A Study of Unit Graphs and Unitary Cayley Graphs Associated With Rings

by

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Abstract

In this thesis, we study the unit graph $G(R)$ and the unitary Cayley graph $\Gamma(R)$ of a ring R, and relate them to the structure of the ring R.

Chapter 1 gives a brief history and background of the study of the unit graphs and unitary Cayley graphs of rings. Moreover, some basic concepts, which are needed in this thesis, in ring theory and graph theory are introduced.

Chapter 2 concerns the unit graph $G(R)$ of a ring R. In Section 2.2, we first prove that the girth $\mathrm{gr}(G(R))$ of the unit graph of an arbitrary ring R is $3, 4, 6$ or ∞. Then we determine the rings R with $R/J(R)$ semipotent and with $\mathrm{gr}(G(R)) = 6$ or ∞, and classify the rings R with $R/J(R)$ right self-injective and with $\mathrm{gr}(G(R)) = 3$ or 4. The girth of the unit graphs of some ring extensions are also investigated. The focus of Section 2.3 is on the diameter of unit graphs of rings. We prove that $\mathrm{diam}(G(R)) \in \{1, 2, 3, \infty\}$ for a ring R with $R/J(R)$ self-injective and determine those rings R with $\mathrm{diam}(G(R)) = 1, 2, 3$ or ∞, respectively. It is shown that, for each $n \geq 1$, there exists a ring R such that $n \leq \mathrm{diam}(G(R)) \leq 2n$. The planarity of unit graphs of rings is discussed in Section 2.4. We completely determine the rings whose unit graphs are planar. In the last section of this chapter, we classify all finite commutative rings whose unit graphs have genus 1, 2 and 3, respectively.

Chapter 3 is about the unitary Cayley graph $\Gamma(R)$ of a ring R. In Section 3.2, it is

proved that $\mathrm{gr}(\Gamma(R)) \in \{3, 4, 6, \infty\}$ for an arbitrary ring R, and that, for each $n \geq 1$, there exists a ring R with $\mathrm{diam}(\Gamma(R)) = n$. Rings R with $R/J(R)$ self-injective are classified according to diameters of their unitary Cayley graphs. In Section 3.3, we completely characterize the rings whose unitary Cayley graphs are planar. In Section 3.4, we prove that, for each $g \geq 1$, there are at most finitely many finite commutative rings R with genus $\gamma(\Gamma(R)) = g$. We also determine all finite commutative rings R with $\gamma(\Gamma(R)) = 1$, 2, 3, respectively.

Chapter 4 is about the isomorphism problem between $G(R)$ and $\Gamma(R)$. We prove that for a finite ring R, $G(R) \cong \Gamma(R)$ if and only if either $char(R/J(R)) = 2$ or $R/J(R) = \mathbb{Z}_2 \times S$ for some ring S.

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List of Symbols

Symbols	Descriptions
$V(G)$	The vertex set of a graph G
$E(G)$	The edge set of a graph G
x—y	Two vertices x and y are adjacent
$N(x)$	The neighborhood of a vertex x in a graph
$d(x, y)$	The distance between two vertices x and y in a graph
$\mathrm{diam}(G)$	The diameter of a graph G
$\mathrm{gr}(G)$	The girth of a graph G
$\gamma(G)$	The genus of a graph G
kG	k copies of a graph G
$G - H$	Deleting a subgraph H form a graph G
$G_1 \cong G_2$	G_1 is isomorphic to G_2 as graphs
K_n	The complete graph on n vertices
$K_{m,n}$	The complete bipartite graph
$K_{m_1,\ldots,m_k}$	The complete k-partite graph
$G \otimes H$	The tensor product of graphs G and H
$\mathbb{S}_g$	The compact surface with genus g
$G(R)$	The unit graph of a ring R
$\Gamma(R)$	The unitary Cayley graph of a ring R

List of Symbols

Symbols	Descriptions
$U(R)$	The unit group of a ring R
$Z(R)$	The zero divisors of a ring R
$J(R)$	The Jacobson radical of a ring R
$\overline{R}$	The factor ring $R/J(R)$
$\overline{x}$	The element $x + J(R) \in \overline{R}$
$R[t]$	The polynomial ring in indeterminate t over a ring R
$R[[t]]$	The power series ring in indeterminate t over a ring R
$R[t_1, \ldots, t_n]$	The polynomial ring in indeterminate $t_1, \ldots, t_n$ over a ring R
$\mathbb{M}_n(R)$	The $n \times n$ matrix ring over a ring R
$\mathbb{Z}_n$	The ring of integers mod n
$\mathbb{F}_q$	The field with q elements
$\mathbf{u}(R)$	The unit sum number of a ring R
$usn(R)$	The smallest integer $n \geq 1$ such that each
	element in R is a sum of at most n units
$\mathbb{T}_n(R)$	The $n \times n$ upper triangular matrix ring over a ring R
$o(x)$	The order of an element x in a group
$char(R)$	The characteristic of a ring R
$R_1 \times \cdots \times R_n$	The direct product of rings $R_1, \ldots, R_n$
S_n	The symmetric group of degree n
A_n	The alternating group of degree n
D_n	The dihedral group of degree n
C_n	The cyclic group of order n
$\|X\|$	The cardinality of a set X

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Chapter 1

Introduction

1.1 Motivation and Background

Associating a graph with an algebraic object is an active research subject in algebraic graph theory, an area of mathematics in which methods of abstract algebra are employed in studying various graph invariants and tools in graph theory are used in studying various properties of the associated algebraic structure. Research in this subject has attracted considerable attention and has a very long history. For instances, the Cayley graph of a finite group was first considered by Arthur Cayley in 1878 [17], and Max Dehn in his unpublished lectures on group theory from 1909-10 reintroduced Cayley graphs of groups under the name Group Diagram, which led to the geometric group theory of today. This thesis concerns graphs associated with rings. Research on graphs associated with rings was started in 1988. In 1988, being the first to associate a graph to a ring, Beck [10] introduced and studied the zero-divisor graph of a commutative ring. Since then, the zero-divisor graphs have been extensively studied, with various investigations on coloring, chromatic number, clique number, diameter, girth, cyclic structure, genus, planarity and the isomorphism problem. Today, there has been a very rich literature on zero-divisor graphs of rings. For a survey and recent results concerning zero-divisor graphs, we refer the reader to [1]. In the literature, there are some other graphs associated with rings, such as the comaximal graph of a ring [52], the total graph of a ring [2], the annihilator-ideal graph of a ring [15] and the Jacobson graph of a ring [4].

Generally speaking, studying the zero divisor graph is a way to investigate the ring through the properties of its zero divisors. This is applicable especially when the zero divisors of the ring can be easily identified. This may explain why most of the publications on the zero divisor graph just concern the finite rings. The units of a ring are key elements in determining the structure of the ring, and many properties of a ring are closely connected to these of its units. So it is natural to associate a ring with a graph whose edge relationships rely on units of the ring instead of zero divisors. The unit graph and the unitary Cayley graph of a ring are such two graphs, which are the topics in this thesis. The study of these graphs has brought out many new and interesting questions to algebraists working in the newly developing area of studying rings by associating various graphs to the ring via its algebraic structure.

Next, we recall the definitions of the unit graph and unitary Cayley graph of a ring, and give a brief account of the various works known about them. The *unit graph* of a ring R, denoted $G(R)$, is the simple graph defined on the elements of R with an edge between distinct vertices x and y if and only if $x + y$ is a unit of R. In 1990, the unit graph was first investigated by Grimaldi for $\mathbb{Z}_n$, the ring of integers modulo n, in [27] where the author considered the degree of a vertex, the Hamilton cycles, the covering number, the independence number and the chromatic polynomial of the graph $G(\mathbb{Z}_n)$. In 2010, Ashrafi, et al. [8] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R and obtained various characterization results for finite (commutative) rings regarding connectedness, chromatic index, diameter, girth, and planarity of $G(R)$. Maimani et al. gave the necessary and sufficient conditions for unit graphs to be Hamiltonian in [51]. Heydari and Nikmehr investigated the unit graph of a left Artinian ring in [33]. Afkhami and Khosh-Ahang studied the unit graphs of rings of polynomials and power series in [7]. Akbari et al. concerned the unit graph of a noncommutative ring in [5].

Determining the genus $\gamma(G)$ of a graph G is one of the fundamental problems in topological graph theory. Until recently it was unknown if the question "Given a graph G and a natural number k, is $\gamma(G) \leq k$?" could be answered using a polynomial-time algorithm. In a recent paper [61] Thomassen announced that the graph genus problem is NP-complete. Das et al. studied the nonplanarity of unit graphs and classified the toroidal ones for finite commutative rings in [21]. Khashyarmanesh and Khorsandi generalized the unit graph of a commutative ring in [38], and later in [3] Asir and Chelvam studied the genus of generalized unit and unitary Cayley graphs of a commutative ring. Many other papers are devoted to this topic (see [48], [49] and [50]). A survey of the study of unit graphs can be found in [47].

Let us turn to unitary Cayley graphs of rings. The *unitary Cayley graph* of a ring R, denoted $\Gamma(R)$, is the simple graph defined on the elements of R with an edge between vertices x and y if and only if $x - y$ is a unit of R. Unitary Cayley graphs are highly symmetric. These graphs have integral spectrum and play an important role in modeling quantum spin networks supporting the perfect state of transfer.

The unitary Cayley graph of a ring was initially investigated for $\mathbb{Z}_n$ by Dejter and Giudici in [20] where some properties of $\Gamma(\mathbb{Z}_n)$ are presented. For instances, if n is prime, then $\Gamma(\mathbb{Z}_n)$ is the complete graph on n vertices; if n is even, then $\Gamma(\mathbb{Z}_n)$ is bipartite; if n is a power of 2, then $\Gamma(\mathbb{Z}_n)$ is complete bipartite and so on. Many publications are devoted to the unitary Cayley graph of $\mathbb{Z}_n$ (see, for example, [12], [13], [24] and [39]).

In 2009, Akhtar, et al. [6] generalized the unitary Cayley graph $\Gamma(\mathbb{Z}_n)$ to $\Gamma(R)$ for a finite ring R and obtained various characterization results regarding connectedness, chromatic index, diameter, girth, and planarity of $\Gamma(R)$. In [43], Lucchini and Maróti proved that the clique number and the chromatic number of $\Gamma(R)$ are equal for an Artinian ring R. In [44], Lanski and Maróti considered the unitary Cayley graph of an Artinian ring R and showed that $\Gamma(R)$ contains 2^{k-1} connected components, each of which is a bipartite graph, where k is the number of summands isomorphic to $\mathbb{Z}_2$ in $R/J(R)$. Recently, Kiani and Aghaei [36] investigated the isomorphism problem for unitary Cayley graphs associated with finite (commutative) rings. They proved that if $\Gamma(R) \cong \Gamma(S)$ where R, S are finite rings, then $\Gamma(R/J(R)) \cong \Gamma(S/J(S))$, and if, in addition, R and S are commutative, then $R/J(R) \cong S/J(S)$. They also proved that if $\Gamma(\mathbb{M}_n(F)) \cong \Gamma(R)$ where F is a finite filed, then $R \cong \mathbb{M}_n(F)$. A number of other papers considered the spectral properties and the energy of unitary Cayley graphs of rings (see [19], [35], [37], [45], [55], [57]). Khashyarmanesh and Khorsandi generalized the definition of the unitary Cayley graphs of rings in [38] and studied the properties of the resulting graph and extended some results in the unit graphs and unitary Cayley graphs. For example, they classified all commutative finite rings whose generalized unitary Cayley graphs are planar.

1.2 Preliminaries

In this section, we state some concepts in graph theory and ring theory, which are needed in the sequel.

A *graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is the *vertex set* and $E(G)$ is the *edge set*. The edge set consists of unordered pairs of distinct elements of $V(G)$. All graphs G in this thesis are *simple*. That is, there are no loops and no repeated edges in a graph. The *neighborhood* of a vertex x in a graph G, denoted $N(x)$, is the set of all vertices adjacent to x. The *closure* of the neighborhood is $\overline{N(x)} = N(x) \cup \{x\}$. For any vertices x, y in a graph G, we write x—y to mean that x and y are adjacent. The *degree* of a vertex x in a graph G, denoted $deg(x)$, is the number of vertices of G adjacent to x, that is the cardinality of $N(x)$. If all vertices in a graph G have the same degree k, then we say that G is *k-regular*.

A *walk* is a sequence of vertices and edges, where each edge's endpoints are the preceding and following vertices in the sequence. The length of a walk is the number of edges that it uses. A *path* in a graph is a walk that has all distinct vertices (except the endpoints). A path that starts and ends at the same vertex is called a *cycle*. The length of a cycle is defined as the number of its edges. The *girth* of a graph G, denoted $\mathrm{gr}(G)$, is the length of a shortest cycle contained in the graph G. If a graph does not contain any cycle, its girth is defined to be ∞. Obviously, the girth of a graph is at least 3.

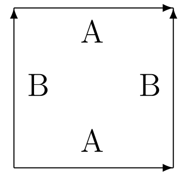
A graph G is *connected* if there is a path between each pair of the vertices of G; otherwise, G is *disconnected*. The *distance* between two vertices x and y, denoted $d(x, y)$, is the length of the shortest path in G beginning at x and ending at y. The largest distance between all pairs of vertices of G is called the *diameter* of G, and is denoted by $\mathrm{diam}(G)$.

A *complete k-partite graph* is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset, and each vertex in a subset is adjacent to every vertex in other subsets. A complete k-partite graph with partitions of size $m_1, m_2, \ldots, m_k$ is denoted by $K_{m_1,m_2,\ldots,m_k}$. The *complete bipartite* (i.e., 2-partite) *graph* is denoted by $K_{m,n}$, where the set of partition has sizes m and n. A *complete* *graph* is a graph where each vertex is adjacent to all other vertices. We denote by K_n the complete graph on n vertices.

An isomorphism of graphs G_1 and G_2 is a bijection ϕ between the vertex sets of G_1 and G_2 such that for any two vertices x and y of G_1, x and y are adjacent in G_1 if and only if $\phi(x)$ and $\phi(y)$ are adjacent in G_2. If an isomorphism exists between two graphs G_1 and G_2, then the graphs are called isomorphic and we write $G_1 \cong G_2$.

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A classical result of Kuratowski says that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see [66, Theorem 6.2.2]), where a *subdivision* of a graph G is a graph obtained from G by subdividing some of the edges. A quick consequence of Kuratowski's Theorem is that if the maximal degree of a graph is less than 3, then this graph must be planar. If a planar graph is finite, then the minimal degree of the graph is at most five. For an infinite graph, however, the situation is quite different. In fact, there exists a k-regular planar infinite graph for any positive integer k (see [26]). Of course, any subgraph of a planar graph is planar.

An orientable surface $\mathbb{S}_g$ is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. A graph G that can be drawn without crossing on a compact surface of genus g, but not on one of genus $g-1$, is called a graph of genus g. We write $\gamma(G)$ for the *genus* of the graph G. A planar graph is a graph with genus zero and a toroidal graph is a graph with genus one. It is clear that if H is a subgraph of a graph G, then $\gamma(H) \leq \gamma(G)$. Each closed surface in the sense of geometric topology can be constructed from an even-sided oriented polygon, called a *fundamental polygon*, by pairwise identification of its edges. For example, a torus (genus one) can be constructed from a quadrangle as shown below.



For other graph theoretical notions and notations adopted in this thesis, please refer to [14] and [70].

All rings R are associative with identity, but not necessarily commutative. If R has finitely many elements, then R is a *finite* ring. If $xy = yx$ for all $x, y \in R$, then R is called *commutative*. A subring of a ring shares the same multiplicative identity and any ring homomorphism preserves the identity. An element $u \in R$ is said to be a *unit* if there exists an element $v \in R$ such that $uv = vu = 1$. We use $J(R)$, $U(R)$ and $char(R)$ to denote the Jacobson radical, the group of units, and the characteristic of a ring R, respectively. We write $\overline{R} = R/J(R)$ and $\bar{a} = a + J(R) \in \overline{R}$ for $a \in R$. As usual, we write $\mathbb{Z}_n$ for the ring of integers modulo n and $\mathbb{F}_p$ for the field of p elements. The $n \times n$ upper triangular matrix ring and the $n \times n$ matrix ring over R are denoted by $\mathbb{T}_n(R)$ and $\mathbb{M}_n(R)$, respectively. The polynomial ring over a ring R in the indeterminate t is denoted by $R[t]$. The power series ring over a ring is denoted by $R[[t]]$. The cardinal of a set X is denoted $|X|$. Let $\mathbf{c} = |\mathbb{R}|$ be the cardinality of the continuum. By a *regular ring*, we mean a von Neumann regular ring, that is, for each $a \in R$, there is an element $b \in R$ such that $a = aba$. Let $x \in R$. If there exists $0 \neq y$ such that $xy = yx = 0$, then x is called a *zero divisor* of R. We use $Z(R)$ for the set of zero divisors of R. An element a is *idempotent* if $a^2 = a$. If all elements in R are idempotent, then R is called a *Boolean ring*. An element a is *nilpotent* if $a^n = 0$ for some positive integer n. A ring is *right Artinian* if it satisfies the descending chain condition on right ideals. It is clear that finite rings are Artinian rings. However, the converse is not true in general. For example, any infinite field is an Artinian ring that is not finite. A ring R is a *local ring* if $R/J(R)$ is a division ring. A ring is said to be (left) *semisimple* if it is semisimple as a left module over itself. A ring is semisimple if and only if it is Artinian and its Jacobson radical is zero. A *semilocal* ring R is a ring for which $R/J(R)$ is a semisimple ring. Recall that a ring R is called *right self-injective* if, for any (principal) right ideal I of R, every homomorphism from I_R to R_R extends to a homomorphism from R_R to R_R. A ring R is called *semipotent* if every left (respectively, right) ideal not contained in $J(R)$ contains a nonzero idempotent. If R is a right Artinian ring, then $R/J(R)$ is a right self-injective ring. Right Artinian rings and right self-injective rings are semipotent. We refer the reader to the book [42] for other basic notations in ring theory.

We also need a few concepts in group theory. Let H be a group. If $|H| = p^n$ for some prime p and integer $n \geq 0$, then H is called a *finite p-group*. For a prime number p, a *Sylow p-subgroup* of a group H is a maximal p-subgroup of H. We use S_n, A_n, D_n and C_n to denote the symmetric group of degree n, the alternating group of degree n, the dihedral group of degree n and the cyclic group of order n, respectively.

Chapter 2

Unit Graphs of Rings

2.1 Introduction

This chapter concerns the unit graph associated with a ring. Recall that the unit graph of a ring R, denoted $G(R)$, is the simple graph defined on the elements of R with an edge between distinct vertices x and y if and only if $x + y$ is a unit of R. We investigate several graph invariants of the unit graphs, including the girth, diameter, genus and planarity, and relate them to the structure of rings.

In Section 2.2, our concentration is on the girth of the unit graph of a ring. Recall that the girth of a graph G, denoted $\mathrm{gr}(G)$, is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles, its girth is defined to be ∞. Here, we prove that the girth $\mathrm{gr}(G(R))$ of the unit graph of an arbitrary ring R is 3, 4, 6 or ∞ (Theorem 2.2.1). This was known in [8] for a finite commutative ring, and in [33] for a left Artinian ring. We also determine the rings R with $R/J(R)$ semipotent and with $\mathrm{gr}(G(R)) = 6$ or ∞ (Theorem 2.2.7), and classify the rings R with $R/J(R)$ right self-injective and with $\mathrm{gr}(G(R)) = 3$ or 4 (Theorem 2.2.12). The girth of unit graphs of some ring extensions is also considered in this section. Especially, we prove that $\mathrm{gr}(G(R)) = \mathrm{gr}(G(R[x]))$ for a commutative ring R (Proposition 2.2.20).

In Section 2.3, we study the diameter of the unit graph of a ring. We first determine when $\mathrm{diam}(G(R/J(R)))$ equals $\mathrm{diam}(G(R))$ (Corollary 2.3.3). It was shown that $\mathrm{diam}(G(R)) \in \{1, 2, 3, \infty\}$ in [8] for a finite ring R and later in [33] for a left Artinian ring R. We extend this result to rings R with $R/J(R)$ self-injective (Theorem 2.3.11) and determine those rings R with $\mathrm{diam}(G(R)) = 1$, 2, 3 or ∞, respectively (Theorem 2.3.12). We show that there exists a ring R such that $3 < \mathrm{diam}(G(R)) < \infty$ (Corollary 2.3.9).

Recall that a graph is said to be planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The planarity is an important invariant in graph theory. In [8], Ashrafi, et al. completely determined the finite commutative rings whose unit graphs are planar. In section 2.4, we completely characterize the rings (not necessarily finite, and not necessarily commutative) whose unit graphs are planar (Theorem 2.4.2). In section 2.5, we classify all finite commutative rings whose unit graphs have genus 1, 2 and 3, respectively (Theorem 2.5.9).

2.2 The Girth of Unit Graphs

It was proved that $\mathrm{gr}(G(R)) \in \{3, 4, 6, \infty\}$ for any finite commutative ring R by Ashrafi, Maimani, Pournaki and Yassemi in [8] and for any one-sided Artinian ring R by Heydari and Nikmehr later in [33]. We now prove this for an arbitrary ring.

Theorem 2.2.1. *Let R be a ring. The following statements hold:*

(1) *If* $|U(R)| = 1$, *then* $\mathrm{gr}(G(R)) = \infty$.

(2) *If* $|U(R)| = 2$, *then* $\mathrm{gr}(G(R)) \in \{4, 6, \infty\}$.

(3) *If* $|U(R)| \geq 3$, *then* $\mathrm{gr}(G(R)) \in \{3, 4\}$.

Therefore, $\mathrm{gr}(G(R)) \in \{3, 4, 6, \infty\}$.

Proof. (1) If $|U(R)| = 1$, then $G(R)$ does not contain a cycle, so $\mathrm{gr}(G(R)) = \infty$.
(2) Let $U(R) = \{1, u\}$. Then $u^2 = 1$. If $G(R)$ does not contain a cycle, then $\mathrm{gr}(G(R)) = \infty$. So we can assume that $G(R)$ contains a cycle. We claim that any cycle in $G(R)$ must be an even cycle. Assume to the contrary that $a_1 — a_2 — \cdots — a_{2k+1} — a_1$ is an odd cycle in $G(R)$. Then $a_1 + a_2 = 1, a_2 + a_3 = u, \ldots, a_{2k} + a_{2k+1} = u, a_{2k+1} + a_1 = 1$, showing that $a_2 = a_{2k+1}$, a contradiction.

So we can assume $\mathrm{gr}(G(R)) = 2m + 2$ where $m \geq 1$. If $r := 1 + u \neq 0$, then $0 — (-1) — r — (-u) — 0$ is a 4-cycle, so $\mathrm{gr}(G(R)) = 4$. So we can further assume that $u = -1$. Let

$$x — a_1 — a_2 — a_3 — \cdots — a_m — y — b_m — \cdots — b_3 — b_2 — b_1 — x$$

be a cycle of length $2m + 2$. Then

$$0 — (a_1 + x) — (a_2 - x) — (a_3 + x) — \cdots — (b_3 + x) — (b_2 - x) — (b_1 + x) — 0$$

is also a cycle of length $2m + 2$. Hence, we may assume

$$0 — a_1 — a_2 — a_3 — \cdots — a_m — y — b_m — \cdots — b_3 — b_2 — b_1 — 0$$

is a cycle of length $2m + 2$, where $a_1 = 1, b_1 = -1$. Since $a_i + a_{i+1} = (-1)^i$ and $b_i + b_{i+1} = (-1)^{i+1}$, we have $a_i = (-1)^{i+1} i$ and $b_i = (-1)^i i$. As $a_m + y$ and $b_m + y$ are distinct units, $(a_m + y) + (b_m + y) = 0$. Thus $2y = 0$. As $a_m + y = (-1)^m$, we have $y = (-1)^m - a_m = (-1)^m - (-1)^{m+1} m = (-1)^m (1 + m)$. So $2y = 0$ implies that $2m + 2 = 0$ in R. Next we show that $m \leq 2$. Assume to the contrary that $m \geq 3$.

Case 1: $m = 2k + 1$. In this case, $a_m = 2k + 1$, so $a_m^2 = k(4k + 4) + 1 = k \cdot (2m + 2) + 1 = k \cdot 0 + 1 = 1$. Thus a_m is a unit, a contradiction.

Case 2: $m = 4k + 2$. Then we have $a_{2k+1} a_{4k+1} = (2k + 1)(4k + 1) = k(8k + 6) + 1 = k \cdot (2m + 2) + 1 = k \cdot 0 + 1 = 1$. This shows that a_{2k+1} and a_{4k+1} are units, a contradiction.

Case 3: $m = 4k$. Then we have $a_{2k+1}b_{4k-1} = (2k+1)(-4k+1) = -k(8k+2)+1 = -k \cdot (2m+2) + 1 = -k \cdot 0 + 1 = 1$. This shows that a_{2k+1} and b_{4k-1} are units, also a contradiction.

Therefore, $m \leq 2$. It follows that $\mathrm{gr}(G(R)) = 2m + 2 = 4$ or 6.

(3) By hypothesis, there exist two distinct elements $u, v \in U(R)$ such that $a := u+v \neq 0$. Then 0—$(-u)$—a—$(-v)$—0 is a 4-cycle in $G(R)$ and hence $\mathrm{gr}(G(R)) \leq 4$. Note that if $a = -u$ (or $a = -v$), then 0, a, $-v$ (or 0, $-u$, a) form a triangle.

In conclusion, $\mathrm{gr}(G(R)) \in \{3, 4, 6, \infty\}$. This completes our proof. $\square$

Next we classify the rings according to the girth of their unit graphs. We need the following lemmas.

Lemma 2.2.2. *The following statements hold for a ring R:*

(1) *If $J(R) \neq 0$ or R contains a nonzero nilpotent element, then* $\mathrm{gr}(G(R)) \in \{3, 4\}$.

(2) *If there exist $u, v \in U(R)$ such that $u \neq v$ and $u + v \in U(R)$, then* $\mathrm{gr}(G(R)) = 3$.

Proof. (1) If $J(R) \neq 0$, take $0 \neq j \in J(R)$. Then 0—1—j—$(1+j)$—0 is a 4-cycle, so $\mathrm{gr}(G(R)) \leq 4$. If R contains a nonzero nilpotent element, then there exists $0 \neq a \in R$ such that $a^2 = 0$. Thus 0—1—a—$(1-a)$—0 is a 4-cycle, so $\mathrm{gr}(G(R)) \leq 4$.

(2) If there exist $u, v \in U(R)$ such that $u \neq v$ and $u + v \in U(R)$, then 0—u—v—0 forms a triangle. So $\mathrm{gr}(G(R)) = 3$. $\square$

Lemma 2.2.3. *If D is a division ring with $|D| \geq 4$, then* $\mathrm{gr}(G(D)) = 3$.

Proof. If D is a division ring with $|D| \geq 4$, then there are distinct non-zero elements u, v in D such that $u + v \neq 0$. So $\mathrm{gr}(G(D)) = 3$ by Lemma 2.2.2(2). $\square$

Lemma 2.2.4. *Let R be a ring. Then* $\mathrm{gr}(G(\mathbb{M}_n(R))) = 3$ *for all $n \geq 2$.*

Proof. We show that, for all $n \geq 2$, the following condition (P_n) is satisfied:

(P_n) there exist $u, v \in U(\mathbb{M}_n(R))$ such that $u \neq v$ and $u + v \in U(\mathbb{M}_n(R))$.

We see that (P_2) holds by letting $u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ and that (P_3) holds by taking $u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. For any even number $n = 2k$, as $\mathbb{M}_n(R) \cong \mathbb{M}_2(\mathbb{M}_k(R))$, (P_n) holds as above. Let $n \geq 5$ be an odd number. Write $n = l+3$ with l even. As (P_l) and (P_3) hold, there exist $u_1, v_1 \in U(\mathbb{M}_l(R))$ such that $u_1 \neq v_1$ and $u_1 + v_1 \in U(\mathbb{M}_l(R))$, and $u_2, v_2 \in U(\mathbb{M}_3(R))$ such that $u_2 \neq v_2$ and $u_2 + v_2 \in U(\mathbb{M}_3(R))$. Then $u := \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$ and $v := \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ are distinct units of $\mathbb{M}_n(R)$ with $u + v \in U(\mathbb{M}_n(R))$. So (P_n) holds. For any $n \geq 2$, we have proved that (P_n) holds, so $\mathrm{gr}(G(\mathbb{M}_n(R))) = 3$ by Lemma 2.2.2(2). □

Lemma 2.2.5. *Let R be a ring and let $\overline{R} = R/J(R)$. Then the following hold:*

(1) $\mathrm{gr}(G(R)) \leq \mathrm{gr}(G(\overline{R}))$.

(2) *If* $\mathrm{gr}(G(\overline{R})) = 3$, *then* $\mathrm{gr}(G(R)) = \mathrm{gr}(G(\overline{R}))$.

(3) *If* $\mathrm{gr}(G(\overline{R})) = 6$ *or* ∞, *then* $\mathrm{gr}(G(R)) = \mathrm{gr}(G(\overline{R}))$ *iff* $J(R) = 0$.

(4) *If* $\mathrm{gr}(G(\overline{R})) = 4$, *then* $\mathrm{gr}(G(R)) = \mathrm{gr}(G(\overline{R}))$ *iff* $J(R) = 0$ *or* $2 \notin U(R)$.

Proof. (1) If $\mathrm{gr}(G(\overline{R})) = \infty$, there is nothing to prove. Suppose $\mathrm{gr}(G(\overline{R})) = n < \infty$ and let $\overline{x_1}—\overline{x_2}—\cdots—\overline{x_n}—\overline{x_1}$ be an n-cycle in $G(\overline{R})$. Then $x_1—x_2—\cdots—x_n—x_1$ is a cycle in $G(R)$, so $\mathrm{gr}(G(R)) \leq n$.

(2) If $\mathrm{gr}(G(\overline{R})) = 3$, then $\mathrm{gr}(G(R)) = 3$ by (1), so $\mathrm{gr}(G(R)) = \mathrm{gr}(G(\overline{R}))$.

(3) This is by Lemma 2.2.2(1).

(4) Suppose $\mathrm{gr}(G(R)) = \mathrm{gr}(G(\overline{R}))$. If $J(R) \neq 0$ and $2 \in U(R)$, then take $0 \neq r \in J(R)$ and we see 0—1—$(1+r)$—0 is a triangle in $G(R)$, contradicting the assumption that $\mathrm{gr}(G(R)) = 4$. Thus $2 \notin U(R)$.

For the converse, assume $\mathrm{gr}(G(R)) = 3 < \mathrm{gr}(G(\overline{R})) = 4$. Then $J(R) \neq 0$, and $G(R)$ contains a triangle, say a_1—a_2—a_3—a_1. Since $\overline{a_1}$—$\overline{a_2}$—$\overline{a_3}$—$\overline{a_1}$ is a walk in $G(\overline{R})$, $\mathrm{gr}(G(\overline{R})) = 4$ ensures that $\overline{a_i} = \overline{a_j}$ for some $i \neq j$. We can assume that $\overline{a_1} = \overline{a_2}$. Then $a_1 - a_2 \in J(R)$, but $a_1 + a_2 \in U(R)$. So $2a_1 \in U(R)$ and hence $2 \in U(R)$. □

A ring R is called *semipotent* if every left (respectively, right) ideal not contained in $J(R)$ contains a nonzero idempotent. A ring R is called *directly finite* if $ab = 1$ in R implies $ba = 1$. A ring is *reduced* if it contains no nonzero nilpotent elements. It is clear that a reduced ring is directly finite, and has all the idempotents central.

