# GEOMETRY OF BANACH SPACES AND SOME FIXED POINT THEOREMS



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16707

# GEOMETRY OF BANACH SPACES AND SOME FIXED POINT THEOREMS

BY

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A THESIS

SUBMITTED TO THE COMMITTEE ON GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF ARTS

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

ST. JOHN'S, NEWFOUNDLAND

AUGUST 1972

This Thesis has been examined and approved by:-

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## ABSTRACT

The aim of this thesis is to study the geometry of Banach spaces, the existence of fixed points and the convergence of iterative sequences of certain mappings in Banach spaces.

We introduce some of the basic definitions and give a brief survey of some well-known results on fixed points for different mappings.

We also introduce and discuss different classifications of Banach spaces. A few results, similar to those of uniformly convex Banach spaces, have been given for weakly uniformly convex and weakly* uniformly convex Banach spaces.

Finally considering more general mappings, of types Diaz and Metcalf [46], Dotson [48], Kirk [87], some new results and various generalizations have been given on the asymptotic regularity and the convergence of the iterative sequences in Banach spaces. We end with mentioning some of the applications of fixed point theory in brief.

## ACKNOWLEDGEMENTS

The author would like to express his sincere thanks to Dr. S.P. Singh for his guidance and valuable suggestions during the preparation of this thesis. He would also like to thank Mrs. Rosalind Wells for typing the manuscript.

The financial support in the form of a University Fellowship from the Memorial University of Newfoundland is gratefully acknowledged.

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# INTRODUCTION

S. Banach [ 7], in 1922, formulated his classical theorem, known as the Banach contraction principle, which may be stated as follows:

"A contraction mapping of a complete metric space X into itself has a unique fixed point".

Because of its widespread applicability in proving the existence and uniqueness of solutions of the differential and integral equations, many extensions of the above principle have been given in recent years by several mathematicians such as Chu and Diaz [36], [37], Edelstein [52], [55], [56], Rakotch [113], Bailey [ 6], Boyd and Wong [17], Browder [28], Sehgal [120] and others.

The main objective of this thesis is to study the geometry of Banach spaces, the existence of fixed points and the convergence of the iterative sequences of certain mappings in Banach spaces.

The preliminaries of metric and normed linear spaces as well as some of the well-known results on fixed points of different mappings are given in chapter I.

In chapter II we discuss different classifications of Banach spaces according to various geometric properties of their unit balls. In 1936, J.A. Clarkson [ 38] introduced the notion of uniform convexity of the norm in a Banach space as follows: a norm is uniformly convex if, whenever the midpoint of a variable chord in the unit sphere of the space approaches the boundary of the sphere, the length of the chord approaches zero. Many generalizations of uniform convexity have been given in recent years, for example see [94], [ 3 ], [60]. As a simple consequence of the definitions of (WUC) and (W*UC) Banach spaces, we obtain a few results which are useful in applications. Moreover, using the concept of normal structure ([18]), we give some interesting results for semi-nonexpansive mappings which are modelled after Zizler [139].

In chapter III we have obtained several results even when the hypothesis of nonexpansiveness of a mapping $T$ is weakened up to the extent that $T$ is required to be nonexpansive only at its fixed points (whenever they exist). Considering a nonexpansive (quasi-nonexpansive) mapping $T$, we show that a more general mapping $S$, of type Kirk [87], is nonexpansive (quasi-nonexpansive) and asymptotically regular or weakly asymptotically regular in (UC) and (WUC) Banach spaces. Thus, we generalize certain results of Browder and Petryshyn [32] as well as of Kirk [87]. We note that, in general, it is not the case for nonexpansive mapping $T$ that the sequence of Picard iterates $\{T^n(x_o)\}$ converges to fixed points of $T$. In case $T$ is a densifying or a densifying nonexpansive mapping, we give some new results for the convergence of the iterative sequences of the mappings $S$ and $T_\lambda$, which in turn generalize and improve certain results of Petryshyn [108], Singh [125], Diaz and Metcalf [46], Edelstein [57], Schaefer [119], Krasnoselskii [90], and Kirk [87].

In the end we mention some of the applications of fixed point theory in brief.

# CHAPTER I

# INTRODUCTORY CONCEPTS

In this chapter we introduce some preliminary definitions and some of the known results primarily on metric spaces. We shall use the conventional shorthand "iff" for "if and only if". Moreover, we shall denote by $K$ the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

## 1.1. Preliminaries.

Definition 1.1.1. Let $X$ be a non-empty set and let $R^+$ denote the positive reals. We define a distance function $d : X \times X \to R^+$ to be a metric iff the following conditions are satisfied for all $x,y,z \in X$:

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0$ iff $x = y$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

A metric space is a pair $(X,d)$ in which $X$ is a non-empty set and $d$ is a metric on $X$. We may denote the metric space by $X$ alone when the metric $d$ is understood.

Definition 1.1.2. The diameter $\delta(A)$ of a non-empty subset $A$ of the metric space $(X,d)$ is defined by

$$\delta(A) = \sup \{d(x,y) : x,y \in A\}.$$

If the diameter of $A$ is finite, i.e. $\delta(A) < \infty$, then $A$ is said to be bounded; if not, i.e. $\delta(A) = \infty$, then $A$ is said to be unbounded.

We define the distance between the point $x$ and the subset $A$ of $(X,d)$ to be

$$d(x,A) = \inf \{d(x,y) : y \in A\}$$

and, in an anologous manner, define the distance between the two subsets B and C of $(X,d)$ to be

$$d(B,C) = \inf \{d(x,y) : x \in B,\ y \in C\}.$$

<u>Definition 1.1.3.</u> A sequence $\{x_n\}$ in a metric space $X$ is said to converge to the point $x_o \in X$ and we write $x_n \to x_o$, iff, for each real number $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $d(x_n,x_o) < \varepsilon$, for all $n \geq N$. In otherwords, $x_n \to x_o$ iff $\lim\limits_{n\to\infty} d(x_n,x_o) = 0$.

<u>Definition 1.1.4.</u> A sequence $\{x_n\}$ is said to be a Cauchy (or, fundamental) sequence iff, for each real number $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $d(x_m,x_n) < \varepsilon$, for all $m,n \geq N$.

<u>Remark 1.1.1.</u> Every convergent sequence is a Cauchy sequence.

<u>Definition 1.1.5.</u> A metric space is said to be complete iff every Cauchy sequence in $X$ converges in $X$.

<u>Definition 1.1.6.</u> A metric space $X$ is said to be separable iff there is a countable subset of $X$ that is dense in $X$.

<u>Definition 1.1.7.</u> Let $X$ be a metric space. The subset $A$ of $X$ is said to be totally bounded if given $\varepsilon > 0$ there exists a finite number of subsets $A_1, A_2, A_3, \ldots, A_n$ of $X$ such that

$$\delta(A_k) < \varepsilon \qquad (k = 1,2, \ldots , n) \quad \text{and} \quad A \subset \bigcup_{k=1}^{n} A_k .$$

<u>Remark 1.1.2.</u> If a subset $A$ of a metric space $X$ is totally bounded then it is bounded but not conversely. However, in $\mathbb{R}$ bounded and totally bounded sets are equivalent.

The following well-known theorem is the most important and useful property of totally bounded sets.

Theorem 1.1.1. Let $X$ be a metric space. Then a subset $A$ of $X$ is totally bounded iff every sequence of points of $A$ contains a Cauchy subsequence.

Definition 1.1.8. A metric space $X$ is said to be compact if from every open covering $\{G_\alpha\}$ of $X$ it is possible to extract a finite subcollection of the $G_\alpha$'s which constitute an open covering of $X$.

Definition 1.1.9. A linear space over $K$ is a quadruple $(X,K,+,\cdot)$ where $X$ is a non-empty set, $+$ is a mapping $(x,y) \to x + y$ of $X \times X$ into $X$, $\cdot$ is a mapping $(\alpha,x) \to \alpha\cdot x$ of $K \times X$ into $X$, such that the following conditions are satisfied for all $x,y,z \in X$ and $\alpha,\beta \in K$:

(i) $x + y = y + x$ ,

(ii) $x + (y + z) = (x + y) + z$ ,

(iii) there exists $\theta \in X$ such that $x + \theta = x$ ,

(iv) for each $x \in X$ there exists $-x \in X$ such that $x + (-x) = \theta$,

(v) $(\alpha + \beta)\cdot x = \alpha\cdot x + \beta\cdot x$ ,

(vi) $\alpha\cdot(x + y) = \alpha\cdot x + \alpha\cdot y$ ,

(vii) $\alpha\cdot(\beta\cdot x) = (\alpha\beta)\cdot x$ ,

(viii) $1\cdot x = x$ .

We shall write the linear space $X$ or $(X,K)$ instead of the linear space $(X,K,+,\cdot)$.

We want to introduce and discuss here a few results related to the notion of convexity. Many important topics in the theory of linear spaces rely on convexity. This notion, stemming from intutive geometric ideas, can be formulated purely analytically.

<u>Definition 1.1.10.</u> Let $x$ and $y$ be two points in the linear space $X$. Then the segment connecting the two points $x$ and $y$ is the totality of all points of the form $\alpha x + \beta y$ where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$.

<u>Definition 1.1.11.</u> A subset $G$ of a linear space $X$ over $K$ is called convex if given two arbitrary points $x$ and $y$ belonging to $G$, the segment connecting them also belongs to $G$.

To illustrate the definition, we give the following examples:

<u>Examples 1.1.1.</u>

(1) The empty set and a set consisting of one point are convex sets. Also, the line segment, plane and triangle are convex sets in 3-dimensional Euclidean space.

(2) Let $M$ be a subset of the space $C[a,b]$ consisting of all continuous functions satisfying the extra condition $|f(t)| \leq 1$. Then $M$ is convex, since $|f(t)| \leq 1$, $|g(t)| \leq 1$ together with $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$ implies

$$|\alpha f(t) + \beta g(t) \leq \alpha + \beta = 1.$$

The following lemma gives some basic properties of convex sets.

<u>Lemma 1.1.1.</u> If $G_1$ and $G_2$ are convex sets, so also are the sets $G_1 \cap G_2$, $\lambda G_1$ and $G_1 + G_2$, where $\lambda$ is a scalar.

<u>Definition 1.1.12.</u> The intersection of all closed convex sets containing a set $G$ is a closed convex set which contains $G$ and which is contained in every closed convex set containing $G$. This set is called the closed convex hull or, convex closure of $G$, and is denoted by $\overline{Co}(G)$.

Alternatively, one can define convex closure of $G$ to be the smallest closed convex set containing $G$.

<u>Definition 1.1.13.</u> Let $X$ be a linear space over $K$. A mapping $x \to ||x||$ of $X$ into the set $R^+$ of positive reals is called a norm on $X$ iff it satisfies the following conditions for all $x,y \in X$ and $\alpha \in K$:

(i) $||x|| \geq 0$ ,

(ii) $||x|| = 0$ iff $x = 0$,

(iii) $||\alpha x|| = |\alpha| \, ||x||$ ,

(iv) $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

A normed linear space over $K$ is a pair $(X, ||\cdot||)$, where $X$ is a linear space over $K$ and $||\cdot||$ is a norm on $X$. We shall write the normed linear space $X$ instead of $(X, ||\cdot||)$.

Every normed linear space $X$ is a metric space with a metric $d$ defined on $X$ as $d(x,y) = ||x - y||$ for all $x,y \in X$.

<u>Definition 1.1.14.</u> Two norms $||\ ||_1$ and $||\ ||_2$ on a normed linear space $X$ are said to be equivalent iff there exist numbers $a$ and $b$ with $0 < a \leq b < \infty$ such that

$$a||x||_1 \leq ||x||_2 \leq b||x||_1 \quad \text{for all } x \in X.$$

<u>Definition 1.1.15.</u> A complete normed linear space is called a Banach space.

Examples of some well-known Banach spaces are the following:

$C[a,b]$ : the space of continuous functions $f$ on the interval $[a,b]$, with $||f|| = \sup\{|f(t)| : t \in [a,b]\}$.

$\ell^p$ for $p \geq 1$ : the space of sequences $x = (x_1, x_2, \ldots)$ for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$ with $||x||_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ .

$L^p$ for $p \geq 1$ : the space of all equivalence classes of functions $f$ that are $\mu$ -measurable and $p^{th}$ power summable on finite set $E$ with $$||f||_p = \left(\int_E |f|^p \, d\mu\right)^{1/p} .$$

<u>Definition 1.1.16.</u> Let $X$ be a linear space over $K$. A mapping $x \times y \to (x,y)$ of $X \times X$ into $K$ is said to be an inner product (or scalar product) on $X$ iff

(i) $(x,x) \geq 0$ for all $x \in X$ ,

(ii) $(x,x) = 0$ iff $x = 0$ , $x \in X$,

(iii) $(x,y) = \overline{(y,x)}$ for all $x,y \in X$ ,

(iv) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$ for all $x,y,z \in X$ and $\alpha,\beta \in K$.

If $X$ is an inner product space, we define a norm $||\cdot||$ in terms of the inner product as $||x|| = (x,x)^{1/2}$ for all $x \in X$.

<u>Definition 1.1.17.</u> If an inner product space $X$ is complete, $X$ is said to be a Hilbert space.

## 1.2. Some Fixed Point Theorems

**Definition 1.2.1.** Let $T$ be a mapping or transformation of a set $X$ into itself. A point $x \in X$ is said to be a fixed point of $T$ if $T(x) = x$. In otherwords, a point which remains invariant under a mapping is known as a fixed point.

**Definition 1.2.2.** A mapping $T$ of a metric space $X$ into itself is said to satisfy Lipschitz condition if there exists a real number $k$ (known as Lipschitz constant) such that

(1) $$d(T(x), T(y)) \leq k\, d(x,y), \quad \text{for all } x,y \in X.$$

If the condition (1) is satisfied with a Lipschitz constant $k$ such that $0 \leq k < 1$, then $T$ is called a contraction mapping.

One of the well-known theorems in connection with the fixed points of a mapping in a metric space is that given by Banach [ 7 ] and known as Banach Contraction Principle. This theorem has been used extensively in proving existence and uniqueness theorems of differential and integral equations.

**Theorem 1.2.1. Banach Contraction Principle:**

Let $(X,d)$ be a complete metric space and $T:X \to X$ be a contraction mapping. Then $T$ has a unique fixed point (i.e. the equation $T(x) = x$ has a unique solution).

**Proof.** By hypothesis there is a real number $k$ with $0 \leq k < 1$ such that $d(T(x), T(y)) \leq k\, d(x,y)$, for all $x,y \in X$. Choose any point $x_o \in X$ and set $x_1 = T(x_o)$, $x_2 = T(x_1) = T^2(x_o)$, and in general, let $x_n = T^n(x_o)$. We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. Let $m, n$ be positive integers with $m > n$. In fact,

$$
\begin{aligned}
d(x_m,x_n) = d(T^m(x_o),\ T^n(x_o)) &\leq k\ d(T^{m-1}(x_o),\ T^{n-1}(x_o)) \\
&\leq k^n\ d(T^{m-n}(x_o),\ \ x_o\ ) \\
&= k^n\ d(x_{m-n},\ x_o) \\
&\leq k^n\{d(x_{m-n},\ x_{m-n-1}) + d(x_{m-n-1},\ x_{m-n-2}) \\
&\qquad + \ \ldots\ldots\ldots \quad + d(x\ ,\ x_o)\} \\
&\leq k^n d(x_1,\ x_o)\ \{\ k^{m-n-1} + k^{m-n-2} + \ \ldots. +1\} \\
&\leq \frac{k^n}{1-k}\ \ d(x_1,\ x_o).
\end{aligned}
$$

Since $k < 1$, $d(x_m,\ x_n)$ is arbitrarily small for sufficiently large $n$. Thus the sequence $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, $\lim\limits_{n\to\infty} x_n$ exists. Let $\lim\limits_{n\to\infty} x_n = u$.
Since $T$ is continuous,

$$T(u) = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

Thus the existence of fixed point is proved. We shall now prove the uniqueness. Let $T(u) = u$ and $T(v) = v$, $u \neq v$. Then $d(u,v) = d(T(u),\ T(v)) \leq k\ d(u,v)$. But $k < 1$, therefore $d(u,v) = 0$ i.e. $u = v$. Hence uniqueness.

<u>Definition 1.2.3.</u> A mapping $T$ of a metric space $X$ into itself is said to be contractive mapping if $d(T(x),\ T(y)) \quad d(x,y)$, for all $x,y \in X$, $x \neq y$.

A contractive mapping on a complete metric space need not have a fixed point as the following example demonstrates:

<u>Example 1.2.1.</u> The map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \pi/2 - \text{arc tan}\ x$ is clearly contractive but has no fixed point.

Edelstein [53] has given the following theorem for the existence of a fixed point for a contractive mapping.

<u>Theorem 1.2.2.</u> Let $X$ be a metric space and let $T$ be a contractive mapping of $X$ into itself. If there exists a point $x_o \in X$ such that its sequence of iterates $\{T^n(x_o)\}$ has a convergent subsequence $\{T^{n_i}(x_o)\}$ converging to a point $\xi$ in $X$, then $\xi$ is a unique fixed point of $T$.

A simpler proof [49] of this theorem may be given as follows:

<u>Proof.</u> Since $\{T^{n_i}(x_o)\}$ converges to $\xi \in X$ and $T$, being a contractive mapping, is continuous on $X$ therefore the sequence $\{T^{n_i+1}(x_o)\}$ converges to $T(\xi)$and consequently the sequence $\{T^{n_i+2}(x_o)\}$ converges to $T^2(\xi)$.

Consider the sequence $\{d(T^n(x_o), T^{n+1}(x_o))\}$ of non-negative real numbers. If for any $n$, $d(T^n(x_o), T^{n+1}(x_o)) = 0$, there remains nothing to prove as $T^n(x_o)$ comes out to be a fixed point of $T$. Thus we may assume without loss of generality that each term of this sequence is positive. Since $T$ is contractive therefore for $x_o \neq T(x_o)$, we have

$$d(x_o, T(x_o)) > d(T(x), T^2(x_o)) > \quad \ldots\ldots \quad > d(T^n(x_o), T^{n+1}(x_o)) \geqslant \ldots$$

i.e. $\{d(T^n(x_o), T^{n+1}(x_o))\}$ is a decreasing sequence of positive real numbers bounded by $d(x_o, T(x_o))$. Hence it converges together with all its subsequences to some real number $\alpha$. Now, assume $\xi \neq T(\xi)$. Then

$$\begin{aligned} d(\xi, T(\xi)) &= d\ (\lim T^{n_i}(x_o),\ \lim T^{n_i+1}(x_o)) \\ &= \lim d\ (T^{n_i}(x_o), T^{n_i+1}(x_o)) \\ &= \alpha \\ &= \lim d\ (T^{n_i+1}(x_o), T^{n_i+2}(x_o)) \\ &= d\ (\lim T^{n_i+1}(x_o), \lim T^{n_i+2}(x_o)) \\ &= d\ (T(\xi), T^2(\xi)) \\ &< d\ (\xi, T(\xi)), \end{aligned}$$

which is absurd. Hence $T(\xi) = \xi$ i.e. is a fixed point of $T$. For uniqueness of $\xi$, let $\bar{\xi} \neq \xi$ be a point in $X$ such that $T(\bar{\xi}) = \bar{\xi}$. Then

$$d(\xi, \bar{\xi}) = d(T(\xi), T(\bar{\xi})) < d(\xi, \bar{\xi})$$

a contradiction. Thus $\xi$ is a unique fixed point of $T$.

<u>Definition 1.2.4.</u> A mapping $T$ of a metric space $X$ into itself is said to be nonexpansive mapping if

$$d(T(x), T(y)) \leq d(x,y), \quad \text{for all } x,y \in X.$$

Cheney and Goldstein [35] proved the following theorem:

<u>Theorem 1.2.3.</u> Let $T$ be a mapping of a metric space $X$ into itself such that

(i) $T$ is nonexpansive,

(ii) if $x \neq T(x)$, then $d(T(x), T^2(x)) < d(x,T(x))$, for all $x \in X$

and (iii) for some $x_o \in X$, the sequence $\{T^n(x_o)\}$ has a subsequence $\{T^{n_i}(x_o)\}$ converging to $u$.

Then the sequence $\{T^n(x_o)\}$ converges to $u$ and $u$ is a fixed point of $T$.

These results have been further generalized by Edelstein [52], [55], [56], Rakotch [113], Chu and Diaz [36], [37], Bailey [6 ], Browder [28], Belluce and Kirk [10], [11], Kirk [84], Boyd and Wong [17], Sehgal [120], Wong [134], and many others. Many of these fixed point theorems have been used to guarantee the existence and uniqueness to solutions of differential and integral equations.

