{5.3.13}$$

$$u_2(t,x) = \int_0^\infty e^{-\beta s} g(u_1(t-s,x))ds. \tag{5.3.14}$$

If system (5.3.13)-(5.3.14) admits a solution with the form $(U_1(x+ct), U_2(x+ct))$, then it is called a traveling wave solution with speed c. The following lemma shows

$(u_1(t,x), u_2(t,x))$ be such a solution. For any given $\rho \in (0, u^*)$, denote by $x_+^\rho(t)$ and $x_-^\rho(t)$ the most right and left points with $u_1(t, x_\pm^\rho(t)) = \rho$, respectively. Clearly, $x_+^\rho(t)$ and $x_-^\rho(t)$ are well defined for all large t because of the two limit formulas in Theorem 5.2.1. We claim that $\lim_{t\to\infty} \frac{x_+^\rho(t)}{t} = c^*$. Indeed, by Theorem 5.2.1, it follows that for any $0 < \varepsilon < \min(\rho, u^* - \rho)$, there exists some $t_0 = t_0(\varepsilon) > 0$ such that $u_1(t,x) < \varepsilon$ for all $t \geq t_0, |x| \geq (c^* + \varepsilon)t$, and $|u_1(t,x) - u^*| < \varepsilon$ for all $t \geq t_0, |x| \leq (c^* - \varepsilon)t$. Therefore, $x_+^\rho(t) < (c^* + \varepsilon)t$ and $x_+^\rho(t) > (c^* - \varepsilon)t$, and hence, $|\frac{x_+^\rho(t)}{t} - c^*| < \varepsilon$, for all $t > t_0$. By a similar argument, we can prove that $\lim_{t\to\infty} \frac{|x_-^\rho(t)|}{t} = c^*$. We will use this observation to compute c^* numerically.

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Consider the system

$$u_1(t,x) = \int_0^\infty ds \int_{\mathbb{R}} f(u_1(t-s, x-y))k(s,y)dy, \tag{5.3.13}$$

$$u_2(t,x) = \int_0^\infty e^{-\beta s} g(u_1(t-s, x))ds. \tag{5.3.14}$$

If system (5.3.13)-(5.3.14) admits a solution with the form $(U_1(x+ct), U_2(x+ct))$, then it is called a traveling wave solution with speed c. The following lemma shows that the existence of traveling wave solutions of system (5.2.3) is equivalent to those of system (5.3.13)-(5.3.14).

Lemma 5.3.1 *If system (5.3.13)-(5.3.14) admits a traveling wave $U(x+ct)$ subject to (5.3.12), then $U(x+ct)$ is also a traveling wave of (5.2.3) subject to (5.3.12). The converse also holds.*

Proof. Let $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ be a traveling wave solution of system (5.3.13)-(5.3.14). Then

$$u_2(t,x) = \int_0^\infty e^{-\beta s} g(u_1(t-s,x))ds = U_2(x+ct).$$

In view of (5.2.10), we have

$$\begin{aligned}
u_1(t,x) &= \int_0^\infty ds \int_{\mathbb{R}} f(u_1(t-s,x-y))k(s,y)dy \\
&= \int_0^\infty ds \int_{\mathbb{R}} dy \int_0^s dr \int_{\mathbb{R}} g(u_1(t-s,x-y))\Gamma_1(s-r,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_r^\infty ds \int_{\mathbb{R}} g(u_1(t-s,x-y))\Gamma_1(s-r,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_0^\infty ds \int_{\mathbb{R}} g(u_1(t-r-s,x-y))\Gamma_1(s,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_0^\infty ds \int_{\mathbb{R}} g(u_1(t-r-s,x-y))\Gamma_1(s,z)K(y-z)e^{-\beta r}dz \\
&= \int_0^\infty ds \int_{\mathbb{R}} \Gamma_1(s,z) \int_0^\infty \int_{\mathbb{R}} g(u_1(t-r-s,x-y))K(y-z)e^{-\beta r}dydrdz \\
&= \int_0^\infty ds \int_{\mathbb{R}} \Gamma_1(s,z) \int_{\mathbb{R}} \int_0^\infty g(u_1(t-r-s,x-y))K(y-z)e^{-\beta r}drdydz \\
&= \int_0^\infty T_1(s) \int_{\mathbb{R}} K(y-z)u_2(t-s,x-y)dyds, \qquad (5.3.15)
\end{aligned}$$

where $T_1(t)$ is the semigroup on $BUC(\mathbb{R},\mathbb{R})$ generated by the parabolic equation $\partial_t u = \triangle u - u$, and $BUC(\mathbb{R},\mathbb{R})$ is the Banach space of all bounded and uniformly that the existence of traveling wave solutions of system (5.2.3) is equivalent to those of system (5.3.13)-(5.3.14).