Lemma 2.2.6. *Let R be a semipotent ring that is reduced with $J(R) = 0$. If $|U(R)| = 1$, then R is a Boolean ring.*

Proof. Assume $a^2 \neq a$ for some $a \in R$. As R is semipotent with $J(R) = 0$, there exists $0 \neq e^2 = e \in (a - a^2)R$. Write $e = (a - a^2)b$. Since e is central, $e = ae \cdot (1 - a)e \cdot be$. Since eR is directly finite, ea and $e(1 - a)$ are units of eR. But $|U(R)| = 1$ implies $|U(eR)| = 1$. So it follows that $ea = e$ and $e(1 - a) = e$. This shows that $e = 0$, a contradiction. So R is a Boolean ring. □

We are ready to determine the rings R with $R/J(R)$ semipotent and $\mathrm{gr}(G(R)) = 6$ or ∞.

Theorem 2.2.7. *Let $R/J(R)$ be a semipotent ring. Then:*

(1) $\mathrm{gr}(G(R)) = 6$ *iff* $R \cong \mathbb{Z}_3 \times B$ *where* B *is a nontrivial Boolean ring.*

(2) $\mathrm{gr}(G(R)) = \infty$ *iff* $R \cong \mathbb{Z}_3$ *or* R *is a Boolean ring.*

Proof. (1)($\Longrightarrow$). Suppose $\mathrm{gr}(G(R)) = 6$. Then R is a reduced ring with $J(R) = 0$ by Lemma 2.2.2 and $|U(R)| = 2$ by Theorem 2.2.1. Thus $U(R) = \{1, u\}$ with $u^2 = 1$. Since $1 - u \neq 0$, there exists $0 \neq e^2 = e \in (1-u)R$. Write $e = (1-u)a$ with $a \in R$. We can assume that $a = ae$. Since eR is a reduced ring, it is directly finite. Thus $e = (1-u)e \cdot a$ implies $e = a(1-u)e = a(1-u)$. So $a(1-u) = (1-u)ea = (1-u)a$. As $u^2 = 1$, we now have $e = e^2 = (1-u)a(1-u)a = (1-u)^2a^2 = 2(1-u)a^2 = 2ea = 2a$. As $2e \neq 0$, $|U(eR)| \geq 2$. Since $R = eR \times (1-e)R$, $|U(R)| = 2$ implies that $|U(eR)| = 2$ and $|U((1-e)R)| = 1$.

Next, we show that $eR \cong \mathbb{Z}_3$. First assume that eR is not a division ring. As eR is semipotent with $J(eR) = 0$, eR contains a nontrivial idempotent, say f; that is, $f \neq 0$ and $f \neq e$. As $f = fe = f(2a) = (2f)a$, $2f \neq 0$. Similarly, $2(e-f) \neq 0$. So $|U(fR)| \geq 2$ and $|U((e-f)R)| \geq 2$. This implies that $|U(eR)| \geq 4$. This contradiction shows that eR is a division ring, and hence $eR \cong \mathbb{Z}_3$ as $|U(eR)| = 2$.

Since $\mathrm{gr}(G(\mathbb{Z}_3)) \neq 6$ (indeed, $\mathrm{gr}(G(\mathbb{Z}_3)) = \infty$), we see $1 - e \neq 0$. As $(1-e)R$ is semipotent with $J((1-e)R) = 0$, $|U((1-e)R)| = 1$ implies that $(1-e)R$ is a Boolean ring by Lemma 2.2.6.

(1)($\Longleftarrow$). As $|U(R)| = 2$, $\mathrm{gr}(G(R)) \in \{4, 6, \infty\}$ by Theorem 2.2.1. Since $(0,0)$—$(1,1)$—$(1,0)$—$(0,1)$—$(2,0)$—$(2,1)$—$(0,0)$ is a cycle in $G(R)$, $\mathrm{gr}(G(R)) \leq 6$. Assume that (a_1, b_1)—(a_2, b_2)—(a_3, b_3)—(a_4, b_4)—(a_1, b_1) is a 4-cycle in $G(R)$. Let u, v be the two units of $\mathbb{Z}_3$. Then, $a_1 + a_2 = u, a_2 + a_3 = v, a_3 + a_4 = u, a_4 + a_1 = v$. Thus, $2u = (a_1 + a_2) + (a_3 + a_4) = (a_2 + a_3) + (a_4 + a_1) = 2v$, so $u = v$, a contradiction. Therefore, $\mathrm{gr}(G(R)) = 6$.

(2)($\Longrightarrow$). Suppose $\mathrm{gr}(G(R)) = \infty$. Then R is a reduced ring with $J(R) = 0$ by Lemma 2.2.2. By Theorem 2.2.1, either $|U(R)| = 1$ or $|U(R)| = 2$. If $|U(R)| = 1$, then R is a Boolean ring by Lemma 2.2.6. So let us assume that $|U(R)| = 2$. Thus

$U(R) = \{1, u\}$ with $u^2 = 1$. As argued in the proof of (1), there exists $e^2 = e \in R$ such that eR is a division ring with $|U(eR)| = 2$ and $|U((1-e)R)| = 1$. So $eR \cong \mathbb{Z}_3$.

If $1 - e \neq 0$, then $(1-e)R$ is a Boolean ring by Lemma 2.2.6, so $\mathrm{gr}(G(R)) = 6$ by (1). Hence $e = 1$. Thus, $R \cong \mathbb{Z}_3$.

(2)($\Longleftarrow$). It is clear that $\mathrm{gr}(G(\mathbb{Z}_3)) = \infty$. For a Boolean ring R, $|U(R)| = 1$, so $\mathrm{gr}(G(R)) = \infty$ by Theorem 2.2.1. □

We remark that the class of the rings R with $R/J(R)$ semipotent is quite large. It contains any ring R with $R/J(R)$ right self-injective. Next we consider the rings R with $\mathrm{gr}(G(R)) = 3$ or 4.

Lemma 2.2.8. *Let R be a ring. If $\mathrm{gr}(G(R)) = 3$, then $|R| \geq 4$ and R has no factor ring isomorphic to $\mathbb{Z}_2$.*

Proof. Suppose $\mathrm{gr}(G(R)) = 3$. Then $|R| \geq 4$ by Theorem 2.2.1. Moreover, $G(R)$ contains a triangle a_1—a_2—a_3—a_1. Then $u := a_1 + a_2, v := a_2 + a_3, w := a_1 + a_3$ are units of R. Thus, $u + v + w = 2(a_1 + a_2 + a_3)$. This shows that R has no factor ring isomorphic to $\mathbb{Z}_2$. □

Lemma 2.2.9. *Let $R = A \times B$ be a direct product of nontrivial rings. Then $\mathrm{gr}(G(R)) = 3$ iff $\mathrm{gr}(G(A)) = 3$ or $2 \in U(A)$, and $\mathrm{gr}(G(B)) = 3$ or $2 \in U(B)$.*

Proof. ($\Longrightarrow$). Suppose $\mathrm{gr}(G(R)) = 3$ and let (a_1, b_1)—(a_2, b_2)—(a_3, b_3)—(a_1, b_1) be a triangle in $G(R)$. If a_1, a_2 and a_3 are distinct, then they form a triangle in $G(A)$ and hence $\mathrm{gr}(G(A)) = 3$. If a_1, a_2 and a_3 are not distinct, say $a_1 = a_2$, then $2a_1$ is a unit in A and so $2 \in U(A)$. Similarly, one can show that $\mathrm{gr}(G(B)) = 3$ or $2 \in U(B)$.

($\Longleftarrow$). If $\mathrm{gr}(G(A) = 3$ and $\mathrm{gr}(G(B) = 3$, then $G(A)$ contains a triangle a_1—a_2—a_3—a_1 and $G(B)$ contains a triangle b_1—b_2—b_3—b_1; so (a_1, b_1)—(a_2, b_2)—(a_3, b_3)—(a_1, b_1) is a triangle in $G(R)$.

If $\mathrm{gr}(G(A) = 3$ and $2 \in U(B)$, then $G(A)$ contains a triangle a_1—a_2—a_3—a_1; so $(a_1, 0)$—$(a_2, 1)$—$(a_3, 1)$—$(a_1, 0)$ is a triangle in $G(R)$.

Similarly, the assumption that $2 \in U(A)$ and $\mathrm{gr}(G(B)) = 3$ implies that $G(R)$ contains a triangle.

If $2 \in U(A)$ and $2 \in U(B)$, then $(1, 0)$—$(0, 1)$—$(1, 1)$—$(1, 0)$ is a triangle in $G(R)$. Therefore, $\mathrm{gr}(G(R)) = 3$. $\square$

A reduced regular ring is called a *strongly regular* ring.

Lemma 2.2.10. *Let R be a strongly regular right self-injective ring. Then* $\mathrm{gr}(G(R)) = 3$ *iff* $|R| \geq 4$ *and R has no factor ring isomorphic to* $\mathbb{Z}_2$.

Proof. ($\Longrightarrow$). This follows from Lemma 2.2.8.
($\Longleftarrow$). First let us assume that $2 \in U(R)$. If R is a division ring, then the assumption $|R| \geq 4$ together with Lemma 2.2.3 show that $\mathrm{gr}(G(R)) = 3$. If R is not a division ring, then R contains a nontrivial idempotent e, as R is strongly regular. Since $R = eR \times (1-e)R$ and $2e \in U(eR)$ and $2(1-e) \in U((1-e)R)$, $\mathrm{gr}(G(R)) = 3$ by Lemma 2.2.9.

Next we assume that $2 \notin U(R)$; that is, $2R \neq R$. Hence $R = 2R \times S$, where $S \neq 0$. Since R has no factor ring isomorphic to $\mathbb{Z}_2$, S is not Boolean, so $|U(S)| \geq 2$. Take $1_S \neq u \in U(S)$, and there exists $0 \neq f^2 = f \in (1_S - u)S$. Write $f = (1_S - u)a$ with $a \in S$. So $f = (1_S - u)f \cdot af$. Since $fS = fR$ is directly finite, it follows that $(1_S - u)f \in U(fS)$. Note $uf \in U(fS)$. As $2S = 0$, $uf \neq (1_S - u)f$. Thus $G(fS)$ contains a triangle 0—uf—$(1-u)f$—0, so $\mathrm{gr}(G(fR)) = \mathrm{gr}(G(fS)) = 3$. If $f = 1$, then $\mathrm{gr}(G(R)) = 3$. So we may assume that $1 - f \neq 0$. Then $R = fR \times (1-f)R$ with $(1-f)R \neq 0$. Note that $(1-f)R$ is a right self-injective ring which has no factor ring isomorphic to $\mathbb{Z}_2$. So, by [40, Lemma 4], $1 - f = v_1 + v_2$ where

$v_1, v_2 \in U((1-f)R)$. Thus, $G(R)$ contains a triangle $(0,0)$—(uf, v_1)—$((1-u)f, v_2)$—$(0,0)$, so $\mathrm{gr}(G(R)) = 3$. $\square$

A nonzero regular right self-injective ring is called *purely infinite* if eRe is not directly finite for all $0 \neq e^2 = e \in R$.

Lemma 2.2.11. [40, Lemma 2] *Let R be a regular right self-injective ring. Then $R = S \times T$, where $S = 0$ or S is a direct product of matrix rings of size ≥ 2 and where $T = 0$ or is a strongly regular right self-injective ring.*

Proof. By the proof of [40, Lemma 2], $R = S \times T$ where $S = A \times B$ with $A = 0$ or A being purely infinite regular right self-injective and with $B = 0$ or B being a direct product of matrix rings of size ≥ 2 and where $T = 0$ or T is strongly regular right self-injective. By [28, 10.21], $R_R \cong (R \oplus R)_R$, so $R \cong \mathrm{End}(R_R) \cong \mathrm{End}((R \oplus R)_R) \cong \mathbb{M}_2(R)$. So $S = 0$ or is a direct product of matrix rings of size ≥ 2. $\square$

We now classify the rings R with $R/J(R)$ right self-injective such that $\mathrm{gr}(G(R)) = 3$ or 4. Note that R right self-injective implies that $R/J(R)$ is right self-injective (see [63]).

Theorem 2.2.12. *Let R be a ring with $R/J(R)$ right self-injective. Then:*

(1) $\mathrm{gr}(G(R)) = 3$ *iff* $|R| \geq 4$ *and R has no factor ring isomorphic to $\mathbb{Z}_2$.*

(2) $\mathrm{gr}(G(R)) = 4$ *iff R satisfies the following conditions:*

- (i) $|R| \geq 4$ *and R has a factor ring isomorphic to $\mathbb{Z}_2$, and*
- (ii) *R is not a Boolean ring, and*
- (iii) *R is not a direct product of $\mathbb{Z}_3$ and a Boolean ring.*

Proof. (1)($\Longrightarrow$). This follows from Lemma 2.2.8.

(1)($\Longleftarrow$). Since $\overline{R}$ is right self-injective, it is a regular right self-injective ring (see [63]). By Lemma 2.2.11, $\overline{R} = S \times T$, where $S = 0$ or is a direct product of matrix rings of size ≥ 2 and where $T = 0$ or is a strongly regular right self-injective ring.

Case 1: $S = 0$. Then $\overline{R}$ is a strongly regular right self-injective ring that has no factor ring isomorphic to $\mathbb{Z}_2$. So $|\overline{R}| \neq 2$. If $|\overline{R}| \geq 4$, then $\mathrm{gr}(G(\overline{R})) = 3$ by Lemma 2.2.10, so $\mathrm{gr}(G(R)) = 3$ by Lemma 2.2.5(2).

If $|\overline{R}| = 3$, then $2 \in U(R)$ and $J(R) \neq 0$ as $|R| \geq 4$. Take $0 \neq j \in J(R)$. Then $G(R)$ contains a triangle 0—1—$(1+j)$—0. So $\mathrm{gr}(G(R)) = 3$.

Case 2: $S \neq 0$. It can easily be shown that the girth of the unit graph of a direct product of rings whose unit graphs all have girth 3 must be 3. In view of Lemma 2.2.4, we have $\mathrm{gr}(G(S)) = 3$. If $T = 0$, then $\mathrm{gr}(G(\overline{R})) = 3$, and so $\mathrm{gr}(G(R)) = 3$ by Lemma 2.2.5(2). So we can assume that $T \neq 0$. Since R has no factor ring isomorphic to $\mathbb{Z}_2$, $|T| \neq 2$.

If $|T| \geq 4$, then $\mathrm{gr}(G(T)) = 3$ by Lemma 2.2.10 since T has no factor ring isomorphic to $\mathbb{Z}_2$. Hence $\mathrm{gr}(G(\overline{R})) = 3$ by Lemma 2.2.9, so $\mathrm{gr}(G(R)) = 3$ by Lemma 2.2.5(2).

If $|T| = 3$, then $2 \in U(T)$. So $\mathrm{gr}(G(\overline{R})) = 3$ by Lemma 2.2.9. Hence $\mathrm{gr}(G(R)) = 3$ by Lemma 2.2.5(2).

(2) This follows from (1) and Theorems 2.2.1 and 2.2.7. $\square$

In the rest of this section, we consider the girth of the unit graphs of some extensions of rings.

Proposition 2.2.13. *Let R be a subring of a ring S with $1_S \in R$. Suppose that R is isomorphic to a factor ring of S and $J(S) \neq 0$. Then:*

(1) $\mathrm{gr}(G(S)) \in \{3, 4\}$.

(2) $\mathrm{gr}(G(S)) = 3$ *iff* $\mathrm{gr}(G(R)) = 3$ *or* $2 \in U(R)$.

(3) $\mathrm{gr}(G(S)) = 4$ *iff* $\mathrm{gr}(G(R)) \neq 3$ *and* $2 \notin U(R)$.

Proof. Let $\theta : S \to R$ be an onto ring homomorphism.

(1) Since $J(S) \neq 0$, $\mathrm{gr}(G(S)) \in \{3, 4\}$ by Lemma 2.2.2(1).

(2) It is clear that $\mathrm{gr}(G(R)) = 3$ implies $\mathrm{gr}(G(S)) = 3$. If $2 \in U(R)$, take $0 \neq j \in J(S)$ and we see 0—1—$(1 + j)$—0 is a 3-cycle in $G(S)$; so $\mathrm{gr}(G(S)) = 3$. Conversely, suppose $\mathrm{gr}(G(S)) = 3$ and let s_1—s_2—s_3—s_1 be a 3-cycle in $G(S)$. If $\theta(s_1), \theta(s_2)$ and $\theta(s_3)$ are distinct, then $\theta(s_1)$—$\theta(s_2)$—$\theta(s_3)$—$\theta(s_1)$ is a 3-cycle in $G(R)$ and hence $\mathrm{gr}(G(R)) = 3$. If $\theta(s_1), \theta(s_2)$ and $\theta(s_3)$ are not distinct, say $\theta(s_1) = \theta(s_2)$, then $2\theta(s_1) = \theta(s_1 + s_2) \in U(R)$, so $2 \in U(R)$.

(3) It is clear from (1) and (2). $\square$

Proposition 2.2.13 has some quick consequences.

Corollary 2.2.14. *Let $R[[x]]$ be the power series ring over a ring R. Then:*

(1) $\mathrm{gr}(G(R[[x]])) \in \{3, 4\}$.

(2) $\mathrm{gr}(G(R[[x]])) = 3$ *iff* $\mathrm{gr}(G(R)) = 3$ *or* $2 \in U(R)$.

(3) $\mathrm{gr}(G(R[[x]])) = 4$ *iff* $\mathrm{gr}(G(R)) \neq 3$ *and* $2 \notin U(R)$.

The *trivial extension* of a ring R by an R-bimodule M is $R \propto M := \{(a, x) : a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$. In fact, $R \propto M$ is isomorphic to the subring $\{\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right) : a \in R, x \in M\}$ of the formal triangular matrix ring $\left(\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right)$.

Corollary 2.2.15. *Let M be a nontrivial bimodule over a ring R and let $T = R \propto M$. Then:*

(1) $\mathrm{gr}(G(T)) \in \{3, 4\}$.

(2) $\mathrm{gr}(G(T)) = 3$ *iff* $\mathrm{gr}(G(R)) = 3$ *or* $2 \in U(R)$.

(3) $\mathrm{gr}(G(T)) = 4$ *iff* $\mathrm{gr}(G(R)) \neq 3$ *and* $2 \notin U(R)$.

Corollary 2.2.16. *Let R be a ring and let $S := \mathbb{T}_n(R)$ be the upper triangular matrix ring where $n \geq 2$. Then:*

(1) $\mathrm{gr}(G(S)) \in \{3, 4\}$.

(2) $\mathrm{gr}(G(S)) = 3$ *iff* $\mathrm{gr}(G(R)) = 3$ *or* $2 \in U(R)$.

(3) $\mathrm{gr}(G(S)) = 4$ *iff* $\mathrm{gr}(G(R)) \neq 3$ *and* $2 \notin U(R)$.

Proof. By Proposition 2.2.13, we only need to show that (2) holds. Let R^n denote the direct product of n copies of R. Then $\mathrm{gr}(G(S)) = 3$ iff $\mathrm{gr}(G(R^n)) = 3$ or $2 \in U(R^n)$ (by Proposition 2.2.13) iff $\mathrm{gr}(G(R)) = 3$ or $2 \in U(R)$ (by Lemma 2.2.9). $\square$

Corollary 2.2.17. *Let A, B be rings and M be a nontrivial (A, B)-bimodule. Let $S := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the formal triangular matrix ring. Then:*

(1) $\mathrm{gr}(G(S)) \in \{3, 4\}$.

(2) $\mathrm{gr}(G(S)) = 3$ *iff* $\mathrm{gr}(G(A)) = 3$ *or* $2 \in U(A)$, *and* $\mathrm{gr}(G(B)) = 3$ *or* $2 \in U(B)$.

(3) $\mathrm{gr}(G(S)) = 4$ *iff* $\mathrm{gr}(G(A)) \neq 3$ *and* $2 \notin U(A)$, *or* $\mathrm{gr}(G(B)) \neq 3$ *and* $2 \notin U(B)$.

Proof. By Proposition 2.2.13, we only need to show that (2) holds. Then $\mathrm{gr}(G(S)) = 3$ iff $\mathrm{gr}(G(A \times B)) = 3$ or $2 \in U(A \times B)$ (by Proposition 2.2.13) iff $\mathrm{gr}(G(A)) = 3$ or $2 \in U(A)$, and $\mathrm{gr}(G(B)) = 3$ or $2 \in U(B)$ (by Lemma 2.2.9). $\square$

The *group ring* of a group H over a ring R is denoted RH.

Proposition 2.2.18. *Let R be a ring, F be a field and H be a nontrivial group. Then:*

(1) $\mathrm{gr}(G(RH)) \in \{3, 4\}$.

(2) $\mathrm{gr}(G(\mathbb{Z}H)) = 4$.

(3) (i) $\mathrm{gr}(G(FH)) = 3$ *iff* $|F| > 2$.

(ii) $\mathrm{gr}(G(FH)) = 4$ *iff* $|F| = 2$.

Proof. (1) Take $1 \neq h \in H$. Then 0— (-1)—$(1+h)$ —$(-h)$—0 is a 4-cycle in $G(RH)$, so $\mathrm{gr}(G(RH)) \in \{3, 4\}$.

(2) Assume $\mathrm{gr}(G(\mathbb{Z}H)) = 3$ and let α— β—γ—α be a 3-cycle in $G(\mathbb{Z}H)$. Write $\alpha = \sum_{i=1}^{n} a_i h_i, \beta = \sum_{i=1}^{n} b_i h_i$ and $\gamma = \sum_{i=1}^{n} c_i h_i$. As $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are units of $\mathbb{Z}H$, it follows that

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \pm 1, \ \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} c_i = \pm 1, \ \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} a_i = \pm 1.$$

Adding the equalities yields $2\sum_{i=1}^{n}(a_i + b_i + c_i) = \pm 1$ or ± 3. This is a contradiction. So $\mathrm{gr}(G(\mathbb{Z}H)) = 4$ by (1).

(3) Suppose $|F| > 2$. Let $1 \neq h \in H$. If $0 \neq 2 \in F$, then 1—$(-1 + h)$—$(-1 - h)$—1 is a triangle in $G(FH)$. If $2 = 0$ in F, take $a \in F$ such that $a \neq 0$ and $a \neq 1$. Then 1—$(1 + ah)$—$(1 + (1 + a)h)$—1 is a triangle in $G(FH)$. So $\mathrm{gr}(G(FH)) = 3$.

Suppose $|F| = 2$. Assume that $\mathrm{gr}(G(FH)) = 3$ and that α— β—γ—α is a 3-cycle in $G(FH)$. Write $\alpha = \sum_{i=1}^{n} a_i h_i, \beta = \sum_{i=1}^{n} b_i h_i$ and $\gamma = \sum_{i=1}^{n} c_i h_i$. As $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are units of FH, it follows that

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} a_i = 1.$$

This shows that $0 = 2\sum_{i=1}^{n}(a_i+b_i+c_i) = 3 = 1$, a contradiction. So $\mathrm{gr}(G(FH)) = 4$ by (1). Hence (3) holds by (1). $\square$

Proposition 2.2.19. *Let V be a nontrivial (right) vector space over a division ring D, and let $R = \mathrm{End}_D(V)$ be the ring of linear transformations of V. Then:*

(1) $\mathrm{gr}(G(R)) \in \{3, \infty\}$.

(2) $\mathrm{gr}(G(R)) = 3$ *iff either* $\dim(V) = 1$ *with* $|D| > 3$, *or* $\dim(V) \geq 2$.

(3) $\mathrm{gr}(G(R)) = \infty$ *iff* $\dim(V) = 1$ *and* $|D| = 2$ *or* 3.

Proof. Let $n = \dim(V)$.

Case 1: $n = 1$. Then $R = D$. If $|D| = 2$ or 3, then $\mathrm{gr}(G(R)) = \infty$ by Theorem 2.2.7(2). If $|D| \geq 4$, then $\mathrm{gr}(G(R)) = 3$ by Lemma 2.2.3.

Case 2: $2 \leq n < \infty$. Then $R \cong \mathbb{M}_n(D)$. By Lemma 2.2.4, $\mathrm{gr}(G(R)) = 3$.

Case 3: $n = \infty$. Then we have $V_D \cong (V \oplus V)_D$, so $R = \mathrm{End}_D(V) \cong \mathrm{End}_D(V \oplus V) \cong \mathbb{M}_2(\mathrm{End}_D(V)) = \mathbb{M}_2(R)$. Thus $\mathrm{gr}(G(R)) = \mathrm{gr}(G(\mathbb{M}_2(R)) = 3$ by Lemma 2.2.4.

The claims have been proved. $\square$

For any polynomial $f(x) \in R[x]$, let $f(0)$ stand for the constant term of $f(x)$. Note that for a commutative ring R, $\sum_{i=0}^{n} a_i x^i \in R[x]$ is a unit if and only if $a_0 \in R$ is a unit and a_i is nilpotent for all $i \geq 1$.

Proposition 2.2.20. *Let R be a commutative ring. Then* $\mathrm{gr}(G(R[x])) = \mathrm{gr}(G(R))$.

Proof. It is clear that $\mathrm{gr}(G(R)) \geq \mathrm{gr}(G(R[x]))$.

Case 1: $\mathrm{gr}(G(R[x])) = 3$. Let $f_1(x)$—$f_2(x)$—$f_3(x)$—$f_1(x)$ be a triangle in $G(R[x])$, where $f_i(x) = a_{i0} + a_{i1}x + \cdots$. If $f_1(0), f_2(0)$ and $f_3(0)$ are distinct, then $f_1(0)$—$f_2(0)$—$f_3(0)$—$f_1(0)$ is a 3-cycle in $G(R)$, so $\mathrm{gr}(G(R)) = 3$. Therefore, without loss of generality we can assume $f_1(0) = f_2(0)$. It follows that $2f_1(0) \in U(R)$, and hence $2 \in U(R)$.

If R is reduced, then $a_{1i} + a_{2i}, a_{2i} + a_{3i}$ and $a_{1i} + a_{3i}$ are all zero (being nilpotent) for all $i \geq 1$. Thus $2a_{1i} = 2(a_{1i} + a_{2i} + a_{3i}) = (a_{1i} + a_{2i}) + (a_{2i} + a_{3i}) + (a_{1i} + a_{3i}) = 0$.

This shows $a_{1i} = 0$ and hence $a_{2i} = 0$ for all $i \geq 1$. We deduce $f_1(x) = f_2(x)$, a contradiction. So R contains a nonzero nilpotent element, say b. Hence 0—1—$(1+b)$—0 is a 3-cycle in $G(R)$, so $\mathrm{gr}(G(R)) = 3$.

Case 2: $\mathrm{gr}(G(R[x])) = 4$. Let $f_1(x)$—$f_2(x)$—$f_3(x)$—$f_4(x)$—$f_1(x)$ be a 4-cycle in $G(R[x])$, where $f_i(x) = a_{i0} + a_{i1}x + \cdots$. If $J(R) \neq 0$ or R contains nonzero nilpotent elements, then, by Lemma 2.2.2 (1), $\mathrm{gr}(G(R)) \leq 4$ and hence $\mathrm{gr}(G(R)) = 4$. So we can assume that $J(R) = 0$ and R is a reduced ring. Then $a_{1i} + a_{2i}, a_{2i} + a_{3i}, a_{3i} + a_{4i}$ and $a_{4i} + a_{1i}$ are all zero (being nilpotent) for $i \geq 1$. Thus $a_{1i} = -a_{2i} = a_{3i} = -a_{4i}$ for all $i \geq 1$. It follows that $f_1(0) \neq f_3(0)$ and $f_2(0) \neq f_4(0)$.

If $f_1(0), f_2(0)$, $f_3(0)$ and $f_4(0)$ are not distinct, we can assume without loss of generality that $f_1(0) = f_2(0)$. Then $f_1(0)$—$f_3(0)$—$f_4(0)$—$f_1(0)$ is a triangle, contradicting the assumption that $\mathrm{gr}(G(R[x])) = 4$. So $f_1(0), f_2(0)$, $f_3(0)$ and $f_4(0)$ are distinct. Then we see that $f_1(0)$—$f_2(0)$—$f_3(0)$—$f_4(0)$—$f_1(0)$ is a 4-cycle in $G(R)$; so $\mathrm{gr}(G(R)) \leq 4$ and hence $\mathrm{gr}(G(R)) = 4$.

Case 3: $\mathrm{gr}(G(R[x])) = n > 4$. Then $R[x]$, and hence R is reduced by Lemma 2.2.2(1). Let $f_1(x)$—$f_2(x)$—$\cdots$—$f_n(x)$—$f_1(x)$ be an n-cycle in $G(R[x])$. If there do not exist three distinct elements in $f_1(0), f_2(0), \ldots, f_n(0)$, then we can assume without loss of generality that $f_1(0) = f_2(0) = \cdots = f_{n-1}(0)$. Since R is reduced, $a_{1i}+a_{2i}, a_{2i}+a_{3i}$ and $a_{1i}+a_{3i}$ are all zero (being nilpotent) for $i \geq 1$. Thus $a_{1i} = a_{3i}$ for all $i \geq 1$. So we have $f_1(x) = f_3(x)$, a contradiction. Hence there exist three distinct elements in $f_1(0), f_2(0), \ldots, f_n(0)$. Thus the walk $f_1(0)$—$f_2(0)$—$\cdots$—$f_n(0)$—$f_1(0)$ in $G(R)$ reduces to a cycle of length between 3 and n; so $\mathrm{gr}(G(R)) \leq n$ and hence $\mathrm{gr}(G(R)) = n$. This completes the proof. $\square$

Remark 2.2.21. (1) Let $R = \mathbb{Z}_6[x]$. As $\mathrm{gr}(G(\mathbb{Z}_6)) = 6$, $\mathrm{gr}(G(R)) = 6$ by Proposition 2.2.20. But in contrast to Theorem 2.2.7(1), R is not a direct product of $\mathbb{Z}_3$ and a Boolean ring.

(2) Let $R = \mathbb{Z}_2[x]$. As $\mathrm{gr}(G(\mathbb{Z}_2)) = \infty$, $\mathrm{gr}(G(R)) = \infty$ by Proposition 2.2.20. But in contrast to Theorem 2.2.7(2), $R \ncong \mathbb{Z}_3$ and R is not Boolean.

(3) Let $R = \mathbb{Z}_3[x]$. As $\mathrm{gr}(G(\mathbb{Z}_3)) = \infty$, $\mathrm{gr}(G(R)) = \infty$ by Proposition 2.2.20. But $|R| \geq 4$ and R has no factor ring isomorphic to $\mathbb{Z}_2$. This shows that the converse of Lemma 2.2.8 is false $\big($also see Theorem 2.2.12(1)$\big)$.

(4) Again consider $R = \mathbb{Z}_6[x]$. Then $\mathrm{gr}(G(R)) = 6 \neq 4$. But in contrast to Theorem 2.2.12(2), $|R| \geq 4$ and R has a factor ring isomorphic to $\mathbb{Z}_2$, R is not a Boolean ring, and R is not a direct product of $\mathbb{Z}_3$ and a Boolean ring.

We remark that the results in this section has been published in from [60].

2.3 The Diameter of Unit Graphs

In this section, we study the diameter of unit graphs. Recall that the distance between two vertices x and y in a graph G, denoted $d(x, y)$, is the length of the shortest path in G beginning at x and ending at y. The diameter of a graph G, denoted $\mathrm{diam}(G)$, is the longest distance between two vertices in graph G. In [33, Remark 1], the authors have pointed out that $\mathrm{diam}(G(\overline{R})) \leq \mathrm{diam}(G(R))$ for any ring R. Here, we first determine when the inequality is strict.

