THE GRAPH RECONSTRUCTION CONJECTURE: SOME NEW RESULTS AND OBSERVATIONS

By

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A Thesis submitted to the

School of Graduate Studies

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

DEPARTMENT OF COMPUTER SCIENCE

Memorial University of Newfoundland

February, 2015

St. John's Newfoundland

ABSTRACT

A vertex-deleted subgraph (or simply a card) of graph G is an induced subgraph of G containing all but one of its vertices. The deck of G is the multiset of its cards. One of the best-known unanswered questions of graph theory asks whether G can be reconstructed in a unique way (up to isomorphism) from its deck. The likely positive answer to this question is known as the Reconstruction Conjecture.

In the first part of the thesis two basic equivalence relations are considered on the set of vertices of the graph G to be reconstructed. The one is card equivalence, better known as removal equivalence, by which two vertices are equivalent if their removal results in isomorphic cards. The other equivalence is similarity, also called automorphism equivalence. Two vertices u and v are automorphism-equivalent (similar) if there exists an automorphism of G taking u to v. These relations are analyzed on various examples with special attention to vertices that are card-equivalent but not similar. Such vertices are called pseudo-similar, and they have been studied very extensively in the literature. The first result of the thesis is a structural characterization of card equivalence in terms of automorphism equivalence. A similar result was obtained by Godsil and Kocay in 1982 on the characterization of pseudo-similar vertices, which result is proved in the thesis as a corollary to the characterization theorem on card equivalence.

In the second part of the thesis, the concept of relative degree-sequence is introduced for subgraphs of a graph G. By "relative" it is meant that each degree in the degree-sequence of the subgraph is coupled up with the original degree of the corresponding vertex in G. A new conjecture is formulated, which says that G is uniquely determined (up to isomorphism) by the multiset of the relative degree-sequences of its induced subgraphs. The new conjecture is then related to the Reconstruction Conjecture in a natural way.

The third part of the thesis contains an original new result on graph reconstruction. Card-minimal graphs are investigated, the deck of which is a set. Thus, the deck of such graphs is free from duplicate cards. It is shown that every card-minimal graph G is reconstructible, provided that G does not have pseudo-similar couples of vertices. This condition is recognizable, that is, it can be checked by looking at the deck of G only.

The results of this thesis have been partially published in [1].

ACKNOWLEDGMENTS

First and foremost, I offer my sincerest gratitude to my supervisor, Dr. Miklós Bartha, for his supervision, motivation, immense support and valuable suggestions throughout the research work. I attribute the level of my Masters degree to his encouragement and effort and without him this thesis, would not have been completed. His guidance was source of inspiration for me and I could not have imagined for a better and friendlier supervisor.

I would like to gratefully acknowledge my supervisor, the School of Graduate Studies (SGS) and the Department of Computer Science, Memorial University of Newfoundland for the financial support throughout this Masters period.

At last I would like to express my gratitude to my parents and siblings for their support and encouragement.

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Chapter 1

Introduction

1.1 Historical overview

The Reconstruction Conjecture is generally regarded as one of the foremost unsolved problems in graph theory. Frank Harary [9] has even classified it as a graphical disease because of its contagious nature. Some sources say that the problem was discovered in Wisconsin in 1941 by Kelly and Ulam, and claimed its first victim (P. J. Kelly) in 1942. Indeed, Kelly's doctoral dissertation [13], which he wrote under the supervision of Ulam, appeared in that year. Later, in 1957, Kelly published the first relevant result on graph reconstruction, showing that trees are reconstructible [14].

Others say that Stanisław Ulam knew about the ideas that later became the Reconstruction Conjecture as early as 1929 when, along with Stanisław Mazur, Stefan Banach, Kazimierz Kuratowski, and others, he was a member of the Lwów School of Mathematics in Poland. (Today Lviv, Ukraine, also known as Lemberg, Galicia, Austria-Hungary.) Ulam had spent many years collecting problems that were posed by fellow graduate students and professors during his years in graduate school in Lwów. These problems have been recorded in the famous *Scottish Book*, which was a thick notebook used by mathematicians of the Lwów School of Mathematics for jotting down problems meant to be solved. The notebook was named after the "Scottish Café" in Lwów where it was kept. Ulam himself contributed 40 problems as a single author to the Scottish Book, another 11 with Banach and Mazur, and an additional 15 with others.

Even though the Reconstruction Conjecture cannot be found in the Scottish Book, it does appear in the first part of a monograph written by Ulam in 1960 under the title "A Collection of Mathematical Problems" [27], and Ulam does say in the preface:

> In spirit, the questions considered in the first part of this collection belong to a complex of problems represented in the *Scottish Book*.... Many of the problems contained here were indeed first inscribed in the Scottish Book, but the greater part of the material is of later origin ... Many of the problems originated through conversations with others and were stimulated by the transistory interests of the moment in various mathematical centers.

The uncertainties above have created a difficulty in trying to determine who should have credit for creating this beautiful problem in graph theory. The commonly accepted solution to this dispute is to call the problem the Kelly-Ulam conjecture. The conjecture itself, as specified below, was given the name "an inductive lemma in combinatorial analysis" by Ulam. The reader should keep in mind that this version of the conjecture is presented here for purely historical reasons; we will never actually use the original metric terminology by Ulam.

Ulam's Statement of the Reconstruction Conjecture [27]

> Suppose that in two sets A and B, each containing n elements, there is defined a distance function ρ for every pair of distinct points, with values either 1 or 2, and $\rho(p, p) = 0$. Assume that for every subset of $n - 1$ points of A, there exists an isometric system of $n - 1$ points of B, and that the number of distinct

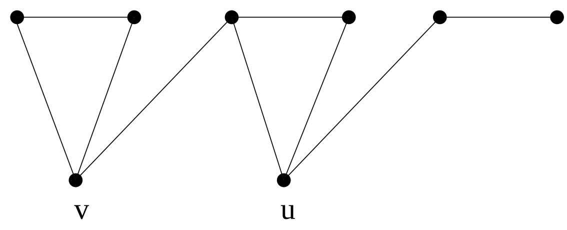

Figure 1.1: Pseudo-similar vertices.

> subsets isometric to any given subset of $n-1$ points is the same in A and in B. Are A and B isometric?

Interestingly, the problem of graph reconstruction has yet another important historical aspect, which is related to the concept of pseudo-similarity. According to the survey [20] by Josef Lauri, graph theorists seem to have stumbled on this concept quite by accident. If two vertices u and v in a graph G are similar, that is, there is an automorphism of G which maps one into the other, then it is clear $G-u$ and $G-v$ are isomorphic graphs. However, the converse is not true, because $G-u$ and $G-v$ can be isomorphic without u and v being similar in G. The smallest graph for which this can happen is shown in Fig. 1.1. Nobody seems to have given this phenomenon any thought until (as reported by Harary and Palmer [11]) someone apparently found a proof of the celebrated Reconstruction Conjecture which depended on the assumption that if $G-u$ and $G-v$ are isomorphic, then u and v must be similar. To Harary and Palmer goes the credit of taking what could simply have remained a curious counter-example, and turning it into a graph theoretic concept worthy of investigation. Their 1965 and 1966 papers proved the first results and set the scene for further studies.

1.2 Basic terminology

In this section we present the collection of basic definitions in graph theory that will be used throughout the thesis. Additional terminology specifically related to the problem of graph reconstruction will follow in later chapters.

An *undirected graph* (or simply a graph) G consists of a finite non-empty set $V(G)$ of *vertices* and a set of unordered pairs $E(G)$ of *edges*. Notice that, by this definition, multiple edges are not allowed to occur in G. Indeed, $E(G)$ is not a multiset. By an unordered pair of vertices we in fact mean a couple $\{u, v\} \subseteq V(G)$, even though we shall simply write $(u, v) \in E(G)$. Loops (i.e., edges connecting vertices with themselves) are therefore excluded from $E(G)$ as well. An edge (u, v) is said to *connect* the vertices u and v, to be *incident* with these two vertices, and vertices u and v are said to be *adjacent*.

Even though it is not consistent with the above definition of graphs, in some situations it is inevitable to accept the "empty graph" also as a graph. This abstraction has no vertices and no edges, so that it is indeed completely empty. Its presence is useful in some graph operations, e.g., in taking the disjoint union of graphs. The relevance of the empty graph is that it becomes a unit element for this operation, rendering the corresponding algebraic structure on graphs a classical (commutative) monoid.

Let G and H be graphs. A one-to-one correspondence ϕ mapping $V(G)$ onto $V(H)$ is called an *isomorphism* if for every pair u, v of vertices in $V(G)$, $(u, v) \in E(G)$ iff $(\phi(u), \phi(v)) \in E(H)$. If such an isomorphism exists, then the graphs G and H are said to be *isomorphic*, in notation $G \cong H$. An isomorphism of graph G onto itself is called an *automorphism* of G.