Following Kannan [81], we define

<u>Definition 1.2.5.</u> A mapping $T$ of a metric space $(X,d)$ into itself is said to be semi-nonexpanding (or semi-nonexpansive [135]) if

$$d(T(x), T(y)) \leq \frac{1}{2}\{d(x,T(x)) + d(y,T(y))\}, \quad x,y \in X.$$

Semi-nonexpansive mappings have been discussed by Kannan [77], [78], [79],

[80], [81], Reich [114], [115], Woodward [135], and others.

Remarks 1.2.1.

(1) In general, a semi-nonexpansive mapping of a complete metric space $X$ into itself does not imply the existence of a fixed point (take the unit circle and $T(z) = -z$, or put $X = \{0\} \cup [1,2]$ and $T(x) = 0$, $x \neq 0$, $T(0) = 1$ [114]).

(2) In some cases semi-nonexpansive maps are nonexpansive. However, the two notions are quite independent; nonexpansive maps must be continuous while semi-nonexpansive maps need not (see Kannan [78], Woodward [135]).

(3) Any semi-nonexpansive map $T$ has at most one fixed point. For, if $T(x) = x$ and $T(y) = y$, then

$$d(x,y) = d(T(x), T(y)) \leq \frac{1}{2}\{d(x,T(x)) + d(y,T(y))\} = 0.$$

(4) A semi-nonexpansive map is continuous at its fixed point (if such a point exists).

For two operators $T_1$ and $T_2$ each mapping a complete metric space $X$ into itself, Kannan [77] investigated a sufficient condition for the existence of a common and unique fixed point in $X$. He has proved the following result which we state without proof.

Theorem 1.2.4. If $T_1$ and $T_2$ are two operators each mapping a complete metric space $(X,d)$ into itself and if

$$d(T_1(x), T_2(y)) \leq \alpha\{d(x,T_1(x)) + d(y, T_2(y))\},$$

where $x,y \in X$ and $0 < \alpha < \frac{1}{2}$, then $T_1$ and $T_2$ have a unique common fixed point in $X$.

In case $T_1$ is identical with $T_2$ in Theorem 1.2.4., we have

<u>Theorem 1.2.5.</u> (Kannan[77]) If $T$ be an operator mapping a complete metric space $(X,d)$ into itself and if

$$d(T(x), T(y)) \leq \alpha\{d(x,T(x)) + d(y, T(y))\} ,$$

where $x,y \in X$ and $0 < \alpha < \frac{1}{2}$ , then $T$ has a unique fixed point in $X$.

Several generalizations of these theorems have been given by Kannan [78], Singh [122], [123], [124], Reich [114], [115], Dube [49], Yadav [136], Fukushima [62], Srivastava and Gupta [129], Zamfirescu [137], and others.

<u>Definition 1.2.6.</u> Let $f$ be a one-to-one mapping of a metric space $(X,d)$ onto a metric space $(Y,d_1)$. The mapping $f$ is said to be a homeomorphism iff the mappings $f$ and $f^{-1}$ are continuous on $X$ and $Y$ respectively. Finally the mapping $f$ is said to be an isometry iff $d(x_1,x_2) = d_1(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

<u>Remark 1.2.2.</u> Clearly every isometry is a nonexpansive mapping.

<u>Definition 1.2.7.</u> A linear topological space is a linear space $X$ with a topology defined in such a way that

(i) the mapping $(x,y) \to x + y$ of $X \times X \to X$ is continuous,

(ii) the mapping $(\alpha,x) \to \alpha x$ of $K \times X \to X$ is continuous.

<u>Definition 1.2.8.</u> A locally convex linear topological space is a linear topological space with a base for its topology consisting of convex sets.

<u>Theorem 1.2.6.</u> <u>Brouwer's fixed point theorem:</u>

Let $C$ be a non-empty compact convex subset of a finite dimensional normed linear space, and let $T$ be a continuous mapping of $C$ into itself. Then $T$ has a fixed point in $C$.

The Brouwer's fixed point theorem in the form stated above does not hold in infinite dimensional spaces as the following example shows:

Example 1.2.2. Consider the space $\ell^2$ of sequences $x = \{x_1, x_2, \ldots\ldots\}$ with $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Define $T$ as a map of the closed solid sphere into itself as follows:

for $x = (x_1, x_2, \ldots)$ let $T(x) = \{\sqrt{1 - ||x||^2}, x_1, x_2, \ldots\}$. $||T(x)||^2 = 1$. Now suppose $x$ is a fixed point. Then $||x|| = ||T(x)|| = 1$. But then $x_1 = 0$ and it can be seen also that $x_2 = 0$, $x_3 = 0$, ...... . Hence $x = 0$. This contradicts the fact $||x|| = 1$. Therefore, $T$ has no fixed point.

The following theorem is an extension to infinite dimensional space of the Brouwer's fixed point theorem.

Theorem 1.2.7. Schauder's fixed point theorem - First form:

A continuous map of a compact convex set $C$ in a normed linear space $X$ into itself has at least one fixed point.

Definition 1.2.9. An operator $T$ which maps a Banach space $X$ into itself is said to be compact if it maps an arbitrary bounded set into a compact set; $T$ is completely continuous if $T$ is continuous and compact.

The second version of Schauder's fixed point theorem, which follows below, is more suitable for the applications.

Theorem 1.2.8. Schauder's Theorem - second form:

Let $T$ be a completely continuous map of a closed convex bounded set $C$ in a complete normed linear space $X$ into itself. Then $T$ has at least one fixed point.

The proof of these theorems, together with a discussion of other related results, may be found in [13]. Schauder's theorem was foreshadowed by the work of Birkhoff and Kellogg [15] on existence theorem in analysis. Afterwords Tychonoff [131] extended Schauder's result from Banach spaces to arbitrary locally convex linear topological spaces. In both cases Brouwer's theorem was used as a starting point.

Theorem 1.2.9. Schauder - Tychonoff's fixed point theorem.

Let C be a non-empty compact convex subset of a locally convex (Hausdorff) linear topological space X, and let T be a continuous mapping of C into itself. Then T has a fixed point in C.

These theorems have been used very often; perhaps the Schauder's theorem is one of the most important theorems for the numerical treatment of equations occurring in analysis. Recently Browder [27] gave generalization of Schauder and Tychonoff fixed point theorems. He has also given several generalizations to Schauder's fixed point theorem (see [121] for references) which centers around the concept of asymptotic fixed point theorems and of deformation of non-compact mappings.

# CHAPTER II

# GEOMETRY OF BANACH SPACES

## 2.1. Some Results on Reflexive Spaces.

Definition 2.1.1. A bounded linear transformation of a normed linear space $X$ over $K$ into $K$ is called a bounded linear functional on $X$.

Remark 2.1.1. The word functional is used to distinguish mappings of a linear space into the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.2. The Banach space consisting of all bounded linear functionals on a normed linear space $X$ over $K$, denoted as $X^*$, is called the dual space (conjugate space) of $X$.

Definition 2.1.3. The dual space $X^*$ has a dual space $(X^*)^*$, called as the second dual of $X$. We usually write $X^{**}$ in place of $(X^*)^*$.

Definition 2.1.4. Let $X$ be a normed linear space. Then

$$U = \{x \in X : ||x|| \leq 1\} \quad \text{and} \quad S = \{x \in X : ||x|| = 1\}$$

are called the unit ball and the unit sphere respectively.

Similarly in the dual space $X^*$ of $X$,

$$U^* = \{f \in X^* : ||f|| \leq 1\} \quad \text{and} \quad S^* = \{f \in X^* : ||f|| = 1\}$$

are called the unit ball and the unit sphere respectively.

Remark 2.1.2. A ball, open or closed, in a normed linear space is convex.

In what follows, we denote by $(f,x)$ the value $f(x)$ of $f$ at $x$.

Definition 2.1.5. For a given $\varepsilon > 0$ and a finite number of elements $f_1, f_2, \ldots, f_n \in X^*$, let

$$V(f_1, f_2, \ldots, f_n; \varepsilon) = \{x \in X : |(f_i,x)| < \varepsilon, \text{ for every } i = 1,2, \ldots, n\}$$

then the family $V$ of all sets $V(f_1, f_2, \ldots , f_n; \varepsilon)$ for every choice of $\varepsilon$ and any finite sequence $f_1, f_2, \ldots , f_n$, defines a basis of neighborhood of zero of a topology which is called the weak (or X*) topology of X.

Remarks 2.1.3.

(1) Under weak topology a normed linear space X is a locally convex linear topological space.

(2) In the sequel by the terms weakly closed set, weakly compact set, weak closure of a set etc., we mean closed set, compact, closure of a set etc., in the weak topology.

(3) The norm topology (or strong topology) and the weak topology of a Banach space X are equivalent iff X is finite dimensional.

Definition 2.1.6. In the dual space X* of a Banach space X the family $V^*$ of sets

$$V^*(x_1, x_2, \ldots , x_n;\varepsilon) = \{f \in X^* : |(f,x_i)| < \varepsilon \text{ for every } i = 1,2, \ldots , n.\},$$

$(\varepsilon > 0 ; x_1, x_2, \ldots , x_n \in X)$

defines a basis of neighborhood of zero of a topology which is called the weak* (or X*) topology in X*.

Remarks 2.1.4.

(1) Under weak* topology, X* is a locally convex linear topological space.

(2) In general, the weak topology in the dual space X* of a Banach space X is finer than the weak* topology in X*.

Some of the importance of the weak* topology stems from the following theorem:

<u>Theorem 2.1.1.</u> (Alaoglu [ 1])

The unit ball $U^*$ of $X^*$ is compact in the weak* topology.

<u>Definition 2.1.7.</u> A sequence $\{x_n\} \subset X$ converges weakly to $x_o \in X$ i.e. $x_n \xrightarrow{w} x_o \in X$ iff $\lim (f,x_n) = (f,x_o)$ for every $f \in X^*$. Geometrically, $x_n \xrightarrow{w} x_o$ means that the distance from $x_n$ to any hyperplane through $x_o$ goes to zero.

We observe that $x_o$ is unique as a weak limit for, if $x_n \xrightarrow{w} y_o$, then $f(x_o - y_o) = 0$ for all $f \in X^*$. Therefore $x_o = y_o$.

<u>Remarks 2.1.5.</u>

(1) Every weakly convergent sequence $\{x_n\}$ is necessarily bounded and moreover, the norm of its limit is less than or equal to $\liminf ||x_n||$.

(2) It is clear that strong convergence implies weak convergence. But the converse implication is not true in general (see [100]).

<u>Definition 2.1.8.</u> A mapping $T$ of a Banach space into itself is said to be demiclosed if for any sequence $\{x_n\}$ such that $x_n \xrightarrow{w} x$ and $T(x_n) \to y$ then $y = T(x)$.

<u>Definition 2.1.9.</u> A mapping $T$ of a Banach space $X$ into itself is said to be strongly continuous if for any sequence $\{x_n\} \subset X$ such that $x_n \xrightarrow{w} x_o \in X$ implies $T(x_n) \to T(x_o)$.

<u>Definition 2.1.10.</u> A sequence $\{f_n\} \subset X^*$ converges weakly* to $f_o \in X^*$ i.e. $f_n \xrightarrow{w^*} f_o \in X^*$ iff $\lim(f_n,x) = (f_o, x)$ for every $x \in X$.

We note that a sequence $\{f_n\} \subset X^*$ cannot have two distinct weak* limits.

We state the following simple property, due to Opial [104], of weakly convergent sequence in a Hilbert space.

Lemma 2.1.1. If the sequence $\{x_n\}$ is weakly convergent to $x_o$ in a Hilbert space $H$, then for any $y_o \neq x_o$ in $H$,

$$(1) \qquad \liminf ||x_n - y_o|| > \liminf ||x_n - x_o||.$$

Proof. Since every weakly convergent sequence is necessarily bounded, both limits in (1) are finite. Thus, to prove this inequality, it suffices to observe that in the equality

$$\begin{aligned} ||x_n - y_o||^2 &= ||x_n - x_o + x_o - y_o||^2 \\ &= ||x_n - x_o||^2 + ||x_o - y_o||^2 + 2\,\mathrm{Re}(x_n - x_o, x_o - y_o), \end{aligned}$$

the last term tends to zero as $n$ tends to infinity.

Theorem 2.1.2. Each closed convex subset of a Banach space is necessarily weakly closed.

The following statement is a simple consequence of the above theorem.

Theorem 2.1.3. The weak closure of every bounded set of a Banach space is contained in its convex closure.

Definition 2.1.11. Let $X$ be a normed linear space. The linear isometry $x \to x^{**}$ of $X$ into its second dual $X^{**}$ is called the canonical mapping.

Definition 2.1.12. A Banach space $X$ is said to be reflexive iff the canonical mapping $x \to x^{**}$ maps $X$ onto $X^{**}$.

Remarks 2.1.6.

(1) It is clear that every Hilbert space is reflexive. For $1 < p < \infty$, the spaces $\ell^p$ and $L^p$ are reflexive. But the converse is not true.

(2) The weak and weak* topologies coincide if the space is reflexive ([76]).

The following theorem due to Gantmakher and Smulian [65], [66], Kakutani

[76], Eberlein [51], Nikaido [99], is one of the fundamental properties of reflexive Banach spaces.

<u>Theorem 2.1.4.</u> A Banach space is reflexive iff its unit ball is weakly compact.

<u>Remark 2.1.7.</u> From Theorem 2.1.2. and Theorem 2.1.4., it follows immediately that in a reflexive Banach space every bounded closed convex set is weakly compact.

In somewhat different way the above theorem may be stated in the following form (see Opial [103]):

<u>Theorem 2.1.5.</u> A Banach space $X$ is reflexive iff every bounded sequence of elements of $X$ contains a subsequence which is weakly convergent.

The following characterization of reflexive spaces, due to Šmulian [126], is worth mentioning:

<u>Theorem 2.1.6.</u> $X$ is reflexive iff every decreasing sequence of non-empty bounded closed convex subsets of $X$ has a non-empty intersection.

The notion of monotone operators was introduced by Zarantonello [138], Minty [97] and Kačurovskii [74], and has been extended to Banach spaces by several authors. The theory of monotone operators and its application to nonlinear partial differential equations, evolution equations, variational inequalities, etc., have evolved into a substantial chapter in nonlinear functional analysis.

In what follows $C$ is a subset of a Banach space $X$.

<u>Definition 2.1.13.</u> A mapping $T:C \to X^*$ is called monotone if

$$(T(x) - T(y), x - y) \geq 0 \quad \text{for all } x,y \text{ in } C,$$

and strictly monotone if

$$(T(x) - T(y), x - y) > 0 \quad \text{for all } x, y (x \neq y) \text{ in } C.$$

It is obvious that the sum, product with a non-negative number of monotone operators are again monotone operators. It is easy to see that a strictly monotone operator has an inverse, which is also a strictly monotone operator.

In Hilbert space an intimate relationship between monotone and non-expansive mappings is expressed by the following:

Proposition 2.1.1. (Minty [97]) Let $C$ be a subset of a Hilbert space $H$ and $T:C \to H$ a nonexpansive mapping. Then the mapping $I-T$ is monotone.

Lemma 2.1.1. enables to prove the following useful property of non-expansive mappings in Hilbert spaces which in [23] has been proved by the means of the theory of monotone mappings.

Proposition 2.1.2. (Opial [104]) In a Hilbert space $H$, for every non-expansive mapping $T:C \to H$ $(C \subset H)$, the mapping $I-T$ is demiclosed.

Proof. Let $\{x_n\} \subset C$ be a sequence which is weakly convergent to an element $x_o$ of $C$ and the sequence $\{x_n - T(x_n)\}$ converges to an element $y_o$ in $X$. Then we have

$$\liminf_{n\to\infty} ||x_n - x_o|| \geq \liminf_{n\to\infty} ||T(x_n) - T(x_o)|| = \liminf_{n\to\infty} ||x_n - y_o - T(x_o)||,$$

so that from Lemma 2.1.1. it follows that $x_o = y_o + T(x_o)$.

Definition 2.1.14. A mapping $T:C \to X^*$ is called strongly monotone if there exists a continuous positive function $d(t)$ defined on $R^+$ with $\lim d(t) = +\infty$ as $t \to \infty$ such that

$$(T(x) - T(y), x - y) \geq d(||x - y||)||x - y||,$$

for all $x,y$ in $C$.

In the following $\rightharpoonup$ denotes the weak* convergence in the dual space $X^*$.

<u>Definition 2.1.15.</u> A mapping $T:C \to X^*$ is called hemicontinuous if for any $x$ in $C$, $y$ in $X$ and any sequence $\{t_n\}$ of positive real numbers, from $x + t_n y \in C$ $(n = 1,2, \dots .)$ and $t_n \to 0$ as $n \to \infty$, it follows that $T(x + t_n y) \rightharpoonup T(x)$.

<u>Definition 2.1.16.</u> A mapping $T:C \to X^*$ is called coercive if

$$\lim \frac{(T(x), x)}{||x||} = +\infty \quad \text{as} \quad ||x|| \to +\infty.$$

The condition of coerciveness of a mapping $T:C \to X^*$ is basically a condition on the behaviour of $T$ at infinity. The following gives the relationship with strongly monotone maps.

<u>Proposition 2.1.3.</u> (See Opial [103]) If $0 \in C$, then every strongly monotone mapping $T:C \to X^*$ is coercive.

One basic property of monotone mappings is expressed by the following fundamental:

<u>Theorem 2.1.7.</u> (Browder [26], Hartmand and Stampacchia [71])

Let $C$ be a closed convex subset of a reflexive Banach space $X$ and $T:C \to X^*$ a monotone hemicontinuous and coercive mapping. Then for each given $u_o$ in $X^*$, there exists an $x_o$ in $C$ such that

$$(T(x_o) - u_o, x - x_o) \geq 0 , \quad \text{for all } x \text{ in } C.$$

In recent years, many results have been given on the surjectivity property of monotone mappings, for example see Minty [97], [98], Browder [19], [20], and the others. For mappings defined on the whole space $X$, one can easily derive from Theorem 2.1.7. the following fundamental:

<u>Theorem 2.1.8.</u> (Minty [98], Browder [19]). Let $T$ be a monotone hemicontinuous and coercive mapping of a reflexive Banach space $X$ into its dual space $X^*$. Then $T$ maps $X$ onto $X^*$. For each given $u_o$ in $X^*$, the set $T^{-1}(u_o) = \{x \in X : T(x) = u_o\}$ is bounded, closed and convex.

For strongly monotone mappings Theorem 2.1.8 can be considerably strengthened by further information on the inverse mapping.

<u>Theorem 2.1.9.</u> (Minty [98], Browder [20]). Let $T$ be a strongly monotone hemicontinuous mapping of a reflexive Banach space $X$ into its dual $X^*$. Then $T$ is one-to-one, maps $X$ onto $X^*$, and the inverse mapping $T^{-1}:X^* \to X$ is continuous and maps bounded sets of $X^*$ onto bounded sets of $X$.

The main results of the theory of monotone mappings can be extended to broader classes of mappings whose consideration is motivated by the theory of partial differential equations.

<u>Definition 2.1.17.</u> A mapping $T:X \to X^*$ of a Banach space $X$ into its dual space $X^*$ is called semimonotone if there exists a mapping $S:X \times X \to X^*$ such that $T(x) = S(x,x)$ for all $x$ in $X$ while $S$ satisfies the three following conditions:

(i) for each fixed $y$ in $X$, the mapping $x \to S(x,y)$ is hemicontinuous,

(ii) for each fixed $x$ in $X$, the mapping $y \to S(x,y)$ is continuous from the weak topology on each weakly compact subset of $X$ to the strong topology of $X^*$,

(iii) for all $x,y$ in $X$,

$$(S(x,y) - S(y,y),\ x - y) \geq 0.$$

Note that every hemicontinuous monotone mapping $T:X \to X^*$ is trivially semimonotone with $S(x,y) = T(x)$, for all $x,y$ in $X$.

The following basic result is a direct generalization of Theorem 2.1.8.

Theorem 2.1.10. (Browder [21]) Let $X$ be a reflexive Banach space and $T:X \to X^*$ a semimonotone coercive mapping. Then $T$ maps $X$ onto $X^*$.

An important example of a monotone mapping from a Banach space $X$ into its dual space $X^*$ is given by the so-called duality mappings. This concept was first introduced and studied by Beurling and Livingston [14]. Later it was generalized, extensively investigated and applied by Browder [24], [3 Further studies were also made by Laursen [92], Kato [82], Asplund [4 ], Dubinsky [50], Petryshyn [106], and others. In addition to their usefulness in the theory of Fourier Analysis and the study of Banach spaces, duality mappings play an essential role in the study of J-monotone, accretive, P-compact and A-proper mappings.