Lemma 2.3.1. *Let R be a ring. If* $\mathrm{diam}(G(R)) \geq 3$*, then* $\mathrm{diam}(G(\overline{R})) = \mathrm{diam}(G(R))$.

Proof. Suppose that $\mathrm{diam}(G(R)) = \infty$. We need to show that $\mathrm{diam}(G(\overline{R})) = \infty$. Assume to the contrary that $\mathrm{diam}(G(\overline{R})) = m < \infty$. For any $x, y \in R$, if $\overline{x} = \overline{y}$, then $x - y \in J(R)$ and hence $1 + x - y \in U(R)$. So we get a walk x—$(1 - y)$—y from x to y, and so $d(x, y) \leq 2$. If $\overline{x} \neq \overline{y}$, then a path from $\overline{x}$ to $\overline{y}$ deduces a path from x to y, which implies that $d(x, y) \leq d(\overline{x}, \overline{y}) \leq m$. So $\mathrm{diam}(G(R)) < \infty$, a contradiction.

Suppose that $\operatorname{diam}(G(R))$ is finite and $\operatorname{diam}(G(R)) = k \geq 3$. By [33, Remark 1], we only need to show that $\operatorname{diam}(G(\overline{R})) \geq k$. There exist $x, y \in R$, such that $d(x, y) = k$. First we claim that $\overline{x} \neq \overline{y}$. For otherwise, if $\overline{x} = \overline{y}$, then as proved above, so $d(x, y) \leq 2$, a contradiction. Assume that $d(\overline{x}, \overline{y}) = l < k$ and $\overline{x}$—$\overline{x_1}$—$\overline{x_2}$—$\cdots$—$\overline{x_{l-1}}$—$\overline{y}$ is a path from $\overline{x}$ to $\overline{y}$. Then x—x_1—x_2—$\cdots$—x_{l-1}—y is path of length l, so $d(x, y) \leq l < k$, a contradiction. Thus, $d(\overline{x}, \overline{y}) \geq k$. This implies $\operatorname{diam}(G(\overline{R})) \geq k$.

Therefore, $\operatorname{diam}(G(\overline{R})) = \operatorname{diam}(G(R))$. □

Now, we determine when $\operatorname{diam}(G(\overline{R})) < \operatorname{diam}(G(R))$.

Theorem 2.3.2. *Let R be a ring. Then the following conditions are equivalent:*

(1) $\operatorname{diam}(G(\overline{R})) < \operatorname{diam}(G(R))$.

(2) *R is a local ring with $J(R) \neq 0$ and $2 \in J(R)$.*

(3) $\operatorname{diam}(G(R)) = 2$ *and* $\operatorname{diam}(G(\overline{R})) = 1$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\operatorname{diam}(G(\overline{R})) < \operatorname{diam}(G(R))$. By Lemma 2.3.1, $\operatorname{diam}(G(R)) \leq 2$. Note that $\operatorname{diam}(G(R)) = 1$ implies $\operatorname{diam}(G(\overline{R})) = 1$. Thus, we have $\operatorname{diam}(G(R)) = 2$ and $\operatorname{diam}(G(\overline{R})) = 1$. So $J(R) \neq 0$, and by [8, Theorem 3.4], $\overline{R}$ is a division ring with $char(\overline{R}) = 2$. Therefore, R is a local ring with $J(R) \neq 0$ and $2 \in J(R)$.

(2) $\Rightarrow$ (3). Suppose that R is a local ring with $J(R) \neq 0$ and $2 \in J(R)$. Then $R/J(R)$ is a division ring and $char(\overline{R}) = 2$. By [8, Theorem 3.4], it follows that $G(\overline{R})$ is a complete graph and hence $\operatorname{diam}(G(\overline{R})) = 1$. On the other hand, for any $r \in R$, either $r \in J(R)$ or $r \in U(R)$. For any two distinct elements $a, b \in R$, if $a + b \in U(R)$, then $d(a, b) = 1$. Suppose that $a + b \in J(R)$. If $a \in J(R)$, then $b \in J(R)$, and we have a path a—1—b, so $d(a, b) = 2$ (note that since $J(R) \neq 0$, such a, b do exist); if

$a \in U(R)$, then $b \in U(R)$, and we have a path a—$(a+b)$—b, so $d(a,b) = 2$. Hence $\mathrm{diam}(G(R)) = 2$.

(3) $\Rightarrow$ (1). It is clear. $\square$

Corollary 2.3.3. *Let R be a ring. Then* $\mathrm{diam}(G(\overline{R})) = \mathrm{diam}(G(R))$ *if and only if one of the following holds:*

(1) *R is not a local ring.*

(2) *R is a local ring with $2 \in U(R)$.*

(3) *R is a division ring.*

As shown in [8], the connectedness of $G(R)$ is relative to whether the ring R is generated additively by its units. So we first recall the following definitions. Let R be a ring and k be a positive integer. An element $r \in R$ is said to be k-*good* if $r = u_1 + \cdots + u_k$ with $u_i \in U(R)$ for each $1 \leq i \leq k$. A ring is said to be k-good if every element of R is k-good. The *unit sum number* of a ring R, denoted by $\mathbf{u}(R)$, is defined to be

(1) $\min\{k \in \mathbb{N} \mid\ R$ is a k-good $\}$, if R is k-good for some $k \geq 1$;

(2) ω, if R is not k-good for every $k \geq 1$, but each element of R is k-good for some k;

(3) ∞, some element of R is not k-good for any $k \geq 1$.

For example, $\mathbf{u}(\mathbb{Z}_3) = 2$, $\mathbf{u}(\mathbb{Z}) = \omega$ and $\mathbf{u}(\mathbb{Z}[t]) = \infty$. It is clear that if $2 \in U(R)$, then $r \in R$ being k-good implies that r is l-good for all $l \geq k$. The investigation of rings generated additively by their units started in 1953-1954 when Wolfson [65] and Zelinsky [71] proved independently that every linear transformation of a vector space V over a division ring D is the sum of two nonsingular linear transformations, except when $dimV = 1$ and $D = \mathbb{Z}_2$. For the unit sum number of rings, we refer the reader to [30], [40], [41] and [64].

Lemma 2.3.4. *Let R be a ring and $r \in R$. Then the following hold:*

(1) *If r is k-good, then $d(r,0) \leq k$;*

(2) *If $r \neq 0$ and $d(r,0) = k$, then r is k-good but not l-good for all $l < k$.*

Proof. (1) Let $r = u_1 + \cdots + u_k$, where each $u_i \in U(R)$. If k is odd, then

$$0 — u_1 — (-u_1 - u_2) — \cdots — (-u_1 - \cdots - u_{k-1}) — (u_1 + \cdots + u_k) = r$$

is a walk of length k. If k is even, then

$$0 — (-u_1) — (u_1 + u_2) — \cdots — (-u_1 - \cdots - u_{k-1}) — (u_1 + \cdots + u_k) = r$$

is a walk of length k. Therefore, $d(r,0) \leq k$.

(2) Let $r = x_0 — x_1 — x_2 — \cdots — x_k = 0$ be a path from r to 0. Then $u_i := x_{i-1} + x_i \in U(R)$ for $1 \leq i \leq k$, so $r = \sum_{i=1}^{k} (-1)^{i+1} u_i$. Thus, r is k-good. By part (1), we know that r is not l-good for all $l < k$. $\square$

Proposition 2.3.5. *Let R be a ring that is not a division ring. If $\mathbf{u}(R) = k$, then* $\operatorname{diam}(G(R)) = k$.

Proof. Let $x, y \in R$. If k is odd, we set $x + y = u_1 + u_2 + \cdots + u_k$, where each u_i is a unit in R. Then there exists a walk

$$x — (-x + u_1) — (x - u_1 - u_2) — \cdots — (x - u_1 - \cdots - u_{k-1}) — (-x + u_1 + \cdots + u_k) = y$$

between x and y, so $d(x,y) \leq k$. If k is even, we set $y - x = u_1 + u_2 + \cdots + u_k$, where each u_i is a unit in R. Then there exists a walk

$$x — (-x - u_1) — (x + u_1 + u_2) — \cdots — (x - u_1 - \cdots - u_{k-1}) — (x + u_1 + \cdots + u_k) = y$$

between x and y, so $d(x,y) \leq k$. Thus, $\operatorname{diam}(G(R)) \leq k$.

On the other hand, as $\mathbf{u}(R) = k \geq 2$ and R is not a division ring, there exists an element $0 \neq r \in R$, such that r is k-good but not l-good for any $l < k$. Then $d(r,0) = k$ by Lemma 2.3.4. Thus, $\operatorname{diam}(G(R)) = k$. $\square$

The converse of Proposition 2.3.5 is not true in general. For example, $\mathrm{diam}(G(\mathbb{Z}_4)) = 2$, but $\mathbf{u}(\mathbb{Z}_4) = \omega$.

Proposition 2.3.6. *Let R be a ring with $\mathrm{diam}(G(R)) = k \geq 2$. If $2 \in U(R)$, then $\mathbf{u}(R) = k$.*

Proof. By [8, Theorem 3.4], R is not a division ring with $char(R) = 2$. If R is a division ring with $char(R) \neq 2$, then $\mathrm{diam}(G(R)) = 2$. Note that, in this case, $\mathbf{u}(R) = 2$. So the result holds.

Now we assume R is not a division ring. Let $0 \neq r \in R$. If $d(r, 0) = l \leq k$, then by Lemma 2.3.4(2), we know that r is l-good. Since 2 is a unit of R, r is k-good and hence R is k-good. By Proposition 2.3.5, R is not l-good for $l < k$, so $\mathbf{u}(R) = k$. □

Proposition 2.3.7. *Let R be a ring and $2 \in U(R)$. Then $\mathrm{diam}(G(R)) = k \geq 2$ if and only if $\mathbf{u}(R) = k$.*

Proof. The "only if" part comes from Proposition 2.3.6. Foe the "if" part, if R is not a division ring, then the result follows from Proposition 2.3.5. If R is a division ring, then $\mathrm{diam}(G(R)) \leq 2$. As $2 \in U(R)$, we have $1 \neq -1$. As $d(1, -1) = 2$, we have $\mathrm{diam}(G(R)) = 2$. □

We note that, however, in the previous example, every element in $\mathbb{Z}_4$ can be expressed as a sum of at most two units. So we recall another slightly different definition which was introduced in [34]. Let $usn(R)$ be the smallest positive integer n such that every element can be written as the sum of at most n units. If some element of R is not k-good for any $k \geq 1$, then $usn(R)$ is defined to be ∞. Note that $usn(R)$ and $\mathbf{u}(R)$ are different. For example, $\mathbf{u}(\mathbb{Z}_2) = \omega$ and $usn(\mathbb{Z}_2) = 2$.

In [33], Heydari and Nikmehr proved that $\mathrm{diam}(G(R)) \in \{1, 2, 3, \infty\}$ for an Artinian ring R. It is interesting to know whether there exists a ring R such that

$3 < \operatorname{diam}(G(R)) < \infty$. In [34, Corollary 4], the authors proved that there exists a ring R such that $usn(R) = n$ for each given $n \geq 2$. This result can be used to show that there exists a ring R such that $3 < \operatorname{diam}(G(R)) < \infty$.

Theorem 2.3.8. *Let R be a ring but not a division ring. If $usn(R) = n$, then $n \leq \operatorname{diam}(G(R)) \leq 2n$.*

Proof. We can assume $usn(R) = n \geq 2$. Since, R is not a division ring, there exists an element $0 \neq r \in R$, such that r is a sum of n units but not a sum of m units for any $m < n$. We claim that $d(r, 0) \geq n$. If $d(r, 0) = k < n$, then, by Lemma 2.3.4(2), r is k-good, a contradiction. So $d(r, 0) \geq n$ and hence $\operatorname{diam}(G(R)) \geq n$. On the other hand, let $x, y \in R$. Suppose that x is k-good and y is l-good. By Lemma 2.3.4(1), $d(x, 0) \leq k \leq n$ and $d(y, 0) \leq l \leq n$, so $d(x, y) \leq k + l \leq 2n$. This implies that $\operatorname{diam}(G(R)) \leq 2n$. □

Corollary 2.3.9. *There exists a ring R such that $3 < \operatorname{diam}(G(R)) < \infty$.*

Proof. This follows from Theorem 2.3.8 and [34, Corollary 4]. □

The condition that R is not a division ring is necessary in Theorem 2.3.8. For example, $usn(\mathbb{F}_4) = 2$, but $\operatorname{diam}(G(\mathbb{F}_4)) = 1$.

Next, we focus on a self-injective ring. In [8], Ashrafi et al. proved that $\operatorname{diam}(G(R))$ is 1, 2, 3 or ∞ for a finite (commutative) ring R, In [33], the authors generalized the result to an Artinian ring R and classified all Artinian rings according to the diameter of their unit graphs. We generalize these results to rings R with $R/J(R)$ self-injective.

Lemma 2.3.10. *Let R be a regular right self-injective ring. Then $\operatorname{diam}(G(R)) \in \{1, 2, 3, \infty\}$.*

Proof. By [40, Theorem 6], $\mathbf{u}(R) = 2, \omega$ or ∞. Suppose that $\mathbf{u}(R) = 2$. If R is not a division ring, then $diam(G(R) = 2$ by Proposition 2.3.5. If R is a division ring, then $\operatorname{diam}(G(R)) \leq 2$.

Suppose that $\mathbf{u}(R) = \omega$. Then, by [40, Theorem 6(2)], we may assume that $R = R_1 \times \mathbb{Z}_2$, where $\mathbf{u}(R_1) = 1$ or 2. If $\mathbf{u}(R_1) = 1$, R_1 is a trivial ring and $R = \mathbb{Z}_2$ and so $\operatorname{diam}(G(R)) = 1$. Now suppose that $\mathbf{u}(R_1) = 2$ and let $x, y \in G(R)$. If $x = (x_1, 0)$ and $y = (y_1, 0)$, then there exists $z_1 \in R_1$, such that $x_1 + z_1$ and $z_1 + y_1$ are units in R_1. So a path $(x_1, 0)$—$(z_1, 1)$—$(y_1, 0)$ from $(x_1, 0)$ to $(y_1, 0)$ deduces $d(x, y) \leq 2$; if $x = (x_1, 1)$ and $y = (y_1, 1)$, a similar argument shows that $d(x, y) \leq 2$; if $x = (x_1, 0)$ and $y = (y_1, 1)$, then there exists $z_1 \in R_1$, such that $x_1 + z_1$ is a unit in R_1. With a similar argument, we have a path $(x_1, 0)$—$(z_1, 1)$—$(w_1, 0)$—$(y_1, 1)$ and hence $d(x, y) \leq 3$. So $diam(G(R)) \leq 3$ always holds.

Suppose that $\mathbf{u}(R) = \infty$. By [8, Theorem 4.3], we know that $G(R)$ is disconnected. So $diam(G(R)) = \infty$. The proof is complete. $\square$

Theorem 2.3.11. *Let R be a ring with $R/J(R)$ right self-injective. Then* $\operatorname{diam}(G(R)) \in \{1, 2, 3, \infty\}$.

Proof. We know that in this case $\overline{R} = R/J(R)$ is a regular right self-injective ring. By Lemma 2.3.10, we have $\operatorname{diam}(\overline{R}) \in \{1, 2, 3, \infty\}$. By Lemma 2.3.1, we get $\operatorname{diam}(G(R)) \in \{1, 2, 3, \infty\}$. $\square$

Theorem 2.3.12. *Let R be a ring with $R/J(R)$ right self-injective. Then the following statements hold:*

(1) $\operatorname{diam}(G(R)) = 1$ *if and only if R is a division ring with $char(R) = 2$.*

(2) $\operatorname{diam}(G(R)) = 2$ *if and only if R is not a division ring with $char(R) = 2$ and one of following holds:*

(i) $\overline{R}$ *has no nonzero Boolean ring as a ring direct summand.*

(ii) $\overline{R} \cong \mathbb{Z}_2$.

(3) $\text{diam}(G(R)) = 3$ *if and only if* $\overline{R} \ncong \mathbb{Z}_2$ *and* $\overline{R}$ *has* $\mathbb{Z}_2$*, but no Boolean ring with more than two elements, as a ring direct summand.*

(4) $\text{diam}(G(R)) = \infty$ *if and only if* $\overline{R}$ *has a Boolean ring with more than two elements as a ring direct summand.*

Proof. (1) This follows from [8, Theorem 3.4].

Next, we assume that R is not a division ring with $char(R) = 2$ and prove (2), (3) and (4) together. Note that $\overline{R}$ is a regular right self-injective ring. So $\mathbf{u}(\overline{R}) = 2$, ω or ∞ by [40, Theorem 6]. To complete the proof, we determine the diameter of $G(R)$ for each case.

Case 1: $\mathbf{u}(\overline{R}) = 2$. In this case, $\overline{R}$ has no nonzero Boolean ring as a ring direct summand or $\overline{R} \cong \mathbb{Z}_2$ by [40, Theorem 6]. Note that $\text{diam}(G(\overline{R})) \in \{1, 2\}$. So $\text{diam}(G(R)) = 2$ by Lemma 2.3.1.

Case 2: $\mathbf{u}(\overline{R}) = \omega$. If $\overline{R} \cong \mathbb{Z}_2$, then $G(R)$ is a complete bipartite graph. So $\text{diam}(G(R)) = 2$. If $\overline{R} \ncong \mathbb{Z}_2$, in this case, we claim that $\text{diam}(G(R)) = 3$. To see this, in view of the proof of Lemma 2.3.10, we know that $\text{diam}(G(\overline{R})) \leq 3$. Note that $d((0, 0), (x, 1)) = 3$ if x is not a unit. So $\text{diam}(G(\overline{R})) = 3$. By Lemma 2.3.1, we have $\text{diam}(G(R)) = 3$.

Case 3: $\mathbf{u}(\overline{R}) = \infty$. Then $G(\overline{R})$ is disconnected by [8, Theorem 4.3], so $\text{diam}(G(\overline{R})) = \infty$. Thus $\text{diam}(G(R)) = \infty$ by Lemma 2.3.1. $\square$

2.4 The Planarity of Unit Graphs

The concentration, in this section, is on the planarity of the unit graph of a ring. Recall that a graph is said to be planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The planarity is an important invariant in graph theory. This work is motivated by the following result of Ashrafi, et al. [8] who completely determined the finite commutative rings whose unit graphs are planar.

Theorem 2.4.1. [8] *Let R be a finite commutative ring. Then $G(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3,\ \mathbb{F}_4,\ \mathbb{Z}_5,\ \mathbb{Z}_3\times\mathbb{Z}_3,\ B,\ \mathbb{Z}_3\times B,\ \mathbb{F}_4\times B,\ \mathbb{Z}_4,\ \tfrac{\mathbb{Z}_2[t]}{(t^2)},\ \mathbb{Z}_4\times B,\ \tfrac{\mathbb{Z}_2[t]}{(t^2)}\times B,$$

where B is a finite Boolean ring. □

A natural question is to characterize the rings (not necessarily finite, and not necessarily commutative) whose unit graphs are planar. This question is settled in this section. Our main result is the following characterization of rings with planar unit graphs.

Theorem 2.4.2. *Let R be a ring. Then $G(R)$ is planar if and only if one of the following holds:*

(1) $|U(R)| \leq 3$ *and* $|R| \leq \mathbf{c}$.

(2) $|U(R)| = 4$, *char*$(R) = 0$ *and* $|R| \leq \mathbf{c}$.

(3) $R \cong \mathbb{Z}_5$.

(4) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

□

To prove Theorem 2.4.2, we proceed with a series of lemmas. The first one is a quick consequence of Theorem 2.4.1.

Lemma 2.4.3. *Let R be a finite commutative ring. If $G(R)$ is planar, then $2 \leq char(R) \leq 6$. Furthermore,*

(1) *If $char(R) = 2$, then $|U(R)| \leq 3$.*

(2) *If $char(R) = 3$, then $|U(R)| \leq 4$.*

(3) *If $char(R) = 4$, then $|U(R)| \leq 2$.*

(4) *If $char(R) = 5$, then $|U(R)| \leq 4$.*

(5) *If $char(R) = 6$, then $|U(R)| \leq 2$.*

□

The next lemma was proved in [8, Proposition 2.4] for a finite ring R and it can be shown by the same argument there.

Lemma 2.4.4. *Let R be a ring with $|U(R)| = k < \infty$. If $2 \notin U(R)$, then $G(R)$ is k-regular.*

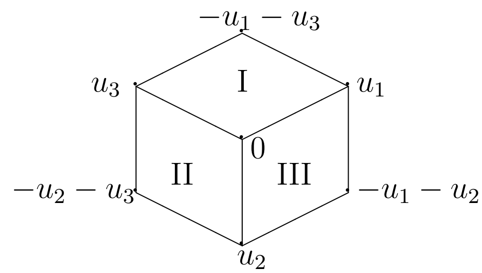
Proof. Let $x \in R$. As $2 \notin U(R)$, $u - x \neq x$ for $u \in U(R)$. Thus, $\phi : U(R) \to N(x)$ given by $\phi(u) = u - x$ is a bijection. So, $deg(x) = |N(x)| = |U(R)| = k$. So $G(R)$ is k-regular. □

Lemma 2.4.5. *Let R be a ring. If $G(R)$ is planar, then $|U(R)| < \infty$.*

Proof. Assume to the contrary that $|U(R)| = \infty$. Take $u_1 \in U(R)$ and $u_2 \in U(R) \backslash \{u_1, -u_1\}$. We show next that there is a contradiction.

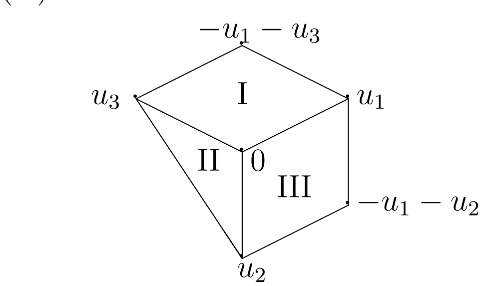
Case 1: $u_1 \neq -u_1 - u_2 \neq u_2$. In this case, we take $u_3 \in U(R) \backslash \{u_1, u_2, -u_1, -u_2, -u_1 - u_2\}$.

Subcase 1.1: $u_1 \neq -u_1 - u_3 \neq u_3$ and $u_2 \neq -u_2 - u_3 \neq u_3$. Then the following graph is a subgraph of $G(R)$:



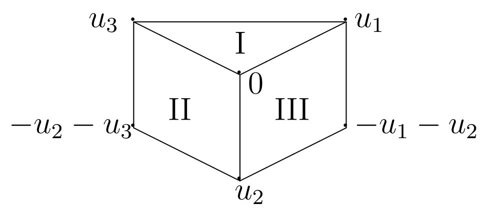
Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_1 - u_3, -u_2 - u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2, u_2 + u_3 - u_1\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices $0, u_1, u_3, -u_1 - u_3$. But this contradicts the choice of v.

Subcase 1.2: $u_1 \neq -u_1 - u_3 \neq u_3$ and $-u_2 - u_3 = u_2$ or u_3. Then the following graph is a subgraph of $G(R)$:



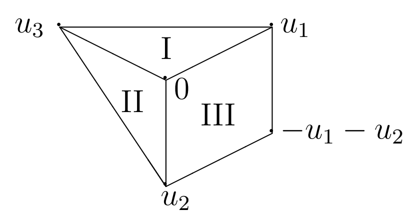
Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_1 - u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices $0, u_1, u_3, -u_1 - u_3$. But this contradicts the choice of v.

Subcase 1.3: $-u_1 - u_3 = u_1$ or u_3, and $u_2 \neq -u_2 - u_3 \neq u_3$. Then the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_2 - u_3, u_1 + u_2 - u_3, u_2 + u_3 - u_1\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices $0, u_1, u_3$. But this contradicts the choice of v.

Subcase 1.4: $-u_1 - u_3 = u_1$ or u_3, and $-u_2 - u_3 = u_2$ or u_3 (of course, it cannot occur that $-u_1 - u_3 = u_3$ and $-u_2 - u_3 = u_3$). Then the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, u_1 + u_2 - u_3\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices $0, u_1, u_3$. But this contradicts the choice of v.

Case 2: $-u_1 - u_2 = u_1$ or u_2. Take $u_3 \in U(R) \backslash \{u_1, u_2, -u_1, -u_2\}$. A similar argument as in Case 1 yields a contradiction. $\square$

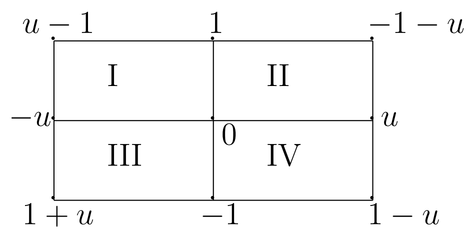
Lemma 2.4.6 is a self-strengthening of Lemma 2.4.5.

Lemma 2.4.6. *Let R be a ring. If $G(R)$ is planar, then $|U(R)| \leq 4$.*

Proof. Assume to the contrary that $|U(R)| \geq 5$. To get a contradiction, we proceed with two cases.

Case 1: $char(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Since $|U(R)| < \infty$ by Lemma 2.4.5, $n \notin U(R)$ for all $\pm 1 \neq n \in \mathbb{Z}$. Take $\pm 1 \neq u \in U(R)$.

Subcase 1.1: $2u \neq -2$ and $2u \neq 2$. That is, $-1 - u \neq 1 + u$ and $u - 1 \neq 1 - u$. In this case, the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R) \backslash \{1, -1, u, -u\}$. Since $G(R)$ is planar and v is adjacent to 0, either v is in one of the regions (I), (II), (III) and (IV), or v is one of the vertices $u - 1$, $-1 - u$, $1 - u$ and $1 + u$.

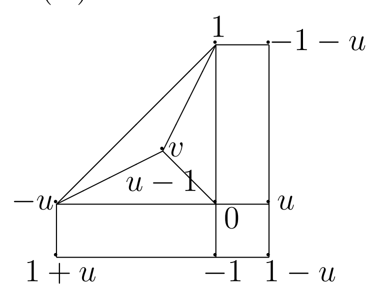
If v is in region (I), consider the vertices $1 - v$ and $-u - v$. As $1 - v$ is adjacent to both v and -1, we have $1 - v = -u$ or $1 - v = u - 1$. As $-u - v$ is adjacent to both v and u, we have $-u - v = 1$ or $-u - v = u - 1$. Thus, we must have a contradiction: If $1 - v = -u$ and $-u - v = 1$, then $2v = 0$, i.e. $2 = 0$; If $1 - v = u - 1$ and $-u - v = 1$, then $3 = 0$; If $1 - v = -u$ and $-u - v = u - 1$, then $3u = 0$, i.e. $3 = 0$; If $1 - v = u - 1$ and $-u - v = u - 1$, then $u = -1$.

If v is in region (II), consider the vertices $1-v$ and $u-v$. Arguing as above, we have $1-v=-1-u$ or $1-v=u$, and $u-v=1$ or $u-v=-1-u$. This clearly leads to a contradiction.

If v is in region (III), consider the vertices $-1-v$ and $-u-v$. Then we have $-1-v=-u$ or $-1-v=1+u$, and $-u-v=1+u$ or $-u-v=-1$. This also leads to a contradiction.

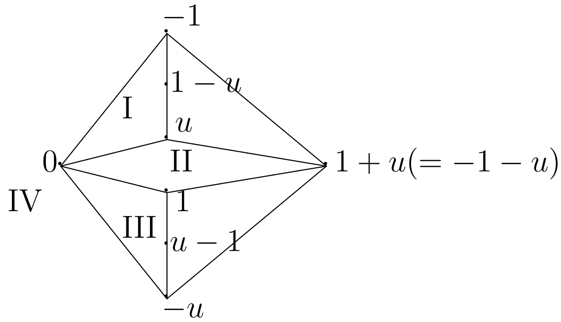
If v is region in (IV), consider the vertices $-1-v$ and $u-v$. Then we have $-1-v=u$ or $-1-v=1-u$, and $u-v=-1$ or $u-v=1-u$, and this also leads to a contradiction.

If v is one of the vertices $u-1$, $-1-u$, $1-u$ and $1+u$, we can assume that $v=u-1$ (the other cases are similar). Note that 1 is adjacent to $-u$. So we have the following subgraph of $G(R)$:



As $-u-v$ is adjacent to both v and u, we must have $-u-v=1$. As $1-v$ is adjacent to both v and -1, we must have $1-v=-u$. Thus, $2v=0$, i.e. $2=0$, a contradiction.

Subcase 1.2: $2u=-2$, i.e. $-1-u=1+u$. In this case, $u-1\neq 1-u$, so the following graph is a subgraph of $G(R)$:



Take $v \in U(R)\backslash\{1, -1, u, -u\}$. Then either v is in one of the regions (I), (II), (III) and (IV) or $v \in \{1-u, u-1, 1+u\}$.

If v is in region (I), consider the vertices $-1-v$ and $u-v$. As $-1-v$ is adjacent to both v and 1, we have $-1-v = u$. As $u-v$ is adjacent to both v and $-u$, we have $u - v = -1$. It follows that $-2v = 0$, i.e. $2 = 0$, a contradiction.

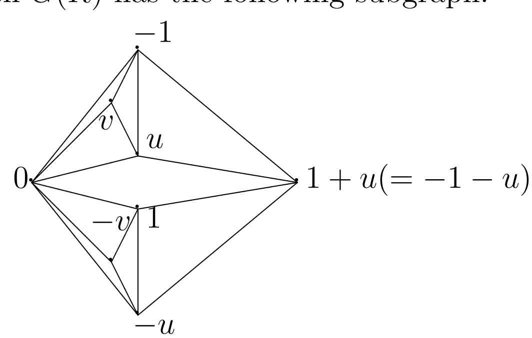
If v is in region (II), consider the vertices $1-v$ and $u-v$. Arguing as above, we have $1-v = u$ and $u-v = 1$, which gives $-2v = 0$, i.e. $v = 0$, a contradiction.

If v is in region (III), consider the vertices $1-v$ and $-u-v$ and we have $1-v = -u$ and $-u-v = 1$, giving $-2v = 0$, i.e. $v = 0$, a contradiction.

If v is in region (IV), consider the vertices $-1-v$ and $-u-v$ and we have $-1-v = -u$ and $-u-v = -1$, giving $-2v = 0$, i.e. $v = 0$, a contradiction.

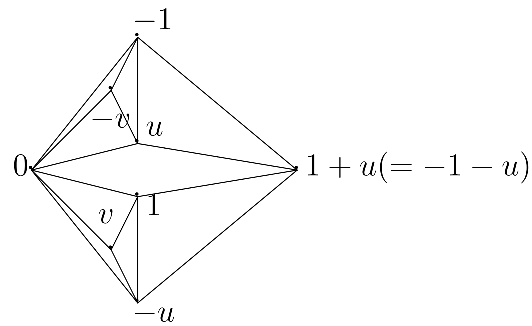
Now assume $v \in \{1-u, u-1, 1+u\}$. If $v = 1+u$, then 0 is adjacent to $1+u$ and 1 is adjacent to u. This is impossible.

If $v = 1-u$, then $G(R)$ has the following subgraph:



In this case, we consider the vertices $-1-v$ and $u-v$. As $-1-v$ is adjacent to both 1 and v, we have $-1-v=u$; as $u-v$ is adjacent to both $-u$ and v, we have $u-v=-1$. So $-2v=0$, i.e. $2=0$, a contradiction.

