EXTENDING THE ISOLATED HORIZON PHASE SPACE TO STRING-INSPIRED GRAVITY MODELS

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Extending the isolated horizon phase space to string-inspired gravity models

by

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Abstract

An isolated horizon (IH) is a null hypersurface at which the geometry is held fixed. This generalizes the notion of an event horizon so that the black hole is an object that is in local equilibrium with its (possibly) dynamic environment. The first law of IH mechanics that arises from the framework relates quantities that are all defined *at the horizon*.

IHs have been extensively studied in Einstein gravity with various matter couplings and rotation, and in asymptotically flat and asymptotically anti-de Sitter (ADS) spacetimes in all dimensions $D \geq 3$. Motivated by the nonuniqueness of black holes in higher dimensions and by the black-hole/string correspondence principle, we devote this thesis to the extension of the framework to include IHs in string-inspired gravity models, specifically to Einstein-Maxwell-Chern-Simons (EM-CS) theory and to Einstein-Gauss-Bonnet (EGB) theory in higher dimensions. The focus is on determining the generic features of black holes that are solutions to the field equations of the theories under consideration. To this end, we construct a covariant phase space for both theories; this allows us to prove that the corresponding weakly IHs (WIHs) satisfy the zeroth and first laws of black-hole mechanics.

For EM-CS theory, we find that in the limit when the surface gravity of the horizon goes to zero there is a topological constraint. Specifically, the integral of the scalar curvature of the cross sections of the horizon has to be positive when the dominant energy condition

is satisfied and the cosmological constant Λ is zero or positive. There is no constraint on the topology of the horizon cross sections when $\Lambda < 0$. These results on topology of IHs are independent of the material content of the stress-energy tensor, and therefore the conclusions for EM-CS theory carry over to theories with arbitrary matter fields (minimally) coupled to Einstein gravity.

In addition, we consider rotating IHs in asymptotically ADS and flat spacetimes, and find the restrictions that are imposed on them if one assumes they are supersymmetric. For the existence of a null Killing spinor in four-dimensional $N = 2$ gauged supergravity we show that ADS supersymmetric isolated horizons (SIHs) are necessarily extremal, that rotating SIHs must have non-trivial electromagnetic fields, and that non-rotating SIHs necessarily have constant curvature horizon cross sections and a magnetic (though not electric) charge. When the cosmological constant is zero then the gravitational angular momentum vanishes identically and the corresponding SIHs are strictly non-rotating. Likewise for the existence of a null Killing spinor in five-dimensional $N = 1$ supergravity, we show that SIHs (in asymptotically flat spacetimes) are strictly non-rotating and extremal.

For EGB theory we restrict our study to non-rotating WIHs and show explicitly that the expression for the entropy appearing in the first law is in agreement with those predicted by the Euclidean and Noether charge methods. By carefully examining a concrete example of two Schwarzschild black holes in a flat four-dimensional spacetime that are merging, we find that the area-increase law can be violated for certain values of the GB parameter. This provides a constraint on the free parameter.

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Notation and Conventions

In this thesis, the geometrical setting is a D-dimensional Lorentzian spacetime manifold $\mathcal{M}$ bounded by two spacelike partial Cauchy surfaces, M^- and M^+, which are asymptotically related by a time translation and extend from the internal boundary Δ (with $\Delta \cap M \cong \mathbb{S}^{D-2}$ for some compact $(D-2)$-dimensional space $\mathbb{S}^{D-2}$) to the boundary at infinity $\mathscr{B}$. Below is a table of symbols that are used in this thesis.

Symbol	Description
$\mathscr{I}$	Timelike conformal boundary of asymptotically anti-de Sitter spacetimes
$\widehat{\mathcal{M}} \cong \mathcal{M} \cup \mathscr{I}$	Conformal completion of asymptotically anti-de Sitter spacetimes
$\mathbb{C}^{D-2}$	Compact $(D-2)$-dimensional cross section of $\mathscr{I}$ such that $\mathscr{I} \cap M \cong \mathbb{C}^{D-2}$
$a, b, \ldots \in \{0, \ldots, D-1\}$	Spacetime indices on $\mathcal{M}$
$I, J, \ldots \in \{0, \ldots, D-1\}$	Internal Lorentz indices in the tangent space of $\mathcal{M}$
$i, j, \ldots \in \{2, \ldots, D-1\}$	Spatial indices on $\mathbb{S}^{D-2}$
$\iota \in \{1, \ldots, \lfloor (D-1)/2 \rfloor\}$	Rotation index; corresponds to $\lfloor (D-1)/2 \rfloor$ independent rotation parameters in D dimensions (with $\lfloor \cdot \rfloor$ denoting "integer value of")

Symbol	Description
$A, B, \ldots \in \{1,2\}$	Spinor indices labelling components of two component spinors in four dimensions
$A', B', \ldots \in \{1,2\}$	Dual spinor indices labelling components of two component spinors in four dimensions
$\alpha, \beta, \ldots \in \{1,2\}$	Spinor indices labelling symplectic spinors in configuration space of spinors in five dimensions
g_{ab}	D-dimensional metric tensor on $\mathcal{M}$; sig. $(- + \cdots +)$
R_{abcd}	Riemann curvature tensor determined by g_{ab}; employing the convention of Wald (1984)
$R_{ab} = R^c{}_{acb}, R = g^{ab}R_{ab}$	Ricci tensor and Ricci scalar
$G_{ab} = R_{ab} - (R/2)g_{ab}, T_{ab}$	Einstein tensor and stress-energy tensor
$A_a, F_{ab} = \partial_a A_b - \partial_b A_a$	Electromagnetic vector potential and Faraday field tensor
$\tilde{g}_{ab}$	D-dimensional metric tensor on $\widehat{\mathcal{M}}$
$\widetilde{C}_{abcd}$	Weyl tensor of $\tilde{g}_{ab}$
$\widetilde{E}_{ab}$	Electric part of $\tilde{C}_{abcd}$
Ω	Conformal factor relating g_{ab} and $\tilde{g}_{ab}$
Λ, L	Cosmological constant and anti-de Sitter radius
$c, \hbar, k_B, G_D$	Physical constants: speed of light, Planck constant, Boltzmann constant, gravitational constant
κ_D, α	Coupling constants: $k_D = 8\pi G_D$, Gauss-Bonnet
λ	Chern-Simons parameter; 1 if D is odd and 0 if D is even
$\eta_{IJ} = \mathrm{diag}(-1, 1, \ldots, 1)$	Internal Minkowski metric in the tangent space of $\mathcal{M}$
$e^I = e_a{}^I dx^a$	Coframe; a set of D orthogonal vectors defined by the condition $g_{ab} = \eta_{IJ} e_a{}^I \otimes e_b{}^J$

Symbol	Description
$\ell_a \in [\ell]$	Null normal to Δ; $\ell' \sim \ell$ if $\ell' = z\ell$, z constant
n_a	Auxiliary null normal to Δ normalized such that $n_a \ell^a = -1$
$\vartheta_{(i)}$	$(D-2)$ spacelike vectors satisfying $\ell \cdot \vartheta_{(i)} = n \cdot \vartheta_{(i)} = 0$ and normalized such that $\vartheta_{(i)} \cdot \vartheta_{(j)} = \delta_{ij}$
$q_{ab} = g_{ab} + \ell_a n_b$	$(D-1)$-dimensional degenerate metric on Δ; sig. $(0 + \ldots +)$
$\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a$	Induced metric on $\mathbb{S}^{D-2}$
$\omega_a, \mathcal{A}, \kappa_{(\ell)} = \ell^a \omega_a, \Phi_{(\ell)} = \ell^a A_a$	Induced normal connection, area, surface gravity and electromagnetic scalar potential on Δ
$\underleftarrow{\cdot}$	Denotes pullback to Δ
$\approx$	Denotes equality restricted to Δ
$A^I{}_J = A_a{}^I{}_J dx^a$	Gravitational $SO(D-1,1)$ connection
$\Omega^I{}_J$	Gravitational curvature two-form defined by $A^I{}_J$
$\Sigma_{I_1 \ldots I_m}$	$(D-m)$-form determined by the coframe
$\epsilon_{a_1 \ldots a_D}$	Spacetime volume element
$\epsilon_{I_1 \ldots I_D}$	Totally antisymmetric Levi-Civita tensor
$\tilde{\epsilon} = \vartheta^{(1)} \wedge \cdots \wedge \vartheta^{(D-2)}$	Area element of $\mathbb{S}^{D-2}$
K, K'	Timelike Killing fields in globally stationary asymptotically anti-de Sitter spacetimes
k^a	Killing vector field that generates a symmetry (i.e. time translation etc) in asymptotically anti-de Sitter spacetime
$\tilde{u}^a$	Unit timelike normal to $\mathbb{C}^{D-2}$
$\tilde{\varepsilon}$	Area form on $\mathbb{C}^{D-2}$
$\tilde{n}^a$	Gradient of Ω on $\mathscr{I}$

Symbol	Description
$\boldsymbol{A} = A_a dx^a, \boldsymbol{F} = d\boldsymbol{A}$	Electromagnetic ($U(1)$) connection one-form and associated curvature two-form
q_e, q_m	Electric and magnetic charges
ϵ	Anticommuting Dirac spinor in four dimensions
ϵ^α	Commuting symplectic Majorana spinors in five dimensions
$\epsilon_{\alpha\beta}$	Spinor structure associated with ϵ^α; norm. $\epsilon_{12} = \epsilon^{12} = 1$
γ^a	Gamma matrices in four dimensions
Γ^I	Gamma matrices in five dimensions
$\gamma_{a_1 \cdots a_D}; \Gamma_{I_1 \cdots I_D}$	Totally antisymmetric product of D gamma matrices
$M^\dagger, M^T$	Hermitian conjugate $\dagger$ and transpose T of the matrix M
$\mathcal{C}$	Charge congujation matrix
$f, g, V^a, W^a, \Psi^{ab}$	Bilinear covariants defined by products of ϵ and γ^a
$\mathcal{F}, \mathcal{V}^I, \Phi^{IJ}$	Bilinear covariants defined by products of ϵ^α and Γ^I
$E_\perp, B_\perp$	Electric and Magnetic fluxes through Δ
$\tilde{X}^a \in T^*(\mathbf{S}_v)$	Function describing flows of electromagnetic radiation along $\Delta \cong R \times \mathbf{S}_v$
$\psi_{AA'} = (\alpha_A, \beta_{A'})$	Killing spinor in four dimensions
ϵ_{AB}	Spinor structure associated with $\psi_{AA'}$; norm. $\epsilon_{12} = \epsilon^{12} = 1$
$\phi_{AB}, \bar{\phi}_{A'B'}$	Maxwell spinor and its complex congujate
$K_{AA'} = \ell_{AA'} + n_{AA'}$	Killing vector which is null on Δ
$\mathcal{K}$	Complex-valued function defined by $\bar{\beta}^A = \mathcal{K}\alpha^A$
ϕ	Complex-valued function defined by $\phi_{AB} = \phi\alpha_A\alpha_B$
$\Psi, \boldsymbol{\Psi}$	Generic field variables: tensor field, differential form

Symbol	Description
$\partial_a\Psi, \nabla_a\Psi, \mathscr{D}\Psi, \mathcal{D}\Psi, d\Psi$	Derivative operators: partial derivative, covariant derivative, gauge covariant derivative, intrinsic covariant derivative on Δ and exterior derivative
$\Psi\lrcorner\Psi = \Psi^a\Psi_a$	Contraction of a vector with a differential form
$\star\Psi$	Hodge dual of Ψ
$\pounds_\ell\Psi = \ell\lrcorner d\Psi + d(\ell\lrcorner\Psi)$	Lie derivative
$\delta\Psi, \delta\boldsymbol{\Psi}$	first variation; also used to denote small changes
$E[\cdot]$	Equations of motion
$J[\cdot,\cdot]$	Surface terms
$\boldsymbol{\Omega}[\cdot,\cdot]$	Symplectic structure
Ω_ι	$\lfloor(D-1)/2\rfloor$ angular velocities of Δ
φ^a_ι	Globally defined rotational spacelike Killing fields on $\mathcal{M}$
ϕ^a_ι	Restrictions of φ^a_ι to Δ
$\xi^a = z\ell^a + \sum_\iota \Omega_\iota\phi^a_\iota$	Evolution vector field; spacelike on Δ
$\mathcal{S}, \mathcal{Q}, \mathcal{J}_\iota$	Conserved charges on Δ: entropy, electric charge and angular momenta
$\mathcal{R}_{abcd}$	Riemann curvature tensor associated with the metric $\tilde{q}_{ab}$
$\mathcal{R}$	Scalar curvature of $\mathbb{S}^{D-2}$
$\chi(\mathbb{S}^2) = 2-2g$	Euler characteristic of $\mathbb{S}^2$ with genus g
$\mathscr{A}, \kappa$	Area and surface gravity of a stationary black hole
$\zeta^a = t^a + \sum_\iota \tilde{\Omega}_\iota m^a_\iota$	Evolution vector field for stationary spacetimes with t^a a timelike Killing vector and m^a_ι globally defined rotational spacelike Killing vectors; the hypersurface at which $\zeta_a\zeta^a = 0$ defines the Killing horizon with angular velocities $\tilde{\Omega}_\iota$

Symbol	Description
$\mathfrak{E}, \mathfrak{Q}, \mathfrak{J}_\iota$	Conserved charges of a stationary black hole (measured at infinity): energy, electric charge and angular momenta
$\mathcal{V}_{(k)N-1} = \pi^{N/2}/\Gamma(N/2+1)$	Volume of an $(N-1)$-dimensional space $S^{N-1} = S^{N-1}_{(k)}$ of constant curvature
$d\Omega^2_{(k)N-1}$	Metric on $\mathbb{S}^{N-1}$
k	curvature index of $\mathbb{S}^{N-1}$: $k=1$ corresponds to positive constant curvature, $k=-1$ corresponds to negative constant curvature, and $k=0$ corresponds to zero curvature
r_+	radius of event horizon
$\mathcal{A}_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$	Surface area of a unit $(N-1)$-sphere; $\Gamma(N/2)$ gamma function
$\mathscr{S}, T$	Thermodynamic parameters: entropy and temperature

Authorship Statement

This thesis is based on the following four articles:

- **Liko T** and Booth I 2007 Isolated horizons in higher-dimensional Einstein-Gauss-Bonnet gravity *Class. Quantum Grav.* **24** 3769
- **Liko T** 2007 Topological deformation of isolated horizons *Phys. Rev.* D **77** 064004
- **Liko T** and Booth I 2008 Supersymmetric isolated horizons *Class. Quantum Grav.* **25** 105020
- Booth I and **Liko T** 2008 Supersymmetric isolated horizons in ADS spacetime *Phys. Lett.* B **670** 61

I am first author of the first three articles. I wrote the manuscripts and Ivan helped with the editing. I submitted and revised these articles to address the issues that were raised by the referees. I initiated the projects and did most of the calculations. The extremality condition (2.54) is due to previous work by Ivan and his collaborator Stephen Fairhurst. The identity (4.18) for the Riemann tensor was proved by Ivan. The latest article was a joint collaboration between Ivan and myself. I initiated the project, but the calculations and preparation of the manuscript were done together.

Chapter 1

Introduction

"What is mind? No matter. What is matter? Never mind." ∼ H J Simpson

1.1 Statement of the problem

It has been appreciated for some time that a black hole behaves as a thermal object and has a macroscopic entropy $\mathscr{S}_{\mathrm{BH}}$, the Bekenstein-Hawking entropy, that is proportional to the surface area $\mathscr{A}$ of the event horizon (Bekenstein 1973; Bekenstein 1974; Hawking 1975). This fact is a very beautiful example of the profound relationship between the classical and quantum aspects of the gravitational field, and is one of the main reasons why the study of black holes continues to be one of the most interesting areas of research in gravitational theory. It is also the main reason to believe that the gravitational field should have a quantum description. One of the goals of all the different approaches to quantum gravity is to identify the microscopic degrees of freedom that account for the entropy, and to obtain the area-entropy relation from first principles using statistical mechanics. If it turned out that gravity cannot be quantized, then this fact would provide a very striking counterexample to our belief that thermal properties *of any object* are described quantum mechanically in terms of the microstates of the corresponding system.

To get a general feeling for the problem, it is worth looking at the entropy with a concrete example. First, let us write down the expression with all physical constants. For a black hole in *Einstein gravity* this is

$$\mathscr{S}_{\mathrm{BH}} = \frac{\mathscr{A} k_{\mathrm{B}} c^3}{4\hbar G_D} = \frac{\mathscr{A} k_{\mathrm{B}}}{4 l_{\mathrm{P}}^2}, \tag{1.1}$$

with k_{B} the Boltzmann constant and l_{P}^2 the Planck "area" defined by the speed of light c, D-dimensional gravitational constant G_D and Planck constant $\hbar$. Let us further consider a Schwarzschild black hole of one solar mass $M_{\odot} = 1.989 \times 10^{30}$ kg in four dimensions. The spacetime for this solution in spherical coordinates is the line element

$$dS^2 = -\left(1 - \frac{2M_{\odot}G_4}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2M_{\odot}G_4}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{1.2}$$

The event horizon has radius $r = 2G_4 M_{\odot}/c^2$ and the surface area is then $\mathscr{A} = 32\pi G_4^2 M_{\odot}^2/c^4$. This gives a numerical value of

$$\mathscr{S} = \frac{8\pi G_4 k_{\mathrm{B}} M_{\odot}^2}{\hbar c} = 2.895 \times 10^{54}\ \mathrm{J} \cdot \mathrm{K}^{-1} \tag{1.3}$$

for the entropy of the black hole. The number of quantum states $\mathscr{N}$ that this entropy corresponds to is therefore

$$\mathscr{N} = \exp\left(\frac{\mathscr{S}}{k_{\mathrm{B}}}\right) = \exp(2.098 \times 10^{77}), \tag{1.4}$$

which is a huge number by any standards. For comparison, we note that the number $\mathscr{S}/k_{\mathrm{B}}$ is on the same order of magnitude as the estimated total number of nucleons in the universe!

The problem is to answer the following question: *What are the microscopic degrees of freedom that account for the entropy of the black hole?* The Schwarzschild solution is static, which implies that the degrees of freedom cannot be gravitons. They must be described by nonperturbative configurations of the gravitational field.

The leading approaches to quantum gravity that have been most successfully applied to the problem of black-hole microstates are loop quantum gravity (LQG) (Ashtekar and Lewandowski 2004) and superstring theory (ST) (Aharoney *et al* 2000).

- *Loop quantum gravity.* Here one counts the states arising from punctures where spin networks traversally intersect a surface that is specified in the quantized phase space with a set of boundary conditions (Ashtekar *et al* 1998; Ashtekar *et al* 2000a). This surface represents the black hole horizon and is intrinsically flat. Curvature is induced at the punctures where the spin networks intersect the surface and give it "quanta of area". The LQG framework has been successful in describing the statistical mechanics of all black holes (with simple topologies) in four dimensions, but up to a single free parameter that enters into the classical phase space as an ambiguity in the choice of real self-dual connection (Barbero 1996; Immirzi 1997; Rovelli and Thiemann 1997). In order for the framework to produce the correct coefficient that matches the one for $\mathscr{S}$ in (1.1), this parameter must be fixed to a specific value which depends on how the state-counting is done. See e.g. (Tamaki and Nomura 2005). It was recently pointed out, however, that if the Newton constant as well as the surface area of the black hole are renormalized then the entropy may be the same for all values of the parameter (Jacobson 2007). Also, certain special properties of four-dimensional spacetimes have to be exploited within the framework, which are crucial for the calculations to work at all. This makes it difficult to extend the framework to higher dimensions.

- *Superstring theory.* There are two (independent) approaches to the problem here. The first is the D-brane picture (Maldacena 1996; and references therein), whereby one counts the states of a particular quantum field theory on a configuration of D-branes which forms a black hole in the limit when the string coupling is increased. The second is the anti-de Sitter/conformal field theory (ADS/CFT) picture (Witten 1998a; 1998b), whereby a black hole in a five-dimensional ADS spacetime is described by a conformally invariant $SU(N)$ super Yang-Mills theory; here the states accounting for the entropy are the quantum states of the CFT. Both of these approaches have been successful in describing the statistical mechanics, with the exact coefficient for the area-entropy relation, but for a very limited class of black holes: extremal and near-extremal in the D-brane picture while very small black holes (corresponding to high temperature limit) in the ADS/CFT picture. In particular, astrophysical black holes such as those described by the solution (1.2) are not among the class of black holes that are described in the ST approaches.

The LQG and ST approaches are very different, both philosophically and in the methods that are used for quantization. LQG on the one hand is a background independent canonically quantized theory of pure gravity in four dimensions, while ST on the other hand is a quantum field theory over a fixed nondynamical background in higher dimensions that is supposed to describe all interactions as well as gravity. It is unclear, and surprising, that such different approaches all lead to the same answer. This is an instance of the "problem of universality" which has been advocated for some time now by Carlip (2007). Essentially, the entropy of a black hole may be fixed universally by the diffeomorphism invariance of general relativity.

1.2 Motivation for thesis research

The fact that the ST approaches give the exact coefficient for the entropy of a black hole is truly remarkable, despite that they do so for such a limited class of solutions. Nevertheless, the ST approaches are the most favored because they explain the entropy *dynamically*, and also from an aesthetically pleasing point of view that ST is a unified theory of all interactions. Despite all successes though, a number of problems remain. Among the more serious ones are black hole nonuniqueness in higher dimensions and an inconsistency that has been overlooked in the black-hole/string correspondence principle.