Lemma 5.3.1 *If system (5.3.13)-(5.3.14) admits a traveling wave $U(x+ct)$ subject to (5.3.12), then $U(x+ct)$ is also a traveling wave of (5.2.3) subject to (5.3.12). The converse also holds.*

Proof. Let $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ be a traveling wave solution of system (5.3.13)-(5.3.14). Then

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In view of (5.2.10), we have

$$\begin{aligned}
u_1(t,x) &= \int_0^\infty ds \int_{\mathbb{R}} f(u_1(t-s,x-y))k(s,y)dy \\
&= \int_0^\infty ds \int_{\mathbb{R}} dy \int_0^s dr \int_{\mathbb{R}} g(u_1(t-s,x-y))\Gamma_1(s-r,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_r^\infty ds \int_{\mathbb{R}} g(u_1(t-s,x-y))\Gamma_1(s-r,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_0^\infty ds \int_{\mathbb{R}} g(u_1(t-r-s,x-y))\Gamma_1(s,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_0^\infty ds \int_{\mathbb{R}} g(u_1(t-r-s,x-y))\Gamma_1(s,z)K(y-z)e^{-\beta r}dz \\
&= \int_0^\infty ds \int_{\mathbb{R}} \Gamma_1(s,z) \int_0^\infty \int_{\mathbb{R}} g(u_1(t-r-s,x-y))K(y-z)e^{-\beta r}dydrdz \\
&= \int_0^\infty ds \int_{\mathbb{R}} \Gamma_1(s,z) \int_{\mathbb{R}} \int_0^\infty g(u_1(t-r-s,x-y))K(y-z)e^{-\beta r}drdydz \\
&= \int_0^\infty T_1(s) \int_{\mathbb{R}} K(y-z)u_2(t-s,x-y)dyds, \qquad (5.3.15)
\end{aligned}$$

where $T_1(t)$ is the semigroup on $BUC(\mathbb{R},\mathbb{R})$ generated by the parabolic equation $\partial_t u = \Delta u - u$, and $BUC(\mathbb{R},\mathbb{R})$ is the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to itself. By [83, Proposition 4.3], it follows that $(u_1(t,\cdot), u_2(t,\cdot))$ satisfies the abstract integral equations

$$u_1(t) = T_1(t-r)u_1(r) + \int_r^t T_1(t-s)\int_{\mathbb{R}} K(y-z)u_2(s)dyds, \tag{5.3.16}$$

$$u_2(t) = e^{-\beta(t-r)}u_2(r) + \int_r^t e^{-\beta(t-s)}g(u_1(s))ds, \quad \forall t \geq r,\ r \in \mathbb{R}. \tag{5.3.17}$$

Clearly, $u_2(t,x)$ satisfies the second equation of system (5.2.3). By the form $u_1(t,x) = U_1(x+ct)$ and the smoothing property of parabolic operators (see, e.g., [88, Corollary 2.2.5] with $r = 0$), it follows that $u_1(t,x)$ satisfies the first equation of system (5.2.3). Thus $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ is a traveling wave solution of system (5.2.3) with speed c.

Conversely, let $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ be a traveling wave solution of system (5.2.3). Then $(u_1(t,x), u_2(t,x))$ is a continuous and bounded solution of (5.3.16)-(5.3.17) on $(-\infty, +\infty)$. By [83, Proposition 4.3], $u_2(t,x)$ satisfies (5.3.14) and $u_1(t,x)$ satisfies

$$u_1(t,x) = \int_0^\infty T_1(s)\int_{\mathbb{R}} K(y-z)u_2(t-s, x-y)dyds.$$

Since the process in formula (5.3.15) is invertible, $u_1(t,x)$ satisfies equation (5.3.13). It follows that $(u_1(t,x), u_2(t,x))$ is a traveling wave solution of system (5.3.13)-(5.3.14) with wave speed c. ∎

Theorem 5.3.1 *Let (M1)-(M3) hold, and let c^*, v^* be defined as in Theorem 5.2.1. Then the following statements are valid.*

(i) There is no traveling wave solution for system (5.2.3) and (5.3.12) with wave speed $c \in (0, c^*)$.

continuous functions from $\mathbb{R}$ to itself. By [83, Proposition 4.3], it follows that $(u_1(t,\cdot), u_2(t,\cdot))$ satisfies the abstract integral equations

$$u_1(t) = T_1(t-r)u_1(r) + \int_r^t T_1(t-s)\int_{\mathbb{R}} K(y-z)u_2(s)dyds, \qquad (5.3.16)$$

$$u_2(t) = e^{-\beta(t-r)}u_2(r) + \int_r^t e^{-\beta(t-s)}g(u_1(s))ds, \quad \forall t \geq r,\ r \in \mathbb{R}. \qquad (5.3.17)$$

Clearly, $u_2(t,x)$ satisfies the second equation of system (5.2.3). By the form $u_1(t,x) = U_1(x+ct)$ and the smoothing property of parabolic operators (see, e.g., [88, Corollary 2.2.5] with $r=0$), it follows that $u_1(t,x)$ satisfies the first equation of system (5.2.3). Thus $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ is a traveling wave solution of system (5.2.3) with speed c.