If $v=u-1$, $G(R)$ has the following subgraph:



In this case, we consider the vertices $1-v$ and $-u-v$. As $1-v$ is adjacent to both -1 and v, we have $1-v=-u$; as $-u-v$ is adjacent to both u and v, we have $-u-v=1$. So $-2v=0$, i.e. $2=0$, a contradiction.

Subcase 1.3: $2u=2$, i.e. $u-1=1-u$. In this case, $-1-u\neq 1+u$. By a similar process as Subcase 1.2, we also can get a contradiction.

Case 2: $char(R)=n\geq 2$. Then R contains $\mathbb{Z}_n$ as a subring. Since $G(\mathbb{Z}_n)$ is planar, we have $n\leq 6$ by Lemma 2.4.3. We need two notations. For any $a\in R$, let $\mathbb{Z}_n[a]$ be the subring of R generated by $\mathbb{Z}_n\cup\{a\}$. Note that $G(\mathbb{Z}_n[a])$ is also planar. For $u\in U(R)$, let $o(u)$ be the order of u in the multiplicative group $U(R)$. Then $o(u)<\infty$ for all $u\in U(R)$ by Lemma 2.4.5.

Subcase 2.1: $n=6$. Take $\pm 1\neq u\in U(R)$. As $o(u)<\infty$, $\mathbb{Z}_6[u]$ is a finite commutative ring. So, by Lemma 2.4.3(5), $|U(\mathbb{Z}_6[u])|\leq 2$. But $\mathbb{Z}_6[u]$ has at least three units, a contradiction.

Subcase 2.2: $n=5$. Take $u\in U(R)\backslash U(\mathbb{Z}_5)$. Then $\mathbb{Z}_5[u]$ is a finite commutative subring of R. So, by Lemma 2.4.3(4), $|U(\mathbb{Z}_5[u])|\leq 4$. But $\mathbb{Z}_5[u]$ has at least five units, a contradiction.

Subcase 2.3: $n=4$. Take $\pm 1\neq u\in U(R)$. Then $\mathbb{Z}_4[u]$ is a finite commutative subring of R. So, by Lemma 2.4.3(3), $|U(\mathbb{Z}_4[u])| \leq 2$. But $\mathbb{Z}_4[u]$ has at least three units, a contradiction.

Subcase 2.4: $n = 3$. Take $\pm 1 \neq u \in U(R)$. As above, $\mathbb{Z}_3[u]$ is a finite commutative subring of R. So, by Lemma 2.4.3(2), we have $|U(\mathbb{Z}_3[u])| \leq 4$. In particular, $o(u) \leq 4$. If $o(u) = 4$ and $u^2 = -1$, then $\mathbb{Z}_3[u]$ contains at least 8 units: $1, -1, u, -u, 1+u, 1-u, -1+u$ and $-1-u$, a contradiction. If $o(u) = 4$ and $u^2 \neq -1$, then $1, 2, u, u^2, u^3$ are five distinct units of $\mathbb{Z}_3[u]$, a contradiction. If $o(u) = 3$, then $1, 2, u, 2u, u^2, 2u^2$ are six distinct units of $\mathbb{Z}_3[u]$, a contradiction.

Hence $o(u) = 2$, and in this case, $U(\mathbb{Z}_3[u]) = \{1, 2, u, 2u\}$. Note that the same argument as above shows that $v^2 = 1$ for all $v \in U(R)$. So the group $U(R)$ is abelian. As $|U(R)| \geq 5$, take $v \in U(R) \backslash U(\mathbb{Z}_3[u])$. Consider the subring $\mathbb{Z}_3[u, v]$ of R generated by $\mathbb{Z}_3[u] \cup \{v\}$. Then $\mathbb{Z}_3[u, v]$ is a finite commutative ring containing at least 5 units: $1, 2, u, 2u, v$. This contradicts Lemma 2.4.3(2).

Subcase 2.5: $n = 2$. Let $H = U(R)$. For $u \in H$, $\mathbb{Z}_2[u]$ is a finite commutative ring. So, by Lemma 2.4.3(1), we have $|U(\mathbb{Z}_2[u])| \leq 3$. In particular, $o(u) \leq 3$. Thus, we have proved that $o(u) \leq 3$ for all $u \in H$.

If $H \cong S_3$, the symmetric group of degree 3, then the subring $\mathbb{Z}_2[H]$ of R generated by $\mathbb{Z}_2 \cup H$ is a finite ring containing exactly six units such that 2 is not a unit of $\mathbb{Z}_2[H]$. Hence, by Lemma 2.4.4, $G(\mathbb{Z}_2[H])$ is 6-regular. In particular, $G(\mathbb{Z}_2[H])$ is not planar, and so $G(R)$ is not planar. This contradiction shows that H is not isomorphic to S_3. To finish the proof, we need the following claim.

Claim: There exist $u, v \in H \backslash \{1\}$ such that $uv = vu$ and $\langle u \rangle \cap \langle v \rangle = \{1\}$.

Proof of Claim. As above, we have $|H| = 2^k 3^l$, where $k, l \geq 0$. Note that $|H| \geq 5$ by hypothesis. If $k = 0$ or $l = 0$, there is nothing to prove because any finite p-group has nontrivial center. If $k > 1$, consider a Sylow 2-subgroup P of H. Being a finite p-group, P contains a non-trivial central element, say u. As $|\langle u \rangle| \leq 3$ and $|P| \geq 2^k \geq 4$, we can take $v \in P\backslash\langle u\rangle$. Then $uv = vu$ and $\langle u\rangle \cap \langle v\rangle = \{1\}$. If $l > 1$, we can consider a Sylow 3-subgroup and a similar argument also shows the existence of such elements u and v. If $k = l = 1$, then $|H| = 6$. As $H \not\cong S_3$, H is a cyclic group of order 6. But this is impossible, as every element of H has order less than or equal to 3. This completes the proof of the Claim.

Now by the Claim, take $u, v \in H\backslash\{1\}$ such that $uv = vu$ and $\langle u\rangle \cap \langle v\rangle = \{1\}$. Then the subring $\mathbb{Z}_2[u, v]$ of R generated by $\mathbb{Z}_2 \cup \{u, v\}$ is a finite commutative ring, containing at least four distinct units $1, u, v, uv$. This contradicts Lemma 2.4.3(1).

The proof is now complete. $\square$

The next lemma is about the genus of a simple graph, which will be used frequently in Section 2.5 and Section 3.4. A surface is said to be of *genus g* if it is topologically homeomorphic to a sphere with g handles. A graph G that can be drawn without crossing on a compact surface of genus g, but not on one of genus $g-1$, is called a graph of genus g. The genus of a graph G is denoted by $\gamma(G)$. Note that a graph is planar if and only if it has genus zero.

Lemma 2.4.7. [70, Corollaries 6.14, 6.15] *Suppose that a simple graph G is connected with $p \geq 3$ vertices and q edges. Then $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, if G has no triangles, then $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$.* $\square$

Now we are ready to prove our main result in this section.

Proof of Theorem 2.4.2. ($\Longrightarrow$). Suppose that $G(R)$ is planar. Then R embeds in $\mathbb{R} \times \mathbb{R}$ as sets, so $|R| \leq \mathbf{c}$. By Lemma 2.4.6, $|U(R)| \leq 4$. If $|U(R)| = 3$, we are done. So we can assume that $|U(R)| = 4$, and we can further assume $n := char(R) > 0$. Then R contains $\mathbb{Z}_n$ as a subring. Being a subgraph of $G(R)$, $G(\mathbb{Z}_n)$ is planar, so $2 \leq n \leq 6$ by Lemma 2.4.3. Take $\pm 1 \neq u \in U(R)$. Then $\mathbb{Z}_n[u]$ is a finite commutative subring of R, and hence $G(\mathbb{Z}_n[u])$ is planar. If $n = 4$ or $n = 6$, then $\mathbb{Z}_n[u]$ contains at least three units; this is impossible by Lemma 2.4.3(3,4). So $n \neq 4$ and $n \neq 6$. Next we prove that $n \neq 2$. Assume that $n = 2$. Then, for any $1 \neq u \in U(R)$, $\mathbb{Z}_2[u]$ is a finite commutative subring of R, and hence $o(u) \leq 3$ by Lemma 2.4.3(1). If $o(u) = 3$, take $v \in U(R)\backslash\{1, u, u^2\}$ and we see $1, u, u^2, v, uv$ are five distinct units of R, contradicting that $|U(R)| = 4$. Hence $o(u) \leq 2$ for all $u \in U(R)$. So $U(R)$ is a commutative multiplicative group. Take $1 \neq u \in U(R)$ and $v \in U(R)\backslash\{1, u\}$. Then $\mathbb{Z}_2[u, v]$ is a finite commutative subring of R containing four units $1, u, v, uv$. But this is impossible by Lemma 2.4.3(1). Hence $n \neq 2$. Thus, we have proved that $n = 3$ or $n = 5$.

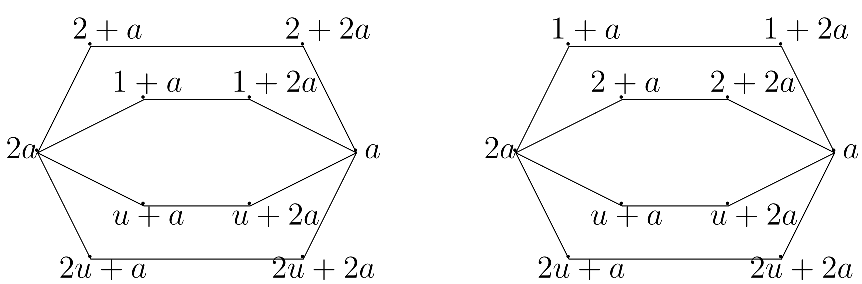
Suppose $n = 3$. We prove that $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Take $\pm 1 \neq u \in U(R)$. Then $\mathbb{Z}_3[u]$ is a finite commutative subring of R, and $U(\mathbb{Z}_3[u]) = \{1, 2, u, 2u\}$ (as $|U(R)| = 4$). If $R \neq \mathbb{Z}_3[u]$, take $a \in R\backslash\mathbb{Z}_3[u]$ and consider the subring $\mathbb{Z}_3[u, a]$ of R generated by $\mathbb{Z}_3 \cup \{u, a\}$. Note that

$$a\text{—}(1+2a)\text{—}(1+a)\text{—}2a\text{—}(u+a)\text{—}(u+2a)\text{—}a$$

and

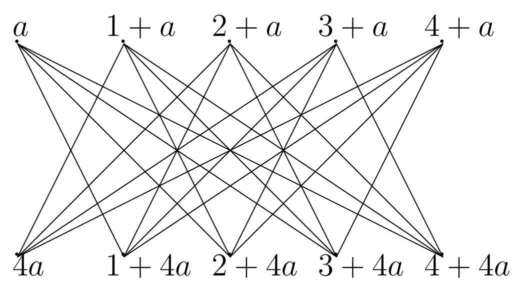
$$a\text{—}(2+2a)\text{—}(2+a)\text{—}2a\text{—}(2u+a)\text{—}(2u+2a)\text{—}a$$

are two 6-cycles in $G(\mathbb{Z}_3[u, a])$. By symmetry, essentially there are two ways to draw the subgraph below:



For the subgraph on the left, as $u+2+2a$ is adjacent to both $1+a$ and $2u+a$, the planarity of $G(R)$ ensures that $u+2+2a=a$. On the other hand, as $u+2+a$ is adjacent to both $1+2a$ and $2u+2a$, the planarity of $G(R)$ ensures that $u+2+a=2a$. So, it follows that $a=-a$, i.e., $2a=0$ or $a=0$, a contradiction. For the subgraph on the right, as $u+1+2a$ is adjacent to both $2+a$ and $2u+a$, the planarity of $G(R)$ ensures that $u+1+2a=a$. On the other hand, as $u+1+a$ is adjacent to both $2+2a$ and $2u+2a$, the planarity of $G(R)$ ensures that $u+1+a=2a$. So, it follows that $a=-a$, i.e., $2a=0$ or $a=0$, a contradiction. Therefore, $R=\mathbb{Z}_3[u]$ with $\mathbb{Z}_3[u]\cong\mathbb{Z}_3\times\mathbb{Z}_3$.

Suppose $n=5$. We prove that $R\cong\mathbb{Z}_5$. We see that R contains $\mathbb{Z}_5$ as a subring. Assume to the contrary that $R\neq\mathbb{Z}_5$. Take $a\in R\backslash\mathbb{Z}_5$. The following graph H is a subgraph of $G(\mathbb{Z}_5[a])$, and hence of $G(R)$:

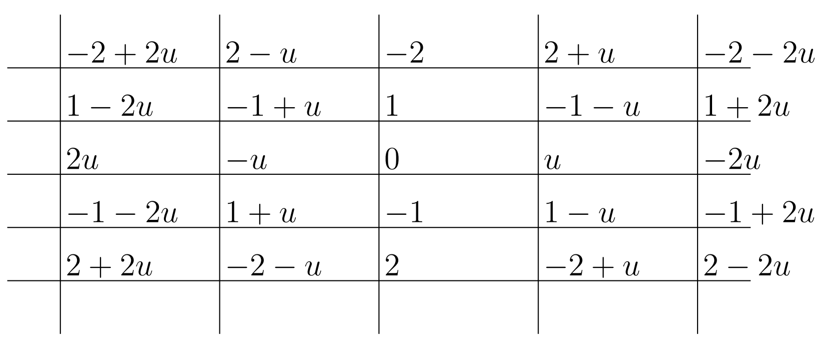


Note that H has 10 vertices and 20 edges with no triangles. So $\gamma(H)\geq 1$ by Lemma 2.4.7. This shows that H is not planar, giving a contradiction that $G(R)$ is planar.

($\Longleftarrow$). We have $|R|\leq\mathbf{c}$. If $R\cong\mathbb{Z}_5$ or $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$, then $G(R)$ is planar by Theorem 2.4.1. If $|U(R)|\leq 2$, then the maximal degree of $G(R)$ is at most two, so $G(R)$ must be planar.

Suppose that $|U(R)|=3$. Then we easily see that $2=0$ in R. So $G(R)$ is 3-regular by Lemma 2.4.4. Let $U(R)=\{u_1,u_2,u_3\}$. For a given $r\in R$, r is adjacent to $u_i - r$ $(i = 1,2,3)$. If $u_1 - r$ is adjacent to one of $u_i - r$ $(i = 2,3)$, say $u_2 - r$, then $(u_1 - r) + (u_2 - r) = u_1 + u_2$ is a unit of R, so it must be that $u_1 + u_2 = u_3$. Thus $u_1 - r$ is also adjacent to $u_3 - r$ and $u_2 - r$ is adjacent to $u_3 - r$. Hence, the vertices $r, u_1 - r, u_2 - r, u_3 - r$ form a complete graph K_4. As $G(R)$ is 3-regular, $G(R)$ must be a disjoint union of copies of K_4, so $G(R)$ is planar. Therefore, we can let the neighborhoods of $u_1 - r$ be r, a, b, where $a, b \notin \{u_2 - r, u_3 - r\}$. We may assume $u_1 - r + a = u_2$ and $u_1 - r + b = u_3$. Then $u_2 - r + a = u_1$ and $u_3 - r + b = u_1$. This means that a is adjacent to $u_2 - r$ and b is adjacent to $u_3 - r$. Let c be the third neighborhood of $u_2 - r$. Then $u_2 - r + c = u_3$, so $u_3 - r + c = u_2$. This means that c is also a neighborhood of $u_3 - r$. Now consider the vertex a. Let the neighborhoods of a be $u_1 - r, u_2 - r, x$. Then $a + x = u_3$. As $b + x = b + u_3 - a = r + u_1 - a = u_1 - r + a = u_2$, x is adjacent to b. Similarly, x is adjacent to c. So, the vertices $r, u_1 - r, u_2 - r, u_3 - r, a, b, c$ and x form a cube, which is 3-regular. As $G(R)$ is 3-regular, $G(R)$ must be a disjoint union of copies of a cube. As a cube is a planar graph, $G(R)$ is planar.

Finally, suppose that $|U(R)| = 4$ and $char(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Take $\pm 1 \neq u \in U(R)$. As $|U(R)| = 4$, we have $U(R) = \{1, -1, u, -u\}$. By Lemma 2.4.4, both $G(\mathbb{Z}[u])$ and $G(R)$ are 4-regular. It follows that $G(R)$ is a disjoint union of $G(\mathbb{Z}[u])$. As shown below, $G(\mathbb{Z}[u])$ is planar, so $G(R)$ is planar.



Graph $G(\mathbb{Z}[u])$

$\square$

We end the section by giving some examples of rings with planar unit graphs.

Example 2.4.8. *Let $\mathbb{T}_2(\mathbb{Z}_2)$ be the 2×2 upper triangular matrix ring over $\mathbb{Z}_2$ and let B be the zero ring or a finite Boolean ring. Then $R = \mathbb{T}_2(\mathbb{Z}_2) \times B$ has a planar unit graph.* $\square$

A ring R is *semilocal* if $R/J(R)$ is semisimple Artinian, where $J(R)$ is the Jacobson radical of R. The next example gives a countable non-semilocal ring whose unit graph is planar. Let D be a ring and C be a subring of D. With addition and multiplication defined componentwise, $\mathcal{R}[D, C] := \{(d_1, \cdots, d_n, c, c, \cdots) : d_i \in D, c \in C, n \geq 1\}$ becomes a ring.

Example 2.4.9. *Let $S = R \propto R/I$ where $R = \mathcal{R}[\mathbb{Z}_2, \mathbb{Z}_2]$ and $I = \mathcal{R}[\mathbb{Z}_2, 0]$. Then S is not semilocal, but $G(S)$ is planar.*

Proof. We easily see that $J(S) = \{(0, x) : x \in R/I\}$, so $|J(S)| = |R/I| = 2$, and $S/J(S) \cong R$ is Boolean. Since $S/J(S)$ is an infinite Boolean ring, S is not semilocal. As $|U(S)| = 2$, $G(S)$ is planar by Theorem 2.4.2. $\square$

Some other examples of rings with planar unit graphs can be constructed through polynomial rings. In [7], the authors determined the finite rings R with $G(R[t])$ planar. By Theorem 2.4.2, we now can characterize the rings R with $G(R[t])$ planar. Remark that, for a reduced ring R, $U(R[t]) = U(R)$ (we cannot find a reference for this, but it can be easily proved).

Corollary 2.4.10. *Let R be a ring, and let $t_1, t_2, \ldots, t_n$ be commuting indeterminates over R. Then $G(R[t_1, t_2, \ldots, t_n])$ is planar if and only if R is reduced with $|R| \leq \mathbf{c}$ such that either $|U(R)| \leq 3$, or $|U(R)| = 4$ with $char(R) = 0$.*

Proof. Without loss of generality, we can assume that $n = 1$.

($\Longleftarrow$). This follows from Theorem 2.4.2 and the Remark above.

($\Longrightarrow$). As $G(R[t])$ is planar, R is reduced by [7, Proposition 6.1(ii)], and $|R[t]| \leq \mathbf{c}$. So $|R| \leq \mathbf{c}$. Moreover, by Theorem 2.4.2, either $|U(R[t])| \leq 3$, or $|U(R[t])| = 4$ with $char(R) = 0$. Since R is reduced, $U(R[t]) = U(R)$. So the result follows. □

We remark that the results in this section were taken from [59].

2.5 Higher Genus Unit Graphs for Finite Commutative Rings

All rings considered in this section are commutative. A planar graph is a graph with genus zero and a toroidal graph is a graph with genus one. Determining the genus of a graph is one of the most fundamental problems in topological graph theory. In [61], Thomassen proved that the graph genus problem is indeed NP-complete. This means that the problem can be solved in Polynomial time using a non-deterministic Turing machine. The genus of graphs associated with rings is the topic of a number of publications. For instances, the planarity of zero divisor graphs were studied in [9], [11] and [56]. The rings with toroidal zero divisor graphs were classified in Wang [67] and Wickham [68, 69]. Genus of two zero divisor graphs of local rings were investigated by Bloomfield and Wickham in [16]. Recently, Maimani et al. [53] determined all isomorphism classes of finite rings whose total graphs have genus at most one, and Tamizh Chelvam and Asir [62] characterized all isomorphism classes of finite rings whose total graphs have genus two. For a finite ring R, the unit graph $G(R)$ is the complement of the total graph of the ring R. In [8, Theorem 5.14], all finite rings having planar unit graphs are completely classified, and in [21] toroidal ones are completely determined. The goal of this section is to classify all finite rings R with $\gamma(G(R)) = 1$, 2, and 3, respectively.

We state some needed basic facts in graph theory and on finite rings. If H is a subgraph of a graph G, then $\gamma(H) \leq \gamma(G)$. The well-known *Euler's formula* says that, if G is a finite connected graph with p vertices, q edges and genus g, then $p - q + f = 2 - 2g$, where f is the number of faces created when G is minimally embedded on a surface of genus g. We refer the reader to [70] for the details on embedding a graph in a surface.

By Ganesan [25], if R is a ring containing n zero divisors with $n > 0$, then $|R| \leq n^2$. For a finite local ring R with maximal ideal $\mathfrak{m}$, there exists a prime p such that $|R/\mathfrak{m}| = p^t$ for some integer $t \geq 1$ and hence $|R| = p^n$ for some integer $n \geq t$. According to [18, P.687], all local rings having order 4, 9, 8 are, respectively, $\mathbb{F}_4$, $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, and $\mathbb{F}_9$, $\mathbb{Z}_9$, $\mathbb{Z}_3[X]/(X^2)$, and $\mathbb{F}_8$, $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$.

The first two lemmas from graph theory will be frequently used.

Lemma 2.5.1. [70, Theorems 6.37, 6.38] *Let $m \geq 2, n \geq 3, p \geq 2$ be integers. Then $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$, $\gamma(K_{m,p}) = \lceil \frac{1}{4}(m-2)(p-2) \rceil$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x.*

Lemma 2.5.2. [70, Corollary 6.19] *The genus of a graph is the sum of the genera of its components.*

The following result, which is of interest in its own right, is needed for the proof of our main theorem.

Theorem 2.5.3. *Let G be the complete 5-partite graph $K_{2,2,2,2,1}$. Then $\gamma(G) = 3$.*

Proof. We know that G is a subgraph of the complete graph K_9, so $\gamma(G) \leq \gamma(K_9) = 3$. On the other hand, $\gamma(G) \geq \frac{32}{6} - \frac{9}{2} + 1 = \frac{11}{6}$ by Lemma 2.4.7, so $2 \leq \gamma(G) \leq 3$. We use the following two facts.

Fact 1: [31, Theorem 1.7] There is no simple triangulation with 9 vertices on $\mathbb{S}_2$.

Fact 2: [54, Theorem 1.1] K_7 has only one embedding on $\mathbb{S}_1$ up to homeomorphism. Moreover, $K_7 - K_2$ has only two embeddings on $\mathbb{S}_1$ up to homeomorphism. When x and y are non-adjacent pair, one has a quadrangle face $x * y*$ and the other has a quadrangle face $* * **$ (i.e. x and y do not appear).

We suppose that there exists an embedding of $G = K_{2,2,2,2,1}$ on $\mathbb{S}_2$. By Euler's formula, such an embedding has 20 triangles and 1 quadrangle.

Let $V(G) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $01x7$ be a quadrangle face. We may assume that 0 and x have degree seven. Suppose that the consecutive neighbors of 0 are $\{1, 2, 3, 4, 5, 6, 7\}$. By Fact 1, we see that x is not 8. By symmetry, we can separate to two cases: $x = 4$ and $x = 3$.

Case 1: $x = 4$. By symmetry, we may assume that vertex 4 has neighbors 7∗503∗1 (if not, it will result in the case $x = 3$). We may also assume that the neighbors are 7250384. Now we see that (0,8) and (4,6) are non-adjacent pairs.

Cut the double torus by essential closed curve 404 (40 is an edge of G and 04 is a diagonal of a quadrangle face of G). Omitting the vertices 0,4 and covering two holes, we then get $K_7 - 2K_2$ on the torus which has two quadrangle faces 3812 and 5276. (It is obvious that the closed curve is not a separating curve.) By connections of edges, we see that $2K_2$ are (13, 57) or (15, 37) (notice that 17 cannot be a non-adjacent pair by Fact 1).

If $2K_2$ are (13, 57), we can get K_7 by adding two edges in quadrangle faces, but it contradicts to Fact 2 (K_7 on the tours cannot have such four triangle faces: 381,812, 527 and 726). If $2K_2$ are (15, 37), vertex 1 has neighbors 48∗ ∗ ∗20, but contradicts wherever 3 is.

Case 2: $x = 3$. Separate to two cases: 3 has neighbors 7y402z1, and 3 has neighbors 7402yz1.

Subcase 2.1: 3 has neighbors 7y402z1. We see that $y = 8$ by connections of edges (notice neighbors of vertex 4 if $y = 5$). Cut the double torus by essential closed curve 303 (30 is an edge of G and 03 is a diagonal of a quadrangle face of G). Omitting the vertices 0,3 and covering two holes, we then get $K_7 - 2K_2$ on the torus which has two faces 2z1 and 48765.

If $z = 6$, we see that non-adjacent pairs are 08, 35, 27 and 14 or 46. If 14 is, neighbors of 6 must be 05813247. Then neighbors of 2 must be 0185463. Then neighbors of 1 must be 863028$*$, a contradiction. If 46 is, we can add an edge 46 in the face 48765. The remaining non-adjacent pair 27 and quadrangle face 4876 contradict Fact 2.

If $z = 5$, we see that non-adjacent pairs are 08, 36. The others are (14, 27) or (14, 57) or (24, 57). If (14, 27) is, we can get a contradiction by searching neighbors of 5, 7 and 8. If (14, 57) is (resp. (24, 57)), we can add the edge 57 in the face 48765. The remaining nonadjacent pair 14 (resp. 24) and quadrangle face 4875 contradict Fact 2.

Subcase 2.2: 3 has neighbors 7402yz1. Cut the double torus by essential closed curve 303. Omitting the vertices 0,3 and covering two holes, we then get $K_7 - 2K_2$ on the torus which has two quadrangle faces 2yz1 and 4765. Separate to three cases: (y, z) = (8,6), (5,8) or (8,5) ((6,8) and (8,5) are the same by symmetry).

If $(y, z) = (8, 6)$, we see that non-adjacent pairs are 08, 35. The others are (14,26), (14,27) or (27,46). If (14,27) is, we can get a contradiction by noticing neighbors of 6. If (14,26) is (resp. (27,46)), we can add edge 26 (resp. 46) in the face 2861 (resp. 4765). The remaining nonadjacent pair 14 (resp. 27) and quadrangle face 4765 (resp. 2861) contradict Fact 2.

If $(y, z) = (5, 8)$, we see that non-adjacent pairs are 08, 36. The others are (15,24), (14,27), (15,27), (14,57) or (24,57). If (14,27) is, we can get a contradiction by noticing neighbors of 5. If (15,24) is (resp. (15,27)), we can add the edge 15 in the face 2581.

The remaining nonadjacent pair 24 (resp. 27) and quadrangle face 4765 contradict Fact 2. If (14,57) (resp. (24,57)), we can add the edge 57 in the face 4765. The remaining nonadjacent pair 14 (resp. 24) and quadrangle face 2581 contradict Fact 2.

If $(y, z) = (8, 5)$, we see that non-adjacent pairs are 08, 36. The others are (14,25), (14,27), (14,57), or (24,57). If (14,27) is, we can get a contradiction by noticing neighbors of 5. If (14,25) is, we can add the edge 25 in the face 2851. The remaining nonadjacent pair 14 and quadrangle face 4765 contradict Fact 2. If (14,57) is (resp. (24,57)), we can add the edge 57 in the face 4765. The remaining nonadjacent pair 14 (resp. 24) and quadrangle face 2851 contradict Fact 2.

All cases result in a contradiction. The proof is complete. $\square$

The next five lemmas are further preparation for proving our main theorem.

Lemma 2.5.4. *The following statements hold:*

(1) *Let $R = \mathbb{Z}_2 \times \mathbb{F}_q$. Then $\gamma(G(R)) \geq 5$ for $q \geq 7$, $\gamma(G(R)) = 1$ for $q = 5$, and $\gamma(G(R)) = 0$ for $q \leq 4$.*

(2) *Let $R = \mathbb{Z}_2 \times S$, where S is a local ring of order eight which is not a field. Then $\gamma(G(R)) = 2$.*

(3) *Let $R = \mathbb{Z}_2 \times S$, where S is a local ring of order nine which is not a field. Then $\gamma(G(R)) \geq 6$.*

Proof. (1) By Lemma 2.4.4, $G(R)$ is $(q-1)$-regular since $2 \notin U(R)$. By [8, Theorem 3.5], $G(R)$ is a bipartite graph, so it contains no triangles. As $G(R)$ contains $2q$ vertices and $q(q-1)$ edges, $\gamma(G(R)) \geq \frac{(q-1)(q-4)}{4}$ by Lemma 2.4.7. If $q \geq 7$, then $\gamma(G(R)) \geq 5$. If $q = 5$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$, and we see that $\gamma(G(R)) \geq 1$. On the other hand, we can embed $G(\mathbb{Z}_2 \times \mathbb{Z}_5)$ into $\mathbb{S}_1$ as shown in **Figure 1**. Hence, $\gamma(G(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 1$. If $q \leq 4$, $\gamma(G(R)) = 0$ by Theorem 2.4.1.

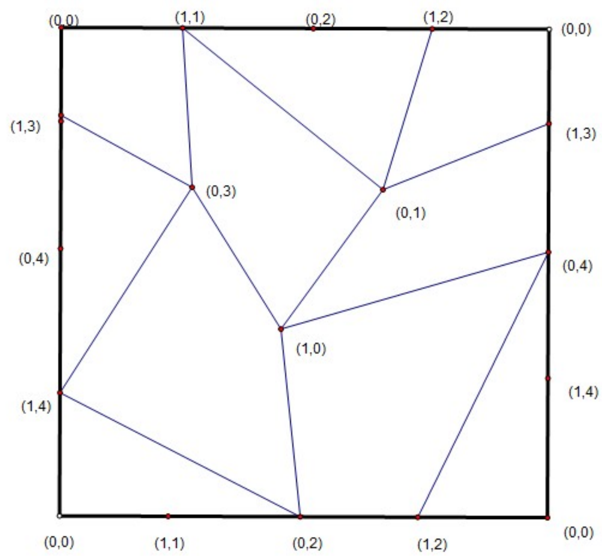


Figure 1: $G(\mathbb{Z}_2 \times \mathbb{Z}_5)$

(2) It is clear that $|U(S)| = 4$ and $|J(S)| = 4$. So $|U(R)| = |J(R)| = 4$. Note that each element in $J(R)$ is adjacent to every element in $U(R)$ and $G(R)$ is 4-regular. Thus, $G(R)$ is two copies of $K_{4,4}$. By Lemmas 2.5.1 and 2.5.2, $\gamma(G(R)) = 2$.

(3) It is clear that $|U(S)| = 6$ and $2 \notin U(R)$. By Lemma 2.4.4, $G(R)$ is 6-regular. As $G(R)$ contains no triangles, the claim follows from Lemma 2.4.7. $\square$

Lemma 2.5.5. *The following statements hold:*

(1) *Let $R = \mathbb{Z}_3 \times S$, where S is a local ring of order four which is not a field. Then $\gamma(G(R)) = 1$.*

(2) *Let $R = \mathbb{Z}_3 \times \mathbb{F}_4$. Then $\gamma(G(R)) = 3$.*

(3) *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_5$. Then $\gamma(G(R)) \geq 6$.*

Proof. (1) Note that $G(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and $G(\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2))$ have the same graph structure. It is clear that $G(R)$ is 4-regular and it contains no triangles. By Lemma 2.4.7,

$\gamma(G(R)) \geq 1$. On the other hand, we can embed $G(R)$ into $\mathbb{S}_1$ as shown in **Figure 2**. Hence, $\gamma(G(R)) = 1$.