- *Black-hole nonuniqueness.* In a four-dimensional asymptotically flat spacetime, a charged and rotating black hole is uniquely described by its conserved charges; the only unique solution is the Kerr-Newman metric (Robinson 1973). This is a statement of the black-hole uniqueness theorem, and is a striking property of the simplicity of black holes in nature. The advent of ST revolutionized our view of the universe, for example with the requirement of extra spatial dimensions. For a long time it was generally assumed that the properties of four-dimensional black holes, particularly the uniqueness theorem, simply carry over to higher dimensions as well. The black ring solution (Emparan and Reall 2002) that describes a rotating black hole with horizon topology $S^1 \times S^2$ in five dimensions, was the first counter-example to the uniqueness of black holes in asymptotically flat spacetimes. Specifically, the uniqueness theorem fails (in five dimensions) because the conserved charges of the ring can coincide with the conserved charges of a rotating black hole with horizon topology S^3 (Myers and Perry 1986). The natural question that should be investigated is therefore the following: *What properties of black holes in four dimensions carry over to higher-dimensional spacetimes?* More specifically, we should ask the following question: *What are the generic features of black holes in higher-dimensional spacetimes in general and within the ST framework in particular?* An ideal method of investigating such questions is to employ a covariant phase space of all solutions to the equations of motion for a given action principle.

- *Black-hole/string correspondence principle.* The methods that are employed in ST lead to a first law of black-hole mechanics that relates quantities at the event horizon and quantities defined at infinity. This "hybrid" relation appears to be unphysical in ST from the point of view of the black-hole/string correspondence principle (Susskind 1993), which states that there is a smooth transition from a black hole to a string in the limit when the string coupling is decreased. For this correspondence principle to work, the entropies of the black hole and string are required to be equal for a particular value of the string coupling constant because the entropy of the black hole is proportional to the *mass squared* and the entropy of the string is proportional to the *mass* (Horowitz and Polchinski 1997). However, while the mass of the black hole is measured at infinity, the mass of the string is determined by the string coupling and tension which are intrinsic quantities of the string state in the sense that no reference needs to be made to infinity at all. Therefore the conserved charges of the black hole state should *not* be defined at infinity.

The moral to be extracted from the above considerations is that a framework for black holes in ST should be employed that is both quasilocal and general enough to allow for a large class of solutions to be investigated. Remarkably, such a framework does exist! This is the isolated horizon (IH) framework (Ashtekar and Krishnan 2004). The classical theory of IHs was motivated by earlier considerations by Hayward (1994), but the framework is considerably different as covariant phase space methods (Witten 1986; Crnković 1987; Crnković and Witten 1987; Crnković 1988; Lee and Wald 1990; Ashtekar *et al* 1991; Wald and Zoupas 2000) are employed in the former case. All the quantities that appear in the first law of IH mechanics are defined intrinsically at the horizon. The concept of such a surface generalizes the notion of a Killing horizon in stationary spacetimes to much more general and therefore physical spacetimes that may include external radiation fields that are dynamical. Examples of such systems in general relativity are given by the so-called Robinson-Trautman spacetimes (Ashtekar *et al* 1999; Lewandowski 2000).

The IH framework may fit naturally into ST and the black-hole/string correspondence principle. The work presented here is a first step towards extending the IH phase space beyond Einstein gravity so that a quasilocal description of black holes may be realized within the context of ST. In this thesis the framework is extended first to Einstein-Maxwell-Chern-Simons (EM-CS) theory (Booth and Liko 2008; Liko and Booth 2008) and then to Einstein-Gauss-Bonnet (EGB) theory (Liko and Booth 2007; Liko 2008) in higher dimensions. There are of course many more theories of gravity in higher dimensions. Some of the modern approaches in five dimensions incorporating a large extra dimension include braneworld cosmology (Brax and van de Bruck 2003; Maartens 2004) and induced-matter theory (Overduin and Wesson 1997; Liko *et al* 2004).

The motivation for extending the IH framework specifically to the two theories presented in this thesis came from their relevance within the context of ST. EM-CS theory is important in ST because, in five dimensions, the corresponding action with negative cosmological constant is the bosonic action of $N = 1$ gauged supergravity (Cremmer 1980); black holes in particular are described by solutions to the bosonic equations of motion with all fermionic fields and their variations vanishing in the vacuum (Gibbons *et al* 1994; Gauntlett *et al* 1999; Gutowski and Reall 2004). In addition, the action in four dimensions reduces to the Einstein-Maxwell (EM) action, which is the bosonic action of $N = 2$ gauged supergravity (Gibbons *et al* 1994). EGB theory is important in ST because the corresponding action contains the only possible combination of curvature-squared interactions for which the linearized equations of motion do not contain any ghosts (Zwiebach 1985; Zumino 1986; Myers 1987). This is particularly important in ST because of the no-ghost theorem (Polchinski 1998), which states that the BRST inner product is positive.

1.3 Overview and main results

In Chapter 2 we consider the phase space of solutions to the equations of motion for the EM-CS action

$$S = \frac{1}{2\kappa_D}\int_{\mathcal{M}} d^{D-1}x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F^2 - \frac{2\lambda}{3\sqrt{3}}\epsilon_{ab_1\cdots b_{D-1}}A^a F^{b_1b_2}\ldots F^{b_{D-2}b_{D-1}}\right] . \tag{1.5}$$

Here, R is the scalar curvature, g is the determinant of the spacetime metric tensor g_{ab} ($a, b, \ldots \in \{0, \ldots, D-1\}$), A_a is the vector potential and $F_{ab} = \partial_a A_b - \partial_b A_a$ (with $F^2 = F_{ab}F^{ab}$) is the field strength. The constants appearing in the action are the gravitational coupling constant $\kappa_D = 8\pi G_D$ and the cosmological constant Λ. The cosmological constant is given by

$$\Lambda = \frac{\varepsilon}{2L^2}(D-1)(D-2)\,, \tag{1.6}$$

where $\varepsilon \in \{-1, 1\}$ and L is the (anti-)de Sitter radius. Also, we set $c = \hbar = 1$ from here on unless otherwise stated. The last term is a Chern-Simons (CS) term for the electromagnetic field; here $\lambda = 0$ if D is even and $\lambda = 1$ if D is odd. The field equations that are derived from the action (1.5) when the metric is varied are the Einstein equations

$$G_{ab} = 2T_{ab} - \Lambda g_{ab} \tag{1.7}$$

with the Einstein tensor G_{ab} and stress-energy tensor T_{ab} given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} \quad \text{and} \quad T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^2\,. \tag{1.8}$$

The field equations that are derived from the action (1.10) when the vector potential is varied are the Maxwell-Chern-Simons equations

$$\nabla_a F^{ab} = \frac{4(D+1)\lambda}{3\sqrt{3}\sqrt{-g}}\epsilon^{bc_1\cdots c_{D-1}}F_{c_1c_2}\cdots F_{c_{D-2}c_{D-1}}\,. \tag{1.9}$$

There are several solutions to these equations that describe black holes. In four dimensions with $\lambda = 0$ the equations are solved by a family of topological Kerr-Newman-ADS (KN-ADS) spacetimes (Kostelecký and Perry 1996; Caldarelli and Klemm 1999). The solutions that are supersymmetric describe: (a) rotating and extremal black holes with horizon cross sections of spherical, cylindrical or toroidal topologies and having non-trivial electromagnetic fields; and (b) non-rotating and extremal black holes with constant curvature horizon cross sections of genus $g > 1$ and with magnetic (but not electric) charge. In five dimensions, with $\Lambda = 0$, the simplest solution is the five-dimensional Reissner-Nordström (RN) spacetime (Tangherlini 1963; Myers and Perry 1986). The equations admit two asymptotically flat solutions that describe supersymmetric black holes. These are the Breckenridge-Myers-Peet-Vafa (BMPV) black hole (Breckenridge *et al* 1997), and the Elvang-Emparan-Mateos-Reall (EEMR) black ring (Elvang *et al* 2004). The Gutowski-Reall (GR) black hole is a generalization to ADS spacetime of the BMPV black hole (Gutowski and Reall 2004). The main purpose of the work in (Booth and Liko 2008; Liko and Booth 2008) was to develop a quasilocal framework for these black holes.

First, we examine the boundary conditions and their consequences. To this end, we consider the action (1.5) in the first-order connection formulation of general relativity, after which we specify the boundary conditions that are imposed onto the inner boundary of $\mathcal{M}$. These boundary conditions capture the notion of a weakly isolated horizon (WIH) that physically corresponds to an isolated black hole in a surrounding spacetime with (possibly dynamical) fields and leads to the zeroth law of black-hole mechanics.

Next we investigate the mechanics of the WIHs. We show that the action principle with boundaries is well defined by explicitly showing that the first variation of the surface term vanishes on the horizon. We then find an expression for the symplectic structure by integrating over a spacelike $(D-1)$-surface the antisymmetrized second variation of the surface term and adding to this the pullback of the resulting two-form to the WIH. This allows us to find an expression for the local version of the (equilibrium) first law of black-hole mechanics in dimensions $D \geq 5$.

Summarizing thus far, we have the following:

Result 1. *A charged and rotating WIH $\Delta \subset \mathcal{M}$ on the phase space of solutions of EM-CS* *theory in D dimensions satisfies the zeroth and first laws of black-hole mechanics.*

After proving that the first law holds, we restrict our study to the stronger notion of (fully) IHs. These are WIHs for which the extrinsic as well as intrinsic geometries are invariant under time translations. For these horizons, the sign of the surface gravity $\kappa_{(\ell)}$ is well defined. The requirement that $\kappa_{(\ell)} \geq 0$ therefore allows us to define a parameter that provides a constraint on the topology of the IHs. We find that the integral of the scalar curvature of the cross sections of the IH (in a spacetime with nonnegative cosmological constant) have to be strictly positive if the dominant energy condition is satified. Furthermore, this integral will be zero if the horizon is extremal and non-rotating, and the stress-energy tensor T_{ab} is of the form such that $T_{ab}\ell^a n^b = 0$ for any two null vectors ℓ and n with normalization $\ell_a n^a = -1$ at the horizon. For negative cosmological constant there is no restriction on the scalar curvature of the cross sections of the IH.

Summarizing now, we have the following:

Result 2. *The IH cross sections in a higher-dimensional spacetime with nonnegative cosmological constant are of positive Yamabe type; if $\Lambda < 0$ then there is no restriction on the sign of the scalar curvature.*

This result is in agreement with recent work on the topological constraints of higher-dimensional black holes in globally stationary spacetimes (Helfgott *et al* 2006; Galloway 2006; Galloway and Schoen 2006). We note that the physical content of the stress-energy tensor at this point is completely arbitrary. Therefore Result 2 implies that the topology considerations are valid for any matter (nonminimally) coupled to Einstein gravity. In the case of electromagnetic fields with or without the CS term, the scalar $T_{ab}\ell^a n^b$ is the square of the electric flux crossing the horizon.

In Chapter 3 we examine the restrictions that are imposed on IHs if one assumes that they are supersymmetric. To do this we specialize to IHs in four dimensions with negative cosmological constant and in five dimensions with vanishing cosmological constant. The former theory is the bosonic part of four-dimensional $N = 2$ gauged supergravity, and the latter theory is the bosonic part of five-dimensional $N = 1$ supergravity. We show that the existence of a Killing spinor in four dimensions requires that the induced (normal) connection ω on the horizon has to be non-zero unless the electric charge (but not magnetic charge) vanishes, and that the surface gravity κ has to be zero. The former condition means that the gravitational component of the horizon angular momentum is non-zero provided that ω is not a closed one-form. The latter condition means that the IH is extremal. Likewise, we show that the existence of a Killing spinor in five dimensions requires that ω vanishes and this immediately also gives $\kappa = 0$.

Summarizing now, we have the following:

Result 3. *A SIH of four-dimensional $N = 2$ gauged supergravity is extremal, and is either: (a) rotating with non-trivial electromagnetic field; or (b) non-rotating with constant curvature horizon cross sections and magnetic (but not electric) charge. A SIH of four-dimensional $N = 2$ supergravity and of five-dimensional $N = 1$ supergravity with zero cosmological constant is non-rotating and extremal.*

The topological KN-ADS family of solutions (Caldarelli and Klemm 1999) describe rotating and extremal supersymmetric black holes in four-dimensional ADS spacetime. Among the special cases is a solution describing a non-rotating and extremal black hole with constant curvature horizon cross sections and magnetic charge. The BMPV solution describes an extremal black hole with nonvanishing angular momentum and non-rotating Killing horizon; this black hole solution is an example of a distorted IH with arbitrary rotations in the bulk fields (Ashtekar *et al* 2004). When the angular momentum vanishes this solution reduces to the extremal RN solution in isotropic coordinates. The conclusions drawn from our Result 2 together with Result 3 are that the only possible horizon topologies for SIHs are S^2 in four dimensions (when $\Lambda = 0$) or S^3 and $S^1 \times S^2$ in five dimensions. Both these topologies have been realized and the corresponding solutions, for example the BMPV black hole (Breckenridge *et al* 1997) and the EEMR black ring (Elvang *et al* 2004), are well known. The torus topology is a special case that can occur only if the stress-energy tensor is of the form such that $T_{ab}\ell^a n^b = 0$ for any two null vectors ℓ and n with normalization $\ell_a n^a = -1$. A solution describing such a black hole has yet to be discovered.

In Chapter 4 we consider the phase space of solutions to the equations of motion for the EGB action

$$S = \frac{1}{2\kappa_D}\int_{\mathcal{M}} d^D x\sqrt{-g}\left[R - 2\Lambda + \alpha\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\right] . \tag{1.10}$$

In addition to the quantities that also appear in the EM-CS action (1.5), the action (1.10) also contains explicit dependence on the Riemann tensor R_{abcd} and Ricci tensor $R_{ab} = R^c{}_{acb}$. The constant α is the GB parameter. The field equations that are derived from the action (1.10) when the metric is varied are the EGB equations

$$\begin{aligned} G_{ab} = -\Lambda g_{ab} \quad &+ \quad \alpha\left\{\frac{1}{2}\left(R^2 - 4R_{cd}R^{cd} + R_{cdef}R^{cdef}\right) g_{ab}\right. \\ &- \quad \left. 2RR_{ab} + 4R_{ac}R_b{}^c + 4R_{acbd}R^{cd} - 2R_{acde}R_b{}^{cde}\right\} . \end{aligned} \tag{1.11}$$

When $\alpha = 0$ these equations reduce to the vacuum Einstein equations. There are several solutions to these equations that describe black holes. The simplest were found independently by Boulware and Deser (1985) and by Wheeler (1986a; 1986b), and describe a static spherically symmetric black hole. The causal structure and thermodynamics of this solution were later studied by Myers and Simon (1988). The solution was subsequently extended to ADS spacetime by Cai (2002) and by Cho and Neupane (2002). The purpose of the work in (Liko and Booth 2007; Liko 2008) was to develop a quasilocal framework for these black holes.

Just as for IHs in EM-CS theory, we begin by examining the boundary conditions and their consequences. It turns out that for the zeroth law to be satisfied, the boundary conditions need to be slightly modified. Specifically, an analogue of the dominant energy condition has to be imposed onto the Ricci tensor instead of the matter stress-energy tensor. The zeroth law then follows naturally from the modified boundary conditions.

Next we investigate the mechanics of the WIHs. In particular, we show that the action principle for WIHs in EGB theory is well defined by explicitly showing that the first variation of the surface term vanishes on the horizon. This turns out to be quite complicated due to the presence of the GB term. Nevertheless, we verify the differentiability of the action for EGB theory by brute force at the expense of restricting the phase space to non-rotating WIHs. We then find an expression for the symplectic structure by integrating over a spacelike $(D-1)$-surface the antisymmetrized second variation of the surface term and adding to this the pullback of the resulting two-form to the WIH. This allows us to find an expression for the local version of the (equilibrium) first law of black-hole mechanics in dimensions $D \geq 5$, with an entropy expression that contains a correction term that is proportional to the surface integral of the scalar curvature of the cross sections of the horizon. We demonstrate the validity of our expression for the quasilocal entropy of WIHs by directly comparing it to those expressions that are obtained by the Euclidean (Cai 2002;

Cho and Neupane 2002) and Noether charge (Clunan *et al* 2004) methods.

Summarizing thus far, we have the following:

Result 4. *A non-rotating WIH $\Delta \subset \mathcal{M}$ on the phase space of solutions of EGB theory in D dimensions satisfies the zeroth and first laws of black-hole mechanics.*

We conclude our investigation of IHs in EGB theory by looking at physical consequences of the correction term in the entropy can have on the area-increase law. In order to make the analysis concrete, the calculation is done for black holes in four dimensions, specifically for the merging of two Schwarzschild black holes in flat spacetime. It turns out that for this very special case the second law of black-hole mechanics will be violated if α is greater than the product of the masses of the black holes before merging minus a small correction due to radiation that may be lost by gravitational waves during the merging process.

Summarizing now, we have the following:

Result 5. *There is a lower bound on α for which the area-increase law will be violated when two black holes merge.*

The calculation of the bound on α is done in four dimensions. However, a similar bound may presumably be derived for specific solutions in higher dimensions as well [although in this case the topologies are not as severely restricted as they are in four dimensions, even for Einstein gravity with $\Lambda = 0$ (Helfgott *et al* 2006; Galloway 2006; Galloway and Schoen 2006)]. Result 2 also corrects a long-held misconception about the GB term, namely that its presence in four dimensions does not lead to any physical effects because the term is a topological invariant and does not show up in the equations of motion.

In Chapter 5 we conclude the thesis with a brief summary of the work that has been done here, and discuss some classical applications of IHs in EM-CS theory and EGB theory.

Chapter 2

Isolated Horizons in EM-CS Theory

"The beginner ... should not be discouraged if ... he finds that he does not have the prerequisite for reading the prerequisites." ~ P Halmos

2.1 First-order action for EM-CS theory

For application to IHs, we work with the "connection-dynamics" formulation of general relativity. For details we refer the reader to the review (Ashtekar and Lewandowski 2004) and references therein. In this formulation, the configuration space consists of the triple $(e^I, A^I{}_J, \boldsymbol{A})$; the coframe $e^I = e_a{}^I dx^a$ $(I, J, \ldots \in \{0, \ldots, D-1\})$ determines the spacetime metric

$$g_{ab} = \eta_{IJ} e_a{}^I \otimes e_b{}^J \,, \tag{2.1}$$

the gravitational $(SO(D-1,1))$ connection $A^I{}_J = A_a{}^I{}_J dx^a$ determines the curvature two-form

$$\Omega^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J \,, \tag{2.2}$$

and the electromagnetic ($U(1)$) connection $\boldsymbol{A}$ determines the curvature

$$\boldsymbol{F} = d\boldsymbol{A} . \tag{2.3}$$

In this thesis, spacetime indices $a, b, \ldots$ are raised and lowered using the metric g_{ab}, while internal Lorentz indices $I, J, \ldots$ are raised and lowered using the Minkowski metric $\eta_{IJ} = \mathrm{diag}(-1, 1, \ldots, 1)$. The curvature Ω defines the Riemann tensor $R^I{}_{JKL}$ [with the convention of Wald (1984)] via

$$\Omega^I{}_J = \frac{1}{2} R^I{}_{JKL} e^K \wedge e^L . \tag{2.4}$$

The Ricci tensor is then $R_{IJ} = R^K{}_{IKJ}$, and the Ricci scalar is $R = \eta^{IJ} R_{IJ}$. The gauge covariant derivative $\mathscr{D}$ acts on generic fields Ψ_{IJ} such that

$$\mathscr{D}\Psi^I{}_J = d\Psi^I{}_J + A^I{}_K \wedge \Psi^K{}_J - A^K{}_J \wedge \Psi^I{}_K . \tag{2.5}$$

The coframe defines the $(D-m)$-form

$$\Sigma_{I_1 \ldots I_m} = \frac{1}{(D-m)!} \epsilon_{I_1 \ldots I_m I_{m+1} \ldots I_D} e^{I_{m+1}} \wedge \cdots \wedge e^{I_D} , \tag{2.6}$$

where the totally antisymmetric Levi-Civita tensor $\epsilon_{I_1 \ldots I_D}$ is related to the spacetime volume element by

$$\epsilon_{a_1 \ldots a_D} = \epsilon_{I_1 \ldots I_D} e_{a_1}{}^{I_1} \cdots e_{a_D}{}^{I_D} . \tag{2.7}$$

In this configuration space, the action (1.5) for EM-CS theory on the manifold $(\mathcal{M}, g_{ab})$ (assumed for the moment to have no boundaries) is given by

$$S = \frac{1}{2\kappa_D} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} - 2\Lambda\epsilon - \frac{1}{4} \boldsymbol{F} \wedge \star \boldsymbol{F} - \frac{2\lambda}{3\sqrt{3}} \boldsymbol{A} \wedge \boldsymbol{F}^{(D-1)/2} . \tag{2.8}$$

Here $\epsilon = e^0 \wedge \cdots \wedge e^{D-1}$ is the spacetime volume element and "$\star$" denotes the Hodge dual.