Definition 2.1.18. A gauge function is a real-valued continuous function $\mu$ defined on $R^+ = \{t \in R : t \geq 0\}$ such that

(i) $\mu(0) = 0$ ,

(ii) $\lim \mu(t) = +\infty$ ,

(iii) $\mu$ is strictly increasing.

An example of a gauge function is $\mu(t) = t$.

Definition 2.1.19. Let $X$ be a Banach space and $X^*$ its dual space. Let $\mu(t)$ be a given gauge function. The duality mapping in $X$ with gauge function $\mu$ is a mapping $J$ from $X$ into the set $2^{X^*}$ of all subsets of $X^*$ such that

$$J(0) = 0$$

and $$J(x) = \{x^* \in X^* : (x^*,x) = ||x^*||\ ||x||,\ ||x^*|| = \mu(||x||)\},\ x \neq 0.$$

Remark 2.1.8. For $x \neq 0$, the set $J(x)$ is nonempty and convex.

The following result is essentially a reformulation of a characterization of reflexivity due to James [72].

Theorem 2.1.11. Let $X$ be a Banach space and $X^*$ its dual space. Let $J$ be the duality mapping in $X$ with a given gauge function $\mu$. Then $X$ is reflexive iff the union of all sets $J(x)$, $x \in X$, covers $X^*$.

Remark 2.1.9. See Laursen [92] for another characterization of reflexivity and Petryshyn [107] for characterization of certain Banach spaces using duality mappings.

## 2.2. Uniformly Convex Spaces and Related Results.

The theory of different spaces which are contained in general Banach spaces, has been developed considerably in the last two decades. In these years, considerable progress has been made in the classification and characterization of Banach spaces according to various geometric properties of their unit spheres. For the reference of these systematic developments, one can see Cudia [39], Zizler [139] and Milman [96]. The different properties, given in this and the following sections, depend on the norm and linear structure, and thus can be defined for arbitrary normed linear spaces (not necessarily complete). But, since our primary concern is with Banach spaces, we have phrased all the definitions in terms of a Banach space X. In some cases, we have listed two or more equivalent formulations of the same property.

In 1936, J.A. Clarkson [38] introduced the notion of uniform convexity of the norm in a Banach space. Expressed in geometric terms this property is simple: a norm is uniformly convex if, whenever the midpoint of a variable chord in the unit sphere of the space approaches the boundary of the sphere, the length of the chord approaches zero.

Definition 2.2.1. Uniformly Convex Spaces (UC):

A Banach space $X$ is called uniformly convex (UC) iff it satisfies any one of the following equivalent conditions:

(I) [38] for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $x, y \in U$ and $||x - y|| \geq \varepsilon$ then

$$\left|\left|\frac{x + y}{2}\right|\right| \leq 1 - \delta(\varepsilon).$$

(II) [116] for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$x_n, y_n \in U$ $(n = 1,2, \ldots)$ and $\left\|\frac{x_n + y_n}{2}\right\| \geq 1 - \delta(\varepsilon)$

imply $\|x_n - y_n\| \leq \varepsilon$.

(III) [16] given $x_n, y_n \in S$ $(n = 1,2, \ldots)$ and $\left\|\frac{x_n + y_n}{2}\right\| \to 1$

imply $\|x_n - y_n\| \to 0$.

For $1 < p < \infty$, the Banach spaces $\ell^p$ and $L^p$ are uniformly convex ([38]).

Every Hilbert space is uniformly convex but the converse is not always true, e.g., spaces $\ell^p$, $1 < p < \infty$ are uniformly convex but none of them is Hilbert space except $\ell^2$.

Remark 2.2.1. It is interesting to note that Lemma 2.1.1. fails to be true for all uniformly convex Banach spaces (see [104]).However, it remains still valid for a larger class of uniformly convex Banach spaces having weakly continuous duality mappings.

Lemma 2.2.1. (Opial [104]).If in a Banach space $X$ having a weakly continuous duality mapping $J$ the sequence $\{x_n\}$ is weakly convergent to $x_o$, then for any $y_o$ in $X$,

$$\liminf_{n\to\infty} \|x_n - y_o\| \geq \liminf_{n\to\infty} \|x_n - x_o\|. \tag{1}$$

If, in addition, the space $X$ is uniformly convex, then the equality in (1) occurs iff $x_o = y_o$.

The following useful result is worth mentioning:

Theorem 2.2.1. (Milman [95], Pettis [111], Kakutani [76])

Every uniformly convex Banach space is reflexive.

Of course, the converse is not true, as can be seen from the following:

Example 2.2.1. ([44]) Consider a finite dimensional Banach space X, in which the surface of the unit ball has a 'flat' part. Such a Banach space is reflexive because of finite dimensionality. But the 'flat' portion in the surface of the ball destroys uniform convexity.

Remark 2.2.2. It might be of interest to note that none of the following spaces can be renormed so as to be uniformly convex:

$$\ell^1,\ \ell^\infty,\ L^1,\ L^\infty,\ C[a,b],$$

and the space C of convergent sequences, since none of them is reflexive.

The following characterizations of a uniformly convex Banach space are due to Bynum [34]. We will omit the proof.

Theorem 2.2.2. A Banach space X is uniformly convex iff for each t in (0,2],

$$\beta(t) = \inf\ \{1 - (f,y) : x,y \in S,\ ||x - y|| \geq t,\quad f \in J(x)\}$$

is positive, where J is the duality map from X into X*.

Theorem 2.2.3. A Banach space X is uniformly convex iff the duality map J of X into X* is uniformly monotone - in the sense that for each $t \in (0,2]$,

$$\gamma(t) = \inf\ \{(f - g,\ x - y) : x,y \in S,\ ||x - y|| \geq t,\ f \in J(x),\ g \in J(y)\}$$

is positive.

The following lemmas, stated without proof, are immediate consequences of the definition of uniform convexity (see [103] for details).

Lemma 2.2.2. Let X be a uniformly convex Banach space. Then for any $d > 0$ and $\varepsilon > 0$, the inequalities $||x|| \leq d$, $||y|| \leq d$, $||x - y|| \geq \varepsilon$ imply

$$\left\| \frac{x+y}{2} \right\| \leq \{1 - \delta(\tfrac{\varepsilon}{d})\} \cdot d.$$

The following lemma is due to Schaefer ([119]).

Lemma 2.2.3. Let $X$ be a uniformly convex Banach space. Then for any $\varepsilon > 0$, $d > 0$ and $\alpha \in (0,1)$, the inequalities $\|x\| \leq d$, $\|y\| \leq d$ and $\|x - y\| \geq \varepsilon$ imply

$$\|\alpha x + \beta y\| \leq \{1 - 2\ \delta(\tfrac{\varepsilon}{d}) \min.\ (\alpha,\beta)\} \cdot d$$

where $\alpha + \beta = 1$.

The following result is useful in applications ([44]).

Proposition 2.2.1. Let $X$ be a uniformly convex Banach space. Suppose that there are given two sequences $\{x_n\}$ and $\{y_n\}$ such that $\|x_n\| \to 1$, $\|y_n\| \leq \|x_n\|$ and $\left\|\frac{x_n + y_n}{2}\right\| \to 1$ as $n \to \infty$. Then

$$\|x_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Construct two new sequences

$$z_n = \frac{x_n}{\|x_n\|} \quad \text{and} \quad w_n = \frac{y_n}{\|x_n\|} .$$

It is easy to see that $\|z_n\| = 1$, $\|w_n\| \leq 1$ and $\left\|\frac{z_n + w_n}{2}\right\| \to 1$. So by uniform convexity it follows that $\|z_n - w_n\| \to 0$, which implies readily that

$$\|x_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Remark 2.2.3. In Proposition 2.2.1. the condition on $\frac{x_n + y_n}{2}$ could be replaced by the analogous one on $\alpha x_n + (1 - \alpha)y_n$, where $\alpha \in (0,1)$.

We mention some interesting results on uniformly convex Banach spaces, which are due to Edelstein [58]. Following Edelstein [58], we define the asymptotic center.

Let $C$ be a closed convex set in a uniformly convex Banach space $X$. Given a bounded sequence $\{u_n : n = 1,2, \ldots\}$ in the set $C$, define

$$r_m(y) = \sup\{||u_k - y|| : k \geq m\}, \quad y \in X.$$

It is well- known that a unique point $c_m \in C$ exists such that

$$r_m(c_m) = \inf\{r_m(y) : y \in C\} = r_m.$$

Clearly, $r_m \geq r_{m+1}$ and $r_m \geq 0$ for all $m = 1,2, \ldots,$ so that sequence $\{r_m : m = 1,2, \ldots,\}$ converges to $r = \inf\{r_m : m = 1,2, \ldots\}$. We note that if $r = 0$, then the sequence $\{u_n\}$ converges.

<u>Definition 2.2.2.</u> If $\{c_m\}$ converges then $c = \lim\limits_{m\to\infty} c_m$ is called the asymptotic center of $\{u_n\}$ (with respect to C) [58].

The following result due to Edelstein [58] shows that the asymptotic center $c$ exists.

<u>Theorem 2.2.4.</u> Let $C$ be a closed convex subset of a uniformly convex Banach space $X$ and $\{u_n\}$ is a bounded sequence in $C$, then the sequence $\{c_m\}$ converges. (Thus the asymptotic center $c$ exists.)

<u>Proof.</u> If $r = 0$ then $\{u_n\}$ is a Cauchy sequence and

$$\lim_{n\to\infty} u_n = \lim_{m\to\infty} c_m = c.$$

We may then assume that $r > 0$. Suppose now, for a contradiction, that $\{c_m\}$ fails to converge. Then an $\varepsilon > 0$ exists such that for any natural number $N$ there are integers $n > m \geq N$ with $||c_m - c_n|| \geq \varepsilon$. From the uniform convexity of $X$ and the fact that

$$||u_k - c_n|| \leq r_n \leq r_m \qquad (k \geq n),$$
$$||u_k - c_m|| \leq r_m \qquad (k \geq m),$$

it follows that

$$(1) \qquad ||u_k - \frac{c_m + c_n}{2}|| \le r_m \quad (1 - \delta(\frac{||c_m - c_n||}{r_m}))$$

$$\le r_m \quad (1 - \delta(\tfrac{\varepsilon}{D})) \qquad (k \ge n),$$

where $D$ is the diameter of $\{u_n\}$. On the other hand, since $\frac{1}{2}(c_m + c_n) \ne c_n$, there is a $k \ge n$ such that

$$(2) \qquad r_n < ||u_k - \frac{c_m + c_n}{2}|| .$$

For such a $k$, (1) and (2) hold simultaneously so that $r_m - r_n \ge r_m \ \delta(\frac{\varepsilon}{D}) \ge r\,\delta(\frac{\varepsilon}{D})$. This, however, is impossible since $\{r_k\}$ converges.

Remark 2.2.4. If $X$ is a Hilbert space then $c$ belongs to the convex closure of $\{u_n\}$.

Using the concept of the asymptotic center, Edelstein [58] proved the following result, which we state without proof.

Theorem 2.2.5. Let $X$, $C$, $\{u_n\}$ and $c(= \lim\limits_{m\to\infty} c_m)$ be as in Theorem 2.2.4. and $T:C \to C$ be a mapping of $C$ into itself satisfying the following conditions:

(1) $u_n = T^n(x)$ for some $x \in C$ and all $n = 1,2, \ldots;$

(2) there exists a positive integer $n_o$ and neighborhood $V$ of $c$ in $C$ such that

$$||T^k(x) - T(v)|| \le ||T^{k-1}(x) - v|| \qquad (k \ge n_o,\ v \in V).$$

Then $T(c) = c$.

An immediate consequence of Theorem 2.2.5. is the following result ([58]).

Corollary 2.2.1. Let $C$ be a bounded closed convex subset of a uniformly convex Banach space $X$ and suppose that $T$ is a continuous mapping of $C$ into itself such that for each $x \in C$ there is a positive integer $N = N(x)$ such that, for all integer $n \geq N$ and all $y \in C$,

$$||T^n(x) - T^n(y)|| \leq ||T^{n-1}(x) - T^{n-1}(y)||.$$

Then $T(\xi) = \xi$ for some $\xi \in C$.

Remark 2.2.5. The well-known theorem of Browder [25], Göhde [70] and Kirk [83], asserting that each nonexpansive mapping of a bounded closed convex subset of a uniformly convex Banach space $X$ into itself has a fixed point, follows from the above Corollary 2.2.1. upon setting $N = 1$ for all $x \in C$.

Definition 2.2.3. Weakly Uniformly Convex Spaces (WUC):

A Banach space $X$ is called weakly uniformly convex, denoted as (WUC), iff it satisfies any one of the following equivalent conditions:

(I) [128] for each $\varepsilon > 0$ and each $g \in S^*$, there is a $\delta(\varepsilon,g) > 0$ such that if $x,y \in S$ then the inequality

$$\left|\left| \frac{x+y}{2} \right|\right| > 1 - \delta(\varepsilon,g)$$

implies $|g(x) - g(y)| < \varepsilon$.

(II) [139] $x_n, y_n \in U$ $(n = 1,2, \ldots)$, $\left|\left| \frac{x_n + y_n}{2} \right|\right| \to 1$ implies $x_n - y_n \xrightarrow{w} 0$.

Remark 2.2.6.

(1) Let $X$ be a (WUC)-space, $M$ be a closed linear subspace of $X$. Then $M$ is a (WUC)-space.

(2) A space $X$ has an equivalent norm which is (WUC) iff $X$ is isomorphic to a (WUC)-space $Y$.

(3) Space $C[0,1]$ has no equivalent (WUC)-norm.

(4) Obviously, uniform convexity implies weak uniform convexity. But the converse is not always true, as can be seen from the following example [139].

Example 2.2.2. Day [41] has constructed a separable reflexive strictly convex (see Definition 2.3.4.) space $X_o$ which is not uniformly convex in any norm. Introduce a norm $|||\cdot|||$ in this space as follows: let $\{f_k\}_{k=1}^{\infty}$ be a countable dense subset of $S^*$. Define the functional $I(x)$ on $X_o$ by

$$I(x) = \left(\sum_{k=1}^{\infty} \frac{1}{2^k} f^2_k(x)\right)^{1/2} .$$

Let $||x||$ denote the norm of $X_o$. Then it is easy to see that norm $|||\cdot|||$ defined as

$$|||x||| = (||x||^2 + I^2(x))^{1/2} ,$$

is the equivalent norm to $||x||$. We see that this norm is (WUC). Let $|||x_n||| = |||y_n||| = 1$ $(n = 1,2, \ldots)$ , $|||\frac{x_n + y_n}{2}||| \to 1$.

We have

$$I^2(x_n + y_n) + I^2(x_n - y_n) = 2 \cdot (I^2(x_n) + I^2(y_n)) .$$

It is easy to see that

$$||x_n + y_n||^2 \leq 2 \cdot (||x_n||^2 + ||y_n||^2).$$

From these facts it follows (by addition) that

$$|||x_n + y_n||| + I^2(x_n - y_n) \leq 2 \cdot (|||x|||^2 + |||y_n|||^2).$$

The right hand side of this inequality is equal to $4$, $|||x_n + y_n|||^2 \to 4$ by assumption. Therefore

$$I^2(x_n - y_n) \to 0.$$

Thus we have $\qquad f_k(x_n - y_n) \to 0, \quad$ as $\quad n \to \infty, \quad$ for every $\quad k.$

Thus sequence $\{x_n - y_n\}$ is bounded in $X$ and $f_k(x_n - y_n) \to 0$ as $n \to \infty$ and $k = 1, 2, \ldots\ldots$ . Hence by the well-known theorem [112],

$$x_n - y_n \overset{w}{\to} 0, \qquad \text{as } n \to \infty.$$

Thus $|||\cdot|||$ is (WUC) but not uniformly convex.

(5) It follows from Zizler [139] that not every (WUC)-Banach space is reflexive.

Definition 2.2.4. Weakly* Uniformly Convex Spaces (W*UC):

A Banach space $X^*$ is called Weakly* Uniformly Convex, denoted as (W*UC), iff it satisfies any one of the following equivalent conditions:

(I) [128] for each $\varepsilon > 0$ and each $x \in S$, there is a $\delta(\varepsilon, x) > 0$ such that if $f, g \in S^*$, then the inequality

$$\left\|\frac{f+g}{2}\right\| > 1 - \delta(\varepsilon, x)$$

implies $\qquad | f(x) - g(x) | < \varepsilon.$

(II) [139] $f_n, g_n \in U^*$ $(n = 1, 2, \ldots)$ , $\quad \left\|\frac{f_n + g_n}{2}\right\| \to 1$

implies $\qquad f_n - g_n \overset{w^*}{\to} 0.$

We state and prove the following lemmas, which are immediate consequences of the definitions of (WUC) and (W*UC) Banach spaces.

Lemma 2.2.4. Let $X$ be a (WUC)-Banach space, then for any $d > 0$, $\varepsilon > 0$ and for every $g \in S^*$ there exists a $\delta(\frac{\varepsilon}{d}, g) = \delta_{\frac{\varepsilon}{d}, g} > 0$ such that

$||x|| \le d$, $||y|| \le d$, $|g(x - y)| \ge \varepsilon$ , $\quad (x, y \in X)$ imply

$$\left|\left|\frac{x + y}{2}\right|\right| \leq \{1 - \delta_{\frac{\varepsilon}{d},g}\} \cdot d.$$

Proof. Zizler [139] showed that if $X$ is (WUC) then the following implication holds: For every $g \in S^*$ and for every $\varepsilon' > 0$, there exists a $\delta_{\varepsilon',g} > 0$ such that $x,y \in X$, $|g(x - y)| \geq \varepsilon' . \max. (||x||,||y||)$ imply

$$\left|\left|\frac{x + y}{2}\right|\right| \leq (1 - \delta_{\varepsilon',g}) . \max (||x||,||y||).$$

Since in this case, $\max. (||x||,||y||) = d$, hence the result follows if we choose $\varepsilon' = \frac{\varepsilon}{d}$.

Lemma 2.2.5. Let $X$ be a (WUC)-Banach space. Then for any $\varepsilon > 0$, $d > 0$, $\alpha \in (0,1)$ and for every $g \in S^*$, the inequalities $||x|| \leq d, ||y|| \leq d$ and $|g(x - y)| \geq \varepsilon, (x,y \in X)$ imply

$$||\alpha x + \beta y|| \leq \{1 - 2\,\delta_{\frac{\varepsilon}{d},g} \min. (\alpha,\beta)\} \cdot d,$$

where $\alpha + \beta = 1$.

Proof. Without loss of generality, we may assume that $0 < \alpha < \frac{1}{2}$. Then

$$\begin{aligned} ||\alpha x + \beta y|| &= ||\alpha(x + y) + (\beta - \alpha)y|| \\ &\leq 2\alpha \left|\left|\frac{x + y}{2}\right|\right| + (\beta - \alpha)||y|| \\ &\leq 2\alpha (1 - \delta_{\frac{\varepsilon}{d},g}) \cdot d + (\beta - \alpha) \cdot d \\ &= 2\alpha \cdot d - 2\alpha \cdot \delta_{\frac{\varepsilon}{d},g} \cdot d + (1 - \alpha) \cdot d - \alpha \cdot d \\ &= \{1 - 2\,\delta_{\frac{\varepsilon}{d},g} \min. (\alpha,\beta)\} \cdot d. \end{aligned}$$

We give the following result for a (WUC)-Banach space, which is useful in applications.

Proposition 2.2.2. Let $X$ be a (WUC)-Banach space. Suppose that there are given two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $||x_n|| \to 1$, $||y_n|| \leq ||x_n||$ and $||\frac{x_n + y_n}{2}|| \to 1$ as $n \to \infty$. Then

$$x_n - y_n \xrightarrow{w} 0, \quad \text{as} \quad n \to \infty.$$

Proof. We construct two sequences

$$z_n = \frac{x_n}{||x_n||} \quad \text{and} \quad w_n = \frac{y_n}{||x_n||} .$$

It is easy to see that $||z_n|| = 1$, $||w_n|| \leq 1$ and $||\frac{z_n + w_n}{2}|| \to 1$. Therefore, by the definition of (WUC), it follows that

$$z_n - w_n \xrightarrow{w} 0, \quad \text{as} \quad n \to \infty .$$

Consequently $x_n - y_n \xrightarrow{w} 0$, as $n \to \infty$.

We also state the following similar results without proof, for (W*UC)-Banach spaces.