Conversely, let $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ be a traveling wave solution of system (5.2.3). Then $(u_1(t,x), u_2(t,x))$ is a continuous and bounded solution of (5.3.16)-(5.3.17) on $(-\infty, +\infty)$. By [83, Proposition 4.3], $u_2(t,x)$ satisfies (5.3.14) and $u_1(t,x)$ satisfies

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Since the process in formula (5.3.15) is invertible, $u_1(t,x)$ satisfies equation (5.3.13). It follows that $(u_1(t,x), u_2(t,x))$ is a traveling wave solution of system (5.3.13)-(5.3.14) with wave speed c. ∎

Theorem 5.3.1 *Let (M1)-(M3) hold, and let c^*, v^* be defined as in Theorem 5.2.1. Then the following statements are valid.*

(i) There is no traveling wave solution for system (5.2.3) and (5.3.12) with wave speed $c \in (0, c^*)$.

(ii) Assume in addition that (M4) holds, and that $|g(u) - g(v)| \leq g'(0)|u - v|, \forall u, v \in [0, u^*]$, *and* $g''(0)$ *exists. Then system (5.2.3) with (5.3.12) admits a monotone traveling wave connecting* $(0,0)$ *and* (u^*, v^*) *with speed* $c \geq c^*$. *Moreover, the monotone traveling wave with speed* $c > c^*$ *is unique up to translation.*

Proof. (i). Note that (M1)-(M3) imply the assumption (B) and (C) in Section 1.3. The result is a straight forward consequence of Theorem 1.4.7.

(ii). Since $g''(0)$ exists, we can find two numbers $\delta > 0, b > 0$ such that $g(u) \geq g'(0)(u - bu^2), \forall u \in [0, \delta]$. By Theorem 1.4.5 and 1.4.6, as applied to equation (5.3.13) with $F(u, s, x) = f(u)k(s, x)$, it follows that for each $c \geq c^*$, (5.3.13) admits a monotone traveling wave $u_1(t, x) = U_1(x + ct)$ connecting 0 and u^*. Define $u_2(t, x)$ as in equation (5.3.14), we then have

$$u_2(t, x) = \int_0^\infty e^{-\beta s} g(U_1(x + c(t - s)))ds := U_2(x + ct), \tag{5.3.18}$$

where $U_2(\xi) = \int_0^\infty e^{-\beta s} g(U_1(\xi - cs))ds$. Obviously, $U_2'(\xi) > 0$. By the dominant convergence theorem, $\lim_{\xi \to -\infty} U_2(\xi) = 0$, and $\lim_{\xi \to \infty} U_2(\xi) = v^*$. Therefore, $(u_1(t, x), u_2(t, x))$ is a traveling wave of system (5.3.13)-(5.3.14), and hence Lemma 5.3.1 implies the result. The uniqueness of traveling waves with $c > c^*$ follows from Theorem 1.4.4, as applied to (5.3.13) with $F(u, s, x) = f(u)k(s, x)$, and Lemma 5.3.1. ∎

Numerical simulation. We numerically simulate system (5.2.3) with $n = 1$. Let the transfer kernel K be the standard normal density function, i.e., $K(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and set $g(u) = \frac{2u}{1+u}$, $\beta = 1$. It is easy to see that system (5.2.3) satisfies assumptions (M1)-(M4) with $u^* = v^* = 1$. By Theorem 5.3.1, for any continuous *(ii) Assume in addition that (M4) holds, and that* $|g(u) - g(v)| \leq g'(0)|u - v|, \forall u, v \in [0, u^*]$, *and* $g''(0)$ *exists. Then system (5.2.3) with (5.3.12) admits a monotone traveling wave connecting* $(0,0)$ *and* (u^*, v^*) *with speed* $c \geq c^*$. *Moreover, the monotone traveling wave with speed* $c > c^*$ *is unique up to translation.*

Proof. (i). Note that (M1)-(M3) imply the assumption (B) and (C) in Section 1.3. The result is a straight forward consequence of Theorem 1.4.7.