The equations of motion are given by $\delta S = 0$, where δ is the first variation; i.e. the stationary points of the action. For this configuration space the equations of motion are derived from independently varying the action with respect to the fields $(e, A, \boldsymbol{A})$. To get the equation of motion for the coframe we note the identity

$$\delta\Sigma_{I_1\ldots I_m} = \delta e^M \wedge \Sigma_{I_1\ldots I_m M} \,. \tag{2.9}$$

This leads to

$$\Sigma_{IJK} \wedge \Omega^{JK} - 2\Lambda\Sigma_I = \mathscr{T}_I \,, \tag{2.10}$$

where $\mathscr{T}_I$ denotes the electromagnetic stress-energy $(D-1)$-form. The equation of motion for the connection A is

$$\mathscr{D}\Sigma_{IJ} = 0 \,; \tag{2.11}$$

this equation says that the torsion $T^I = \mathscr{D}e^I$ is zero. The equation of motion for the connection $\boldsymbol{A}$ is

$$d \star \boldsymbol{F} - \frac{4(D+1)\lambda}{3\sqrt{3}} \boldsymbol{F}^{(D-1)/2} = 0 \,. \tag{2.12}$$

The second term in this equation is the contribution due to the CS term in the action. In even dimensions the equation reduces to the standard Maxwell equation $d \star \boldsymbol{F} = 0$. The equations (2.10) and (2.11) are equivalent to the field equations (1.7) and (1.9) in the metric formulation, with the components of $\mathscr{T}_I$ identified with the electromagnetic stress-energy tensor.

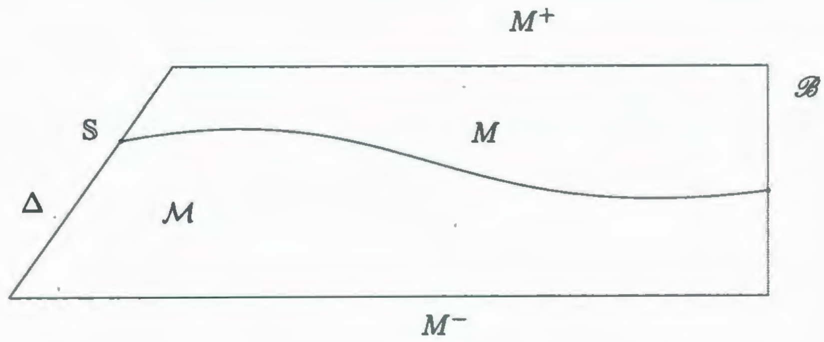

Figure 2.1: The spacetime manifold $\mathcal{M}$ and its boundaries. The region of the D-dimensional spacetime $\mathcal{M}$ being considered has an internal boundary Δ representing the event horizon, and is bounded by two $(D-1)$-dimensional spacelike hypersurfaces $M^{\pm}$ which extend from the inner boundary Δ to the boundary at infinity $\mathscr{B}$. M is a partial Cauchy surface that intersects Δ in a compact $(D-2)$-space $\mathbb{S}$.

2.2 Boundary conditions

Let us from here on consider the manifold $(\mathcal{M}, g_{ab})$ to contain boundaries; the conditions that we will impose on the inner boundary will capture the notion of an isolated black hole that is in local equilibrium with its (possibly) dynamic surroundings. We follow the general recipe that was developed in (Ashtekar *et al* 2000c).

First we give some general comments about the structure of the manifold. Specifically, $\mathcal{M}$ is a D-dimensional Lorentzian manifold with topology $R \times M$, contains a $(D-1)$-dimensional null surface Δ as inner boundary (representing the horizon), and is bounded by $(D-1)$-dimensional spacelike manifolds $M^{\pm}$ that extend from Δ to infinity. The topology of Δ is $R \times \mathbb{S}^{D-2}$, with $\mathbb{S}^{D-2}$ a compact $(D-2)$-space. M is a partial Cauchy surface such that $\mathbb{S}^{D-2} \cong \Delta \cap M$. See Figure 1.

The outer boundary $\mathscr{B}$ is some arbitrary $(D-1)$-dimensional surface. With the exception of §2.6, we consider the purely quasilocal case in this chapter and neglect any subleties that are associated with the outer boundary. Including this contribution in the phase space amounts to imposing fall-off conditions on the fields for fixed Λ [e.g. asymptotically flat (Ashtekar *et al* 2000c) or asymptotically ADS (Ashtekar *et al* 2007)] as they approach $\mathscr{B}$. In §2.6 we briefly discuss rotation in asymptotically ADS spacetimes.

Δ is a WIH, which is defined in the following way:

Definition I. *A WIH Δ is a null surface and has a degenerate metric q_{ab} with signature $0+\ldots+$ (with $D-2$ non-degenerate spatial directions) along with an equivalence class of null normals $[\ell]$ (defined by $\ell \sim \ell' \Leftrightarrow \ell' = z\ell$ for some constant z) such that the following conditions hold: (a) the expansion $\theta_{(\ell)}$ of ℓ_a vanishes on Δ; (b) the field equations hold on Δ; (c) the stress-energy tensor is such that the vector $-T^a{}_b\ell^b$ is a future-directed and causal vector; (d) $\pounds_\ell \omega_a = 0$ and $\pounds_\ell \underleftarrow{\mathbf{A}} = 0$ for all $\ell \in [\ell]$ (see below).*

The first three conditions determine the intrinsic geometry of Δ. Since ℓ is normal to Δ the associated null congruence is necessarily twist-free and geodesic. By condition (a) that congruence is non-expanding. Then the Raychaudhuri equation implies that $T_{ab}\ell^a\ell^b = -\sigma_{ab}\sigma^{ab}$, with σ_{ab} the shear tensor, and applying the energy condition (c) we find that $\sigma_{ab} = 0$. Thus, together these conditions tell us that the intrinsic geometry of Δ is "time-independent" in the sense that all of its (two-dimensional) cross sections have identical intrinsic geometries.

Next, the vanishing of the expansion, twist and shear imply that (Ashtekar *et al* 2000c)

$$\nabla_{\underleftarrow{a}}\ell_b \approx \omega_a \ell_b\,, \tag{2.13}$$

with "$\approx$" denoting equality restricted to Δ and the underarrow indicating pull-back to

Δ. Thus the one-form ω is the natural connection (in the normal bundle) induced on the horizon. These conditions also imply that (Ashtekar *et al* 2000c)

$$\underleftarrow{\ell \lrcorner \boldsymbol{F}} = 0 \,. \tag{2.14}$$

With the field equations (2.12) and the Bianchi identity $d\boldsymbol{F} = 0$, it then follows that

$$\pounds_\ell \underleftarrow{\boldsymbol{F}} \approx \ell \lrcorner \underleftarrow{d\boldsymbol{F}} + d(\underleftarrow{\ell \lrcorner \boldsymbol{F}}) = 0 \,. \tag{2.15}$$

This implies that the electric charge is independent of the choice of cross sections $\mathbb{S}^{D-2}$ (Ashtekar *et al* 2000b). Similarly (in four dimensions) the magnetic charge is also a constant.

From (2.13) we find that

$$\ell^a \nabla_a \ell^b = (\ell \lrcorner \omega) \ell^b \,, \tag{2.16}$$

and define the surface gravity $\kappa_{(\ell)} = \ell \lrcorner \omega$ as the inaffinity of this geodesic congruence. Note that it is certainly dependent on specific element of $[\ell]$ as under the transformation $\ell \rightarrow z\ell$:

$$\kappa_{(\ell)} \rightarrow z\kappa_{(\ell)} \,. \tag{2.17}$$

In addition to the surface gravity, we also define the electromagnetic scalar potential $\Phi_{(\ell)} = -\ell \lrcorner A$ for each $\ell \in [\ell]$ and this has a similar dependence.

Now, it turns out that if the first three conditions hold, then one can always find an equivalence class $[\ell]$ such that (d) also holds. Hence this last condition does not further restrict the geometries under discussion, but only the scalings of the null normal. However, making such a choice ensures that (Ashtekar *et al* 2000c):

$$d\kappa_{(\ell)} = d(\ell \lrcorner \omega) = 0 \quad \text{and} \quad d\Phi_{(\ell)} = d(\ell \lrcorner \boldsymbol{A}) = 0 \,. \tag{2.18}$$

These conditions follow from the Cartan identity, (2.14) and the property that $d\omega$ is proportional to $\tilde{\epsilon}$ [defined below in (2.34)] (Ashtekar *et al* 2000c). This establishes the zeroth law of WIH mechanics: the surface gravity and scalar potential are constant on Δ.

2.3 Variation of the boundary term

Let us now look at the variation of the action (1.10). Denoting the triple $(e, A, \boldsymbol{A})$ collectively as a generic field variable Ψ, the first variation gives

$$\delta S = \frac{1}{2\kappa_D}\int_{\mathcal{M}} E[\Psi]\delta\Psi - \frac{1}{2\kappa_D}\int_{\partial\mathcal{M}} J[\Psi,\delta\Psi] \,. \tag{2.19}$$

Here $E[\Psi]=0$ symbolically denotes the equations of motion and

$$J[\Psi,\delta\Psi] = \Sigma_{IJ}\wedge\delta A^{IJ} - \boldsymbol{\Phi}\wedge\delta\boldsymbol{A} \tag{2.20}$$

is the surface term with $(D-2)$-form

$$\boldsymbol{\Phi} = \star\boldsymbol{F} - \frac{4(D+1)\lambda}{3\sqrt{3}}\boldsymbol{A}\wedge\boldsymbol{F}^{(D-3)/2} \,. \tag{2.21}$$

If the integral of J on the boundary $\partial\mathcal{M}$ vanishes then the action principle is said to be differentiable. We must show that this is the case. Because the fields are held fixed at $M^{\pm}$ and at $\mathscr{B}$, J vanishes there. Therefore it suffices to show that J vanishes at the inner boundary Δ. To show that this is true we need to find an expression for J in terms of Σ, A and $\boldsymbol{A}$ pulled back to Δ. As for the gravitational variables, this is accomplished by fixing an internal basis consisting of the (null) pair (ℓ, n) and $D-2$ spacelike vectors $\vartheta_{(i)}$ $(i\in\{2,\ldots,D-1\})$ such that

$$e_0 = \ell\,,\quad e_1 = n\,,\quad e_i = \vartheta_{(i)}\,, \tag{2.22}$$

together with the conditions

$$\ell \cdot n = -1\,, \quad \ell \cdot \ell = n \cdot n = \ell \cdot \vartheta_{(i)} = n \cdot \vartheta_{(i)} = 0\,, \quad \vartheta_{(i)} \cdot \vartheta_{(j)} = \delta_{ij}\,. \tag{2.23}$$

This basis represents a higher-dimensional analogue of the Newman-Penrose (NP) formalism (Pravda *et al* 2004). The coframe ${e_a}^I$ can be decomposed in terms of the vectors in the basis (2.22) such that

$$ {e_a}^I = -\ell^I n_a - \ell_a n^I + \vartheta_{(i)}^{\ I}\vartheta^{(i)}_a\,; \tag{2.24}$$

summation is understood over repeated spacelike indices (i, j, k etc). The pullback of the coframe to Δ is therefore

$$ e_{\underleftarrow{a}}^{\ I} \approx -\ell^I n_a + \vartheta_{(i)}^{\ I}\vartheta^{(i)}_a\,, \tag{2.25}$$

whence the $(D-2)$-form

$$\begin{aligned} \underleftarrow{\Sigma}_{IJ} \;\approx\;& -\frac{1}{(D-3)!}\epsilon_{IJA_1\ldots A_{D-2}}\ell^{A_1}\vartheta_{(i_1)}^{\ A_2}\cdots\vartheta_{(i_{D-3})}^{\ A_{D-2}}\left(n\wedge\vartheta^{(i_1)}\wedge\cdots\wedge\vartheta^{(i_{D-3})}\right)\\ &+\frac{1}{(D-2)!}\epsilon_{IJA_1\ldots A_{D-2}}\vartheta_{(i_1)}^{\ A_1}\cdots\vartheta_{(i_{D-2})}^{\ A_{D-2}}\left(\vartheta^{(i_1)}\wedge\cdots\wedge\vartheta^{(i_{D-2})}\right)\,. \end{aligned} \tag{2.26}$$

To find the pull-back of A we first note that

$$\begin{aligned} \nabla_{\underleftarrow{a}}\ell_I &\approx \nabla_{\underleftarrow{a}}\left({e^b}_I\ell_b\right)\\ &\approx (\nabla_{\underleftarrow{a}}{e^b}_I)\ell_b + {e^b}_I\nabla_{\underleftarrow{a}}\ell_b\\ &\approx {e^b}_I\omega_a\ell_b\\ &\approx \omega_a\ell_I\,, \end{aligned} \tag{2.27}$$

where we used $\nabla_a {e^b}_I = 0$ in going from the second to the third line (a consequence of the metric compatibility of the connection). Then, taking the covariant derivative of ℓ acting on internal indices gives

$$\nabla_a \ell_I = \partial_a \ell_I + A_{aIJ}\ell^J \,, \tag{2.28}$$

with ∂ representing a flat derivative operator that is compatible with the internal coframe on Δ. Thus $\partial_a \ell_I \approx 0$ and

$$\nabla_{\underleftarrow{a}} \ell_I \approx A_{\underleftarrow{a}IJ}\ell^J \,. \tag{2.29}$$

Putting this together with (2.27) we have that

$$A_{\underleftarrow{a}IJ}\ell^J \approx \omega_a \ell_I \,, \tag{2.30}$$

and this implies that the pull-back of A to the horizon is of the form

$$A_{\underleftarrow{a}}{}^{IJ} \approx -2\ell^{[I}n^{J]}\omega_a + a_a^{(i)}\ell^{[I}\vartheta_{(i)}^{\;J]} + b_a^{(ij)}\vartheta_{(i)}^{\;[I}\vartheta_{(j)}^{\;J]} \,, \tag{2.31}$$

where the $a_a^{(i)}$ and $b_a^{(ij)}$ are one-forms in the cotangent space $T^*(\Delta)$. It follows that the variation of (2.31) is

$$\delta A_{\underleftarrow{a}}{}^{IJ} \approx -2\ell^{[I}n^{J]}\delta\omega_a + \delta a_a^{(i)}\ell^{[I}\vartheta_{(i)}^{\;J]} + \delta b_a^{(ij)}\vartheta_{(i)}^{\;[I}\vartheta_{(j)}^{\;J]} \,. \tag{2.32}$$

Finally, by direct calculation, it can be shown that the gravitational part J_{Grav} of the surface term (2.20) reduces to

$$J_{\mathrm{Grav}}[\Psi, \delta\Psi] \approx \tilde{\epsilon} \wedge \delta\omega \,. \tag{2.33}$$

Here,

$$\tilde{\epsilon} = \vartheta^{(1)} \wedge \cdots \wedge \vartheta^{(D-2)} \tag{2.34}$$

is the area element of the cross sections $\mathbb{S}^{D-2}$ of the horizon.

Now we make use of the fact that, because ℓ is normal to the surface, its variation will also be normal to the surface. That is, $\delta\ell \propto \ell$ for some ℓ fixed in $[\ell]$. This together with $\pounds_\ell \omega = 0$ then implies that $\pounds_\ell \delta\omega = 0$. However, ω is held fixed on $M^\pm$ which means that $\delta\omega = 0$ on the initial and final cross-sections of Δ (i.e. on $M^- \cap \Delta$ and on $M^+ \cap \Delta$), and because $\delta\omega$ is Lie dragged on Δ it follows that $J_{\mathrm{Grav}} \approx 0$. The same argument also holds for the electromagnetic part J_{EM} of the surface term (2.20). In particular, because the electromagnetic field is in a gauge adapted to the horizon, $\pounds_\ell \underleftarrow{\boldsymbol{A}} = 0$, and with $\delta\ell \propto \ell$ we also have that $\pounds_\ell \delta \underleftarrow{\boldsymbol{A}} = 0$. This is sufficient to show that $J_{\mathrm{EM}} \approx 0$ as well. Therefore the surface term $J|_{\partial\mathcal{M}} = 0$ for the Einstein-Maxwell theory with electromagnetic CS term, and we conclude that the equations of motion $E[\Psi] = 0$ follow from the action principle $\delta S = 0$.

2.4 Covariant phase space

The derivation of the first law involves two steps. First we need to find the symplectic structure on the covariant phase space $\boldsymbol{\Gamma}$ consisting of solutions $(e, A, \boldsymbol{A})$ to the field equations (2.10), (2.11) and (2.12) on $\mathcal{M}$. Once we have a suitable (closed and conserved) symplectic two-form, we then need to specify an evolution vector field ξ^a. In this section we derive the symplectic two-form. In the next section we will specify the evolution vector field which will also serve to introduce an appropriate notion of horizon angular momentum.

The antisymmetrized second variation of the surface term gives the symplectic current, and integrating over a spacelike hypersurface M gives the symplectic structure $\boldsymbol{\Omega} \equiv \boldsymbol{\Omega}(\delta_1, \delta_2)$ (with the choice of M being arbitrary). Following (Ashtekar *et al* 2000c), we find that the second variation of the surface term (2.20) gives

$$J[\Psi, \delta_1\Psi, \delta_2\Psi] = \delta_1\Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1\mathbf{\Phi} \wedge \delta_2\mathbf{A} - \delta_2\mathbf{\Phi} \wedge \delta_1\mathbf{A} \,. \tag{2.35}$$

Whence integrating over M defines the *bulk* symplectic structure

$$\mathbf{\Omega}_{\mathrm{B}} = \frac{1}{2\kappa_D}\int_M \left[\delta_1\Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1\mathbf{\Phi} \wedge \delta_2\mathbf{A} + \delta_2\mathbf{\Phi} \wedge \delta_1\mathbf{A}\right] \,. \tag{2.36}$$

We also need to find the pull-back of J to Δ and add the integral of this term to $\mathbf{\Omega}_{\mathrm{B}}$ so that the resulting symplectic structure on $\mathbf{\Gamma}$ is conserved. If we define potentials ψ and χ for the surface gravity $\kappa_{(\ell)}$ and electric potential $\Phi_{(\ell)}$ such that

$$\pounds_\ell\psi \approx \ell \lrcorner \omega = \kappa_{(\ell)} \quad \text{and} \quad \pounds_\ell\chi \approx \ell \lrcorner \mathbf{A} = -\Phi_{(\ell)} \,, \tag{2.37}$$

then the pullback to Δ of the symplectic structure will be a total derivative; using the Stokes theorem this term becomes an integral over the cross sections $\mathbb{S}^{D-2}$ of Δ. Hence the full symplectic structure is given by

$$\begin{aligned}\mathbf{\Omega} \;=\; & \frac{1}{2\kappa_D}\int_M \left[\delta_1\Sigma_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\Sigma_{IJ} \wedge \delta_1 A^{IJ} - \delta_1\mathbf{\Phi} \wedge \delta_2\mathbf{A} + \delta_2\mathbf{\Phi} \wedge \delta_1\mathbf{A}\right] \\ & + \frac{1}{\kappa_D}\oint_{\mathbb{S}^{D-2}} \left[\delta_1\bar{\epsilon} \wedge \delta_2\psi - \delta_2\bar{\epsilon} \wedge \delta_1\psi + \delta_1\mathbf{\Phi} \wedge \delta_2\chi - \delta_2\mathbf{\Phi} \wedge \delta_1\chi\right] \,. \end{aligned} \tag{2.38}$$

2.5 Angular momentum and the first law

In D dimensions, there are $\lfloor (D-1)/2 \rfloor$ rotation parameters given by the Casimir invariants of the rotation group $SO(D-1)$. Here, "$\lfloor\cdot\rfloor$" denotes the "integer value of". For a multidimensional WIH rotating with angular velocities Ω_ι ($\iota = 1, \ldots, \lfloor (D-1)/2 \rfloor$), a suitable evolution vector field on the covariant phase space is given by (Ashtekar *et al* 2001; Ashtekar *et al* 2007)

$$\xi^a = z\ell^a + \sum_{\iota=1}^{\lfloor (D-1)/2 \rfloor} \Omega_\iota \phi_\iota^a \,. \tag{2.39}$$

Here, ϕ_ι^a are spacelike rotational vector fields that satisfy

$$\pounds_\phi q_{ab} = 0\,, \quad \pounds_\phi \ell_a = 0\,, \quad \pounds_\phi \omega_a = 0\,, \quad \pounds_\phi \underleftarrow{\boldsymbol{A}} = 0\,, \quad \pounds_\phi \underleftarrow{\boldsymbol{F}} = 0\,. \tag{2.40}$$

The vector field ξ is similar to the linear combination $\zeta = t + \sum_\iota \Omega_\iota m_\iota$ (with t a timelike Killing vector and m_ι spacelike Killing vectors) for the KN solution. By contrast, we note that ξ is spacelike in general and becomes null when all angular momenta are zero, while ζ is null in general and becomes timelike when all angular momenta are zero.