The number of edges in graph G incident with a concrete vertex v is called the *degree* of v and denoted by $d_G(v)$ ($d(v)$, if G is understood). The *degree-sequence* of G is the sequence of degrees of G's vertices in a non-decreasing order. A graph in which all degrees are equal to k is said to be *k-regular*, and if G is k-regular for some k, we simply say that G is *regular*. A *complete* graph is one in which every two distinct vertices are connected by an edge.

An alternating sequence of vertices and edges, beginning and ending with vertices such that each edge in the sequence is incident with the vertex immediately preceding it and with the one immediately following it, is called a *walk*. If all edges in a walk are distinct, then the walk is a *trail*, and if, in addition, the vertices are also distinct, then the trail is a *path*. The *length* of a walk is the number of occurrences of edges in it. A walk or trail in which the first vertex coincides with the last one is called *closed*. A *cycle* is a closed trail of length at least three that consists of a path together with an edge connecting the first and last vertices of the path. A graph not containing cycles is called a *tree*.

If G and H are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph* of G. If H is a subgraph of G and if every edge $(u, v) \in E(G)$ is also in $E(H)$, provided that its endpoints u and v are in $V(H)$, then we call H an *induced* subgraph of G. If $Y \subseteq V(G)$, then $G[Y]$, the subgraph of G *induced by* Y, is the induced subgraph of G having Y as its vertices. With some ambiguity, for $X \subseteq V(G)$ we shall simply write $G - X$ to denote the induced subgraph $G[V(G) \setminus X]$. Sometimes we even write $G - x$ or $G - x - y$ to mean $G - \{x\}$ or $G - \{x, y\}$ if x and y are individual vertices of G.

A graph is *connected* if every two vertices are joined by a path. A maximal connected subgraph of G is called a *component* of G. A *disconnected* graph is one that has more than one components.

If the edges of a graph have a direction assigned to them, then we speak of a directed graph. More precisely, a *directed graph*, or *digraph* for short, G consists of a set $V(G)$ of vertices and a set $E(G)$ of *ordered* pairs of vertices, called edges. The definition of walk, trail, path, and cycle must be modified somewhat in digraphs, saying that each edge $e = (u, v)$ in these constructs (as alternating sequences of vertices and edges) connects the vertex u before e to the vertex v after e. An *acyclic* digraph is one containing no directed cycles. A digraph is *strongly connected* if for every ordered pair (u, v) of vertices there is a (directed) path joining u to v.

A *tournament* is a digraph obtained by assigning a direction for each edge in an undirected complete graph. In other words, a tournament is an orientation of a complete graph, or equivalently, a directed graph in which every pair of distinct vertices is connected by a single directed edge. A tournament T is called *transitive* if $(u, v) \in E(T)$ and $(v, w) \in E(T)$ imply $(u, w) \in E(T)$ for all distinct vertices u, v, w. If $(u, v) \in E(T)$, then we also say that vertex u *dominates* vertex v in T.

At the end of Chapter 4 we shall also need to use some basic linear algebraic terminology to address the connection between the Reconstruction Conjecture and our new conjecture on sets of relative degree-sequences. Since the introduction of this terminology would take us too far in this short introduction, we just refer the reader to any standard text in linear algebra for the concepts involved in those arguments.

Chapter 2

Definitions, and some easily recoverable data

In this chapter we review the most important definitions relating to the problem of graph reconstruction, and provide a brief summary of some of the best-known elementary results.

2.1 Definitions

Definition 2.1.1 For a graph G and vertex $v \in V(G)$, $G - v$ is called a *vertex-deleted subgraph* of G, or the *card* associated with vertex v in G. We do not distinguish between isomorphic cards, though. The multiset of cards collected from G in this way is called the *deck* of G, denoted $D(G)$.

See Fig 2.1 for the deck of a small graph G.

In the language of modern graph theory, the *Reconstruction Conjecture*, introduced in Section 1.1 as the Kelly-Ulam conjecture, states that an arbitrary graph G having at least three vertices can be reconstructed in a unique way (up to isomorphism) from its deck.

Ever since its inception, this problem has remained a mystery. Trying to solve it is similar to conducting a criminal investigation. There is a suspect, the graph G, who leaves plenty of evidence (i.e., the deck $D(G)$) on the crime

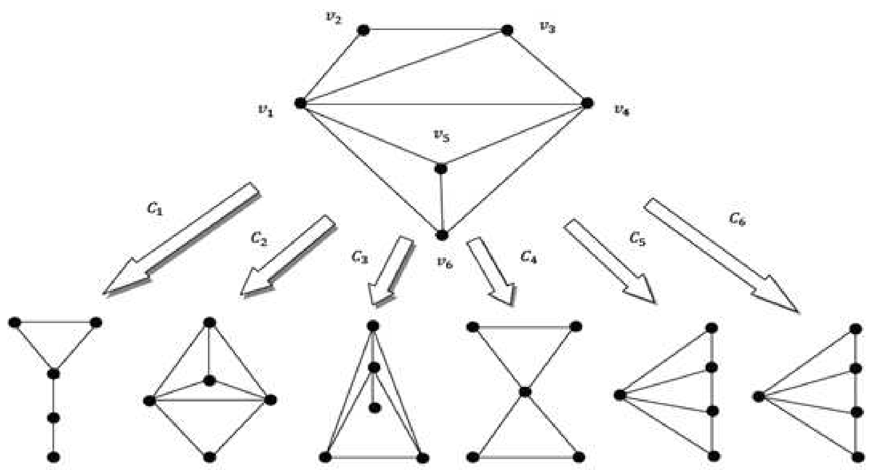

Figure 2.1: Constructing the deck of a graph.

scene. Yet, no brilliant detective has been able to track down the suspect for over 70 years, and the number of works on the case is rapidly decreasing year by year. The reconstruction problem was, however, very popular in the past. According to [25], more than 300 research papers had been published on graph reconstruction between 1950 and 2004.

Definition 2.1.2 Two vertices $u, v \in V(G)$ are called *hypomorphic* or *card-equivalent* (c-equivalent, for short) if the card associated with u is identical to the one associated with v, i.e., $G - u \cong G - v$. (Remember that we do not distinguish between isomorphic cards.)

Yet another name for card equivalence is *removal equivalence*, which is often used in the literature. Card equivalence will be denoted by $\sim_c$. Clearly, $\sim_c$ is indeed an equivalence relation on $V(G)$.

Definition 2.1.3 Two graphs G and H are *hypomorphic* if $D(G)$ and $D(H)$ are identical as multisets, that is, each card appears in $D(G)$ and $D(H)$ the

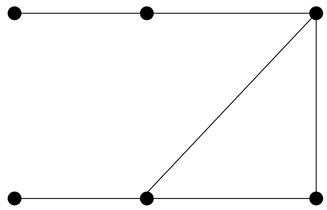

Figure 2.2: The smallest card-minimal graph.

same number of times. (Recall that $D(G)$ denotes the deck of G.) If G and H are hypomorphic, then a *hypomorphism* of G onto H is a bijection $\phi : V(G) \to V(H)$ such that $G - v \cong H - \phi(v)$ holds for every $v \in V(G)$.

A *reconstruction* of G is a graph G' such that G and G' are hypomorphic, or, equivalently, there exists a hypomorphism of G onto G'. Using this terminology, the Reconstruction Conjecture simply says that two graphs G and H are hypomorphic iff they are isomorphic. In other words, all reconstructions of G are isomorphic (to G, of course). Clearly, every isomorphism of G onto H is a hypomorphism, but the converse is not true, even if the Reconstruction Conjecture holds. Indeed, the Reconstruction Conjecture only says that if there is a hypomorphism between G and H, then there is one which is also an isomorphism.

Definition 2.1.4 Graph G is called *card-minimal* if $D(G)$ is a set, that is, each card is unique in $D(G)$.

Our aim in this thesis is to show that the Reconstruction Conjecture holds true for a large subclass of card-minimal graphs. Notice that both graphs on two vertices have two identical cards, therefore every card-minimal graph has at least three vertices. In fact it is easy to see by trying out all graphs on five or less vertices that the smallest card-minimal graph has six vertices. See Fig. 2.2.

One might think that card-minimal graphs are trivially reconstructible, since there is a unique hypomorphism between any two hypomorphic card-minimal graphs G and H. While this is certainly true, we have no direct information on $E(G)$ and $E(H)$, therefore the given unique hypomorphism may not be an isomorphism. Reconstructing G from $D(G)$ is still a very complex issue for such graphs. As we shall see, any duplication of cards in $D(G)$ indicates a kind of symmetry in the internal structure of G. Consequently, the class of card-minimal graphs is really large. Our result is therefore in accordance with the observation in [3] saying that the probability that a randomly chosen graph on n vertices is not reconstructible goes to 0 as n goes to infinity.

Definition 2.1.5 A function defined on a class $\mathcal{G}$ of graphs is *reconstructible* if, for each graph G in $\mathcal{G}$, it takes the same value on all reconstructions of G.