Lemma 2.2.6. Let $X^*$ be a (W*UC)-Banach space, then for any $d > 0$, $\varepsilon > 0$ and for every $x \in S$, there exists a $\delta_{\frac{\varepsilon}{d},x} > 0$ such that inequalities $||f|| \leq d$, $||g|| \leq d$ and $|f(x) - g(x)| \geq \varepsilon$, $(f,g \in X^*)$ imply

$$||\frac{f + g}{2}|| \leq (1 - \delta_{\frac{\varepsilon}{d},x})\cdot d.$$

Lemma 2.2.7. Let $X^*$ be a (W*UC)-Banach space. Then for any $x \in S$, $d > 0$, for every $\varepsilon > 0$ and $\alpha \in (0,1)$, the inequalities $||f|| \leq d$, $||g|| \leq d$ and $|f(x) - g(x)| \geq \varepsilon$, $(f,g \in X^*)$ imply

$$||\alpha x + \beta y|| \le \{1 - \delta_{\frac{\varepsilon}{d}, x} \cdot \min(\alpha,\beta)\}\cdot d,$$

where $\alpha + \beta = 1$.

Proposition 2.2.3. Let $X^*$ be a (W*UC)-Banach space. Suppose that there are given two sequences $\{f_n\}$ and $\{g_n\}$ in $X^*$ such that $||f_n|| \to 1$, $||g_n|| \le ||f_n||$ and $||\frac{f_n + g_n}{2}|| \to 1$, as $n \to \infty$. Then

$$f_n - g_n \xrightarrow{w^*} 0, \quad \text{as} \quad n \to \infty.$$

Remark 2.2.7. In Proposition 2.2.2. (Proposition 2.2.3) the condition on $\frac{x_n + y_n}{2}$ (on $\frac{f_n + g_n}{2}$) could be replaced by the analogous one on $\alpha x_n + (1 - \alpha)y_n$ (on $\alpha f_n + (1 - \alpha)g_n$), where $0 < \alpha < 1$.

Using the modification of one method of Kadec [75], Zizler [139] has proved the following results, which we state without proof.

Theorem 2.2.6. Let $X^*$ be separable space. Then $X$ is (WUC)-Banach space.

Corollary 2.2.2. Let $X$ be a reflexive separable Banach space, then $X$ is (WUC)-Banach space.

Theorem 2.2.7. Let $X$ be a separable Banach space. Then $X^*$ is (W*UC)-Banach space.

## 2.3. Some Other Spaces in General Banach Spaces.

A.R. Lovaglia [94], in 1955, considered a weaker type of convexity which is known as local uniform convexity. Geometrically, this differs from uniform convexity in that it is required that one end point of the variable chord remains fixed.

**Definition 2.3.1. Locally Uniformly Convex Spaces (LUC):**

A Banach space $X$ is called locally uniformly convex (LUC) iff for given $\varepsilon > 0$ and an element $x_o \in X$ with $||x_o|| = 1$, there exists a $\delta(\varepsilon,x_o) > 0$ such that

$$|| \frac{x_o + y}{2} || \leq 1 - \delta(\varepsilon,x_o),$$

whenever $||x_o - y|| \geq \varepsilon$ and $||y|| = 1$.

**Remarks 2.3.1.**

(1) It is clear from the definitions that uniform convexity implies local uniform convexity. But the converse is not true in general, as can be seen in [94].

(2) If we introduce in the space $C[0,1]$ an equivalent (LUC)-norm by a method if Kadec [75], we obtain an example of a (LUC)-space which has no equivalent (WUC)-norm [140].

M.M. Day [42] defines the notion of local uniform convexity near a point. Geometrically, this differs from uniform convexity in that the variable chord in the unit sphere is contained in a sphere about some point $b_o$, where as local uniform convexity requires only that one end point of the chord remains fixed.

**Definition 2.3.2. Locally Uniformly Convex Spaces near a point:**

If $||b_o|| = 1$, a Banach space $X$ is said to be locally uniformly near $b_o$ if there is a sphere about $b_o$ in which the condition for uniform convexity holds.

Remark 2.3.2. In his paper Day [42] proves that if a Banach space X is locally uniformly convex near a point $b_o$ then X is isomorphic to a uniformly convex space. Hence local uniform convexity near a point $b_o$ implies isomorphism of the space X with an uniformly convex space. However, Lovaglia [94] showed that there exist locally uniformly convex Banach spaces which are not isomorphic to any uniformly convex Banach spaces. Thus the notion of local uniform convexity and Day's notion of local uniform convexity near a point are essentially different.

In 1960, K.W. Anderson [3] investigated another type of convexity which is called midpoint locally uniform convexity (MLUC). Geometrically, it states that if the midpoint of a variable chord in the unit sphere approaches a fixed point on the unit sphere, then the length of the chord approaches zero. In fact, this property has been known and considered for sometime by other people, notably G. Lumer and M.M. Day, but its relations to other convexities were investigated by Anderson [3].

Definition 2.3.3. Midpoint Locally Uniformly Convex Spaces (MLUC):

A Banach space X is called midpoint locally uniformly convex (MLUC) iff it satisfies one of the following equivalent conditions:

(I) given $\varepsilon > 0$ and an element $x_o \in X$ with $||x_o|| = 1$, there exists a $\delta(\varepsilon, x_o) > 0$ and $||x|| \leq 1$, $||y|| \leq 1$, $||x - y|| \geq \varepsilon$ such that

$$||x + y - 2x_o|| \geq \delta.$$

(II) $||x_n|| = ||y_n|| = ||x_o|| = 1$ and $||x_n + y_n - 2x_o|| \to 0$ imply either

$$||x_n - y_n|| \to 0$$

or, $$||x_n - x_o|| \to 0$$

or, $||y_n - x_o|| \to 0$

Remark 2.3.3. Anderson [ 3] has shown that local uniform convexity implies midpoint local uniform convexity. But the converse is not true as can be seen from example given in [3 P.11].

Definition 2.3.4. Strictly Convex Spaces:

A Banach space $X$ is called strictly convex iff $||x + y|| = ||x|| + ||y||$ implies $x = \lambda y$, $\lambda > 0$ whenever $x \neq 0$, $y \neq 0$, $x,y \in X$.

We state without proof the following result which gives necessary and sufficient conditions for a Banach space to be strictly convex.

Theorem 2.3.1. (Ruston [117]). Either of the following conditions is necessary and sufficient for a Banach space $X$ to be strictly convex:

(1) whenever $||x|| = ||y|| = 1$ and $x \neq y$ $(x,y \in X)$

$$\Longrightarrow \left|\left| \frac{x + y}{2} \right|\right| < 1.$$

(2) for any non-vanishing functional $f$ over $X$ there exists at most one (there may not be any) element $x_o \in X$ such that $||x_o|| = 1$ and $f(x_o) = ||f||$.

Remarks 2.3.4.

(1) In Theorem 2.3.1. the restriction on $\frac{x + y}{2}$ could be replaced by the analogous one on $\alpha x + (1 - \alpha)y$ where $\alpha \in (0,1)$.

(2) All Hilbert spaces, (UC)-spaces, (LUC)-spaces and (MLUC)-spaces are strictly convex. The converse is not true in general for any of these spaces, (see [ 38], [ 94], [ 3 ]).

(3) The spaces $\ell^1$, $L^1$, $\ell^\infty$ are not strictly convex.

(4) If $X$ is a reflexive Banach space, then $X$ has an equivalent

strictly convex norm [93].

Combining the results in [38], [94], Anderson [3] has proved the following general theorem, which we state without proof.

<u>Theorem 2.3.2.</u> For any Banach space the following implication holds:

$$\text{UC} \rightarrow \text{LUC} \rightarrow \text{MLUC} \rightarrow \text{strict convexity.}$$

Furthermore, none of these implications can be reversed.

We state without proof the following lemma which is very useful in application.

<u>Lemma 2.3.1.</u> Let $X$ be a strictly convex Banach space and $C$ a weakly compact convex subset of it. Then, for every $y \notin C$, there exists a unique $x_o \in C$ such that

$$||x_o - y|| = \inf_{x \in C} ||x - y|| .$$

In general, the duality mapping $J$ is multi-valued. However, if the dual space $X^*$ is strictly convex, then the set $J(x)$ consists of exactly one point. We state the following propositions without proof.

<u>Proposition 2.3.1</u>. ([44]) Let $X$ be a Banach space with a strictly convex dual space $X^*$. Let $J$ be the duality mapping in $X$ with the gauge function $\mu$. Then the set $J(x)$ consists of precisely one point.

<u>Proposition 2.3.2</u>. ([44]) Let $X$ be a Banach space with a strictly convex dual space $X^*$. Then the duality mapping $J$ in $X$ with gauge function $\mu$ is monotone (strictly monotone if $x \neq y$).

We mention the following characterizations of strictly convex Banach spaces by means of duality mapping $J$.

<u>Theorem 2.3.3.</u> (Torrance [130]). A Banach space $X$ is strictly convex iff for $x$ and $y$ in $S$ such that $x \neq y$ and for $f$ in $J(x)$, $1 - (f,y) > 0$.

<u>Proof.</u> Suppose that $X$ is strictly convex and let $x,y$, and $f$ be as above. Then it follows from Theorem 2.3.1. that

$$1 - (f,y) \geq 2 - ||x + y|| > 0.$$

Now suppose that the second condition of the Theorem is satisfied and that $X$ is not strictly convex. Then, there exist $x,y \in S$ $(x \neq y)$ such that $||x + y|| = 2$. Let $z = \frac{x + y}{2}$ and $h \in J(z)$. Since $||h|| = 1 = ||x|| = ||y||$ and $(h,x + y) = 2$, $(h,x) = 1$, a contradiction, since $z \neq x$.

<u>Theorem 2.3.4.</u> (Petryshyn [107]). A Banach space $X$ is strictly convex iff the duality mapping $J$ of $X$ into $X^*$ is strictly monotone.

A simpler proof [34] of this theorem may be given as follows:

<u>Proof.</u> Suppose that $X$ is strictly convex. Let $x,y \in X$, $f \in J(x)$, and $g \in J(y)$. Then,

$$||f||\ ||y|| - (f,y) \geq ||f||\ (||x|| + ||y|| - ||x + y||)$$

and

$$||g||\ ||x|| - (g,x) \geq ||g||\ (||x|| + ||y|| - ||x + y||)$$

and by the use of equation

$$(f - g,x - y) = [||f|| - ||g||][||x|| - ||y||] + [||f||\ ||y|| - (f,y)] + [||g||\ ||x|| - (g,x)]$$

(with each of the three terms on the right being non-negative), we have

$$(f - g,\ x - y) \geq (||x|| - ||y||)^2 + (||x|| + ||y||)(||x|| + ||y|| - ||x + y||).$$

If $x \neq y$ and $||x|| = ||y||$, then $||x|| > 0$ and

$$||x|| + ||y|| - ||x + y|| = ||x|| \left(2 - \left|\left| \frac{x}{||x||} + \frac{y}{||x||}\right|\right|\right),$$

which is positive by the strict convexity of $X$. Consequently, $J$ is strictly monotone.

Now, suppose that $J$ is strictly monotone and that $X$ is not strictly convex. Then by Theorem 2.3.3, there exist $x,y \in S$ $(x \neq y)$ and an $f \in J(x)$ such that $1 - (f,y) = 0$. As before, $1 - (f,y) \geq 2 - ||x + y||$, so $||x + y|| = 2$. If $z = \frac{x + y}{2}$ and $h \in J(z)$, then $(h, x + y) = 2$ and $||h|| = 1 = ||x|| = ||y||$, so $(h,x) = 1$. Consequently, $(h - f, z - x) = 1 - (h,x) + 1 - (f,z) = 0$, which contradicts the fact that $z \neq x$.

The concepts of fully k-convex and weakly fully k-convex spaces have been introduced and studied by Fan and Glicksberg [60], [61].

<u>Definition 2.3.5.</u> Let $X$ be a real normed linear space. For any integer $k \geq 2$ a sequence $\{x_n\}$ of elements of $X$ will be called a $(k.i)$-sequence for $i = 1,2,3$, or $4$, respectively, if

(k.1) $\lim\limits_{n\to\infty} ||x_n|| = 1$ and $\left|\left| \frac{1}{k} \sum\limits_{i=1}^{k} x_{n_i} \right|\right| \geq 1$ for any $k$ indices $n_1 \leq n_2 \leq \ldots \leq n_k$ ;

(k.2) $\lim\limits_{n_1,\ldots,n_k\to\infty} \left|\left| \frac{1}{k} \sum\limits_{i=1}^{k} x_{n_i} \right|\right| = 1$ ;

(k.3) $\{x_n\}$ is bounded, and $||x_{n_1}|| \leq \left|\left| \frac{1}{k} \sum\limits_{i=1}^{k} x_{n_i} \right|\right|$ for any $k$ indices $n_1 \leq n_2 \leq \ldots \leq n_k$ ;

or ,

(k.4) $\{x_n\}$ is bounded, and, for every $n_1$, there exists an $M = M(n_1)$ such that $||x_{n_1}|| \leq \left|\left| \frac{1}{k} \sum\limits_{i=1}^{k} x_{n_i} \right|\right|$ for $M \leq n_2 \leq \ldots \leq n_{k-1} \leq n_k$.

For any integer $k \geq 2$ and for $i = 1,2,3,4$, let us consider the following conditions concerning $X$:

(F.k.C.i) Every $(k\ i)$-sequence $\{x_n\}$ in $X$ is a Cauchy sequence.

(W.F.k.C.i) Every $(k\ i)$-sequence $\{x_n\}$ in $X$ is weakly convergent.

We note that condition (F.2.C.2) has been previously considered by Smulian [127].

We state without proof the following result due to Fan and Glicksberg [60].

Theorem 2.3.5. For any fixed integer $k \geq 2$ and for any normed linear space $X$, the four conditions (F.k.C.i), $1 \leq i \leq 4$, are mutually equivalent. Also, the four conditions (W.F.k.C.i), $1 \leq i \leq 4$, are mutually equivalent.

Definition 2.3.6. A Banach space $X$ is said to be fully k-convex (weakly fully k-convex) if it satisfies any one of the equivalent conditions (F. k. C. i) $1 \leq i \leq 4$ (conditions (W.F.k.C.i), $1 \leq i \leq 4$).

Remark 2.3.5. It follows from

$$\frac{k+1}{k}\left\|\frac{1}{k+1}\sum_{i=1}^{k+1} x_{n_i}\right\| - \frac{1}{k}\|x_{n_{k+1}}\| \leq \left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}\right\| \leq \frac{1}{k}\sum_{i=1}^{k}\|x_{n_i}\|,$$

that every $((k+1).2)$-sequence is also a $(k.2)$-sequence. Hence every fully k-convex (weakly fully k-convex) space is also fully $(k+1)$-convex (weakly fully $(k+1)$-convex). Every uniformly convex space is easily seen to be fully 2-convex and therefore fully k-convex space for any $k \geq 2$.

We state without proof a property not shared by uniformly convex spaces.

Theorem 2.3.6. (Fan and Glicksberg [60]) Let $k$ be an integer $\geq 2$, and let $p$ be a real number $>1$. Let $\{X^{(i)}\}$ be a sequence of fully k-convex Banach spaces. If $X$ denotes the Banach space of all those sequences

$x = \{ \xi^{(i)} \}$ with $\xi^{(i)} \in X^{(i)}$ $(i = 1,2,3, \ldots)$ and

$||x|| = ( \sum_{i=1}^{\infty} ||\xi^{(i)}||^p )^{1/p} < \infty$, then $X$ is fully k-convex.

Remarks 2.3.6.

(1) In Theorem 2.3.6., if we take $X^{(i)} = \ell^{i+1}$ $(i = 1,2,3, \ldots)$, then, by a result of Day [41], the resulting fully 2-convex Banach space $X$ (for any $p > 1$) is not uniformly convex in any topologically equivalent norm.

(2) Fully k-convex Banach spaces are reflexive. Also, a weakly fully k-convex space is reflexive iff it is weakly complete.

We define the concept of uniformly non-squareness originally introduced by James [73].

Definition 2.3.7. Uniformly Non-square spaces (UNS):

A Banach space $X$ is uniformly non-square iff there is a positive number $\delta > 0$ such that there do not exist $x$ and $y$ in $X$ for which $||x|| \leq 1$, $||y|| \leq 1$, $||\frac{x + y}{2}|| > 1 - \delta$ and $|| \frac{x - y}{2}|| > 1 - \delta$.

Remark 2.3.7. Obviously, a uniformly convex space is uniformly non-square, but it is not known whether uniform non-squareness and uniform convexity are isomorphically equivalent.

We state without proof the following result due to James [73].

Theorem 2.3.7. A Banach space is reflexive if it is uniformly non-square.

Remark 2.3.8. The converse of Theorem 2.3.7. is not true. James [73] describes a class of reflexive Banach spaces, no one of which is isomorphic to any uniformly non-square space. It is interesting to note that this gives an alternative method of proving Day's [41] theorem that there exist a reflexive Banach space that is not isomorphic to any uniformly convex Banach space.

The notion of uniformly convex in every direction (UCED) was first used by A.L. Garkavi [67], [68], to characterize normed linear spaces for which every bounded subset has at most one Cebysev center. The geometrical significance of this concept is that the collection of all chords of the unit ball that are parallel to a fixed direction and whose lengths are bounded below by a positive number has the property that the midpoints of the chords lie uniformly deep inside the unit ball.

<u>Definition 2.3.8.</u> <u>Uniformly Convex in Every Direction Spaces (UCED):</u>

A Banach space $X$ is uniformly convex in every direction (UCED) iff for any $\varepsilon > 0$ and every nonzero $z \in X$, there exists a number $\delta(\varepsilon, z) > 0$ such that, if $x - y = \lambda z$, $||x|| = ||y|| = 1$ and $||\frac{x+y}{2}|| > 1 - \delta$ then $|\lambda| \leq \varepsilon$.

The following theorem, stated without proof, gives several properties that are equivalent to uniform convexity in every direction for a normed linear space.

<u>Theorem 2.3.8.</u> (Day-James-Swaminathan [43])

Each of the following is a necessary and sufficient condition for a normed linear space $X$ to be (UCED).

(I) If there are sequences $\{x_n\}$ and $\{y_n\}$ and a nonzero member $z$ of $X$ for which

(a) $||x_n|| = ||y_n|| = 1$, for every $n$,

(b) $x_n - y_n = \alpha_n z$, for every $n$,

(c) $||x_n + y_n|| \to 2$,

then $\alpha_n \to 0$.

(II) If there are sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(α) $||x_n|| \leq 1$ and $||y_n|| \leq 1$, for every n,

(β) $x_n - y_n \to z$.

(γ) $||x_n + y_n|| \to 2$,

then $z = 0$.

(III) For no nonzero $z$ is there a bounded sequence $\{x_n\}$ in X such that

$$2^{p-1}(||x_n + z||^p + ||x_n||^p) - ||2x_n + z||^p \to 0,$$

where $p$ is any number for which $2 \leq p < \infty$.

(IV) For each nonzero $z$ in X, there is a positive number $\delta$ such that $||x + \frac{1}{2} z|| < 1 - \delta$, whenever $||x|| \leq 1$ and $||x + z|| \leq 1$.

Remarks 2.3.9.

(1) It might be noted that in Theorem 2.3.8. condition (I) '$\leq 1$' can be substituted for '$=1$' in the restrictions on $x_n$ and $y_n$.

(2) A uniformly convex space is (UCED)-space but the converse is not always true. In fact, there are spaces - even reflexive Banach spaces - that are (UCED), but not isomorphic to a uniformly convex Banach space. However, it is not known whether every reflexive Banach space can be renormed so as to be (UCED).

(3) If X is (UCED), then X is strictly convex. The converse is not true as can be seen from the following example [68] (also see [43]):

Example 2.3.1. The space C[0,1] of all real continuous functions on the unit interval with the norm

$$||f|| = \sup \{|f(t)|\} + \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}$$

is strictly convex, but this space is not (UCED).

(4) It has been shown by Zizler [141] that $X$ can be renormed so as to be (UCED) if there is a continuous one-to-one linear map of $X$ into a space $Y$ that is (UCED).

We state without proof the following result.

Proposition 2.3.3. (Zizler [141]). Every separable Banach space has an equivalent norm which is (UCED).

It is interesting to give the following example, due to Zizler [141], of a Banach space which is (UCED), but not (WUC).

Example 2.3.2. Consider the space $C[0,1]$ with an equivalent norm

$$|||f||| = (||f||^2_{C[0,1]} + ||T(f)||^2_{L^2[0,1]})^{1/2}$$

where $T$ is the natural 'identity mapping' of $C[0,1]$ into $L^2[0,1]$. Then $|||\cdot|||$ is (UCED). But space $C[0,1]$ does not have any equivalent (WUC)-norm (see Remark 2.2.6 (3)).