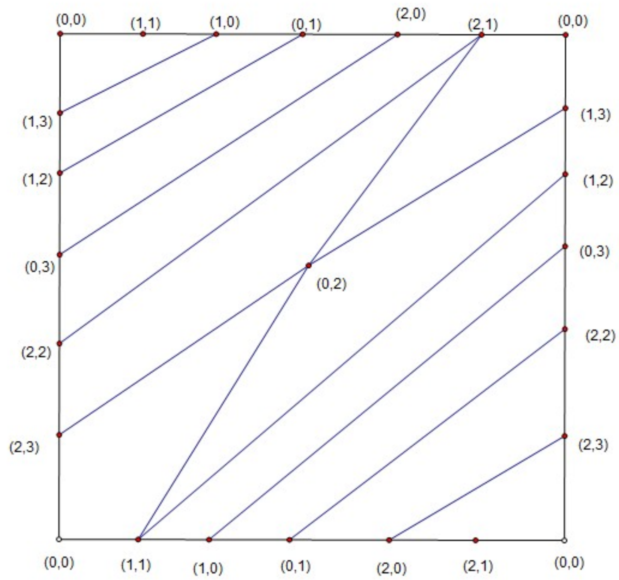


Figure 2: $G(\mathbb{Z}_3 \times \mathbb{Z}_4)$

(2) Write $\mathbb{F}_4 = \{0, 1, x, x+1\}$. Let $G = G(\mathbb{Z}_3 \times \mathbb{F}_4)$ and $G' = G - E$, where $E = \{(1,1)—(1,x), (1,1)—(1,1+x), (1,x)—(1,x+1), (2,1)—(2,x), (2,1)—(2,1+x), (2,x)—(2,x+1)\}$. So G' has 30 edges and 12 vertices and has no triangles. Thus $\gamma(G') \geq 3$ by Lemma 2.4.7. Hence G has genus at least three. On the other hand, we can embed G into $\mathbb{S}_3$ as shown in **Figure 3**. Therefore, $\gamma(G) = 3$.

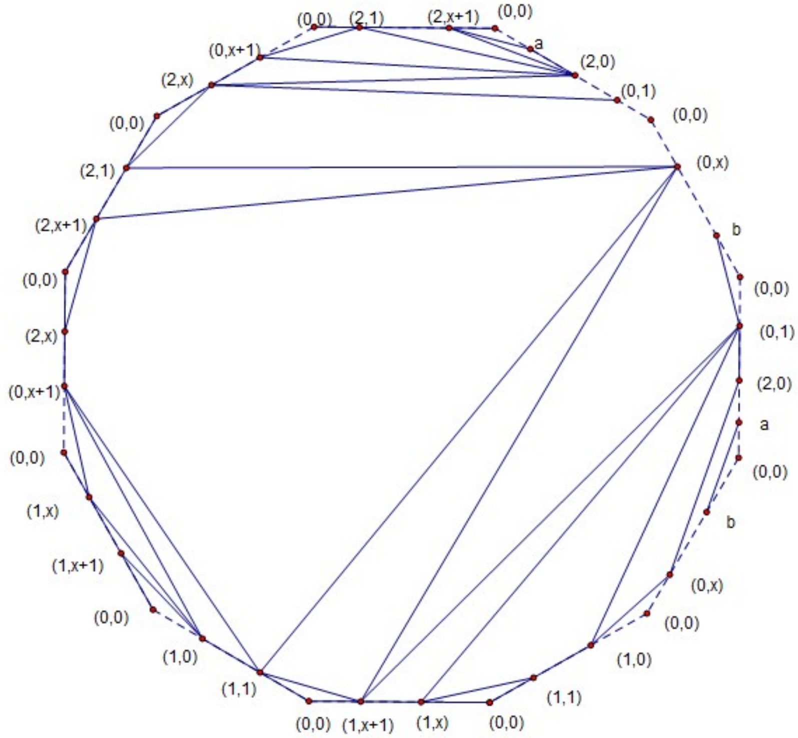


Figure 3: $G(\mathbb{Z}_3 \times \mathbb{F}_4)$

(3) Let $G' = G(R) - E$, where $E = \{(1,1)$—(1,2), (1,1)—(1,3), (1,2)—(1,4), (1,3)—(1,4), (2,1)—(2,2), (2,1)—(2,3), (2,2)—(2,4), (2,3)—(2,4)$\}$. Then G' has 48 edges and no triangles, so $\gamma(G') \geq 6$ by Lemma 2.4.7. Hence, $\gamma(G(R)) \geq 6$. □

Lemma 2.5.6. *The following statements hold:*

(1) *Let $R = S \times T$, where S and T are local rings of order four which are not fields. Then $\gamma(G(R)) = 2$.*

(2) *Let $R = \mathbb{F}_4 \times S$, where S is a local ring of order four which is not a field. Then $\gamma(G(R)) \geq 5$.*

Proof. (1) It is clear that $|U(S)| = |U(T)| = 2$ and $|J(S)| = |J(T)| = 2$. So $|U(R)| = |J(R)| = 4$. Note that each element in $J(R)$ is adjacent to every element in $U(R)$ and that $G(R)$ is 4-regular. Thus, $G(R)$ is two copies of $K_{4,4}$, so $\gamma(G(R)) = 2$ by Lemmas 2.5.1 and 2.5.2.

(2) Since $2 \notin U(R)$ and $|U(R)| = 6$, $G(R)$ is 6-regular by [8, Proposition 2.4]. As $G(R)$ contains no triangles, $\gamma(G(R)) \geq 5$ by Lemma 2.4.7. $\square$

Lemma 2.5.7. *The following statements hold:*

(1) *Let R be a local ring of order nine which is not a field. Then $\gamma(G(R)) = 2$.*

(2) *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\gamma(G(R)) = 1$.*

Proof. (1) We see $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[X]/(X^2)$ (see [18, P. 687]). The unit graphs of the two rings have the same graph structure. Without loss of generality we assume that $R = \mathbb{Z}_9$. According to [8, Proposition 2.4], $G(R)$ has 24 edges, so $\gamma(G(R)) \geq 1$ by Lemma 2.4.7. Assume to the contrary that $\gamma(G(R)) = 1$. Then, by Euler's formula, $G(R)$ has 15 faces as $G(R)$ has 9 vertices and 24 edges. Fix a representation of $G(R)$ on the surface of a torus and let $\{F_1, \ldots, F_{15}\}$ be the set of faces of $G(R)$ corresponding to the representation. Let $G' = G(R) - E_1$, where $E_1 = \{1—4, 1—7, 4—7, 2—5, 2—8, 5—8\}$. Suppose that $\{F_1', \ldots, F_n'\}$ is the set of faces of G' from the representation corresponding to $\{F_1, \ldots, F_{15}\}$. Note that $G' \cong K_{3,6}$ and $K_{3,6}$ has 9 faces, so $n = 9$. Furthermore, all face boundaries in $K_{3,6}$ are 4-cycles. This means that the boundary of each F_i' is a 4-cycle. Now $\{F_1, \ldots, F_{15}\}$ can be recovered by adding all edges in E_1 into the representation corresponding to $\{F_1', \ldots, F_9'\}$. There are two triangles in E_1, and it is easily seen that the edges in E_1 cannot be linked into G' without crossings, a contradiction. Therefore, we can conclude that $\gamma(G(R)) \geq 2$. On the other hand, we can embed $G(\mathbb{Z}_9)$ into $\mathbb{S}_2$ as shown in **Figure 4**. Hence, $\gamma(G(R)) = 2$.

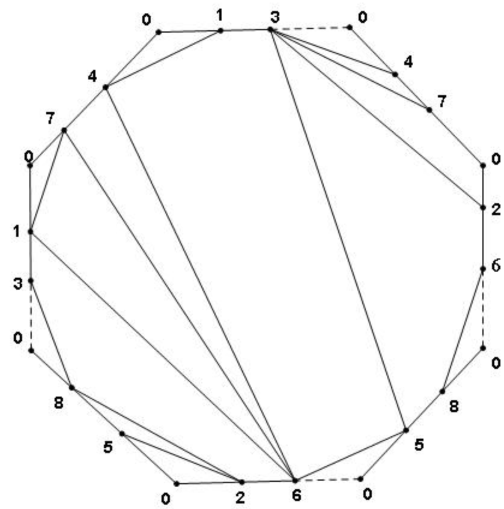


Figure 4: $G(\mathbb{Z}_9)$

(2) We have $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_6$. By [8, Proposition 2.4], $G(R)$ is 4-regular since $2 \notin U(R)$. As $G(R)$ has no triangles, $\gamma(G(R)) \geq 1$ by Lemma 2.4.7. On the other hand, we can embed $G(\mathbb{Z}_3 \times \mathbb{Z}_6)$ into $\mathbb{S}_1$ as shown in **Figure 5**, so $\gamma(G(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 1$. $\square$

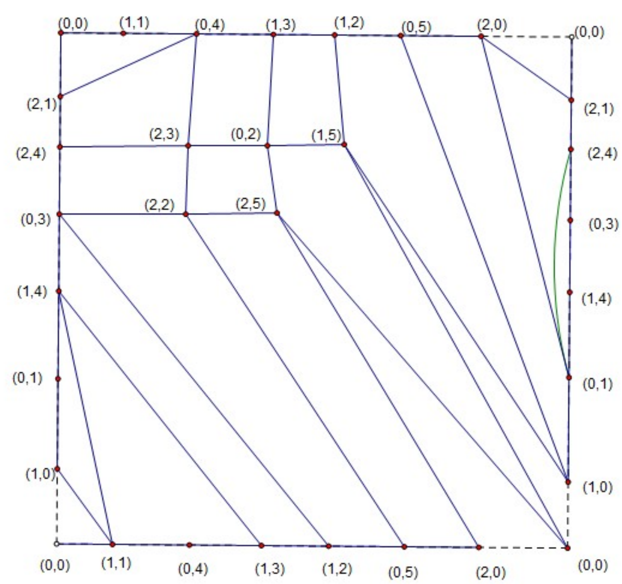


Figure 5: $G(\mathbb{Z}_3 \times \mathbb{Z}_6)$

Lemma 2.5.8. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times S$, where S is a ring. Then $G(R)$ is two copies of $G(\mathbb{Z}_2 \times S)$.*

Proof. It is straightforward. $\square$

The finite rings with planar unit graph were classified in [8] (see Theorem 2.4.1). We now prove the main result of this section.

Theorem 2.5.9. *Let R be a finite commutative ring. Then:*

(1) $\gamma(G(R)) = 1$ *if and only if R is isomorphic to one of the following rings:*

$\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, $\mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$.

(2) $\gamma(G(R)) = 2$ *if and only if R is isomorphic to one of the following rings:*

$\mathbb{F}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_3[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$.

(3) $\gamma(G(R)) = 3$ *if and only if $R \cong \mathbb{F}_9$ or $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$.*

Proof. Assume $\gamma(G(R)) \leq 3$. If $|J(R)| \geq 6$, pick up six distinct elements in $J(R)$, say x_i, $i = 1, 2, \ldots, 6$. Then $1 - x_i \in U(R)$ for $i = 1, 2, \ldots, 6$, so $G(R)$ contains a complete bipartite subgraph $K_{6,6}$, and this gives $\gamma(G(R)) \geq \gamma(K_{6,6}) = 4$. Therefore, $|J(R)| \leq 5$. On the other hand, by [8, Proposition 2.4], the minimum degree of vertices of $G(R)$ is $|U(R)| - 1$. If $|U(R)| \geq 10$, then the number of edges $\geq \frac{9}{2}|R|$, and so $\gamma(G(R)) \geq \frac{1}{4}|R| + 1 \geq 4$ by Lemma 2.4.7. Hence, $|U(R)| \leq 9$. To complete the proof, we consider only the finite rings R satisfying $|J(R)| \leq 5$ and $|U(R)| \leq 9$, and we determine the genus of the unit graph of each of such rings.

We first assume that R is a finite local ring. Let $\mathfrak{m} = J(R)$ be the unique maximal ideal of R. We proceed with five cases.

Case 1: $|\mathfrak{m}| = 1$; that is, R is a field. Only the following rings need be considered: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{F}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{F}_8$, $\mathbb{F}_9$. Since the genus of a complete graph K_9 equals 3, the unit graphs of these rings have genus at most 3.

After simply checking, we can conclude that $\gamma(G(\mathbb{Z}_2)) = \gamma(G(\mathbb{Z}_3)) = \gamma(G(\mathbb{F}_4)) = \gamma(G(\mathbb{Z}_5)) = 0$ and $\gamma(G(\mathbb{F}_8)) = \gamma(K_8) = 2$. For $R = \mathbb{Z}_7$, $G(R)$ is a subgraph of K_7, so $\gamma(G(R)) \leq \gamma(K_7) = 1$. On the other hand, $\gamma(G(R)) \geq \frac{1}{2}$ by Lemma 2.4.7. Thus, $\gamma(G(\mathbb{Z}_7)) = 1$. For $R = \mathbb{F}_9$, $G(R)$ is the complete 5-partite graph $K_{2,2,2,2,1}$, so $\gamma(G(\mathbb{F}_9)) = 3$ by Theorem 2.5.3.

Case 2: $|\mathfrak{m}| = 2$. Then $|R| \leq 4$ and $|R| = 2^t$, so $|R| = 4$. Thus, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. In either case, $\gamma(G(R)) = 0$.

Case 3: $|\mathfrak{m}| = 3$. Then $|R| \leq 9$ and $|R| = 3^t$, so $|R| = 9$. Thus, $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[X]/(X^2)$. By Lemma 2.5.7(1), $\gamma(G(R)) = 2$.

Case 4: $|\mathfrak{m}| = 4$. Then $|R| \leq 16$ and $|R| = 2^t$, so $|R| = 8$ or 16. The condition that $|R| = 16$ would give $|U(R)| = 12 > 9$. Hence $|R| = 8$. As mentioned in the introduction, we know that R is one of the following rings:

$$\mathbb{Z}_8,\ \mathbb{Z}_2[X]/(X^3),\ \mathbb{Z}_4[X]/(2X, X^2),\ \mathbb{Z}_4[X]/(2X, X^2 - 2) \text{ and } \mathbb{Z}_2[X, Y]/(X, Y)^2.$$

In all these cases, R is a local ring with maximal ideal $\mathfrak{m}$ and $R/\mathfrak{m} = \mathbb{Z}_2$. Thus $G(R) = K_{m,n}$, where $m = |\mathfrak{m}|$ and $n = |R - \mathfrak{m}|$. So $G(R) = K_{4,4}$, which implies that $\gamma(G(R)) = 1$.

Case 5: $|\mathfrak{m}| = 5$. Then $|R| \leq 25$ and $|R| = 5^t$, so $|R| = 25$. This gives $|U(R)| = 20 > 9$. Thus, this case is ruled out.

We next assume that R is not a finite local ring. Write $R = R_1 \times \cdots \times R_s$, where each R_i is local and $s \geq 2$. We proceed with five cases.

Case A: $|J(R)| = 1$. Then R_i is a field for all i. We can assume that $|U(R_1)| \leq |U(R_2)| \leq \cdots \leq |U(R_s)|$. Since $|U(R)| \leq 9$, we have $|U(R_i)| \leq 9$ for all i. Thus, R is isomorphic to one of the following rings:

(i) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \ \ times} \times S$, where $l \geq 1$ and $S \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{F}_8, \mathbb{F}_9\}$.

(ii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \ \ times} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, where $l \geq 0$.

(iii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \ \ times} \times \mathbb{Z}_3 \times \mathbb{F}_4$, where $l \geq 0$.

(iv) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \ \ times} \times \mathbb{Z}_3 \times \mathbb{Z}_5$, where $l \geq 0$.

(v) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \ \ times} \times \mathbb{F}_4 \times \mathbb{F}_4$, where $l \geq 0$.

(vi) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \ \ times} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, where $l \geq 0$.

If R is isomorphic to one of the rings that appear in (i), then $G(R)$ is a union of 2^{l-1} copies of vertex-disjoint graph $G(\mathbb{Z}_2 \times S)$ by Lemma 2.5.8. Therefore, $\gamma(G(R)) = 2^{l-1}\gamma(G(\mathbb{Z}_2 \times S))$ by Lemma 2.5.2. If $|S| \leq 4$, i.e., $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$ or $S \cong \mathbb{F}_4$, we have $\gamma(G(R)) = 0$ by Lemma 2.5.4(1). If $S \cong \mathbb{Z}_5$, we have $\gamma(G(R)) = 2^{l-1}\gamma(G(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 2^{l-1}$ by Lemma 2.5.4(1). As $\gamma(G(R)) \leq 3$, $l \leq 2$. If $l = 1$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$, so $\gamma(G(R)) = 1$. If $l = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, so $\gamma(G(R)) = 2$. If $S \cong \mathbb{Z}_7$, $S \cong \mathbb{F}_8$, or $S \cong \mathbb{F}_9$, then $\gamma(G(R)) \geq 5$ by Lemma 2.5.4(1).

Let R be a ring appearing in (ii). For $l = 0$, $G(R) = G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is planar. For $l = 1$, $\gamma(G(R)) = \gamma(G(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 1$ by Lemma 2.5.7(2). For $l = 2$, $\gamma(G(R)) = \gamma(G(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$ by Lemma 2.5.2. Consequently, $l \leq 2$ by Lemma 2.5.2.

For a ring R appearing in (iii), since $G(\mathbb{Z}_3 \times \mathbb{F}_4)$ has genus three by Lemma 2.5.5(2), l must be zero. So, $\gamma(G(R)) = \gamma(G(\mathbb{Z}_3 \times \mathbb{F}_4)) = 3$.

The graph $G(\mathbb{Z}_3 \times \mathbb{Z}_5)$ has genus at least 6 by Lemma 2.5.5(3). Thus, all rings appearing in (iv) are ruled out.

The graph $G(\mathbb{F}_4 \times \mathbb{F}_4)$ is 9-regular, and has genus at least 5 by Lemma 2.4.7. Thus, all rings appearing in (v) are ruled out.

The graph $G(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ has 104 edges, so has genus at least 5 by Lemma 2.4.7. Thus, all rings appearing in (vi) are ruled out.

Case B: $|J(R)| = 2$. We may assume that R_i is a field for all $1 \leq i \leq s-1$ and R_s is a local ring with $|J(R_s)| = 2$. Then $R_s \cong \mathbb{Z}_4$ or $R_s \cong \mathbb{Z}_2[X]/(X^2)$. As $|U(R)| \leq 9$ and $|U(R_s)| = 2$, R is isomorphic to one of the following rings:

(i) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l+1 \quad times} \times S$;

(ii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \quad times} \times \mathbb{Z}_3 \times S$;

(iii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \quad times} \times \mathbb{F}_4 \times S$;

(iv) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \quad times} \times \mathbb{Z}_5 \times S$;

(v) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \quad times} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times S$,

where $l \geq 0$ and $S \in \{\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2)\}$.

If R is a ring appearing in (i), then $\gamma(G(R)) = 0$ for all $l \geq 0$ by [8, Theorem 5.14].

Let R be a ring appearring in (ii). Then $l \leq 1$ by Lemma 2.5.5(1) and Lemma 2.5.8. If $l = 0$, then $\gamma(G(R)) = 1$; in this case, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$. If $l = 1$, then $\gamma(G(R)) = 2$; in this situation, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$.

For $R \cong \mathbb{F}_4 \times S$, $\gamma(G(R)) \geq 5$ by Lemma 2.5.6(2). For $R \cong \mathbb{Z}_5 \times S$, $G(R)$ is 8-regular and so $\gamma(G(R)) \geq 5$ by Lemma 2.4.7. For $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times S$, $G(R)$ is 8-regular and so $\gamma(G(R)) \geq 7$ by Lemma 2.4.7. Note that for rings R appearring in (iii), (iv) and (v), $G(R)$ is some copies of $G(\mathbb{F}_4 \times S)$, $G(\mathbb{Z}_5 \times S)$ and $G(\mathbb{Z}_3 \times \mathbb{Z}_3 \times S)$, respectively. Thus, all rings appearing in (iii), (iv) and (v) are ruled out.

Case C: $|J(R)| = 3$. We may assume that R_i is a field for all $1 \leq i \leq s-1$ and R_s is a local ring with $J(R_s) = 3$. Thus, $R_s \cong \mathbb{Z}_9$ or $R_s \cong \mathbb{Z}_3[X]/(X^2)$. Since $|U(R)| \leq 9$ and $|U(R_s)| = 6$, we have $R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l \quad times} \times S$, where $l \geq 1$ and $S \cong \mathbb{Z}_9$ or $S \cong \mathbb{Z}_3[X]/(X^2)$. Thus, by Lemma 2.5.4(3), $G(R)$ has genus at least 6. So this case is ruled out.

Case D: $|J(R)| = 4$. There are two subcases for consideration.

Subcase D.1: Suppose that R_i is a field for all $1 \leq i \leq s-1$ and R_s is a local ring with $|J(R_s)| = 4$. Since $|U(R)| \leq 9$, we have $|U(R_s)| \leq 9$. Thus, $|R_s| \leq 13$. It follows that $|R_s| = 8$. There are five rings satisfying this condition, namely, $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$ and $\mathbb{Z}_2[X,Y]/(X,Y)^2$. In all cases, $|U(R_s)| = 4$. Therefore, R is isomorphic to one of the following rings:

(i) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l\ \ times} \times S$, where $l \geq 1$ and S is one of $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$ and $\mathbb{Z}_2[X,Y]/(X,Y)^2$.

(ii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l\ \ times} \times \mathbb{Z}_3 \times S$, where $l \geq 0$ and S is one of $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{Z}_4[X]/(2X, X^2-2)$ and $\mathbb{Z}_2[X,Y]/(X,Y)^2$.

If R is a ring appearing in (i), then $l = 1$ and $\gamma(G(R)) = 2$ by Lemmas 2.5.4(2) and 2.5.8.

If $R \cong \mathbb{Z}_3 \times S$, where S is given in (ii), then $G(R)$ is 8-regular, and so $\gamma(G(R)) \geq \frac{96}{6} - \frac{24}{2} + 1 = 5$ by Lemma 2.4.7. Thus, all rings appearing in (ii) are ruled out.

Subcase D.2: Suppose that R_i is a field for all $1 \leq i \leq s-2$, and R_{s-1}, R_s are local rings with $|J(R_{s-1})| = |J(R_s)| = 2$. Then $R_{s-1}, R_s \in \{\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2)\}$. Since $|U(R)| \leq 9$ and $|U(R_{s-1})| = |U(R_s)| = 2$, R is isomorphic to one of the following rings:

(i) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l\ \ times} \times S \times T$, where $l \geq 0$ and S and T are local rings of order 4.

(ii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l\ \ times} \times \mathbb{Z}_3 \times S \times T$, where $l \geq 0$ and S and T are local rings of order 4.

If R is a ring appearing in (i), then $l = 0$ and $\gamma(G(R)) = 2$ by Lemmas 2.5.6(1) and 2.5.8.

If $R \cong \mathbb{Z}_3 \times S \times T$, where S and T are local rings of order 4, then $G(R)$ is 8-regular, and so $\gamma(G(R)) \geq \frac{192}{6} - \frac{48}{2} + 1 = 9$ by Lemma 2.4.7. Thus, all rings appearing in (ii) are ruled out.

Case E: $|J(R)| = 5$. Then R_i is a field for all $1 \leq i \leq s-1$ and R_s is a local ring with $J(R_s) = 5$. Since $|U(R)| \leq 9$, we have $|U(R_s)| \leq 9$. Thus, $|R_s| \leq 14$. But no local ring satisfies this condition, so this case is ruled out.

The proof is now complete. $\square$

We remark that the results in this section were taken from [58].

Chapter 3

Unitary Cayley Graphs of Rings

3.1 Introduction

This chapter concerns the unitary Cayley graphs of rings. Let R be a ring. Recall that the unitary Cayley graph of the ring R, denoted $\Gamma(R)$, is the simple graph with vertex set R, where two vertices x and y are adjacent if and only if $x - y$ is a unit of R.

Section 3.2 is about the girth and the diameter of unitary Cayley graphs. It is proved that $\mathrm{gr}(\Gamma(R)) \in \{3, 4, 6, \infty\}$ for an arbitrary ring R (Theorem 3.2.1), and that for each integer $n > 0$, there exists a ring R with $\mathrm{diam}(\Gamma(R)) = n$ (Theorem 3.2.4). We also show that $\mathrm{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$ for a ring R with $R/J(R)$ self-injective (Theorem 3.2.10) and classify all those rings with $\mathrm{diam}(\Gamma(R)) = 1$, 2, 3 and ∞, respectively (Theorem 3.2.11).

In [6] Akhtar et al. gave a list of finite commutative rings whose unitary Cayley graphs are planar (see [6, Theorem 8.2]). However, in view of the proof of [6, Theorem 8.2], the rings $\frac{\mathbb{Z}_2[t]}{(t^2)}$ and $\frac{\mathbb{Z}_2[t]}{(t^2)} \times B$ (where B is a finite Boolean ring) should be added to the list. We restate the result as follows.

Theorem 3.1.1. [6] *Let R be a finite commutative ring. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3,\ \mathbb{F}_4,\ B,\ \mathbb{Z}_3 \times B,\ \mathbb{F}_4 \times B,\ \mathbb{Z}_4,\ \tfrac{\mathbb{Z}_2[t]}{(t^2)},\ \mathbb{Z}_4 \times B,\ \tfrac{\mathbb{Z}_2[t]}{(t^2)} \times B,$$

where B is a finite Boolean ring.

Theorem 3.1.1 is the motivation of Section 3.3, where we completely characterize the rings (not necessarily finite and not necessarily commutative) whose unitary Cayley graphs are planar (Theorem 3.3.6). If a finite graph G is planar, then the minimal degree of G is at most five. However, by Theorem 3.3.6, there exist 6-regular planar unitary Cayley graphs.

In Section 3.4, motivated by Theorem 3.1.1, we study finite commutative rings with higher genus unitary Cayley graphs. It is proved that for each integer $g \geq 1$, there are at most finitely many finite commutative rings whose unitary Cayley graphs have genus g (Theorem 3.4.6). We also determine all finite commutative rings whose unitary Cayley graphs have genus 1, 2, 3, respectively (Theorem 3.4.16).

3.2 The Girth and Diameter of Unitary Cayley Graphs

It was proved by Akhtar et al. in [6, Theorem 3.2] that $\mathrm{gr}(\Gamma(R)) \in \{3, 4, 6, \infty\}$ for any finite commutative ring R. In fact, this is true for an arbitrary ring.

Theorem 3.2.1. *Let R be a ring. Then* $\mathrm{gr}(\Gamma(R)) \in \{3, 4, 6, \infty\}$*.*

Proof. If $|U(R)| = 1$, then $\Gamma(R)$ does not contain a cycle and hence $\mathrm{gr}(\Gamma(R)) = \infty$. If $|U(R)| \geq 3$, then there exist two distinct elements $u, v \in U(R)$ such that $r := u + v \neq 0$. Then 0—u—r—v—0 is a 4-cycle in $\Gamma(R)$ and hence $\mathrm{gr}(\Gamma(R)) \leq 4$.

So we may assume that $|U(R)| = 2$ and $\Gamma(R)$ contains a cycle. Let $U(R) = \{1, u\}$. If $u \neq -1$, i.e., $r := 1+u \neq 0$, then $0—1—r—u—0$ is a 4-cycle in $\Gamma(R)$, so $\mathrm{gr}(\Gamma(R)) \leq 4$. So we can further assume $U(R) = \{1, -1\}$. Let $a_1—a_2—\cdots—a_k—a_1$ be a cycle in $\Gamma(R)$.

Case 1: $k = 2m+1$ is odd. Then $a_i - a_{i+1} = 1$ for all $1 \leq i \leq k$ or $a_i - a_{i+1} = -1$ for all $1 \leq i \leq k$ (we let $a_{k+1} = a_1$). Either case gives $k = 2m+1 = 0$. So 2 is a unit in R, and hence $2 = -1$. So $0—1—2—0$ is a cycle in $\Gamma(R)$ and thus $\mathrm{gr}(\Gamma(R)) = 3$.

Case 2: k is even. Let $k = 2m+2$ and

$$x—a_1—a_2—a_3—\cdots—a_m—y—b_m—\cdots—b_3—b_2—b_1—x$$

be a cycle of length $2m+2$. Then

$$0—(a_1-x)—(a_2-x)—(a_3-x)—\cdots—(b_3-x)—(b_2-x)—(b_1-x)—0$$

is also a cycle of length $2m+2$. Hence, we may assume that

$$0—a_1—a_2—a_3—\cdots—a_m—y—b_m—\cdots—b_3—b_2—b_1—0$$

is a cycle of length $2m+2$, where $a_1 = 1, b_1 = -1$. In this case, we must have $a_{i+1} - a_i = 1$ and $b_{i+1} - b_i = -1$ for all $1 \leq i \leq m-1$. It follows that $a_i = i$ and $b_i = -i$ for $i = 1, 2, \ldots, m$. As $y - a_m = 1$ and $y - b_m = -1$, we have $2m+2 = 0$ in R. Next we show that $m \leq 2$. Assume on the contrary that $m \geq 3$.

Subcase 2.1: $m = 2k+1$. Then $a_m = 2k+1$, so $a_m^2 = k(4k+4)+1 = k \cdot (2m+2)+1 = k \cdot 0 + 1 = 1$. Hence, a_m is a unit, a contradiction.

Subcase 2.2: $m = 4k+2$. Then we have $a_{2k+1}a_{4k+1} = (2k+1)(4k+1) = k(8k+6)+1 = k \cdot (2m+2)+1 = k \cdot 0 + 1 = 1$. This shows that a_{2k+1} and a_{4k+1} are units, a contradiction.

Subcase 2.3: $m = 4k$. Then we have $a_{2k+1}b_{4k-1} = (2k+1)(-4k+1) = -k(8k+2)+1 = k \cdot (2m+2)+1 = k \cdot 0 + 1 = 1$. This shows that a_{2k+1} and b_{4k-1} are units, also a contradiction.

Therefore, $m \leq 2$. It follows that $\mathrm{gr}(\Gamma(R)) = 4$ or 6. In conclusion, $\mathrm{gr}(\Gamma(R)) \in \{3, 4, 6, \infty\}$. This completes our proof. $\square$

Similarly, as did in Section 2.2, we can classify the semipotent rings with $\mathrm{gr}(\Gamma(R)) = 6$ or ∞, and determine the self-injective rings with $\mathrm{gr}(\Gamma(R)) = 3$ or 4. We omit the details here.

Next, we investigate the diameter of unitary Cayley graphs of rings. We first show that, for each integer $n \geq 1$, there exists a ring R such that $\mathrm{diam}(\Gamma(R)) = n$. Some lemmas are needed.