Definition 2.1.6 A class (or property) $\mathcal{G}$ of graphs is *recognizable* if, for each $\mathcal{G}$-graph G, every reconstruction of G is also in $\mathcal{G}$.

Definition 2.1.6 essentially says that a property of graphs is recognizable if its presence is already indicated by the deck of such graphs. For example, as we shall immediately see, the degree-sequence of graphs is a recognizable property. To spell it out, the class of graphs having the same concrete degree-sequence (as a property) is recognizable. By the same token, according to Definition 2.1.5, the degree-sequence is a reconstructible function of (general) graphs. Definitions 2.1.5 and 2.1.6 are therefore somewhat ambiguous, and the words "reconstructible" and "recognizable" can be interchanged, depending on the context (namely, the class $\mathcal{G}$ of graphs in hand). To add even more to the confusion, we shall sometimes say that a property or some data obtained about graph G is *recoverable* if it can be recovered from $D(G)$.

2.2 A few known elementary results

In general, it is trivial that $|V(G)|$, the number of vertices of G, is recoverable from $D(G)$. It is still easy to see that $|E(G)|$ is also recoverable. Indeed, add up the numbers of edges appearing on the cards of $D(G)$, and observe that this sum is equal to

$$(|V(G)| - 2) \cdot |E(G)|.$$

See [23, Theorem 2.1] for the details of this simple combinatorial argument.

Once $|E(G)|$ is given, calculating the degree $d(v)$ of vertex v for card $G - v$ is straightforward:

$$d(v) = |E(G)| - |E(G - v)|.$$

Clearly, the degree of any vertex c-equivalent with v is the same as that of v. We thus have managed to recover the degree-sequence of G from $D(G)$. (Recall that the degree-sequence of G is the sequence of degrees of G's vertices in a non-decreasing order.)

A similar combinatorial argument leads to the following result, known as Kelly's Lemma [14], see also [23, Theorem 2.4].

Proposition 2.2.1 *For any graph Q, let $s_Q(G)$ denote the number of subgraphs of G isomorphic to Q. Then $s_Q(G) = s_Q(H)$ whenever G and H are hypomorphic and $|V(Q)| < |V(G)|$.*

Proof. There exists a hypomorphism ϕ of G onto H. Since each subgraph of G isomorphic to Q is contained in $|V(G)| - |V(Q)|$ vertex-deleted subgraphs of G, and a similar remark applies to H, and since $G - v \cong H - \phi(v)$ for every $v \in V(G)$, it follows that

$$\begin{aligned} s_Q(G) \cdot (|V(G)| - |V(Q)|) &= \sum_{v \in V(G)} s_Q(G - v) \\ &= \sum_{v \in V(G)} s_Q(H - \phi(v)) = \sum_{w \in V(H)} s_Q(H - w) \\ &= s_Q(H) \cdot (|V(H) - |V(Q)||), \end{aligned}$$

from which we infer that $s_Q(G) = s_Q(H)$, since $|V(Q)| < |V(G)| = |V(H)|$. □

The following result is also due to Kelly [14].

Theorem 2.2.2 *Regular graphs are reconstructible.*

Proof. Let G be a k-regular graph. Since the degree-sequence is recognizable (reconstructible/recoverable), all reconstructions of G are k-regular. Also, all k-regular reconstructions of G are isomorphic, because each can be obtained, up to isomorphism, from an arbitrary card $G - v$ by adding a vertex and connecting it to every vertex of degree $k - 1$ in $G - v$. □

Kelly [14] also applied his lemma (Proposition 2.2.1 above) to show that disconnected graphs and trees are reconstructible. The reader is referred to either of the surveys [4] or [23] for a proof of these results.

Nash-Williams [23] has shown that the so-called degree-sequence sequence of G is recoverable from $D(G)$. Essentially this means that, not only $d(v)$ can be read from the card $G - v$ as above, but also the degrees of the neighbors of v are recoverable in this way. We shall reformulate Nash-Williams' proof in Chapter 4 in terms of relative degree-sequences. A natural question to ask at this point is whether the degrees of the neighbors of the neighbors of v are also recoverable, and so on, moving away further and further from vertex v. This question is already a lot more difficult to answer, mainly because the desired degrees and degree-sequences are no longer c-equivalence invariant. In other words, the answer depends on the representant vertex v chosen for the card $G - v$. For card-minimal graphs, however, these data should be recoverable, even though probably very difficult to obtain.

One of the last true champions of graph reconstruction was F. Harary. He suggested a natural analogue [8] of the Reconstruction Conjecture, which says that every graph having at least four vertices is uniquely reconstructible from the deck of its *edge-deleted* subgraphs. Others have come up with similar conjectures for directed graphs, cf. [24, 26], and have obtained partial results proving or disproving them. See again [23] and [4] for more details.

Chapter 3

Card equivalence and pseudo-similarity

3.1 Characterizing card equivalence

The simple results discussed so far are of a strictly combinatorial nature, and they do not even touch on the structural properties of card equivalence. In this section we present a real structural characterization of this equivalence relation, which is our first main result. In this characterization, card equivalence is compared with another important equivalence relation on V(G), namely automorphism equivalence.

Definition 3.1.1 Two vertices $u, v \in V(G)$ are *automorphism-equivalent* (a-equivalent, for short) if there exists an automorphism of G taking u to v. The (vertex-)*orbits* of G are the equivalence classes of $V(G)$ by automorphism equivalence.

Automorphism equivalence will be denoted by $\sim_a$. In the literature, two automorphism-equivalent vertices are usually called *similar*. It is obvious that $\sim_a$ is an equivalence relation, but its relationship to $\sim_c$ is not clear for the first sight. Remember that $\sim_c$ denotes card-equivalence.

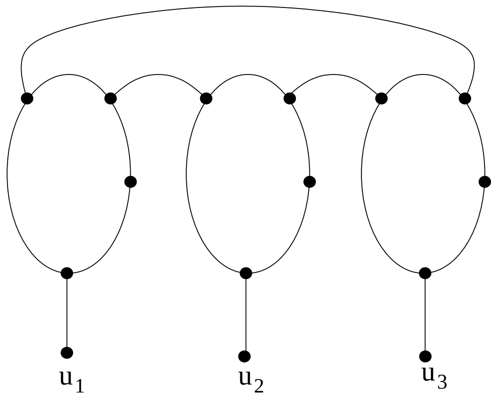

Figure 3.1: The graph of Example 3.1.2.

Example 3.1.2 Let G be the graph in Fig. 3.1, and consider the vertices u_1, u_2, u_3 in G. It is easy to see that $u_i \sim_c u_j$ and $u_i \sim_a u_j$ both hold for any $1 \leq i, j \leq 3$.

In general, it is clear by the definitions that $\sim_a \subseteq \sim_c$. Example 3.1.3 below shows, however, that $\sim_c \not\subseteq \sim_a$.

Example 3.1.3 Let G be the graph of Fig. 3.2, and consider again the vertices u_1, u_2, u_3. As it turns out, $u_1 \sim_c u_3$, but $u_1 \not\sim_a u_3$. Furthermore, G has no automorphisms other than the identity.

Vertices that are c-equivalent but not a-equivalent are called *pseudo-similar* in the literature. See [20] for an extensive survey on pseudo-similarity. Vertices u_1 and u_3 in the graph G of Fig. 3.2 are typical pseudo-similar ones. The graph G itself arises from deleting a leaf vertex u_4 from an appropriate graph H — shown in Fig. 3.3 — in which u_4 is the "natural continuation" of the sequence of vertices u_1, u_2, u_3. The graph H has a non-trivial automorphism θ, which extends an appropriate automorphism ψ of the "kernel" subgraph $G - \{u_1, u_2, u_3\}$ in such a way that the vertices u_i, $1 \leq i \leq 4$, are mapped by θ into each other in the cyclic order

$$u_i \mapsto u_{(i\,(mod\,4))+1}.$$

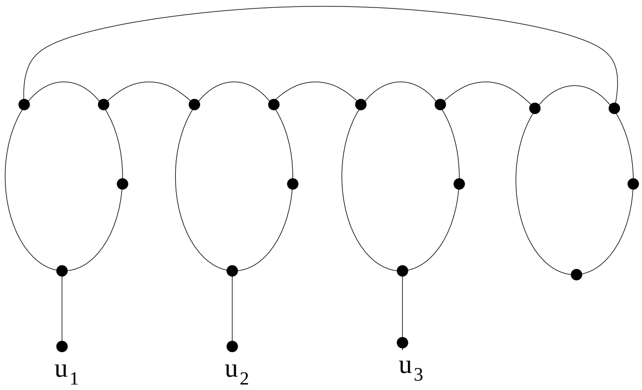

Figure 3.2: The graph of Example 3.1.3.

The vertices u_1 and u_3 must of course be at the two ends of the "tail" sequence u_1, u_2, u_3 of G in order to maintain card equivalence.