(ii). Since $g''(0)$ exists, we can find two numbers $\delta > 0, b > 0$ such that $g(u) \geq g'(0)(u - bu^2), \forall u \in [0, \delta]$. By Theorem 1.4.5 and 1.4.6, as applied to equation (5.3.13) with $F(u, s, x) = f(u)k(s, x)$, it follows that for each $c \geq c^*$, (5.3.13) admits a monotone traveling wave $u_1(t, x) = U_1(x + ct)$ connecting 0 and u^*. Define $u_2(t, x)$ as in equation (5.3.14), we then have

$$u_2(t, x) = \int_0^\infty e^{-\beta s} g(U_1(x + c(t - s)))ds := U_2(x + ct), \tag{5.3.18}$$

where $U_2(\xi) = \int_0^\infty e^{-\beta s} g(U_1(\xi - cs))ds$. Obviously, $U_2'(\xi) > 0$. By the dominant convergence theorem, $\lim_{\xi \to -\infty} U_2(\xi) = 0$, and $\lim_{\xi \to \infty} U_2(\xi) = v^*$. Therefore, $(u_1(t, x), u_2(t, x))$ is a traveling wave of system (5.3.13)-(5.3.14), and hence Lemma 5.3.1 implies the result. The uniqueness of traveling waves with $c > c^*$ follows from Theorem 1.4.4, as applied to (5.3.13) with $F(u, s, x) = f(u)k(s, x)$, and Lemma 5.3.1. ∎

Numerical simulation. We numerically simulate system (5.2.3) with $n = 1$. Let the transfer kernel K be the standard normal density function, i.e., $K(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and set $g(u) = \frac{2u}{1+u}$, $\beta = 1$. It is easy to see that system (5.2.3) satisfies assumptions (M1)-(M4) with $u^* = v^* = 1$. By Theorem 5.2.1, for any continuous initial functions ϕ_1, ϕ_2 with compact supports, we have

$$\lim_{t\to\infty, |x|\geq ct} u(t,x) = (0,0), \quad \forall c > c^*,$$

and

$$\lim_{t\to\infty, |x|\leq ct} u(t,x) = (1,1), \quad \forall c \in (0, c^*).$$

We discretise system (5.2.3) by the finite difference method coupled with composite integration formulas on a finite spatial interval $[-L, L]$ with the Neumann boundary condition, where $L > 0$ is sufficiently large in comparison with the domain in which the solutions rapidly change shapes. Let

$$\phi_1(x) = \phi_2(x) = \begin{cases} 0, & \text{if } x \leq -\pi/2, \\ \frac{1}{2}\cos x, & \text{if } x \in (-\pi/2, \pi/2), \\ 0, & \text{if } x \geq \pi/2. \end{cases} \tag{5.3.19}$$

Figure 5.1 and 5.2 illustrates the corresponding numerical solution $u(t,x) = (u_1(t,x), u_2(t,x))$. Obviously, the result is consistent with the above two limit formulas. In order to get the asymptotic speed c^*, we use Remark 5.2.1 to approximate c^*. Figure 5.3 shows the curves $x_+^{0.25}(t)/t$ and $x_-^{0.25}(t)/t$ versus t. Thus, $c^* \approx 1.0$. To get a traveling wave, we choose the initial condition as

$$\phi_1(x) = \phi_2(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ \frac{1}{2}(1+x), & \text{if } x \in (-1, 1), \\ 1, & \text{if } x \geq 1. \end{cases} \tag{5.3.20}$$

The evolution of the solution is shown in Figure 5.4 and 5.5. The solution becomes smooth immediately. The shape of the solution promptly converges to a traveling initial functions ϕ_1, ϕ_2 with compact supports, we have

$$\lim_{t\to\infty, |x|\geq ct} u(t,x) = (0,0), \quad \forall c > c^*,$$

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The evolution of the solution is shown in Figure 5.4 and 5.5. The solution becomes smooth immediately. The shape of the solution promptly converges to a traveling wave. The wave moves in the negative x-direction as the time t increases (shown as in Figure 5.6), and the wave speed is about 1.0, which coincides with the spreading speed.