Lemma 3.2.2. *Let R be a ring and $r \in R$. Then the following statements hold:*

(1) *If r is k-good, then $d(r, 0) \leq k$ in $\Gamma(R)$.*

(2) *If $r \neq 0$ and $d(r, 0) = k$ in $\Gamma(R)$, then r is k-good but not l-good for all $l < k$.*

(3) *For any $x, y, z \in R$, $d(x, y) = k$ if and only if $d(x + z, y + z) = k$.*

Proof. (1) Let $r = u_1 + u_2 + \cdots + u_k$ with each $u_i \in U(R)$ and let $x_i = u_1 + \cdots + u_i$, $i = 1, \ldots, k$. Then $0 — x_1 — x_2 — \cdots — x_{k-1} — x_k = r$ is a path from 0 to r, so $d(r, 0) \leq k$.
(2) Let $0 = x_0 — x_1 — x_2 — \cdots — x_k = r$ be a path from 0 to r. Then $u_i := x_i - x_{i-1} \in U(R)$ for $1 \leq i \leq k$. It is easy to check that $r = \sum_{i=1}^{k} u_i$. So, r is k-good. By part (1), we know that r is not l-good for all $l < k$.
(3) Let $d(x, y) = k$. Suppose that $x = x_0 — x_1 — x_2 — \cdots — x_k = y$ is a path from x to y. Then $x + z = (x_0 + z) — (x_1 + z) — (x_2 + z) — \cdots — (x_{k-1} + z) — (x_k + z) = y + z$ is a path from $x + z$ to $y + z$. So $d(x + z, y + z) \leq k$. Similarly, $d(x + z, y + z) = k$ implies $d(x, y) \leq k$. Thus, $d(x, y) = k$ if and only if $d(x + z, y + z) = k$. $\square$

Lemma 3.2.3. *Let R be a ring and $k \geq 3$ be an integer. Then $usn(R) = k$ if and only if* $\mathrm{diam}(\Gamma(R)) = k$.

Proof. ($\Longrightarrow$). For $x \neq y \in R$, as $usn(R) = k$, $x - y$ can be expressed as a sum of m ($\leq k$) units. Let $x - y = u_1 + u_2 + \cdots + u_m$ with each $u_i \in U(R)$. Set $x_i = u_1 + \cdots + u_i + y$, $i = 1, \ldots, m$. Then y—x_1—x_2—$\cdots$—$x_m = x$ is a walk from y to x, so $d(x, y) \leq m \leq k$, implying $\operatorname{diam}(\Gamma(R)) \leq k$.

By assumption, there exists an element $r \in R$, such that r is a sum of k units but not a sum of m units for any $m < k$. Then $d(r, 0) \leq k$. We claim that $d(r, 0) = k$. If $d(r, 0) = l < k$, then, by Lemma 3.2.2(2), r is l-good, a contradiction. So $d(r, 0) = k$. Hence $\operatorname{diam}(\Gamma(R)) = k$.

($\Longleftarrow$). It is clear that 0 is 2-good. For any $0 \neq r \in R$, as $\operatorname{diam}(\Gamma(R)) = k$, we have $d(r, 0) = l \leq k$. It follows that r is l-good by Lemma 3.2.2(2). Again as $\operatorname{diam}(\Gamma(R)) = k$, there exist x and y with $d(x, y) = k$. Then $d(x - y, 0) = k$. By Lemma 3.2.2, $x - y$ is k-good, but not l-good for any $l < k$. So $usn(R) = k$. □

Theorem 3.2.4. *For each integer $n \geq 1$, there is a ring R such that* $\operatorname{diam}(\Gamma(R)) = n$.

Proof. In [34, Corollary 4], the authors proved that there exists a ring R such that $usn(R) = n$ for each $n \geq 2$. So, the theorem holds for $n \geq 3$ by Lemma 3.2.3. It is clear that $\operatorname{diam}(\Gamma(\mathbb{Z}_2)) = 1$ and $\operatorname{diam}(\Gamma(\mathbb{Z}_4)) = 2$. This completes the proof. □

Corollary 3.2.5. *Let R be a ring. Then $\Gamma(R)$ is connected if and only if $\mathbf{u}(R) \leq \omega$.*

In [6, Theorem 3.1], the authors proved that $\operatorname{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$ for a left Artinian ring R and classified all left Artinian rings according to the diameter of their unitary Cayley graphs. Next, we generalize the results to self-injective rings. To do so, we first study the relationship between $\operatorname{diam}(\Gamma(\overline{R}))$ and $\operatorname{diam}(\Gamma(R))$. Note that r is a unit in R if and only if $\overline{r}$ is a unit in $\overline{R}$. Using the idea of Remark 1 in [33], we have $\operatorname{diam}(\Gamma(\overline{R}) \leq \operatorname{diam}(\Gamma(R))$. Indeed, suppose $\operatorname{diam}(\Gamma(R)) = m$. Then for any $\overline{x} \neq \overline{y} \in \overline{R}$, we have $d(x, y) \leq m$. As a path from x to y gives a walk from $\overline{x}$ to $\overline{y}$, $d(\overline{x}, \overline{y}) \leq d(x, y) \leq m$. Thus, $\operatorname{diam}(\Gamma(\overline{R})) \leq m$.

Lemma 3.2.6. *Let R be a ring. If* $\mathrm{diam}(\Gamma(R)) \geq 3$, *then* $\mathrm{diam}(\Gamma(\overline{R})) = \mathrm{diam}(\Gamma(R))$.

Proof. It suffices to show that $\mathrm{diam}(\Gamma(R)) \leq \mathrm{diam}(\Gamma(\overline{R}))$ when $\mathrm{diam}(\Gamma(R)) \geq 3$.

Suppose $\mathrm{diam}(\Gamma(R)) = \infty$. We show that $\mathrm{diam}(\Gamma(\overline{R})) = \infty$. Assume to the contrary that $\mathrm{diam}(\Gamma(\overline{R})) = m < \infty$. For any $x, y \in R$, if $\overline{x} = \overline{y}$, then $x - y \in J(R)$ and hence $1 + x - y \in U(R)$. So we get a path x—$(y-1)$—y from x to y. So $d(x, y) \leq 2$. If $\overline{x} \neq \overline{y}$, then a path form $\overline{x}$ to $\overline{y}$ deduces a path from x to y. This implies that $d(x, y) \leq d(\overline{x}, \overline{y}) \leq m$. So, $\mathrm{diam}(\Gamma(R)) \leq m$, a contradiction.

Assume that $\mathrm{diam}(\Gamma(R))$ is finite and $k := \mathrm{diam}(\Gamma(R)) \geq 3$. There exist $x, y \in R$, such that $d(x, y) = k$. First we claim that $\overline{x} \neq \overline{y}$. In fact, if $\overline{x} = \overline{y}$, then as just shown above, $d(x, y) \leq 2$, a contradiction. Assume $m := d(\overline{x}, \overline{y}) < k$ and $\overline{x}$—$\overline{x_1}$—$\overline{x_2}$—$\cdots$—$\overline{x_{m-1}}$—$\overline{y}$ is a path from $\overline{x}$ to $\overline{y}$. Then x—x_1—x_2—$\cdots$—x_{m-1}—y is path of length m, so $d(x, y) \leq m < k$, a contradiction. Thus, $d(\overline{x}, \overline{y}) = k$. This proves $\mathrm{diam}(\Gamma(\overline{R})) \geq k$. Hence, $\mathrm{diam}(\Gamma(\overline{R})) = \mathrm{diam}(\Gamma(R))$. □

Theorem 3.2.7. *Let R be a ring. Then the following conditions are equivalent:*

(1) $\mathrm{diam}(\Gamma(\overline{R})) < \mathrm{diam}(\Gamma(R))$.

(2) *R is a local ring with $J(R) \neq 0$.*

(3) $\mathrm{diam}(\Gamma(R)) = 2$ *and* $\mathrm{diam}(\Gamma(\overline{R})) = 1$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathrm{diam}(\Gamma(\overline{R})) < \mathrm{diam}(\Gamma(R))$. By Lemma 3.2.6, $\mathrm{diam}(\Gamma(R)) \leq 2$. Since $\mathrm{diam}(\Gamma(R)) = 1$ implies $\mathrm{diam}(\Gamma(\overline{R})) = 1$, we have $\mathrm{diam}(\Gamma(R)) = 2$ and $\mathrm{diam}(\Gamma(\overline{R})) = 1$. It then follows that $J(R) \neq 0$. It is easy to see that $\overline{R}$ is a division ring. Therefore, R is a local ring with $J(R) \neq 0$.

(2) $\Rightarrow$ (3). Suppose that R is a local ring with $J(R) \neq 0$. Then $R/J(R)$ is a division ring. It is clear that $\Gamma(\overline{R})$ is a complete graph and hence $\mathrm{diam}(\Gamma(\overline{R})) = 1$. On the other hand, for any $r \in R$, either $r \in J(R)$ or $r \in U(R)$. For any two distinct

elements $a, b \in R$, if $a - b \in U(R)$, then $d(a, b) = 1$. Suppose that $a - b \in J(R)$. If $a \in J(R)$, then $b \in J(R)$ too. So we have a path a—1—b and hence $d(a, b) = 2$ (note that since $J(R) \neq 0$, such a, b do exist); if $a \in U(R)$, then $b \in U(R)$, we have a path a—$(a + b)$—b, so $d(a, b) = 2$. Hence $\mathrm{diam}(\Gamma(R)) = 2$.

(3) $\Rightarrow$ (1). It is clear. $\square$

Corollary 3.2.8. *Let R be a ring. Then* $\mathrm{diam}(\Gamma(\overline{R})) = \mathrm{diam}(\Gamma(R))$ *if and only if one of the following holds:*

(1) *R is not a local ring.*

(2) *R is a division ring.*

Lemma 3.2.9. *Let R be a regular right self-injective ring. Then* $\mathrm{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$.

Proof. By [40, Theorem 6], $\mathbf{u}(R) = 2, \omega$ or ∞.

If $\mathbf{u}(R) = 2$, then $\mathrm{diam}(\Gamma(R)) \leq 2$ by Lemma 3.2.2.

If $\mathbf{u}(R) = \omega$, then, by [40, Theorem 6(2)], we may assume that $R = R_1 \times \mathbb{Z}_2$, where $\mathbf{u}(R_1) = 1$ or 2. If $\mathbf{u}(R_1) = 1$, R_1 is a trivial ring and $R = \mathbb{Z}_2$ and so $\mathrm{diam}(\Gamma(R)) = 1$. Now suppose that $\mathbf{u}(R_1) = 2$. For any vertices $x, y \in \Gamma(R)$, if $x = (x_1, 0)$ and $y = (y_1, 0)$, then there exists $z_1 \in R_1$, such that $x_1 - z_1$ and $z_1 - y_1$ are units in R_1. So a path $(x_1, 0)$—$(z_1, 1)$—$(y_1, 0)$ from$(x_1, 0)$ to $(y_1, 0)$ deduces $d(x, y) \leq 2$; if $x = (x_1, 1)$ and $y = (y_1, 1)$, a similar argument shows that $d(x, y) \leq 2$; if $x = (x_1, 0)$ and $y = (y_1, 1)$, then there exists $z_1 \in R_1$, such that $x_1 - z_1$ is a unit in R_1. With a similar argument, we have a path $(x_1, 0)$—$(z_1, 1)$—$(w_1, 0)$—$(y_1, 1)$ and hence $d(x, y) \leq 3$. So $\mathrm{diam}(\Gamma(R)) \leq 3$.

If $\mathbf{u}(R) = \infty$, then $\Gamma(R)$ is disconnected by Corollary 3.2.5, so $\mathrm{diam}(\Gamma(R)) = \infty$. The proof is complete. $\square$

Theorem 3.2.10. *Let R be a ring with $R/J(R)$ right self-injective. Then* $\mathrm{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$.

Proof. We know that in this case $\overline{R} = R/J(R)$ is a regular right self-injective ring. By Lemma 3.2.9, we have $\mathrm{diam}(\Gamma(\overline{R})) \in \{1, 2, 3, \infty\}$. By Lemma 3.2.6, we get $\mathrm{diam}(\Gamma(R)) \in \{1, 2, 3, \infty\}$. □

Theorem 3.2.11. *Let R be a ring with $R/J(R)$ right self-injective. Then the following statements hold:*

(1) $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if R is a division ring.*

(2) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if R is not a division ring and one of following holds:*

  (i) *$\overline{R}$ has no nonzero Boolean ring as a ring direct summand.*

  (ii) $\overline{R} \cong \mathbb{Z}_2$.

(3) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if $\overline{R} \ncong \mathbb{Z}_2$ and $\overline{R}$ has $\mathbb{Z}_2$, but no Boolean ring with more than two elements, as a ring direct summand.*

(4) $\mathrm{diam}(\Gamma(R)) = \infty$ *if and only if $\overline{R}$ has a Boolean ring with more than two elements as a ring direct summand.*

Proof. (1) If $\mathrm{diam}(\Gamma(R)) = 1$, then $\Gamma(R)$ is a complete graph. For any nonzero element r in R, the vertex 0 is adjacent to r, so r is a unit and hence R is a division ring. Conversely, suppose that R is a division ring. Then for any two distinct vertices x and y, $0 \neq x - y \in R$ is a unit of R. So $d(x, y) = 1$ and hence $\mathrm{diam}(\Gamma(R)) = 1$.

Next, we assume that R is not a division ring and prove (2), (3) and (4) together. Note that $\overline{R}$ is a regular right self-injective ring. So $\mathbf{u}(\overline{R}) = 2$, ω or ∞ by [40, Theorem 6]. To complete the proof, we determine the diameter $\Gamma(R)$ for each case.

Case 1: $\mathbf{u}(\overline{R}) = 2$. In this case, $\overline{R}$ has no nonzero Boolean ring as a ring direct summand or $\overline{R} \cong \mathbb{Z}_2$ by [40, Theorem 6]. Note that $\mathrm{diam}(\Gamma(\overline{R})) \in \{1, 2\}$. So $\mathrm{diam}(\Gamma(R)) = 2$ by Lemma 3.2.6.

Case 2: $\mathbf{u}(\overline{R}) = \omega$. If $\overline{R} \cong \mathbb{Z}_2$, then $\Gamma(R)$ is a complete bipartite graph. So $\mathrm{diam}(\Gamma(R)) = 2$. If $\overline{R} \ncong \mathbb{Z}_2$, in this case, $usn(\overline{R}) = 3$ by [40, Theorem 6], so $\mathrm{diam}(\Gamma(\overline{R})) = 3$ by Lemma 3.2.3. Thus $\mathrm{diam}(\Gamma(R)) = 3$ by Lemma 3.2.6.

Case 3: $\mathbf{u}(\overline{R}) = \infty$. Then $\Gamma(\overline{R})$ is disconnected by Corollary 3.2.5. So $\mathrm{diam}(\Gamma(\overline{R})) = \infty$. Thus $\mathrm{diam}(\Gamma(R)) = \infty$ by Lemma 3.2.6. □

Proposition 3.2.12. *Let R be a commutative ring. Then* $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[[t]]))$.

Proof. We first prove that $\mathrm{diam}(\Gamma(R)) \leq \mathrm{diam}(\Gamma(R[[t]]))$. If $\mathrm{diam}(\Gamma(R[[t]])) = \infty$, there is nothing to prove. Suppose that $\mathrm{diam}(\Gamma(R[[t]])) = n < \infty$. Let $a, b \in R$. Then we have $k := d(a, b) \leq n$ in $\Gamma(R[[t]])$. Let

$$a—f_1(t)—f_2(t)—\cdots—f_k(t) = b$$

be a path from a to b. Then

$$a—f_1(0)—f_2(0)—\cdots—f_k(0) = b$$

is a walk from a to b in $\Gamma(R)$. So, $d(a, b) \leq k \leq n$ in $\Gamma(R)$ and hence $\mathrm{diam}(\Gamma(R)) \leq n$.

Now we prove that $\mathrm{diam}(\Gamma(R)) \geq diam(\Gamma(R[[t]]))$. If $\mathrm{diam}(\Gamma(R)) = \infty$, there is nothing to prove. Suppose that $\mathrm{diam}(\Gamma(R)) = n < \infty$. Let $f(t), g(t) \in R[[t]]$. Then we have $k := d(f(0), g(0)) \leq n$ in $\Gamma(R)$. Let

$$f(0)—a_1—a_2—\cdots—a_k—g(0)$$

be a path from $f(0)$ to $g(0)$ in $\Gamma(R)$. Then

$$f(t)—a_1—a_2—\cdots—a_k—g(t)$$

is a path from $f(t)$ to $g(t)$ in $\Gamma(R[[t]])$. So, $d(f(t), g(t)) = k \leq n$ in $\Gamma(R[[t]])$ and hence $\text{diam}(\Gamma(R[[t]])) \leq n$. $\square$

3.3 The Planarity of Unitary Cayley Graphs

The study of groups admitting planar Cayley graphs can be traced back over almost 120 years, when Maschke [46] in 1896 proved that the only finite groups with planer Cayley graphs are exactly C_n (the cyclic group of order n), $C_2 \times C_n, D_n, A_4, S_4$ and S_5. There is a long history for studying infinite planar Cayley graphs which satisfy additional special conditions (see, for example, [22] and [29]). The authors of [6] gave a list of finite commutative rings whose unitary Cayley graphs are planar (see [6, Theorem 8.2]). This result only deals with finite graphs, and the main algebraic tool used in its proof is the Wedderburn-Artin Theorem. In this section, the graphs under consideration are mostly infinite, and they are the unitary Cayley graphs of arbitrary rings. As one can expect, the techniques dealing with the planarity of a finite graph and an infinite graph are quite different. For example, for a finite planar graph, the minimal degree of the graph is at most five. However, there exists a k-regular planar infinite graph for any positive integer k (see [26]). With Theorem 3.1.1 served as the starting point and through a thorough analysis of the (groups of) units of the associated rings, we obtain a complete characterization of the rings whose unitary Cayley graphs are planar (Theorem 3.3.6). As an application of this result, the semilocal rings with planar unitary Cayley graphs are completely determined (Corollary 3.3.8).

We first show that a ring with a planar unitary Cayley graph has either at most 4 units or exactly 6 units. This is the key to characterize the rings with planar unitary Cayley graphs. Lemma 3.3.1 below is a consequence of Theorem 3.1.1.

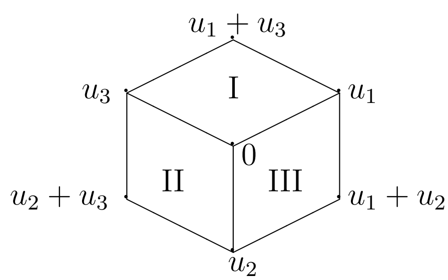
Lemma 3.3.1. *Let R be a finite commutative ring such that $\Gamma(R)$ is planar. Then $char(R) \in \{2, 3, 4, 6\}$. Furthermore,*

(1) *If $char(R) = 2$, then $|U(R)| = 1, 2$ or 3.*

(2) *If $char(R) \in \{3, 4, 6\}$, then $|U(R)| = 2$.*

Lemma 3.3.2. *Let R be a ring. If $\Gamma(R)$ is planar, then $|U(R)|$ must be finite.*

Proof. Assume to the contrary that $|U(R)| = \infty$. Take $u_1 \in U(R)$, $u_2 \in U(R) \backslash \{u_1, -u_1\}$, and $u_3 \in U(R) \backslash \{u_1, u_2, -u_1, -u_2, u_2 - u_1, u_1 - u_2, u_1 + u_2\}$. Then the following graph is a subgraph of $\Gamma(R)$:



Now take $v \in U(R) \backslash S$, where $S = \{\pm u_1, \pm u_2, \pm u_3, \pm(u_2 - u_1), \pm(u_3 - u_2), \pm(u_1 - u_3), u_1 + u_2, u_1 + u_3, u_2 + u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2, u_2 + u_3 - u_1\}$. Since $\Gamma(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, we can assume that v is in region (I). Note that $v + u_2$ is adjacent to both v and u_2. As $\Gamma(R)$ is planar, $v + u_2$ must be one of the vertices 0, u_1, u_3 and $u_1 + u_3$. This contradicts our choice of v. Therefore, $|U(R)|$ is finite. $\square$

Lemma 3.3.3. *Let R be a ring with $char(R) \neq 0$. If $\Gamma(R)$ is planar, then $|U(R)| \leq 3$.*

Proof. Suppose that $char(R) = n \geq 2$. Then R contains $\mathbb{Z}_n$ as a subring. As a subgraph of $\Gamma(R)$, $\Gamma(\mathbb{Z}_n)$ is planar. Hence $n \in \{2, 3, 4, 6\}$ by Lemma 3.3.1. We need two notations. For $a \in R$, let $\mathbb{Z}_n[a]$ be the subring of R generated by $\mathbb{Z}_n \cup \{a\}$. For $u \in U(R)$, let $o(u)$ be the order of u in the multiplicative group $U(R)$. Then $o(u) < \infty$

for all $u \in U(R)$ by Lemma 3.3.2, and $\Gamma(\mathbb{Z}_n[a])$ is planar for all $a \in R$. Assume to the contrary that $|U(R)| \geq 4$. We proceed with two cases.

Case 1: $n \in \{3, 4, 6\}$. Take $\pm 1 \neq u \in U(R)$. Then $\mathbb{Z}_n[u]$ is a finite commutative subring of R. Since $\Gamma(R)$ is planar, $\Gamma(\mathbb{Z}_n[u])$ is also planar. Thus, $|U(\mathbb{Z}_n[u])| = 2$ by Lemma 3.3.1(2). But $\mathbb{Z}_n[u]$ has at least three units, a contradiction.

Case 2: $n = 2$. Let $H = U(R)$. For $u \in H$, $\mathbb{Z}_2[u]$ is a finite commutative subring of R. The planarity of $\Gamma(\mathbb{Z}_2[u])$ implies $|U(\mathbb{Z}_2[u])| \leq 3$ by Lemma 3.3.1(1). Thus, $o(u) \leq 3$ for all $u \in H$, and we have $|H| = 2^k 3^l$, where $k, l \geq 0$.

If $k > 1$, consider a Sylow 2-subgroup P of H. Being a finite 2-group, P contains a non-trivial central element, say u. As $|P| = 2^k \geq 4$, there exists $v \in P \backslash \{1, u, u^{-1}\}$. Then the subring $\mathbb{Z}_2[u, v]$ of R generated by $\mathbb{Z}_2 \cup \{u, v\}$ is a finite commutative ring, containing at least four distinct units $1, u, v, uv$. This contradicts Lemma 3.3.1(1), as $\Gamma(\mathbb{Z}_2[u, v])$ is planar. So $k \leq 1$. Similarly, $l \leq 1$. Therefore, we must have $k = l = 1$, i.e., $|H| = 6$. Since H is not a cyclic group of order 6, we deduce $H \cong S_3$, the symmetric group of degree 3. Thus the subring $\mathbb{Z}_2[H]$ of R generated by $\mathbb{Z}_2 \cup H$ is a finite ring containing m units with $m \geq 6$. Hence, by [6, Proposition 2.2], $\Gamma(\mathbb{Z}_2[H])$ is m-regular. In particular, the minimum degree of $\Gamma(\mathbb{Z}_2[H])$ is $m \geq 6$, and hence $\Gamma(\mathbb{Z}_2[H])$ is not planar. It follows that $\Gamma(R)$ is not planar, a contradiction. □

Lemma 3.3.4. *Let R be a ring with char$(R) = 0$. If $\Gamma(R)$ is planar, then either $|U(R)| \leq 4$ or $|U(R)| = 6$.*

Proof. By Lemma 3.3.2, $U(R)$ is a finite set, say $|U(R)| = n < \infty$. Let $u \in U(R)$. Then $u^n = 1 = (-u)^n$. If n is odd, then $u^n = -u^n$, which implies that $1 = -1$, i.e., $2 = 0$. This contradiction shows that n is even. So it suffices to prove that $|U(R)| < 8$. Assume to the contrary that $U(R) = \{u_1, u_2, \ldots, u_n\}$ with $n \geq 8$. Note that 0 is a vertex of degree n whose neighbor set is $U(R)$. Without loss of generality, we may draw the vertices $u_1, u_2, \ldots, u_n$ in clockwise order, and may further assume that $u_1 + u_3 \neq 0$ (if $u_1 + u_3 = 0$, then $u_{n-1} + u_1 \neq 0$, so we can relabel $u_{n-1}, u_n, u_1, \ldots, u_{n-2}$ as $u_1, u_2, u_3, \ldots, u_n$). We next show that $u_1 + u_3 = u_2$. First we show that $r := u_1 + u_3 \in U(R)$. In fact, if $r \notin U(R)$, then $0, u_1, r, u_3, 0$ form a bounded face F of the planar graph which either contains u_2 or $A = \{u_4, \ldots, u_n\}$ (but not both). Since $|A| \geq 5$, there exists some $u_j \in A$ such that $u_2 + u_j \notin \{0, u_1, u_3, r\}$. Thus, $u_2 + u_j$ is adjacent to both u_2 and u_j, but, however, one cannot draw theses edges without a crossing. Hence, $u_1 + u_3 \in U(R)$. If $u_1 + u_3 = u_i$ for some $i \geq 4$, then $0, u_1, u_i, 0$ form a bounded face F of the planar graph which either contains u_2 or $B = \{u_4, u_5, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n\}$ (but not both). Since $|B| \geq 4$, there exists some $u_j \in B$ such that $u_2 + u_j \notin \{0, u_1, u_3\}$. Thus, $u_2 + u_j$ is adjacent to both u_2 and u_j, but, however, one cannot draw theses edges without a crossing. Therefore, we have proved $u_1 + u_3 = u_2$.

Now we consider $u_2 + u_4$. We first show $u_2 + u_4 \neq 0$. Assume $u_2 + u_4 = 0$. Then $u_n + u_2 \neq 0$. As argued above (with u_n replacing u_1 and u_2 replacing u_3), we obtain $u_n + u_2 = u_1$. It follows that $u_3 + u_n = 0$, and so $u_3 + u_5 \neq 0$. Again as above, $u_3 + u_5 = u_4$. Moreover, as $u_2 + u_4 = 0$, $u_4 + u_6 \neq 0$. So, as above again, $u_4 + u_6 = u_5$. It follows that $u_3 + u_6 = 0$. But, since $u_3 + u_n = 0$, we have $u_n = u_6$. This contradiction shows that $u_2 + u_4 \neq 0$. As argued in the previous paragraph, we have $u_2 + u_4 = u_3$.

By repeating the process we can show that $u_i + u_{i+2} = u_{i+1}$ for $1 \leq i < n - 1$. Especially, we have that $u_1 + u_3 = u_2$ and $u_2 + u_4 = u_3$, implying $u_1 + u_4 = 0$, and that $u_4 + u_6 = u_5$ and $u_5 + u_7 = u_6$, implying $u_4 + u_7 = 0$. This shows $u_1 = u_7$, a contradiction. $\square$

Lemma 3.3.5. [6, Proposition 2.2] *Let R be a ring with $|U(R)| = k < \infty$. Then $\Gamma(R)$ is k-regular.*

We are now ready to characterize the rings with planar unitary Cayley graphs.

Theorem 3.3.6. *Let R be a ring. Then $\Gamma(R)$ is planar if and only if one of the following holds:*

(1) $|U(R)| \leq 3$ *and* $|R| \leq \mathbf{c}$.

(2) $|U(R)| = 4$, $char(R) = 0$ *and* $|R| \leq \mathbf{c}$.

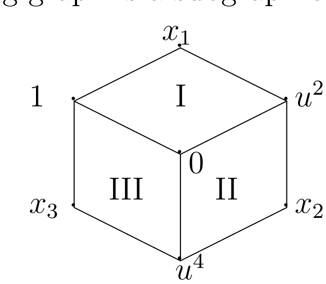
(3) $|U(R)| = 6$ *and R contains a subring isomorphic to* $\frac{\mathbb{Z}[t]}{(t^2-t+1)}$ *with* $|R| \leq \mathbf{c}$.

Proof. ($\Longrightarrow$). Suppose that $\Gamma(R)$ is planar. Then R embeds in $\mathbb{R} \times \mathbb{R}$ as sets, so $|R| \leq \mathbf{c}$. By Lemmas 3.3.3 and 3.3.4, $|U(R)| \leq 4$ or $|U(R)| = 6$. If $|U(R)| = 3$, we are done. If $|U(R)| = 4$, then by Lemma 3.3.3, we have $char(R) = 0$. This is case (2).

Suppose $|U(R)| = 6$. We need to show that R contains a subring isomorphic to $\frac{\mathbb{Z}[t]}{(t^2-t+1)}$. By Lemma 3.3.3, we have $char(R) = 0$. Note that there are only two groups of order 6. If $U(R) \cong S_3$, then $U(R) = \{1, u, u^2, v, uv, u^2v\}$ with $u^3 = v^2 = 1$. But, the non-abelianness of $U(R)$ shows that $-1 \neq v$. Since $char(R) = 0$, $-1 \neq 1$. Moreover, it is clear that $-1 \notin \{u, u^2, uv, u^2v\}$. Thus, $-1 \notin U(R)$. This contradiction shows that $U(R)$ is a cyclic group of order 6. So $U(R) = \{1, u, u^2, u^3, u^4, u^5\}$ with $u^3+1 = 0$.

Claim: $1 + u^2 \in U(R)$.

Proof of Claim. Assume to the contrary that $1 + u^2 \notin U(R)$. Let $x_1 = 1 + u^2$, $x_2 = u^2 + u^4$ and $x_3 = 1 + u^4 = u^4(1 + u^2)$. Then x_1, x_2, x_3 are distinct non-unit elements in R, so the following graph is a subgraph of $\Gamma(R)$:



Since $\Gamma(R)$ is planar and u is adjacent to 0, u must be in one of the regions (I), (II) and (III). If u is in region (II), as $1+u \neq 0$ is adjacent to both 1 and u, the planarity of $\Gamma(R)$ ensures that $1+u = u^2$ or $1+u = u^4$ or $1+u = x_2$. Note that $1+u = u^2$ implies $u+u^2 = u^3 = -1$; so $2u^2 = 0$, i.e., $2 = 0$, a contradiction. Moreover, $1+u = u^4$ implies $1+u = -u$; so $2u = 1$, a contradiction (note that, since $|U(R)| < \infty$, $n \notin U(R)$ for all $\pm 1 \neq n \in \mathbb{Z}$). Furthermore, $1+u = x_2$ means $1+u = u^2+u^4$, which implies $u^2+u^3 = u^4+1$, i.e., $u^2 = u^4+2$. It follows that $1+u = (u^4+2)+u^4 = 2u^4+2 = -2u+2$, i.e., $3u = 1$, a contradiction. If u is in region (III), as $u+u^2 \neq 0$ is adjacent to both u and u^2, the planarity of $\Gamma(R)$ ensures that $u+u^2 = 1$ or $u+u^2 = u^4$ or $u+u^2 = x_3$. Note that $u+u^2 = 1$ implies $u^2+u^3 = u$; so $2u = 0$, i.e., $2 = 0$, a contradiction. Moreover, $u+u^2 = u^4$ implies $u+u^2 = -u$; so $2 = -u$, a contradiction. Furthermore, $u+u^2 = x_3$ means $u+u^2 = 1+u^4$, which implies $u^3+u^4 = u^2+1$, i.e., $-1+u^4 = u^2+1$; it follows that $-1+u+u^2-1 = u^2+1$, i.e., $3 = u$, a contradiction. Therefore, u must be in region (I). A similar discussion shows that u^3 must be in region (II). Now as $u+u^3 \neq 0$ is adjacent to both u and u^3, the planarity of $\Gamma(R)$ ensures that $u+u^3 = u^2$. Then $1+u^2 = u$ is a unit, a contradiction. So the Claim is proved.