The following example is somewhat different, yet it illustrates the same kernel-tail decomposition idea with the kernel subgraph being disconnected.

Example 3.1.4 Consider the graph G of Fig. 3.4. Clearly, the vertices u_1 and u_2 are pseudo-similar. At the first glance it appears that one cannot extend G to a graph H having an automorphism that builds on an appropriate automorphism of the kernel $G - \{u_1, u_2\}$ in such a way that u_1 is mapped to u_2. A look at Fig. 3.5, however, shows immediately that this is possible, and the solution follows the exact same pattern as in Example 3.1.3.

The above examples show that the equivalence $\sim_c$ is rather inconvenient to deal with in a direct way. We need to find a characterization of $\sim_c$ that brings it in line with the much better structured equivalence $\sim_a$. The basis of this characterization is the following lemma.

Lemma 3.1.5 *Let u and v be two distinct vertices of G. Then $u \sim_c v$ iff there exists a sequence of vertices $x_0, x_1, \ldots, x_k$ $(k \geq 1)$ in G satisfying the conditions (i) and (ii) below.*

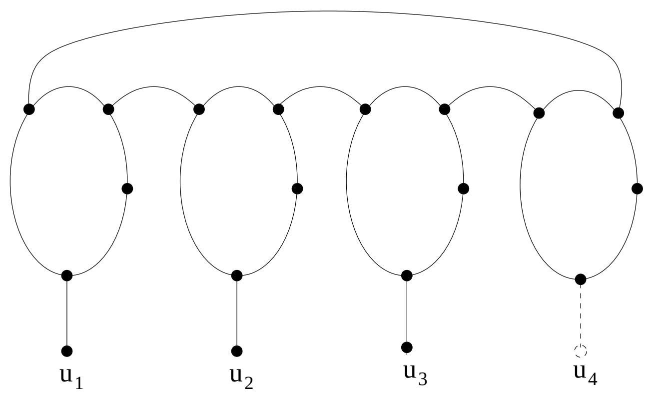

Figure 3.3: The graph H of Example 3.1.3.

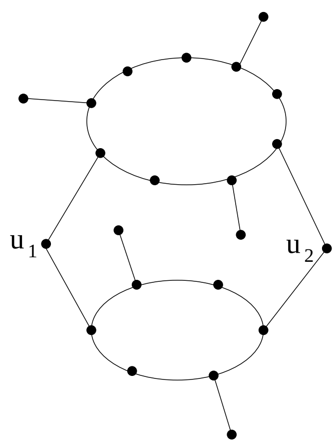

Figure 3.4: The graph of Example 3.1.4.

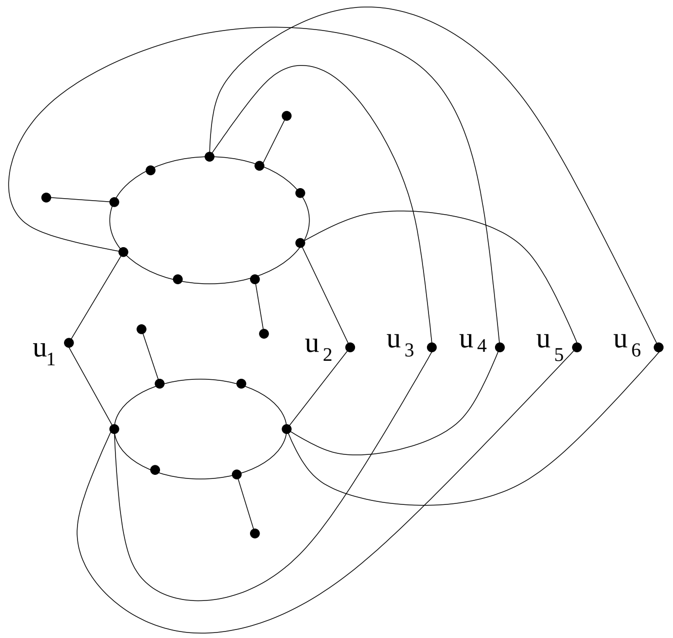

Figure 3.5: The graph of Example 3.1.4 extended.

(i) $x_0 = v$ *and* $x_k = u$*;*

(ii) there exists an isomorphism ϕ *of* $G-u$ *onto* $G-v$ *such that* $\phi(x_i) = x_{i+1}$ *for every* $0 \leq i < k$*.*

Proof. Notice first that the graphs $G-u$ and $G-v$ are not separated in the lemma, they both use the vertices of the common supergraph G. The lemma therefore establishes a link between two c-equivalent vertices u and v in G through a sequence of (necessarily distinct) vertices $x_1, \ldots, x_{n-1}$ in $G-u-v$. These vertices, however, need not be c-equivalent with u or each other in G. For example, in the graph of Fig. 3.2, if $v = u_1$ and $u = u_3$, then $k = 2$, $x_1 = u_2$, and ϕ can be derived from the automorphism of $G - \{u_1, u_2, u_3\}$ that determines a cyclic permutation of the four small cycles of G from left to right. Clearly, $u_1 \not\sim_c u_2$.

Sufficiency of condition (ii) alone for having $u \sim_c v$ is trivial. Assuming that $u \sim_c v$, choose an arbitrary isomorphism $\phi : G-u \to G-v$. Let $x_1 = \phi(v)$, $x_2 = \phi(x_1)$, and so on, until $u = x_k = \phi(x_{k-1})$ is reached. Vertex u must indeed be encountered at some point along this line, since ϕ, being an isomorphism, is an injective mapping $V(G) \setminus \{u\} \to V(G) \setminus \{v\}$. Consequently, the vertices $x_1, \ldots, x_{k-1}$ in $V(G) \setminus \{u, v\}$ will all be different until $x_k = u$ stops this necessarily finite sequence. Mind that $x_{i+1} = \phi(x_i) \neq v$, since v is not a vertex of $G - v$. Therefore the sequence $x_0, \ldots, x_k$ cannot return to itself. The proof is complete. □

Theorem 3.1.6 *Let* u *and* v *be two distinct vertices of* G*. Then* $u \sim_c v$ *iff there exists a sequence of pairwise distinct vertices* $x_0, x_1, \ldots, x_k$ *(*$k \geq 1$*) and an automorphism* ψ *of the subgraph* $G - \{x_0, x_1, \ldots, x_k\}$ *which satisfy the following three conditions.*

(i) $x_0 = v$ *and* $x_k = u$*;*

(ii) if $X = \{x_0, x_1, \ldots, x_k\}$*, then for every* $0 \leq i < k$ *and vertex* $w_i \in V(G) \setminus X$ *adjacent to* x_i *in* G *(or, equivalently, in* $G - u$*), the vertex*

$w_{i+1} = \psi(w_i)$ is adjacent to x_{i+1} in G (i.e., in $G - v$);

(iii) for every $0 \le i < j < k$,

$$x_i \text{ is adjacent to } x_j \text{ iff } x_{i+1} \text{ is adjacent to } x_{j+1}$$

(in G, of course).

Vertices u and v are a-equivalent iff the sequence of vertices $x_0, \dots, x_k$ and the automorphism ψ can be chosen in such a way that the assignments $x_i \mapsto x_{i+1}$, $u \mapsto v$ extend ψ to an automorphism of G.

Proof. Intuitively, condition (ii) says that for every $0 \le i < k$, the neighbors of x_i in $G - X$ are matched up with those of x_{i+1} in $G - X$ by the automorphism ψ. Condition (iii) settles the issue of how the vertices x_i themselves are connected in G. Clearly, the question whether u is connected to v is irrelevant.

The first statement of the theorem, regarding the existence of X and ψ, is in fact a simple consequence of Lemma 3.1.5. Concerning sufficiency, if ψ is an automorphism of $G - X$ satisfying (ii), then by (i) and (iii) it can be extended to an isomorphism ϕ of $G - u$ onto $G - v$ satisfying (ii) of Lemma 3.1.5. Thus, $u \sim_c v$. Conversely, if $u \sim_c v$, then the required automorphism ψ can be chosen as the restriction of the isomorphism ϕ – guaranteed by Lemma 3.1.5 – to $G - X$. Notice that the subgraph $G - X$ may turn out to be empty.

As to the second statement of the theorem (regarding the a-equivalence of u and v), if the given extension of ψ becomes an automorphism of G, then clearly $u \sim_a v$. On the other hand, if χ is an automorphism of G taking u to v, then the restriction of χ to the vertices $V(G) - \{u\}$ defines an isomorphism ϕ of $G - u$ onto $G - v$. Apply Lemma 3.1.5 to obtain the vertices X from ϕ, and construct the automorphism ψ of $G - X$ by restricting ϕ to that subgraph. Clearly, the assignments $x_i \mapsto x_{i+1}$ and $u \mapsto v$, when extending ψ, will simply reconstruct the original automorphism χ. □

At this point the reader may want to have a second look at Examples 3.1.2 and 3.1.3, and identify the underlying automorphism ψ in the graphs of Fig. 3.1 and Fig. 3.2. One important point is that, given the fact $v \sim_a u$ (and therefore $u \sim_c v$), one must not jump to the conclusion that $x_0 = v$ and $x_1 = u$ will do for $X = \{x_0, x_1\}$ in Theorem 3.1.6, and then be taken by surprise that the desired automorphism ψ cannot be located in $G - X$. For example, in the graph G of Fig. 3.1, if $v = u_1$ and $u = u_2$, then $x_1 = u_3$! Consequently, $X = \{u_1, u_2, u_3\}$, and the automorphism ψ is just the one taking the three small cycles into one another following a cyclic permutation with offset 2 from left to right.