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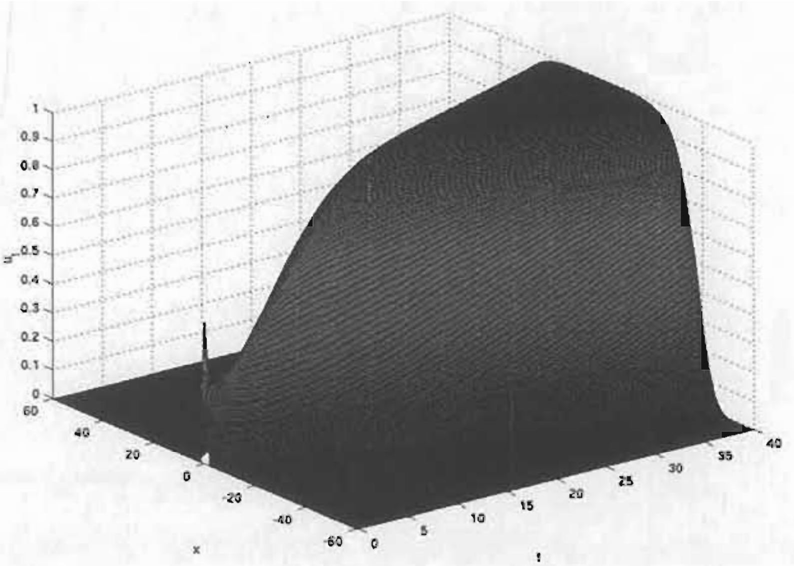


Figure 5.1: The evolutionary graph of $u_1(t,x)$ for the solution $(u_1(t,x),u_2(t,x))$ with initial function (5.3.19)

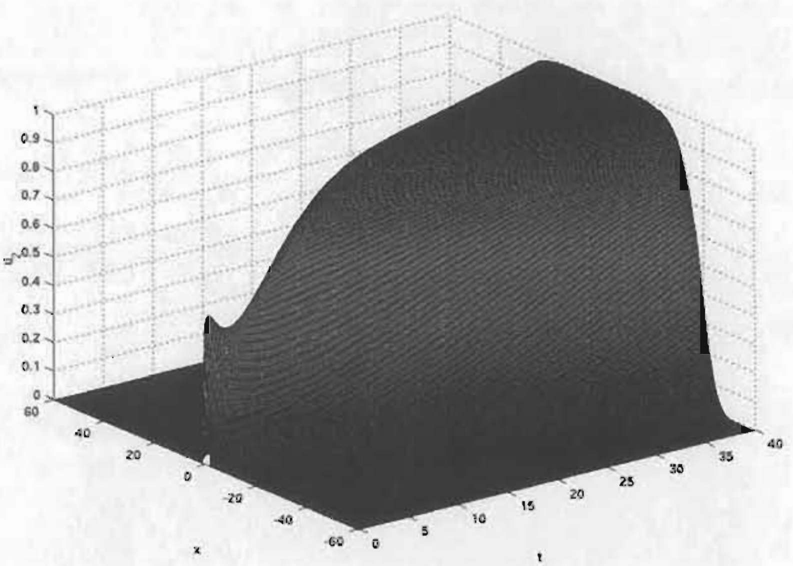


Figure 5.2: The evolutionary graph of $u_2(t,x)$ for the solution $(u_1(t,x),u_2(t,x))$ with initial function (5.3.19).

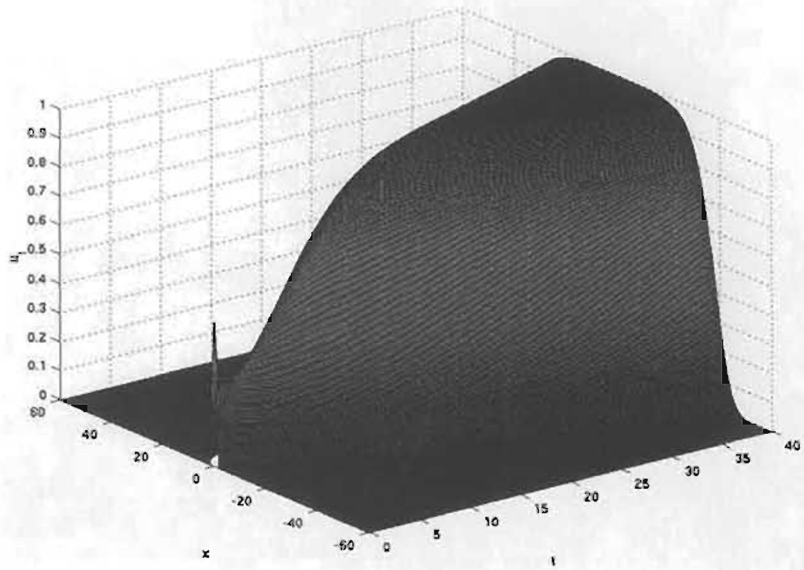


Figure 5.1: The evolutionary graph of $u_1(t,x)$ for the solution $(u_1(t,x), u_2(t,x))$ with initial function (5.3.19)