Moving on, the first law now follows directly from evaluating the symplectic structure at (δ, δ_ξ) (Ashtekar *et al* 2007). This gives two surface terms: one at infinity (which is identified with the ADM energy), and one at the horizon. We find that the surface term at the horizon is given by

$$\boldsymbol{\Omega}|_\Delta = \frac{\kappa_{(z\ell)}}{\kappa_D} \delta \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} + \frac{\Phi_{(z\ell)}}{\kappa_D} \delta \oint_{\mathbb{S}^{D-2}} \boldsymbol{\Phi} + \sum_{\iota=1}^{\lfloor (D-1)/2 \rfloor} \frac{\Omega_\iota}{\kappa_D} \delta \oint_{\mathbb{S}^{D-2}} \left[(\phi_\iota \lrcorner \omega) \tilde{\epsilon} + (\phi_\iota \lrcorner \boldsymbol{A}) \boldsymbol{\Phi} \right] , \tag{2.41}$$

where we used $\kappa_{(z\ell)} = \pounds_{z\ell} \psi = z\ell \lrcorner \omega$ and $\Phi_{(z\ell)} = \pounds_{z\ell} \chi = z\ell \lrcorner \boldsymbol{A}$. These potentials are constant for any given horizon, but in general vary across the phase space from one point to another. This implies that (2.41) is *not* in general a total variation. However, if $\kappa_{(z\ell)}$, $\Phi_{(z\ell)}$ and Ω_ι can be expressed as functions of the surface area $\mathcal{A}$, charge $\mathcal{Q}$ and angular momenta $\mathcal{J}_\iota$ defined by

$$\mathcal{S} = \frac{1}{4G_D}\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \tag{2.42}$$

$$\mathcal{Q} = \frac{1}{8\pi G_D}\oint_{\mathbb{S}^{D-2}} \mathbf{\Phi} \tag{2.43}$$

$$\mathcal{J}_\iota = \frac{1}{8\pi G_D}\oint_{\mathbb{S}^{D-2}} [(\phi_\iota \lrcorner \omega)\tilde{\epsilon} + (\phi_\iota \lrcorner \boldsymbol{A})\mathbf{\Phi}] \tag{2.44}$$

and satisfy the integrability conditions

$$\frac{\partial \kappa}{\partial \mathcal{J}} = \frac{\partial \Omega}{\partial \mathcal{A}}, \quad \frac{\partial \kappa}{\partial \mathcal{Q}} = \frac{\partial \Phi}{\partial \mathcal{A}}, \quad \frac{\partial \Omega}{\partial \mathcal{Q}} = \frac{\partial \Phi}{\partial \mathcal{J}}, \tag{2.45}$$

then there exists a function $\mathcal{E}$ such that (Ashtekar *et al* 2001; Ashtekar *et al* 2007)

$$\mathbf{\Omega}|_\Delta(\delta, \delta_\xi) = \delta\mathcal{E} \,. \tag{2.46}$$

In this case (2.41) becomes

$$\delta\mathcal{E} = \frac{\kappa_{(z\ell)}}{\kappa_D}\delta\mathcal{A} + \Phi_{(z\ell)}\delta\mathcal{Q} + \sum_{\iota=1}^{\lfloor (D-1)/2 \rfloor} \Omega_\iota \delta\mathcal{J}_\iota \,, \tag{2.47}$$

which is the first law (for a quasi-static process). Therefore WIHs in D-dimensional EM-CS theory satisfy the first law (and the zeroth law) of black-hole mechanics. This is in agreement with (Gauntlett *et al* 1999), but with a very important difference. Here, all the quantities appearing in the first law are defined at the horizon; no reference was made to the boundary at infinity.

Remarks. Several remarks are in order here.

1. The expression (2.44) implies that the horizon angular momentum contains contributions from both gravitational *and* electromagnetic fields, here referred to as $\mathcal{J}_{\text{Grav}}$ and $\mathcal{J}_{\text{EM}}$. This is in contrast to the standard angular momentum expressions at infinity,

such as the Komar expression. One can show (Ashtekar *et al* 2001; Ashtekar *et al* 2007) that $\mathcal{J}_{\text{Grav}}$ is equivalent to the (quasilocal) Komar integral

$$\mathcal{J}_{\text{K}} = -\frac{1}{8\pi G_D} \oint_{S^{D-2}} \star d\phi \,, \tag{2.48}$$

and this matches the expression for Killing horizons at infinity.

2. It would appear that if $\mathcal{J}_{\text{Grav}} = 0$ then there is still a non-zero contribution to (2.44) from $\mathcal{J}_{\text{EM}}$. However, it can be shown (Ashtekar *et al* 2001) that if ϕ is the restriction to Δ of a *global* rotational Killing field φ contained in $\mathcal{M}$, then $\mathcal{J}_{\text{EM}}$ is actually the angular momentum of the electromagnetic radiation in the bulk. What happens is that the bulk integral $\int_M T_{ab}\varphi^a dS^b$ can be written as the sum of a surface term at Δ and a surface term at $\mathscr{B}$. Therefore we say that a non-rotating WIH is one for which $\mathcal{J}_{\text{Grav}} = 0$.

3. The charges at $\mathscr{B}$ are the charges of the spacetime and are independent of the charges at Δ.

2.6 Rotation in ADS spacetime

Currently there is a lot of interest in the ADS/CFT correspondence (Maldacena 1998; Witten 1998a; Witten 1998b; Aharony *et al* 2000). A significant amount of effort on the gravity side has been focused on finding charged and rotating black hole solutions in five-dimensional ADS spacetime, both non-extremal in general (Hawking *et al* 1999; Chamblin *et al* 1999; Hawking and Reall 1999; Klemm and Sabra 2001; Cvetič *et al* 2004; Chong *et al* 2005) and supersymmetric in particular (Gutowski *et al* 2004; Kunduri *et al* 2006; Kunduri *et al* 2007a; Kunduri and Lucietti 2007).

For these black holes, however, there is an ambiguity in how the conserved charges are defined. This was first pointed out by Caldarelli *et al* (2000). The ambiguity arises because for rotating black holes in ADS spacetime there are two distinct natural choices for the timelike Killing field. To see this, let's compare the KN solution (with $\Lambda = 0$) and the KN-ADS solution. The KN solution contains the vector $K = \partial/\partial t$ with which one can define the charges. When $\Lambda < 0$, however, there is another timelike Killing vector in addition to K that appears and is given by $K' = \partial/\partial t + (a/L^2)\partial/\partial\phi$. For the KN-ADS solution, K remains timelike *everywhere* outside the event horizon which implies that if K is chosen as the generator of time translations then there is no ergoregion present. By contrast, K' becomes spacelike near the event horizon which implies that if K' is chosen as the generator of time translations then there is an ergoregion in the neighbourhood of the event horizon. Physically, this means that defining the conserved charges with respect to K corresponds to a frame at infinity that is co-rotating, whereas defining the conserved charges with respect to K' corresponds to a frame at infinity that is non-rotating.

The original motivation for defining the conserved charges with respect to K was that the corresponding boundary CFT conserved charges satisfy the first law of thermodynamics (Hawking *et al* 1999); but this comes at the cost that the bulk conserved charges do not (Caldarelli *et al* 2000; Gibbons *et al* 2005). This claim has by now been corrected. As was shown in (Gibbons *et al* 2006), one can always pass from the bulk conserved charges to the boundary conserved charges in such a way that both sets separately satisfy the first law. The key to this resolution is that the conserved charges of a rotating black hole in ADS spacetime have to be measured with respect to the timelike vector which corresponds to a frame that is non-rotating at infinity.

From the above considerations, it is clear that rotation in ADS spacetime should be independent of the coordinates that are used. This is especially crucial when considering supersymmetric black holes in ADS spacetime (the extremal limit of a non-rotating ADS black hole results in a naked singularity). In this section we will briefly discuss how the IH framework provides a resolution to the above pathology.

To begin, we define an asymptotically ADS spacetime. Following (Ashtekar and Das 2000; Ashtekar *et al* 2007), we have the following:

Definition II. *A spacetime* $(\mathcal{M}, g_{ab})$ *is said to be asymptotically ADS if there exists a spacetime* $(\widehat{\mathcal{M}}, \tilde{g}_{ab})$ *with outer boundary* $\mathscr{I}$ *such that* $\widehat{\mathcal{M}} - \mathscr{I}$ *is diffeomorphic to* $\mathcal{M}$ *and the following conditions hold: (a) there exists a function* Ω *on* $\widehat{\mathcal{M}}$ *for which* $\tilde{g}_{ab} = \Omega^2 g_{ab}$ *on* $\mathcal{M}$*; (b)* Ω *vanishes on* $\mathscr{I}$ *but the gradient* $\nabla_a \Omega$ *is nowhere vanishing on* $\mathscr{I}$*; (c) the stress-energy tensor* T_{ab} *on* $\mathcal{M}$ *is such that* $\Omega^{-(D-2)} T_{ab}$ *has a smooth limit to* $\mathscr{I}$*; and (d) the Weyl tensor* $\tilde{C}_{abcd}$ *of* $\tilde{g}_{ab}$ *is such that* $\Omega^{-(D-4)} \tilde{C}_{abcd}$ *is smooth on* $\mathcal{M}$ *and vanishes on* $\mathscr{I}$.

These are the standard boundary conditions which have been tailored to ensure that a spacetime will be asymptotically ADS. Their meaning is discussed in detail in (Ashtekar and Das 2000).

In the presence of a negative cosmological constant and with no matter fields, the covariant phase space of WIHs is modified to include a set of conserved charges at $\mathscr{I}$ (Ashtekar *et al* 2007). These are the Ashtekar-Magnon-Das (AMD) charges (Ashtekar and Magnon 1984; Ashtekar and Das 2000)

$$\mathscr{Q}_{\xi}^{(\mathscr{I})} = \frac{L}{8\pi G_D} \oint_{C^{D-2}} \widetilde{E}_{ab} k^a \tilde{u}^b \tilde{\varepsilon}\,, \tag{2.49}$$

with k^a a Killing vector field that generates a symmetry (i.e. time translation etc), $\tilde{u}^a$ the unit timelike normal to $\mathbb{C}^{D-2}$, $\tilde{\varepsilon}$ the area form on $\mathbb{C}^{D-2}$ and $\widetilde{E}_{ab}$ the leading-order electric part of the Weyl tensor. Explicitly we have that

$$\widetilde{E}_{ab} = \frac{1}{D-3}\Omega^{3-D}\widetilde{C}_{abcd}\tilde{n}^c\tilde{n}^d\,, \tag{2.50}$$

where $\tilde{n}^a = \tilde{\nabla}^a\Omega$. As was shown in Appendix B of (Hollands *et al* 2005), inclusion of antisymmetric tensor fields in the action does not contribute anything to the charges at $\mathscr{I}$ because the fields fall off too quickly. Therefore the charges at infinity for EM-CS theory are precisely the AMD charges (2.49).

Gibbons *et al* (2005) showed that the asymptotic time translation Killing field for an exact solution has to be chosen in such a way that the frame at infinity is non-rotating. If this is done then the AMD charge evaluated for the solution will result in an expression for mass that satisfies the first law. Moreover, Gibbons *et al* (2006) showed that using this definition for the asymptotic time translation has to be used for a consistent transition to the conserved charges of the boundary CFT.

Let us summarize. The IH framework provides a coherent physical picture whereby two sets of conserved charges arise in ADS spacetime: the charges measured at infinity and the local charges measured at the horizon. The local conserved charges at the horizon then satisfy the first law. When evaluated on exact solutions to the field equations, the charges at infinity correspond to asymptotic symmetries that are measured with respect to a *non-rotating* frame at infinity.

The description of ADS black holes presented here is somewhat different from the description of black holes in globally stationary spacetimes where an ambiguity appears that manifests itself as a choice of whether the conserved charges are measured with respect to a frame at infinity that is rotating or non-rotating. This ambiguity does not appear in the IH framework essentially because the conserved charges of the black hole are measured *at the horizon*, and the corresponding first law is intrinsic to the horizon with no mixture of quantities there and at infinity!

2.7 A topological constraint from extremality

One of the properties of an extremal black hole is that its surface gravity is zero. Another property is that its horizons are degenerate: the inner and outer horizons coincide. As a result, an extremal black hole is one for which there are no trapped surfaces "just inside" the horizon. This property was recently used (Booth and Fairhurst 2008) to define an extremality condition for quasilocal horizons. We note here the evolution equation for the expansion of the null normal n^a (Booth and Fairhurst 2008):

$$\pounds_\ell \theta_{(n)} + \kappa\theta_{(n)} + \frac{1}{2}\mathcal{R} = d_a\tilde{\omega}^a + \|\tilde{\omega}\|^2 + (T_{ab} - \Lambda g_{ab})\ell^a n^b \,. \tag{2.51}$$

Here, $\mathcal{R}$ is the scalar curvature of $\mathbb{S}^{D-2}$, d_a is the covariant derivative operator that is compatible with the metric

$$\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a \tag{2.52}$$

on $\mathbb{S}^{D-2}$, and $\|\tilde{\omega}\|^2 = \tilde{\omega}_a\tilde{\omega}^a$ where

$$\tilde{\omega}_a = \tilde{q}_a{}^b\omega_b = \omega_a + \kappa_{(\ell)} n_a \tag{2.53}$$

is the projection of ω onto $\mathbb{S}^{D-2}$. $\tilde{\omega}$ is referred to as the rotation one-form.

Our desire is to apply the expression (2.51) to *black holes*, and in order to do this we need to impose some restrictions on the WIHs. In order to proceed we now restrict our attention to fully isolated horizons (IHs). These are WIHs for which there is a scaling of the null normals for which the commutator $[\mathcal{L}_\ell, \mathcal{D}] = 0$, where $\mathcal{D}$ is the intrinsic covariant derivative on the horizon. This means that not only is condition (d) of Definition I satisfied, but also it implies that $[\mathcal{L}_\ell, \mathcal{D}]n^a = 0$ (Ashtekar *et al* 2002). In contrast to the condition (d) for WIHs, then, this stronger condition cannot always be met and geometrically such horizons not only have time-invariant intrinsic geometry, they also have time-invariant extrinsic geometry. That said it is clear that this condition similarly fixes ℓ only up to a constant scaling. As such it does not uniquely determine the value of the surface gravity $\kappa_{(\ell)}$ but does fix its sign. In particular this allows us to invariantly say whether or not $\kappa_{(\ell)}$ vanishes. This then gives rise to an invariant characterization of extremality that is intrinsic to the horizon: a horizon is sub-extremal if $\kappa > 0$ ($\theta_{(n)} < 0$) and extremal (with degenerate horizons) if $\kappa = 0$. Further, $\mathcal{L}_\ell \theta_{(n)} = 0$ and combining this with the fact that the inward expansion $\theta_{(n)}$ should always be less than zero, an integration of (2.51) gives

$$\eta \equiv \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(T_{ab}\ell^a n^b + \Lambda + \|\tilde{\omega}\|^2) - \frac{1}{2}\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R} \leq 0 \,. \tag{2.54}$$

(Here we used $-\Lambda g_{ab}\ell^a n^b = \Lambda$ and the fact that $\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} d_a \tilde{\omega}^a = 0$.) This inequality provides an alternative characterization of extremal IHs: if $\eta < 0$ ($\kappa > 0$ and $\theta_{(n)} < 0$) then Δ is nonextremal, and if $\eta = 0$ ($\kappa = 0$) then Δ is extremal. However, this inequality also provides a topological constraint on the cross sections of Δ. To see this, rewrite (2.54) so that

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(\mathcal{R} - 2\Lambda) \geq 2\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(T_{ab}\ell^a n^b + \|\tilde{\omega}\|^2) \,. \tag{2.55}$$

Now, observe that the dominant energy condition requires that $T_{ab}\ell^a n^b \geq 0$. In addition, $\|\tilde{\omega}\|^2$ is manifestly non-negative. The inequality (2.55) therefore restricts the topology of the cross sections of the horizon. The condition (2.55) is the same as the one that was found in four dimensions for marginally trapped surfaces (Hayward 1994), nonexpanding horizons (Pawlowski *et al* 2004) and dynamical horizons (Ashtekar and Krishnan 2003; Booth and Fairhurst 2007).

For nonextremal horizons, $\eta < 0$, and the constraint (2.55) splits into two possibilities, depending on the nature of the cosmological constant:

- $\Lambda \geq 0$. The integral of the scalar curvature is strictly positive. In four dimensions the GB theorem says that $\oint_{\mathbb{S}^2} \tilde{\epsilon}\mathcal{R} = 8\pi(1-g)$, with g the genus of the surface $\mathbb{S}^2$. In this case $\eta < 0$ implies that $g = 0$ and hence the only possibility is that the cross sections are two-spheres S^2. In five dimensions $\eta < 0$ implies that the cross sections are of positive Yamabe type; this implies that topologically $\mathbb{S}^3$ can only be a finite connected sum of the three-sphere S^3 or of the ring $S^1 \times S^2$ (Schoen and Yau 1979; Galloway and Schoen 2006; Galloway 2006). Both these topologies have been realized and the corresponding solutions, for example the Myers-Perry black hole (Myers and Perry 1986) and the Emparan-Reall black ring (Emparan and Reall 2002), are well known.

- $\Lambda < 0$. The integral of the scalar curvature can have either sign, or even vanish, and the inequality will always be satisfied. The only restriction is that

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(\mathcal{R} + 2|\Lambda|) \geq 2 \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}(T_{ab}\ell^a n^b + \|\tilde{\omega}\|^2) . \tag{2.56}$$

  There is no constraint on the topology of $\mathbb{S}^{D-2}$ except that the space has to be compact. Owing to this special property, many such black holes have been found with exotic topologies in $D \geq 3$ dimensions. See e.g. (Bañados *et al* 1993; Åminneborg *et al* 1996; Vanzo 1997; Åminneborg *et al* 1998; Bañados 1998; Bañados *et al* 1998; Klemm *et al* 1998).

For extremal horizons, $\eta = 0$, and the constraint (2.55) becomes an equality. In this case the same restrictions apply to $\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R}$ as for nonextremal horizons. However, there is also a special case that occurs:

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R} = 0 \tag{2.57}$$

for an extremal and non-rotating ($\omega = \tilde{\omega} = 0$) horizon when the scalar $T_{ab}\ell^a n^b$ vanishes on the horizon. This case corresponds to the torus topology T^{D-2}.

Remark. Although the expression (2.56) does not constrain the topology of ADS black holes explicitly, there is an interesting area-topology relation that comes out. The cosmological term can be integrated out, and upon rearranging to isolate the surface area $\mathcal{A}_{\mathbb{S}^2} \equiv \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}$ gives

$$\mathcal{A}_{\mathbb{S}^{D-2}} \geq \frac{1}{|\Lambda|} \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} \left(-\frac{1}{2}\mathcal{R} + T_{ab}\ell^a n^b + \|\tilde{\omega}\|^2 \right) . \tag{2.58}$$

In four dimensions, the GB theorem then implies that

$$\mathcal{A}_{\mathbb{S}^2} \geq \frac{1}{|\Lambda|} \left[4\pi(g-1) + \oint_{\mathbb{S}^2} \tilde{\epsilon}(T_{ab}\ell^a n^b + \|\tilde{\omega}\|^2) \right] . \tag{2.59}$$

This implies that the maximum allowed angular momentum is bound by the genus and area of the horizon; see (Booth and Fairhurst 2008; Hennig *et al* 2008) for discussions of the corresponding result for asymptotically flat spacetimes and appendix B of (Booth and Fairhurst 2008) for a particular discussion of Kerr-ADS.

Alternatively, reversing the inequality, one can view it as bounding the allowed area of isolated horizons from below by the scale of the cosmological curvature and the genus of the horizon: higher genus horizons necessarily have larger areas. Similar bounds have previously been discovered for stationary ADS black holes (Gibbons 1999; Woolgar 1999; Cai and Galloway 2001).

Chapter 3

Supersymmetric isolated horizons

"String theorists listening to talks on loop quantum gravity are often puzzled by the lack of interest in supersymmetry and higher dimensions, which string theory has shown seem to be required to satisfy certain criteria for a good theory." ~ L Smolin

3.1 Black holes and Killing spinors

Until now we have discussed the mechanics of WIHs in arbitrary dimensions. We now specialize to supersymmetric horizons and in particular we focus on the bosonic sector of four-dimensional $N = 2$ gauged supergravity and the bosonic sector of five-dimensional $N = 1$ supergravity. In both cases, black holes are solutions to the bosonic equations of motion and so the fermion fields vanish. By definition, supersymmetric solutions are invariant under the full supersymmetry transformations that are generated by spinor fields. This means that for black hole solutions, these transformations should leave the fermion fields unchanged (and vanishing). Therefore any such black hole solutions must admit a Killing spinor field.

For full stationary black hole solutions such as those discussed in (Caldarelli and Klemm 1999; Gauntlett *et al* 1999; Gutowski and Reall 2003), the Killing spinor gives rise to a (timelike) time-translation Killing vector field in the region outside of the black hole horizon. However, in the quasilocal spirit of the isolated horizon programme we will only assume the existence of a Killing spinor *on the horizon itself.* In this case the spinor will generate a null geodesic vector field that has vanishing twist, shear, and expansion and this is an allowed ℓ on the WIH.

As we did in §2.7, we will consider fully IHs, which allows for a clear difference between nonextremal and extremal IHs. Finally we define a *supersymmetric isolated horizon* (SIH) as an IH on which the null vector generated by the Killing spinor coincides (up to a free constant) with the preferred null vector field arising from the IH structure. As we shall now see these are necessarily extremal as well as having restricted geometry, rotation, and matter fields.

3.2 Killing spinors in four dimensions

We will first consider the four-dimensional action. With $D = 4$ and $\Lambda = -3/L^2$ the action (2.8) is the bosonic action of $N = 2$ gauged supergravity. The (extremal) KN-ADS black hole, which is a solution to the $N = 2$ supergravity with the fermion fields set to zero. As was shown in (Kostelecký and Perry 1996), the condition for a supersymmetric KN-ADS black hole in four dimensions to have positive energy is that

$$\mathfrak{M} = |\mathfrak{Q}| \left(1 \pm \frac{a}{L}\right) , \tag{3.1}$$

which is the extremality condition for the KN-ADS black hole relating the mass $\mathfrak{M}$, total charge $\mathfrak{Q} \equiv \sqrt{q_e^2 + q_m^2}$ (with q_e and q_m the electric and magnetic charges) and angular momentum $\mathfrak{J} = a\mathfrak{M}$ at infinity. This is also the saturated Bogomol'ny-Prasad-Sommerfeld (BPS) inequality. When $\Lambda = 0$ the equality (3.1) reduces to (Gibbons and Hull 1982)

$$\mathfrak{M} = |\mathfrak{Q}| \,, \tag{3.2}$$

which is the extremality condition for the KN black hole.