## 2.4. Normal and Complete Normal Structures.

The concepts of normal structure and complete normal structure have been of fundamental importance in some recent investigations concerned with determining fixed points of different mappings, for example see Belluce and Kirk [8], [9], [10], Kirk [85], Kannan [80], [81], Zizler[139], Woodward [135] and others.

Definition 2.4.1. Let $C$ be a bounded convex set in a Banach space $X$, of diameter $d$.

A point $x \in X$ is said to be diametral point for $C$ if $\sup_{y \in C} ||x - y|| = d$.

Example 2.4.1. In the Banach space $C[0,1]$ every point of the bounded and convex set

$$\{f(t): 0 \leq f(t) \leq 1, \quad f(0) = 0, \quad f(1) = 1\}$$

is diametral.

The notion of normal structure was introduced by Brodskii and Milman [18] as follows:

Definition 2.4.2. A convex set $G$ in a Banach space $X$ is said to have normal structure if for each bounded convex subset $C$ of $G$, which contains more than one point, there exists a point $x \in C$ which is not diametral for $C$.

Geometrically, $G$ has normal structure if for every bounded and convex subset $C$ of $G$, there exists a ball of radius less than the diameter of $C$ centered at a point of $C$ and containing $C$.

Remark 2.4.1. We say that a Banach space has normal structure if each of its bounded convex subsets has normal structure. But there are Banach spaces which do not possess normal structure.

Example 2.4.2. The Banach spaces $C[0,1]$, $\ell^1$, $L^1$ do not have normal structure.

We state the following results without proof, which give larger class of sets with normal structure.

Theorem 2.4.1. (Brodskii and Milman [18]) Every convex and compact subset of a Banach space has normal structure.

Theorem 2.4.2. (Edelstein [54], Browder [25]) Every uniformly convex Banach space has normal structure.

Theorem 2.4.3. (Zizler [141]). A Banach space has normal structure if it is uniformly convex in every direction.

Theorem 2.4.4. (Zizler [139]). Every bounded closed convex subset of a (WUC)-Banach space has normal structure.

Theorem 2.4.5. (Zizler [139]). Every bounded closed convex subset of a (W*UC)-Banach space has normal structure.

The notion of complete normal structure was introduced by Belluce and Kirk [ 9 ] in the following way:

For bounded subsets $H$ and $S$ of a Banach space $X$, let

$$r_s(H) = \sup \{||s - x|| : x \in H\}$$

$$r(H,S) = \inf \{r_s(H) : s \in S\}$$

$$C(H,S) = \{s \in S : r_s(H) = r(H,S)\}.$$

The members of $C(H,S)$ are called the Cebysev centers of $H$ in $S$.

Definition 2.4.3. ([ 9 ]). Let $G$ be a closed convex subset of a Banach space $X$. Then $G$ has complete normal structure iff each bounded closed convex subset $W$ of G, which contains more than one point, has the property that the closure of $\bigcup_{\alpha \in \Lambda} C(W_\alpha, W)$ is a nonempty proper subset of $W$ whenever $\{W_\alpha : \alpha \in \Lambda\}$ is a decreasing net of subsets of $W$ such that $r(W_\alpha, W) = r(W,W)$ for each $\alpha \in \Lambda$ ($\Lambda$-index-set).

Remark 2.4.2. Complete normal structure implies normal structure is eeen by taking $W_\alpha = W$ in the above definition.

We state the following results without proof.

Theorem 2.4.6. (Belluce and Kirk [ 9]) If $C$ is a convex, compact subset of a Banach space then $C$ has complete normal structure.

Theorem 2.4.7. (Belluce and Kirk [ 9]) If $C$ is a nonempty bounded closed convex subset of a uniformly convex Banach space then $C$ has complete normal structure.

Theorem 2.4.8. (Day, James and Swaminathan [43]) A reflexive Banach space has complete normal structure if it is uniformly convex in every direction.

There have been a number of recent results on fixed points of nonexpansive and semi-nonexpansive mappings in Banach spaces, using the notion of normal structure. Brodskii and Milman [18] have considered isometries $T$ of a bounded closed convex subset $C$ of a Banach space $X$ into itself. They were able to prove the existence of a fixed point for $T$ proveded $X$ is reflexive and $C$ has normal structure. An argument similar to the one in Brodskii and Milman [18] was used by Kirk [83] to prove the following fundamental result, which we state without proof.

Theorem 2.4.9. Let $X$ be a reflexive Banach space, and $C$ a bounded closed convex subset of $X$ with normal structure. Then a nonexpansive mapping $T$ of

C into itself has a fixed point.

Remark 2.4.3. It is worth mentioning that the restriction on C and the space X in Theorem 2.4.9., are necessary, as shown by means of examples in [44]. The necessity for normal structure of C is illustrated by the following example (Browder [25]):

Example 2.4.3. Let $X = C_0$, the space of sequences converging to zero, U the unit ball in the maximum norm, $e_1$ the unit vector with first component 1 and other zero, $S(x) = (0, x_1, x_2, \ldots)$. Then the mapping

$$T(x) = e_1 + S(x)$$

maps U into itself, is nonexpansive, and has no fixed point in U.

An immediate consequence of Theorem 2.4.9. is the following result of Browder [25], Göhde [70], Kirk [83].

Theorem 2.4.10. Let $T:C \to C$ be a nonexpansive mapping on a bounded closed convex subset C of a uniformly convex Banach space X. Then T has a fixed point in C.

The following result follows from Theorem 2.4.9. and Theorem 2.4.4.

Theorem 2.4.11. (Zizler [139]). Let X be a reflexive (WUC)-Banach space, C a bounded closed convex subset of X, T a nonexpansive mapping of C into itself. Then T has a fixed point in C.

The following is an immediate consequence of Theorem 2.4.11. and Corollary 2.2.2.

Theorem 2.4.12. (Zizler [139]).Let X be a separable, reflexive Banach space. Then X is isomorphic to a space Y with the following property:

Every nonexpansive mapping $T$ of a bounded closed convex subset $C$ into itself has a fixed point.

We give the following result for (UCED)-Banach spaces.

Theorem 2.4.13. Let $X$ be a reflexive (UCED)-Banach space and $C$ a bounded closed convex subset of $X$. Then a nonexpansive mapping $T$ of $C$ into itself has a fixed point in $C$.

Proof. It follows from Theorem 2.4.3. that $C$ has normal structure, hence result follows from Theorem 2.4.9.

Remark 2.4.4. It is of interest to see that Theorem 2.4.12 can be obtained as an immediate consequence of Theorem 2.4.13, and Proposition 2.3.3.

We state the following results without proof:

Theorem 2.4.14. (Belluce and Kirk [ 8]) Let $C$ be a bounded closed convex subset of a Banach space $X$ and suppose that $C$ has normal structure. Let $M$ be a weakly compact subset of $X$. Assume $T$ is a nonexpansive mapping of $C$ into itself with the property that for each $x \in C$, the closure of $\{T^n(x) : n = 1,2, \ldots\}$ contains a point of $M$. Then there is an $x \in M$ such that $T(x) = x$.

Theorem 2.4.15. (Kirk [85]) Let $C$ be a non-empty weakly compact, convex subset of a Banach space $X$, and suppose $C$ has normal structure. Then every nonexpansive mapping $T:C \to C$ has a fixed point.

Efforts to generalize Theorem 2.4.15. by weakening the assumption of normal structure have been unsuccessful, although an apparent slight weakening does yield a result for contractive mappings.

Definition 2.4.4. ([85]) A bounded convex subset $C$ of a Banach space $X$

is said to have normal structure relative to $F$, $F \subseteq X$, if for each bounded convex subset $H$ of $C$ which contains more than one point, there is a point $x$ in $F$ such that

$$\sup\{\|x - y\| : y \in H\} < \delta(H) .$$

<u>Theorem 2.4.16</u>. (Kirk [85]) Let $C$ be a nonempty weakly compact, convex subset of a Banach space $X$ and suppose $C$ has normal structure relative to $C$. If $T:C \to C$ is contractive mapping, then $T$ has a fixed point in $C$.

The following generalization of Theorem 2.4.15. is due to Kirk [85], which we state without proof.

<u>Theorem 2.4.17.</u> Suppose $C$ is a nonempty weakly compact, convex subset of a Banach space $X$, and let $T:C \to C$ be nonexpansive map. If for each $x$ in $C$ it is the case that $\operatorname{conv}\{x, Tx, T^2x, \dots\}$ has normal structure, then $T$ has a fixed point in $C$.

If one merely assumes that for some positive integer $N$ the Nth iterate, $T^N$, of $T$ is nonexpansive then $T$ need not have a fixed point since, in particular, a periodic homeomorphism of the unit ball of a Hilbert space may be fixed point free (Klee [88]). Goebel [69] obtained sufficient conditions to guarantee existence of fixed points for mapping $T$ such that $T^N$ is nonexpansive. Using normal structure Kirk [86] proved the following result which we state without proof.

<u>Theorem 2.4.18.</u> Let $X$ be a reflexive Banach space which has strictly convex norm and suppose $C$ is a nonempty bounded closed convex subset of $X$ which possesses normal structure. Suppose the mapping $T:C \to C$ has the property that for some integer $N > 1$, $T^N$ is nonexpansive, and suppose further that there is a constant $k$ satisfying

$$N^{-2}[(N-1)(N-2)k^2 + 2(N-1)k] < 1$$

such that $||T^j(x) - T^j(y)|| \leq k\, ||x - y||$, for all $x, y \in C$, $1 \leq j \leq N - 1$. Then $T$ has a fixed point in $C$.

A more thorough study of the concept of normal structure has been initiated in Belluce - Kirk - Steiner [12], where examples of noncompact convex subsets of non-strictly convex spaces which possess normal structure were obtained. Also spaces shown by Day [41] to be strictly convex, reflexive, and not isomorphic to any uniformly convex space, were shown to have the property that each of their bounded convex subsets has normal structure. It is interesting to note the example ([12]) which shows that normal structure is not implied by reflexivity. We state the following two results due to Belluce - Kirk - Steiner [12] without proof.

Theorem 2.4.19. There exists a Banach space which is reflexive , strictly convex, and which possesses normal structure, but which is not isomorphic to any uniformly convex Banach space.

Theorem 2.4.20. Let $X_1$ and $X_2$ be Banach spaces with norms $||\cdot||_1$ and $||\cdot||_2$ respectively. Let $X = X_1 \oplus X_2$ with the norm of $X$ given by $||\cdot|| = \sup(||\cdot||_1, ||\cdot||_2)$. If $X_1$ and $X_2$ have normal structure, then $X$ has normal structure.

Some interesting results, using normal structure, have been given for semi-nonexpansive mappings by Kannan [80], [81], and Woodward [135]). Following Kannan [81], we define

Definition 2.4.5. A mapping $T$ of a bounded subset $C$ of a Banach space $X$ into itself is said to have property $B$ on $C$ if for every closed convex subset $F$ of $C$, mapped into itself by $T$ and containing more than one element, there exists $x \in F$ such that

$$||x - T(x)|| < \sup_{y \in F} ||y - T(y)||.$$

Remark 2.4.5. If $C$ has normal structure then a semi-nonexpansive mapping $T$ of $C$ into itself must have property $B$ on $C$. But the converse is not true as can be seen from the following example [81]:

Example 2.4.4. Let $m$ be the space of bounded sequences of numbers with the supremum norm [3, p. 5'] and let $C = \{ x \in m : ||x|| \leq 2 \}$. Clearly $C$ is a bounded convex set in $m$. Now let $F$ be the subset of $C$ such that $F = [x_1, x_2, \ldots]$ where $x_k = \{0, 0, \ldots, 1, 0, \ldots\}$, (1 in the kth place). Evidently $\delta(F) = 1$. Also $\sup_{y \in F} ||x - y|| = 1$ for every $x \in F$. Hence $C$ does not have normal structure. But the operator $T : C \to C$ defined by $T(x) = \frac{x}{3}$, $x \in C$ is such that

$$||T(x) - T(y)|| \leq \frac{1}{2} [||x - T(x)|| + ||y - T(y)||], \quad x, y \in C,$$

and for every closed subset $F'$ of $C$ mapped into itself by $T$ and containing more than one element, there exists $x \in F'$ such that

$$||x - T(x)|| < \sup_{y \in F'} ||y - T(y)||.$$

We state the following result without proof.

Theorem 2.4.21. (Kannan [80]) Let $T$ be a continuous semi-nonexpansive mapping of a bounded closed convex subset $C$ of a reflexive Banach space $X$ into itself and let $T$ have property $B$ over $C$. Then $T$ has a unique fixed point.

We give the following result:

<u>Theorem 2.4.22.</u> Let $T$ be a continuous semi-nonexpansive mapping of a bounded closed convex subset $C$ of a reflexive (WUC)-Banach space (of a reflexive (UCED)-Banach space) $X$ into itself. Then $T$ has a unique fixed point.

<u>Proof.</u> It follows from Theorem 2.4.4. (Theorem 2.4.3.) that $C$ has normal structure. Hence result follows from Remark 2.4.5. and Theorem 2.4.21.

Combining Corollary 2.2.2.(or Proposition 2.3.3.) and Theorem 2.4.22., we get the following:

<u>Theorem 2.4.23.</u> Let $X$ be a separable reflexive Banach space. Then $X$ is isomorphic to a Banach space $Y$ with the following property:

Every continuous semi-nonexpansive mapping $T$ of a bounded closed convex subset $C$ into itself has a unique fixed point.

# CHAPTER III

# SOME FIXED POINT THEOREMS

Let $X$ be a Banach space with norm $||\cdot||$ and $D$ a subset of $X$. Throughout this chapter, if $T$ is a self-mapping of $D$, we use $F(T)$ to denote the set of fixed points of $T$ in $D$.

## 3.1. Nonexpansive and Quasi-Nonexpansive Mappings.

We recall that a mapping $T:D \to X$ is called nonexpansive if $||T(x) - T(y)|| \leq ||x - y||$ for all $x,y$ in $D$. It is well-known, however, that the fundamental property of contraction mappings, expressed by the Banach contraction principle, does not extend to nonexpansive mappings (see Example 2.4.3.). It is of great importance in the applications (see Browder [22]) to find out if nonexpansive mappings have fixed points. In order to obtain existence of fixed points for such mappings some restriction has to be made on the Banach space $X$ and on the subset $D$. In case $X$ is uniformly convex Banach space (see Theorem 2.4.10) or more generally $X$ is a reflexive Banach space with normal structure (see Theorem 2.4.9.) then a nonexpansive mapping from a bounded closed convex subset $D$ of $X$ into itself has a fixed point. A general situation of nonexpansive mapping $T:D \to X$ has been considered by Petryshyn [105] for the Hilbert space case (see also Browder and Petryshyn [33]). Many geometric properties of Banach spaces under consideration are constantly involved in obtaining fixed points from different mappings, viz. convexity, uniform convexity, strict convexity, normal structure, complete normal structure, structure of Hilbert spaces and so on.

In this section we show that some of the results can be obtained, in the general setting of a Banach space, even when the hypothesis of nonexpansiveness is considerably weakened. Essentially, we show that part of the analysis which does not require the full force of nonexpansiveness, but requires only the existence of at least one fixed point together with nonexpansiveness only about each fixed point.

Definition 3.1.1. (Dotson [48]) A self-mapping $T$ of a subset $D$ of a normed linear space $X$ is said to be quasi-nonexpansive provided $T$ has at least one fixed point in $D$, i.e. $F(T)$ is non-empty, and if $p \in F(T)$ then

$$||T(x) - p|| \leq ||x - p|| \quad \text{holds for all } x \in D.$$

This concept which Dotson [47] has labelled quasi-nonexpansive, was essentially introduced (along with some related ideas) by Diaz and Metcalf [45]. One notes that in assuming $T:D \to D$ a quasi-nonexpansive mapping, we also assume the existence of a fixed point of $T$ in $D$ and thus a nonexpansive mapping $T:D \to D$ with at least one fixed point in $D$ is quasi-nonexpansive and that a linear quasi-nonexpansive mapping on a subspace is nonexpansive on that subspace. But there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings which are not nonexpansive. Following is an example, due to Dotson [48], of a continuous quasi-nonexpansive mapping which is not nonexpansive.

Example 3.1.1. The mapping $T$ from the reals to the reals defined by

$$\begin{aligned} T(x) &= \frac{x}{2} \sin \frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0. \end{aligned}$$

Following Browder and Petryshun [32], we define

Definition 3.1.2. A mapping $T$ from a Banach space $X$ into itself is said to be asymptotically regular if $T^{n+1}(x) - T^n(x) \to 0$, as $n \to \infty$, for all $x \in X$.

Definition 3.1.3. A mapping $T$ from a Banach space $X$ into itself is said to be weakly asymptotically regular if $T^{n+1}(x) - T^n(x) \overset{w}{\to} 0$, as $n \to \infty$, for all $x \in X$.

Remark 3.1.1. Obviously every asymptotically regular mapping is weakly asymptotically regular.

In general, a nonexpansive mapping $T$ is not necessarily asymptotically regular. However, in some cases the determination of the fixed points of $T$ can be replaced by the same problem for an asymptotically regular mapping. Namely, the following result due to Browder and Petryshyn [32] holds, which we state without proof.

Theorem 3.1.1. Let $X$ be a uniformly convex Banach space and $T:X \to X$ a nonexpansive mapping. If $F(T)$ is non-empty then the mapping

$$T_\lambda = \lambda I + (1 - \lambda)T \quad , \qquad 0 < \lambda < 1,$$

is nonexpansive and asymptotically regular. Moreover, $F(T) = F(T_\lambda)$.

Considering a more general mapping than $T_\lambda$, Kirk [87] has proved the following two results which we state without proof.

Theorem 3.1.2. Let $D$ be a convex subset of a Banach space $X$ and $T$ a nonexpansive mapping of $D$ into itself. Define the mapping $S:D \to D$ by

$$\text{(K)} \qquad S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \cdots\cdots + \alpha_k T^k,$$

$$\text{where} \quad \alpha_i \geq 0,\ \alpha_1 > 0, \quad \text{and} \quad \sum_{i=0}^{k} \alpha_i = 1 \ .$$

Then $S(x) = x$ iff $T(x) = x$, i.e, $F(T) = F(S)$.

Remark 3.1.2. Assumption $\alpha_1 > 0$ in Theorem 3.1.2. is necessary to rule out the possibility that a fixed point of $S$ is merely a point at which $T$ is periodic.

Theorem 3.1.3. Let $D$ be a convex subset of a uniformly convex Banach space $X$ and $T$ a nonexpansive mapping of $D$ into itself. Define the mapping $S:D \to D$ as (K) in Theorem 3.1.2. If $F(T)$ is non-empty then mapping $S$ is asymptotically regular.

We remark that Theorem 3.1.3. is true even if $T$ is a nonexpansive mapping of $X$ into itself. Hence the following holds:

Theorem 3.1.4. Let $X$ be a uniformly convex Banach space and $T$ a nonexpansive mapping of $X$ into itself. Define $S:X \to X$ as (K) in Theorem 3.1.2. If $F(T)$ is non-empty then the mapping $S$ is nonexpansive and asymptotically regular.

Proof. Same as of Theorem 3.1.3. with the fact that $p \in F(T)$ implies $p \in F(S)$.

Remark 3.1.3. In case $\alpha_o = \lambda$, $\alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$, we have $S = T_\lambda$ and $F(T) = F(T\ )$. Hence we obtain Theorem 3.1.1. as a particular case of Theorem 3.1.4.

We give the following generalization of Theorem 3.1.4. for quasi-nonexpansive mappings.

Theorem 3.1.5. Let $X$ be a uniformly convex Banach space and $T$ a quasi-nonexpansive mapping of $X$ into itself. Define the mapping $S:X \to X$ as (K) in Theorem 3.1.2. Then the mapping $S$ is quasi-nonexpansive and asymptotically regular.

Proof. Let $x \in X$. Define the sequence $\{x_n\}$ by $x_n = S^n(x)$, $n = 1,2, \ldots$ . Suppose $p \in F(T)$. Then the sequence $\{||x_n - p||\}$ is nonincreasing, since $S$ is quasi-nonexpansive and $S(p) = p$, we may suppose $\lim_{n\to\infty} ||x_n - p|| = d \geq 0$. If $d = 0$, there is nothing to prove. Therefore, assume $d > 0$. Then (adopting the notion $T^o = I$) we have

$$\begin{aligned} x_{n+1} - p &= S(x_n) - p \\ &= \sum_{i=0}^{k} \alpha_i T^i(x_n) - p \\ &= \alpha_o(x_n - p) + (1 - \alpha_o)z_n, \end{aligned}$$

where $z_n = \frac{1}{1 - \alpha_o} \sum_{i=1}^{k} \alpha_o(T^i(x_n) - p)$.