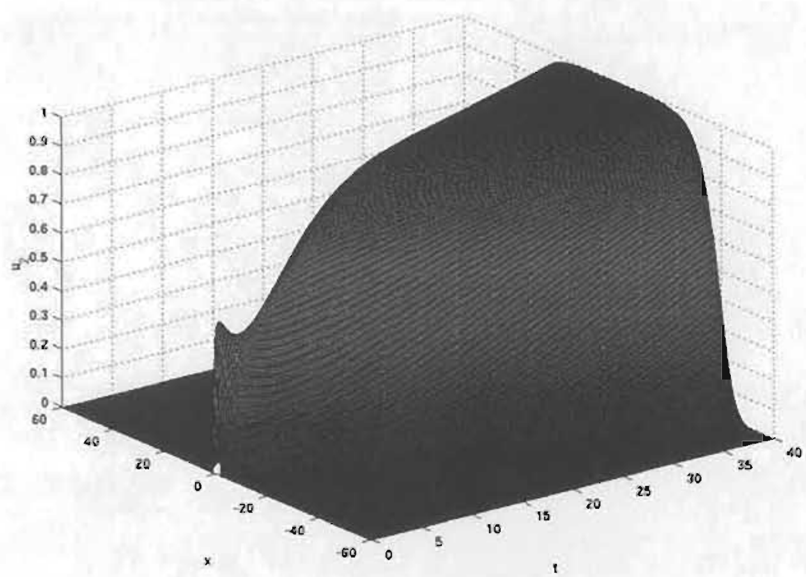


Figure 5.2: The evolutionary graph of $u_2(t,x)$ for the solution $(u_1(t,x), u_2(t,x))$ with initial function (5.3.19).

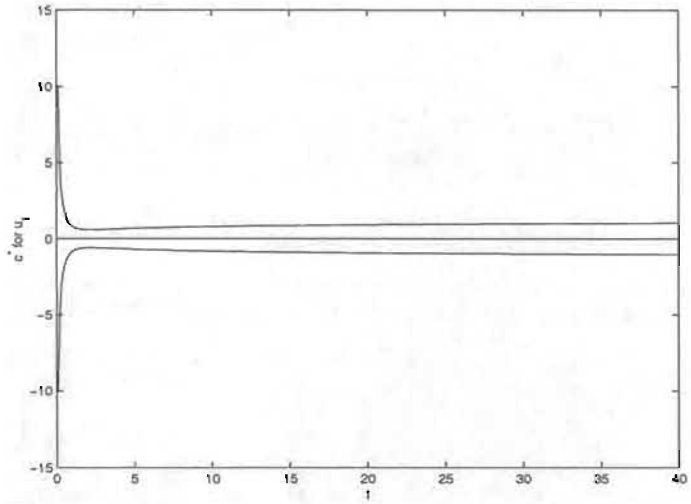


Figure 5.3: The curves $x_+^{0.25}(t)/t$ (upper) and $x_-^{0.25}(t)/t$ (lower) versus t.

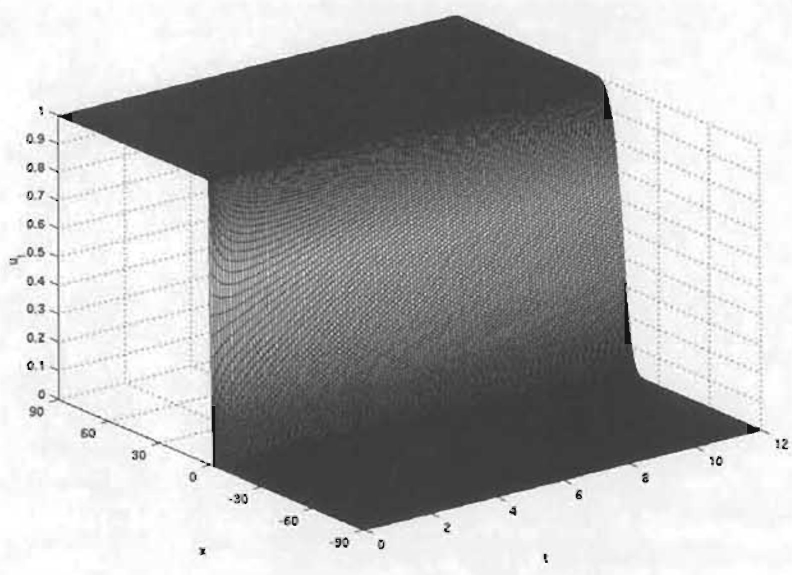
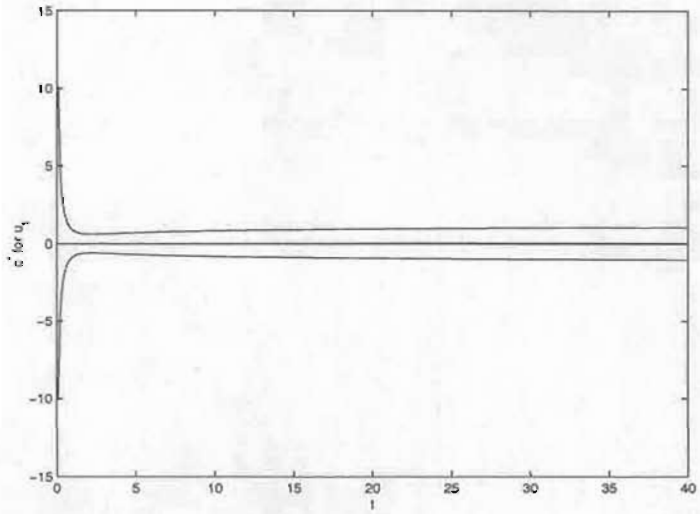