For four-dimensional $N = 2$ gauged supergravity, we shall employ the conventions of (Caldarelli and Klemm 2003). The corresponding (bosonic) action is

$$S = \frac{1}{16\pi G_4} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} + \frac{6}{L^2}\epsilon - \frac{1}{4}\boldsymbol{F} \wedge \star\boldsymbol{F} \,. \tag{3.3}$$

The necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor ϵ^α such that

$$\left[\nabla_a + \frac{i}{4}F_{bc}\gamma^{bc}\gamma_a + \frac{1}{L}\gamma_a\right]\epsilon = 0 \,. \tag{3.4}$$

Here, γ^a are a set of gamma matrices that satisfy the anticommutation rule

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab} \tag{3.5}$$

and the antisymmetry product

$$\gamma_{abcd} = \epsilon_{abcd} \,. \tag{3.6}$$

$\gamma_{a_1 \dots a_D}$ denotes the antisymmetrized product of D gamma matrices. The spinor ϵ satisfies the reality condition

$$\bar{\epsilon} = i(\epsilon)^\dagger \gamma_0 \,; \tag{3.7}$$

overbar denotes complex conjugation and † denotes Hermitian conjugation.

From ϵ one can construct five bosonic bilinears f, g, V^a, W^a and $\Psi^{ab} = \Psi^{[ab]}$ where

$$f = \bar{\epsilon}\epsilon\,, \quad g = i\bar{\epsilon}\gamma^5\epsilon\,, \quad V^a = \bar{\epsilon}\gamma^a\epsilon\,, \quad W^a = i\bar{\epsilon}\gamma^5\gamma^a\epsilon\,, \quad \Psi^{ab} = \bar{\epsilon}\gamma^{ab}\epsilon\,. \tag{3.8}$$

These are inter-related by several algebraic relations (from the Fierz identities) and differential equations (from the Killing equation (3.4)) (Caldarelli and Klemm 2003). For our purposes the significant ones are:

$$V_aV^a = -W_aW^a = -(f^2+g^2)\,, \tag{3.9}$$

$$V^aW_a = 0\,, \tag{3.10}$$

$$gW_a = \Psi_{ab}V^b\,, \tag{3.11}$$

$$f\Psi_{ab} = -\epsilon_{abcd}V^cW^d + \frac{1}{2}g\epsilon_{abcd}\Psi^{cd}\,, \tag{3.12}$$

$$\nabla_a f = F_{ab}V^b\,, \tag{3.13}$$

$$\nabla_a g = -\frac{1}{L}W_a - \frac{1}{2}\epsilon_{abcd}V^bF^{cd}\,, \tag{3.14}$$

$$\nabla_a V_b = \frac{1}{L}\Psi_{ab} - fF_{ab} + \frac{g}{2}\epsilon_{abcd}F^{cd}\,, \tag{3.15}$$

$$\nabla_a W_b = -\frac{g}{L}g_{ab} - F_{(a}{}^{c}\epsilon_{b)cde}\Psi^{de} + \frac{1}{4}g_{ab}\epsilon_{cdef}F^{cd}\Psi^{ef} \text{ and} \tag{3.16}$$

$$\nabla_c\Psi_{ab} = \frac{2}{L}g_{c[a}V_{b]} + 2F_{[a}{}^{d}\epsilon_{b]dce}W^e + F_c{}^{d}\epsilon_{dabe}W^e + g_{c[a}\epsilon_{b]def}W^dF^{ef}\,. \tag{3.17}$$

These are general relations for the existence of a Killing spinor in spacetime. Although the Killing spinor may exist in a neighbourhood of the horizon, we only require that it exist *on the horizon itself.* Henceforth we specialize by setting $f = g = 0$ and at the same time require that the relations hold on Δ. Thus, the differential equations (3.13)-(3.17) are only required to hold when the derivatives are pulled-back onto the horizon.

With $f = g = 0$, equation (3.9) implies that V^a and W^a are both null. On an SIH we identify $\ell^a = V^a$ and so condition (2.13) together with the differential constraint (3.15) implies that

$$\nabla_{\underleftarrow{a}} \ell_b = \omega_a \ell_b = \frac{1}{L} \Psi_{\underleftarrow{a} b} \,, \tag{3.18}$$

and using the skew-symmetry of Ψ_{ab} we can write

$$\Psi_{ab} = L(\omega_a \ell_b - \omega_b \ell_a) \,. \tag{3.19}$$

Then by equation (3.11)

$$\ell \lrcorner \omega = 0 \Leftrightarrow \kappa_{(\ell)} = 0 \,. \tag{3.20}$$

Thus, an SIH is necessarily extremal.

For ease of presentation we now assume that the SIH is foliated into spacelike two-surfaces $\mathbb{S}_v$. One can always construct such a foliation (and its labelling) so that the associated null normal $n \equiv dv$ satisfies $\ell \lrcorner n = -1$ (Ashtekar *et al* 2002). Then the two-metric on the $\mathbb{S}_v$ is given by (2.52) and area form on the $\mathbb{S}_v$ can be written as

$$\tilde{\epsilon}_{cd} = -\ell^a n^b \epsilon_{abcd} \,. \tag{3.21}$$

Now we note that with $\kappa_{(\ell)} = 0$ it follows from (2.53) that $\omega_a = \tilde{\omega}_a$ and hence $\omega_a \in T^\star(\mathbb{S}_v)$. Finally, with respect to this foliation, the usual restriction (2.14) and (redundantly) equation (3.13) implies that the electromagnetic field takes the form

$$F_{ab} = 2E_\perp \ell_{[a} n_{b]} + B_\perp \tilde{\epsilon}_{ab} + 2\tilde{X}_{[a} \ell_{b]} \,, \tag{3.22}$$

on Δ. Here, $E_\perp$ and $B_\perp$ are the electric and magnetic fluxes through the surface and $\tilde{X}^a \in T(\mathbb{S}_v)$ describes flows of electromagnetic radiation along (but not through) the horizon.

With these preliminaries in hand we can consider the properties of SIHs in asymptotically ADS spacetimes. First, relations (3.10) and (3.12) tell us that

$$W^a = L\beta V^a \tag{3.23}$$

for some function β (the factor of L has been included for later convenience). Then the pull-back of (3.14) trivially vanishes without giving us any new information but (3.16) provides a differential equation for β on each $\mathbb{S}_v$

$$d_a\beta + \beta\tilde{\omega}_a = B_\perp\tilde{\omega}_a - E_\perp\tilde{\epsilon}_a{}^b\tilde{\omega}_b\,, \tag{3.24}$$

where d_a is the intrinsic covariant derivative on $\mathbb{S}_v$, along with its time-invariance: $\pounds_\ell\beta = 0$.

Next applying the various properties of extremal IHs, one can show that the pull-back of (3.17) is

$$\nabla_{\underleftarrow{c}}\Psi_{ab} = 2L\left(\frac{1}{L^2} - \beta B_\perp\right)\tilde{q}_{c[a}\ell_{b]} + 2L\beta E_\perp\tilde{\epsilon}_{c[a}\ell_{b]}\,, \tag{3.25}$$

and combining this with (3.19) we find that

$$d_a\tilde{\omega}_b + \tilde{\omega}_a\tilde{\omega}_b = \left(\frac{1}{L^2} - \beta B_\perp\right)\tilde{q}_{ab} + \beta E_\perp\tilde{\epsilon}_{ab}\,. \tag{3.26}$$

Now as was seen in (2.44), the gravitational angular momentum associated with a rotational Killing field ϕ^a is

$$\mathcal{J}_{\text{Grav}} = \frac{1}{8\pi G_4}\oint_{\mathbb{S}_v}\tilde{\epsilon}\phi\lrcorner\tilde{\omega}\,, \tag{3.27}$$

and so a necessary condition for non-zero angular momentum is a non-vanishing rotation one-form $\tilde{\omega}_a$. That said, this is not quite sufficient as it is possible for a non-vanishing $\phi\lrcorner\tilde{\omega}$ to integrate to zero. For example, consider the case where $\mathbb{S}_v$ has topology S^2 and ϕ^a is a

Killing field (and so divergence-free). Then for some function ζ we can write $\phi^a = \tilde{\epsilon}^{ab} d_b \zeta$ and

$$\oint_{\mathbb{S}_v} \tilde{\epsilon} \phi \lrcorner \tilde{\omega} = \oint_{\mathbb{S}_v} \zeta d\tilde{\omega} \, . \tag{3.28}$$

Thus, for all closed rotational one-forms ($d\tilde{\omega} = 0$) the associated gravitational angular momentum will vanish. As such, it is standard in the isolated horizon literature [see e.g. (Ashtekar *et al* 2001)] to take $d\tilde{\omega} \neq 0$ as the defining characteristic of a rotating isolated horizon. In our case

$$d_{[a} \tilde{\omega}_{b]} = \beta E_{\perp} \tilde{\epsilon}_{ab} \, , \tag{3.29}$$

and so an SIH is rotating if and only if $\beta E_{\perp} \neq 0$. Thus, a rotating horizon must have a non-trivial electromagnetic field. This is in agreement with known exact solutions: rotating supersymmetric Kerr-Newmann-AdS black holes as well as those with cylindrical or higher genus horizons all have non-trivial EM fields (Caldarelli and Klemm 1999).

3.3 Killing spinors in five dimensions

We will now consider the five-dimensional action. With $D = 5$ and $\Lambda = 0$ the action (2.8) is the bosonic sector of $N = 1$ gauged supergravity. As in the four-dimensional EM theory, solutions to the bosonic equations of motion require the existence of a Killing spinor to ensure that supersymmetry is preserved. For black holes, the positive energy theorem together with this requirement imply that the mass and charge at infinity are constrained such that (Gibbons *el al* 1994)

$$\mathfrak{M} = \frac{\sqrt{3}}{2} |\mathfrak{Q}| \, . \tag{3.30}$$

As can be verified, the equality (3.30) is satisfied by the (5D) extremal RN black hole (Myers and Perry 1986), the BMPV black hole (Breckenridge *et al* 1997) and the EEMR black ring (Elvang *et al* 2004).

The strategy for finding supersymmetric solutions to the bosonic equations of motion based on Killing spinors is essentially the same in five dimensions as it is in four dimensions. However, a problem arises specifically in five dimensions – spinors satisfying certain reality conditions cannot be consistently defined unless they come in pairs and are equipped with a symplectic structure. For details we refer the interested reader to the excellent Les Houches lectures by van Nieuwenhuizen (1984).

For five-dimensional $N = 1$ supergravity, we shall employ the conventions of (Gauntlett *et al* 2003). The corresponding (bosonic) action is

$$S = \frac{1}{16\pi G_5} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} - \frac{1}{4} \boldsymbol{F} \wedge \star \boldsymbol{F} - \frac{2}{3\sqrt{3}} \boldsymbol{A} \wedge \boldsymbol{F} \wedge \boldsymbol{F} \, . \tag{3.31}$$

The necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor ϵ^α $(\alpha, \beta, \ldots \in \{1, 2\})$ such that

$$\left[\nabla_I + \frac{1}{4\sqrt{3}} (\Gamma_I{}^{JK} - 4\delta_I{}^J \Gamma^K) F_{JK} \right] \epsilon^\alpha = 0 \, . \tag{3.32}$$

Here, Γ^I are a set of gamma matrices that satisfy the anticommutation rule

$$\Gamma^I \Gamma^J + \Gamma^J \Gamma^I = 2\eta^{IJ} \tag{3.33}$$

and the antisymmetry product

$$\Gamma_{IJKLM} = \epsilon_{IJKLM} \, . \tag{3.34}$$

$\Gamma_{I_1 \ldots I_D}$ denotes the antisymmetrized product of D gamma matrices. The spinors ϵ^α satisfy the reality condition

$$\bar{\epsilon}^{\alpha} \equiv (\epsilon^{\alpha})^{\dagger}\Gamma_0 = (\epsilon^{\alpha})^T \mathcal{C}\,; \tag{3.35}$$

the second equality is the symplectic Majorana condition, where T denotes matrix transpose and $\mathcal{C}$ the charge conjugation operator satisfying

$$\mathcal{C}(\Gamma_I)^T \mathcal{C}^{-1} = \Gamma_I\,. \tag{3.36}$$

Spinor indices are raised and lowered using the symplectic structure $\epsilon_{\alpha\beta}$ which is defined such that $\epsilon_{12} = \epsilon^{12} = +1$.

Using ϵ^{α} we can construct bosonic bilinears $\mathcal{F}$, $\mathcal{V}^I$ and $\Phi^{\alpha\beta} = \Phi^{(\alpha\beta)}$ such that

$$\mathcal{F}\epsilon^{\alpha\beta} = \bar{\epsilon}^{\alpha}\epsilon^{\beta}\,, \quad \mathcal{V}^I\epsilon^{\alpha\beta} = \bar{\epsilon}^{\alpha}\Gamma^I\epsilon^{\beta}\,, \quad \Phi^{\alpha\beta IJ} = \bar{\epsilon}^{\alpha}\Gamma^{IJ}\epsilon^{\beta}\,. \tag{3.37}$$

As in §3.2, these bilinears are inter-related by algebraic relations and differential equations. For our present purposes we only need the following (Gauntlett *et al* 2003):

$$\mathcal{V}_I\mathcal{V}^I = \mathcal{F}^2\,, \tag{3.38}$$

$$\nabla_I\mathcal{V}_J = \frac{2}{\sqrt{3}}F_{IJ}\mathcal{F} + \frac{1}{2\sqrt{3}}\epsilon_{IJKLM}F^{KL}\mathcal{V}^M\,. \tag{3.39}$$

For IHs, $\mathcal{F} = 0$ and $\mathcal{V}^I$ is null. Then, using (2.13) together with (3.39), and again making the identification $\mathcal{V}^a = \ell^a$, we find that an IH of five-dimensional $N = 1$ supergravity equipped with a null normal ℓ will be supersymmetric if

$$\nabla_{\underleftarrow{a}}\ell_b \approx \frac{1}{2\sqrt{3}}e_{\underleftarrow{a}}{}^I e_b{}^J \epsilon_{IJKLM}F^{KL}\ell^M\,. \tag{3.40}$$

It follows that the RHS in (3.40) vanishes because of the IH condition (2.14) on $\boldsymbol{F}$ and the pullback expression (2.25) for e, and therefore that $\omega = 0$.

3.4 Interpretation

In this chapter we examined the restrictions that are imposed on IHs when they are assumed to be supersymmetric. The necessary and sufficient condition for supersymmetry in four-dimensional ADS spacetime is that there exists a Killing spinor ϵ that satisfies the conditions (3.4). For four-dimensional SIHs in asymptotically ADS spacetimes we found that the surface gravity vanishes identically from the algebraic conditions that are implied by the Killing spinor equation. This means that the SIHs are necessarily extremal. A further constraint that we found for these SIHs is that the corresponding connection one-form is generically non-zero and is not closed. Then it follows from (3.29) that these SIHs are rotating for non-trivial electromagnetic fields. As we will see below, such a SIH can be non-rotating if the horizon cross sections are constant curvature surfaces and there is magnetic (but not electric) charge.

The necessary and sufficient condition for supersymmetry in five dimensions is that there exists a Killing spinor ϵ^{α} that satisfies (3.32). For five-dimensional SIHs in asymptotically flat spacetimes we found that ω vanishes identically. This implies that the corresponding SIHs are non-rotating. The condition also implies that κ is zero. The corresponding SIHs are therefore non-rotating and extremal.

These properties impose additional constraints on the topology of the corresponding IHs. For SIHs in ADS spacetime there is still no constraint on the possible topologies. The topologies become severely restricted, however, when $\Lambda = 0$. For SIHs in asymptotically flat spacetimes the connection ω vanishes both in four dimensions (see the appendix) and in five dimensions. In this case the topology constraint (2.55) gives

$$\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R} = 2\oint_{\mathbb{S}^{D-2}} \tilde{\epsilon} T_{ab}\ell^a n^b \,. \tag{3.41}$$

In four dimensions we find that there are two possibilities for the topology of a SIH:

- If $T_{ab}\ell^a n^b > 0$ then $\oint_{\mathbb{S}^2} \tilde{\epsilon}\mathcal{R} > 0$. In this case $\mathbb{S}^2 \cong S^2$;
- If $T_{ab}\ell^a n^b = 0$ then $\oint_{\mathbb{S}^2} \tilde{\epsilon}\mathcal{R} = 0$. In this case $\mathbb{S}^2 \cong T^2$.

In five dimensions we find that there are three possibilities:

- If $T_{ab}\ell^a n^b > 0$ then $\oint_{\mathbb{S}^3} \tilde{\epsilon}\mathcal{R} > 0$. In this case $\mathbb{S}^3 \cong S^3$ or $\mathbb{S}^3 \cong S^1 \times S^2$;
- If $T_{ab}\ell^a n^b = 0$ then $\oint_{\mathbb{S}^3} \tilde{\epsilon}\mathcal{R} = 0$. In this case $\mathbb{S}^3 \cong T^3$.

Exact solutions to the field equations for the cases where $\mathbb{S}^2 \cong S^2$, $\mathbb{S}^3 \cong S^3$ and $\mathbb{S}^3 \cong S^1 \times S^2$ are known. The torus topologies, which are classically allowed topologies, have not been found as of yet.

As is the case for spacetimes with no cosmological constant, SIHs in ADS spacetime have vanishing surface gravity and so are always extremal. However, in contrast to the asymptotically flat case, ADS SIHs in four dimensions can be either rotating or non-rotating with strong constraints linking the rotation to the electromagnetic and Killing spinor fields. To give a taste of their application, let us apply these constraints to the case when $\tilde{\omega} = 0$. Then, the Maxwell equations along with the extremal IH conditions tell us that $E_\perp$ and $B_\perp$ are both constant in time ($\pounds_\ell E_\perp = \pounds_\ell B_\perp = 0$). In addition, the Maxwell equations

$$\nabla_a F^{ab} = 0 \tag{3.42}$$

$$\nabla_{[a} F_{bc]} = 0 \tag{3.43}$$

can be projected to $\mathbb{S}_v$, respectively, such that

$$\tilde{q}_a{}^b\tilde{q}^{cd}\nabla_c F_{db} - \tilde{q}_a{}^b n^c \ell^d \nabla_c F_{db} + \tilde{q}_a{}^b n^c \ell^d \nabla_d F_{bc} = 0 \tag{3.44}$$

$$\tilde{q}_a{}^b n^c \ell^d \nabla_b F_{cd} + \tilde{q}_a{}^b n^c \ell^d \nabla_c F_{db} + \tilde{q}_a{}^b n^c \ell^d \nabla_d F_{bc} = 0\,. \tag{3.45}$$

Adding these two equations together, and using the decomposition (3.22) for F_{ab} along with the assumption $\tilde{\omega} = 0$ then gives

$$\tilde{q}_a{}^b n^c \ell^d \nabla_b F_{cd} + \tilde{q}_a{}^b \tilde{q}^{cd} \nabla_c F_{db} + 2\tilde{q}_a{}^b n^c \ell^d \nabla_d F_{bc} = d_a B_\perp + \tilde{\epsilon}_a{}^b d_b E_\perp = 0\,. \tag{3.46}$$

Hence $E_\perp$ and $B_\perp$ are also constant on each $\mathbb{S}_v$. Next the supersymmetry constraint (3.26) says that

$$\beta B_\perp = \frac{1}{L^2} \text{ and } \beta E_\perp = 0\,. \tag{3.47}$$

Thus, $E_\perp = 0$ while $B_\perp \neq 0$ – that is, these SIHs necessarily have magnetic, but not electric, charges. Further, applying the extremality condition (2.55):

$$\frac{1}{2}\mathcal{R} = d_a\tilde{\omega}^a + \tilde{\omega}_a\tilde{\omega}^a + T_{ab}\ell^a n^b - \frac{3}{L^2} \tag{3.48}$$

$$= B_\perp^2 - \frac{3}{L^2}\,. \tag{3.49}$$

[This equation has been solved by Kunduri and Lucietti (2008) for vacuum gravity in the context of near-horizon geometries.] It is clear that the two-dimensional Ricci curvature $\mathcal{R}$ of the $\mathbb{S}_v$ is constant in this case – unfortunately the sign of that curvature does not seem to be determined by the equations. Consulting a listing of exact supersymmetric black hole solutions (Caldarelli and Klemm 1999) we see that such solutions are known: specifically there is a supersymmetric asymptotically ADS black hole in four dimensions which can be non-rotating if the horizon cross sections have genus $g > 1$. As prescribed by our formalism, these solutions have magnetic but not electric charge.

The quasilocal picture that we have presented is in excellent agreement with the results that are known for stationary spacetimes (Gibbons *et al* 1994; Gauntlett *et al* 1999; Gutowski and Reall 2004). In that context a supersymmetric black hole in a spacetime with $\Lambda = 0$ (in four and in five dimensions) contains an extremal and non-rotating horizon. Extremality is a consequence of the BPS bounds being saturated, which implies that there exists a Killing spinor. Non-rotation is then a consequence of the fact that a vector cannot be constructed in the neighbourhood of a supersymmetric Killing horizon that is spacelike, as can be seen from the algebraic conditions (3.9) and (3.38). Therefore the spacetime of such a black hole cannot contain an ergoregion, which means that the horizon must be non-rotating. On the other hand, a supersymmetric black hole with $\Lambda < 0$ contains a rotating and extremal horizon (with non-trivial electromagnetic field). The rotation is required, otherwise the extremal limit would result in a naked singularity.