Since $||T^i(x_n) - p|| \leq ||x_n - p||$, and $\sum_{i=0}^{k} \alpha_i = 1$, it follows that

$$\limsup_{n\to\infty} ||z_n|| \leq d.$$

Also

$$\lim_{n\to\infty} ||x_n - p|| = d, \quad \lim_{n\to\infty} ||x_{n+1} - p|| = d.$$

Because $X$ is uniformly convex it must be the case that

$$\lim_{n\to\infty} ||x_n - p - z_n|| = 0.$$

However, $x_{n+1} - x_n = (1 - \alpha_o)(x_n - p - z_n)$,

and so $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, completing the proof.

In case $\alpha_o = \lambda, \alpha_2 = \alpha_3 = \ldots\ldots. = \alpha_k = 0$, we have $S = T_\lambda$ and $F(T) = F(T_\lambda)$. Hence the following result is a particular case of Theorem 3.1.5.

<u>Theorem 3.1.6.</u> Let $X$, $T$ be as in Theorem 3.1.5. Then the mapping $T_\lambda : X \to X$ defined by

$$T_\lambda = \lambda I + (1 - \lambda)T, \qquad 0 < \lambda < 1,$$

is quasi-nonexpansive and asymptotically regular. Moreover, $F(T) = F(T\ )$.

We remark that Theorem 3.1.6. shows that Theorem 3.1.1. is true for quasi-nonexpansive mappings.

We give the following result for (WUC)-Banach spaces.

Theorem 3.1.7. Let $X$ be a (WUC)-Banach space and $T$ a nonexpansive mapping of $X$ into itself. Define the mapping $S:X \to X$ as (K) in Theorem 3.1.2. If $F(T)$ is non-empty then the mapping $S$ is nonexpansive and weakly asymptotically regular.

Proof. On the same lines as of Theorem 3.1.3., except, since $X$ is (WUC)-Banach space, Proposition 2.2.2. implies that

$$||x_n - p - z_n|| \xrightarrow{w} 0.$$

However, $x_{n+1} - x_n = (1 - \alpha_o)(x_n - p - z_n)$, and so $x_{n+1} - x_n \xrightarrow{w} 0$, completing the proof.

In case $\alpha_o = \lambda$, $\alpha_2 = \alpha_3 = \dots = \alpha_k = 0$, we have $S = T_\lambda$ and $F(T) = F(T_\lambda)$. Hence the following result is a particular case of Theorem 3.1.7.

Theorem 3.1.8. Let $X$ and $T$ be as in Theorem 3.1.7. If $F(T)$ is nonempty then the mapping $T_\lambda:X \to X$ defined by

$$T_\lambda = \lambda I + (1 - \lambda)T, \quad 0 < \lambda < 1,$$

is nonexpansive and weakly asymptotically regular. Moreover $F(T) = F(T_\lambda)$.

We give the following generalization of Theorem 3.1.7. for quasi-nonexpansive mappings.

Theorem 3.1.9. Let $X$ be a (WUC)-Banach space and $T$ a quasi-nonexpansive mapping of $X$ into itself. Define the mapping $S:X \to X$ as (K) in Theorem 3.1.2. Then the mapping $S$ is quasi-nonexpansive and weakly asymptotically regular.

Proof. Same as of Theorem 3.1.5., except, since $X$ is (WUC)-Banach space, Proposition 2.2.2. implies that

$$||x_n - p - z_n|| \xrightarrow{w} 0.$$

However, $x_{n+1} - x_n = (1 - \alpha_o)(x_n - p - z_n)$ ,

and so $x_{n+1} - x_n \xrightarrow{w} 0$, completing the proof.

Remark 3.1.4. In Theorem 3.1.9., if $\alpha_o = \lambda$, $\alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$, we have $S = T_\lambda$ and $F(T) = F(T_\lambda)$. Hence, in particular, we see that Theorem 3.1.8. holds for quasi-nonexpansive mappings.

Browder and Petryshyn [32] have proved the following result for asymptotically regular mapping. We omit the proof.

Theorem 3.1.10. Let $T$ be a nonexpansive asymptotically regular mapping of a Banach space $X$ into itself. Suppose that a subsequence $\{T^{n(j)}(x_o)\}$ , converges strongly to some point $y$. Then $y$ is a fixed point of $T$ and the whole sequence $\{T^n(x_o)\}$ converges strongly to $y$.

In the following, we just assume the nonexpansiveness of $T$ about the fixed points.

Theorem 3.1.11. Let $X$ be a Banach space and $T$ a continuous asymptotically regular mapping of $X$ into itself such that

(i) whenever $p \in F(T)$, $||T(x) - p|| \leq ||x - p||$, for all $x \in X$.

Suppose that a subsequence $\{T^{n(j)}(x_o)\}$ , $x_o \in X$, converges strongly to some point $y$. Then $y$ is a fixed point of $T$ and the whole sequence $\{T^n(x_o)\}$ converges strongly to $y$.

Proof. We first show that $y$ is a fixed point of $T$, i.e., $F(T)$ is nonempty. We have

$$T^{n(j)}(x_o) \to y \quad \text{implies} \quad (I - T)T^{n(j)}(x_o) \to (I - T)y.$$

On the other hand,

$$(I - T)T^{n(j)}(x_o) = T^{n(j)}(x_o) - T^{n(j)+1} \to 0,$$

since $T$ is asymptotically regular. Thus $(I - T)y = 0$. Hence $y \in F(T)$. Therefore, from (i), we see that the whole sequence converges to $y$ because

$$||T^{n+1}(x_o) - y|| \leq ||T^n(x_o) - y|| , \quad \text{for all } n = 1,2, \ldots .$$

For weakly asymptotically regular mappings, Browder and Petryshyn [ ] have proved the following result which we state without proof.

<u>Theorem 3.1.12.</u> Let $X$ be a Banach space, $T$ a nonexpansive mapping of $X$ into itself. For a given $f \in X$, let $T_f(u) = T(u) + f$, and suppose that the mapping $T_f$ is weakly asymptotically regular. Let $x_n = T_f^n(x_o)$ be the sequence of Picard iterates for the equation $u = T(u) - f$ starting with $x_o$, and suppose that an infinite subsequence of the sequence $\{x_n\}$ converges strongly to an element $y$ of $X$. Then $y$ is a solution of $u - T(u) = f$ and the whole sequence $\{x_n\}$ converges strongly to $y$.

For weakly asymptotically regular mappings which are nonexpansive on $F(T)$, we give the following:

<u>Theorem 3.1.13.</u> Let $X$ be a Banach space and $T$ a continuous weakly asymptotically regular mapping of $X$ into itself such that

(i) whenever $p \in F(T)$, $||T(x) - p|| \leq ||x - p||$, for all $x \in X$.

Suppose that a subsequence $\{T^{n(j)}(x_o)\}$, $x_o \in X$, converges strongly to some point $y$. Then $y$ is a fixed point of $T$ and the whole sequence $\{T^n(x_o)\}$ converges strongly to $y$.

Proof. We first show that $y$ is a fixed point of $T$, i.e., $F(T)$ is non-empty. We have

$$T^{n(j)}(x_o) \to y \quad \text{implies} \quad (I - T)T^{n(j)}(x_o) \to (I - T)y.$$

On the other hand,

$$(I - T)T^{n(j)}(x_o) = T^{n(j)}(x_o) - T^{n(j)+1}(x_o) \xrightarrow{w} 0,$$

since $T$ is weakly asymptotically regular. Since strong convergence implies weak convergence and weak limit of a sequence is unique, therefore $(I - T)y = 0$. Hence $y \in F(T)$. We see, from (i), that the whole sequence converges to $y$ because

$$||T^{n+1}(x_o) - y|| \leq ||T^n(x_o) - y||, \quad \text{for all } n = 1,2, \dots .$$

We state without proof the following result on metric spaces.

Theorem 3.1.14. (Diaz and Metcalf [46]) Let $T:M \to M$ be continuous map, where $M$ is a non-empty metric space. Suppose

(i) $F(T)$ is non-empty

(ii) for each $x \in M$, with $x \notin F(T)$, and each $p \in F(T)$, one has $d(T(x),p) < d(x,p)$.

Let $x_o \in M$. Then either the sequence $\{T^n(x_o)\}$ contains no convergent subsequence, or $\lim\limits_{n\to\infty} T^n(x_o)$ exists and belongs to $F(T)$.

Corollary 3.1.1. (Diaz and Metcalf [46]). Suppose, in addition to the hypotheses of Theorem 3.1.14., that, for some $x_o \in M$, the sequence of iterates $\{T^n(x_o)\}$ contains a convergent subsequence. Then $\lim\limits_{n\to\infty} T^n(x_o)$

exists and belongs to $F(T)$. Thus, under the above assumptions, the sequence $\{T^n(x_o)\}$ converges to a fixed point of $T$.

In general, it is not the case for nonexpansive mappings $T$ that the sequences of Picard iterates $\{T^n(x)\}$ converge to fixed points of $T$, and thus when such fixed points exist other approximation techniques are needed. One such technique is to form the mapping

$$T_\lambda = \lambda I + (1 - \lambda)T, \qquad 0 < \lambda < 1,$$

and then show that under certain circumstances the Picard iterates of $T_\lambda$ converges to a fixed point of $T$. The first such result (for $\lambda = \frac{1}{2}$) was obtained by Krasnoselskii [90]. Schaefer [119] has proved Krasnoselskii's result for arbitrary $\lambda \in (0,1)$ which we state below without proof.

<u>Theorem 3.1.15.</u> Let $D$ be a closed convex subset of a uniformly convex Banach space $X$, $T:D \to D$ a nonexpansive mapping, and suppose $T(D)$ is contained in a compact subset of $D$. Let $x$ be an arbitrary point of $D$. Then the sequence defined by

$$x_{n+1} = \lambda x_n + (1 - \lambda)x_n, \qquad n = 0,1,2, \ldots ; \ \lambda \in (0,1),$$

converges to a fixed point of $T$ in $D$.

Edelstein [57] established Theorem 3.1.15. (for $\lambda = \frac{1}{2}$) in a strictly convex Banach space, which is, recently, proved by Diaz and Metcalf [46] for arbitrary $\lambda \in (0,1)$. We will omit the proof.

<u>Theorem 3.1.16.</u> Let $D$ be a closed convex subset of a strictly convex Banach space $X$, $T:D \to D$ a nonexpansive mapping, and suppose $T(D)$ is contained in a compact subset of $D$. Then, for $x_o \in D$, the sequence $\{T^n_\lambda(x_o)\}$, where $T_\lambda:D \to D$ is the mapping defined by

$T_\lambda = \lambda I + (1 - \lambda)T$, $\lambda \in (0,1)$, converges to a fixed point of $T$ in $D$.

Considering a more general mapping $S$ than $T_\lambda$, Kirk [87] has proved the following two results which we state without proof.

<u>Theorem 3.1.17.</u> Let $X$ be a uniformly convex Banach space and $T$ a nonexpansive compact mapping of $X$ into itself which has at least one fixed point. Define the mapping $S:X \to X$ as (K) in Theorem 3.1.2. Then, for each $x_0 \in X$, the sequence $\{S^n(x_0)\}$ converges to a fixed point of $T$.

<u>Theorem 3.1.18.</u> Let $X$ be a uniformly convex Banach space, $D$ a bounded closed convex subset of $X$, and $T$ a nonexpansive mapping of $D$ into $D$. Define the mapping $S:D \to D$ as (K) in Theorem 3.1.2. Suppose $T$ has at most one fixed point $p$ in $D$. Then, for each $x_0 \in D$, the sequence $\{S^n(x_0)\}$ converges weakly to $p$ in $D$.

We state without proof the following result due to Browder and Petryshyn [32].

<u>Theorem 3.1.19.</u> Let $T:X \to X$ be a nonexpansive asymptotically regular mapping in a Banach space $X$. Suppose the set $F(T)$ of fixed points of $T$ is non-empty. Suppose $T$ satisfies the following condition:

($\Theta$) $(I - T)$ maps bounded closed subsets of $X$ into closed subsets of $X$.

Then, for each $x_0 \in X$, the sequence $\{T^n(x_0)\}$ converges strongly to some point in $F(T)$.

<u>Remark 3.1.5.</u> Let $T_\lambda = \lambda I + (1 - \lambda)T$, where $\lambda \in (0,1)$. Then $T$ satisfies condition ($\Theta$) iff $T_\lambda$ also does. To see this observe that $I - T_\lambda = (1 - \lambda)(I - T)$.

The following corollary of Theorem 3.1.19. follows from Remark 3.1.5.

and Theorem 3.1.1.

Corollary 3.1.2. Let $T$ be a nonexpansive mapping of a uniformly convex Banach space $X$ into itself. Suppose that the set $F(T)$ of fixed points of $T$ is non-empty. Suppose $T$ satisfies the following condition:

($\Theta$) $(I - T)$ maps bounded closed subsets of $X$ into closed subsets of $X$.

Then, for each $x_o \in X$, the sequence $\{x_{n+1}\} = \{T^n_\lambda(x_o)\}$ determined by the iteration method

$$x_{n+1} = \lambda x_n + (1 - \lambda)T(x_n) , \quad \lambda \in (0,1), \; n = 0,1,2, \ldots ,$$

converges strongly to a fixed point of $T$.

Remark 3.1.6. Since every completely continuous mapping satisfies hypothesis ($\Theta$) (see [32]), with the use of Theorem 3.1.1. one obtains Theorem 3.1.5. (Schaefer [119] for arbitrary $\lambda \in (0,1)$ and Krasnoselskii [90] for $\lambda = \frac{1}{2}$) as a corollary to Theorem 3.1.19.

Following [105], we define the following class of operators.

Definition 3.1.4. A continuous mapping $T$ from a Banach space into itself is said to be demicompact if every bounded sequence $\{x_n\}$, such that $\{(I - T)x_n\}$ converges strongly, contains a strongly convergent subsequence $\{x_{n_j}\}$.

Petryshyn [105] has proved that the class of demicompact operators contains, among others, all compact (completely continuous) operators.

We state the following result without proof.

Theorem 3.1.20. (de Figueiredo [44], p.47) A demicompact mapping $T$ of a Banach space $X$ into itself satisfies condition ($\Theta$).

Remark 3.1.7. It was stated in [32] that the converse of Theorem 3.1.20. holds. But, in general, this is not true. For example, the mapping $T = I$ satisfies trivially condition $(\Theta)$, but it is not demicompact [44].

We reframe a result of Diaz and Metcalf [46] in terms of quasi-nonexpansive mappings as follows:

Theorem 3.1.21. Let $T:X \to X$ be a continuous quasi-nonexpansive asymptotically regular mapping of a Banach space $X$ into itself. Suppose

$(\Theta')$ the (continuous) real-valued function $f$, defined by $f(x) = ||x - T(x)||$ for $x \in X$, maps bounded closed subsets of $X$ into closed sets of real numbers.

Then, for $x_o \in X$, the sequence $\{T^n(x_o)\}$ converges to some point in $F(T)$.

Remark 3.1.8. It is of interest to note that the condition of nonexpansiveness in Theorem 3.1.19. is weakened in Theorem 3.1.21. But at the same time hypothesis $(\Theta')$ of Theorem 3.1.21. is stronger than hypothesis $(\Theta)$ of Theorem 3.1.19. To see this, suppose, in accordance with $(\Theta')$, that $f$ maps bounded closed subsets of $X$ into closed sets of real numbers. Let $D$ be a bounded closed set in $X$. Then, by $(\Theta')$, the set $f(D) = \{||(I - T)(x)|| \; ; \; x \in D\}$ is a closed set of real numbers. But, the norm function is a continuous function on $X$ to the real numbers, while the set $f(D)$, by $(\Theta')$, is a closed set of real numbers. Consequently, the inverse image of the set $f(D)$, with respect to the norm function, namely, the set $(I - T)(D)$, must be a closed subset of $X$. But this means that every bounded closed set $D$ is mapped by $(I - T)$ into a closed set, which is just hypothesis $(\Theta)$. ([46]).

The next result due to Diaz and Metcalf [46], stated without proof, shows that the hypothesis ($\Theta'$) of Theorem 3.1.21. can be weakened to that of ($\Theta$) of Theorem 3.1.19., without altering the conclusion of Theorem 3.1.21. We again reframe the statement in terms of quasi-nonexpansive mappings.

Theorem 3.1.22. Let $T:X \to X$ be a continuous quasi-nonexpansive asymptotically regular mapping of a Banach space $X$ into itself. Suppose $T$ satisfies the following:

($\Theta$) $(I - T)$ maps bounded closed subsets of $X$ into closed subsets of $X$.

Then, for $x_o \in X$, the sequence $\{T^n(x_o)\}$ converges to some point in $F(T)$.

We give the following result for quasi-nonexpansive mappings.

Theorem 3.1.23. Let $T$ be a continuous quasi-nonexpansive mapping of a uniformly convex Banach space $X$ into itself. Also if $T$ satisfies condition:

($\Theta$) $(I - T)$ maps bounded closed subsets of $X$ into closed subsets of $X$.

Define the mapping $T_\lambda:X \to X$ by $T_\lambda = \lambda I + (1 - \lambda)T$, $\lambda \in (0,1)$. Then, for $x_o \in X$, the sequence $\{T^n_\lambda(x_o)\}$ converges strongly to some point in $F(T)$.

Proof. It follows from Theorem 3.1.6. that $T_\lambda$ is a continuous quasi-nonexpansive asymptotically regular mapping and $F(T) = F(T_\lambda)$. Moreover, Remark 3.1.5. implies that $T_\lambda$ satisfies hypothesis ($\Theta$). Hence $T_\lambda$ satisfies all the hypotheses of Theorem 3.1.22., thus result follows.

Remark 3.1.9. Theorem 3.1.23. is true, in particular, if we replace hypothesis ($\Theta$) by the complete continuity or demicompactness of mapping $T$, since completely continuous or demicompact mappings always satisfy hypothesis ($\Theta$).

## 3.2. Measure of Noncompactness and Some Fixed Point Theorems.

The concept of measure of noncompactness is due to Kuratowski [91]. Let D be a bounded subset of a metric space X (in case X is a Banach space we always mean a real Banach space). Following [91] we define $\alpha(D)$, the (set) measure of noncompactness of D, as follows:

<u>Definition 3.2.1.</u> By the real number $\alpha(D)$ we denote the infimum of all numbers $\varepsilon > 0$ such that D admits a finite covering consisting of subsets of diameter less than $\varepsilon$.

Some useful properties of $\alpha$ are the following ones (see Nussbaum [101] for detailed discussion and proof).

<u>Theorem 3.2.1.</u> (Nussbaum [101], Darbo [40]). Let A and B be bounded subsets of a metric space X, and let

$$B_r(A) = \{x \in A : d(x,A) < r\}.$$

Then, we have

(i) $0 < \alpha(A) \leq \delta(A)$,

(ii) $\alpha(A) > 0$ and $\alpha(A) = 0$ iff A is precompact,

(iii) $\alpha(\lambda A) = |\lambda| \, \alpha(A)$, where $\lambda$ is a real number,

(iv) if $A \subset B$ then $\alpha(A) \leq \alpha(B)$,

(v) $\alpha(B_r(A)) \leq \alpha(A) + 2r$,

(vi) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{Co}\ A)$,

(vii) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$,

and (viii) $\alpha(A \cup B) = \max. \{\alpha(A), \alpha(B)\}$.

We state without proof the following result due to Nussbaum [101].

Theorem 3.2.2. Let $X$ be an infinite dimensional Banach space, let $B = \{x \in X : ||x|| \leq 1\}$ and $S = \{x \in X : ||x|| = 1\}$. Then $\alpha(B) = \alpha(S) = 2$.

We also define $\chi(D)$, the measure of noncompactness of $D$ used by Sadovsky [118], to be

$$\chi(D) = \inf. \{\varepsilon > 0 : D \text{ admits a finite } \varepsilon\text{-net}\}.$$

Although these two measures have a good deal in common but $\chi(D)$ does not have all the properties of $\alpha(D)$ since $\chi(D)$ does not depend intrinsically on the bounded set $D$. In fact, if $D \subset B \subset X$, where $X$ is a metric space and $D$ is bounded subset, then the $\alpha(D)$ is independent of whether $D$ is considered as a subset of $B$ or of $X$. But, in general, this is not true for $\chi(D)$, as the following example shows ([64], see also [101]):

Example 3.2.1. Let $D$ be an infinite orthonormal system of a Hilbert space $H$. We have $\chi(D) = 1$ if we consider $D$ as a subset of $H$ and $\chi(D) = \sqrt{2}$ if we consider $D$ as a subset of itself.

Closely associated with the notion of the measure of noncompactness, is the concept of 'k-set-contraction', introduced by Kuratowski [91] and further studied in [40], [118], [101] (see Nussbaum [101] for other references).