The pair (s, ψ), where s is the sequence $(x_0, \ldots x_k)$ constructed in Lemma 3.1.5 and ψ is the automorphism of the graph $G - X$ according to Theorem 3.1.6, is called a *kernel-tail decomposition* of G with respect to the c-equivalent pair of vertices (u, v). The sequence s is the *tail* and the subgraph $G - X$ is the *kernel* of this decomposition. Clearly, the decomposition is not unique regarding the sequence s. Moreover, even with s being fixed, the corresponding automorphism ψ of the kernel $G - X$ may not be uniquely determined.

Following [20], for the rest of this chapter we shall concentrate on the concept of pseudo-similarity in graphs. The most general construction from which all pairs of pseudo-similar vertices can be obtained was found by Godsil and Kocay [7], who showed that such pairs can in fact always be captured by destroying some circular symmetry in a larger graph. We have already elaborated on this idea to some extent in Examples 3.1.3 and 3.1.4. Below we give a proof of [7, Theorem 2.2] as a corollary to our Theorem 3.1.6. The reader is advised to follow the main steps of the proof on Figures 3.3 and 3.4.

Corollary 3.1.7 ([7]) *Let G be a graph with pseudo-similar vertices u and v. Then there is a graph H with the following properties:*

(i) G is an induced subgraph of H.

(ii) There exists an automorphism θ of H which maps $G-u$ onto $G-v$ in such a way that $u=\theta^k(v)$ for some $k\geq 1$.

(iii) $V(H)\setminus V(G)=\{y_1,\ldots,y_r\}$, where $y_i=\theta^{k+i}(v)$ and $\theta(y_r)=v$.

Proof. By Theorem 3.1.6, G has a kernel-tail decomposition (s,ψ) with respect to (u,v), where $s=(x_0,\ldots,x_k)$, $k\geq 1$ is the tail with $v=x_0$ and $u=x_k$, and ψ is an appropriate automorphism of the kernel $G-X$ ($X=\{x_0,\ldots,x_k\}$). Let M denote the set of neighbors of v belonging to the kernel. Even though it is an aside at this point, observe that M cannot be empty. Indeed, if $M=\emptyset$, then Theorem 3.1.6 (iii) would imply that that the vertices x_i and x_{k-i} are exchange-equivalent for each $0\leq i\leq k$, contradicting the fact that $x_0=v$ and $x_k=u$ are pseudo-similar. For the same reason, it cannot happen either that M is a singleton and $\psi(M)=M$. Nevertheless, the construction we are going to present will extend G to an appropriate H in these two special cases as well. As we shall see, the chiral symmetry of the vertices x_i will simply be turned into a circular one by adding an extra vertex x_{k+1}.

Consider the sets of vertices $M,\psi(M),\psi^2(M),\ldots$, where by definition, $\psi(M)=\{\psi(w)|w\in M\}$. By Theorem 3.1.6 (ii) we know that these sets of vertices, up to power k, are exactly the kernel-neighbors of $x_0,\ldots,x_k$. Since $G-X$ is finite and ψ is an automorphism, there exists a smallest integer $j\geq 1$ such that $\psi^j(M)=M$ and the sets $\psi^i(M)$, $1\leq i\leq j$ are pairwise distinct. Consequently, there exists a smallest $l>k$ such that $\psi^{l+1}(M)=M$. Notice that the choice $l>k$ is crucial. Choosing $l=k$ – if at all possible – would allow G to be extended to a seemingly appropriate graph H by adding edges connecting some of $x_0,\ldots,x_k$ only, and so missing the requirement that G be an induced subgraph of the extension. For example, in the graph of Fig. 3.1, connect u_1 with u_2 and u_2 with u_3 to obtain graph G. Then u_1 and u_3 are no longer a-equivalent, but they are still pseudo-similar. If $v=u_1$ and $u=u_3$, then the number j in the argument above is clearly 3. Simply connecting u_3 with u_1 by a new edge in this situation would result in a graph

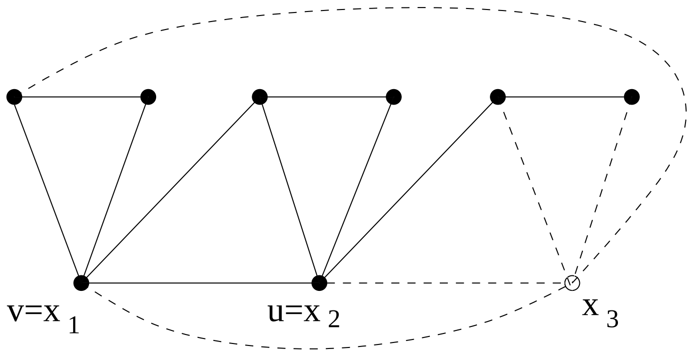

Figure 3.6: Extending graph G to H, a small example.

H satisfying the requirements of the theorem, except for the condition that G be an induced subgraph of H.

Let $r = l - k$, and adjoin to $V(G)$ r new vertices $y_1, \ldots, y_r$. For each $1 \leq i \leq r$, install a new edge from y_i to each vertex of $\psi^{k+i}(M)$. See Fig. 3.5. Finally, by letting $x_{k+1} = y_1, \ldots x_{k+r} = y_r$ in the sequence of vertices

$$x_0, x_1, \ldots, x_k, x_{k+1}, \ldots x_{k+r},$$

add new edges between x_i and x_j recursively to the minimum extent in order to maintain the condition (iii) of Theorem 3.1.6 in a circular fashion. That is, the condition:

$$x_i \text{ is adjacent to } x_j \text{ iff } x_{i+1\,(mod\,(k+r))} \text{ is adjacent to } x_{j+1\,(mod\,(k+r))}$$

must hold for each $0 \leq i < j \leq k + r$. The details of this recursive procedure are straightforward, and left to the reader. See Fig. 3.6 for a small example.

The resulting graph after the above extensions is H. It is obvious that G is a proper induced subgraph of H, and H satisfies the conditions (i) and (ii) of the corollary with the extension θ of ψ by which the vertices $x_0, x_1, \ldots, x_{k+r}$ are mapped to each other in the way:

$$x_i \mapsto x_{i+1\,(mod\,(k+r))},\ 1 \leq i \leq k + r. \qquad \square$$

A reader familiar with the original proof of [7, Theorem 2.2] by Godsil and Kocay will notice that their proof (for a finite graph G) is essentially the combination of our Theorem 3.1.6 and Corollary 3.1.7, just as we presented them above. For this reason, Theorem 3.1.6 cannot be considered a completely new contribution. There is, however, a major difference in the nature of these two results. While our Theorem 3.1.6 is primarily a decomposition tool, [7, Theorem 2.2] tends to blow up the graph, sometimes quite significantly. Refer to Fig. 3.5 for an evidence of this fact. The importance of the kernel-tail decomposition lies in the fact that it could be used as a lemma in inductive reasonings about the structure of symmetries in G.

One disturbing observation arising from Theorem 3.1.6 is that it does not reflect the transitivity of card equivalence. Indeed, if (s, ψ) and (t, χ) are kernel-tail decompositions of G with respect to c-equivalent pairs (u, v) and (v, w), respectively, then the theorem provides no clue for finding a kernel-tail decomposition with respect to (u, w). Working out such a method will be the subject of a future study.

3.2 Mutually pseudo-similar sets of vertices

According to [20], the most interesting open questions on pseudo-similarity are related to the issue of finding large sets of mutually pseudo-similar vertices in graphs. Lauri does not explain in his survey why one must say "mutually pseudo-similar" in this context. The reason is that pseudo-similarity by itself is not an equivalence relation on $V(G)$ as one might suspect, and it is not true that card equivalence is simply the union of automorphism equivalence and pseudo-similarity. Mutually pseudo-similar therefore means pairwise pseudo-similar in standard mathematical terms.

The problem of finding graphs with more than two mutually pseudo-similar vertices has been investigated by a number of authors [6, 15, 16, 17, 18, 19, 21, 22]. Among these, the construction of [16] is particularly simple to describe in terms of our kernel-tail decomposition. The basic idea of the construction is the observation that coming up with a number of mutually pseudo-similar vertices is much easier in directed graphs. Notice that the concept of card equivalence and that of of pseudo-similarity is meaningful in directed graphs, literally by the original definition of these concepts.