In this chapter we focused on null Killing spinors that are defined at the horizon itself. However, the results obtained here for black holes will not be affected if we assume that the spinors are defined globally. We note that there are many other solutions with such spinors that are defined for the entire spacetime, which do not describe black holes. These include pp-waves (plane-fronted gravitational waves with parallel rays) – spacetimes with vanishing expansion, shear and twist, and are a subset of a wider class of solutions that share this property, known as Kundt spacetimes. For details, see e.g. (Stephani *et al* 2003). If a spinor is defined globally then ℓ, which is hypersurface orthogonal everywhere, is also defined globally. Therefore such spacetimes can actually be foliated by SIHs (Pawlowski *et al* 2004).

3.5 Relation to asymptotically flat solutions

In five dimensions, there are two supersymmetric solutions with the property that the bulk electromagnetic field contains angular momentum while the horizon is non-rotating. These are the BMPV black hole (Breckenridge *et al* 1997) and the EEMR black ring (Elvang *et al* 2004).

The BMPV black hole was first discovered in (Breckenridge *et al* 1997) as a solution to the equations that result from the truncation of $D = 6$ supergravity. Later it was shown that this spacetime is a solution to the $D = 5$ EM-CS theory (Gauntlett *et al* 1999). As was shown in (Reall 2003), the BMPV black hole is the unique asymptotically flat extremal solution to EM-CS theory with horizon topology S^3. The line element of this spacetime can be written in the following form:

$$ds^2 = -\left(1+\frac{\mu}{r^2}\right)^{-2}\left(dt+\frac{j\sigma_3}{2r^2}\right)^2 + \left(1+\frac{\mu}{r^2}\right)\left(dr^2 + r^2 d\Omega_3^2\right), \tag{3.50}$$

with

$$d\Omega_3^2 = \frac{1}{4}(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi) \tag{3.51}$$

$$\sigma_3 = d\psi + \cos\theta d\phi\,; \tag{3.52}$$

the angular coordinates have the ranges

$$0 \le \theta < \pi\,, \quad 0 \le \phi < 2\pi\,, \quad 0 \le \psi < 4\pi\,. \tag{3.53}$$

The $(U(1))$ vector potential that solves the EM-CS equations of motion with the metric (3.50) is

$$\boldsymbol{A} = \frac{\sqrt{3}}{2}\left[\left(1+\frac{\mu}{r^2}\right)^{-1}(dt + \frac{j\sigma_3}{2r^2}) - dt\right]. \tag{3.54}$$

The parameters μ and j are related to the total mass $\mathfrak{M}$, charge $\mathfrak{Q}$ and angular momentum $\mathfrak{J}$ at infinity via

$$\mathfrak{M} = \frac{3\pi\mu}{4G_5}, \quad \mathfrak{Q} = \frac{\sqrt{3}\pi\mu}{2G_5}, \quad \mathfrak{J} = -\frac{\pi j}{2G_5}. \tag{3.55}$$

Here, the mass is related to the total charge via

$$\mathfrak{M} = \frac{\sqrt{3}}{2}\mathfrak{Q}, \tag{3.56}$$

which implies that the BPS bound is saturated. This is a typical characteristic of supersymmetric solitons in $D = 5$ supergravity (Gibbons *et al* 1994).

The BMPV black hole has one independent rotation parameter. With $\mathcal{J}_{\mathrm{Grav}} = 0$ this corresponds to a SIH with one angular momentum given by

$$\mathcal{J}_{\mathrm{EM}} = \frac{1}{8\pi G_5}\oint_{S^3}(\phi\lrcorner\boldsymbol{A})\boldsymbol{\Phi} = \frac{j\pi}{4G_5}\left(1 - \frac{j^2}{\mu^3}\right). \tag{3.57}$$

The spacetime of the BMPV black hole is described by a non-rotating spherical horizon with angular momentum stored in the electromagnetic fields. In addition, the distribution of angular momentum of the spacetime is such that there is a non-zero fraction on the horizon as well as a negative fraction behind the horizon (Gauntlett *et al* 1999). The net result is that the horizon geometry is that of a squashed three-sphere. Within our framework, these interesting physical properties correspond to IHs with arbitrary distortions and rotations in their neighbourhoods. Such IHs have been studied using multipole moments (Ashtekar *et al* 2004). When the angular momentum of the BMPV black hole is zero the solution reduces to the extremal RN solution in isotropic coordinates.

The generalization of the BMPV black hole to the case where the solution has two independent angular momentum parameters is the EEMR black ring (Elvang *et al* 2004).

The solution describes an extremal black hole with horizon topology $S^1 \times S^2$. The solution was discovered by Elvang *et al* (2004); the properties and stringy origin were studied in detail in (Elvang *et al* 2005). The line element of this spacetime can be written in the following form:

$$dS^2 = -f^2(dt + \omega_\psi d\psi + \omega_\phi d\phi)^2 + f^{-1} dS_4^2 \,, \tag{3.58}$$

with

$$\begin{aligned}
dS_4^2 &= \frac{R^2}{(x-y)^2}\left[\frac{dx^2}{1-x^2} + \frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + (1-x^2)d\phi^2\right] && (3.59)\\
f^{-1} &= 1 + \frac{Q-q^2}{2R^2}(x-y) - \frac{q^2}{4R^2}(x^2-y^2) && (3.60)\\
\omega_\psi &= -\frac{q}{8R^2}(1-x^2)[3Q - q^2(3+x+y)] && (3.61)\\
\omega_\phi &= \frac{3q}{2}(1+y) + \frac{q}{8R^2}(1-y^2)[3Q - q^2(3+x+y)]\,; && (3.62)
\end{aligned}$$

the spatial coordinates have the ranges

$$-1 \le x \le 1\,, \quad -\infty < y \le -1\,, \quad 0 \le \psi < 2\pi\,, \quad 0 \le \phi < 2\pi\,. \tag{3.63}$$

The parameters q and Q are positive constants that are proportional to the magnetic dipole moment and total charge, and $R > 0$ is the radius of a circle that is parametrized by ψ at $y = -\infty$. Also note that the set of coordinates (x, ϕ) parametrize a two-sphere. Therefore the horizon is "blown up" to a ring (topologically $S^1 \times S^2$) in the $5D$ geometry. The $(U(1))$ vector potential that solves the EM-CS equations of motion with the metric (3.58) is

$$\boldsymbol{A} = \frac{\sqrt{3}}{2}\left[f(dt + \omega_\psi d\psi + \omega_\phi d\phi) - \frac{q}{2}[(1+x)d\phi + (1+y)d\psi]\right]\,. \tag{3.64}$$

The parameters q and Q are related to the total mass $\mathfrak{M}$ and total charge $\mathfrak{Q}$ at infinity via

$$\mathfrak{M} = \frac{3\pi Q}{4G_5}\,, \quad \mathfrak{Q} = \frac{\sqrt{3}\pi Q}{2G_5}\,, \tag{3.65}$$

and to the angular momenta $\mathfrak{J}_\psi$ and $\mathfrak{J}_\phi$ at infinity via

$$\mathfrak{J}_\psi = \frac{\pi q}{8G_5}(6R^2 + 3Q - q^2)\,, \quad \mathfrak{J}_\phi = \frac{\pi q}{8G_5}(3Q - q^2)\,. \tag{3.66}$$

Note that $\mathfrak{M} = (\sqrt{3}/2)\mathfrak{Q}$ as for the BMPV black hole. Also note that when $R = 0$ the angular momenta $\mathfrak{J}_\psi$ and $\mathfrak{J}_\phi$ are equal. This would suggest that the BMPV black hole with equal angular momenta is essentially the EEMR black ring in the limit when $R = 0$. However, in this limit there is then an apparent singularity in the four-metric and also in the connection one-form.

The EEMR black ring has two independent rotation parameters. This corresponds to a SIH with two angular momenta given by

$$\mathcal{J}_j = \frac{1}{8\pi G_5}\oint_{S^1\times S^2}(\phi_j \lrcorner \boldsymbol{A})\boldsymbol{\Phi} \quad (j \in \{1,2\})\,. \tag{3.67}$$

In addition, a black ring can also have dipole charges which are naturally defined on the horizon (Astefanesei and Radu 2006; Copsey and Horowitz 2006; Emparan and Reall 2006). For an IH with ring topology a definition for the dipole charge $\mathcal{P}$ can be realized by integrating the electromagnetic field strength plus the CS contribution over S^2:

$$\mathcal{P} = \frac{1}{2\pi}\int_{S^2}\star\boldsymbol{\Phi}\,. \tag{3.68}$$

Charges of this type appear in the first law for a black ring (Astefanesei and Radu 2006; Copsey and Horowitz 2006). However, this is not the case with the first law (2.47) that we derived. This is probably due to the fact that the dipole charges $\mathcal{P}$ are associated with the presence of *magnetic charge*, which we have not incorporated into our current phase space.

Appendix: an alternate formalism in four dimensions

In four dimensions, there is an alternative formalism that can be used to derive the supersymmetry conditions for IHs as was originally done for flat spacetime (Liko and Booth 2008). This is a NP-type spinor formalism, in which the necessary and sufficient condition for supersymmetry with vanishing fermion fields is that there exists a Killing spinor $\psi_{AA'} = (\alpha_A, \beta_{A'})$ ($A, B, \ldots \in \{1, 2\}$ and $A', B', \ldots \in \{1, 2\}$) such that (Tod 1983; Tod 1995)

$$\begin{aligned} \nabla_{AA'}\alpha_B + \sqrt{2}\phi_{AB}\beta_{A'} &= 0 &(3.69)\\ \nabla_{AA'}\beta_{B'} - \sqrt{2}\bar{\phi}_{A'B'}\alpha_A &= 0\,. &(3.70)\end{aligned}$$

Here, ϕ_{AB} is the Maxwell spinor and $\bar{\phi}_{A'B'}$ is its complex conjugate. These are related to the field strength via

$$\boldsymbol{F} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}\,; \tag{3.71}$$

the spinor symplectic structure is defined such that $\epsilon^{12} = -\epsilon^{21} = 1$. Using the spinors α and β we can define the following set of null vectors:

$$\ell_{AA'} = \alpha_A\bar{\alpha}_{A'}\,, \quad n_{AA'} = \bar{\beta}_A\beta_{A'}\,, \quad \vartheta_{AA'} = \alpha_A\beta_{A'}\,, \quad \bar{\vartheta}_{AA'} = \bar{\beta}_A\bar{\alpha}_{A'}\,. \tag{3.72}$$

It can be shown that the vector

$$K_{AA'} \equiv \ell_{AA'} + n_{AA'} \tag{3.73}$$

is a Killing vector; the norm of this vector is given by

$$\|K\| = 2V\bar{V}\,, \tag{3.74}$$

where we defined the scalar $V = \alpha_A\bar{\beta}^A$. It follows that $K_{AA'}$ can be either timelike (referred to as nondegenerate) when $V \neq 0$ or null (referred to as degenerate) when $V = 0$.

For IHs, the case of interest is the one for which the Killing spinor is null. This is a particularly special case because $V = \alpha_A \bar{\beta}^A = 0$ implies that

$$\bar{\beta}^A = \mathcal{K}\alpha^A \tag{3.75}$$

for some function $\mathcal{K}$. Putting this into the conditions (3.69) and (3.70) allows one to find an expression for the covariant derivative in terms of the spinors (Tod 1983; Tod 1995):

$$\nabla_{AA'}\alpha_B = -\sqrt{2}\bar{\mathcal{K}}\phi\alpha_A\alpha_B\bar{\alpha}_{A'} \,. \tag{3.76}$$

Here, ϕ is a function defined by $\phi_{AB} = \phi\alpha_A\alpha_B$. Let us use this form of the covariant derivative to find the covariant derivative of the null normal ℓ of an IH. We find that

$$\nabla_a \ell_b = -\sqrt{2}(\bar{\mathcal{K}}\phi + \mathcal{K}\bar{\phi})\ell_a\ell_b \,. \tag{3.77}$$

This immediately implies that

$$\nabla_{\underleftarrow{a}}\ell_b \approx 0 \,, \tag{3.78}$$

and with (2.13) it follows that $\omega = 0$.

Chapter 4

Isolated Horizons in EGB Theory

"Higher-derivative theories are frequently avoided because of undesirable properties, yet they occur naturally as corrections to general relativity and cosmic strings." ~ J Z Simon

4.1 First-order action for EGB theory

As in Chapter 2, we work in the first-order connection-dynamics formulation. Here, the configuration space consists of the pair $(e^I, A^I{}_J)$ (with electromagnetic fields zero). In this configuration space, the action (1.10) for EGB theory on the manifold $(\mathcal{M}, g_{ab})$ (assumed for the moment to have no boundaries) is given by

$$S = \frac{1}{2\kappa_D} \int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} - 2\Lambda\epsilon + \alpha\Sigma_{IJKL} \wedge \Omega^{IJ} \wedge \Omega^{KL} . \tag{4.1}$$

The equation of motion for the connection is

$$\mathscr{D}(\Sigma_{IJ} + 2\alpha\Sigma_{IJKL} \wedge \Omega^{KL}) = 0 . \tag{4.2}$$

This equation says that, in general, there exists a non-vanishing torsion $T^I = \mathscr{D}e^I$. To see what constraints are imposed on T, we can use the Bianchi identity $\mathscr{D}\Omega^{IJ} = 0$ together with the identity

$$\mathscr{D}\Sigma_{I_1\ldots I_m} = \mathscr{D}e^M \wedge \Sigma_{I_1\ldots I_m M} \,. \tag{4.3}$$

Substituting these into equation (2.11) gives

$$T^I \wedge (\Sigma_{IJK} + 2\alpha\Sigma_{IJKLM} \wedge \Omega^{LM}) = 0 \,. \tag{4.4}$$

In analogy with Einstein gravity, we assume directly that the torsion in (4.4) vanishes. (The torsion in Einstein gravity is zero, but this is not an assumption. The condition follows directly from the equation of motion for the connection.) To get the equation of motion for the co-frame we note that the variation of Σ is given by

$$\delta\Sigma_{I_1\ldots I_m} = \delta e^M \wedge \Sigma_{I_1\ldots I_m M} \,. \tag{4.5}$$

This leads to

$$\Sigma_{IJK} \wedge \Omega^{JK} - 2\Lambda\Sigma_I + \alpha\Sigma_{IJKLM} \wedge \Omega^{JK} \wedge \Omega^{LM} = 0 \,. \tag{4.6}$$

The equations (4.2) and (4.6) for the connection and co-frame are equivalent to the field equations (1.11) in the metric formulation.

4.2 Modified boundary conditions and the zeroth law

Let us now turn to the case for which the manifold $(\mathcal{M}, g_{ab})$ has boundaries; the region of spacetime that we consider for EGB theory is essentially the same as that which we considered in Chapter 2 for EM-CS theory. Namely, $(\mathcal{M}, g_{ab})$ is a D-dimensional Lorentzian manifold with topology $R \times M$, contains a $(D-1)$-dimensional null surface Δ as inner boundary (representing the horizon), and is bounded by $(D-1)$-dimensional manifolds $M^{\pm}$

that extend from Δ to infinity. As in Chapter 3, we consider the purely quasilocal case and neglect any subleties that are associated with the outer boundary.

Let us now state the modification that is required for EGB theory. First, we note that for IHs in general relativity the dominant energy condition can be imposed interchangably onto the Ricci tensor or the stress-energy tensor. This is because the Einstein field equations $G_{ab} = 2T_{ab}$ imply that $R_{ab}v^a v^b = 2T_{ab}v^a v^b$ for any null vector v^a. For IHs in EGB theory this is no longer true because $G_{ab} \neq \kappa_D T_{ab}$. [This is also the reason why the topology constraint (4.4) is not applicable to IHs in EGB theory.] It turns out that condition (c) of Definition I for IHs in EGB theory has to be imposed onto the Ricci tensor in order for the shear tensor to vanish when the Raychaudhuri equation is employed. Thus we have the following modified definition for a WIH in EGB theory:

Definition III. *A WIH Δ in EGB theory is a null surface and has a degenerate metric q_{ab} with signature $0+\ldots+$ (with $D-2$ nondegenerate spatial directions) along with an equivalence class of null normals $[\ell]$ (defined by $\ell \sim \ell' \Leftrightarrow \ell' = z\ell$ for some constant z) such that the following conditions hold: (a) the expansion $\theta_{(\ell)}$ of ℓ_a vanishes on Δ; (b) the field equations hold on Δ; (c) the Ricci tensor is such that the vector $-R^a{}_b\ell^b$ is a future-directed and causal vector; (d) $\pounds_\ell \omega_a = 0$ and $\pounds_\ell \underleftarrow{A} = 0$ for all $\ell \in [\ell]$.*

The condition (c) on the Ricci tensor is the only modification that needs to be made for WIHs in EGB theory. In particular the zeroth law now follows just as it does for IHs in EM-CS theory. Remarkably, the zeroth law is essentially independent of the functional content of the action, and follows from the boundary conditions alone. This is the same conclusion that was obtained for globally stationary spacetimes (Wald 1993; Iyer and Wald 1994; Jacobson *et al* 1994).

4.3 Variation of the boundary term

We have seen that the definition for a nonexpanding horizon needs to be modified for EGB gravity by imposing the analogue of the dominant energy condition directly on the Ricci tensor. In the action principle, the main modification to the formalism is the appearance of an additional surface term. Let us therefore reconsider the action (4.1) but for a region of the manifold $\mathcal{M}$ that is bounded by null surface Δ and spacelike surfaces $M^{\pm}$ which extend to the (arbitrary) boundary $\mathscr{B}$.

Denoting the pair (e, A) collectively as a generic field variable Ψ, the first variation gives

$$\delta S = \frac{1}{2\kappa_D}\int_{\mathcal{M}} E[\Psi]\delta\Psi + \frac{(-1)^D}{2\kappa_D}\int_{\partial\mathcal{M}} J[\Psi, \delta\Psi] \,. \tag{4.7}$$

Here $E[\Psi] = 0$ symbolically denotes the equations of motion and

$$J[\Psi, \delta\Psi] = \widetilde{\Sigma}_{IJ} \wedge \delta A^{IJ} \tag{4.8}$$

is the surface term, with $(D-2)$-form

$$\widetilde{\Sigma}_{IJ} = \Sigma_{IJ} + 2\alpha\Sigma_{IJKL} \wedge \Omega^{KL} \,. \tag{4.9}$$

If the integral of J on the boundary $\partial\mathcal{M}$ vanishes then the action principle is said to be differentiable. We must show that this is the case. Because the fields are held fixed at $M^{\pm}$ and at $\mathscr{B}$, J vanishes there. So we only need to show that J vanishes at the inner boundary Δ. To show that this is true we need to find an expression for J in terms of A and $\widetilde{\Sigma}$ pulled back to Δ. As for EM-CS theory we do this by fixing an internal basis consisting of the (null) pair (ℓ, n) and $D-2$ spacelike vectors $\vartheta_{(i)}$ given by (2.22) together with the conditions (2.23).

The pull-back of A is the same as for EM-CS theory, and is given by (2.31). For EGB theory we also need the pull-back of the curvature Ω to Δ, which can be obtained either by direct calculation from (2.31) or by construction from the definition of the Riemann tensor. We will take the second approach here. We will use the definition (2.52) of the metric on $\mathbb{S}^{D-2}$ and the definition (2.53) of the connection projected onto $\mathbb{S}^{D-2}$.

In thinking about these quantities it is useful to keep in mind the case where Δ is foliated into spacelike $(D-2)$-surfaces $\mathbb{S}_v$ which are labelled by a parameter v and n is chosen to be $-dv$. Then ℓ^a evolves the foliation surfaces while ℓ^a and n^a together span the normal bundle $T^{\perp}(\mathbb{S}_v)$ on which $\tilde{\omega}_a$ is the connection. Furthermore, the $\vartheta^a_{(i)}$ span the tangent bundle $T(\mathbb{S}_v)$ and $\tilde{q}_{ab}$ is the metric tensor for the $\mathbb{S}_v$. Then, it is clear that Δ is non-rotating when $\tilde{\omega} = 0$ provided that the rotational vector field ϕ is tangential to Δ.