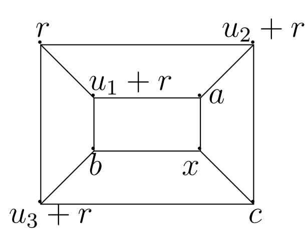
It is easy to check that $1+u^2 \notin \{1, u^2, u^3, u^4, u^5\}$. So, by the Claim, $1+u^2 = u$. Consider the subring $\mathbb{Z}[u]$ of R generated by $\mathbb{Z} \cup \{u\}$. We now prove that $\mathbb{Z}[u] \cong \frac{\mathbb{Z}[t]}{(t^2-t+1)}$. The mapping $\phi : \mathbb{Z}[t] \to \mathbb{Z}[u]$ given by $\phi(f(t)) = f(u)$ is an epimorphism of rings. So it suffices to show that $\ker\phi = (t^2-t+1)$. As $\phi(t^2-t+1) = u^2-u+1 = 0$, $(t^2-t+1) \subseteq \ker\phi$. Let $f(t) \in \ker\phi$. There exist $a, b \in \mathbb{Z}$ and $q(t) \in \mathbb{Z}[t]$ such that $f(t) = q(t)(t^2-t+1)+(at+b)$. As $f(u) = 0$, we have $au+b = 0$. So $b^2 = (-b)^2 = (au)^2 = a^2u^2 = a^2(u-1) = a(au)-a^2 = a(-b)-a^2$. That is, $a^2+b^2+ab = 0$ in $\mathbb{Z}$, which implies that $a = b = 0$. So $f(t) = q(t)(t^2-t+1)$. It is proved that $\ker\phi = (t^2-t+1)$. Therefore, $\mathbb{Z}[u] \cong \frac{\mathbb{Z}[t]}{(t^2-t+1)}$.

($\Longleftarrow$). By assumption, we have $|R| \leq \mathbf{c}$. If $|U(R)| \leq 2$, then the maximal degree of $\Gamma(R)$ is at most two, so $\Gamma(R)$ must be planar.

Suppose that $|U(R)| = 3$. Then $2 = 0$ in R, and $\Gamma(R)$ is 3-regular by Lemma 3.3.5. Write $U(R) = \{u_1, u_2, u_3\}$. For a given $r \in R$, r is adjacent to $u_i + r$ $(i = 1, 2, 3)$. There are two situations.

Case 1: $u_1 + r$ is adjacent to $u_2 + r$. Then $(u_1 + r) - (u_2 + r) = u_3$. So $u_1 + r$ is adjacent to $u_3 + r$. In fact, $u_1 + r$ is adjacent to $u_2 + r \Leftrightarrow u_1 + r$ is adjacent to $u_3 + r$ $\Leftrightarrow u_2 + r$ is adjacent to $u_3 + r$. So in this case, $r, u_1 + r, u_2 + r, u_3 + r$ form a complete graph K_4, which is 3-regular.

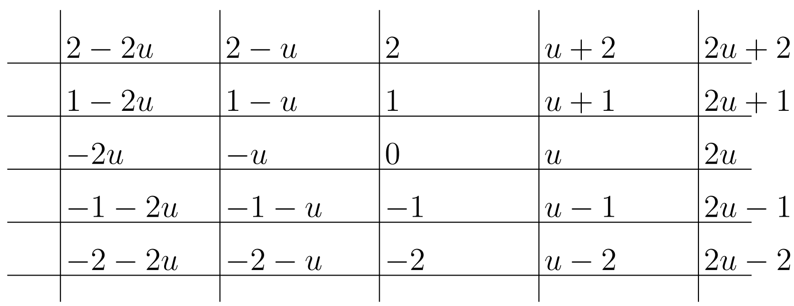
Case 2: $u_i + r$ is not adjacent to $u_j + r$ whenever $i \neq j$. Let the neighborhoods of $u_1 + r$ be r, a, b. We may assume $u_1 + r - a = u_2$ and $u_1 + r - b = u_3$. Then $u_2 + r - a = u_1$ and $u_3 + r - b = u_1$. This means that a is adjacent to $u_2 + r$ and b is adjacent to $u_3 + r$. Let c be the third neighborhood of $u_2 + r$. One can verify that $c \notin \{u_1 + r, u_2 + r, u_3 + r, a, b, r\}$. Moreover, it must be that $u_2 + r - c = u_3$, so $u_3 + r - c = u_2$. This means that c is also a neighborhood of $u_3 + r$. Now consider the vertex a. Let x be the third neighborhood of a. Then it must be that $x - a = u_3$. As $x - b = u_3 + a - b = u_1 + r - b + a - b = u_1 + r - a = u_2$, x is adjacent to b. As $x - c = (u_3 + a) - (u_3 + r - u_2) = u_2 + r - a = u_1$, x is adjacent to c. One can verify that $x \notin \{a, b, u_1 + r, u_2 + r, u_3 + r, r, c\}$. So, the vertices $r, u_1 - r, u_2 - r, u_3 - r, a, b, c$ and x form a cube (see the diagram below), which is 3-regular.



We notice that $\Gamma(R)$ can not contain both K_4 and a cube as subgraphs. In fact, if $r, u_1 + r, u_2 + r, u_3 + r, a, b, c, x$ form a cube (as shown in the diagram above), then

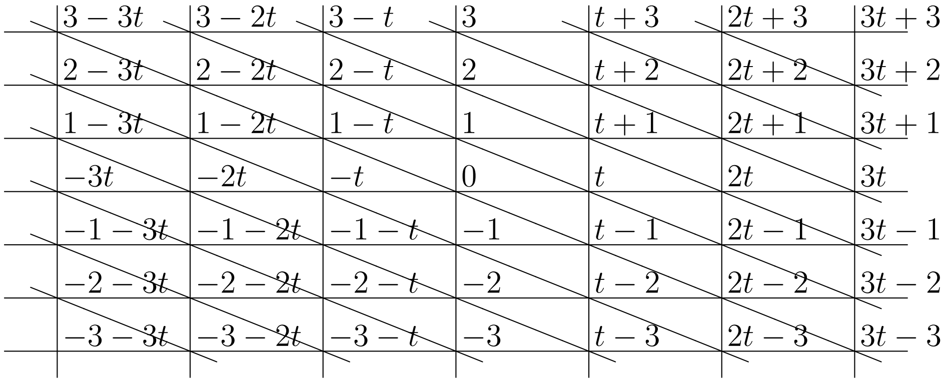
$u_1 - u_2 = (u_1 + r) - (u_2 + r)$ is not a unit; but if $s, u_1 + s, u_2 + s, u_3 + s$ form a complete graph K_4, then $u_1 - u_2 = (u_1 + s) - (u_2 + s) = u_3$ is a unit. Hence, as $\Gamma(R)$ is 3-regular, either $\Gamma(R)$ is a disjoint union of copies of a cube, or $\Gamma(R)$ is a disjoint union of copies of K_4. As a cube and K_4 are planar graphs, $\Gamma(R)$ is planar.

Suppose that $|U(R)| = 4$ and $char(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Take $\pm 1 \neq u \in U(R)$. As $|U(R)| = 4$, we have $U(R) = \{1, -1, u, -u\} = U(\mathbb{Z}[u])$. By Lemma 3.3.5, both $\Gamma(\mathbb{Z}[u])$ and $\Gamma(R)$ are 4-regular. For any $a \in R$, the graph with vertex set $a + \mathbb{Z}[u] = \{a + r | r \in \mathbb{Z}[u]\}$ is isomorphic to the graph $\Gamma(\mathbb{Z}[u])$. It follows that $\Gamma(R)$ is a disjoint union of copies of $\Gamma(\mathbb{Z}[u])$. As shown below, $\Gamma(\mathbb{Z}[u])$ is planar, so $\Gamma(R)$ is planar.



Graph $\Gamma(\mathbb{Z}[u])$

Finally, suppose that $|U(R)| = 6$ and R has a subring $S \cong \frac{\mathbb{Z}[t]}{(t^2-t+1)}$. It is easy to check that $U(\frac{\mathbb{Z}[t]}{(t^2-t+1)}) = \{1, -1, t, -t, 1-t, t-1\}$. So by Lemma 3.3.5, $\Gamma(\frac{\mathbb{Z}[t]}{(t^2-t+1)})$ (and hence $\Gamma(S)$) and $\Gamma(R)$ are 6-regular. As shown below, $\Gamma(\frac{\mathbb{Z}[t]}{(t^2-t+1)})$ is planar. So $\Gamma(S)$ is planar.



Graph $\Gamma(\frac{\mathbb{Z}[t]}{(t^2-t+1)})$

For any $a \in R$, the graph with vertex set $a + S = \{a + r | r \in S\}$ is isomorphic to the graph $\Gamma(S)$. Therefore, $\Gamma(R)$ is a disjoint union of copies of $\Gamma(S)$, and hence is planar. The proof is complete. $\square$

A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian. As a corollary of Theorem 3.3.6, we determine all semilocal rings whose unitary Cayley graphs are planar. A lemma is needed.

Lemma 3.3.7. *Let S be a semilocal ring. Then $S/J(S)$ is a Boolean ring with $|J(S)| = 2$ if and only if $S \cong A$ or $S \cong A \times B$, where $A \in \left\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[t]}{(t^2)}, \mathbb{T}_2(\mathbb{Z}_2)\right\}$ and B is a finite Boolean ring.*

Proof. ($\Longleftarrow$). This is clear.
($\Longrightarrow$). The assumptions on S clearly show that S is a finite ring. Write $S = T_1 \times \cdots \times T_k$ where $k \geq 1$ and each T_i is indecomposable. As $J(S) = J(T_1) \times \cdots \times J(T_k)$ has only two elements, we can assume that $|J(T_1)| = 2$ and $J(T_i) = 0$ for $i > 1$. So $S/J(S) \cong T_1/J(T_1) \times T_2 \times \cdots \times T_k$. As $S/J(S)$ is Boolean, we see $T_i \cong \mathbb{Z}_2$ for $i = 2, \ldots, k$ and $T_1/J(T_1)$ is Boolean. So it suffices to show that $T_1 \in \left\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[t]}{(t^2)}, \mathbb{T}_2(\mathbb{Z}_2)\right\}$. Therefore, we can assume that $k = 1$, i.e., S is indecomposable.

If $\overline{S} := S/J(S)$ is indecomposable, then $\overline{S} \cong \mathbb{Z}_2$; so $S \cong \mathbb{Z}_4$ or $S \cong \frac{\mathbb{Z}_2[t]}{(t^2)}$ (as $|J(S)| = 2$). Let us assume that $\overline{S}$ is not indecomposable. Then $\overline{S}$ has a nontrivial central idempotent, say $\bar{e}$. Since $J(S)^2 = 0$, idempotents of $\overline{S}$ can be lifted to idempotents of S. Hence we can assume $e^2 = e \in S$. Since S is indecomposable, e is not central, so either $eS(1-e) \neq 0$ or $(1-e)Se \neq 0$. Without loss of generality, we can assume $eS(1-e) \neq 0$. Let $e' = 1-e$. Then, as $|U(S)| = 2$, $e'Se = 0$ and $|eSe'| = 2$. So the Peirce decomposition of S with respect to e is

$$S = \begin{pmatrix} eSe & eSe' \\ 0 & e'Se' \end{pmatrix}.$$

Since $J(S) = \begin{pmatrix} J(eSe) & eSe' \\ 0 & J(e'Se') \end{pmatrix}$ and $|J(S)| = 2$, it follows that $J(eSe) = 0$ and $J(e'Se') = 0$. So $eSe \times e'Se' \cong S/J(S)$ is a finite Boolean ring, and hence eSe is a finite Boolean ring. We claim $eSe \cong \mathbb{Z}_2$. If not, then eSe is not indecomposable, so $eSe = A \times B$ where $A \neq 0$ and $B \neq 0$. Write $eSe' = \{0, r\}$. As $r = er = (1_A + 1_B)r = 1_A r + 1_B r$, either $1_A r \neq 0$ or $1_B r \neq 0$. Without loss of generality, we may assume $1_A r \neq 0$. Then $1_A r = r$ and so $1_B r = 0$. Thus $BJ(S) = 0$. Consequently, $S = \begin{pmatrix} A \times B & eSe' \\ 0 & e'Se' \end{pmatrix} \cong \begin{pmatrix} A & eSe' \\ 0 & e'Se' \end{pmatrix} \times B$, and the isomorphism is given by $\begin{pmatrix} (a,b) & x \\ 0 & y \end{pmatrix} \mapsto \Big(\begin{pmatrix} a & x \\ 0 & y \end{pmatrix}, b \Big)$. This contradicts the indecomposability of S. Hence we have proved $eSe \cong \mathbb{Z}_2$. Similarly, $e'Se' \cong \mathbb{Z}_2$. Therefore, $S \cong \mathbb{T}_2(\mathbb{Z}_2)$. $\square$

The next result extends Theorem 3.1.1 from a finite commutative ring to a semilocal ring.

Corollary 3.3.8. *Let R be a semilocal ring. Then $\Gamma(R)$ is planar if and only if R is* *isomorphic to one of the following rings:*

$$\mathbb{Z}_3,\ \mathbb{F}_4,\ B,\ \mathbb{Z}_3 \times B,\ \mathbb{F}_4 \times B, \mathbb{Z}_4,\ \tfrac{\mathbb{Z}_2[t]}{(t^2)},\ \mathbb{T}_2(\mathbb{Z}_2),\ \mathbb{Z}_4 \times B,\ \tfrac{\mathbb{Z}_2[t]}{(t^2)} \times B,\ \mathbb{T}_2(\mathbb{Z}_2) \times B,$$

where B is a finite Boolean ring.

Proof. ($\Longleftarrow$). All rings appearing in the corollary have at most 3 units, so $\Gamma(R)$ is planar by Theorem 3.3.6.

($\Longrightarrow$). We proceed with two cases.

Case 1: $J(R) = 0$. As R is semilocal, $R \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_s}(D_s)$, where each D_i is a division ring. By Lemma 3.3.2, $|U(R)| < \infty$. So $|R| < \infty$ and hence $char(R) \neq 0$. Thus, $|U(R)| \leq 3$ by Lemma 3.3.3, and so $R \cong D_1 \times \cdots \times D_s$. It follows that $R \in \{\mathbb{Z}_3,\ \mathbb{F}_4,\ B,\ \mathbb{Z}_3 \times B,\ \mathbb{F}_4 \times B\}$, where B is a finite Boolean ring.

Case 2: $J(R) \neq 0$. If $|J(R)| \geq 3$, take three distinct elements in $J(R)$, say a_1, a_2, a_3. For any i, j with $1 \leq i, j \leq 3$, a_i is adjacent to $1 + a_j$. So $\Gamma(R)$ contains $K_{3,3}$ as a subgraph, contradicting the planarity of $\Gamma(R)$. Therefore, $|J(R)| = 2$. Note that $\Gamma(\overline{R})$ is planar. In fact, for each element $x \in \overline{R}$, fix $a_x \in R$ such that $x = \overline{a_x}$. Thus, the map $f : \Gamma(\overline{R}) \to \Gamma(R)$ given by $f(x) = a_x$ is one-to-one. For $x, y \in \overline{R}$, $x + y \in U(\overline{R})$ if and only if $a_x + a_y \in U(R)$. Thus, $\Gamma(\overline{R})$ is isomorphic to the subgraph of $\Gamma(R)$ with vertex set $f(\overline{R})$, showing that $\Gamma(\overline{R})$ is planar. Since $\overline{R}$ is semilocal, as shown in Case 1, we have $\overline{R} \in \{\mathbb{Z}_3,\ \mathbb{F}_4,\ B,\ \mathbb{Z}_3 \times B,\ \mathbb{F}_4 \times B\}$ where B is a finite Boolean ring. As $|J(R)| = 2$, we have $|R| < \infty$, so $|U(R)| \leq 3$ by Lemma 3.3.3. Moreover, $|J(R)| = 2$ also implies that $|U(R)| = 2|U(\overline{R})|$. It follows that $|U(\overline{R})| = 1$. Hence, $\overline{R} \cong B$. Then by Lemma 3.3.7, $R \in \{\mathbb{Z}_4, \frac{\mathbb{Z}_2[t]}{(t^2)}, \mathbb{T}_2(\mathbb{Z}_2), \mathbb{Z}_4 \times B, \frac{\mathbb{Z}_2[t]}{(t^2)} \times B, \mathbb{T}_2(\mathbb{Z}_2) \times B\}$. $\square$

As another application of Theorem 3.3.6, we characterize the rings R with $\Gamma(R[t])$ planar. We remark that, for a reduced ring R, $U(R[t]) = U(R)$ (we cannot find a reference for this, but it can be easily proved).

Corollary 3.3.9. *Let R be a ring. Then $\Gamma(R[t])$ is planar if and only if $\Gamma(R)$ is planar and R is reduced.*

Proof. ($\Longleftarrow$). This follows from Theorem 3.3.6 and the remark above.
($\Longrightarrow$). Suppose that $\Gamma(R[t])$ is planar. Then $\Gamma(R)$ is planar as $\Gamma(R)$ is a subgraph of $\Gamma(R[t])$. If R is not reduced, then take $a \in R$ with $a^2 = 0$. As each vertex in $\{0, a, at\}$ is adjacent to every vertex in $\{1, 1 + a, 1 + at\}$, $\Gamma(R[t])$ contains a $K_{3,3}$, a contradiction. Thus, R is reduced, as desired. $\square$

Corollary 3.3.10. *Let R be a ring, and let $t_1, t_2, \ldots, t_n$ be commuting indeterminates over R. Then $\Gamma(R[t_1, t_2, \ldots, t_n])$ is planar if and only if $\Gamma(R)$ is planar and R is reduced.*

3.4 Higher Genus Unitary Cayley Graphs for Finite Commutative Rings

The genus of graphs associated with rings is the topic of many publications. For instances, the planarity of zero divisor graphs were studied in [9], [11] and [56]. The rings with toroidal zero divisor graphs were classified in Wang [67] and Wickham [68], [69]. Genus two zero divisor graphs of local rings were investigated by Bloomfield and Wickham in [16]. Recently, Maimani et al. [53] determined all isomorphism classes of finite rings whose total graphs have genus at most one, and Tamizh Chelvam and Asir [62] characterized all isomorphism classes of finite rings whose total graphs have genus two. In [6, Theorem 8.2], all finite commutative rings having planar unitary Cayley graphs are completely classified. The goal of this section is to classify all finite commutative rings whose unitary Cayley graphs have genus 1, 2, and 3, respectively.

Khashyarmanesh and Khorsandi generalized the definition of the unitary Cayley graphs of rings in [38], studied the properties of the resulting graphs and extended some results of the unit graphs and unitary Cayley graphs. Especially, they classified all commutative finite rings whose generalized unitary Cayley graphs are planar.

Our first result says that for each $g \geq 1$, there are at most finitely many finite commutative rings R with $\gamma(\Gamma(R)) = g$. The proof relies on several lemmas.

Lemma 3.4.1. [69, Proposition 2.1] *Let G be a graph with n (≥ 3) vertices. Then $\delta(G) \leq 6 + \frac{12(\gamma(G)-1)}{n}$, where $\delta(G)$ is the minimal degree of G.*

Lemma 3.4.2. *Let R be a finite commutative ring with $\gamma(\Gamma(R)) = g > 0$. Then either $|R| \leq 12(g-1)$ or $|U(R)| \leq 6$.*

Proof. If $|R| > 12(g-1)$, then, by Lemma 3.4.1, $\delta(\Gamma(R)) \leq 6 + \frac{12(g-1)}{|R|} < 7$. But $\delta(\Gamma(R))$ is an integer, so we have $\delta(\Gamma(R)) \leq 6$. By Lemma 3.3.5, the result follows. □

Lemma 3.4.3. *Let R be a finite commutative local ring. Then $|U(R)| \neq 5$. Furthermore,*

(1) *If $|U(R)| = 2$, then $R \in \{\mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}\}$.*

(2) *If $|U(R)| = 3$, then $R = \mathbb{F}_4$.*

(3) *If $|U(R)| = 4$, then $R \in \{\mathbb{Z}_5, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_4[x]}{(2x, x^2-2)}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2x,x^2)}\}$.*

(4) *If $|U(R)| = 6$, then $R \in \{\mathbb{Z}_7, \mathbb{Z}_9, \frac{\mathbb{Z}_3[x]}{(x^2)}\}$.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. By the assumption, we can set $|R| = p^n$ and $|\mathfrak{m}| = |Z(R)| = p^m$ for some prime p and integers $n > m \geq 0$. So $|U(R)| = p^n - p^m = p^m(p^{n-m} - 1)$. It is clear that $|U(R)| \neq 5$.

(1) If $|U(R)| = 2$, then $p = 2$, $m = 1$ and $n = 2$, or $p = 3$, $n = 1$ and $m = 0$. So, $R \in \{\mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}\}$.

(2) If $|U(R)| = 3$, then $p = 2$, $m = 0$ and $n = 2$. So, $R = \mathbb{F}_4$.

(3) If $|U(R)| = 4$, then $p = 2$, $m = 2$ and $n = 3$, or $p = 5$, $n = 1$ and $m = 0$. So, $R \in \{\mathbb{Z}_5, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2x,x^2)}\}$.
(4) If $|U(R)| = 4$, then $p = 3$ $m = 1$ and $n = 2$, or $p = 7$, $n = 1$ and $m = 0$. So $R \in \{\mathbb{Z}_7, \mathbb{Z}_9, \frac{\mathbb{Z}_3[x]}{(x^2)}\}$. □

In [44, Theorem 1.2], the authors proved that if R is an Artinian ring such that $R/J(R)$ has s summands isomorphic to $\mathbb{Z}_2$, then $\Gamma(R)$ contains 2^{s-1} connected components. Applying Lemma 2.5.2, we have:

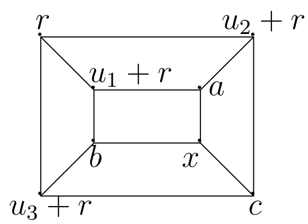
Lemma 3.4.4. *Let R be a finite commutative ring. Then $\gamma(\Gamma((\mathbb{Z}_2)^s \times R)) = 2^{s-1}\gamma(\Gamma(\mathbb{Z}_2 \times R))$.*

Lemma 3.4.5. *Let R be a finite commutative ring. If $|U(R)| \leq 3$, then $\Gamma(R)$ is planar.*

Proof. If $|U(R)| \leq 2$, then the maximal degree of $\Gamma(R)$ is at most two, so $\Gamma(R)$ must be planar.

Suppose that $|U(R)| = 3$. Then $\Gamma(R)$ is 3-regular by Lemma 3.3.5. Note that $2 = 0$ in R. Let $U(R) = \{u_1, u_2, u_3\}$. For a given $r \in R$, r is adjacent to $u_i + r$ ($i = 1, 2, 3$). If $u_1 + r$ is adjacent to $u_2 + r$, then $(u_1 + r) - (u_2 + r) = u_3$. So $u_1 + r$ is adjacent to $u_3 + r$ too. It follows that $u_2 + r$ is adjacent to $u_3 + r$. Hence, $r, u_1 + r, u_2 + r, u_3$ form a complete graph K_4. As $\Gamma(R)$ is 3-regular, $\Gamma(R)$ must be a disjoint union of copies of K_4, so $\Gamma(R)$ is planar. Let the neighborhoods of $u_1 + r$ be r, a, b. We may assume $u_1 + r - a = u_2$ and $u_1 + r - b = u_3$. Then $u_2 + r - a = u_1$ and $u_3 + r - b = u_1$. This means that a is adjacent to $u_2 + r$ and b is adjacent to $u_3 + r$. Let c be the third neighborhood of $u_2 + r$. Then $u_2 + r - c = u_3$, so $u_3 + r - c = u_2$. This means that c is also a neighborhood of $u_3 + r$. Now consider the vertex a. Let the neighborhoods of a be $u_1 + r, u_2 + r, x$. Then $x - a = u_3$. As $x - b = u_3 + a - b = u_1 + r - b + a - b = u_1 + r - a = u_2$, x is adjacent to b. Similarly,

x is adjacent to c. So, the vertices $r, u_1 - r, u_2 - r, u_3 - r, a, b, c$ and x form a cube, which is 3-regular. The graph is shown as follows:



As $\Gamma(R)$ is 3-regular, $\Gamma(R)$ must be a disjoint union of copies of a cube. As a cube is a planar graph, $\Gamma(R)$ is planar. $\square$

We are ready to prove our first main result in this section.

Theorem 3.4.6. *For each integer $g \geq 1$, there are at most finitely many finite commutative rings R with $\gamma(\Gamma(R)) = g$.*

Proof. Let R be a finite commutative ring with $\gamma(\Gamma(R)) = g$. It suffices to prove that $|R|$ is bounded above by a constant depending only on g.

If R is a field, then $\Gamma(R)$ is a complete graph $K_{|R|}$. So, $g = \gamma(\Gamma(R)) = \gamma(K_{|R|}) = \lceil \frac{(|R|-3)(|R|-4)}{12} \rceil$ by Lemma 2.5.1. This gives $(|R| - 3)(|R| - 4) \leq 12g$, or $|R| \leq \frac{7+\sqrt{49+48(g-1)}}{2}$, as needed.

If R is a local ring which is not a field, then $\mathfrak{m} = Z(R)$ is the maximal ideal of R and $|R| \leq |Z(R)|^2$ by [25]. Note that every element in $\mathfrak{m}$ is adjacent to each element in $1 + \mathfrak{m} = \{1 + a \mid a \in \mathfrak{m}\}$. So $K_{|\mathfrak{m}|,|\mathfrak{m}|}$ is a subgraph of $\Gamma(R)$. Thus, we have $g = \gamma(\Gamma(R)) \geq \gamma(K_{|\mathfrak{m}|,|\mathfrak{m}|}) = \lceil \frac{(|\mathfrak{m}|-2)^2}{4} \rceil$ by Lemma 2.5.1. This implies that $(|\mathfrak{m}| - 2)^2 \leq 4g$ or $|\mathfrak{m}| \leq 2\sqrt{g} + 2$. So, $|R| \leq (2\sqrt{g} + 2)^2$, as needed.

Now suppose that R is not a local ring. We may assume that $R = (\mathbb{Z}_2)^s \times R_1 \times \cdots \times R_t$, where $s \geq 0$ and each R_i is a local ring with at least three elements. By Lemma 3.4.4, we have $s \leq 1 + \log_2 g$ as $g > 0$. If $|R| \leq 12(g - 1)$, we are done. Otherwise, by

Lemmas 3.4.2 and 3.4.5, we have $4 \leq |U(R)| \leq 6$. As $|U(R)| = |U(R_1)| \times \cdots \times |U(R_t)|$, we have the following possibilities:

(1) $|U(R)| = 4$. In this case, we have $t = 1$ and $|U(R_1)| = 4$, or $t = 2$ and $|U(R_1)| = |U(R_2)| = 2$. By Lemma 3.4.3, either $R \cong (\mathbb{Z}_2)^s \times R_1$ with $s \geq 1$, where $R_1 \in \{\mathbb{Z}_5, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}, \frac{\mathbb{Z}_4[x]}{(2x,x^2)}, \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}\}$, or $R \cong (\mathbb{Z}_2)^s \times R_1 \times R_2$ with $s \geq 0$, where $R_1, R_2 \in \{\mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}\}$.

(2) $|U(R)| = 5$. In this case, $t = 1$ and $|U(R_1)| = 5$. This is impossible, because there is no local ring with 5 units by Lemma 3.4.3.

(3) $|U(R)| = 6$. In this case, we have $t = 1$ and $|U(R_1)| = 6$, or $t = 2$ and $|U(R_1)| = 2$ and $|U(R_2)| = 3$. By Lemma 3.4.3, either $R \cong (\mathbb{Z}_2)^s \times R_1$ with $s \geq 1$, where $R_1 \in \{\mathbb{Z}_7, \mathbb{Z}_9, \frac{\mathbb{Z}_3[x]}{(x^2)}\}$, or $R \cong (\mathbb{Z}_2)^s \times R_1 \times \mathbb{F}_4$ with $s \geq 0$, where $R_1 \in \{\mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}\}$.

In conclusion, in each case, we always have $|R_1 \times \cdots \times R_t| \leq 16$. It then follows that $|R| \leq 2^s \cdot 16 \leq 2^{(1+\log_2 g)} \cdot 16 = 32g$, as required. $\square$

Next, we determine the candidates whose unitary Cayley graphs have genus at most three. Lemma 2.4.7 from graph theory is very useful to determine the lower bound of the genus of a graph. We first present some lemmas.

Lemma 3.4.7. *Let R be a finite ring. If $\gamma(\Gamma(R)) \leq 3$, then R has at most 8 units.*

Proof. Suppose that $|R| = n$ with k units. Then $\Gamma(R)$ is k-regular with n vertices. So $\Gamma(R)$ has $\frac{1}{2}kn$ edges. By Lemma 2.4.7, we have $\gamma(\Gamma(R)) \geq \frac{kn}{12} - \frac{n}{2} + 1$. If $k \geq 9$, then $\gamma(\Gamma(R)) \geq 4$, a contradiction. So $|U(R)| \leq 8$. $\square$

Lemma 3.4.8. *Let R be a finite commutative local ring. If $\gamma(\Gamma(R)) \leq 3$, then R has at most 13 elements.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of R. As each element in $\mathfrak{m}$ is adjacent to every element in $1 + \mathfrak{m} = \{1 + a | a \in \mathfrak{m}\}$, $\Gamma(R)$ contains a subgraph $K_{|\mathfrak{m}|,|\mathfrak{m}|}$. If

$|\mathfrak{m}| \geq 6$, then by Lemma 2.5.1, $\gamma(\Gamma(R)) \geq 4$, a contradiction. So $|\mathfrak{m}| \leq 5$. Thus $|R| = |U(R)| + |\mathfrak{m}| \leq 8 + 5 = 13$ by Lemma 3.4.7. $\square$

Lemma 3.4.9. *The following statements hold:*

(1) *Let $R = \mathbb{Z}_2 \times S$, where S is a local ring of order eight which is not a field. Then $\gamma(\Gamma(R)) = 2$.*

(2) *Let $R = \mathbb{Z}_3 \times S$, where S is a local ring of order four which is not a field. Then $\gamma(\Gamma(R)) = 1$.*

(3) *Let $R = S \times T$, where S and T are local rings of order four which are not fields. Then $\gamma(\Gamma(R)) = 2$.*

Proof. (1) It is clear that $|U(S)| = 4$ and $|J(S)| = 4$. So $|U(R)| = |J(R)| = 4$. Note that each element in $J(R)$ is adjacent to every element in $U(R)$ and $\Gamma(R)$ is 4-regular. Thus, $\Gamma(R)$ is two copies of $K_{4,4}$. By Lemmas 2.5.1 and 2.5.2, $\gamma(G(R)) = 2$.
(2) Note that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)})$ have the same graph structure. It is clear that $\Gamma(R)$ is 4-regular and it contains no triangles. By Lemma 2.4.7, $\gamma(\Gamma(R)) \geq 1$. On the other hand, we can embed $\Gamma(R)$ into $\mathbb{S}_1$ as shown in **Figure 6**. Hence, $\gamma(\Gamma(R)) = 1$.

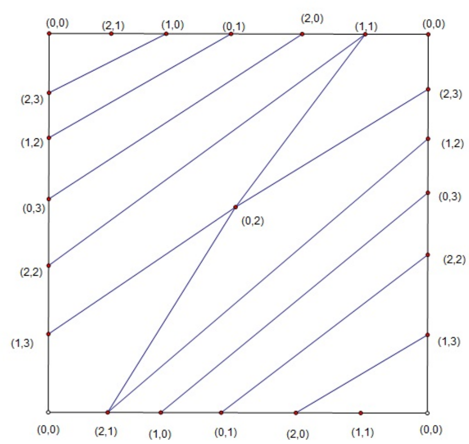


Figure 6: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$

(3) It is clear that $|U(S)| = |U(T)| = 2$ and $|J(S)| = |J(T)| = 2$. So $|U(R)| = |J(R)| = 4$. Note that each element in $J(R)$ is adjacent to every element in $U(R)$ and that $\Gamma(R)$ is 4-regular. Thus, $\Gamma(R)$ is two copies of $K_{4,4}$, so $\gamma(\Gamma(R)) = 2$ by Lemmas 2.5.1 and 2.5.2. $\square$

Lemma 3.4.10. *The following statements hold:*

(1) *Let* $R = \mathbb{Z}_2 \times \mathbb{Z}_7$. *Then* $\gamma(\Gamma(R)) \geq 5$.