The transitive tournament T_k on k vertices $\{1, \ldots k\}$, in which vertex i dominates vertex j iff $i < j$, is a perfect example of a directed graph having all of its vertices mutually pseudo-symmetric. The problem of finding an undirected graph with k mutually pseudo-symmetric vertices therefore reduces to transforming (i.e., blowing up) T_k into an undirected graph while preserving the pseudo-similarity of the old vertices $\{1, \ldots, k\}$. We shall work out the solution for the simplest case $k = 3$ below.

Take an arbitrary graph H with two pseudo-similar vertices u, v, so that H has a kernel-tail decomposition (s, ψ) with $s = (v, u)$. The graph H with its distinguished ordered pair (v, u) of vertices can be considered as a directed (meta-)edge in any graph, connecting two (not necessarily distinct) vertices x and y, so that the source x of the meta-edge is identified with v and the target y is identified with u. If $x = y$, then vertices u and v will be joined in H, which might lead to a multigraph. The exact mathematical formalism that corresponds to this technique is the general operation of graph composition described e.g. in [2, 5]. For our present purposes, however, the heuristic idea of creating a meta-edge from H is completely adequate.

Take, for example, the graph of Fig. 3.7 as H, and assemble the transitive triangle T_3 from three "H-edges" as shown in Fig. 3.8. The "direction" of each edge in the resulting graph G is $v \to u$. It is now obvious that the vertices x, y, z corresponding to the three vertices of T_3 are pseudo-similar. As proved in [16], this technique works for an arbitrary $k \geq 3$. One could as well use Theorem 3.1.6 to provide an independent proof of this fact. It is essential, however, that the tail s of the decomposition (s, q) of H has length two, otherwise the construction does not work.

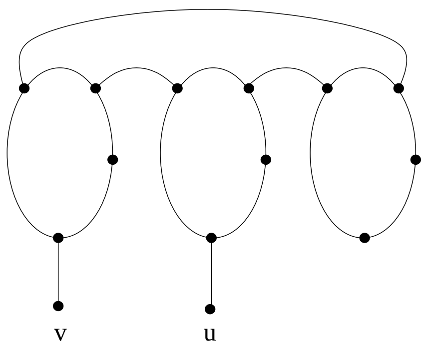

Figure 3.7: The graph H used as a meta-edge.

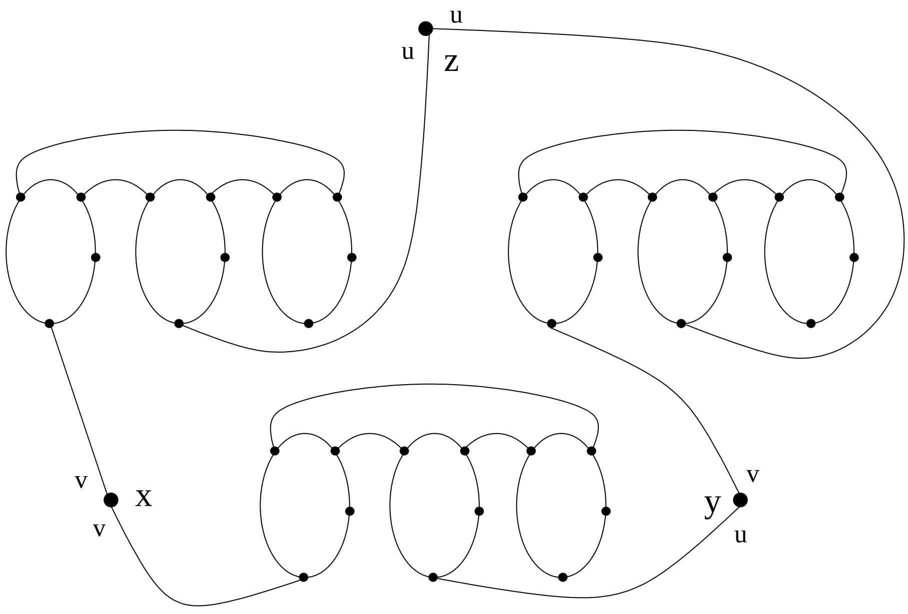

Figure 3.8: The triangle T_3 built up from three H-edges.

As a generalization of Corollary 3.1.7, Kocay [17] has obtained the following result on mutually pseudo-similar vertices.

Theorem 3.2.1 ([17]) *Let G be a graph with a set $U = \{u_0, u_1, \ldots u_{k-1}\}$ of k mutually pseudo-similar vertices. Let $\phi_i : G - u_i \to G - u_0$ be isomorphisms for $1 \leq i \leq k-1$. Then G can be extended to a graph H, and each ϕ_i to an endomorphism θ_i of H such that:*

(i) G is an induced subgraph of H;

(ii) the vertices of U, together with those in $V(H) \setminus V(G)$, are in the same orbit in H.

Theorem 3.2.1 is an analogue of Corollary 3.1.7 with one important difference: the graph H in Theorem 3.2.1 is *infinite*, even when G is finite. The full analogue of Corollary 3.1.7, as well as that of our Theorem 3.1.6 regarding the kernel-tail decomposition of a graph relative to a complete c-equivalence class of vertices is still unknown.

We end this section by a definition that will be used in Chapter 5. A *2-vertex-deleted* subgraph of a graph G having at least three vertices is an induced subgraph of G containing all but two of its vertices. On the analogy of cards, a 2-vertex-deleted subgraph of G is called a *2-card*, and the multiset of G's 2-cards is called the *2-deck* of G. As in the case of cards, we do not distinguish between isomorphic 2-cards. Clearly, the 2-deck of G is computable from $D(G)$. Indeed, one must include each vertex-deleted subgraph of every card in $D(G)$, and correct by observing that each 2-card has thus been counted twice.

Definition 3.2.2 Let $\{u, v\}$ and $\{x, y\}$ be two couples of vertices in $V(G)$. These two couples are *2-card-equivalent* if they generate the same 2-card, that is, $G - u - v \cong G - x - y$. The couples $\{u, v\}$ and $\{x, y\}$ are *pseudo-similar* if they are 2-card-equivalent, but none of u, v is similar (i.e., a-equivalent) to any of x, y.

Observe that, by definition, if $\{u, v\}$ and $\{x, y\}$ are pseudo-similar, then the vertices u, v, x, y must be pairwise distinct.

The creation of the 2-deck of G raises the following natural question: is it possible to mark each 2-card $G - u - v$ of G with the cards of the vertices u and v? The question is not ambiguous, since the correspondence between the multiset of 2-cards and pairs of vertices $\{u, v\}$ in G is a bijection. It may happen, though, that a given pair of (not necessarily distinct) 1-cards will mark several distinct 2-cards, and different instances of the same 2-card will be marked by different pairs of 1-cards. As we shall see in Chapter 4, answering the above question is very difficult, almost as hard as proving the Reconstruction Conjecture itself.

Chapter 4

Relative degree-sequences

4.1 A few combinatorial observations

Recall from Chapter 1 that the degree-sequence of graph G is the sequence of degrees of its vertices in a non-decreasing order. Let Q be a subgraph of G. The degree of a vertex $v \in V(Q)$ relative to G is a pair (r, d), where d (r) is the degree of v in G (respectively, Q). We shall use the notation r^d for the pair (r, d), and say that v has relative degree r out of d.

Definition 4.1.1 The *relative degree-sequence* of subgraph Q (with respect to G) is the sequence of relative degrees r^d of its vertices in an order that is non-decreasing regarding the superscripts d and also non-decreasing in r among those degrees that have the same superscript d.

The degree-sequence of G and the relative degree-sequence of Q with respect to G will be denoted by $ds(G)$ and $rds_G(Q)$, respectively. In order to ensure that $ds(G)$ and $rds_G(Q)$ have the same length, we shall include a relative "degree" $\emptyset^d$ in $rds_G(Q)$ for each vertex $v \in V(G) \setminus V(Q)$ with degree d. The "number" $\emptyset$ is treated as 0, but the notation $\emptyset$ will distinguish between a vertex that has been deleted and one that is still present but isolated. This distinction is purely technical, however, because one can easily fill in the $\emptyset^d$ relative degrees in $rds_G(Q)$ once $ds(G)$ is known.

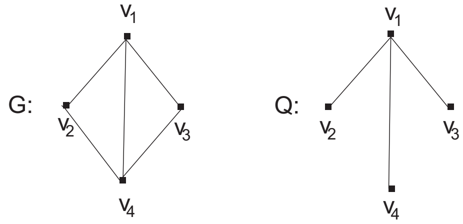

Figure 4.1: Graph G and its subgraph Q.

Example 4.1.2 Consider the graph G and its subgraph Q in Fig. 4.1. The degree-sequence of G is $2, 2, 3, 3$, while the relative degree sequence of Q with respect to G is $1^2, 1^2, 1^3, 3^3$.