Definition 3.2.2. If $G$ is a subset of $X$ and $T$ a continuous mapping of $G$ into $X$, then $T$ is said to be k-set-contraction if

$$\alpha(T(D)) \leq k \ \alpha(D),$$

for some $k \geq 0$ and for any bounded set $D$ in $G$.

Remark 3.2.1. It is easy to see that mappings of Lipschitz's class are k-set-contractions with the same constant $k$. The class of k-set-contractions with $k < 1$ contains the class of completely continuous mappings and contraction mappings. It was shown in [101]that a more general example of a k-set-contraction with $k \leq 1$, is the class of semi-contractive type mappings (see Definition 3.2.6., also see [108]).

We state without proof the following elementary properties about k-set-contractions.

Theorem 3.2.3. (Nussbaum [101]).

(a) Let $X_i$, $i = 1,2,3$, be metric spaces. Assume that

$$T_1 : X_1 \to X_2 \quad \text{is a } k_1\text{-set-contraction,}$$

and

$$T_2 : X_2 \to X_3 \quad \text{is a } k_2\text{-set-contraction.}$$

Then $T_2T_1$ is a $k_1k_2$-set-contraction.

(b) Let $X$ be a metric space and $Y$ a Banach space. Assume that

$$T_1 : X \to Y \quad \text{is a } k_1\text{-set-contraction,}$$

and

$$T_2 : X \to Y \quad \text{is a } k_2\text{-set-contraction.}$$

Then $T_1 + T_2 : X \to Y$ is a $(k_1 + k_2)$-set-contraction.

We state without proof the following result for k-set-contraction mappings.

Theorem 3.2.4. (Darbo [40]) Let $D$ be a bounded closed convex subset of a Banach space $X$, and $T:D \to D$ a k-set-contraction, $k < 1$. Then $T$ has a fixed point in $D$.

The following generalization of above result is due to Nussbaum [101] which we state without proof.

Theorem 3.2.5. Let $D$ be a bounded closed convex subset of a Banach space and $T:D \to D$ a continuous mapping. Let $D_1 = \overline{co}(T(D))$, and

$D_n = \overline{Co}(T(D_{n-1}))$, for $n > 1$. Furthermore, assume that $\alpha(D_n) \to 0$, as $n \to \infty$. Then $F(T) \neq \phi$.

Following [63], we define

Definition 3.2.3. A continuous mapping $T$ of $G \subset X$ into $X$ is said to be densifying, if for any bounded set $D \subset G$ such that $\alpha(D) > 0$,

$$\alpha(T(D)) < \alpha(A).$$

Remarks 3.2.2.

(1) Using the notion of the measure of noncompactness $X$; Sadovsky [118] defines the concept of 'condensing mappings' same as definition 3.2.3.

(2) Obviously, every k-set-contraction with $k < 1$ is a densifying mapping but the converse is not true, as can be seen from the following example ([132]):

Example 3.2.2. Let $\phi:[0, +\infty] \to [0, +\infty]$ be a right continuous nondecreasing function such that $\phi(r) < r$ for $r > 0$, and let $T:X \to X$ satisfies $||T(x) - T(y)|| \leq \phi(||x - y||)$, for every $x,y \in X$. Then $T$ is densifying. On the otherhand $T$ is not a k-set-contraction.

Note that contraction mappings and completely continuous mappings are densifying; also sums of contraction mappings and completely continuous mappings defined on Banach spaces are densifying.

Definition 3.2.4. A continuous mapping $T$ of $G \subset X$ into $X$ is said to be 1-set-contraction, if

$$\alpha(T(D)) \leq \alpha(D)$$

for any bounded set $D \subset G$.

<u>Remark 3.2.3.</u> Obviously, every k-set-contraction mapping with $k \leq 1$ is 1-set-contraction. In particular, the class of densifying maps and the class of nonexpansive maps are contained in the class of 1-set-contraction.

We state without proof the following basic result.

<u>Theorem 3.2.6.</u> (Furi and Vignoli [64], Nussbaum [102]).

Let $D$ be a nonempty bounded closed convex subset of a Banach space $X$, and let $T:D \to D$ be a densifying mapping. Then $T$ has at least one fixed point in $D$.

<u>Remark 3.2.4.</u> Furi and Vignoli [63] were first to introduce formally the notion of densifying mappings. It seems that Theorem 3.2.6. has been established independently by Furi and Vignoli [64] and Nussbaum [102]. In case $T$ is a condensing mapping, Theorem 3.2.6. has been established by Sadovsky [118].

Recently, Petryshyn [108] has proved the following generalization of Theorem 3.1.15. and Theorem 3.1.16., which we state without proof.

<u>Theorem 3.2.7.</u> Let $X$ be a strictly convex Banach space, $D$ a bounded closed convex subset of $X$, and $T$ be a densifying nonexpansive mapping of $D$ into $D$. For each constant $\lambda$ with $0 < \lambda < 1$, let

$$T_\lambda = \lambda I + (1 - \lambda)T.$$

Then, for each $x_o$ in $D$, the sequence $\{x_{n+1}\} = \{T_\lambda^n(x_o)\}$ determined by the iteration method

$$x_{n+1} = \lambda x_n + (1 - \lambda)T(x_n), \quad n = 0,1,2, \dots ; \quad x_o \in D,$$

converges strongly to a fixed point of $T$ in $D$.

<u>Remark 3.2.5.</u> Theorem 3.2.7. certainly holds if the nonexpansive mapping

$T$ of $D$ into $D$ is of the form $T = H + C$, with $C$ completely continuous on $D$ and $H$ such that $||H(x) - H(y)|| \leq q||x - y||$ for all $x$ and $y$ in $D$ and some $q$ with $0 < q < 1$. We note that in case $T = H + C$ with $H = 0$ or $q = 0$, Theorem 3.2.7. yields Theorem 3.1.16. In case $T = H + C$ with $H = 0$ or $q = 0$, and $\lambda = \frac{1}{2}$, Theorem 3.2.7. yields the result of Edelstein [57]. In case $0 < q \leq 1$, Theorem 3.2.7. for $T = H + C$ improves the corresponding result in [110].

The following generalization of Theorem 3.2.7. is due to Singh [125] which we state without proof.

<u>Theorem 3.2.8.</u> Let $X$ be a Banach space, $D$ a bounded closed convex subset of $X$, and $T$ be a densifying mapping of $D$ into $D$. Define a mapping $T_\lambda : D \to D$ by $T_\lambda = \lambda I + (1 - \lambda)T$, $0 < \lambda < 1$. Let $T_\lambda$ be such that

(i) whenever $p \in F(T_\lambda)$, $||T_\lambda(x) - p|| < ||x - p||$ for all $x \in D - F(T_\lambda)$.

Then, for each $x_0$ in $D$, the sequence $\{T_\lambda^n(x_0)\}$ converges strongly to a fixed point of $T$ in $D$.

We prove the following general result.

<u>Theorem 3.2.9.</u> Let $D$ be a bounded closed subset of a Banach space $X$ and $T : D \to D$ a densifying mapping such that $F(T)$ is nonempty. Define a mapping $T_\lambda : D \to D$ by

$$T_\lambda = \lambda I + (1 = \lambda)T, \qquad 0 < \lambda < 1,$$

such that

(a) whenever $p \in F(T_\lambda)$, $||T_\lambda(x) - p|| < ||x - p||$, for all $x \in D - F(T_\lambda)$.

Then, for each $x_0$ in $D$, the sequence $\{T_\lambda^n(x_0)\}$ converges strongly to a fixed point of $T$.

<u>Proof.</u> It is obvious that $F(T) = F(T_\lambda)$. Therefore $F(T_\lambda)$ is nonempty since $F(T)$ is nonempty by assumption. Also $T_\lambda$ is a densifying mapping of $D$ into $D$, this being a consequence of the facts that $T$ is densifying and that $\lambda$ lies in $(0,1)$.

In view of the inequality (a) above and Corollary 3.1.1., to show that the sequence $\{T_\lambda^n(x_o)\}$ converges strongly to a point in $F(T)$, it suffices to show that $\{T_\lambda^n(x_o)\}$ contains a convergent subsequence $\{T_\lambda^{n_j}(x_o)\}$ . Now, for each $x_o$ in $D$, the sequence $S_o \equiv \{T^n(x_o):n=0,1,2,..\}$ is bounded and its transformed into the sequence $S_1 \equiv \{T^n(x\ ): n=1,2,...\}$. Hence $\alpha(S_o) = \alpha(S_1)$, and therefore $\alpha(S_o) = 0$, since $T$ is densifying. Thus sequence $\{T_\lambda^n(x_o)\}$ contains a convergent subsequence.

Thus all the hypotheses of Corollary 3.1.1. are fulfilled, and $F(T) = F(T_\lambda)$, hence the result follows.

<u>Remarks 3.2.6.</u>

(1) We can obtain a number of well-known results such as Theorem 3.2.8., Theorem 3.2.7., Theorem 3.1.16. ([46] for arbitrary $\lambda \in (0,1)$ and [57] for $\lambda = \frac{1}{2}$), and Theorem 3.1.15. ([119] for arbitrary $\lambda \in (0,1)$ and [90] for $\lambda = \frac{1}{2}$), as corollaries to Theorem 3.2.9. as follows (we will omit the detailed discussion): Since with the given hypotheses in any one of these theorems, $T$ is a densifying mapping from a bounded closed subset of a Banach space into itself. Moreover $F(T)$ is nonempty and the mapping $T_\lambda$, $0 < \lambda < 1$, always satisfies hypothesis (a) of Theorem 3.2.9. Thus all the hypotheses of Theorem 3.2.9. are fulfilled, hence the result follows.

(2) We observe that Theorem 3.2.9. is valid for any $x_o \in \overline{D}$, even if $D$ is a bounded open subset of $X$ and $T:\overline{D} \to \overline{D}$ is a densifying mapping.

We give the following generalization of Theorem 3.2.7. in which $T_\lambda$ is required to be nonexpansive only about its fixed points.

Theorem 3.2.10. Let $X$ be a strictly convex Banach space, $D$ a bounded closed convex subset of $X$, and $T$ be a densifying mapping of $D$ into $D$. Define a mapping $T_\lambda : D \to D$ by

$$T_\lambda = \lambda I + (1 - \lambda)T, \qquad 0 < \lambda < 1,$$

such that

(i) whenever $p \in F(T_\lambda)$, $||T_\lambda(x) - p|| \leq ||x - p||$ for all $x \in D$.

Then, for each $x_o$ in $D$, the sequence $\{T_\lambda^n(x_o)\}$ converges strongly to a fixed point of $T$ in $D$.

Proof. It follows from Theorem 3.2.6. that the set $F(T)$ is nonempty. It is obvious that $F(T) = F(T_\lambda)$. It is also easy to see that $T_\lambda$ is a densifying nonexpansive map of $D$ into $D$, this being a consequence of the facts that $T$ is densifying nonexpansive and that $\lambda \in (0,1)$. Next, we show that strict convexity of $X$ and condition (i) implies

$$||T_\lambda(x_o) - p|| < ||x_o - p||, \qquad p \in F(T_\lambda) \text{ and } x_o \in D - F(T_\lambda).$$

If $x_o \in D - F(T_\lambda)$, i.e. $x_o$ is not a fixed point of $T_\lambda$ then $x_o \neq p$, and the open line segment joining the point $x_o$ and $T_\lambda(x_o)$ must, by strict convexity, be contained in the open sphere of radius $||x_o - p||$ and centered at $p$. Since $T_\lambda(x_o)$ is an interior point of this line segment, one has

$$||T_\lambda(x_o) - p|| < ||x_o - p||.$$

Thus all the hypotheses of Theorem 3.2.9. are fulfilled, and hence result follows.

Remark 3.2.7. Theorem 3.2.10. generalizes Theorem 3.2.7. in the following sense:

With all the hypotheses of Theorem 3.2.7., we have $F(T_\lambda) = F(T) \neq \phi$, and $T_\lambda$ nonexpansive together imply $T_\lambda$ satisfies condition (i) of Theorem 3.2.10. Thus all the hypotheses of Theorem 3.2.10. are fulfilled and hence result follows. But with all the hypotheses of Theorem 3.2.10., mapping $T_\lambda$ is not nonexpansive on $D$.

We consider a more general mapping $S$ of type Kirk [87], and give the following general unified approach on convergence of the sequences of iterates of $S$.

Theorem 3.2.11. Let $X$ be a Banach space, $D$ a bounded closed convex subset of $X$, and $T$ be a densifying nonexpansive mapping of $D$ into $D$. Define the mapping $S:D \to D$ by

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots\ldots + \alpha_k T^k,$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. Let $S$ be such that

(A) whenever $p \in F(S)$, $||S(x) - p|| < ||x - p||$ for all $x \in D - F(S)$.

Then, for each $x_0$ in $D$, the sequence $\{S^n(x_0)\}$ converges strongly to a fixed point of $T$ in $D$.

Proof. It follows from Theorem 3.2.6. that the set $F(T)$ is nonempty and from Theorem 3.1.2. that $F(T) = F(S)$. It is also easy to see that $S$ is a densifying nonexpansive mapping of $D$ into $D$, this being consequence of the facts that $T$ is densifying nonexpansive mapping and that $\alpha_i \geq 0$, $\alpha_1 > 0$, $\sum_{i=0}^{k} \alpha_i = 1$

In view of the inequality (A) and Corollary 3.1.1. to show that the sequence $\{S^n(x_o)\}$ converges strongly to a point in $F(T)$, it suffices to show that the sequence $\{S^n(x_o)\}$ contains a convergent subsequence $\{S^{n_j}(x_o)\}$. Now, for each $x_o$ in $D$, the sequence $S_o \equiv \{S^n(x_o) : n = 0,1,2, ..\}$ is bounded and it's transformed into the sequence $S_1 \equiv \{S^n(x_o) : n = 1,2, ..\}$. Hence $\alpha(S_o) = \alpha(S_1)$, and therefore $\alpha(S_o) = 0$, since $S$ is densifying. Thus the seqüence $\{S^n(x_o)\}$ contains a convergent subsequence $\{S^{n_j}(x_o)\}$.

Thus all the hypotheses of Corollary 3.1.1. are fulfilled, and $F(T) = F(T_k)$, hence the result follows.

Remark 3.2.8. It is of interest to observe that in case $\alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$, we can delete the nonexpansiveness of mapping $T$ in Theorem 3.2.11., since then, obviously, we have $F(T) = F(S)$.

We obtain some new and other well-known theorems as corollaries to Theorem 3.2.11., in the following way.

Corollary 3.2.1. Let $X$ be a strictly convex Banach space, $D$ and $T$ as defined in Theorem 3.2.11. Define the mapping $S:D \to D$ by

$$S = \alpha_o I + \alpha_1 T + \alpha_2 T^2 + \ldots\ldots + \alpha_k T^k,$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. Then, for each $x_o$ in $D$, the sequence $\{S^n(x_o)\}$ converges strongly to a fixed point of $T$ in $D$.

Proof. Obviously, $S$ is nonexpansive and $F(S)$ is non-empty. Since $X$ is a strictly convex Banach space, using the argument of Theorem 3.2.10., it can be seen that $S$ satisfies hypothesis (A) and hence all the hypotheses of Theorem 3.2.11. Therefore result follows from Theorem 3.2.11.

Remark 3.2.9. Theorem 3.2.11. and Corollary 3.2.1. certainly hold if the nonexpansive mapping $T$ of $D$ into $D$ is of the form $T = G + H$, where $G$

compact (completely continuous) on D and H such that $||H(x) - H(y)|| \leq q||x - y||$ for all x and y in D and some $q \in (0,1)$.

Corollary 3.2.2. Theorem 3.2.8. becomes a particular case of Theorem 3.2.11. as follows: In case $\alpha_o = \lambda, \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$ in Theorem 3.2.11., we have $S = T_\lambda$, hence the result follows from Theorem 3.2.11. and Remark 3.2.8.

Corollary 3.2.3. Theorem 3.2.7. can be derived from Theorem 3.2.11. as follows: Take $\alpha_o = \lambda, \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$ in Corollary 3.2.1., then we have $S = T_\lambda$ and hence Theorem 3.2.7. follows from Corollary 3.2.1.

Corollary 3.2.4. Theorem 3.2.10. becomes a particular case of Theorem 3.2.11. as follows: In case $\alpha_o = \lambda, \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$ in Theorem 3.2.11., we have $S = T_\lambda$. Moreover, strict convexity of X and hypothesis (i) of Theorem 3.2.10. imply hypothesis (A) of Theorem 3.2.11. Hence the result follows from Theorem 3.2.11. and Remark 3.2.8.

Corollary 3.2.5. Theorem 3.1.16. can be derived from Theorem 3.2.11. as follows: Take $\alpha_o = \lambda, \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$ in Corollary 3.2.1. then we have $S = T_\lambda$. Hence the result follows from Corollary 3.2.1. and Remark 3.2.9. with $H = 0$.

If in addition to above, we assume that $\alpha_o = \lambda = \frac{1}{2}$, we get a result due to Edelstein [57].

Corollary 3.2.6. Theorem 3.1.15. can be derived from Theorem 3.2.11. as follows: Take $\alpha_o = \lambda, \alpha_2 = \alpha_3 = \ldots\ldots = \alpha_k = 0$ in Corollary 3.2.1., then we have $\{S^n(x_o)\} = \{x_{n+1}\}$. Since every uniformly convex Banach space is strictly convex, hence the result follows from Corollary 3.2.1. and Remark 3.2.9. with $H = 0$.

If in addition to above, we assume that $\alpha_o = \lambda = \frac{1}{2}$, we get a result due to Kransnoselskii [90].

Remarks 3.2.10.

(1) In view of Remark 3.2.9., Theorem 3.2.11 and Corollary 3.2.1., for $H = 0$ improve Theorem 3.1.17.

(2) In view of Remark 3.2.9., Theorem 3.2.11. and Corollary 3.2.1. for $T = H + G$ with $0 < q \leq 1$ improve the corresponding result in Petryshyn and Tucker [110].

We will now discuss the mappings with a boundary condition. Let $X$ be a real Banach space and $D$ an open bounded subset of $X$, with $\overline{D}$ and $\dot{D}$ denoting its closure and boundary respectively. In particular, we will denote by

$$B = \{x \in X : ||x|| < r\}$$

the open ball about the origin,

$$\dot{B} = \{x \in X : ||x|| = r\}$$

its boundary and $\overline{B} = B \cup \dot{B}$ its closure.

Following [29], we define

Definition 3.2.5. Let $X$ be a Banach space, $D$ a subset of $X$, $T$ a mapping of $D$ into $X$. Then $T$ is said to be semicontractive if there exists a mapping $V$ of $D \times D$ into $X$ such that $T(x) = V(x,x)$ for $x$ in $D$, while

(i) for each fixed $y$ in $D$, $V(\cdot,y)$ is nonexpansive from $D$ to $X$,

(ii) for each fixed $x$ in $D$, $V(x,\cdot)$ is strongly continuous from $D$ to $X$, uniformly for $x$ in bounded subsets of $D$.

Following [29], [108], we also define

Definition 3.2.6. The mapping $T:D \to X$ is of semicontractive type with constant $k \leq 1$ if there exists a continuous mapping $V$ of $D \times D$ into $X$ such that $T(x) = V(x,x)$ for all $x$ in $D$, while

$$||V(x,z) - V(y,z)|| \leq k||x - y||, \quad x,y,z \in D,$$

and the map $x \to V(\cdot,x)$ is completely continuous from $D$ to the space of maps from $D$ to $X$ with the uniform metric.

Imposing the well-known Leray-Schauder condition Browder [29] has proved the following two results which we state without proof.

Theorem 3.2.12. Let $X$ be a uniformly convex Banach space, $D$ a bounded closed convex subset of $X$ with 0 in the interior of $D$. Let $T$ be a semicontractive mapping of $D$ into $X$ such that for each $x$ in $\dot{D}$, $T(x) \neq \lambda x$ for any $\lambda > 1$. Then $T$ has a fixed point in $D$.

Theorem 3.2.13. Let $X$ be a Banach space, $D$ a bounded closed convex subset of $X$ having 0 in the interior, $T$ a mapping of $D$ into $X$ such that for each $x$ in $\dot{D}$, $T(x) \neq \lambda x$ for any $\lambda > 1$. Suppose that $T$ is a semicontractive type with constant $k$ such that

(a) If $k < 1$, $T$ has a fixed point in $D$.

(b) If $k \leq 1$ and $(I - T)D$ is closed in $X$, then $T$ has a fixed point in $D$.