We now turn to the Riemann tensor with the first two indices pulled back to Δ. By definition,

$$R_{\underleftarrow{ab}}{}^{c}{}_{d}\ell^{d} = -q_a{}^{e} q_b{}^{f} (\nabla_e \nabla_f - \nabla_f \nabla_e)\ell^{c} \,, \tag{4.10}$$

and with the horizon identity $\nabla_{\underleftarrow{a}} \ell^b = \omega_a \ell^b$ along with the decomposition (2.53), a few lines of algebra gives

$$R_{\underleftarrow{ab}}{}^{c}{}_{d}\ell^{d} = \left(-2n_{[a}d_{b]}\kappa_{(\ell)} + 2\tilde{q}_{[a}{}^{e}\tilde{q}_{b]}{}^{f} d_e \tilde{\omega}_f - 2n_{[a}\tilde{q}_{b]}{}^{f} \mathcal{L}_{\ell}\tilde{\omega}_f\right)\ell^{c} \,, \tag{4.11}$$

where d_a is the covariant derivative that is compatible with the metric $\tilde{q}_{ab}$. For a weakly isolated horizon the zeroth law ensures that $d_a \kappa_{(\ell)} = 0$ and if the horizon is non-rotating then $\tilde{\omega} = 0$ also, whence

$$R_{\underleftarrow{ab}}{}^{c}{}_{d}\ell^{d} = 0 \,. \tag{4.12}$$

Finally, using this result and (2.52) it is straightforward to see that

$$R_{\underleftarrow{ab}cd}\vartheta^c_{(i)}\vartheta^d_{(j)} = \tilde{q}_a{}^e\tilde{q}_b{}^f R_{efcd}\vartheta^c_{(i)}\vartheta^d_{(j)} \,. \tag{4.13}$$

From here one can use the fact that

$$d_a d_b \vartheta^{(i)}_c = \tilde{q}_a{}^d\tilde{q}_b{}^e\tilde{q}_c{}^f\nabla_d\left(\tilde{q}_e{}^g\tilde{q}_f{}^h\nabla_g\vartheta^{(i)}_h\right) , \tag{4.14}$$

and the identity for the Riemann tensor $\mathcal{R}_{abcd}$ associated with $\tilde{q}_{ab}$

$$\mathcal{R}_{abcd}\vartheta^{(i)d} = \left(d_a d_b - d_b d_a\right)\vartheta^{(i)}_c , \tag{4.15}$$

along with (2.52) to show the Gauss relation

$$\tilde{q}_a{}^e\tilde{q}_b{}^f\tilde{q}_c{}^g\tilde{q}_d{}^h R_{efgh} = \mathcal{R}_{abcd} + \left(k^{(\ell)}_{ac}k^{(n)}_{bd} + k^{(n)}_{ac}k^{(\ell)}_{bd}) - (k^{(\ell)}_{bc}k^{(n)}_{ad} + k^{(n)}_{bc}k^{(\ell)}_{ad}\right) . \tag{4.16}$$

Here $k^{(\ell)}_{ab} = \tilde{q}_a{}^c\tilde{q}_b{}^d\nabla_c\ell_d$ and $k^{(n)}_{ab} = \tilde{q}_a{}^c\tilde{q}_b{}^d\nabla_c n_d$ are the extrinsic curvatures associated with ℓ_a and n_a. However, $k^{(\ell)}_{ab} = (1/2)\vartheta_{(\ell)}\tilde{q}_{ab} + \sigma_{ab}$, and on a non-expanding horizon both the expansion and shear vanish. Thus for the cases in which we are interested

$$\tilde{q}_a{}^e\tilde{q}_b{}^f\tilde{q}_c{}^g\tilde{q}_d{}^h R_{efgh} = \mathcal{R}_{abcd} \,. \tag{4.17}$$

Then upon expanding the frame indices of $\Omega_{\underleftarrow{ab}}{}^{IJ}$ in terms of the ℓ^I, n^I and $\vartheta^I_{(i)}$, and applying (4.12) and (4.17), it follows that on any non-rotating WIH the pull-back of the associated curvature is

$$\Omega_{\underleftarrow{ab}}{}^{IJ} \approx \vartheta^{(k)}_a\vartheta^{(l)}_b\mathcal{R}_{kl}{}^{ij}\vartheta_{(i)}{}^I\vartheta_{(j)}{}^J + 2\ell^{[I}\vartheta_{(i)}{}^{J]}\Omega_{\underleftarrow{ab}}{}^{KL}\vartheta^{(i)}{}_K n_L , \tag{4.18}$$

where $\mathcal{R}_{kl}{}^{ij}$ is the Riemann tensor associated with the $(D-2)$ metric $\tilde{q}_{ab} = g_{ab} + \ell_a n_b + n_a\ell_b$. That is, given a foliation of Δ into spacelike $(D-2)$-surfaces, the spacelike $\vartheta^a_{(i)}$ give an orthonormal basis on those surfaces and ${\mathcal{R}_{kl}}^{ij}$ is the corresponding curvature tensor; for a non-expanding horizon, these quantities are independent of both the slice of the foliation and the particular foliation itself.

To find the pull-back to Δ of $\widetilde{\Sigma}$, we use the decomposition (2.25), whence the $(D-2)$-form given by (2.26) and in $D \geq 5$ dimensions, the $(D-4)$-form

$$\begin{aligned} \underleftarrow{\Sigma}_{IJKL} \approx & -\frac{1}{(D-5)!}\epsilon_{IJKLA_1 \dots A_{D-4}} \ell^{A_1} \vartheta_{(i_1)}^{A_2} \dots \vartheta_{(i_{D-5})}^{A_{D-4}} \left[n \wedge \vartheta^{(i_1)} \wedge \dots \wedge \vartheta^{(i_{D-5})} \right] \\ & + \frac{1}{(D-4)!}\epsilon_{IJKLA_1 \dots A_{D-4}} \vartheta_{(i_1)}^{A_1} \dots \vartheta_{(i_{D-4})}^{A_{D-4}} \left[\vartheta^{(i_1)} \wedge \dots \wedge \vartheta^{(i_{D-4})} \right] . \end{aligned} \tag{4.19}$$

In four dimensions $\underleftarrow{\Sigma}_{IJKL} = \epsilon_{IJKL}$.

These expressions are somewhat formidable but on combining them to find $\widetilde{\Sigma}_{IJ} \wedge \delta A^{IJ}$ there is significant simplification. The key is to note that each term includes a total contraction of $\epsilon_{I_1 \dots I_D}$. This contraction must include one copy of each of ℓ^I, n^I, and the $\vartheta_{(i)}^I$ – else that term will be zero. Similarly the resulting $(D-1)$ form must be proportional to $n \wedge \vartheta^{(2)} \wedge \dots \wedge \vartheta^{(D-1)}$. Then (4.8) becomes

$$\begin{aligned} J[\Psi, \delta\Psi] \approx \tilde{\epsilon} \wedge \delta\omega \quad + \quad & \frac{2\alpha}{(D-4)!} \left[\epsilon_{IJKLA_1 \dots A_{D-4}} \ell^I n^J \vartheta_{(k)}^K \vartheta_{(l)}^L \vartheta_{(i_1)}^{A_1} \dots \vartheta_{i_{D-4}}^{A_{D-4}} \right] \\ & \times {\mathcal{R}_{mn}}^{kl} \vartheta^{(i_1)} \wedge \dots \wedge \vartheta^{(i_{D-4})} \wedge \vartheta^{(m)} \vartheta^{(n)} \wedge \delta\omega . \end{aligned} \tag{4.20}$$

The first and second terms respectively come from the $\underleftarrow{\Sigma}_{IJ}$ and $\underleftarrow{\Sigma}_{IJKL}$ parts of $\underleftarrow{\widetilde{\Sigma}}_{IJ}$, $\tilde{\epsilon}$ is the area element defined by (2.34), and we keep in mind that the horizon is non-rotating so that $\omega_a = -\kappa_{(\ell)} n_a$. The second term therefore also simplifies. Given that there are only $(D-4)$ elements in the spacelike basis it is reasonably easy to see that this term sums over cases where (m, n) and (i, j) are the same set of indices. That is (up to a numerical factor) the second term amounts to contracting m with i and n with j so that the full surface term reduces to

$$J[\Psi, \delta\Psi] \approx \tilde{\epsilon}(1 + 2\alpha\mathcal{R}) \wedge \delta\omega \ . \tag{4.21}$$

It follows that $J \approx 0$ because $\delta\omega = 0$ on the initial and final cross sections of Δ (i.e. on $M^- \cap \Delta$ and on $M^+ \cap \Delta$), and because $\delta\omega$ is Lie dragged on Δ. Therefore the surface term $J|_{\partial\mathcal{M}} = 0$ for EGB gravity, and we conclude that the equations of motion $E[\Psi] = 0$ follow from the action principle $\delta S = 0$.

4.4 Covariant phase space and the first law

In order to derive the first law we need to find the symplectic structure on the covariant phase space $\mathbf{\Gamma}$ consisting of solutions (e, A) to the EGB field equations on $\mathcal{M}$. We find that the second variation of the EGB surface term (4.8) gives

$$J[\Psi, \delta_1\Psi, \delta_2\Psi] = (-1)^D \left[\delta_1\tilde{\Sigma}_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\tilde{\Sigma}_{IJ} \wedge \delta_1 A^{IJ}\right] \ ; \tag{4.22}$$

integrating over M defines the *bulk* symplectic structure

$$\mathbf{\Omega}_{\mathrm{B}}(\delta_1, \delta_2) = \frac{(-1)^D}{2\kappa_D} \int_M \left[\delta_1\tilde{\Sigma}_{IJ} \wedge \delta_2 A^{IJ} - \delta_2\tilde{\Sigma}_{IJ} \wedge \delta_1 A^{IJ}\right] \ . \tag{4.23}$$

We also need to find the pull-back of J to Δ and add the integral of this term to $\mathbf{\Omega}_{\mathrm{B}}$ to determine the full symplectic structure. From (4.21) we have

$$\mathbf{\Omega}_{\mathrm{S}} \approx \frac{(-1)^D}{\kappa_D} \int_\Delta \left[\delta_1 \left[\tilde{\epsilon}(1 + 2\alpha\mathcal{R})\right] \wedge \delta_2\omega - \delta_2 \left[\tilde{\epsilon}(1 + 2\alpha\mathcal{R})\right] \wedge \delta_1\omega\right] \ . \tag{4.24}$$

which is a total derivative. Now, with the definition (2.37) for ψ, the surface symplectic structure $\Omega_{\mathbb{S}}$ is a total derivative, and hence upon using the Stokes theorem, becomes an integral over $\mathbb{S}^{D-2}$. The full symplectic structure for EGB gravity is therefore

$$\begin{aligned}\mathbf{\Omega}(\delta_1,\delta_2) &= \frac{1}{2\kappa_D}\int_M \left[\delta_1\widetilde{\Sigma}_{IJ}\wedge\delta_2 A^{IJ} - \delta_2\widetilde{\Sigma}_{IJ}\wedge\delta_1 A^{IJ}\right] \\ &+\frac{1}{\kappa_D}\oint_{\mathbb{S}^{D-2}} \left[\delta_1\left[\tilde{\epsilon}(1+2\alpha\mathcal{R})\right]\wedge\delta_2\psi - \delta_2\left[\tilde{\epsilon}(1+2\alpha\mathcal{R})\right]\wedge\delta_1\psi\right]\,, \end{aligned} \tag{4.25}$$

where we have absorbed the overall (irrelevant) factor of $(-1)^D$.

We can now proceed to derive the first law. As before, this follows upon evaluating the symplectic structure at (δ,δ_ξ), giving a surface term at infinity (which is identified with the ADM energy) and a surface term at the horizon. We find that the surface term at the horizon is given by

$$\mathbf{\Omega}|_{\Delta}(\delta,\delta_t) = \frac{\kappa_{(z\ell)}}{\kappa_D}\delta\oint_{\mathbb{S}^{D-2}}\tilde{\epsilon}(1+2\alpha\mathcal{R})\,. \tag{4.26}$$

where we used $\kappa_{(z\ell)} = \pounds_{z\ell}\psi = z\ell\lrcorner\omega$. Finally, assuming that this is a total variation, i.e. that there exists a function $\mathcal{E}$ such that $\mathbf{\Omega}|_{\Delta}(\delta,\delta_\xi) = \delta\mathcal{E}$, we find that

$$\delta\mathcal{E} = \frac{\kappa_{(z\ell)}}{\kappa_D}\delta\oint_{\mathbb{S}^{D-2}}\tilde{\epsilon}(1+2\alpha\mathcal{R})\,. \tag{4.27}$$

This is the first law for a WIH in EGB theory. Comparing this to the standard first law identifies the entropy as

$$\mathcal{S} = \frac{1}{4G_D}\oint_{\mathbb{S}^{D-2}}\tilde{\epsilon}(1+2\alpha\mathcal{R})\,. \tag{4.28}$$

Therefore WIHs in D-dimensional EGB theory satisfy the first law (and the zeroth law) of black-hole mechanics.

4.5 Comparison with Euclidean and Noether charge methods

The quasilocal expression for the entropy is in exact agreement with the Noether charge expression that was derived by Clunan *et al* (2004). As in that approach, no assumptions about the cross sections $\mathbb{S}^{D-2}$ of the horizon need to be made. An important difference, however, is that we did not assume the existence of a globally-defined Killing vector. Instead we had to specify the existence of a time translation vector field which mimics the properties of a Killing vector but *is not defined for the entire spacetime.*

In order to compare the entropy to the Euclidean method, we need to reference a black hole solution. The EGB equations admit the following class of (static) black hole solutions (Cai 2002; Cho and Neupane 2002):

$$\begin{aligned} dS^2 &= -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2_{(k)D-2} \\ h(r) &= k + \frac{r^2}{2\tilde{\alpha}}\left(1 - \sqrt{1 - \frac{8\tilde{\alpha}\Lambda}{(D-1)(D-2)} + \frac{8\kappa_D\tilde{\alpha}M}{(D-2)\mathscr{V}_{(k)D-2}r^{D-1}}}\right) . \end{aligned} \tag{4.29}$$

Here, $\mathscr{V}_{(k)N-1} = \pi^{N/2}/\Gamma(N/2+1)$ is the volume of an $(N-1)$-dimensional space $S^{N-1} = S^{N-1}_{(k)}$ of constant curvature with metric $d\Omega^2_{(k)N-1}$; k is the curvature index with $k = 1$ corresponding to positive constant curvature, $k = -1$ corresponding to negative constant curvature, and $k = 0$ corresponding to zero curvature. M is the mass of the black hole, and $\tilde{\alpha}$ is related to the GB parameter via

$$\tilde{\alpha} = (D-3)(D-4)\alpha . \tag{4.30}$$

The singular surfaces with radii r_* are given by the roots to the equation $h(r = r_*) = 0$. We denote the event horizon by r_+. The location of this surface depends on the sign of the cosmological constant: if $\Lambda \leq 0$ then the largest root r_+ is the event horizon, and if $\Lambda > 0$ then the largest root is the cosmological horizon and therefore the second largest root is the event horizon.

The thermodynamics of the black hole is determined in the usual way using path integral methods (Hawking 1979). In particular, the average energy $\langle \mathfrak{E} \rangle$ and entropy $\mathscr{S}$ are given by

$$\langle \mathfrak{E} \rangle = -\frac{\partial}{\partial \beta}(\ln \mathcal{Z}) \quad \text{and} \quad \mathscr{S} = \beta \langle \mathfrak{E} \rangle + \ln \mathcal{Z}\,, \tag{4.31}$$

where $\ln \mathcal{Z}$ is the (zero-loop) partition function and β is the inverse temperature. The partition function is determined via $\ln \mathcal{Z} = -\tilde{I}[g]$ by evaluating the Euclidean action $\tilde{I}[g]$ (in the stationary phase approximation where g are solutions to the equations of motion $\delta \int \tilde{I} = 0$), and the inverse temperature is determined by requiring that the Euclidean manifold does not contain any conical singularities at r_+ where the manifold closes up. For the black hole solution (4.29) one finds that (Cai 2002; Cho and Neupane 2002)

$$\langle \mathfrak{E} \rangle = M \quad \text{and} \quad \mathscr{S} = \frac{\mathscr{A}_{D-2} r_+^{D-2}}{4G_D}\left[1 + \left(\frac{D-2}{D-4}\right)\frac{2\tilde{\alpha}k}{r_+^2}\right]\,. \tag{4.32}$$

Here, $\mathscr{A}_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of a unit $(N-1)$-sphere. This shows that the entropy acquires a correction due to the presence of the GB term. For the solution (4.29), the Ricci scalar is $\mathcal{R} = (D-2)(D-3)k/r_+^2$, and (4.28) reduces to (4.32). Our entropy expression is therefore in agreement with the Euclidean expression as well. In our derivation, however, the entropy (4.28) automatically satisfies the first law (4.27).

An interesting consequence of the GB term is that it is possible for black holes to have negative entropies when $2\alpha\mathcal{R} < 1$. This was first discovered by Cvetič *et al* (2002) and later confirmed by Clunan *et al* (2004). For non-rotating horizons, the first law (4.27) implies that the energy is also negative; this is not surprising, as negative-energy solutions are possible when $\Lambda < 0$ (Horowitz and Myers 1999). We will now proceed to show that the presence of the GB term also has consequences for the area-increase law during the merging of two black holes.

4.6 Decrease of black-hole entropy in EGB theory

For any EGB black hole in D dimensions, the Ricci scalar $\mathcal{R}$ integrates to a constant over $\mathbb{S}^{D-2}$. It is not too surprising then, that the area always increases in any physical process involving just one black hole with an entropy of the form (4.28) (Jacobson *et al* 1995). However, this will not be the case for a system with *dynamical topologies* such as black-hole mergers (Witt 2007). This is a form of topology change, which for a space with a degenerate metric is unavoidable even in classical general relativity (Horowitz 1991). This fact is relevant to the current problem because the entropy expression (4.28) holds for Killing horizons and WIHs, both of which are null surfaces on which the induced metrics are degenerate.

As an example, let us consider the merging of two black holes – one with mass m_1 and entropy $\mathcal{S}_1 = [\mathcal{A}_1 + 2\alpha X(\mathbb{S}_1)]/4G_D$, the other with mass m_2 and entropy $\mathcal{S}_2 = [\mathcal{A}_2 + 2\alpha X(\mathbb{S}_2)]/4G_D$. Here, we have defined the surface area $\mathcal{A} = \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}$ and the correction term $X(\mathbb{S}) = \oint_{\mathbb{S}^{D-2}} \tilde{\epsilon}\mathcal{R}$. Before the black holes merge, the total entropy is

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_1 + \mathcal{S}_2 \\ &= \frac{1}{4G_D}[\mathcal{A}_1 + \mathcal{A}_2 + 2\alpha(X(\mathbb{S}_1) + X(\mathbb{S}_2))] . \end{aligned} \tag{4.33}$$

After the black holes merge, the total entropy of the resulting black hole is

$$\mathcal{S}' = \frac{1}{4G_D}[\mathcal{A}' + 2\alpha X(\mathbb{S}')] \ . \tag{4.34}$$

The expressions (4.33) and (4.34) imply that $\mathcal{S}' > \mathcal{S}$ if and only if

$$\alpha < \frac{-(\mathcal{A}_1 + \mathcal{A}_2 - \mathcal{A}')}{2[X(\mathbb{S}_1) + X(\mathbb{S}_2) - X(\mathbb{S}')]} \ . \tag{4.35}$$

Without knowing the specific details of the black holes in question, nothing further can be said about $\mathcal{S}$ and $\mathcal{S}'$, or about the upper bound (4.35). Let us therefore consider for concreteness the simplest case – the merging of two Schwarzschild black holes in four-dimensional flat spacetime. This is a particularly special case as the topologies are much more restricted than could be hoped for. First, the GB theorem [see e.g. (Hatfield 1992)] relates the correction term to the Euler characteristic $\chi(\mathbb{S})$ via

$$X(\mathbb{S}) = \oint_{\mathbb{S}^2} \tilde{\epsilon}\mathcal{R} = 4\pi\chi(\mathbb{S}) \ . \tag{4.36}$$

Then the Hawking topology theorem (Hawking 1972) restricts the horizon cross sections to be two-spheres for which $\chi(\mathbb{S}) = 2$. For a Schwarzschild black hole the correction term is therefore $X(\mathbb{S}) = 8\pi$. Furthermore, the surface area of a Schwarzschild black hole is related to its mass via

$$\mathcal{A} = 16\pi m^2 \, , \tag{4.37}$$

whence the surface areas of the initial and final black-hole states are

$$\mathcal{A}_1 = 16\pi m_1^2 \, , \quad \mathcal{A}_2 = 16\pi m_2^2 \, , \quad \text{and} \quad \mathcal{A}' = 16\pi(m_1 + m_2 - \gamma)^2 \ . \tag{4.38}$$

Here, a small mass parameter $\gamma \geq 0$ for the surface area of the final black-hole state has been included. This parameter corresponds to any mass that may be carried away by gravitational radiation during merging. With these expressions for the areas, the bound (4.35) in terms of the masses becomes

$$\alpha < 2m_1m_2 - \gamma[2(m_1 + m_2) - \gamma] \ . \tag{4.39}$$

Therefore the second law will be violated if α is greater than twice the product of the masses of the black holes before merging minus a correction due to gravitational radiation.

To summarize, the validity of the second law of black-hole mechanics was examined for a physical process in which the topology is not constant. As was shown, the correction term appearing in the entropy (4.32) can lead to a violation of the second law for certain values of the GB parameter during the merging of two black holes. The calculation was done for two Schwarzschild black holes in four-dimensional flat spacetime. However, a similar bound to (4.39) may presumably be derived for specific solutions in higher dimensions as well [although in this case the topologies are not as severely restricted as they are in four dimensions, even for Einstein gravity with $\Lambda = 0$ (Helfgott *et al* 2006; Galloway 2006; Galloway and Schoen 2006)]. Incidently, the result obtained here shows that the presence of the GB term in the action for gravity can have nontrivial physical effects even in four dimensions, when the term is a topological invariant of the manifold. This is in sharp contrast to the commonly held belief within the literature that the term is only significant in spacetimes with dimension $D \geq 5$.