The following simple combinatorial observation is equivalent to Nash-Williams' result [23, Corollary 3.5] on degree-sequence sequences, also dealt with in Chapter 1.

Proposition 4.1.3 *For every vertex $v \in V(G)$, $rds_G(G - v)$ is recoverable from $D(G)$.*

Proof. We have seen in Chapter 1 that $d(v)$ and $ds(G)$ are recoverable from $D(G)$. Write the sequence $ds(G - v)$ underneath $ds(G)$ by inserting the "degree" $\emptyset$ in $ds(G - v)$ right under the position of the first occurrence of $d(v)$ in $ds(G)$. For example:

$$\begin{array}{llllllll} ds(G): & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ ds(G-v)): & 1 & 1 & 2 & 2 & 3 & \emptyset & 3 \\ rds_G(G-v): & 1^2 & 1^2 & 2^2 & 2^3 & 3^3 & \emptyset^4 & 3^4 \end{array}$$

Observe that the "true" degrees in $ds(G - v)$ will lag behind those in $ds(G)$, so that the difference between two degrees in aligned positions is at most 1. Therefore it is trivial to fill out the missing superscripts in $ds(G - v)$, so that the resulting sequence becomes $rds_G(G - v)$. □

Proposition 4.1.3 basically says that, for every card $G - v$, the degrees of the vertices adjacent to v in G are uniquely determined by $ds(G)$ and

$ds(G-v)$. Indeed, these are exactly the degrees $r+1$ appearing in $rds_G(G-v)$ as r^{r+1}. Of course, we still have no information about the actual position of v's neighbors in $G-v$.

We immediately generalize Proposition 4.1.3 to find out the relative degree-sequence of all 2-vertex-deleted subgraphs of G. Notice that, for two distinct vertices $u, v \in V(G)$, the subgraph $G-u-v$ is no longer determined by the cards $G-u$ and $G-v$ in a unique way, since the cards themselves do not uniquely identify the vertices u, v. Moreover, the subgraph $G-u-v$, too, can be isomorphic to other subgraphs $G-u'-v'$ in which u' and v' are associated with some different cards.

Theorem 4.1.4 *Let u and v be two distinct vertices of G. Given the degree-sequence of the subgraph $G-u-v$, $rds_G(G-u-v)$ is uniquely determined by the data $ds(G)$, $ds(G-u)$, and $ds(G-v)$. Moreover, the question whether u and v are adjacent in G or not turns out from the data $ds(G)$, $ds(G-u-v)$, $d(u)$ and $d(v)$.*

Proof. We use the same alignment argument as in the proof of Proposition 4.1.3. Write the degree-sequences $ds(G)$, $ds(G-u)$, and $ds(G-v)$ under each other, inserting the $\emptyset$ symbol in the appropriate positions of $ds(G-u)$ and $ds(G-v)$. Furthermore, insert two $\emptyset$'s in $ds(G-u-v)$ aligned with the ones already inserted in $ds(G-u)$ and $ds(G-v)$. If $d(u) = d(v) = d$, then insert two consecutive $\emptyset$'s aligned with the beginning of the block marked by degree d in $ds(G)$. For example:

$$\begin{array}{lccc|cc|cccccc}
ds(G): & 2 & 2 & 2 & 3 & 3 & 4 & 4 & \dots & \dots & \dots & \dots \\
ds(G-u): & 1 & 1 & 2 & 2 & 3 & 4 & 4 & \dots & \emptyset & \dots & \dots \\
ds(G-v): & 1 & 2 & 2 & 2 & 2 & 3 & 4 & \dots & \dots & \emptyset & \dots \\
ds(G-u-v): & 0 & 1 & 1 & 2 & 2 & 3 & 4 & \dots & \emptyset & \emptyset & \dots \\
 & & & \rightarrow & \leftarrow & & & & & & & \\
rds_G(G-u-v): & 0^2 & 1^2 & 2^2 & 1^3 & 2^3 & 3^4 & 4^4 & \dots & \emptyset & \emptyset & \dots
\end{array}$$

Let $n_G(d)$ ($n_{G,Q}(r^d)$) denote the number of occurrences of d (r^d) in $ds(G)$ (respectively, $rds_G(Q)$). Assume, for simplicity, that the smallest degree in

G is $d_0 \geq 2$. Then, clearly:

$$n_{G,Q}((d_0-2)^{d_0}) = n_Q(d_0-2).$$

It follows that:

$$n_{G,Q}((d_0-1)^{d_0}) = n_{G-u}(d_0-1) + n_{G-v}(d_0-1) - 2 \cdot n_Q(d_0-2), \quad \text{and}$$

$$n_{G,Q}(d_0^{d_0}) = n_G(d_0) - n_{G,Q}((d_0-2)^{d_0}) - n_{G,Q}((d_0-1)^{d_0}),$$

provided that neither of the degrees $d(u)$ and $d(v)$ equals d_0. If either or both does, then the above calculation changes in a straightforward way regarding the numbers $n_{G,Q}((d_0-1)^{d_0})$ and $n_{G,Q}(d_0^{d_0})$. Observe that some adjacent degrees $d-1$ and d ($d \geq 1$) in $ds(G-u-v)$, showing up as relative degrees $(d-1)^{d+1}$ and d^d in $rds_G(G-u-v)$, must be interchanged in the latter sequence, since the ascending order with respect to the superscripts has priority over that of the actual subgraph degrees. See the example above regarding the relative degrees 1^3 and 2^2.

One can then carry on in the same way, calculating the numbers $n_{G,Q}((d_0-1)^{d_0+1})$, $n_{G,Q}(d_0^{d_0+1})$, $n_{G,Q}((d_0+1)^{d_0+1})$, and so on. Details are left to the reader.

As to the second statement of the theorem, if

$$|E(G)| - |E(G-u-v)| = d(u) + d(v),$$

then u and v are not connected in G, otherwise they are. The numbers $|E(G)|$ and $|E(G-u-v)|$ are determined by $ds(G)$ and $ds(G-u-v)$, respectively. The proof is complete. $\square$

4.2 The relative degree-sequence conjecture

Proposition 4.1.3 and Theorem 4.1.4 show that the concept of relative degree-sequence is rather fundamental in the study of graph reconstruction. To provide yet another evidence of this observation, let $Rds(G)$ denote the multiset

$$\{rds_G(Q) | Q \text{ is an induced subgraph of } G\}.$$

Thus, relative degree-sequences of subgraphs count with multiplicity in $Rds(G)$. We put forward the following conjecture, which is very closely related to the Reconstruction Conjecture.

Conjecture 4.2.1 *For every graph G, $Rds(G)$ identifies G up to isomorphism.*

Conjecture 4.2.1 is especially interesting for several reasons.

1. It appears to hold for all graphs with no exceptions.
2. It provides a characterization of graph isomorphism, which has been sought for a very long time.
3. Algebraically, if $G = G_1 + G_2$, where $+$ denotes disjoint union of graphs, then
$$Rds(G) = Rds(G_1) \times Rds(G_2). \tag{4.2.1}$$

In equation 4.2.1 above, $\times$ stands for concatenation of sets of relative degree-sequences in the formal language sense (taking the quotient of the product by commutativity). In terms of polynomials, one can think of a relative degree r^d as a formal variable. Let X denote the set of all such variables. Then $Rds(G)$ becomes a polynomial P_G of the variables X over the integer ring $\mathbf{Z}$, in which all coefficients are non-negative. Indeed, the coefficient of a term $x_1^{p_1} \dots x_k^{p_k}$ in P_G is the number of times the relative degree-sequence $x_1^{p_1} \dots x_k^{p_k}$, that is, the sequence:

$$(x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_k, \dots, x_k),$$

comes up in $Rds(G)$. The polynomial P_G itself is simply the sum of these terms for all relative degree-sequences. Notice that the empty graph translates into the polynomial 1, that is, the empty sequence (product) of variables, and *not* to 0. The polynomial 0 is not the image of any graph at all.

Let $\mathbf{Z}[X]$ denote the commutative $\mathbf{Z}$-module (in fact algebra) of X-polynomials over $\mathbf{Z}$. (Mind that addition of polynomials is commutative in $\mathbf{Z}[X]$.) Our fundamental observation is that the operation $\times$ in equation 4.2.1 translates naturally into product of polynomials in the algebra $\mathbf{Z}[X]$. This product makes the algebra $\mathbf{Z}[X]$ associative and commutative, therefore a commutative ring. In this way we could establish an embedding of the additive commutative monoid structure of graphs with disjoint union into the multiplicative structure of the commutative ring $\mathbf{Z}[X]$, which would be a very strong result, indeed.