Figure 5.4: The first component of the solution $(u_1(t,x), u_2(t,x))$ with initial function (5.3.20).



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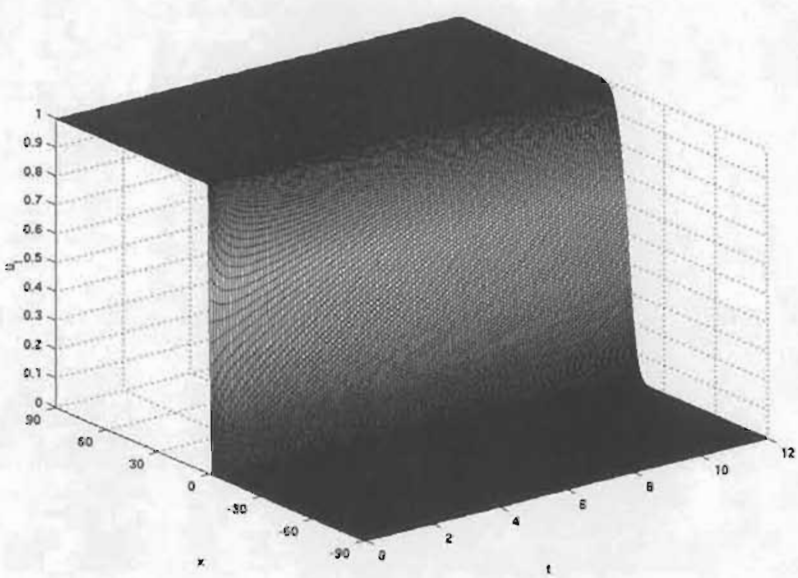

Figure 5.4: The first component of the solution $(u_1(t,x), u_2(t,x))$ with initial function (5.3.20).

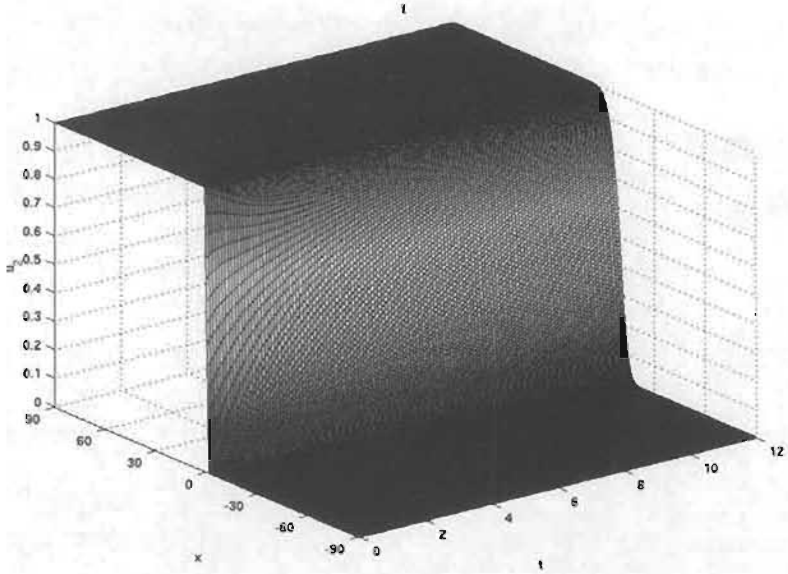

Figure 5.5: The second component of the solution $(u_1(t,x), u_2(t,x))$ with initial function (5.3.20).