Chapter 5

Prospects

"Try to see through fainted views. As reality disappears in a haze. A journey between eternal walls. The senses unfold before my eyes." ~ T G Fischer

5.1 Summary

In this thesis we presented two extensions of the IH framework, first to EM-CS theory and then to EGB theory in D dimensions. In particular, we derived the local version of the zeroth and first laws of black-hole mechanics for general WIHs on the phase space of the corresponding theories. In addition, for EM-CS theory in five dimensions and for EM theory in four dimensions we derived the conditions that are required by supersymmetry. We then turned to EGB theory, for which we showed that the quasilocal entropy is in exact agreement with the expressions that are obtained by the Euclidean and Noether charge methods. Finally, we showed that the GB term can have physical consequences in four dimensions even though it is a topological invariant and does not contribute to the equations of motion.

As was stated in §1.2, our intention was to employ the IH framework with suitable extensions to higher dimensions in order to determine the generic properties of black holes in string-inspired gravity models. A summary of the main five results are given in §1.3.

5.2 Classical applications to EM-CS and EGB theories

There are a number of classical applications of IHs to EM-CS and EGB black holes that can be explored. Here we briefly discuss four problems that are worth investigating.

Let us first consider applications to EM-CS theory, and in particular to the corresponding supersymmetric black holes.

- *BPS bounds.* The general method for deriving the BPS bound for stationary spacetimes is to construct an expression for the energy of the spacetime using spinor identities and the Einstein field equations. This method was pioneered by Witten (1981) and Nester (1981) to prove the positive energy theorem. The method was applied in four dimensions (Gibbons and Hull 1982) and in five dimensions (Gibbons *et al* 1994) to calculate the BPS bounds for the corresponding spacetimes. How can one derive these bounds for IHs? The bounds are saturated when the spinors are super-covariantly constant, which is associated with extremality. This suggests that the extremality condition (2.54) can be used for IHs. This is straight-forward to do for undistorted horizons. Let us consider the four-dimensional EM theory for illustration. Here the contraction $T_{ab}\ell^a n^b$ is the square of the electric flux $E_\perp$ crossing the surface (Ashtekar *et al* 2000c). For any IH this quantity is constant over S^2 and can therefore be moved outside the integral. The result can be used to relate the charge Q on the horizon to its surface area $\mathcal{A}$ via $\mathcal{Q} = E_\perp \mathcal{A}/(4\pi)$. For the RN solution one finds that $\eta = \mathcal{Q}^2/R^2 - 1 \leq 0$ with $R = \sqrt{\mathcal{A}/(4\pi)}$ the areal radius (Booth and Fairhurst 2007b). When the surface gravity vanishes $\eta = 0$ and $\mathcal{Q} = R = M$ with M the mass. This is the condition for supersymmetry in four dimensions. The situation is not as obvious for distorted horizons in five dimensions. This is because the contraction $T_{ab}\ell^a n^b$, which for EM-CS theory is again the square of the electric flux, may not be constant over the horizon cross sections in general. However, for the BMPV black hole in particular we know that the cross sections are S^3 which have constant curvature, and therefore $E_\perp^2$ is constant on Δ. From here, a charge-areal radius relation follows along the same lines as the derivation that was outlined above for the RN black hole in four dimensions.

- *Supersymmetry and horizon geometries.* It was shown (Lewandowski and Pawlowski 2003) that the IH constraints for extremal IHs of four-dimensional EM theory are satisfied iff the intrinsic geometry of the horizon coincides with that of the extremal Kerr-Newman (KN) solution. An extension of that analysis to IHs of five-dimensional EM-CS theory would be of interest, particularly because it would provide a method for deriving the geometries of the corresponding extremal IHs. This would complement a recently developed method (Astefanesei and Yavartanoo 2007; Kunduri *et al* 2007b; Kunduri and Lucietti 2007) for deriving the *near-horizon* geometries of extremal black holes. While speculating on the local uniqueness theorems in five dimensions we need to keep in mind that black holes in five dimensions are much less constrained than in four dimensions, mainly because in five dimension there are two possible topologies

(S^3 and $S^1 \times S^2$), and also because there are two independent rotation parameters. As a consequence of this richer structure, it is possible that two distinct black holes in five dimensions can have the same asymptotic charges (Emparan and Reall 2002). This is a striking example of black-hole nonuniqueness in higher dimensions. Nevertheless, uniqueness has been established for supersymmetric black holes in five dimensions (Reall 2003). Therefore it is expected that the five-dimensional analogues of the local uniqueness theorem of (Lewandowski and Pawlowski 2003) do exist, but for SIHs. We also note that, while supersymmetry constrains the geometry (i.e. connection one-form), the dominant energy condition and the Einstein field equations are still required to constrain the topology. Therefore we expect that there should be a unique horizon geometry for a given topology. For example, if the topology of a SIH is $S^1 \times S^2$, then the geometry should coincide uniquely with the induced metric and vector potential of the EEMR black ring solution. We also expect that, if the topology is S^3, then the geometry should coincide uniquely with that of the BMPV black hole *in general*, and the extremal RN black hole as a limiting case when the angular momentum of the Maxwell fields vanishes. It would be of considerable interest to try solving the IH constraints for a SIH when $T_{ab}\ell^a n^b = 0$ (at the horizon); the resulting geometry would provide the first explicit solution of a supersymmetric black hole with toroidal topology.

Now let us consider applications to EGB theory.

- *Rotation.* One of the main assumptions that we made in our calculations was that the horizons are non-rotating. It would be interesting to extend the IH phase space to include rotation by relaxing the condition $\tilde{\omega} = 0$.

- *Torsion.* The formalism presented here can be further extended by including torsion. Recall that in §5.2 we assumed $T^I = 0$ directly, which became crucial when we derived the pull-back to Δ of the connection. However, as the equation of motion (4.2) for A indicates, the torsion-free condition is not imposed in $D \geq 5$ dimensions. If the torsion is non-zero then the pull-back to Δ of A will not be given by (2.31). In order to derive the modified pull-back of A in the presence of torsion we would need to find $\nabla_{\underleftarrow{a}} e^b{}_I$ explicitly. In addition, the Raychaudhuri equation would be different as well, and so the boundary conditions would require a more careful analysis. The effects of torsion on IHs should therefore lead to some interesting consequences. This would be a particularly interesting project to work out in five dimensions, for which a solution has recently been found that describes a supersymmetric black hole (Canfora *et al* 2008). More interestingly, there is also a solution of the equations with non-zero torsion that describes a constant-curvature black hole with entropy that is proportional to the surface area of the *inner* horizon rather than the event horizon (Bañados 1998). This curious interchange of thermodynamic parameters, namely the outer and inner horizons, may be a consequence of the torsion that is present in the equations of motion. The IH framework could be employed in order to test this hypothesis.

There are many more avenues to explore. We hope that others will consider some of them.

Appendix A

Black Hole Mechanics: An Overview

A.1 Thermodynamics

The study of macroscopic properties of materials without knowing their internal structure is the science of thermodynamics. This is the branch of science concerned with the dynamics of materials where thermal effects are important. Because no reference is made to the internal structure, the formalism involved is very general and therefore very powerful. Let us quickly review the four laws of thermodynamics (Reif 1965; Poisson 2000); this will make the connection between the laws of black-hole mechanics and the four laws of thermodynamics apparent.

Let us now state the four laws of thermodynamics and discuss some of their physical consequences.

- *Zeroth law.* If two systems are each in thermal equilibrium with a third system, then they are in thermal equilibrium with each other. This implies that the temperature remains constant throughout the systems that are in thermal equilibrium with each other.

- *First law.* The internal energy $\mathscr{U}$ of a system that interacts with its surroundings will undergo a change given by

$$d\mathscr{U} = d\mathscr{Q} - d\mathscr{W}, \tag{A.1}$$

where $d\mathscr{Q}$ is the amount of heat absorbed by the system and $d\mathscr{W}$ is the amount of work done by the system during its change of macrostate. This is really just the statement of conservation of energy. It implies that a gain of heat to a system can do physical work on its surroundings.

- *Second law.* Heat flows spontaneously from higher temperatures to lower temperatures. This means that: (i) the spontaneous tendancy of a system to go toward thermal equilibrium cannot be reversed without changing some organized energy (work) into some disorganized energy (heat); (ii) it is not possible to convert heat from a hot reservoir into work in a cyclic process without transferring some heat to a colder reservoir; (iii) the change in entropy $d\mathscr{S} = d\mathscr{Q}/T$ (with T the temperature) of a system and its surroundings is positive and approaches zero for any process that approaches reversibility.

- *Third law.* The difference in entropy $\delta\mathscr{S}$ between states connected by a reversible process goes to zero as the temperature T goes to absolute zero. Unlike the other laws which are based on classical considerations, the third law is a consequence of quantum mechanics. The third law implies that a system at absolute zero will drop to its lowest quantum state and thus become completely ordered.

The first law expresses the change in internal energy of a system in terms of inexact differentials. This means that, unlike the difference $d\mathscr{Q} - d\mathscr{W}$, seperately $d\mathscr{Q}$ and $d\mathscr{W}$ depend on the path that is taken from the initial state to final state. The first law can be expressed in terms of exact differentials though. First, we note that in a quasi-static process where the volume of a fluid changes but the pressure remains approximately unchanged, the physical work done by the system is given by $d\mathscr{W} = Pd\mathscr{V}$. Then, modulo the second law we can write the first law in the more familiar form:

$$d\mathscr{U} = Td\mathscr{S} - Pd\mathscr{V}. \tag{A.2}$$

This is the form of the first law that appears in most references on gravitational physics that discuss the laws of black-hole mechanics.

A.2 Black-hole mechanics: global equilibrium

Now we will review the corresponding laws of black hole mechanics. A beautiful introduction is given in the book by Poisson (2004). The laws of black-hole mechanics were first formulated for globally stationary spacetimes in four-dimensional Einstein gravity (Bardeen *et al* 1973), and later extended using covariant phase space methods to D-dimensional spacetimes in arbitrary diffeomorphism-invariant theories (Wald 1993; Iyer and Wald 1994; Jacobson *et al* 1994). This seminal work has revealed, among other properties of black holes, that the area-entropy relation will only be modified in cases when gravity is supplemented with nonminimally coupled matter or higher-curvature interactions. This is a consequence of the fact that such terms modify the gravitational surface term in the symplectic structure. Such terms naturally arise in the effective actions of superstrings and supergravity. For now we will restrict our review to D-dimensional black holes of EM theory.

Let us first state some facts about Killing horizons. We shall consider the black hole region

$$B = \mathcal{M} - J^{-}(\mathcal{I}^{+}) \tag{A.3}$$

of a spacetime manifold $\mathcal{M}$; i.e. a region of spacetime that excludes all events that belong to the causal past of future null infinity. The event horizon is the boundary ∂B of the region B of spacetime. If the black hole is stationary, then the event horizon coincides with the Killing horizon – a hypersurface at which the vector

$$\zeta^a = t^a + \sum_{\iota}^{\lfloor (D-1)/2 \rfloor} \widetilde{\Omega}_\iota m_\iota^a \tag{A.4}$$

is null. Here, t^a is a timelike Killing vector, m_ι^a are $\lfloor (D-1)/2 \rfloor$ rotational spacelike Killing vectors and the coefficients $\widetilde{\Omega}_\iota$ are the angular velocities of the black hole. The vector ζ^a is tangent to the null generators of the Killing horizon and therefore satisfies the geodesic equation

$$\zeta^a \nabla_a \zeta^b = \kappa \zeta^b \,. \tag{A.5}$$

This defines the surface gravity κ of the black hole. Equivalent definitions are given by

$$2\kappa\zeta_a = (-\zeta_b\zeta^b)_{;a} \quad \text{and} \quad \kappa^2 = -\frac{1}{2}\zeta_{a;b}\zeta^{a;b} \,. \tag{A.6}$$

For the Schwarzschild solution (1.2) it can be verified by direct computation that $\kappa = 1/(4M_\odot)$.

We now state the four laws of black-hole mechanics:

- *Zeroth law.* The surface gravity κ is constant over the entire event horizon.

- *First law.* For a stationary black hole with energy $\mathfrak{E}$, surface area $\mathscr{A}$, electric charge $\mathfrak{Q}$ and angular momenta $\mathfrak{J}_\iota$, the change in mass during a quasi-static process is given by

$$\delta\mathfrak{E} = \frac{\kappa}{\kappa_D}\delta\mathscr{A} + \widetilde{\Phi}\delta\mathfrak{Q} + \sum_{\iota=1}^{\lfloor(D-1)/2\rfloor} \widetilde{\Omega}_\iota \delta\mathfrak{J}_\iota\,, \tag{A.7}$$

where $\widetilde{\Phi} = -A_a\zeta^a|_{r=r_+}$ is the electric potential at the horizon ($r = r_+$) and A_a is the vector potential.

- *Second law.* The surface area $\mathscr{A}$ can never decrease in a physical process if the stress-energy tensor T_{ab} satisfies the dominant energy condition $T_{ab}\zeta^a\zeta^b \geq 0$.

- *Third law.* The surface gravity cannot be reduced to zero by any physical process in a finite period of time.

The laws of black-hole mechanics are very similar to the laws of thermodynamics. If one makes the identification $\mathscr{U} = \mathfrak{E}$, then the first law of thermodynamics (A.2) and the first law of black-hole mechanics (A.7) require that $\mathscr{S} \propto \mathscr{A}k_\mathrm{B}/l_\mathrm{P}^2$ (with physical constants restored for the moment) and $\kappa \propto T$. However, the latter does not seem to make sense from a classical point of view, because a black hole presumably has zero temperature which would imply that the entropy is infinite. Is the similarity just a mathematical coincidence, or is nature telling us something deep about the interplay between gravitational phenomena and quantum mechanics? Indeed, the identification $\mathscr{S} \propto \mathscr{A}k_\mathrm{B}/l_\mathrm{P}^2$ requires the presence of c and $\hbar$ in order for the entropy to be dimensionless. It was Bekenstein (Bekenstein 1973; Bekenstein 1974) who first recognized the physical significance of this similarity. Bekenstein also recognized that the irreversable process of dropping matter into a black hole leads to the *generalized* second law of thermodynamics:

$$\delta\mathscr{S}_{\text{Universe}} + \delta\mathscr{S}_{\text{BH}} \geq 0 \ . \tag{A.8}$$

Using a semiclassical approach, Hawking (1975) then fixed the free parameter to 1/4 and the temperature to $T = \kappa/2\pi$. Thus a black hole is not eternal, but rather has a thermodynamical temperature and radiates. (It should be noted that the final state of this process is unknown, because the temperature goes to infinity as the surface area decreases. One of the many goals of all theories of quantum gravity is to give a detailed first-principles description of the evapouration process and to determine what the final state of a black hole should be.) Therefore, the laws of black-hole mechanics *are* the laws of thermodynamics, applied to an object of special character.

Appendix B

Differential forms

B.1 Definitions

Definition B.I: *A differential p-form is a totally antisymmentric tensor of type* $(0, p)$, *i.e.* $\omega_{a_1 \cdots a_p}$ *is a p-form if*

$$\omega_{a_1 \cdots a_p} = \omega_{[a_1 \cdots a_p]}.$$

Denote the vector space of p-forms at a point x by Λ_x^p and the collection of p-form fields by Λ^p. Taking the outer product of a p-form $\omega_{a_1 \cdots a_p}$ and a q-form $\mu_{b_1 \cdots b_q}$, results in a tensor of type $(0, p+q)$, which will not be a $(p+q)$-form since this tensor is not generally antisymmetric.

Definition B.II: *The wedge product on an m-dimensional manifold* $\mathcal{M}$ *is a map* $\wedge$: $\Lambda_x^p \times \Lambda_x^q \rightarrow \Lambda_x^{p+q}$ *such that the tensor product*

$$(\omega \otimes \mu)_{a_1 \cdots a_p b_1 \cdots b_q}$$

is totally antisymmetric.

Note that this tensor is zero if $p+q>m$. Thus the wedge product of two one-forms on $\mathbb{R}^{m\geq 2}$ is $\boldsymbol{\omega}\wedge\boldsymbol{\mu}=-\boldsymbol{\mu}\wedge\boldsymbol{\omega}$. Now define the vector space of all differential forms at x to be the direct sum of the Λ_x^p such that

$$\Lambda_x=\bigoplus_{p=0}^{n}\Lambda_x^p.$$

The map $\wedge:\Lambda_x^p\times\Lambda_x^q\longrightarrow\Lambda_x^{p+q}$ gives Λ_x the structure of a Grassmann algebra over the vector space of one-forms.

Definition B.III: *The exterior derivative on an m-dimensional manifold $\mathcal{M}$ is a map from the space of p-forms to the space of $(p+1)$-forms:*

$$d:\Lambda^p\longrightarrow\Lambda^{p+1},$$

together with the following properties:

1. $d(\boldsymbol{\omega}+\boldsymbol{\mu})=d\boldsymbol{\omega}+d\boldsymbol{\mu}$;
2. $d(c\boldsymbol{\omega})=cd\boldsymbol{\omega}$;
3. $d(\boldsymbol{\omega}\wedge\boldsymbol{\lambda})=d\boldsymbol{\omega}\wedge\boldsymbol{\lambda}+(-1)^p\boldsymbol{\omega}\wedge d\boldsymbol{\lambda}$;
4. $d(d\boldsymbol{\omega})=0$;

$\forall\ \boldsymbol{\omega},\boldsymbol{\mu}\in\Lambda^p(\mathcal{M})$, $\boldsymbol{\lambda}\in\Lambda^q(\mathcal{M})$, *and* $c\in\mathbb{R}$.

The last property can be easily shown as follows. Consider the p-form $\boldsymbol{\omega}$ such that

$$\boldsymbol{\omega}=\frac{1}{p!}\omega_{a_1\cdots a_p}dx^{a_1}\wedge\cdots\wedge dx^{a_p}.$$

The exterior derivative acting on $\boldsymbol{\omega}$ is given by

$$d\boldsymbol{\omega}=\frac{1}{p!}\partial_\mu\omega_{a_1\cdots a_p}dx^\mu\wedge dx^{a_1}\wedge\cdots\wedge dx^{a_p},$$

and the second exterior derivative is

$$dd\omega = \frac{1}{p!}\partial_\nu \partial_\mu \omega_{a_1 \cdots a_p} dx^\nu \wedge dx^\mu \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_p}.$$

Since the functions $\omega_{a_1 \cdots a_p}$ are by definition smooth, the partial derivatives acting on them commute and the operator $\partial_\mu \partial_\nu$ is symmetric. The two-form $dx^\mu \wedge dx^\nu$, however, is antisymmetric. Thus we have $dd\omega = 0$. This important property is often written simply $d^2 = 0$.

In general, if $\alpha \in \Lambda^{p+1}(\mathcal{M})$ and $\beta \in \Lambda^p(\mathcal{M})$ then: α is called an exact form iff $d\alpha = 0$; α is called a closed form iff $\alpha = d\beta$. By the last property $d^2 = 0$ of the exterior derivative, a closed form is automatically an exact form as well. However, the converse is not true, and the study of this is called DeRham cohomology.

Definition B.IV: *The interior product of a p-form ω with vector field X on an m-dimensional manifold $\mathcal{M}$ is a map $i_X : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ such that*

$$i_X \omega(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1})$$

together with the "anti-derivation" property with respect to the wedge product

$$i_X(\omega \wedge \eta) = (i_X \omega) \wedge \eta + (-1)^p \omega \wedge (i_X \eta)$$

where $\omega \in \Lambda^p(\mathcal{M})$ and $\eta \in \Lambda^q(\mathcal{M})$. If $\omega \in \Lambda^0(\mathcal{M})$ then $i_X \omega = 0$.

The interior product is also called the contraction, and is also denoted $X \lrcorner \omega$. As an example, consider $\omega \in \Lambda^2(\mathcal{M})$ and $Z = X^\alpha(\partial/\partial x^\alpha) + Y^\alpha(\partial/\partial y^\alpha)$ a vector field on $\mathcal{M}$. The interior product of ω and Z is given by

$$Z \lrcorner \omega = -Y_\alpha dx^\alpha + X_\alpha dy^\alpha.$$

Definition B.V: *The Lie derivative of a p-form $\boldsymbol{\omega}$ with respect to a vector field X is given by*

$$\pounds_X \boldsymbol{\omega} = X \lrcorner d\boldsymbol{\omega} + d(X \lrcorner \boldsymbol{\omega}) \, .$$

Definition B.VI: *The Hodge star operator on an m-dimensional manifold $\mathcal{M}$ is a linear map $* : \Lambda^p(\mathcal{M}) \to \Lambda^{m-p}(\mathcal{M})$ given by*

$$*\left(\boldsymbol{e}_{a_1} \wedge \cdots \wedge \boldsymbol{e}_{a_p}\right) = \frac{1}{(m-p)!} \epsilon_{a_1 \cdots a_p}{}^{a_{p+1} \cdots a_m} \boldsymbol{e}_{a_{p+1}} \wedge \cdots \wedge \boldsymbol{e}_{a_m},$$

where $\{\boldsymbol{e_a}\}_{a=1}^m$ is a positively-oriented set of one-forms on some chart of $\mathcal{M}$.

As a simple example, consider a two-form on $\mathbb{R}^3$. Choosing a basis $\{\boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \boldsymbol{e}_1 \wedge \boldsymbol{e}_3, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\}$, the definition gives $*(\boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \epsilon_{ij}{}^k \boldsymbol{e}_k$, or $*(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = \boldsymbol{e}_3$, $*(\boldsymbol{e}_1 \wedge \boldsymbol{e}_3) = -\boldsymbol{e}_2$ and $*(\boldsymbol{e}_2 \wedge \boldsymbol{e}_3) = \boldsymbol{e}_1$.

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