Conjecture 4.2.1 was the starting point of the present study, and the observation in the previous paragraph served as a motivation for it. Even more generally, our ambitious goal is to find an embedding of the free traced monoidal category [12] of flowchart schemes, which is practically a graph structure with the tensor operation being disjoint union of graphs, into the compact closed category of free modules over the commutative ring $\mathbf{Z}[X]$, in which tensor and trace are the standard matrix operations. Conjecture 4.2.1 is the key to this very general result. We do not wish to elaborate on algebraic and category theoretic issues in the present study, however, and the arguments above are included for the sake of revealing the real motives in formulating Conjecture 4.2.1 only. As it turned out very soon, the key to Conjecture 4.2.1 is in fact the Reconstruction Conjecture, therefore the focus of research has changed from algebra to combinatorics.

Naturally enough, Conjecture 4.2.1 also has an "edge" version, in which $Rds(G)$ is defined as the set of relative degree-sequences of *all* subgraphs of G. This version, too, appears to hold for all graphs G with no exceptions.

The connection between Conjecture 4.2.1 and the Reconstruction Conjecture is the following. If one could compute $Rds(G)$ from $D(G)$, then Conjecture 4.2.1 would imply the Reconstruction Conjecture. As our second main result in Chapter 5 shows, however, computing the whole multiset $Rds(G)$ appears to be far too much work in order to reconstruct G. Therefore this reconstruction argument probably does not hold much water, indicating that Conjecture 4.2.1 is even tougher than the Reconstruction Conjecture.

On the other hand, if, given $Rds(G)$, one could isolate $Rds(G-v)$ for each vertex-deleted subgraph of G, then the Reconstruction Conjecture would imply Conjecture 4.2.1 through a straightforward induction argument. Since our concern is eventually Conjecture 4.2.1, and the construction of the multiset of multisets

$$\{Rds(G-v)|v \in V(G)\}$$

from $Rds(G)$ looks promising, we shall try to prove the Reconstruction Conjecture first.

Chapter 5

The reconstruction of card-minimal graphs

In this chapter we present our second main result, which aims at the reconstruction of card-minimal graphs not containing pseudo-similar couples of vertices.

5.1 The role of pseudo-similar couples of vertices in card-minimal graphs

Recall from Chapter 2 that graph G is card-minimal if $D(G)$ is a set of $|V(G)|$ different cards. According to Definition 3.2.2, two couples of vertices $\{u, v\}$ and $\{x, y\}$ in a card-minimal graph G are pseudo-similar iff:

$$G - u - v \cong G - x - y,$$

and u, x, y, z are pairwise distinct. Indeed, given the fact that there are no distinct a-equivalent vertices in a card-minimal graph (not even pseudo-similar ones), the condition that neither of u, v is a-equivalent to any of x, y is equivalent to the condition that these vertices are pairwise distinct.

To shed some light on the intuition behind pseudo-similarity between couples of vertices in a card-minimal graph G, let Q be an arbitrary graph

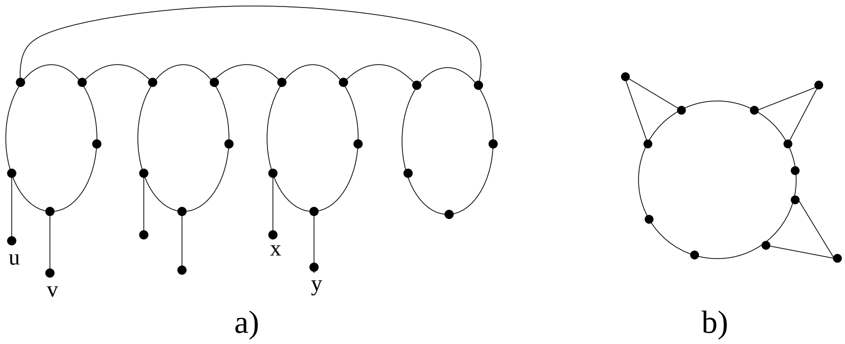

Figure 5.1: Graphs with and without pseudo-similar couples of vertices.

having $|V(G)| - 2$ vertices. Consider the set C of cards in $D(G)$ in which Q is isomorphic to at least one vertex-deleted subgraph. Construct the graph G_Q which has C as its set of vertices, and any two cards $G - u$, $G - v$ are connected in G_Q iff $G - u - v \cong Q$. (Remember that G is card-minimal, therefore the definition of G_Q is correct.) Then G is free from pseudo-similar couples of vertices iff G_Q is either a triangle or a star graph for every 2-card Q of G. In other words, either $|C| = 2$ and G_Q is a single edge, or $|C| > 2$ and the following two conditions are met:

1. the subgraph Q occurs $k \geq 2$ times as a vertex-deleted subgraph in some card $G - u \in C$;
2. $|C| = k + 1$ and the cards in C different from $G - u$ all have a single occurrence of Q in them, with the possible exception that $k = 2$ and all the three cards in C have two occurrences of Q in them.

See Fig. 5.1a for a card-minimal graph G which does, and Fig. 5.1b for one which does not contain pseudo-similar couples of vertices. The smallest card-minimal graph of Fig. 2.2 does have a pair of pseudo-similar couples, as shown by Fig. 5.2

It is clear by the above characterization that the property of not having pseudo-similar couples of vertices is recognizable for card-minimal graphs,

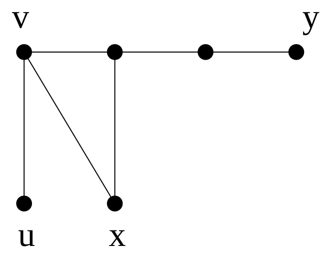

Figure 5.2: The smallest card-minimal graph, revisited.

i.e., the property is decidable by looking at the deck only. (Of course, the property of being card-minimal is immediately recognizable.) Indeed, for each 2-card Q of G, one must construct the graph G_Q and check if it is a triangle or a star graph. If so, then G is free from pseudo-similar couples of vertices, otherwise it is not. Remember from Chapter 3 that the 2-deck of G is constructible from $D(G)$.

5.2 The reconstruction result

In this section we present our second main result on the reconstruction of card-minimal graphs not containing pseudo-similar couples of vertices.

Theorem 5.2.1 *Every card-minimal graph G free from pseudo-similar couples of vertices is reconstructible.*

Proof. Let Q be an arbitrary 2-card of G, and find the set C of cards in which Q is isomorphic to at least one vertex-deleted subgraph. Clearly, C has at least two elements. If there are exactly two cards $G - u$ and $G - v$ in C, then conclude that $Q \cong G - u - v$, and use Theorem 4.1.4 to decide if u and v are adjacent in G or not. If C has more than two elements, then two cases are possible.
Case A: $|C| = 3$, and each card in C has two subgraphs isomorphic to Q.
Case B: there is exactly one card $G - u \in C$ that contains more than one subgraph isomorphic to Q.

In case A, $Q \cong G - u - v$ for any pair $G - u$, $G - v$ of distinct cards in C, while in case B, $Q \cong G - u - v$ for all vertices $v \neq u$ such that $G - v \in C$. Furthermore, in case B, Q is not isomorphic to any other 2-card of G. (In other words, $Q \ncong G - u_1 - u_2$, where $G - u_1$ and $G - u_2$ are both in C but $u_i \neq u$ for either $i = 1$ or 2.) In both cases, use Theorem 4.1.4 to find out if u is adjacent to v in G, knowing that $Q \cong G - u - v$. It is evident that the above procedure will decide for each pair of cards $G - u$, $G - v$ in $D(G)$ if the vertices u and v are adjacent in G or not. Graph G is thus reconstructed, and the proof is complete. $\square$

Chapter 6

Conclusion

Motivated by an independent study in algebra and category theory, we have presented a structural analysis of graphs with the aim of being able to reconstruct them from some partial information. The basis of the reconstruction of graph G could either be the classical multiset of G's vertex-deleted subgraphs, or the multiset of relative degree-sequences of all induced subgraphs of G.

In order to better understand the problem of graph reconstruction, in Chapter 3 we have considered two basic equivalence relations on the set of vertices of a graph G: card equivalence and automorphism equivalence. Card equivalence is the one that is directly related to the Reconstruction Conjecture. Our examples have shown, however, that this equivalence is sometimes rather inconvenient to deal with. Automorphism equivalence has a much more transparent structure, and it has turned out to be very closely related to card equivalence. To demonstrate this fact, we have worked out a characterization theorem for card equivalence, which shows how to bring card equivalence in line with automorphism equivalence.

With respect to relative degree sequences, in Chapter 4 we have provided a generalization of an earlier observation by Nash-Williams on the degree-sequence sequence of graphs. Recognizing the importance of relative degree-sequences in graph reconstruction, we have proposed a new conjecture saying that every graph G is uniquely determined (up to isomorphism) by the multiset of relative degree-sequences of its induced subgraphs. This conjecture directly connects the motivating algebraic study to the problem of graph reconstruction, showing that the new conjecture is probably even more difficult to prove than the Reconstruction Conjecture.

In Chapter 5 we have investigated the class of card-minimal graphs, the deck of which is a set. We have also generalized the concept of peudo-similarity to couples of vertices in any graph G. As a result we have shown that every card-minimal graph G not having pseudo-similar couples of vertices is reconstructible.

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