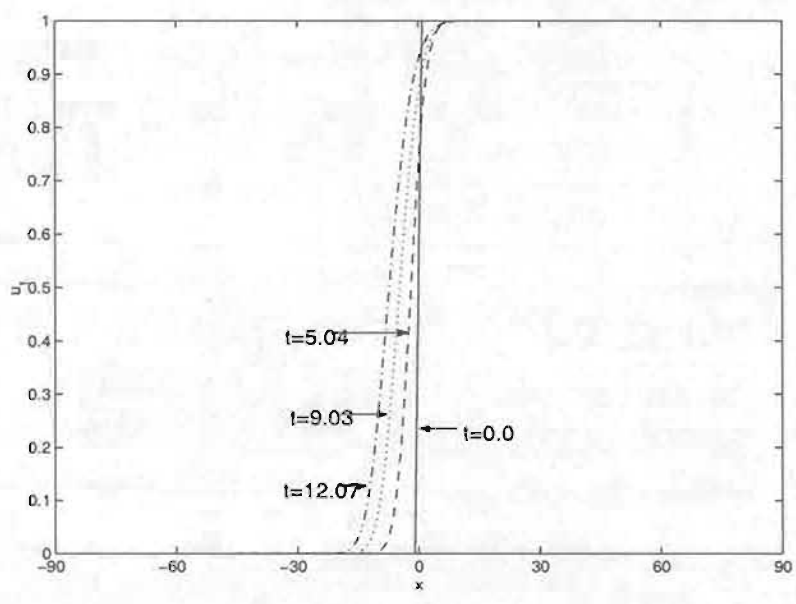

Figure 5.6: The first component of the solution at some specific times.

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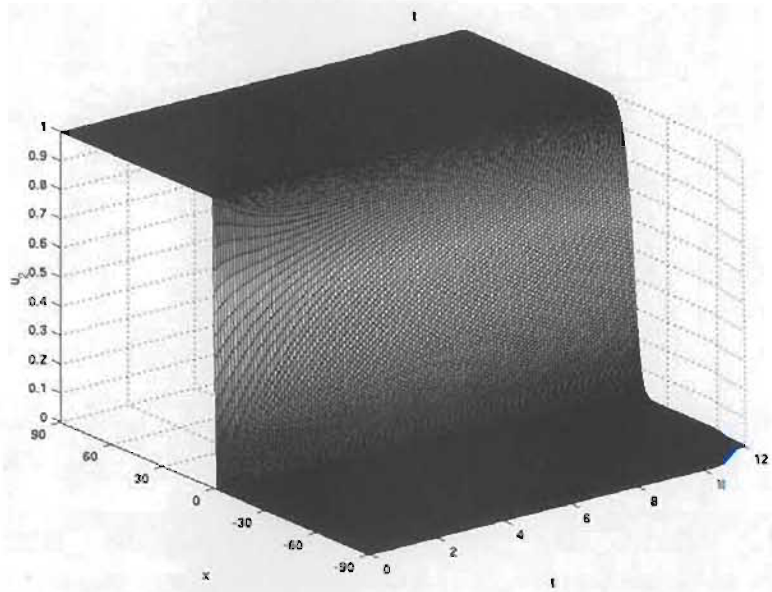
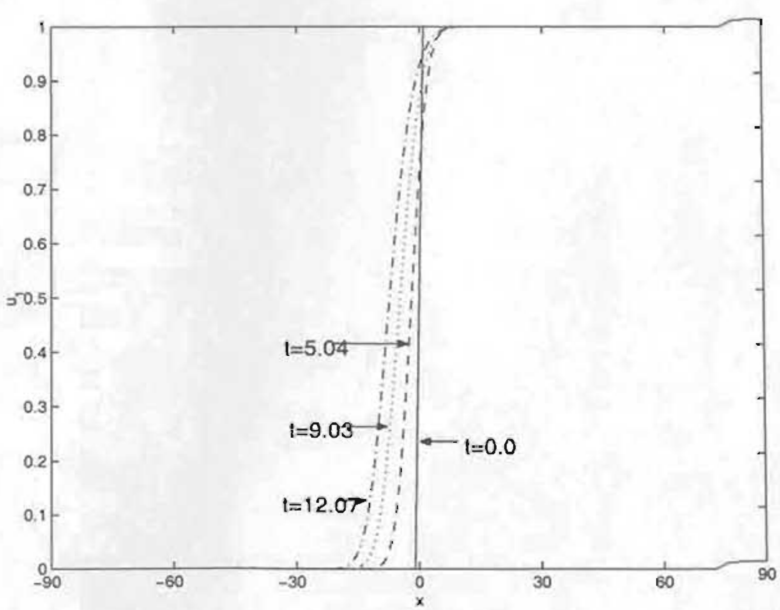


Figure 5.5: The second component of the solution $(u_1(t,x), u_2(t,x))$ with initial function (5.3.20).



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