(2) *Let* $R = \mathbb{Z}_2 \times S$, *where* S *is a local ring of order nine which is not a field. Then* $\gamma(\Gamma(R)) \geq 6$.

(3) *Let* $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$. *Then* $\gamma(\Gamma(R)) \geq 7$.

(4) *Let* $R = \mathbb{F}_4 \times S$, *where* S *is a local ring of order four which is not a field. Then* $\gamma(\Gamma(R)) \geq 5$.

Proof. (1) $\Gamma(R)$ is 6-regular. Note that $\Gamma(R)$ is a bipartite graph, so it contains no triangles. As $\Gamma(R)$ contains 14 vertices and 42 edges, $\gamma(\Gamma(R)) \geq 5$ by Lemma 2.4.7.
(2) It is clear that $|U(S)| = 6$. So $\Gamma(R)$ is 6-regular. As $\Gamma(R)$ has 54 edges and 18 vertices and contains no triangles, the claim follows from Lemma 2.4.7.
(3) It is clear that $|U(R)| = 6$. So $\Gamma(R)$ is 6-regular. As $\Gamma(R)$ has 72 edges and 24 vertices and contains no triangles, the claim follows from Lemma 2.4.7.
(4) It is clear that $|U(S)| = 6$. So $\Gamma(R)$ is 6-regular. As $\Gamma(R)$ has 48 edges and 16 vertices and contains no triangles, the claim follows from Lemma 2.4.7. $\square$

We now are ready to prove the second main result in this section.

Theorem 3.4.11. *Let* R *be a finite commutative ring. If* $1 \leq \gamma(\Gamma(R)) \leq 3$, *then* R *is isomorphic to one of following rings:*

(1) $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{F}_8$, $\mathbb{F}_9$.

(2) $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$.

(3) $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(4) $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^3)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

Proof. Suppose that R is a filed. As $\gamma(\Gamma(R)) \leq 3$, in view of the proof of Theorem 3.4.6, we have $|R| \leq 9$. By Lemma 3.4.5, the graphs $\Gamma(\mathbb{Z}_2)$, $\Gamma(\mathbb{Z}_3)$ and $\Gamma(\mathbb{F}_4)$ are planar, so $R \in \{\mathbb{Z}_5, \mathbb{Z}_7, \mathbb{F}_8, \mathbb{F}_9\}$.

Suppose that R is a local ring which is not a filed. As $\gamma(\Gamma(R)) \leq 3$, by Lemma 3.4.8, $|R| \leq 13$. So $|R| \in \{4, 8, 9\}$. If $|R| = 4$, then $\Gamma(R)$ is planar by Lemma 3.4.5. So $|R| \in \{8, 9\}$. It follows that R is isomorphic to one of rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$.

Suppose that R is not a local ring. As $\gamma(\Gamma(R)) \leq 3$, in view of the proof of Theorem 3.4.6, we have $|R| \leq 24$ or $R \cong (\mathbb{Z}_2)^s \times R_1 \times \cdots \times R_t$ with $0 \leq s \leq 2$ and $|U(R_1 \times \cdots \times R_t)| = 4$ or 6.

For the case of $s = 0$, we may assume that $R \cong R_1 \times \cdots \times R_t$, where each R_i has at least three elements. As $|R| \leq 24$, we have $t = 2$. It follows that R is one of the following rings:

$\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_5$.

As $|U(R)| \leq 8$ by Lemma 3.4.7, we have $R \neq \mathbb{Z}_3 \times \mathbb{Z}_7$ and $R \neq \mathbb{F}_4 \times \mathbb{F}_4$. If $R = \mathbb{Z}_3 \times \mathbb{Z}_5$, then $\Gamma(R)$ is a graph with 15 vertices and 60 edges. It then follows that $\gamma(\Gamma(R)) \geq 4$ by Lemma 2.4.7. If $R = \mathbb{Z}_4 \times \mathbb{Z}_5$, then $\Gamma(R)$ is a graph with 20 vertices and 80 edges. It then follows that $\gamma(\Gamma(R)) \geq 5$ by Lemma 2.4.7. If $R = \mathbb{Z}_4 \times \mathbb{F}_4$ or $R = \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4$), then these cases are ruled out by Lemma 3.4.10(4).

Thus, $R \in \{\, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}, \frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)} \}$.

For the case of $1 \leq s \leq 2$, according the proof of Theorem 3.4.6, R is one of the following rings:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$;

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^3)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2)}$;

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^3)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2)}$;

(iv) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$;

(v) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$;

(vi) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$;

(vii) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$;

(viii) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$;

(ix) $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$;

(x) $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$;

(xi) $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$;

(xii) $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{(x^2)}$;

(xiii) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$;

(xiv) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{F}_4$;

(xv) $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4$.

By Lemmas 3.4.9(1)(2) and 2.5.2, the rings appearing in (iii) and (vi) are ruled out. By Lemmas 3.4.9(3) and 2.5.2, the rings appearing in (vii)–(ix) are ruled out. By Lemma 3.4.10, the rings appearing in (x)–(xv) are ruled out. So, in this case, R is one of rings appearing in (i), (ii), (iv) and (v). This completes the proof. $\square$

Our next goal is to classify all finite commutative rings whose unitary Cayley graphs have genus 1, 2, 3, respectively. By the previous theorem, we need only to determine the genus of the unitary Cayley graphs of all the rings appeared in Theorem 3.4.11.

Lemma 3.4.12. *The following statements hold:*

(1) $\gamma(\Gamma(\mathbb{Z}_5)) = \gamma(\Gamma(\mathbb{Z}_7)) = 1$.

(2) $\gamma(\Gamma(\mathbb{F}_8)) = 2$.

(3) $\gamma(\Gamma(\mathbb{F}_9)) = 3$.

Proof. Note that the unitary Cayley graph of a field is a complete graph. The results follow by Lemma 2.5.1. □

Lemma 3.4.13. [70, Theorem 6.39] *Let m, n be positive integers. Then $\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}$. In particular, $\gamma(K_{3,3,3}) = 1$.*

Lemma 3.4.14. *The following statements hold:*

(1) *If R is a local ring of order 8 which is not a field, then $\gamma(\Gamma(R)) = 1$.*

(2) *If R is a local ring of order 9 which is not a field, then $\gamma(\Gamma(R)) = 1$.*

Proof. (1) It is clear that R has four units and $|J(R)| = 4$. So $\Gamma(R)$ is the complete bipartite graph $K_{4,4}$. The claim follows by Lemma 2.5.1.
(2) Note that $\Gamma(R)$ is a complete 3-partite graph $K_{3,3,3}$. So $\gamma(\Gamma(R)) = 1$ by Lemma 3.4.13. □

Lemma 3.4.15. *The following statements hold:*

(1) *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\gamma(\Gamma(R)) = 1$.*

(2) *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_5$. Then $\gamma(\Gamma(R)) = 1$.*

(3) *Let* $R = \mathbb{Z}_3 \times \mathbb{F}_4$. *Then* $\gamma(\Gamma(R)) = 3$.

Proof. (1) By Theorem 3.1.1, $\Gamma(R)$ is not planar, so we have $\gamma(\Gamma(R)) \geq 1$. The following **Figure 7** shows that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ can be embedded into $\mathbb{S}_1$. Thus $\gamma(\Gamma(R)) = 1$.

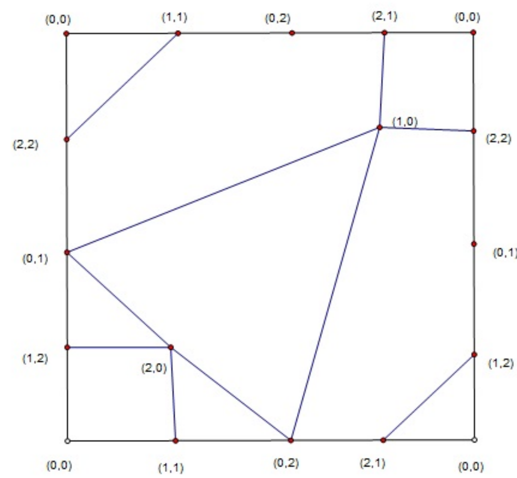


Figure 7: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$

(2) By Lemma 2.4.7, we obtain $\gamma(\Gamma(R)) \geq 1$. On the other hand, we can embed $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$ into $\mathbb{S}_1$ as shown in **Figure 8**. Hence, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 1$.

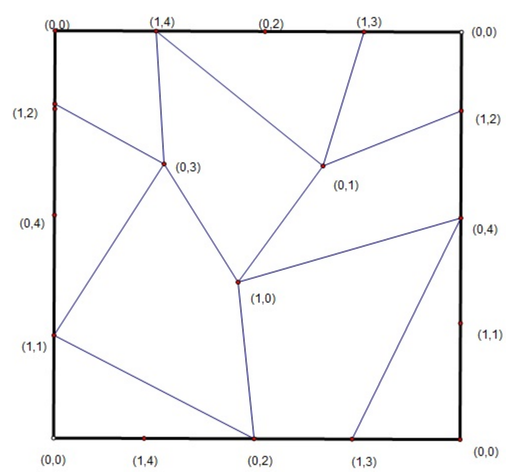


Figure 8: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$

(3) Write $\mathbb{F}_4 = \{0, 1, a, b\}$. Note that $\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$ has 36 edges and 12 vertices. So $\gamma(G) \geq 1$ by Lemma 2.4.7. On the other hand, we can embed G into $\mathbb{S}_1$ as shown in **Figure 9**. Therefore, $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)) = 1$.

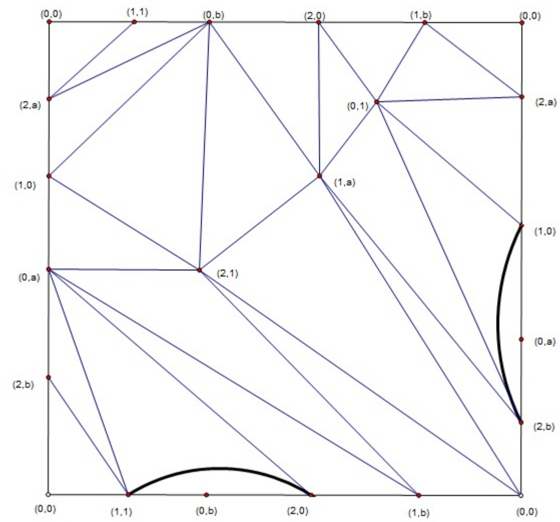


Figure 9: $\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$

$\square$

We now prove the last main result of this section.

Theorem 3.4.16. *Let R be a finite commutative ring. Then the following hold:*

(1) $\gamma(\Gamma(R)) = 1$ *if and only if R is isomorphic to one of the following rings:*

$\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

(2) $\gamma(\Gamma(R)) = 2$ *if and only if R is isomorphic to one of the following rings:*

$\mathbb{F}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^3)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(3) $\gamma(\Gamma(R)) = 3$ *if and only if* $R \cong \mathbb{F}_9$.

Proof. By Theorem 3.4.11, it suffices to determine $\gamma(\Gamma(R))$ for the rings R appearing in Theorem 3.4.11.

In view of Lemmas 3.4.9, 3.4.12, 3.4.14 and 3.4.15, the three rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ are the remaining uncertain cases. By Lemma 3.4.15(2) and Lemma 3.4.4, we know that $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5)) = 2\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 2$.

As $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ has 36 edges and 18 vertices with no triangles, $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) \geq 1$ by Lemma 2.4.7. Note that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_6)$. On the other hand, we can embed $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_6)$ into $\mathbb{S}_1$ shown in **Figure 10**. So $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = 1$.

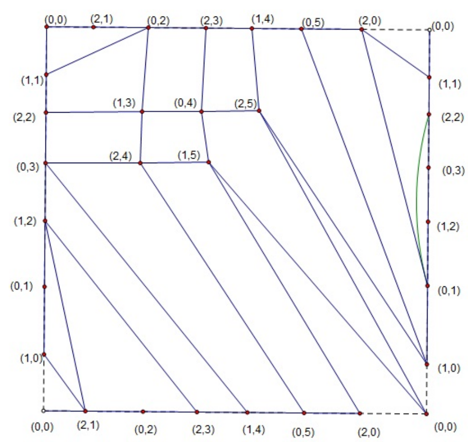


Figure 10: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_6)$

By Lemma 3.4.4, we have $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) = 2$.

This completes the proof. $\square$

As mentioned earlier, Khashyarmanesh and Khorsandi generalized the definition of the unitary Cayley graphs of rings in [38]. Let G be a multiplicative subgroup of $U(R)$ and S be a non-empty subset of G such that $S^{-1} = \{s^{-1} : s \in S\} \subseteq S$. They define a graph $\Gamma(R, G, S)$ with vertex set R and in which two distinct vertices x and y are adjacent if there exists $s \in S$ such that $x + sy \in G$. If $G = U(R)$, they use $\Gamma(R, S)$ to replace $\Gamma(R, G, S)$. We end this section with a remark.

Remark 3.4.17. In [3, Theorem 4.2], the authors classified all commutative Artinian rings whose generalized unit and unitary Cayley graphs are toroidal (genus is one). As its application, they got a list of commutative rings whose unitary Cayley graphs are toroidal [3, Corollary 4.5]. However, their proof has some gaps, and it turns out that their list is not right. In fact, in the case that $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$, one can easily check that $\Gamma(R, S)$ is a cube. So $\Gamma(R, S)$ is planar. For the case that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, they used the contracted techniques to get a minor subgraph whose genus is greater than one. But they made a mistake that the vertices 002, 121 and 001 cannot contract as 002 is not adjacent to 121 and 001. The same mistake occurs in the case that $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$. As a matter of fact, we can embed the graphs $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_6)$ ($\cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$) and $\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$ into $\mathbb{S}_1$ as shown in Figures 4 and 5, respectively. Therefore, the complete list in [3, Corollary 4.5] should contain two more rings $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{F}_4$, with the ring $\mathbb{Z}_2 \times \mathbb{F}_4$ removed.

Chapter 4

Isomorphism Between Unit Graphs and Unitary Cayley Graphs

4.1 Introduction

As we have seen in Chapter 2 and Chapter 3, the unit graph and the unitary Cayley graph of a same ring share many common properties. For instances, the two graphs have the same range of grith, which is $\{3, 4, 6, \infty\}$; and the two graphs of a self-injective ring also have the same range of diameters, which is $\{1, 2, 3, \infty\}$. Thus, it is interesting to know when the two graphs are isomorphic to each other. This question is the topic of this Chapter. We give an answer for any finite ring.

4.2 Isomorphism Between $G(R)$ and $\Gamma(R)$

If we omit the word "distinct" in the definition of unit graph, we obtain the closed unit graph denoted $\overline{G}(R)$; this graph may have loops. Note that if $2 \notin U(R)$, then $\overline{G}(R) = G(R)$.

In graph theory, the tensor product $G \otimes H$ of graphs G and H is a graph such that the vertex set of $G \otimes H$ is the Cartesian product $V(G) \times V(H)$, and any two vertices (u, u') and (v, v') are adjacent in $G \otimes H$ if and only if u is adjacent to v and u' is adjacent to v'. Clearly, for given rings R_1 and R_2, two distinct vertices $(x_1, x_2), (y_1, y_2) \in R_1 \times R_2$ are adjacent if and only if x_1 is adjacent to y_1 in $\overline{G}(R_1)$ and x_2 is adjacent to y_2 in $\overline{G}(R_2)$. From this, we obtain $G(R_1 \times R_2) = \overline{G}(R_1) \otimes \overline{G}(R_2)$.

By Lemmas 2.4.4 and 3.3.5, we easily get the following.

Lemma 4.2.1. *Let R be a ring with finitely many units. If $G(R) \cong \Gamma(R)$, then $2 \notin U(R)$.*

The following lemma is an easy observation.

Lemma 4.2.2. *Let R be a ring. If char$(R) = 2$, then $G(R) \cong \Gamma(R)$.*

Proof. Define $\phi : R \to R$ by $\phi(x) = x$ for all $x \in R$. As $2R = 0$, $x + y = x - y$. So $x + y$ is a unit iff $x - y$ is a unit. So $G(R) \cong \Gamma(R)$. $\square$

Lemma 4.2.3. *Let D be a finite field. Then $G(D) \cong \Gamma(D)$ iff char$(D) = 2$.*

Proof. The necessity follows from Lemma 4.2.1 and the sufficiency follows from Lemma 4.2.2. $\square$

Proposition 4.2.4. *For $n \geq 2$, $G(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_n)$ iff n is even.*

Proof. If $G(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_n)$, then n must be even by Lemma 4.2.1. Conversely, suppose n is even. Define $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\phi(x) = x$ if x is even and $\phi(x) = -x$ if x is odd. It is not difficult to verify that ϕ is an isomorphism between $G(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_n)$. $\square$

Lemma 4.2.5. *Let R be a ring. If $G(\overline{R}) \cong \Gamma(\overline{R})$, then $G(R) \cong \Gamma(R)$.*

Proof. Let ϕ be the isomorphism from graph $G(\overline{R})$ to $\Gamma(\overline{R})$. That is, ϕ is a bijection from $\overline{R}$ to $\overline{R}$ such that $\overline{x}+\overline{y} \in U(\overline{R})$ if and only if $\phi(\overline{x})-\phi(\overline{y}) \in U(\overline{R})$ for all $\overline{x}, \overline{y} \in \overline{R}$. Let $\{x_i : i \in I\}$ be a set of fixed representatives of the distinct cosets of $J(R)$ in R. Then $R = \bigcup_{i\in I}(x_i + J(R))$ is a disjoint union.

For each $i \in I$, there exists a unique $i' \in I$ such that $\phi(\overline{x_i}) = \overline{x_{i'}}$. We define a map $\sigma_i : x_i+J(R) \to x_{i'}+J(R)$ by $\sigma_i(x_i+j) = x_{i'}+j$ for all $j \in J(R)$. Then σ_i is a bijection and the map $\sigma : R \to R$ whose restriction on $x_i + J(R)$ is σ_i is clearly a bijection. We claim that σ is a graph isomorphism from $G(R)$ to $\Gamma(R)$. Suppose x is adjacent to y in $G(R)$. That is, $x + y \in U(R)$. Let us assume $x = x_i + j_1 \in x_i + J(R)$ and $y = x_k+j_2 \in x_k+J(R)$. As $x+y = (x_i+x_k)+(j_1+j_2) \in U(R)$, $x_i+x_k \in U(R)$. Hence, $\overline{x_i}+\overline{x_k} \in U(\overline{R})$. So $\phi(\overline{x_i})-\phi(\overline{x_k}) \in U(\overline{R})$, i.e., $\overline{x_{i'}}+\overline{x_{k'}} \in U(\overline{R})$. Thus $x_{i'}-x_{k'} \in U(R)$. Then $\sigma(x) - \sigma(y) = (x_{i'} + j_1) - (x_{k'} + j_2) = (x_{i'} + x_{k'}) + (j_1 - j_2) \in U(R)$. So $\sigma(x)$ is adjacent to $\sigma(y)$ in $\Gamma(R)$.

Suppose $\sigma(x)$ is adjacent to $\sigma(y)$ in $\Gamma(R)$. That is, $\sigma(x) - \sigma(y) \in U(R)$. Let us assume $x = x_i + j_1 \in x_i + J(R)$ and $y = x_k + j_2 \in x_k + J(R)$. Then $\sigma(x) = x_{i'} + j_1$ and $\sigma(y) = x_{k'} + j_2$. Note that $\phi(\overline{x_i}) = \overline{x_{i'}}$ and $\phi(\overline{x_k}) = \overline{x_{k'}}$. As $\sigma(x) - \sigma(y) = (x_{i'} + x_{k'}) + (j_1 + j_2) \in U(R)$, $x_{i'} + x_{k'} \in U(R)$. Hence, $\overline{x_{i'}} + \overline{x_{k'}} \in U(\overline{R})$. That is $\phi(\overline{x_i}) - \phi(\overline{x_k}) \in U(\overline{R})$. So $\overline{x_i} + \overline{x_k} \in U(\overline{R})$. Thus $x_i + x_k \in U(R)$. Then $x + y = (x_i + j_1) + (x_k + j_2) = (x_i + x_k) + (j_1 + j_2) \in U(R)$. So x is adjacent to y in $G(R)$. This completes our proof. $\square$

Lemma 4.2.6. *Let R be a ring. The following statements hold:*

(1) *If $x - y \in J(R)$, then $N(x) = N(y)$ in $G(R)$.*

(2) *If $x - y \in J(R)$, then $N(x) = N(y)$ in $\Gamma(R)$.*

Proof. (1) Let $z \in N(x)$. Then $x+z \in U(R)$. It follows that $y+z = (x+z)-(x-y) \in U(R)$. So $z \in N(y)$. Hence, $N(x) \subseteq N(y)$. Similarly, $N(y) \subseteq N(x)$.

(2) The proof is similar to (1). □

The next lemma was proved by Kiani and Aghaei [36] for a finite ring. We extend this result to a ring with stable range one. Recall that a ring R has *stable range one* provided that $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in U(R)$. Note that every semilocal ring has stable range one (see [23]). Let R be a ring. Note that $J(R) = \{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}$.

Lemma 4.2.7. *Let R be a ring with stable range one. Then $a + U(R) = U(R)$ iff $a \in J(R)$.*

Proof. Note that $a \in J(R)$ clearly implies $a+U(R) = U(R)$. Now assume $a+U(R) = U(R)$. For $x \in R$, Let $b := 1 - ax$. Then $ax + b = 1$, so $aR + bR = R$. As R has stable range one, there exists $y \in R$ such that $u := a + by \in U(R)$. It follows that $a + (-u) = b(-y)$ is a unit. So b is a unit. Hence $a \in J(R)$. □

Lemma 4.2.8. *Let R be a ring with stable range one. The following statements hold:*

(1) *If $N(x) = N(y)$ in $G(R)$, then $x - y \in J(R)$.*

(2) *If $N(x) = N(y)$ in $\Gamma(R)$, then $x - y \in J(R)$*

Proof. (1) Note that $N(x) = -x + U(R)$ and $N(y) = -y + U(R)$. So $N(x) = N(y)$ deduces $-x + U(R) = -y + U(R)$, i.e., $(x - y) + U(R) = U(R)$. So $x - y \in J(R)$ by Lemma 4.2.7.
(2) Here $N(x) = x+U(R)$ and $N(y) = y+U(R)$. So $N(x) = N(y)$ implies $x+U(R) = y + U(R)$, i.e., $(x - y) + U(R) = U(R)$. So $x - y \in J(R)$ by Lemma 4.2.7. □

Theorem 4.2.9. *Let R be a ring with stable range one. Then $G(R) \cong \Gamma(R)$ iff $G(\overline{R}) \cong \Gamma(\overline{R})$.*

Proof. The sufficiency is follows from Lemma 4.2.5. To show the necessity, let $\phi : G(R) \to \Gamma(R)$ be an isomorphism. Define $\overline{\phi} : \overline{R} \to \overline{R}$ by $\overline{\phi}(\overline{x}) = \overline{\phi(x)}$. Next we verify $\overline{\phi}$ is an isomorphism between $G(\overline{R})$ and $\Gamma(\overline{R})$.

(1) $\overline{\phi}$ is a mapping. Let $\overline{x} = \overline{y}$. Then $x - y \in J(R)$ and thus $N(x) = N(y)$ in $G(R)$ by Lemma 4.2.6. As ϕ is an isomorphism, we have $N(\phi(x)) = N(\phi(y))$ in $\Gamma(R)$. So $\phi(x) - \phi(y) \in J(R)$ by Lemma 4.2.8. Thus, $\overline{\phi(x)} = \overline{\phi(y)}$.

(2) $\overline{\phi}$ is injective. Let $\overline{\phi(x)} = \overline{\phi(y)}$. Then $\phi(x) - \phi(y) \in J(R)$, so $N(\phi(x)) = N(\phi(y))$ in $\Gamma(R)$ by Lemma 4.2.6. As ϕ is an isomorphism, we have $N(x) = N(y)$ in $G(R)$. Then $x - y \in J(R)$ by Lemma 4.2.8. So, $\overline{x} = \overline{y}$.

(3) It is clear that $\overline{\phi}$ is surjective.

(4) $\overline{\phi} : G(\overline{R}) \to \Gamma(\overline{R})$ is an isomorphism. In fact, $\overline{x}$ is adjacent to $\overline{y}$ in $G(\overline{R})$ iff $\overline{x} + \overline{y} \in U(\overline{R})$ iff $x + y \in U(R)$ iff x is adjacent to y in $G(R)$ iff $\phi(x)$ is adjacent to $\phi(y)$ in $\Gamma(R)$ iff $\phi(x) - \phi(y) \in U(R)$ iff $\overline{\phi(x)} - \overline{\phi(y)} \in U(\overline{R})$ iff $\overline{\phi(x)}$ is adjacent to $\overline{\phi(y)}$ in $\Gamma(\overline{R})$. $\square$

Proposition 4.2.10. *Let R be a local ring with $|U(R)| < \infty$. Then $G(R) \cong \Gamma(R)$ iff $2 \notin U(R)$.*

Proof. The necessity follows from Lemma 4.2.1. To see the sufficiency, let $2 \notin U(R)$. Then $\overline{2} \notin U(\overline{R})$ and thus $\overline{2} = \overline{0}$. So by Lemma 4.2.2, $G(\overline{R}) \cong \Gamma(\overline{R})$ and hence $G(R) \cong \Gamma(R)$ by Lemma 4.2.5. $\square$

Proposition 4.2.11. *Let R be a ring with $\overline{R} = \mathbb{M}_n(D)$, where D is a division ring. Then $G(R) \cong \Gamma(R)$ iff $char(D) = 2$.*

Proof. If $char(D) \neq 2$, then $\overline{2}$ is a unit in $\mathbb{M}_n(D)$ and thus 2 is a unit in R. But $G(R) \ncong \Gamma(R)$ by Lemma 4.2.1. For the sufficiency, let $char(D) = 2$. Then $\overline{2} = \overline{0}$, so $G(\overline{R}) \cong \Gamma(\overline{R})$, and hence $G(R) \cong \Gamma(R)$ by Lemma 4.2.5. $\square$

To prove our main result in this section, we need a result from graph theory.

Lemma 4.2.12. [32, Theorem 9.10] *Let A, B and C are finite simple graphs (may have loops). If $A \otimes C \cong B \otimes C$ and C has an odd cycle, then $A \cong B$.*

Theorem 4.2.13. *Let R be a finite ring. Then $G(R) \cong \Gamma(R)$ iff $char(\overline{R}) = 2$ or $\overline{R} \cong \mathbb{Z}_2 \times S$, where S is a finite ring.*

Proof. ($\Leftarrow$). If $char(\overline{R}) = 2$, then $G(\overline{R}) \cong \Gamma(\overline{R})$ by Lemma 4.2.2. The result follows from Theorem 4.2.9. Let $\overline{R} \cong \mathbb{Z}_2 \times S$, where S is a finite ring. Define $\phi : \overline{R} \to \overline{R}$ by $\phi(0, s) = (0, s)$ and $\phi(1, s) = (1, -s)$ for all $s \in S$. Then ϕ is clearly a bijection. Now we show that ϕ is a graph isomorphism from $G(\overline{R})$ to $\Gamma(\overline{R})$.

Assume that x is adjacent to y in $G(\overline{R})$. Then $x + y \in U(\overline{R})$. We may assume, without loss of generality, that $x = (0, s_1)$ and $y = (1, s_2)$ where $s_1 + s_2 \in U(S)$. Then $\phi(x) - \phi(y) = (0, s_1) - (1, -s_2) = (1, s_1 + s_2) \in U(\overline{R})$. So $\phi(x)$ is adjacent to $\phi(y)$ in $\Gamma(\overline{R})$.

Assume that x is adjacent to y in $\Gamma(\overline{R})$. Then $x - y \in U(\overline{R})$. We may assume, without loss of generality, that $x = (0, s_1)$ and $y = (1, s_2)$ where $s_1 - s_2 \in U(S)$. Then, with $x' := \phi^{-1}(x) = (0, s_1)$ and $y' := \phi^{-1}(y) = (0, -s_1)$, we have $x' + y' = (0, s_1) + (1, -s_2) = (1, s_1 - s_2) \in U(\overline{R})$. So x' is adjacent to y' in $G(\overline{R})$.
Therefore, $G(\overline{R}) \cong \Gamma(\overline{R})$. By Theorem 4.2.9, we have $G(R) \cong \Gamma(R)$.

($\Rightarrow$). By Theorem 4.2.9, we have $G(\overline{R}) \cong \Gamma(\overline{R})$. Let $\overline{R} \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_t}(D_t)$, where each D_i is a finite field. As $\overline{2} \notin U(\overline{R})$ by Lemma 4.2.1, there must have some i such that $char(D_i) = 2$. If for all i, $char(D_i) = 2$, then $char(\overline{R}) = 2$. Now assume that $char(D_i) = 2$ for $1 \leq i \leq s$ and $char(D_j) \neq 2$ for $s + 1 \leq j \leq t$.
Note that

$$G(\overline{R}) \cong \overline{G}(\mathbb{M}_{n_1}(D_1)) \otimes \cdots \otimes \overline{G}(\mathbb{M}_{n_s}(D_s)) \otimes \overline{G}(\mathbb{M}_{n_{s+1}}(D_{s+1})) \otimes \cdots \otimes \overline{G}(\mathbb{M}_{n_t}(D_t))$$

$$\cong G(\mathbb{M}_{n_1}(D_1)) \otimes \cdots G(\mathbb{M}_{n_s}(D_s)) \otimes \overline{G}(\mathbb{M}_{n_{s+1}}(D_{s+1})) \cdots \otimes \overline{G}(\mathbb{M}_{n_t}(D_t))$$

and

$$\Gamma(\overline{R}) \cong \Gamma(\mathbb{M}_{n_1}(D_1)) \otimes \cdots \otimes \Gamma(\mathbb{M}_{n_s}(D_s)) \otimes \Gamma(\mathbb{M}_{n_{s+1}}(D_{s+1})) \otimes \cdots \otimes \Gamma(\mathbb{M}_{n_t}(D_t)).$$

If $|(M_{n_i}(D_i))| > 2$ for all $1 \leq i \leq s$, then $gr(G((M_{n_i}(D_i))) = 3 = gr(\Gamma((M_{n_i}(D_i)))$ by Lemmas 2.2.3 and 2.2.4. So $G(\overline{R}) \cong \Gamma(\overline{R})$ implies that $\overline{G}(\mathbb{M}_{n_{s+1}}(D_{s+1})) \otimes \cdots \otimes \overline{G}(\mathbb{M}_{n_t}(D_t)) \cong \Gamma(\mathbb{M}_{n_{s+1}}(D_{s+1})) \otimes \cdots \otimes \Gamma(\mathbb{M}_{n_t}(D_t))$ by Lemma 4.2.12. This is impossible by Lemma 4.2.1. Therefore, there is some D_i such that $D_i = \mathbb{Z}_2$. The proof is complete. $\square$

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