Recently, a number of interesting results have been given for the class of densifying maps and the class of 1-set-contraction $T$ under the assumption that $T$ satisfies the weaker boundary condition:

$$(\Pi \tfrac{\leq}{1}) : \text{ If } T(x) = \alpha x \text{ for some } x \text{ in } \dot{D}, \text{ then } \alpha \leq 1,$$

for example see Petryshyn [108], [109], Nussbaum [101], Webb [133], Edmunds and Webb [59], and others. These results have been used to deduce a number of new, as well as other well-known fixed point theorems for various classes of mappings which have been extensively studied recently.

Remark 3.2.11. In fact, condition $(\Pi\frac{\leq}{1})$ is equivalent to Leray-Schauder condition - for every $x$ in $\dot{D}$, $T(x) \neq \alpha x$, for any $\alpha > 1$.

Definition 3.2.7. Let $B$ be an open ball in $X$ with center $0$ and radius $r > 0$. A mapping $R:X \to \bar{B}$ defined by the formula

$$R(x) = \begin{cases} x & \text{if } ||x|| \leq r \\ \dfrac{rx}{||x||} & \text{if } ||x|| \geq r, \end{cases}$$

is said to be the radial retraction of $X$ onto $\bar{B}$.

The following lemma is due to Nussbaum [102], which we state without proof.

Lemma 3.2.1. Let $X$ be a Banach space and $B$ the open unit ball of $X$ about the origin. Then the radial retraction $R:X$ $\bar{B}$ is a 1-set-contraction.

Theorem 3.2.6. for densifying mappings admits the following practically useful generalization in case D is a ball.

Theorem 3.2.14. (Petryshyn [108]). Let $B$ be an open ball about the origin in a general Banach space $X$. If $T:\bar{B} \to X$ is a densifying mapping (and, in particular, a k-set-contraction with $k < 1$) which satisfies the boundary condition.

$(\Pi\frac{\leq}{1})$: If $T(x) = \alpha x$ for some $x$ in $\dot{B}$, then $\alpha \leq 1$, then $F(T)$, the set of fixed points of $T$ in $\bar{B}$, is nonempty and compact.

Proof. Since every k-set-contraction with $k < 1$ is a densifying map, it suffices to prove Theorem 3.1.14. for the case when $T$ is densifying.

Let $B$ be an open ball in $X$ with center $0$ and radius $r > 0$. Let $R$ be the retraction of $X$ onto $\bar{B}$ as given in Lemma 3.2.1. Then $R$ is a 1-set-contraction of $X$ onto $\bar{B}$. Now, if for all $x$ in $\bar{B}$ we define the mapping $T_1(x) = R(T(x))$, then $T_1$ is a continuous map of $\bar{B}$ into $\bar{B}$ which is also densifying: since, $T:\bar{B} \to X$ is densifying, $R:X \to \bar{B}$ is a 1-set-contraction and, therefore, $\alpha(T_1(\bar{B})) \leq \alpha T(\bar{B}) < \alpha(\bar{B})$. Hence it follows from Theorem 3.2.6. that $T_1$ has at least one fixed point $x_o$ in $\bar{B}$. But then $x_o$ is also a fixed point of $T$. Indeed, if $x_o \in B$, then $T(x_o) = x_o$, since the assumption of the equality $T(x_o) = \frac{||T(x_o)||}{r} x_o$ would contradict the fact that $||x_o|| < r$. If $x_o \in \dot{B}$ and $x_o$ is not a fixed point of $T$, then $\alpha = \dfrac{||T(x_o)||}{r} > 1$, in contradiction to condition $(\Pi \frac{\leq}{1})$. Thus $x_o$ is a fixed point of $T$, and hence $F = F(T)$ is a nonempty set in $\bar{B}$. Since $T$ is continuous, $F$ is obviously a closed subset of $\bar{B}$ such that $T(F) = F$. This also shows that $F$ is compact, for otherwise the assumption $\alpha(F) > 0$ would lead to the contradictory inequality $\alpha(F) = \alpha(T(F)) < \alpha(F)$, which follows from the densifying property of $T$.

<u>Remark 3.2.12.</u> If instead of the boundary condition $(\Pi \frac{\leq}{1})$, we assume that $T$ satisfies condition $(\Pi \frac{<}{1})$ on $\dot{B}$ (i.e., if $T(x) = \alpha x$ for some $x$ in $\dot{B}$, then $\alpha < 1$), then the nonempty compact set $F(T)$ is contained in $B$ and hence lies at the positive distance from $\dot{B}$.

Petryshyn [108] has derived the following corollaries to Theorem 3.2.14. We will omit the proof (see [108] for detailed discussion and proof).

<u>Corollary 3.2.7.</u> Let $T$ be a densifying mapping (and, in particular, a k-set-contraction with $k < 1$) of $\bar{B}$ into $X$, and suppose that $T$

satisfies any one of the following conditions:

(1.1) $T(\overline{B}) \subset \overline{B}$.

(1.2) $T(\dot{B}) \subset \overline{B}$.

(1.3) $||T(x) - x||^2 \geq ||T(x)||^2 - ||x||^2$ for all $x$ in $\dot{B}$.

(14.) $(T(x), J(x)) \leq (x, J(x))$ for all $x$ in $\dot{B}$, where $J$ is a duality mapping of $X$ into the set $2^{X^*}$ of all subsets of $X^*$ such that

$$(J(x), x) = ||x||^2 \text{ and } ||J(x)|| = ||x|| \qquad \text{for all } x \in X.$$

Then the set of fixed points $F(T)$ of $T$ is nonempty and compact.

Remark 3.2.13. In case $X$ is a Hilbert space $H$, then for $J$ we can take the identity mapping $I$ and therefore in this case conditions (1.3) and (1.4) reduce to the condition

$$(T(x), x) \leq (x,x) \qquad \text{for all } x \text{ in } \dot{B},$$

employed by Krasnoselskii [89], Altman [ 2] and others for completely continuous mapping $T$.

The following new comparison result, which may prove to be useful in the solvability of nonlinear equations in $H$, is also valid.

Corollary 3.2.8. Let $T$ be a mapping of $\overline{B}$ into $H$, and $T_o$ a densifying mapping of $\overline{B}$ into $H$, such that

$$(T(x), x) \leq ||x||^2,$$
$$||T(x) - T_o(x)|| \leq ||x - T(x)|| \qquad \text{for all } x \text{ in } \dot{B}.$$

Then $F(T_o) \subset \overline{B}$ is nonempty and compact.

The following special case of Theorem 3.2.14. is useful in application.

Corollary 3.2.9. Let $T = H + G$ be a map from $\overline{B}$ to $X$ such that $H$

$H$ is a contraction on $\overline{B}$ and $G$ is completely continuous on $\overline{B}$. Suppose that $T$ satisfies condition $(\Pi \frac{<}{1})$ on $\dot{B}$. Then $F(T) \neq \phi$ and $F(T)$ is compact.

For semicontractive type maps, Theorem 3.2.14. yields a generalization of Corollary 3.2.9. whose first part (i.e. $F(T) \neq \phi$) has been obtained in [29].

Corollary 3.2.10. If $T:\overline{B} \to X$ is a semicontractive type map with constant $k < 1$ such that $(\Pi \frac{<}{1})$ holds on $\dot{B}$, then $F(T)$ is nonempty and compact.

Remark 3.2.14. Under condition (1.1), Corollary 3.2.7. has been obtained in [40] when $T$ is a k-set-contraction, and in [118] when $T$ is a condensing map, while, under condition (1.2), Corollary 3.2.7. has been obtained in [101] when $T$ is a k-set-contraction with $k < 1$.

Petryshyn [108] has investigated the structure of fixed point sets $F(T)$ of certain densifying maps and demicompact 1-set-contractions $T:\overline{D} \to X$. We state without proof the following result due to Petryshyn [108] in case $D$ is a ball.

Theorem 3.2.15. Let $B$ be an open ball about the origin in a general Banach space $X$. Suppose $T$ is a densifying mapping of $B$ into $X$ which satisfies condition $(\Pi \frac{<}{1})$ on $\dot{B}$; i.e.

$$(\Pi \tfrac{<}{1}): \quad \text{If } T(x) = \alpha x \quad \text{for some } x \text{ in } \dot{B}, \text{ then } \alpha < 1.$$

Suppose there exists a sequence $\{T_n\}$ of densifying mappings of $\overline{B}$ into $X$ such that

(a) $\delta_n = \sup \{||T_n(x) - T(x)|| : x \in \overline{B}\} \to 0$, as $n \to \infty$,

and (b) the equation $x = T_n(x) + y$ has at most one solution if $||y|| \leq \delta_n$.

Then $T$ has a continuum $F(T)$ for its set of fixed point in $B$.

Theorem 3.2.15. is obtained as a corollary of the following more general result which we state without proof.

Theorem 3.2.16. (Petryshyn [108]). Let $B$ be an open ball about the origin in a general Banach space $X$. Suppose $T$ is a 1-set-contraction of $\overline{B}$ into $X$ which is demicompact and which satisfies condition $(\Pi_1^{\leq})$ on $\dot{B}$. Suppose further that there exists a sequence of densifying mappings $\{T_n\}$ of $\overline{B}$ into $X$ such that the hypotheses (a) and (b) of Theorem 3.2.15. hold. Then $T$ has a continuum $F(T) \subset B$ for its set of fixed points.

Remark 3.2.15. Since every densifying mapping is 1-set-contraction and demicompact, Theorem 3.2.15. follows as a special case of Theorem 3.2.16.

In his study of k-set-contractions with $k \leq 1$, and under certain additional conditions on $D$ and/or $T$, Nussbaum [101] succeeded in defining the notions of fixed point index for $T$ and of topological degree for $(I - T)$. Nussbaum [101] used these in obtaining a number of interesting results, and, in particular, in generalizing the fixed point theorems of Darbo [40], Sadovsky [118], Browder [29], but under somewhat stronger "boundary conditions" (e.g., $T(\dot{D}) \subset \overline{D}$). Petryshyn [108], [109], used the degree argument of [101] to obtain more general fixed point theorems for certain densifying maps and 1-set-contractions under the weaker boundary condition $(\Pi_1^{\leq})$. In fact, Petryshyn [108] has proved the following generalization of Theorem 3.2.14. which we state without proof.

Theorem 3.2.17. If $D$ is a bounded open subset of a Banach space $X$ with $0$ in $D$ and $T$ a densifying mapping of $\overline{D}$ into $X$ which satisfies the boundary condition $(\Pi_1^{\leq})$ on $\dot{D}$, then $F(T) \subset \overline{D}$ is nonempty and compact.

Using Theorem 3.2.17., Petryshyn [108] proved the following general fixed point theorem. We omit the proof.

<u>Theorem 3.2.18.</u> Let $D$ be a bounded open subset of a Banach space $X$ with $0$ in $D$ and let $T:\overline{D} \to X$ be a 1-set-contraction satisfying $(\Pi^{\leq}_{1})$ on $\dot{D}$. Then, if $(I - T)(\overline{D})$ is closed, $F(T) \neq \phi$. In particular, if $T$ is demicompact and 1-set-contraction, then $F(T)$ is nonempty and compact.

<u>Remarks 3.2.16.</u>

(1) The set $(I - T)(\overline{D})$, is certainly closed/if $T$ is densifying and, in particular, if $T$ is k-set-contraction with $k < 1$.

(2) If $D$ is also convex, then condition $(\Pi^{\leq}_{1})$ holds on $\dot{D}$ if $T(\dot{D}) \subseteq \overline{D}$ and, in particular, if $T(\overline{D}) \subseteq \overline{D}$.

In case $0 \notin D$, the following generalization of Theorem 3.2.18. holds.

<u>Theorem 3.2.19.</u> (Petryshyn [108]).Let $D$ be a bounded open subset of a Banach space $X$ and $T:\overline{D} \to X$ a 1-set-contraction such that $T$ satisfies any one of the following conditions:

(a) There exists an $x_0$ in $D$ such that $T(x) - x_0 = \alpha(x - x_0)$ holds for some $x$ in $\dot{D}$, then $\alpha \leq 1$.

(b) $D$ is convex and $T(\dot{D}) \subseteq \overline{D}$.

Then, if $(I - T)(\overline{D})$ is closed, we have $F(T) \neq \phi$. In particular, if $T$ is demicompact and 1-set-contraction, then $F(T)$ is nonempty and compact.

As a consequence of Theorem 3.2.9., Theorem 3.2.18, and Theorem 3.2.19, we give the following result on convergence of the sequences of iterates for de... densifying maps.

<u>Theorem 3.2.20.</u> Let $D$ be a bounded open subset of a Banach space $X$ and

$T:\bar{D} \to \bar{D}$ a densifying mapping such that $T$ satisfies any one of the following conditions:

(a') If $0 \in D$, $(\Pi \frac{\leq}{1})$ holds on $\dot{D}$.

(b') If $0 \notin D$, there exists an $y_o$ in $D$ such that if $T(x) - y_o = \alpha(x - y_o)$ holds for some $x$ in $\dot{D}$, then $\alpha \leq 1$.

(c') If $0 \notin D$, $D$ is convex.

Define a mapping $T_\lambda : \bar{D} \to \bar{D}$ by $T_\lambda = \lambda I + (1 - \lambda)T$, $0 < \lambda < 1$. Let $T_\lambda$ be such that

(a) whenever $p \in F(T_\lambda)$, $||T_\lambda(x) - p|| < ||x - p||$ for all $x \in \bar{D} - F(T_\lambda)$.

Then, for each $x_o$ in $D$, the sequence $\{T_\lambda^n(x_o)\}$ converges strongly to a fixed point of $T$ in $\bar{D}$.

Proof. By Theorem 3.2.9., it suffices to show that each of the conditions (a'), (b') and (c') along with the other hypotheses imply $F(T) \neq \phi$. Indeed, with the given hypotheses if follows from Theorem 3.2.18, Theorem 3.2.19., and Remarks 3.2.16. that (a'), (b') and (c'), each separately implies $F(T) \neq \phi$. Therefore $F(T_\lambda)$ is nonempty. Hence result follows from Theorem 3.2.9. and Remark 3.2.6(2).

Now, we shall discuss some of the applications of fixed point theory. Many applications of fixed point theorems occur in differential and integral equations, nonlinear vibrations, calculus of variation, optimal control theory, nonlinear optimization, nonlinear approximation and many other fields. Those most frequently used are the contraction mapping principle and Schauder's principle. Other results concerning completely continuous operators are also used. Recently, Browder [ 30] gave a survey of the applications to partial differential equations, mostly about existence and uniqueness of solutions and about iteration procedures, also in the case of noncompact operators and of nonexpansive mappings.

In recent years densifying (condensing) mappings proved to be very useful in the study of certain differential and integral equations. In particular, we discuss the existence of solutions of the following differential equation of neutral type:

$$x'(t) = f[t, x(t), x(t - h_1(t)), x'(t - h_2(t))] \qquad (1)$$

If the function $f(t,x,y,z)$ satisfies a Lipschitz condition in the variable x,y and z, with constant $k_x$, $k_y$ and $k_z$, respectively, with $k_z < 1$, then, under minor additional assumptions, the question of the existence of solution is easily reduced to the contraction mapping principle. As it was noted in [ 5], this problem can also be reduced to the Schauder's principle, by another method. Here we shall dispense with the Lipschitz condition in the variables x and y. To prove the existence theorem in this case, Badoev and Sadovsky [ 5] have used the fixed point principle for condensing mappings (Theorem 3.2.6.) which we state without going into much detail:

Let E be a Banach space and $D \subset E$. We shall consider (1) in conjunction with the initial condition

$$x(t) = x_o(t) \quad (-h < t \leq 0), \qquad (1')$$

where $x_o(t)$ is a fixed function defined on the (finite or infinite) semi-interval $(-h, 0]$. By a solution of the problem (1) - (1') we shall mean a function $x(t)$ $(-h < t \leq H)$ that satisfies the initial condition (1') and the following three requirements: a) $x(t)$ is continuous on $(-h,H]$; b) $x'(t)$ exists almost everywhere on $(-h,H]$ and is pth power integrable, $p \geq 1$; c) almost everywhere on $[0,H]$

$$x'(t) = f[t,x(t), x(t - h_1(t)), x'(t - h_2(t))].$$

We shall denote by $\mathfrak{C}(0,H)$ the set of continuous functions on $[0,H]$

having a derivative that is pth power integrable; this set becomes a Banach space with the natural linear operations if we put $||x||_E = ||x||_C + ||x'||_{L_p}$. For any function $x(t) \in E(0,H)$ we put

$$\tilde{x}(t) = \begin{cases} x_0(t), & -h < t < 0, \\ x(t), & 0 \leq t \leq H. \end{cases}$$

Together with the problem (1) - (1') we consider the following operator equation in the space $E(0,H)$:

$$y = Iy, \tag{2}$$

where the operator $I$ is defined by the formula

$$Iy(t) = x_0 + \int_0^t f[s, y(s), \tilde{y}(s - h_1(s)), \tilde{y}'(s - h_2(s))]ds$$
$$(x_0 = x_0(0)).$$

It is not difficult to verify that if the function $x_0(t)$ is continuous and its derivative is pth power integrable, then the equation (2) is equivalent to the problem (1) - (1') in the following sense: if $x(t)$ is a solution of the problem (1) - (1') then its restriction $y(t)$ to the segment $[0,H]$ is a solution of the equation (2) and, conversely, if $y(t)$ is a solution of the equation (2) then the function $x(t) = \tilde{y}(t)$ is a solution of the problem (1) - (1').

Badoev and Sadovsky [ 5] have established the following properties of the operator I.

Lemma 3.2.2. Let $E_0$ be the set of functions in $E(0,H)$ that satisfy the condition $x(0) = x_0$. Suppose that the functions $x_0(t)$, $h_1(t)$, $h_2(t)$ and $f(t, x, y, z)$ satisfy the following requirements: (I) $x_0(t)$ is continuous and bounded ; moreover $x_0'(t)$ is pth power integrable on $(-H,0]$; (II) $-H + t \leq h_i(t) < h + t (i = 1,2,; 0 \leq t \leq H)$; (III) $h_1(t)$ and $h_2(t)$

are measurable on $[0,H]$; (IV) the function $q(t) = t - h_2(t)$ is such that a) the inverse image of every set of measure zero is measurable and b) for any measurable set $E \subseteq [0,H]$ satisfying the condition $q(E) \subseteq [0,H]$, we have the inequality $\mu E \leq r\, \mu q(E)$ (where the number $r$ does not depend on $E$); (V) $f(t, x, y, z)$ is defined for $0 \leq t \leq H$ and all real $x$, $y$ and $z$; (VI) $f(t, x, y, z)$ is measurable in $t$ for any fixed $x$, $y$ and $z$; (VII) $f(t, x, y, z)$ is measurable in the pair $x$, $y$ for fixed $t$ and $z$; (VIII) $f(t, x, y, z)$ satisfies a Lipschitz condition in $z$:

$$|f(t, x, y, z_1) - f(t, x, y, z_2)| \leq k|z_1 - z_2|;$$

(IX) for any $R > 0$ we can find a function $m_R(D) \in L_p(0,H)$ such that

$$|f(t, x, y, z)| \leq m_R(t)$$

$$(0 \leq t \leq H;\ |x - x_o|,\quad |y - x_o| \leq R;\ -\infty < z < \infty).$$

Then the operator $I$ is continuous from $E_o$ into $E_o$.

<u>Lemma 3.2.3</u>. If the conditions of Lemma 3.2.2. are satisfied, and in addition suppose that the following condition is satisfied:

$$\text{(X)}\quad kr^{1/p} < \begin{cases} 1, & \text{if } p > 1, \\ 1/2, & \text{if } p = 1. \end{cases}$$

Then the operator $I$ is condensing on $D$, if $H$ is sufficiently small.

From Lemmas 3.2.2, 3.2.3 and Theorem 3.2.6., Badoev and Sadovsky [5] have obtained the following theorem on the solvability of the problem (1) - (1').

<u>Theorem 3.2.21.</u> Let the functions $x_o(t)$, $h_1(t)$, $h_2(t)$ and $f(t, x, y, z)$ satisfy the conditions (I) - (X). Then the problem (1) - (1') has a solution $x(t)$ that is defined on some semi-interval $(-h,H]$ $(H > 0)$.

We also mention that there are many areas of physics and economics where the fixed point theory is applicable. Many of the methods used are based mainly on proving the convergence of the iterative sequence $\{x_n\}$, with $x_{n+1} = T(x_n)$. There are many different ways of setting up an iteration scheme to obtain a fixed point of $T$, and which may converge faster than the iteration sequence $x_{n+1} = T(x_n)$. For example, we may replace $T$ by $T_\lambda$ (see chapter III) or we may take the mapping $T^{1/2} x^{1/2}$ (if this is defined). Another method is Newton's method (suitably extended to infinite dimensional spaces).

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